

## Well-posedness and error estimates for coupled systems of nonlocal conservation laws

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This article deals with the error estimates for numerical approximations of the entropy solutions of coupled systems of nonlocal hyperbolic conservation laws. The systems can be strongly coupled through the nonlocal coefficient present in the convection term. A fairly general class of fluxes is being considered, where the local part of the flux can be discontinuous at infinitely many points, with possible accumulation points. The aims of the paper are threefold: (1) Establishing existence of entropy solutions with rough local flux for such systems, by deriving a uniform BV bound on the numerical approximations; (2) Deriving a general Kuznetsov-type lemma (and hence uniqueness) for such systems with both smooth and rough local fluxes; (3) Proving the convergence rate of the finite volume approximations to the entropy solutions of the system as  $1/2$  and  $1/3$ , with homogeneous (in any dimension) and rough local parts (in one dimension), respectively. Numerical experiments are included to illustrate the convergence rates.

**Keywords:** nonlocal conservation laws; traffic flow; convergence rate; hyperbolic systems; adapted entropy.

### 1. Introduction

We here study the initial-value problem (IVP) for the coupled system of nonlocal hyperbolic conservation laws with discontinuous flux given by

$$\begin{aligned} \partial_t U + \partial_x (\mathbf{F}(\boldsymbol{\sigma}(x) \circ U) \circ \mathbf{v}(\boldsymbol{\mu} \circledast U)) &= \mathbf{0}, \quad (t, x) \in (0, T) \times \mathbb{R}, \\ U(0, x) &= U_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

Here the unknown is  $U = (U^1, \dots, U^N) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^N$ , with  $\circ$  denoting the Hadamard (component-wise) product. Furthermore,  $\boldsymbol{\sigma} = (\sigma^1, \dots, \sigma^N) : \mathbb{R} \rightarrow \mathbb{R}^N$ , where each component  $\sigma^k$  is allowed to have infinitely many discontinuities, including accumulation points. The system is coupled due to the nonlocal terms  $\boldsymbol{\mu} \circledast U : \mathbb{R} \rightarrow \mathbb{R}^{N^2}$ , with  $\boldsymbol{\mu}$  a smooth  $N \times N$  matrix, and  $(\boldsymbol{\mu} \circledast U)^k := (\mu^{k,1} * U^1, \dots, \mu^{k,N} * U^N)$  with convolution  $*$  over space only. We have  $\mathbf{v} = (v^1, \dots, v^N)$ , with each  $v^k : \mathbb{R}^N \rightarrow \mathbb{R}$ . The function  $\mathbf{F} = (f^1, \dots, f^N) : \mathbb{R} \rightarrow \mathbb{R}^N$  is Lipschitz in each component, with  $\mathbf{F}(\mathbf{0}) = \mathbf{0}$  and  $\mathbf{v}(\mathbf{0}) = \mathbf{0}$ , and  $|\boldsymbol{\sigma}|$  is strictly bounded away from zero. Precise assumptions are given below.

Hyperbolic conservation laws are partial differential equations (PDEs) that describe the conservation of  $U$  in a physical system, where the temporal rate of change of  $U$  is exactly balanced by the spatial rate of change of the flux function. The scalar case, i.e., with  $N = 1$ , is well understood, also in multidimensions, while the case of systems, i.e.,  $N > 1$ , is considerably harder even in one dimension. The use of Hadamard products restricts the coupling in (1.1) to the convolution term  $v(\mu \otimes U)$ , allowing us to use the scalar theory. Hyperbolic conservation laws were originally introduced in the context of fluid dynamics; however, recently these equations have been applied as models covering a wide range of phenomena, and this has necessitated a considerably wider class of flux functions. Here we include two effects—we let the flux function depend on space in a discontinuous manner through the function  $\sigma(x)$ , and, furthermore, we let the flux function depend on a local average of (the full vector)  $U$  in the form of the convolution term  $v(\mu \otimes U)$ .

Some of the recent applications include crowds (Colombo & Lécureux-Mercier, 2011; Colombo *et al.*, 2011, 2012), traffic (Colombo *et al.*, 2011; Blandin & Goatin, 2016), opinion formation (Aletti *et al.*, 2007), sedimentation models (Betancourt *et al.*, 2011), structured population dynamics (Perthame, 2007), supply chain models (Colombo *et al.*, 2011), granular material dynamics (Shen, 2012) and conveyor belt dynamics (Göttlich *et al.*, 2014). The case of smooth  $F$  has been well studied in literature, a nonexhaustive list being Colombo & Lécureux-Mercier (2011); Colombo *et al.* (2011, 2012); Aggarwal *et al.* (2015); Amorim *et al.* (2015); Aggarwal & Goatin (2016); Blandin & Goatin (2016); Friedrich *et al.* (2022), and various references therein.

Let us explain why it is necessary to study the extensions given in this paper. As we know, roads may experience rapidly changing local conditions, and when we model traffic flow by hyperbolic conservation laws, we allow for this by having a nonsmooth, spatially depending function  $\sigma(x)$  in the flux function. Furthermore, more practical problems like the multi-agent systems, as well as laser technology, dynamics of multiple crowds/cars, lane formation, sedimentation and conveyor belts, see, for example, Colombo & Marcellini (2015); Colombo *et al.* (2016); Gugat *et al.* (2016), one needs to analyse coupled systems of nonlocal conservation laws with discontinuous flux.

The literature of the following *local* counterpart of (1.1) with  $N = 1$ , viz.

$$U_t + (F(x, U)v(U))_x = 0,$$

is quite rich, with the results spanning to  $F$  being a nonlinear function of  $U$  and having general spatial discontinuities; see, for example, Adimurthi & Veerappa Gowda (2004, 2005, 2007); Perthame (2007); Andreianov *et al.* (2010); Aggarwal *et al.* (2020, 2021); Ghoshal *et al.* (2020, 2022a,b) and references therein.

However, the nonlocal counterpart (1.1) remains far from being settled, with the only exceptions being Keimer & Pflug (2021), Aggarwal & Vaidya (2023) and Chiarello *et al.* (2023). References Keimer & Pflug (2021); Chiarello *et al.* (2023) settle the problem for the case  $F(x, U) = s(x)U$ , with Keimer & Pflug (2021) establishing this via a fixed-point argument with a bounded discontinuous  $s$  and monotone nonsymmetric kernels, and Chiarello *et al.* (2023) settling the same via a convergent numerical scheme for monotonically decreasing  $s$ , with a single spatial discontinuity and  $\mu$  being a downstream kernel. The question of well-posedness for the cases of  $F(x, U) = s(x)U$  with  $s$  having nondiscrete discontinuities, and for the case of  $F(x, U) = s(x)f(U)$  with a nonlinear  $f$  and  $s$  having single or finitely many spatial discontinuities, with or without any constraints on monotonicity of  $v$  or  $\mu$ , remained unexplored and unsettled for a long time. This question was recently settled in Aggarwal & Vaidya (2023), where the well-posedness was explored in a very general setup with  $F(x, U) = f(s(x)U)$ , where  $f$  is nonlinear and there are no additional constraints on  $v$ ,  $\mu$  and the discreteness or monotonicity of spatial discontinuities.

The existence of the entropy solutions for (1.1) for  $N \geq 1$  with sufficiently smooth  $\sigma$  has been studied in [Betancourt et al. \(2011\)](#); [Göttlich et al. \(2014\)](#); [Aggarwal et al. \(2015\)](#); [Colombo & Marcellini \(2015\)](#); [Chiarello & Goatin \(2023\)](#), while the uniqueness has been settled in [Colombo & Marcellini \(2015\)](#); [Colombo et al. \(2016\)](#); [Colombo & Rossi \(2018\)](#), for example. The question of existence as well as uniqueness of the entropy solutions for the system (1.1) with  $\sigma$  having any kind of spatial discontinuities remains unexplored and unsettled as of now. This article settles these questions for IVPs (1.1), which we can write on component form as

$$\partial_t U^k + \partial_x \left( f^k(\sigma^k(x) U^k) v^k((\mu * U)^k) \right) = 0, \quad (t, x) \in Q_T := (0, T) \times \mathbb{R}, \quad (1.2)$$

$$U^k(0, x) = U_0^k(x), \quad x \in \mathbb{R}, \quad (1.3)$$

for each  $k = 1, 2, \dots, N$ . Here  $T$  is a given positive time. The assumptions that  $\sigma^k \in \text{BV}(\mathbb{R}, \mathbb{R}^+)$  and  $\inf_{x \in \mathbb{R}} \sigma^k(x) > 0$  are justified, being the essential technical assumptions required for well-posedness of scalar *local* counterpart of (1.2)–(1.3), see [Panov \(2009\)](#); [Towers \(2020\)](#); [Ghoshal et al. \(2022a,b\)](#) and references therein. As stated earlier, the system (1.2)–(1.3) has been studied in [Keimer & Pflug \(2021\)](#) only for the linear case  $f^k(u) = u$ , using a fixed point argument. However, in the setup of this paper,  $f^k$  is nonlinear and consequently, there can be multiple weak solutions, like in the local case. Hence, an entropy condition is required to single out the unique solution.

In this article, this question will be settled in Section 4, where it will be shown that any two *entropy* solutions (cf. (2)) of (1.2) corresponding to the same initial data (1.3) are in fact equal, and the approximate solutions obtained from the finite volume schemes proposed in Section 3 satisfy the discrete form of the entropy inequality (2). As a result, one will obtain that the approximate solutions converge to the unique entropy solution of (1.2)–(1.3). This uniqueness result is novel and is not dealt with in existing uniqueness results of [Bürger et al. \(2004\)](#); [Blandin & Goatin \(2016\)](#); [Colombo et al. \(2016\)](#); [Colombo & Rossi \(2018\)](#); [Keimer & Pflug \(2021\)](#), which deal with either local part of the flux being linear in state variable and possibly discontinuous in space variable, or nonlinear smooth *local* fluxes only. At the time of writing of this article, this is the first result establishing the existence and uniqueness of solutions of such coupled systems with discontinuous flux. It is to be noted that no technical restrictions on monotonicity of kernel or  $\sigma$  have been made in the analysis, and hence the article presents the first result in a very general setting. However, to exercise this method, we need a uniform BV bound on the entropy solution  $U$  and its numerical approximation  $U_\Delta$ , which is one of the objectives of this article.

The final objective of this article is the error analysis of our finite volume approximations, proposed in Section 3. To the best of our knowledge, this is the first result towards error analysis of numerical approximations for coupled systems of nonlocal conservation laws, including the case of smooth  $\sigma$ . However, the literature on error estimates for local conservation laws is quite rich, see [Kuznetsov \(1976\)](#); [Karlsen \(1994\)](#); [Sabac \(1997\)](#); [Karlsen & Risebro \(2001\)](#); [Cancès & Seguin \(2012\)](#); [Badwaik & Ruf \(2020\)](#); [Ghoshal et al. \(2022b\)](#) for  $N = 1$ . This is achieved by following the techniques of [Kuznetsov \(1976\)](#), where the so-called doubling of variables argument is exploited to express the error between the entropy solution  $U = u$  (i.e.,  $N = 1$ ) and its numerical approximation  $u_\Delta$  in terms of their relative entropy. A first result towards scalar nonlocal conservation laws, i.e., with  $N = 1$ , with a homogeneous smooth local part of the flux, was recently done in [Aggarwal et al. \(2023\)](#). In general, for any  $N > 1$ , the local counterpart of (1.2) is fairly less explored with a limited list being [Bressan et al. \(2021\)](#); [Bressan & Shen \(2021\)](#). This paper establishes error estimates for  $N > 1$  for the nonlocal counterpart (1.2) with smooth local fluxes in any dimension, and with a fairly large class of rough local fluxes (precise assumptions described later) in one dimension.

Thus, the novelties of this article for the system of nonlocal conservation laws (1.1), where  $k^{\text{th}}$  equation is given by (1.2), are as follows:

- Existence of the *adapted entropy* solutions via finite volume approximations.
- A Kuznetsov-type lemma and thereby the uniqueness of the entropy solution.
- Convergence rate estimates for the proposed numerical schemes.

The paper is organized as follows. In Section 2, we introduce some definitions and notation to be used in the article. In Section 3, we present the convergence of the Godunov type and the Lax–Friedrichs type scheme of the approximation of the IVP (1.2)–(1.3). In Section 4, we prove the uniqueness result, and, in Section 5, we discuss the convergence rates. Finally, in Section 6, we present some numerical experiments that illustrate the theory.

## 2. Definitions and notation

Throughout this paper, we use the following notation:

- (1) For  $z := (z^1, \dots, z^N) \in \mathbb{R}^N$ , let  $\|z\| := \sum_{k=1}^N |z^k|$  denote the usual 1-norm.
- (2) For  $f := (f^1, \dots, f^N) \in (L^1(\mathbb{R}))^N$ ,  $\|f\|_{(L^1(\mathbb{R}))^N} := \sum_{k=1}^N \|f^k\|_{L^1(\mathbb{R})}$  be the norm on the product space.
- (3) For the matrix valued function  $\mu \in L^\infty(\mathbb{R}; \mathbb{R}^{N^2})$ ,

$$\|\mu\|_{(L^\infty(\mathbb{R}))^{N^2}} := \max_{1 \leq i, j \leq N} \|\mu^{ij}\|_{L^\infty(\mathbb{R})}.$$

In addition, if  $\mu \in C^2(\mathbb{R}; \mathbb{R}^{N^2})$ , then  $\mu' \in C^1(\mathbb{R}; \mathbb{R}^{N^2})$  and  $\mu'' \in C(\mathbb{R}; \mathbb{R}^{N^2})$  denote the component-wise derivative and second derivative, respectively.

- (4) For  $u : \bar{Q}_T \rightarrow \mathbb{R}$ ,  $U : \bar{Q}_T \rightarrow \mathbb{R}^N$  and  $\tau > 0$ ,

$$\begin{aligned} |u|_{\text{Lip}_t L_x^1} &:= \sup_{0 \leq t_1 < t_2 \leq T} \frac{\|u(t_1) - u(t_2)\|_{L^1(\mathbb{R})}}{|t_1 - t_2|}, \\ |U|_{(\text{Lip}_t L_x^1)^N} &:= \max_{1 \leq k \leq N} |U^k|_{\text{Lip}_t L_x^1}, \\ |U|_{(L_t^\infty \text{BV}_x)^N} &:= \max_{1 \leq k \leq N} \sup_{t \in [0, T]} TV(U^k(t, \cdot)), \\ \|U\|_{(L^\infty(\bar{Q}_T))^N} &:= \max_{1 \leq k \leq N} \|U^k\|_{L^\infty(\bar{Q}_T)}, \\ \|U\|_{(L^1(\bar{Q}_T))^N} &:= \sum_{k=1}^N \|U^k\|_{L^1(\bar{Q}_T)}, \\ \gamma(U, \tau) &:= \max_{1 \leq k \leq N} \sup_{\substack{|t_1 - t_2| \leq \tau \\ 0 \leq t_1 \leq t_2 \leq T}} \|U^k(t_1) - U^k(t_2)\|_{L^1(\mathbb{R})}. \end{aligned} \tag{2.1}$$

We will make the following assumptions throughout this work:

- (H1)  $f^k \in \text{Lip}(\mathbb{R})$  with  $f^k(0) = 0$ ;
- (H2)  $v^k \in (C^2 \cap \text{BV} \cap W^{2,\infty})(\mathbb{R}^N, \mathbb{R}^+)$ ,  $\mu \in (C^2 \cap \text{BV} \cap W^{2,\infty})(\mathbb{R}, \mathbb{R}^{N^2})$ , with  $v^k(\mathbf{0}) = 0$ ;
- (H3)  $\sigma^k \in (\text{BV} \cap L^1)(\mathbb{R})$  with  $\inf_{x \in \mathbb{R}} \sigma^k(x) > 0$ .

DEFINITION 2.1 A function  $\mathbf{U} \in (C([0, T]; L^1(\mathbb{R})) \cap L^\infty([0, T]; \text{BV}(\mathbb{R})))^N$  is an entropy solution of IVP (1.2)–(1.3), if for all  $k \in \{1, \dots, N\}$ , and for all  $\alpha \in \mathbb{R}$ ,

$$\begin{aligned} & \int_{Q_T} \left| U^k(t, x) - \frac{\alpha}{\sigma^k(x)} \right| \phi_t(t, x) \, dx \, dt \\ & + \int_{Q_T} \text{sgn}(\bar{U}^k(t, x) - \alpha) v^k((\mu * \mathbf{U})^k(t, x)) (f^k(\bar{U}^k(t, x)) - f^k(\alpha)) \phi_x(t, x) \, dx \, dt \\ & - \int_{Q_T} f^k(\alpha) (\text{sgn}(\bar{U}^k(t, x) - \alpha)) \partial_x v^k((\mu * \mathbf{U})^k(t, x)) \phi(t, x) \, dx \, dt \\ & + \int_{\mathbb{R}} \left| U_0^k(x) - \frac{\alpha}{\sigma^k(x)} \right| \phi(0, x) \, dx \geq 0, \quad (2.2) \end{aligned}$$

for all non-negative  $\phi \in C_c^\infty([0, T] \times \mathbb{R})$ , and where for every  $(t, x) \in \bar{Q}_T$  and for every  $k \in \{1, \dots, N\}$ ,  $\bar{U}^k(t, x) := U^k(t, x)\sigma^k(x)$ .

The following section is dedicated to the finite volume schemes approximating the above defined adapted entropy solutions for the IVP (1.2)–(1.3), and establishes their existence via the convergence of the finite volume scheme.

### 3. Existence of the entropy solution

For  $\Delta x, \Delta t > 0$  and  $\lambda = \Delta t / \Delta x$ , consider equidistant spatial grid points  $x_i := i\Delta x$  for  $i \in \mathbb{Z}$  and temporal grid points  $t^n := n\Delta t$  for integers  $n \in \{0, \dots, N_T\}$ , such that  $T = N_T \Delta t$ . Let  $\mathbb{1}_i(x)$  denote the indicator function of  $C_i := [x_{i-1/2}, x_{i+1/2})$ , where  $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$  and let  $\chi^n(t)$  denote the indicator function of  $C^n := [t^n, t^{n+1})$ , and let  $C_i^n := C^n \times C_i$ . For each  $k \in \{1, \dots, N\}$ , we approximate the initial data and  $\sigma^k$  according to the following:

$$\begin{aligned} U_\Delta^k(0, x) & := \sum_{i \in \mathbb{Z}} \mathbb{1}_i(x) U_i^{k,0}, \quad x \in \mathbb{R}, \text{ where } U_i^{k,0} = \int_{C_i} U_0^k(x) \, dx, \quad i \in \mathbb{Z}, \\ \sigma_\Delta^k(x) & := \sum_{i \in \mathbb{Z}} \mathbb{1}_i(x) \sigma_i^k, \quad x \in \mathbb{R}, \text{ where } \sigma_i^k = \sigma^k(x_i), \quad i \in \mathbb{Z}. \end{aligned} \quad (3.1)$$

Further, we define a piecewise constant approximate solution  $\mathbf{U}_\Delta = (U_\Delta^1, \dots, U_\Delta^N)$ , i.e.,  $\mathbf{U}_\Delta \approx \mathbf{U}$ , to the IVP (1.2)–(1.3) by

$$U_\Delta^k(t, x) = U_i^{k,n} \text{ for } (t, x) \in C_i^n, \text{ where } n \in \{0, \dots, N_T\}, i \in \mathbb{Z}, k \in \{1, \dots, N\}.$$

The approximate solution  $U_\Delta$  is updated through the following marching formula:

$$\begin{aligned} U_i^{k,n+1} &:= U_i^{k,n} - \lambda \left[ F^k \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right), \bar{U}_i^{k,n}, \bar{U}_{i+1}^{k,n} \right) - F^k \left( v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right), \bar{U}_{i-1}^{k,n}, \bar{U}_i^{k,n} \right) \right] \\ &:= U_i^{k,n} - \lambda \left[ F_{i+1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_{i+1}^{k,n} \right) - F_{i-1/2}^{k,n} \left( \bar{U}_{i-1}^{k,n}, \bar{U}_i^{k,n} \right) \right] \\ &= H^k \left( v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right), v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right), \bar{U}_{i-1}^{k,n}, \bar{U}_i^{k,n}, \bar{U}_{i+1}^{k,n} \right), \end{aligned} \quad (3.2)$$

where

- (1) for each  $n \in \{0, \dots, N_T\}$ ,  $i \in \mathbb{Z}$  and  $k \in \{1, \dots, N\}$ ,  $\bar{U}_i^{k,n} := \sigma_i^k U_i^{k,n} \approx \sigma^k(x_i) U^k(x_i, t^n)$  (pointwise value of the right continuous representative) and which is extended to the function defined in  $Q_T$ , via

$$\bar{U}_\Delta^k(t, x) = \sum_{n=0}^{N_T-1} \sum_{i \in \mathbb{Z}} \mathbb{1}_i(x) \chi^n(t) \bar{U}_i^{k,n};$$

- (2) the convolution term is updated through the following formula:

$$\mathbf{c}_{i+1/2}^{k,n} = \left( c_{i+1/2}^{1,k,n}, \dots, c_{i+1/2}^{N,k,n} \right), \quad (3.3)$$

with, for each  $j \in \{1, \dots, N\}$ , we have  $c_{i+1/2}^{j,k,n}$  being computed through the following quadrature formula:

$$c_{i+1/2}^{j,k,n} = \Delta x \sum_{p \in \mathbb{Z}} \mu_{i+1/2-p}^{j,k} U_{p+1/2}^{j,n} \approx c^{j,k} \left( t^n, x_{i+1/2} \right) = \int_{\mathbb{R}} \mu^{j,k}(x_{i+1/2} - y) U_\Delta^j(t^n, y) dy,$$

with  $U_{p+1/2}^{j,k,n}$  being any convex combination of  $U_p^{j,k,n}$  and  $U_{p+1}^{j,k,n}$ , and  $\mu_{i+1/2}^{j,k} = \mu^{j,k}(x_{i+1/2})$ .

- (3)  $F^k$  is chosen so as to make  $H^k$  increasing in its last three arguments.

The convolution terms satisfy the following estimates:

LEMMA 3.1 For for each  $n \in \{0, \dots, N_T\}$ ,  $i \in \mathbb{Z}$  and  $k \in \{1, \dots, N\}$ , the convolution term (3.3) satisfies

$$\begin{aligned} \left\| \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right\| &\leq \mathcal{K}_1 \Delta x, \\ \left\| \mathbf{c}_{i+3/2}^{k,n} - 2\mathbf{c}_{i+1/2}^{k,n} + \mathbf{c}_{i-1/2}^{k,n} \right\| &\leq \mathcal{K}_2 \Delta x^2, \end{aligned}$$

where

$$\begin{aligned} \mathcal{K}_1 &= \left\| \boldsymbol{\mu}' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| \mathbf{U}_\Delta(t^n) \right\|_{(L^1(\mathbb{R}))^N}, \\ \mathcal{K}_2 &= 2 \left\| \boldsymbol{\mu}'' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| \mathbf{U}_\Delta(t^n) \right\|_{(L^1(\mathbb{R}))^N}. \end{aligned}$$

*Proof.* The proof follows by repeating the arguments of Amorim *et al.* (2015, Prop. 2.8) and Aggarwal *et al.* (2015, Lemma A.2) in every component of the vector  $\mathbf{c}_{i+1/2}^{k,n}$ .  $\square$

For each  $k \in \{1, \dots, N\}$ , we modify the schemes proposed in [Aggarwal \*et al.\* \(2015\)](#); [Aggarwal & Vaidya \(2023\)](#) and define the numerical fluxes that are the nonlocal extensions of the well-known monotone fluxes for local conservation laws as described below:

(1) **Lax–Friedrichs type flux:**

$$F_{LF}^k(a, b, c) = \frac{a}{2} \left( f^k(b) + f^k(c) \right) - \frac{\theta_k}{2\lambda} (c - b),$$

$$\theta_k \in \left( 0, \frac{2}{3 \|\sigma^k\|_{L^\infty(\mathbb{R})}} \right).$$
(3.4)

Further,  $\Delta t$  is chosen in order to satisfy the following Courant–Friedrichs–Lewy (CFL) like condition:

$$\lambda \leq \min_k \left\{ \frac{\min \left( 1, 4 - 6\theta_k \|\sigma^k\|_{L^\infty(\mathbb{R})}, 6\theta_k \|\sigma^k\|_{L^\infty(\mathbb{R})} \right)}{1 + 6 \|\sigma^k\|_{L^\infty(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} \|v^k\|_{L^\infty(\mathbb{R})}} \right\}.$$
(3.5)

(2) **Godunov type flux:**

$$F_{\text{Godunov}}^k(a, b, c) = a F_{\text{Godunov}}^k(b, c),$$

where  $F_{\text{Godunov}}^k$  is the Godunov flux for the corresponding local conservation law with discontinuous flux  $f^k(x, u) = f^k(\sigma^k(x)u)$ . Further,  $\Delta t$  is chosen in order to satisfy the following CFL condition:

$$\lambda \max_k \left\{ \|\sigma^k\|_{L^\infty(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} \|v^k\|_{L^\infty(\mathbb{R})} \right\} \leq \frac{1}{6}.$$
(3.6)

The scheme (3.2) can now be rewritten in an incremental form as (see [Ghoshal \*et al.\*, 2022a](#); [Aggarwal & Vaidya, 2023](#))

$$\begin{aligned} \bar{U}_i^{k,n+1} &= \bar{U}_i^{k,n} + \bar{U}_i^{k,n+1} - \bar{U}_i^{k,n} \\ &= \bar{U}_i^{k,n} - \lambda \sigma_i^k \left[ F_{i+1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_{i+1}^{k,n} \right) - F_{i-1/2}^{k,n} \left( \bar{U}_{i-1}^{k,n}, \bar{U}_i^{k,n} \right) \right] \\ &= \bar{U}_i^{k,n} - a_{i-1/2}^{k,n} \Delta_- \bar{U}_i^{k,n} + b_{i+1/2}^{k,n} \Delta_+ \bar{U}_i^{k,n} \\ &\quad - \lambda \sigma_i^k \left( F_{i+1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_i^{k,n} \right) - F_{i-1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_i^{k,n} \right) \right), \end{aligned}$$
(3.7)

where  $\Delta_+ z_i^{k,n} = z_{i+1}^{k,n} - z_i^{k,n} = \Delta_- z_{i+1}^{k,n}$ , for any vector  $z$ , and

$$a_{i+1/2}^{k,n} = \lambda \sigma_i^k \frac{F_{i+1/2}^{k,n} \left( \bar{U}_{i+1}^{k,n}, \bar{U}_{i+1}^{k,n} \right) - F_{i+1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_{i+1}^{k,n} \right)}{\Delta_+ \bar{U}_i^{k,n}},$$

$$b_{i+1/2}^{k,n} = \lambda \sigma_i^k \frac{F_{i+1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_i^{k,n} \right) - F_{i+1/2}^{k,n} \left( \bar{U}_i^{k,n}, \bar{U}_{i+1}^{k,n} \right)}{\Delta_+ \bar{U}_i^{k,n}}.$$

The following theorem establishes the existence and the regularity of the entropy solutions of IVP (1.1):

**THEOREM 3.2** [Stability of the finite volume approximations] Assume that (H1)–(H3) hold. For  $0 < t \leq T$ , fix a non-negative initial data  $U_0 \in ((L^1 \cap BV)(\mathbb{R}))^N$ . Then, there exist constants  $\mathcal{K}_i$ ,  $i = 3, \dots, 6$ , independent of the mesh size  $\Delta x$  such that the sequence of approximations  $U_\Delta$  defined by (3.2) under an appropriate CFL condition (cf. (3.5) and (3.6) for the Lax–Friedrichs and Godunov type scheme, respectively), satisfy:

(1) Positivity:

$$U_\Delta(t, x) \geq 0. \quad (3.8)$$

(2)  $L^1$  estimate:

$$\|U_\Delta(t)\|_{(L^1(\mathbb{R}))^N} = \|U_0\|_{(L^1(\mathbb{R}))^N}. \quad (3.9)$$

(3)  $L^\infty$  estimate:

$$\|\bar{U}_\Delta(t)\|_{(L^\infty(\mathbb{R}))^N} \leq \exp(\mathcal{K}_3 t) \|\bar{U}_\Delta(0)\|_{(L^\infty(\mathbb{R}))^N}. \quad (3.10)$$

Furthermore,

$$\|U_\Delta(t)\|_{(L^\infty(\mathbb{R}))^N} \leq \frac{\|\sigma\|_{(L^\infty(\mathbb{R}))^N}}{\inf_{x \in \mathbb{R}} \sigma^k(x)} \exp(\mathcal{K}_3 t) \|U_0\|_{(L^\infty(\mathbb{R}))^N}.$$

(4) BV estimate: for each  $n \in \{0, \dots, N_T\}$  and  $k \in \{1, \dots, N\}$ ,

$$\sum_{i \in \mathbb{Z}} \left| \Delta_+ \bar{U}_i^{k,n} \right| \leq \exp(\mathcal{K}_4 t) \left( \sum_{i \in \mathbb{Z}} |\Delta_+ \bar{U}_i^{k,0}| + \mathcal{K}_5 t \right). \quad (3.11)$$

(5) Time estimate: for each  $n \in \{0, \dots, N_T - 1\}$  and  $k \in \{1, \dots, N\}$ ,

$$\Delta x \sum_{i \in \mathbb{Z}} \left| U_i^{k,n+1} - U_i^{k,n} \right| \leq \mathcal{K}_6 \Delta t. \quad (3.12)$$

(6) Discrete entropy inequality: for each  $n \in \{0, \dots, N_T - 1\}$ ,  $i \in \mathbb{Z}$ , and  $k \in \{1, \dots, N\}$ , and for all  $\alpha \in \mathbb{R}$ ,

$$\begin{aligned} & \left| U_i^{k,n+1} - \frac{\alpha}{\sigma_i^k} \right| - \left| U_i^{k,n} - \frac{\alpha}{\sigma_i^k} \right| + \lambda \left( G_{i+1/2}^{k,n} \left( U_i^{k,n}, U_{i+1}^{k,n}, \alpha \right) - G_{i-1/2}^{k,n} \left( U_{i-1}^{k,n}, U_i^{k,n}, \alpha \right) \right) \\ & + \lambda \operatorname{sgn} \left( U_i^{k,n+1} - \frac{\alpha}{\sigma_i^k} \right) f^k(\alpha) \left( v^k \left( c_{i+1/2}^{k,n} \right) - v^k \left( c_{i-1/2}^{k,n} \right) \right) \leq 0, \end{aligned}$$



where

$$G_{i+1/2}^{k,n}(p, q, \alpha) := F_{i+1/2}^{k,n} \left( \sigma_i^k \left( p \vee \frac{\alpha}{\sigma_i^k} \right), \sigma_{i+1}^k \left( q \vee \frac{\alpha}{\sigma_{i+1}^k} \right) \right) - F_{i+1/2}^{k,n} \left( \sigma_i^k \left( p \wedge \frac{\alpha}{\sigma_i^k} \right), \sigma_{i+1}^k \left( q \wedge \frac{\alpha}{\sigma_{i+1}^k} \right) \right),$$

with  $u \vee v = \max(u, v)$ ,  $u \wedge v = \min(u, v)$ .

- (7) Up to a subsequence, the numerical approximations  $U_\Delta$  converge to an entropy solution  $U \in (C([0, T]; L^1(\mathbb{R})) \cap L^\infty([0, T]; BV(\mathbb{R})))^N$ .

*Proof.* The proof of the above theorem follows by invoking the incremental form of the scheme and can be done in the spirit of Aggarwal *et al.* (2015); Amorim *et al.* (2015); Aggarwal & Vaidya (2023) with appropriate modifications. We skip the detailed proof and present only the key estimates below. Let  $n \in \{0, \dots, N_T\}$ ,  $i \in \mathbb{Z}$  and  $k \in \{1, \dots, N\}$ .

- (1) Using the CFL condition, cf. (3.5) and (3.6), it is straightforward to see that the incremental coefficients satisfy,  $0 \leq a_{i+1/2}^{k,n}, b_{i+1/2}^{k,n} \leq \frac{1}{3}$ . Moreover, using  $U^{k,n} \geq 0$ ,  $\bar{U}^{k,n} \geq 0$  and Lemma 3.1, the CFL condition (cf. (3.5) and (3.6)) can again be invoked to show:

$$\begin{aligned} \sigma_i^k \left| F_{i+1/2}^{k,n}(\bar{U}_i^{k,n}, \bar{U}_i^{k,n}) - F_{i-1/2}^{k,n}(\bar{U}_i^{k,n}, \bar{U}_i^{k,n}) \right| &= \sigma_i^k \bar{U}_i^{k,n} \left| v^k(c_{i+1/2}^{k,n}) - v^k(c_{i-1/2}^{k,n}) \right| \\ &\leq 2\sigma_i^k \bar{U}_i^{k,n} \|v^k\|_{L^\infty(\mathbb{R})} \\ &\leq \frac{1}{3\lambda} \bar{U}_i^{k,n}. \end{aligned}$$

Finally,

$$\begin{aligned} \bar{U}_i^{k,n+1} &= \left(1 - a_{i-1/2}^{k,n} - b_{i+1/2}^{k,n}\right) \bar{U}_i^{k,n} + a_{i-1/2}^{k,n} \bar{U}_{i-1}^{k,n} + b_{i+1/2}^{k,n} \bar{U}_{i+1}^{k,n} \\ &\quad - \lambda \sigma_i^k \left( F_{i+1/2}^{k,n}(\bar{U}_i^{k,n}, \bar{U}_i^{k,n}) - F_{i-1/2}^{k,n}(\bar{U}_i^{k,n}, \bar{U}_i^{k,n}) \right) \\ &\geq \left(\frac{2}{3} - a_{i-1/2}^{k,n} - b_{i+1/2}^{k,n}\right) \bar{U}_i^{k,n} + a_{i-1/2}^{k,n} \bar{U}_{i-1}^{k,n} + b_{i+1/2}^{k,n} \bar{U}_{i+1}^{k,n} \geq 0. \end{aligned}$$

Since  $\sigma^k > 0$ , the result follows.

- (2) Since the scheme is conservative, the result follows from the positivity of the scheme, cf. (3.8).  
 (3) Using Lemma 3.1, we obtain:

$$\begin{aligned} F_{i+1/2}^{k,n}(\bar{U}_i^{k,n}, \bar{U}_i^{k,n}) - F_{i-1/2}^{k,n}(\bar{U}_i^{k,n}, \bar{U}_i^{k,n}) &= f^k(\bar{U}_i^{k,n}) \left( v^k(c_{i+1/2}^{k,n}) - v^k(c_{i-1/2}^{k,n}) \right) \\ &\leq |f^k|_{\text{Lip}(\mathbb{R})} \bar{U}_i^{k,n} |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_1 \Delta x. \end{aligned} \tag{3.13}$$

Now, using the incremental form of the scheme, cf. (3.7), and the inequality (3.13), we get

$$\begin{aligned}
\left| \overline{U}_i^{k,n+1} \right| &\leq \left( 1 - a_{i-1/2}^{k,n} - b_{i+1/2}^{k,n} \right) \overline{U}_i^{k,n} + a_{i-1/2}^{k,n} \overline{U}_{i-1}^{k,n} \\
&\quad + b_{i+1/2}^{k,n} \overline{U}_{i+1}^{k,n} + \Delta t \left\| \sigma^k \right\|_{L^\infty(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_1 \overline{U}_i^{k,n} \\
&\leq \left( 1 - a_{i-1/2}^{k,n} - b_{i+1/2}^{k,n} \right) \left\| \overline{U}_\Delta^k(t^n) \right\|_{L^\infty(\mathbb{R})} \\
&\quad + a_{i-1/2}^{k,n} \left\| \overline{U}_\Delta^k(t^n) \right\|_{L^\infty(\mathbb{R})} + b_{i+1/2}^{k,n} \left\| \overline{U}_\Delta^k(t^n) \right\|_{L^\infty(\mathbb{R})} \\
&\quad + \Delta t \left\| \sigma^k \right\|_{L^\infty(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_1 \left\| \overline{U}_\Delta^k(t^n) \right\|_{L^\infty(\mathbb{R})} \\
&\leq \left\| \overline{U}_\Delta^k(t^n) \right\|_{L^\infty(\mathbb{R})} \left( 1 + \Delta t \mathcal{K}_1 \left\| \sigma^k \right\|_{L^\infty(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\| \overline{U}_\Delta^k(t^n) \right\|_{L^\infty(\mathbb{R})} &\leq \left\| \overline{U}_\Delta^k(0) \right\|_{L^\infty(\mathbb{R})} \left( 1 + \Delta t \mathcal{K}_1 \left\| \sigma^k \right\|_{L^\infty(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \right)^n \\
&\leq \exp(\mathcal{K}_3 t^n) \left\| \overline{U}_\Delta^k(0) \right\|_{L^\infty(\mathbb{R})},
\end{aligned}$$

so that (3.10) holds. Furthermore, since  $\overline{U}_\Delta$  is bounded and  $\sigma^k > 0$ , the approximate function  $U_\Delta$  is also bounded.

- (4) Because of the incremental form of the scheme, we get, by taking the forward difference in the spatial direction, that

$$\begin{aligned}
\Delta_+ \overline{U}_i^{k,n+1} &= \Delta_+ \overline{U}_i^{k,n} \left( 1 - a_{i+1/2}^{k,n} - b_{i+1/2}^{k,n} \right) + a_{i-1/2}^{k,n} \Delta_- \overline{U}_i^{k,n} + b_{i+3/2}^{k,n} \Delta_+ \overline{U}_{i+1}^{k,n} \\
&\quad - \lambda \sigma_{i+1}^k \left( F_{i+3/2}^{k,n} \left( \overline{U}_{i+1}^{k,n}, \overline{U}_{i+1}^{k,n} \right) - F_{i+1/2}^{k,n} \left( \overline{U}_{i+1}^{k,n}, \overline{U}_{i+1}^{k,n} \right) \right) \\
&\quad + \lambda \sigma_i^k \left( F_{i+1/2}^{k,n} \left( \overline{U}_i^{k,n}, \overline{U}_i^{k,n} \right) - F_{i-1/2}^{k,n} \left( \overline{U}_i^{k,n}, \overline{U}_i^{k,n} \right) \right) \\
&:= A_1 + A_2 + A_3.
\end{aligned}$$

Since the incremental coefficients satisfy  $0 \leq a_{i+1/2}^{k,n}, b_{i+1/2}^{k,n} \leq \frac{1}{3}$ , we can repeat Harten's argument (see Holden & Risebro, 2015, Lemma 3.12) to get

$$\sum_{i \in \mathbb{Z}} |A_1| \leq \sum_{i \in \mathbb{Z}} \left| \Delta_+ \overline{U}_i^{k,n} \right|.$$

Adding and subtracting  $\lambda \sigma_i^{k,f^k} \left( \overline{U}_i^{k,n} \right) \left( v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right)$ , we obtain

$$A_2 + A_3 = -\lambda \left( \sigma_{i+1}^k f^k \left( \overline{U}_{i+1}^{k,n} \right) - \sigma_i^k f^k \left( \overline{U}_i^{k,n} \right) \right) \left( v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right) \\ - \lambda \sigma_i^k f^k \left( \overline{U}_i^{k,n} \right) \left( v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - 2v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) + v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right).$$

Further,

$$|A_2 + A_3| \leq \lambda \left| \sigma_{i+1}^k f^k \left( \overline{U}_{i+1}^{k,n} \right) - \sigma_i^k f^k \left( \overline{U}_i^{k,n} \right) \right| |v^k|_{\text{Lip}(\mathbb{R})} \left\| \mathbf{c}_{i+3/2}^{k,n} - \mathbf{c}_{i+1/2}^{k,n} \right\| \\ + \lambda \left| \sigma_i^k \right| |f^k|_{\text{Lip}(\mathbb{R})} \left| \overline{U}_i^{k,n} \right| \left| v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - 2v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) + v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right|.$$

Consider next

$$\left| \sigma_{i+1}^k f^k \left( \overline{U}_{i+1}^{k,n} \right) - \sigma_i^k f^k \left( \overline{U}_i^{k,n} \right) \right| \\ \leq |f^k|_{\text{Lip}(\mathbb{R})} \left| \Delta_+ \overline{U}_i^{k,n} \right| \left\| \sigma_\Delta^k \right\|_{L^\infty(\mathbb{R})} + |f^k|_{\text{Lip}(\mathbb{R})} \left\| \overline{U}_\Delta^{k,n} \right\|_{L^\infty(\mathbb{R})} \left| \Delta_+ \sigma_i^k \right|.$$

Now we estimate the term

$$v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - 2v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) + v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \\ = \nabla v^k \left( \xi_{i+1}^{k,n} \right) \cdot \left( \mathbf{c}_{i+3/2}^{k,n} - \mathbf{c}_{i+1/2}^{k,n} \right) - \nabla v^k \left( \xi_i^{k,n} \right) \cdot \left( \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right),$$

where  $\nabla$  denotes the gradient and  $\xi_{i+1}^{k,n} \in I(\mathbf{c}_{i+1/2}^{k,n}, \mathbf{c}_{i+3/2}^{k,n})$  (here  $I(\mathbf{a}, \mathbf{b})$  denotes a rectangular box with sides parallel to the coordinate planes and  $\mathbf{a}$  and  $\mathbf{b}$  at opposite corners) and  $\xi_i^{k,n} \in I(\mathbf{c}_{i-1/2}^{k,n}, \mathbf{c}_{i+1/2}^{k,n})$ . Now, adding and subtracting  $\nabla v^k \left( \xi_{i+1}^{k,n} \right) \cdot \left( \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right)$ , we have

$$v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - 2v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) + v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \\ = \nabla v^k \left( \xi_{i+1}^{k,n} \right) \cdot \left( \mathbf{c}_{i+3/2}^{k,n} - 2\mathbf{c}_{i+1/2}^{k,n} + \mathbf{c}_{i-1/2}^{k,n} \right) \\ - \left( \nabla v^k \left( \xi_{i+1}^{k,n} \right) - \nabla v^k \left( \xi_i^{k,n} \right) \right) \cdot \left( \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right).$$

This implies, using Lemma 3.1,

$$\left| \left( \nabla v^k \left( \xi_{i+1}^{k,n} \right) - \nabla v^k \left( \xi_i^{k,n} \right) \right) \cdot \left( \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right) \right| \\ \leq |\nabla v^k|_{\text{Lip}(\mathbb{R})} \left\| \xi_{i+1}^{k,n} - \xi_i^{k,n} \right\| \left\| \mathbf{c}_{i+1/2}^{k,n} - 2\mathbf{c}_{i-1/2}^{k,n} \right\| \\ \leq |\nabla v^k|_{\text{Lip}(\mathbb{R})} \left\| \xi_{i+1}^{k,n} - \xi_i^{k,n} \right\| \mathcal{K}_1 \Delta x. \tag{3.14}$$

Now using Lemma 3.1,

$$\begin{aligned}
\|\xi_{i+1}^{k,n} - \xi_i^{k,n}\| &= \left\| \left( \xi_{i+1}^{k,n} - \mathbf{c}_{i+1/2}^{k,n} \right) + \left( \mathbf{c}_{i+1/2}^{k,n} - \xi_i^{k,n} \right) \right\| \\
&\leq \left\| \xi_{i+1}^{k,n} - \mathbf{c}_{i+1/2}^{k,n} \right\| + \left\| \mathbf{c}_{i+1/2}^{k,n} - \xi_i^{k,n} \right\| \\
&\leq \left\| \mathbf{c}_{i+3/2}^{k,n} - \mathbf{c}_{i+1/2}^{k,n} \right\| + \left\| \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right\| \\
&\leq 2\mathcal{K}_1 \Delta x.
\end{aligned} \tag{3.15}$$

Finally, using Lemma 3.1, as well as (3.14) and (3.15), we have

$$\begin{aligned}
&\left\| v^k \left( \mathbf{c}_{i+3/2}^{k,n} \right) - 2v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) + v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right\| \\
&\leq \left\| \nabla v^k \left( \xi_{i+1}^{k,n} \right) \right\| \left\| \left( \mathbf{c}_{i+3/2}^{k,n} - 2\mathbf{c}_{i+1/2}^{k,n} + \mathbf{c}_{i-1/2}^{k,n} \right) \right\| \\
&\quad + \left\| \nabla v^k \left( \xi_{i+1}^{k,n} \right) - \nabla v^k \left( \xi_i^{k,n} \right) \right\| \left\| \left( \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right) \right\| \\
&\leq |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_2 \Delta x^2 + 2|\nabla v^k|_{\text{Lip}(\mathbb{R})} (\mathcal{K}_1)^2 (\Delta x)^2.
\end{aligned} \tag{3.16}$$

Consequently,

$$\begin{aligned}
\sum_{i \in \mathbb{Z}} |A_2 + A_3| &\leq \lambda \sum_{i \in \mathbb{Z}} \left| \sigma_{i+1}^k f^k \left( \bar{U}_{i+1}^{k,n} \right) - \sigma_i^k f^k \left( \bar{U}_i^{k,n} \right) \right| |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_1 \Delta x \\
&\quad + \lambda \Delta x^2 |f^k|_{\text{Lip}(\mathbb{R})} \left\| \sigma_\Delta^k \right\|_{L^\infty(\mathbb{R})} \left( |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_2 + 2|\nabla v^k|_{\text{Lip}(\mathbb{R})} (\mathcal{K}_1)^2 \right) \sum_{i \in \mathbb{Z}} \bar{U}_i^{k,n} \\
&\leq \Delta t |f^k|_{\text{Lip}(\mathbb{R})} \left\| \sigma_\Delta^k \right\|_{L^\infty(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_1 \sum_{i \in \mathbb{Z}} \left| \Delta_+ \bar{U}_i^{k,n} \right| \\
&\quad + \Delta t |f^k|_{\text{Lip}(\mathbb{R})} \left\| \bar{U}_\Delta^k \right\|_{L^\infty(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_1 \left\| \sigma_\Delta^k \right\|_{BV(\mathbb{R})} \\
&\quad + \Delta t |f^k|_{\text{Lip}(\mathbb{R})} \left\| \sigma_\Delta^k \right\|_{L^\infty(\mathbb{R})} \left( |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_2 + 2|\nabla v^k|_{\text{Lip}(\mathbb{R})} (\mathcal{K}_1)^2 \right) \left\| \bar{U}_\Delta^k \right\|_{L^1(\mathbb{R})} \\
&= \Delta t \sum_{i \in \mathbb{Z}} \left| \Delta_+ \bar{U}_i^{k,n} \right| \mathcal{K}_4 + \Delta t \mathcal{K}_5,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_4 &= |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \left\| \sigma_\Delta^k \right\|_{L^\infty(\mathbb{R})} \mathcal{K}_1, \\
\mathcal{K}_5 &= \mathcal{K}_1 \left\| \sigma_\Delta^k \right\|_{BV(\mathbb{R})} |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \left\| \bar{U}_\Delta^k \right\|_{L^\infty(\mathbb{R})} \\
&\quad + |f^k|_{\text{Lip}(\mathbb{R})} \left\| \sigma_\Delta^k \right\|_{L^\infty(\mathbb{R})} \left( |v^k|_{\text{Lip}(\mathbb{R})} \mathcal{K}_2 + 2|\nabla v^k|_{\text{Lip}(\mathbb{R})} (\mathcal{K}_1)^2 \right) \left\| \bar{U}_\Delta^k \right\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Finally, using the above estimates, we get the following:

$$\sum_{i \in \mathbb{Z}} \left| \Delta_+ \bar{U}_i^{k,n+1} \right| \leq \sum_{i \in \mathbb{Z}} \left| \Delta_+ \bar{U}_i^{k,n} \right| (1 + \mathcal{K}_4 \Delta t) + \Delta t \mathcal{K}_5,$$

completing the proof.

- (5) Owing to the total variation bound of  $\bar{U}_\Delta$ , and following Aggarwal *et al.* (2015); Aggarwal & Vaidya (2023), the proof follows by repeating the arguments in every component of the vector.
- (6) For  $k \in \{1, \dots, N\}$  fixed, the discrete entropy inequality follows by repeating the arguments in Aggarwal *et al.* (2015); Aggarwal & Vaidya (2023) in every component of the vector  $\mathbf{U}_\Delta$ , using appropriate bounds on the matrix  $c$ .
- (7) Follows by repeating the Lax–Wendroff type argument (see Betancourt *et al.*, 2011, Thm. 5.1) equation by equation. □

The following corollary is a direct consequence of the above theorem.

**COROLLARY 3.3** (Regularity). For  $0 < t \leq T$ , the limit  $\mathbf{U}$  of the numerical approximation  $\mathbf{U}_\Delta$ , which is an entropy solution of the IVP (1.2)–(1.3) for the system of nonlocal conservation laws, satisfies the following:

$$\begin{aligned} \|\mathbf{U}(t, \cdot)\|_{(L^\infty(\mathbb{R}))^N} &\leq \frac{\|\sigma\|_{(L^\infty(\mathbb{R}))^N}}{\min_{k \in \{1 \dots N\}} \inf_{x \in \mathbb{R}} \sigma^k(x)} \exp(\mathcal{K}_3 T) \|\mathbf{U}_0(\cdot)\|_{(L^\infty(\mathbb{R}))^N}, \\ \|\mathbf{U}(t, \cdot)\|_{(L^1(\mathbb{R}))^N} &= \|\mathbf{U}_0\|_{(L^1(\mathbb{R}))^N}, \\ |\sigma(\cdot) \circ \mathbf{U}(t, \cdot)|_{(BV(\mathbb{R}))^N} &\leq \exp(\mathcal{K}_4 t) (|\sigma(\cdot) \circ \mathbf{U}_0|_{(BV(\mathbb{R}))^N} + \mathcal{K}_5 t), \\ \|\mathbf{U}(t_2, \cdot) - \mathbf{U}(t_1, \cdot)\|_{(L^1(\mathbb{R}))^N} &\leq N \mathcal{K}_6 |t_2 - t_1|. \end{aligned}$$

In the next section, we prove that the entropy solutions are in fact unique.

#### 4. Kuznetsov-type estimate and the uniqueness

For  $\epsilon, \epsilon_0 > 0$ , define  $\Phi : \bar{Q}_T^2 \rightarrow \mathbb{R}$  by

$$\Phi(t, x, s, y) := \Phi^{\epsilon, \epsilon_0}(t, x, s, y) = \omega_\epsilon(x - y) \omega_{\epsilon_0}(t - s),$$

where  $\omega_a(x) = \frac{1}{a} \omega\left(\frac{x}{a}\right)$ ,  $a > 0$  and  $\omega$  is a standard symmetric mollifier with  $\text{supp}(\omega) \in [-1, 1]$ . Furthermore, we assume that  $\int_{\mathbb{R}} \omega_a(x) dx = 1$  and  $\int_{\mathbb{R}} |\omega'_a(x)| dx = \frac{1}{a}$ . It is straight forward to see that  $\Phi$  is symmetric,  $\Phi_x = -\Phi_y$  and  $\Phi_t = -\Phi_s$ . Further,

$$\int_{Q_T} \left( \int_{C^n} \Phi(s, y, t, x_{i+1/2}) dt - \lambda \int_{C_i} \Phi(s, y, t^{n+1}, x) dx \right) ds dy = \mathcal{O} \left( \frac{\Delta x^2}{\epsilon} + \frac{\Delta t^2}{\epsilon_0} \right), \tag{4.1}$$

$$\int_{Q_T} \left( \int_{C_{i+1}} \Phi(s, y, t^{n+1}, x) dx - \int_{C_i} \Phi(s, y, t^{n+1}, x) dx \right) ds dy = \mathcal{O} \left( \frac{\Delta x^2}{\epsilon} \right). \tag{4.2}$$

Details of these estimates can be found in [Holden & Risebro \(2015, Sec. 3.3\)](#). We now list some of the notation to be used in the sequel. For  $U, V \in ((L^1 \cap L^\infty)(\mathbb{R}))^N$ , all non-negative  $\phi \in C_c^\infty(\overline{Q_T})$ , for each  $(t, x), (s, y) \in Q_T$  and for each  $k \in \{1, \dots, N\}$ , we define the following:

$$\mathcal{U}^k(t, x) := v^k((\boldsymbol{\mu} * U)^k(t, x)), \quad (t, x) \in Q_T,$$

$$\mathcal{V}^k(s, y) := v^k((\boldsymbol{\mu} * V)^k(s, y)), \quad (s, y) \in Q_T,$$

$$G^k(a, b) := \operatorname{sgn}(a - b)(f^k(a) - f^k(b)), \quad a, b \in \mathbb{R},$$

$$\begin{aligned} \Lambda_T^k \left( U^k, \phi, \frac{\alpha}{\sigma^k} \right) &:= \int_{Q_T} \left( \left| U^k(t, x) - \frac{\alpha}{\sigma^k(x)} \right| \phi_t + G^k \left( U^k(t, x), \frac{\alpha}{\sigma^k(x)} \right) \mathcal{U}^k(t, x) \phi_x \right) dt dx \\ &\quad - \int_{Q_T} \operatorname{sgn} \left( U^k(t, x) - \frac{\alpha}{\sigma^k(x)} \right) f^k(\alpha) \mathcal{U}_x^k(t, x) \phi dt dx \\ &\quad - \int_{\mathbb{R}} \left| U^k(T, x) - \frac{\alpha}{\sigma^k(x)} \right| \phi(T, x) dx + \int_{\mathbb{R}} \left| U_0^k(x) - \frac{\alpha}{\sigma^k(x)} \right| \phi(0, x) dx, \end{aligned}$$

$$\Lambda_{\epsilon, \epsilon_0}^k(U^k, V^k) := \int_{Q_T} \Lambda_T^k \left( U^k(\cdot, \cdot), \Phi(\cdot, \cdot, s, y), \frac{\overline{V}^k(s, y)}{\sigma^k(\cdot)} \right) ds dy,$$

where

$$\overline{V}^k(s, y) = \sigma^k(y) V^k(s, y),$$

$$\begin{aligned} K := \left\{ U : \overline{Q_T} \rightarrow \mathbb{R}^N : \quad &\|U\|_{(L^\infty(\overline{Q_T}))^N} + |U|_{(L^\infty([0, T]; \text{BV}(\mathbb{R})))^N} < \infty \right. \\ &\left. \text{and } \|U^k(t)\|_{L^1(\mathbb{R})} = \|U^k(0)\|_{L^1(\mathbb{R})} \text{ for } t \geq 0, 1 \leq k \leq N \right\}. \end{aligned}$$

We now state and prove the Kuznetsov-type lemma for the coupled system of nonlocal conservation laws (1.1).

**LEMMA 4.1** [A Kuznetsov-type lemma for nonlocal systems of conservation laws] Let  $U$  be an entropy solution of the IVP of the system (1.2)–(1.3) and let  $V \in K$ . Then,

$$\begin{aligned} &\|U(T, \cdot) - V(T, \cdot)\|_{(L^1(\mathbb{R}))^N} \\ &\leq C \sum_{k=1}^N \left( \gamma(V^k, \epsilon_0) - \Lambda_{\epsilon, \epsilon_0}^k(V^k, U^k) \right) \\ &\quad + C \left( \|U_0 - V_0\|_{(L^1(\mathbb{R}))^N} + N(\epsilon + \epsilon_0) + N \frac{\epsilon}{\epsilon_0} \left| \frac{1}{\sigma} \right|_{(\text{BV}(\mathbb{R}))^N} \right), \end{aligned} \quad (4.3)$$

where  $\gamma$  is defined by (2.1) and  $C$  depends only on  $f, \boldsymbol{\mu}, \boldsymbol{v}, \boldsymbol{\sigma}, U, V$  and  $T$ .

*Proof.* For any  $k \in \{1, \dots, N\}$ , add the entropy functionals  $\Lambda_{\epsilon, \epsilon_0}^k(U^k, V^k)$  and  $\Lambda_{\epsilon, \epsilon_0}^k(V^k, U^k)$ , and invoke the symmetry of  $\Phi$  to get

$$\Lambda_{\epsilon, \epsilon_0}^k(U^k, V^k) + \Lambda_{\epsilon, \epsilon_0}^k(V^k, U^k) = I_{\Phi_t}^k + I_{\Phi_x}^k + I_{\Phi}^k + I_0^k - I_T^k,$$

with

$$\begin{aligned} I_{\Phi_t}^k &= \int_{Q_T^2} \left| U^k(t, x) - \frac{\bar{V}^k(s, y)}{\sigma^k(x)} \right| \Phi_t(t, x, s, y) \, dx \, dt \, dy \, ds \\ &\quad - \int_{Q_T^2} \left| V^k(s, y) - \frac{\bar{U}^k(t, x)}{\sigma^k(y)} \right| \Phi_t(t, x, s, y) \, dx \, dt \, dy \, ds, \\ I_{\Phi_x}^k &= \int_{Q_T^2} G^k(\bar{U}^k(t, x), \bar{V}^k(s, y)) (\mathcal{U}^k(t, x) - \mathcal{V}^k(s, y)) \Phi_x(t, x, s, y) \, dt \, dx \, ds \, dy, \\ I_{\Phi}^k &= - \int_{Q_T^2} \operatorname{sgn}(\bar{U}^k(t, x) - \bar{V}^k(s, y)) f^k(\bar{V}^k(s, y)) \mathcal{U}_x^k(t, x) \Phi(t, x, s, y) \, dx \, dt \, dy \, ds \\ &\quad + \int_{Q_T^2} \operatorname{sgn}(\bar{U}^k(t, x) - \bar{V}^k(s, y)) f^k(\bar{U}^k(t, x)) \mathcal{V}_y^k(s, y) \Phi(t, x, s, y) \, dx \, dt \, dy \, ds, \\ I_T^k &= \int_{Q_T} \int_{\mathbb{R}} \left| U^k(T, x) - \frac{\bar{V}^k(s, y)}{\sigma^k(x)} \right| \Phi(t, x, T, y) \, dy \, dx \, dt \\ &\quad + \int_{Q_T} \int_{\mathbb{R}} \left| V^k(T, y) - \frac{\bar{U}^k(t, x)}{\sigma^k(y)} \right| \Phi(t, x, T, y) \, dy \, dx \, dt, \\ I_0^k &= \int_{Q_T} \int_{\mathbb{R}} \left( \left| U_0^k(x) - \frac{\bar{V}^k(s, y)}{\sigma^k(x)} \right| + \left| V_0^k(y) - \frac{\bar{U}^k(t, x)}{\sigma^k(y)} \right| \right) \Phi(t, x, 0, y) \, dx \, dy \, dt. \end{aligned}$$

Since  $U$  is an entropy solution of (1.2)–(1.3), we have that  $\Lambda_{\epsilon, \epsilon_0}^k(U^k, V^k) \geq 0$ , and hence

$$I_T^k \leq -\Lambda_{\epsilon, \epsilon_0}^k(V^k, U^k) + I_{\Phi_t}^k + I_{\Phi_x}^k + I_{\Phi}^k + I_0^k. \tag{4.4}$$

Since  $U^k(t, x) = \frac{\bar{U}^k(t, x)}{\sigma^k(x)}$ , and  $|a - b| - |c - d| \leq |a - c| + |b - d|$ , we have

$$\begin{aligned} I_{\Phi_t}^k &\leq \int_{Q_T^2} \left( |\bar{U}^k(t, x)| + |\bar{V}^k(s, y)| \right) \left| \frac{1}{\sigma^k(x)} - \frac{1}{\sigma^k(y)} \right| |\omega'_{\epsilon_0}(t - s)| \omega_{\epsilon}(x - y) \, dt \, dx \, ds \, dy \\ &\leq T \left( \|\bar{U}\|_{(L^\infty(Q_T))^N} + \|\bar{V}\|_{(L^\infty(Q_T))^N} \right) \frac{\epsilon}{\epsilon_0} \left| \frac{1}{\sigma} \right|_{(\text{BV}(\mathbb{R}))^N}. \end{aligned}$$

The terms  $I_0^k$  and  $I_T^k$  appear for local conservation laws with Panov type flux as well and can be estimated as in Ghoshal *et al.* (2022b, Lemma 2) to get

$$\begin{aligned} I_T^k &\geq \left\| U^k(T, \cdot) - V^k(T, \cdot) \right\|_{L^1(\mathbb{R})} - C_1 \left( \epsilon + \epsilon_0 + \gamma \left( V^k, \epsilon_0 \right) \right), \\ I_0^k &\leq \left\| U_0^k - V_0^k \right\|_{L^1(\mathbb{R})} + C_1 \left( \epsilon + \epsilon_0 + \gamma \left( V^k, \epsilon_0 \right) \right), \end{aligned}$$

where  $C_1 = C_1(|U|_{(L^\infty([0,T];BV(\mathbb{R})))^N}, |V|_{(L^\infty([0,T];BV(\mathbb{R})))^N}, |U|_{\text{Lip}([0,T];L^1(\mathbb{R}))^N})$ . It is to be noted that  $I_{\Phi_x}^k$  and  $I_\Phi^k$  comprise of the coupled terms from the system, which we now estimate. Using integration by parts, we have

$$\begin{aligned} I_{\Phi_x}^k + I_\Phi^k &= - \int_{Q_T^2} \Phi G_x^k(\bar{U}^k(t,x), \bar{V}^k(s,y)) (\mathcal{U}^k(t,x) - \mathcal{V}^k(s,y)) \, dt \, dx \, ds \, dy \\ &\quad - \int_{Q_T^2} \Phi \operatorname{sgn}(\bar{U}^k(t,x) - \bar{V}^k(s,y)) f^k(\bar{U}^k(t,x)) \mathcal{U}_x^k(t,x) \, dt \, dx \, ds \, dy \\ &\quad + \int_{Q_T^2} \Phi \operatorname{sgn}(\bar{U}^k(t,x) - \bar{V}^k(s,y)) f^k(\bar{U}^k(t,x)) \mathcal{V}_y^k(s,y) \, dx \, dt \, ds \, dy \\ &:= I_{\mathcal{U}^k} + I_{\mathcal{U}_x^k}. \end{aligned}$$

Let  $|\bar{U}_x^k|$  denote the absolute value of the Radon measure  $\bar{U}_x^k$ , which is finite as it is a distributional derivative of a BV function. Owing to the Lipschitz continuity of the entropy flux  $G$ , we have  $|\partial_x G^k(\bar{U}^k(t,x), \bar{V}^k(s,y))| \leq |f^k|_{\text{Lip}(\mathbb{R})} |\bar{U}_x^k|$  (see [Lemma A2.1] Bouchut & Perthame, 1998, for details), implying

$$I_{\mathcal{U}^k} \leq |f^k|_{\text{Lip}(\mathbb{R})} \int_{Q_T^2} \Phi |\bar{U}_x^k| |\mathcal{V}^k(s,y) - \mathcal{U}^k(t,x)| \, dx \, dy \, dt \, ds.$$

Further, the nonlocal weights  $\mathcal{V}^k(s,y), \mathcal{U}^k(t,x)$  satisfy

$$\begin{aligned} &|\mathcal{V}^k(s,y) - \mathcal{U}^k(t,x)| \\ &\leq |\mathcal{V}^k(s,y) - \mathcal{V}^k(s,x)| + |\mathcal{V}^k(s,x) - \mathcal{U}^k(t,x)| \\ &= |v^k((\boldsymbol{\mu} * \mathbf{V})^k(s,y)) - v^k((\boldsymbol{\mu} * \mathbf{V})^k(s,x))| \\ &\quad + |v^k((\boldsymbol{\mu} * \mathbf{V})^k(s,x)) - v^k((\boldsymbol{\mu} * \mathbf{U})^k(t,x))| \\ &\leq |v^k|_{\text{Lip}(\mathbb{R})} \left\| (\boldsymbol{\mu} * \mathbf{V})^k(s,y) - (\boldsymbol{\mu} * \mathbf{V})^k(s,x) \right\| \\ &\quad + |v^k|_{\text{Lip}(\mathbb{R})} \left\| (\boldsymbol{\mu} * \mathbf{V})^k(s,x) - (\boldsymbol{\mu} * \mathbf{U})^k(t,x) \right\| \\ &= |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu}' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| \mathbf{V}(s, \cdot) \right\|_{(L^1(\mathbb{R}))^N} |y - x| \\ &\quad + |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| \mathbf{V}(s, \cdot) - \mathbf{U}(t, \cdot) \right\|_{(L^1(\mathbb{R}))^N}. \end{aligned}$$



It can be seen that the estimates on nonlocal weights for any  $k$  depend on all the  $N$  components of  $\mathbf{U}$  and  $\mathbf{V}$ . Consequently, we get

$$\begin{aligned}
 I_{\mathcal{U}^k} &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T^2} \Phi \left| \overline{U}_x^k \right| \|\mathbf{V}(s, \cdot)\|_{(L^1(\mathbb{R}))^N} |y-x| \, dx \, dy \, dt \, ds \\
 &\quad + |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T^2} \Phi \left| \overline{U}_x^k \right| \|\mathbf{V}(s, \cdot) - \mathbf{U}(t, \cdot)\|_{(L^1(\mathbb{R}))^N} \, dx \, dy \, dt \, ds \\
 &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T^2} \Phi \left| \overline{U}_x^k \right| \|\mathbf{V}(s, \cdot)\|_{(L^1(\mathbb{R}))^N} |y-x| \, dx \, dy \, dt \, ds \\
 &\quad + |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T^2} \Phi \left| \overline{U}_x^k \right| \|\mathbf{U}(s, \cdot) - \mathbf{U}(t, \cdot)\|_{(L^1(\mathbb{R}))^N} \, dx \, dy \, dt \, ds \\
 &\quad + |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T^2} \Phi \left| \overline{U}_x^k \right| \|\mathbf{V}(s, \cdot) - \mathbf{U}(s, \cdot)\|_{(L^1(\mathbb{R}))^N} \, dx \, dy \, dt \, ds, \\
 &:= I_{\mathcal{U}^k}^1 + I_{\mathcal{U}^k}^2 + I_{\mathcal{U}^k}^3,
 \end{aligned}$$

where

$$\begin{aligned}
 I_{\mathcal{U}^k}^1 &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T} \int_{Q_T} \omega_\epsilon(x-y) \omega_{\epsilon_0}(t-s) \left| \overline{U}_x^k \right| \|\mathbf{V}(s, \cdot)\|_{(L^1(\mathbb{R}))^N} \, dx \, dy \, dt \, ds \\
 &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}'\|_{(L^\infty(\mathbb{R}))^{N^2}} |\overline{U}^k|_{(L^\infty([0,T];\text{BV}(\mathbb{R})))} \|\mathbf{V}\|_{(L^1(Q_T))^N} \epsilon, \\
 I_{\mathcal{U}^k}^2 &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} |\mathbf{U}|_{(\text{Lip}(Q_T))^N} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_{Q_T^2} \omega_\epsilon(x-y) \omega_{\epsilon_0}(t-s) \left| \overline{U}_x^k \right| |t-s| \, dx \, dy \, dt \, ds \\
 &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} |\overline{U}^k|_{(L^\infty([0,T];\text{BV}(\mathbb{R})))} |\mathbf{U}|_{(\text{Lip}(Q_T))^N} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_0^T \int_0^T \omega_{\epsilon_0}(t-s) \epsilon_0 \, dt \, ds \\
 &= |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} |\overline{U}^k|_{(L^\infty([0,T];\text{BV}(\mathbb{R})))} |\mathbf{U}|_{(\text{Lip}(Q_T))^N} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} T \epsilon_0, \\
 I_{\mathcal{U}^k}^3 &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} |\overline{U}^k|_{(L^\infty([0,T];\text{BV}(\mathbb{R})))} \|\boldsymbol{\mu}\|_{(L^\infty(\mathbb{R}))^{N^2}} \\
 &\quad \times \int_0^T \|\mathbf{V}(s, \cdot) - \mathbf{U}(s, \cdot)\|_{(L^1(\mathbb{R}))^N} \, ds.
 \end{aligned}$$

Collectively, we get

$$\begin{aligned} I_{\mathcal{U}^k} &\leq |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} |\overline{U}^k|_{(L^\infty([0,T];\text{BV}(\mathbb{R})))} \|\mu'\|_{(L^\infty(\mathbb{R}))^{N^2}} \|\mathbf{V}\|_{(L^1(Q_T))^N} \epsilon \\ &\quad + |f^k|_{\text{Lip}(\mathbb{R})} |v^k|_{\text{Lip}(\mathbb{R})} |\overline{U}^k|_{(L^\infty([0,T];\text{BV}(\mathbb{R})))} \|\mu\|_{(L^\infty(\mathbb{R}))^{N^2}} \\ &\quad \times \left( \|U\|_{(L^1(Q_T))^N} T \epsilon_0 + \int_0^T \|\mathbf{V}(s, \cdot) - U(s, \cdot)\|_{(L^1(\mathbb{R}))^N} ds \right) \\ &\leq \mathcal{K}_1(\epsilon + \epsilon_0) + \mathcal{K}_2 \int_0^T \|\mathbf{V}(s, \cdot) - U(s, \cdot)\|_{(L^1(\mathbb{R}))^N} ds. \end{aligned}$$

Similarly, we can show that the following estimate holds:

$$I_{\mathcal{U}_x^k} \leq \mathcal{K}_3(\epsilon + \epsilon_0) + \mathcal{K}_4 \int_0^T \|\mathbf{V}(s, \cdot) - U(s, \cdot)\|_{(L^1(\mathbb{R}))^N} ds.$$

Substituting the above estimates in (4.4) and summing over  $k$ , we get

$$\begin{aligned} &\|U(T) - V(T)\|_{(L^1(\mathbb{R}))^N} \\ &\leq \|V_0 - U_0\|_{(L^1(\mathbb{R}))^N} + \mathcal{C}_2 \int_0^T \|\mathbf{V}(t, \cdot) - U(t, \cdot)\|_{(L^1(\mathbb{R}))^N} dt \\ &\quad + \mathcal{C}_1 N \left( \epsilon + \epsilon_0 + \frac{\epsilon}{\epsilon_0} \left| \frac{1}{\sigma} \right|_{(\text{BV}(\mathbb{R}))^N} \right) + \mathcal{C}_1 \sum_{k=1}^N \left( \gamma(V^k, \epsilon_0) - \Lambda_{\epsilon, \epsilon_0}^k(V^k, U^k) \right). \end{aligned}$$

Now, the lemma follows by invoking Gronwall's inequality.  $\square$

**THEOREM 4.2** [Uniqueness of the entropy solution for nonlocal systems] For any time  $T > 0$ , let  $U, V$  be the entropy solutions of the IVP for the system (1.2) with initial data  $U_0, V_0$ , respectively. Then, the following holds:

$$\|U(T, \cdot) - V(T, \cdot)\|_{(L^1(\mathbb{R}))^N} \leq \mathcal{C} \|U_0 - V_0\|_{(L^1(\mathbb{R}))^N},$$

where  $\mathcal{C}$  is the same as in Lemma 4.1. In particular, if  $U_0 = V_0$  then  $U = V$  a.e. in  $\overline{Q}_T$ .

*Proof.* Since  $U, V$  are the entropy solutions, we have  $-\sum_{k=1}^N \Lambda_{\epsilon, \epsilon_0}^k(V^k, U^k) \leq 0$ , for all  $\epsilon, \epsilon_0 > 0$ . Now, the proof follows from Lemma 4.1 by sending  $\epsilon \rightarrow 0^+$  and then  $\epsilon_0 \rightarrow 0^+$ .  $\square$

**REMARK 4.3** Theorem 4.2, in combination with Theorem 3.2 (2), implies that in fact the entire sequence  $U_\Delta$  of the finite volume approximations converge to the unique entropy solution  $U$  in  $(L^1_{\text{loc}}(\overline{Q}_T))^N$  and pointwise a.e. in  $\overline{Q}_T$ .

In the next section, we establish the convergence rates for the proposed numerical schemes.

## 5. Error estimate

For every  $n \in \{0, \dots, N_T\}$ ,  $i \in \mathbb{Z}$ ,  $k \in \{1, \dots, N\}$ ,  $(t, x) \in \overline{Q}_T$  and  $\alpha \in \mathbb{R}$ , we define the following:

$$\text{i. } \eta_i^{k,n}(\alpha) := \left| U_i^{k,n} - \frac{\alpha}{\sigma_i^k} \right|,$$

- ii.  $p_i^{k,n}(\alpha) := \text{sgn}(\bar{U}_i^{k,n} - \alpha) \left( f^k(\bar{U}_i^{k,n}) - f^k(\alpha) \right),$
- iii.  $\mathcal{U}_\Delta^k(t, x) := v^k((\boldsymbol{\mu} * \mathbf{U}_\Delta)^k(t, x)).$

It is to be noted that owing to the discontinuous flux,  $\Lambda_{\epsilon, \epsilon_0}^k$  has nonpiecewise constant terms arising from  $\sigma(x)$ , unlike the classical cases considered in [Holden & Risebro \(2015\)](#). In comparison to a similar case considered in [Ghoshal \*et al.\* \(2022a\)](#), the PDE (1.2)–(1.3) consists of nonlocal coupled terms in addition, making it nontrivial. To address this issue, we first prove an estimate on  $\Lambda_{\Delta, \epsilon, \epsilon_0}^k$ , an appropriate approximation of  $\Lambda_{\epsilon, \epsilon_0}^k$ , (with  $\sigma_\Delta$  in place of  $\sigma$ ) in [Lemma 5.1](#), and then estimate the difference between  $\Lambda_{\epsilon, \epsilon_0}^k$  and  $\Lambda_{\Delta, \epsilon, \epsilon_0}^k$ . If the local part of the flux is homogeneous (i.e.,  $\text{TV}(s) = 0$ ), then this step is not required and in fact,  $\Lambda_{\epsilon, \epsilon_0}^k = \Lambda_{\Delta, \epsilon, \epsilon_0}^k$ .

LEMMA 5.1 For every  $k \in \{1, \dots, N\}$ , the relative entropy functional  $\Lambda_{\epsilon, \epsilon_0}^k(U_\Delta^k, U^k)$  satisfies

$$-\Lambda_{\epsilon, \epsilon_0}^k(U_\Delta^k, U^k) \leq \mathcal{M} \left( \frac{\Delta x}{\epsilon} + \frac{\Delta t}{\epsilon_0} + \Delta t + \Delta x \right),$$

where  $\mathcal{M}$  is a constant independent of  $\Delta x, \Delta t$ .

*Proof.* For any  $k \in \{1, \dots, N\}$ , we define

$$\begin{aligned} & -\Lambda_{\Delta, \epsilon, \epsilon_0}^k(U_\Delta^k, U^k) \\ &:= -\int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} \eta_i^{k,n}(\bar{U}^k(s, y)) \Phi_t(s, y, t, x) \, dt \, dx \, ds \, dy \\ & \quad - \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} p_i^{k,n}(\bar{U}^k(s, y)) \mathcal{U}_\Delta^k(t, x) \Phi_x(s, y, t, x) \, dt \, dx \, ds \, dy \\ & \quad + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} \text{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \\ & \quad \quad \times f^k(\bar{U}^k(s, y)) \partial_x \mathcal{U}_\Delta^k(t, x) \Phi(s, y, t, x) \, dt \, dx \, ds \, dy \\ & \quad - \int_{Q_T} \sum_{i \in \mathbb{Z}} \int_{C_i} \eta_i^{k,0}(\bar{U}^k(s, y)) \Phi(s, y, 0, x) \, dx \, ds \, dy \\ & \quad + \int_{Q_T} \sum_{i \in \mathbb{Z}} \int_{C_i} \eta_i^{k, N_T}(\bar{U}^k(s, y)) \Phi(s, y, T, x) \, dx \, ds \, dy. \end{aligned}$$

Consequently,

$$\begin{aligned}
& \left| \Lambda_{\Delta, \epsilon, \epsilon_0}^k (U_{\Delta}^k, U^k) - \Lambda_{\epsilon, \epsilon_0}^k (U_{\Delta}^k, U^k) \right| \\
& \leq C \int_{Q_T^2} \left| \frac{\alpha}{\sigma^k(x)} - \frac{\alpha}{\sigma_{\Delta}^k(x)} \right| (|\Phi_t| + \|\mathcal{U}\|_{L^\infty(Q_T)} |\Phi_x|) \, dx \, dt \, ds \, dy \\
& \quad + C \int_{Q_T^2} \left| \frac{\alpha}{\sigma^k(x)} - \frac{\alpha}{\sigma_{\Delta}^k(x)} \right| \|\mathcal{U}_x\|_{L^\infty(Q_T)} |\Phi| \, dx \, dt \, ds \, dy \\
& \quad + C \int_{Q_T \times \mathbb{R}} \left| \frac{\alpha}{\sigma^k(x)} - \frac{\alpha}{\sigma_{\Delta}^k(x)} \right| (|\Phi(\cdot, \cdot, T, \cdot)| + |\Phi(\cdot, \cdot, 0, \cdot)|) \, dx \, ds \, dy \\
& \leq C \left( \Delta x + \frac{\Delta x}{\epsilon} + \frac{\Delta x}{\epsilon_0} \right).
\end{aligned}$$

In what follows, we estimate  $\Lambda_{\Delta, \epsilon, \epsilon_0}^k (U_{\Delta}^k, U^k)$ , which is enough to get the desired result. To begin with, we invoke fundamental theorem of calculus followed by summation by parts to get

$$\begin{aligned}
& \Lambda_{\Delta, \epsilon, \epsilon_0}^k (U_{\Delta}^k, U^k) \\
& = \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( \eta_i^{k, n+1} (\bar{U}^k(s, y)) - \eta_i^{k, n} (\bar{U}^k(s, y)) \right) \int_{C_i} \Phi(s, y, t^{n+1}, x) \, dx \, ds \, dy \\
& \quad - \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} p_i^{k, n} (\bar{U}^k(s, y)) \mathcal{U}_{\Delta}^k(t, x) \Phi_x(s, y, t, x) \, dt \, dx \, ds \, dy \\
& \quad + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} \operatorname{sgn} \left( U_i^{k, n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \\
& \quad \quad \quad \times f^k(\bar{U}^k(s, y)) \partial_x \mathcal{U}_{\Delta}^k(t, x) \Phi(s, y, t, x) \, dt \, dx \, ds \, dy \\
& := \lambda_1 + \lambda_2 + \lambda_3.
\end{aligned}$$

Now, we make the following estimates:

$$\begin{aligned}
\lambda_2 & = - \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} p_i^{k, n} (\bar{U}^k(s, y)) v^k \left( \mathbf{c}_{i+1/2}^{k, n} \right) \Phi_x(s, y, t, x) \, dt \, dx \, ds \, dy \\
& \quad - \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} p_i^{k, n} (\bar{U}^k(s, y)) \\
& \quad \quad \quad \times \Phi_x(s, y, t, x) \left( \mathcal{U}_{\Delta}^k(t, x) - v^k \left( \mathbf{c}_{i+1/2}^{k, n} \right) \right) \, dt \, dx \, ds \, dy \\
& := \lambda_{2'} + \mathcal{E}_2.
\end{aligned}$$

For  $(t, x) \in C_i^n$ , we have

$$\begin{aligned} & \left| \mathcal{U}_\Delta^k(t, x) - v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right| \\ & \leq \left| \mathcal{U}_\Delta^k(t, x) - \mathcal{U}_\Delta^k(t^n, x) \right| + \left| \mathcal{U}_\Delta^k(t^n, x) - \mathcal{U}_\Delta^k \left( t^n, x_{i+1/2} \right) \right| \\ & \quad + \left| \mathcal{U}_\Delta^k \left( t^n, x_{i+1/2} \right) - v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right|. \end{aligned}$$

Furthermore,

$$\begin{aligned} & \left| \mathcal{U}_\Delta^k \left( t^n, x_{i+1/2} \right) - v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right| \\ & \leq \Delta x |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu}' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| \mathbf{U}_\Delta \right\|_{(L^1(Q_T))^N} := \mathcal{M}_1 \Delta x. \end{aligned} \tag{5.1}$$

Now using the Lipschitz continuity of  $\mathcal{U}_\Delta^k$  in the space variable, we have

$$\left| \mathcal{U}_\Delta^k(t^n, x) - \mathcal{U}_\Delta^k \left( t^n, x_{i+1/2} \right) \right| \leq \mathcal{M}_2 \Delta x.$$

Also for  $t \in C^n$ , using (3.12), we have

$$\left| \mathcal{U}_\Delta^k(t^n, x) - \mathcal{U}_\Delta^k(t, x) \right| \leq |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \mathcal{K}_6 \Delta t := \mathcal{M}_3 \Delta x.$$

Finally, combining all the above estimates, we get

$$\begin{aligned} |\mathcal{E}_2| & \leq \mathcal{M}_4 \Delta x \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( \left| f^k \left( \bar{U}_i^{k,n} \right) \right| + |f^k(\bar{U}^k(s, y))| \right) \\ & \quad \times \int_{C_i^n} |\Phi_x(s, y, t, x)| \, dt \, dx \, ds \, dy \\ & = \mathcal{M}_4 \Delta x \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} |f^k \left( \bar{U}_i^{k,n} \right)| \int_{Q_T} |\Phi_x(s, y, t, x)| \, dt \, dx \, ds \, dy \\ & \quad + \mathcal{M}_4 \Delta x \int_{Q_T} \int_{Q_T} |f^k(\bar{U}^k(s, y)) \Phi_x(s, y, t, x)| \, dt \, dx \, ds \, dy \\ & \leq \tilde{\mathcal{M}}_4 |f^k|_{\text{Lip}(\mathbb{R})} \left\| \bar{U}_\Delta^k \right\|_{L^1(Q_T)} \frac{\Delta x}{\epsilon} + \tilde{\mathcal{M}}_4 |f^k|_{\text{Lip}(\mathbb{R})} \left\| \bar{U}^k \right\|_{L^1(Q_T)} \frac{\Delta x}{\epsilon}. \end{aligned}$$

Thus, we have

$$\lambda_2 = \lambda'_2 + \mathcal{O} \left( \frac{\Delta x}{\epsilon} \right).$$

Now, we consider

$$\begin{aligned}
\lambda_3 &= \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
&\quad \times f^k(\bar{U}^k(s,y)) \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, dt \, ds \, dy \\
&+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) f^k(\bar{U}^k(s,y)) \int_{C_i^n} \partial_x \mathcal{U}_\Delta^k(t,x) \\
&\quad \times \left( \Phi(s,y,t,x) - \Phi(s,y,t,x_{i+1/2}) \right) \, dx \, ds \, dt \, dy \\
&+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) f^k(\bar{U}^k(s,y)) \int_{C_i^n} \Phi(s,y,t,x_{i+1/2}) \\
&\quad \times \left( \partial_x \mathcal{U}_\Delta^k(t,x) - \frac{v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n})}{\Delta x} \right) \, dx \, ds \, dt \, dy \\
&:= \lambda'_3 + \mathcal{E}_{31} + \mathcal{E}_{32}.
\end{aligned}$$

Further,

$$\begin{aligned}
|\mathcal{E}_{31}| &\leq \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} |f^k(\bar{U}^k(s,y))| \\
&\quad \times \int_{C_i^n} \left| \partial_x \mathcal{U}_\Delta^k(t,x) \right| \left| \Phi(s,y,t,x) - \Phi(s,y,t,x_{i+1/2}) \right| \, dt \, dx \, ds \, dy \\
&\leq |v^k|_{\operatorname{Lip}(\mathbb{R})} \|\mathbf{u}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \|\mathbf{U}_0\|_{(L^1(\mathbb{R}))^N} \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} |f^k(\bar{U}^k(s,y))| \\
&\quad \times \int_{C_i^n} \left| \Phi(s,y,t,x) - \Phi(s,y,t,x_{i+1/2}) \right| \, dx \, ds \, dt \, dy \\
&\leq |v^k|_{\operatorname{Lip}(\mathbb{R})} \|\mathbf{u}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \|\mathbf{U}_0\|_{(L^1(\mathbb{R}))^N} \int_{Q_T} |f^k(\bar{U}^k(s,y))| \\
&\quad \times \sum_{i \in \mathbb{Z}} \int_{C_i} |\omega_\epsilon(y-x) - \omega_\epsilon(y-x_{i+1/2})| \, dx \, ds \, dy \\
&\leq |v^k|_{\operatorname{Lip}(\mathbb{R})} \|\mathbf{u}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \|\mathbf{U}_0\|_{(L^1(\mathbb{R}))^N} |\omega_\epsilon|_{BV(\mathbb{R})} \Delta x \int_{Q_T} |f^k(\bar{U}^k(s,y))| \, ds \, dy \\
&\leq |v^k|_{\operatorname{Lip}(\mathbb{R})} \|\mathbf{u}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \|\mathbf{U}_0\|_{(L^1(\mathbb{R}))^N} \|f^k(\bar{U}^k)\|_{L^1(Q_T)} \frac{\Delta x}{\epsilon},
\end{aligned}$$

since

$$\begin{aligned} |\partial_x \mathcal{U}_\Delta^k(t, x)| &= |v_x^k((\boldsymbol{\mu} * \mathbf{U})^k)(t, x)(\boldsymbol{\mu}' * \mathbf{U})^k(t, x)| \\ &\leq |v^k|_{\text{Lip}(\mathbb{R})} \|\boldsymbol{\mu}'\|_{(L^\infty(\mathbb{R}))^{N^2}} \|\mathbf{U}_0\|_{(L^1(\mathbb{R}))^N}. \end{aligned}$$

Next, we consider

$$\begin{aligned} |\mathcal{E}_{32}| &= \left| \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} f^k(\bar{U}^k(s, y)) \int_{C^n} \Phi(s, y, t, x_{i+1/2}) \right. \\ &\quad \left. \times \int_{C_i} \partial_x \mathcal{U}_\Delta^k(t, x) - \frac{v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n})}{\Delta x} dx ds dt dy \right|. \end{aligned}$$

Applying the fundamental theorem of calculus and rearranging the terms, we get

$$\begin{aligned} |\mathcal{E}_{32}| &= \left| \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( \mathcal{U}_\Delta^k(t^n, x_{i+1/2}) - \mathcal{U}_\Delta^k(t^n, x_{i-1/2}) - v^k(\mathbf{c}_{i+1/2}^{k,n}) + v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \right. \\ &\quad \times \int_{C^n} f^k(\bar{U}^k(s, y)) \Phi(s, y, t, x_{i+1/2}) ds dt dy \\ &\quad + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} f^k(\bar{U}^k(s, y)) \int_{C^n} \Phi(s, y, t, x_{i+1/2}) \\ &\quad \times \left( \left( \mathcal{U}_\Delta^k(t, x_{i+1/2}) - \mathcal{U}_\Delta^k(t^n, x_{i+1/2}) \right) - \right. \\ &\quad \left. - \left( \mathcal{U}_\Delta^k(t, x_{i-1/2}) - \mathcal{U}_\Delta^k(t^n, x_{i-1/2}) \right) \right) ds dt dy \Big|. \end{aligned}$$

Applying summation by parts in  $i$ , we get

$$\begin{aligned} |\mathcal{E}_{32}| &\leq \left| \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( \mathcal{U}_\Delta^k(t^n, x_{i+1/2}) - v^k(\mathbf{c}_{i+1/2}^{k,n}) \right) \right. \\ &\quad \left. \times \int_{C^n} f^k(\bar{U}^k(s, y)) (\Phi(s, y, t, x_{i+1/2}) - \Phi(s, y, t, x_{i-1/2})) ds dt dy \right| \\ &\quad + \left| \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C^n} f^k(\bar{U}^k(s, y)) \left( \mathcal{U}_\Delta^k(t, x_{i+1/2}) - \mathcal{U}_\Delta^k(t^n, x_{i+1/2}) \right) \right. \\ &\quad \left. \times (\Phi(s, y, t, x_{i+1/2}) - \Phi(s, y, t, x_{i-1/2})) ds dt dy \right| \\ &:= \mathcal{E}_{321} + \mathcal{E}_{322}. \end{aligned}$$

Now, using (5.1), we have

$$\begin{aligned}
\mathcal{E}_{321} &\leq \int_{Q_T} |f^k(\bar{U}^k(s, y))| |v^k|_{\text{Lip}(\mathbb{R})} \left\| U_{\Delta}^k \right\|_{L^1(Q_T)} \left\| \boldsymbol{\mu}' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \Delta x |\omega_\epsilon|_{BV(\mathbb{R})} \\
&\quad \times \int_0^T |\omega_{\epsilon_0}(s-t)| \, ds \, dt \, dy \\
&\leq \int_{Q_T} |f^k(\bar{U}^k(s, y))| |v^k|_{\text{Lip}(\mathbb{R})} \left\| U_{\Delta}^k \right\|_{L^1(Q_T)} \left\| \boldsymbol{\mu}' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \frac{\Delta x}{\epsilon} \, ds \, dy \\
&= |v^k|_{\text{Lip}(\mathbb{R})} \left\| U_{\Delta}^k \right\|_{L^1(Q_T)} \left\| \boldsymbol{\mu}' \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| f^k(\bar{U}^k) \right\|_{L^1(Q_T)} \frac{\Delta x}{\epsilon}.
\end{aligned}$$

Using (3.12), for  $t \in \mathcal{C}^n$ , we have

$$\begin{aligned}
&\left| \mathcal{U}_{\Delta}^k(t, x_{i+1/2}) - \mathcal{U}_{\Delta}^k(t^n, x_{i+1/2}) \right| \\
&\leq |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \left\| \mathbf{U}_{\Delta}(t, \cdot) - \mathbf{U}_{\Delta}(t^n, \cdot) \right\|_{(L^1(\mathbb{R}))^N} \\
&\leq |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \mathcal{K}_6 \Delta t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{E}_{322} &\leq \int_{Q_T} |f^k(\bar{U}^k(s, y))| |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \mathcal{K}_6 \Delta t |\omega_\epsilon|_{BV(\mathbb{R})} \\
&\quad \times \int_0^T |\omega_{\epsilon_0}(s-t)| \, ds \, dt \, dy \\
&\leq \int_{Q_T} |f^k(\bar{U}^k(s, y))| |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \mathcal{K}_6 \Delta t \frac{1}{\epsilon} \, ds \, dy \\
&= |v^k|_{\text{Lip}(\mathbb{R})} \left\| \boldsymbol{\mu} \right\|_{(L^\infty(\mathbb{R}))^{N^2}} \mathcal{K}_6 \left\| f^k(\bar{U}^k) \right\|_{L^1(Q_T)} \frac{\Delta t}{\epsilon}.
\end{aligned}$$

Finally, combining the above estimates, we get

$$\lambda_3 = \lambda'_3 + \mathcal{O}\left(\frac{\Delta x}{\epsilon}\right).$$

Thus, so far we have proved

$$- \Lambda_{\Delta, \epsilon, \epsilon_0}^k(U_{\Delta}^k, U^k) = \lambda_1 + \lambda'_2 + \lambda'_3 + \mathcal{O}\left(\frac{\Delta x}{\epsilon}\right). \quad (5.2)$$



Now, we estimate  $\lambda_1 + \lambda'_2 + \lambda'_3$  as follows:

$$\begin{aligned}
 & \lambda_1 + \lambda'_2 + \lambda'_3 \\
 &= \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( \eta_i^{k,n+1} (\bar{U}^k(s, y)) - \eta_i^{k,n} (\bar{U}^k(s, y)) \right) \\
 & \quad \times \left( \int_{C_i} \Phi(s, y, t^{n+1}, x) dx \right) ds dy \\
 & \quad - \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} p_i^{k,n} (\bar{U}^k(s, y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) \Phi_x(s, y, t, x) dt dx ds dy \\
 & \quad + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C^n} \Phi(s, y, t, x_{i+1/2}) dt \right) ds dy.
 \end{aligned}$$

Using the discrete entropy inequality, cf. Theorem 3.2 (6), we get

$$\begin{aligned}
 & \lambda_1 + \lambda'_2 + \lambda'_3 \\
 & \leq -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( G_{i+1/2}^{k,n} \left( U_i^{k,n}, U_{i+1}^{k,n}, \bar{U}^k(s, y) \right) - G_{i-1/2}^{k,n} \left( U_{i-1}^{k,n}, U_i^{k,n}, \bar{U}^k(s, y) \right) \right) \\
 & \quad \times \left( \int_{C_i} \Phi(s, y, t^{n+1}, x) dx \right) ds dy \\
 & \quad - \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n+1} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C_i} \Phi(s, y, t^{n+1}, x) dx \right) ds dy \\
 & \quad - \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C_i^n} p_i^{k,n} (\bar{U}^k(s, y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) \Phi_x(s, y, t, x) dt dx ds dy \\
 & \quad + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C^n} \Phi(s, y, t, x_{i+1/2}) dt \right) ds dy
 \end{aligned}$$

$$\begin{aligned}
&= -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( G_{i+1/2}^{k,n} \left( U_i^{k,n}, U_{i+1}^{k,n}, \bar{U}^k(s,y) \right) - G_{i-1/2}^{k,n} \left( U_{i-1}^{k,n}, U_i^{k,n}, \bar{U}^k(s,y) \right) \right) \\
&\quad \times \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
&- \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n+1} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right) \\
&\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
&+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \int_{C^n} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right) \\
&\quad \times \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt \right) ds dy \\
&+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right) \\
&\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt \right) ds dy \\
&= A_2 + A_3 + \lambda'_2 + \lambda'_3,
\end{aligned}$$

by using the fundamental theorem of calculus followed by summation by parts in  $\lambda'_2$ .

**Claim 1.**  $A_2 + \lambda'_2 = \mathcal{O} \left( \frac{\Delta x}{\epsilon} + \frac{\Delta t}{\epsilon_0} \right)$ . Adding and subtracting

$$\begin{aligned}
&\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right) \\
&\quad \times \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy,
\end{aligned}$$

we have

$$\begin{aligned}
&A_2 + \lambda'_2 \\
&= -\lambda \int_{Q_T} \left( G_{i+1/2}^{k,n} \left( U_i^{k,n}, U_{i+1}^{k,n}, \bar{U}^k(s,y) \right) - G_{i-1/2}^{k,n} \left( U_{i-1}^{k,n}, U_i^{k,n}, \bar{U}^k(s,y) \right) \right) \\
&\quad \times \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
&+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right)
\end{aligned}$$

$$\begin{aligned}
 & + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 & + \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 = & \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) - G_{i+1/2}^{k,n}(U_i^{k,n}, U_{i+1}^{k,n}, \bar{U}^k(s,y)) \right) \\
 & \quad \times \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 & - \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i-1/2}^{k,n}) - G_{i-1/2}^{k,n}(U_{i-1}^{k,n}, U_i^{k,n}, \bar{U}^k(s,y)) \right) \\
 & \quad \times \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 & + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy.
 \end{aligned}$$

Applying summation by parts, we get

$$\begin{aligned}
 A_2 + \lambda'_2 = & \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( G_{i+1/2}^{k,n}(U_i^{k,n}, U_{i+1}^{k,n}, \bar{U}^k(s,y)) - p_i^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) \right) \\
 & \quad \times \left( \int_{C_{i+1}} \Phi(s,y,t^{n+1},x) dx - \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 & + \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( p_i^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i+1/2}^{k,n}) - p_{i-1}^{k,n}(\bar{U}^k(s,y)) v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 & \quad \times \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy,
 \end{aligned}$$

where

$$\left| G_{i+1/2}^{k,n} \left( U_i^{k,n}, U_{i+1}^{k,n}, \bar{U}^k(s,y) \right) - v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) p_i^{k,n} \left( \bar{U}^k(s,y) \right) \right| \leq \mathcal{M}_5 |\Delta_+ U_i^{k,n}|, \quad (5.3)$$

and

$$\left| \Delta_- \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) p_i^{k,n} \left( \bar{U}^k(s,y) \right) \right) \right| \leq \mathcal{M}_6 \left[ \left| \Delta_+ U_i^{k,n} \right| + \left| \Delta_- \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right) \right| \right]. \quad (5.4)$$

Using (5.3)–(5.4), we have

$$\begin{aligned} A_2 + \lambda'_2 &\leq \mathcal{M}_5 \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left| U_{i+1}^{k,n} - U_i^{k,n} \right| \\ &\quad \times \left( \int_{C_{i+1}} \Phi(s,y,t^{n+1},x) \, dx - \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right) \, ds \, dy \\ &\quad + \mathcal{M}_6 \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left[ \left| \Delta_+ U_i^{k,n} \right| + \left| \Delta_- \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) \right) \right| \right] \\ &\quad \times \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right) \, ds \, dy. \end{aligned}$$

Now as  $\mu$  is a smooth function with compact support, the claim follows because of (4.1)–(4.2) and the following estimates:

$$\begin{aligned} &\sum_{i \in \mathbb{Z}} \left| v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right| \\ &\leq |v^k|_{\text{Lip}(\mathbb{R})} \sum_{j=1}^N \sum_{i,p} \Delta x \left| \left( \mu_{i+1/2-p}^{j,k} - \mu_{i-1/2-p}^{j,k} \right) U_{p+1/2}^{j,n} \right| \\ &\leq |v^k|_{\text{Lip}(\mathbb{R})} \sum_{j=1}^N \sum_{i,p} \Delta x \left| U_{p+1/2}^{j,n} \right| \int_{C_{i-p}} \left| \mu^{j,k'}(x) \right| \, dx \\ &= |v^k|_{\text{Lip}(\mathbb{R})} \sum_{j=1}^N \sum_p \Delta x \left| U_{p+1/2}^{j,n} \right| \int_{\mathbb{R}} \left| \mu^{j,k'}(x) \right| \, dx \\ &\leq |v^k|_{\text{Lip}(\mathbb{R})} \|U_0\|_{(L^1(\mathbb{R}))^N} \|\mu\|_{(BV(\mathbb{R}))^N} := \mathcal{M}_7, \end{aligned} \quad (5.5)$$

which is true because of (3.9).

**Claim 2.**  $A_3 + \lambda'_3 = \mathcal{O} \left( \frac{\Delta x}{\epsilon} + \frac{\Delta t}{\epsilon_0} + \Delta t \right)$ .

Observe that

$$\begin{aligned}
 & A_3 + \lambda'_3 \\
 &= -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n+1} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 &+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt \right) ds dy \\
 &= -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n+1} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy \\
 &+ \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \int_{C_i} \Phi(s,y,t^{n+1},x) dx ds dy \\
 &- \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \int_{C_i} \Phi(s,y,t^{n+1},x) dx ds dy \\
 &+ \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \int_{C^n} \Phi(s,y,t,x_{i+1/2}) ds dt dy \\
 &= \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) f^k(\bar{U}^k(s,y)) \\
 &\quad \times \left( \int_{C^n} \Phi(s,y,t,x_{i+1/2}) dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds dy
 \end{aligned}$$

$$\begin{aligned}
& -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=0}^{N_T-1} \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right) f^k \left( \bar{U}^k(s, y) \right) \\
& \quad \times \left( \operatorname{sgn} \left( U_i^{k,n+1} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) - \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \right) \\
& \quad \times \left( \int_{C_i} \Phi(s, y, t^{n+1}, x) \, dx \right) \, ds \, dy \\
& := \tilde{\mathcal{E}}_1 + \tilde{\mathcal{E}}_2.
\end{aligned}$$

Now, using (5.5) and (4.1),

$$\tilde{\mathcal{E}}_1 \leq T \|f^k(\bar{U}^k)\|_{L^1(Q_T)} \mathcal{M}_7 \left( \frac{\Delta x}{\epsilon} + \frac{\Delta t}{\epsilon_0} \right).$$

Furthermore,

$$\begin{aligned}
\tilde{\mathcal{E}}_2 &= -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n-1} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n-1} \right) \right) \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \\
& \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C_i} \Phi(s, y, t^n, x) \, dx \right) \, ds \, dy \\
& \quad + \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n} \right) \right) \\
& \quad \times f^k(\bar{U}^k(s, y)) \int_{C_i} \Phi(s, y, t^{n+1}, x) \, dx \, ds \, dy \\
& \quad + \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \left( v^k \left( \mathbf{c}_{i+1/2}^{k,0} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,0} \right) \right) \operatorname{sgn} \left( U_i^{k,0} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \\
& \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C_i} \Phi(s, y, t^1, x) \, dx \right) \, ds \, dy \\
& \quad - \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \left( v^k \left( \mathbf{c}_{i+1/2}^{k,N_T} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,N_T} \right) \right) \operatorname{sgn} \left( U_i^{k,N_T} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \\
& \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C_i} \Phi(s, y, t^{N_T+1}, x) \, dx \right) \, ds \, dy.
\end{aligned}$$

Adding and subtracting

$$\begin{aligned}
& -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left( v^k \left( \mathbf{c}_{i+1/2}^{k,n-1} \right) - v^k \left( \mathbf{c}_{i-1/2}^{k,n-1} \right) \right) \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s, y)}{\sigma_i^k} \right) \\
& \quad \times f^k(\bar{U}^k(s, y)) \left( \int_{C_i} \Phi(s, y, t^{n+1}, x) \, dx \right) \, ds \, dy,
\end{aligned}$$

we have

$$\begin{aligned}
 \tilde{\mathcal{E}}_2 &= -\lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left( v^k(\mathbf{c}_{i+1/2}^{k,n-1}) - v^k(\mathbf{c}_{i-1/2}^{k,n-1}) \right) \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \int_{C_i} \left( \Phi(s,y,t^n,x) - \Phi(s,y,t^{n+1},x) \right) dx \, ds \, dy \\
 &+ \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds \, dy \\
 &- \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left( v^k(\mathbf{c}_{i+1/2}^{k,n-1}) - v^k(\mathbf{c}_{i-1/2}^{k,n-1}) \right) \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) dx \right) ds \, dy \\
 &+ \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \left( v^k(\mathbf{c}_{i+1/2}^{k,0}) - v^k(\mathbf{c}_{i-1/2}^{k,0}) \right) \left( \operatorname{sgn} \left( U_i^{k,0} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^1,x) dx \right) ds \, dy \\
 &- \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \left( v^k(\mathbf{c}_{i+1/2}^{k,N_T}) - v^k(\mathbf{c}_{i-1/2}^{k,N_T}) \right) \operatorname{sgn} \left( U_i^{k,N_T} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \\
 &\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{N_T+1},x) dx \right) ds \, dy \\
 &:= \tilde{\mathcal{E}}_{21} + \tilde{\mathcal{E}}_{22} + \tilde{\mathcal{E}}_{23} + \tilde{\mathcal{E}}_{24} + \tilde{\mathcal{E}}_{25}.
 \end{aligned}$$

Now, let us estimate  $\tilde{\mathcal{E}}_{21}$ :

$$\begin{aligned}
 \tilde{\mathcal{E}}_{21} &= -\lambda \int_{Q_T} \left( \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left( v^k(\mathbf{c}_{i+1/2}^{k,n-1}) - v^k(\mathbf{c}_{i-1/2}^{k,n-1}) \right) \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \right. \\
 &\quad \left. \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \left( \Phi(s,y,t^n,x) - \Phi(s,y,t^{n+1},x) \right) dx \right) \right) ds \, dy \\
 &\leq \Delta x \lambda \left\| f^k(\bar{U}^k) \right\|_{L^\infty(Q_T)} \int_0^T \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left| v^k(\mathbf{c}_{i+1/2}^{k,n-1}) - v^k(\mathbf{c}_{i-1/2}^{k,n-1}) \right| \\
 &\quad \times \left| \omega_{\epsilon_0}(s-t^n) - \omega_{\epsilon_0}(s-t^n - \Delta t) \right| ds \\
 &\leq \Delta t \mathcal{M}_7 \left\| f^k(\bar{U}^k) \right\|_{L^\infty(Q_T)} \sum_{n=1}^{N_T} \int_0^T \left| \omega_{\epsilon_0}(s-t^n) - \omega_{\epsilon_0}(s-t^n - \Delta t) \right| ds \\
 &\leq \mathcal{M}_7 \Delta t \left\| f^k(\bar{U}^k) \right\|_{L^\infty(Q_T)} \sum_{n=1}^{N_T} \left| \omega_{\epsilon_0} \right|_{BV(\mathbb{R})} \Delta t := \mathcal{M}_8 \frac{\Delta t}{\epsilon_0}.
 \end{aligned}$$

Next, we consider  $\tilde{\mathcal{E}}_{22} + \tilde{\mathcal{E}}_{23}$ :

$$\begin{aligned}
\tilde{\mathcal{E}}_{22} + \tilde{\mathcal{E}}_{23} &= \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \left( v^k(\mathbf{c}_{i+1/2}^{k,n}) - v^k(\mathbf{c}_{i-1/2}^{k,n}) \right) \\
&\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right) \, ds \, dy \\
&\quad - \lambda \int_{Q_T} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left( v^k(\mathbf{c}_{i+1/2}^{k,n-1}) - v^k(\mathbf{c}_{i-1/2}^{k,n-1}) \right) \operatorname{sgn} \left( U_i^{k,n} - \frac{\bar{U}^k(s,y)}{\sigma_i^k} \right) \\
&\quad \times f^k(\bar{U}^k(s,y)) \left( \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right) \, ds \, dy \\
&\leq \lambda |v^k|_{\operatorname{Lip}(\mathbb{R})} \left\| f^k(\bar{U}^k) \right\|_{L^\infty(Q_T)} \\
&\quad \times \int_{Q_T} \left( \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left| \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} - \mathbf{c}_{i+1/2}^{k,n-1} + \mathbf{c}_{i-1/2}^{k,n-1} \right| \right. \\
&\quad \quad \left. \times \int_{C_i} |\Phi(s,y,t^{n+1},x)| \, dx \right) \, ds \, dy \\
&= \Delta t |v^k|_{\operatorname{Lip}(\mathbb{R})} \left\| f^k(\bar{U}^k) \right\|_{L^\infty(Q_T)} \\
&\quad \times \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N_T} \left| \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} - \mathbf{c}_{i+1/2}^{k,n-1} + \mathbf{c}_{i-1/2}^{k,n-1} \right|.
\end{aligned}$$

Further,

$$\begin{aligned}
&\sum_{i \in \mathbb{Z}} \left| \left( \mathbf{c}_{i+1/2}^{k,n} - \mathbf{c}_{i-1/2}^{k,n} \right) - \left( \mathbf{c}_{i+1/2}^{k,n-1} - \mathbf{c}_{i-1/2}^{k,n-1} \right) \right| \\
&\leq \Delta x \sum_{i \in \mathbb{Z}} \left| \sum_{j=1}^N \sum_p \left( \boldsymbol{\mu}_{i+1/2-p}^{j,k} - \boldsymbol{\mu}_{i-1/2-p}^{j,k} \right) U_{p+1/2}^{j,n} \right. \\
&\quad \quad \left. - \sum_{j=1}^N \sum_p \left( \boldsymbol{\mu}_{i+1/2-p}^{j,k} - \boldsymbol{\mu}_{i-1/2-p}^{j,k} \right) U_{p+1/2}^{j,n-1} \right| \\
&\leq \Delta x \sum_{j=1}^N \sum_{i,p} \left| U_{p+1/2}^{j,n} - U_{p+1/2}^{j,n-1} \right| \left| \boldsymbol{\mu}_{i+1/2-p}^{j,k} - \boldsymbol{\mu}_{i-1/2-p}^{j,k} \right| \\
&\leq \Delta x |\boldsymbol{\mu}|_{(BV(\mathbb{R}))^{N^2}} \sum_{j=1}^N \sum_p \left| U_{p+1/2}^{j,n} - U_{p+1/2}^{j,n-1} \right| \\
&\leq N |\boldsymbol{\mu}|_{(BV(\mathbb{R}))^{N^2}} \mathcal{K}_6 \Delta t,
\end{aligned}$$



see (3.12). Therefore,

$$\tilde{\mathcal{E}}_{22} + \tilde{\mathcal{E}}_{23} \leq \mathcal{M}_8 \Delta t.$$

The estimates on the remaining terms  $\tilde{\mathcal{E}}_{24}$  and  $\tilde{\mathcal{E}}_{25}$  follow easily from (5.5). Finally, the lemma follows combing all the above assertions.  $\square$

**THEOREM 5.2** Let  $U$  be the entropy solution and  $U_\Delta$  be the numerical approximation, then we have the following convergence rate estimates hold:

$$\|U - U_\Delta\|_{(L^1(\mathbb{R}))^N} = \mathcal{O}(\Delta t)^{1/3}. \tag{5.6}$$

In particular, if  $\text{TV}(s) = 0$ , then we have

$$\|U - U_\Delta\|_{(L^1(\mathbb{R}))^N} = \mathcal{O}(\Delta t)^{1/2}. \tag{5.7}$$

*Proof.* Owing to the CFL condition (cf. (3.5) and (3.6) for the Lax–Friedrichs and Godunov type scheme, respectively),  $\Delta x = \mathcal{O}(\Delta t)$ . Furthermore, the initial approximation (3.1) implies  $\|U_0^\Delta - U_0\|_{(L^1(\mathbb{R}))^N} = \mathcal{O}(\Delta t)$ . Observe that for  $\text{TV}(s) \neq 0$ , the Kuznetsov-type estimate, cf. (4.3), contains  $\epsilon/\epsilon_0$ . Hence, we choose  $\epsilon = (\Delta t)^{2/3}$  and  $\epsilon_0 = (\Delta t)^{1/3}$  and finally substitute the assertions of Lemma 5.1 and the estimate  $\gamma(U^\Delta, (\Delta t)^{1/3}) \leq \mathcal{O}(\Delta t)^{1/3}$  (cf. Theorem 3.2(5)) in (4.3) to get the convergence rate estimate (5.6). On the other hand, if  $\text{TV}(s) = 0$ , then the term  $\frac{\epsilon}{\epsilon_0}$  will not appear in the Kuznetov-type estimate, cf. (4.3), allowing us to choose  $\epsilon = \epsilon_0 = \sqrt{\Delta t}$  and the estimate  $\gamma(U^\Delta, (\Delta t)^{1/2}) \leq \mathcal{O}(\Delta t)^{1/2}$  to get the better convergence rate estimate (5.7).  $\square$

**REMARK 5.3** Kuznestov-type lemma 4.1 remains valid in several space dimensions as well; however, for  $\sigma$  discontinuous, a uniform BV bound on the finite volume approximation  $U_\Delta$  is not available. On the other hand, for the following system in 2 space dimensions with  $\sigma \equiv 1$  and with the  $k^{\text{th}}$  being:

$$\partial_t U^k + \partial_x \left( f^k(U^k) v^k((\mu * U)^k) \right) + \partial_y \left( g^k(U^k) \bar{v}^k((\bar{\mu} * U)^k) \right) = 0, \quad k = 1, 2, \dots, N, \tag{5.8}$$

such bounds on the numerical approximation can be derived, see Aggarwal *et al.* (2015), and consequently Lemma 5.1 and Theorem 5.2 can be reproduced using dimension splitting arguments (see Holden *et al.*, 2010, Sec. 5; Holden & Risebro, 2015, Sec. 4.3) for any space dimension. We illustrate the experimental convergence rates numerically for two space dimensions in the next section.

### 6. Numerical experiments

We now present some numerical experiments to illustrate the theory presented in the previous sections. We show the results for the Lax–Friedrichs scheme. The results obtained by the Godunov scheme are similar and are not shown here. Throughout the section,  $\theta = \theta_x = \theta_y$  is chosen to be  $1/3$  (satisfying the condition (3.4) for the choices of  $\sigma$  in the upcoming examples), and  $\lambda$  and  $\lambda_x = \lambda_y$  are chosen so as to satisfy the CFL condition (3.5).

### 6.1 One dimension

We consider the *discontinuous* generalization of the nonlocal Keyfitz–Kranzer system introduced in Aggarwal *et al.* (2015) in one space dimension, which reads as

$$\begin{cases} \partial_t U^1 + \partial_x (\sigma(x)U^1 v(\mu * U^1, \mu * U^2)) = 0, \\ \partial_t U^2 + \partial_x (\sigma(x)U^2 v(\mu * U^1, \mu * U^2)) = 0. \end{cases} \tag{6.1}$$

We choose the kernel

$$\mu(x) = L(-x(0.125 + x))^{5/2} \mathbb{1}_{(-0.125,0)}(x),$$

where  $L$  is such that  $\int_{\mathbb{R}} \mu(x) dx = 1$ , the velocity  $v(a, b) = (1 - a^2 - b^2)^3$  and a BV function

$$\sigma(x) = \begin{cases} \frac{1}{3}a_1, & x \leq a_1, \\ \frac{1}{3} \sum_{n=2}^{\infty} a_n \mathbb{1}_{[a_n, a_{n+1})}(x), & x \in (a_1, 3], a_n = 3(1 - 0.8^n), \\ 1, & x > 3, \end{cases}$$

which admits infinitely many discontinuities at the points  $\{a_n\}_{n \in \mathbb{N}}$ . The system (6.1) fits into the setup of this article (cf. (1.1)) with  $N = 2$ ,  $\boldsymbol{\mu} = \begin{bmatrix} \mu & \mu \\ \mu & \mu \end{bmatrix}$ ,  $v^k = v$ ,  $f^k(u) = u$  and  $\sigma^k = \sigma$ .

The local counterpart of (6.1), viz.

$$\partial_t U^k + \partial_x (\sigma(x)U^k v(U^1, U^2)) = 0, \quad k \in \{1, 2\}, \tag{6.2}$$

has not been studied yet in the literature and is of independent interest. However, the case  $\sigma \equiv 1$  is well understood, see, for example, Koley & Risebro (2013).

We present the numerical integration of (6.1) on the domain  $[0, 4]$  and the time interval  $[0, 0.3]$  with

$$U_0^1(x) = 0.25 \mathbb{1}_{(1,3)}(x), \quad U_0^2(x) = \mathbb{1}_{(1,3)}(x). \tag{6.3}$$

Figure 1 displays the numerical approximations of (6.1) and (6.3) generated by the numerical scheme (3.2), with decreasing grid size  $\Delta x$ , starting with  $\Delta x = 0.00625$ . It can be seen that the numerical scheme is able to capture both shocks and rarefactions well. Let  $e_{\Delta x}(T) = \left\| U_{\Delta x}(T, \cdot) - U_{\Delta x/2}(T, \cdot) \right\|_{(L^1(\mathbb{R}))^N}$ . The convergence rate reads  $\alpha = \log_2(e_{\Delta x}(T)/e_{\Delta x/2}(T))$ . Errors and the convergence rates at time  $T = 0.3$  obtained in the above experiment are recorded in Fig. 2. The convergence rates thus obtained obey the claims of Theorem 5.2. The results recorded in Fig. 2 show that the observed convergence rates exceed 1.

### 6.2 Two dimensions

As pointed out earlier, the analysis of the paper fits in multiple space dimensions with  $\sigma \equiv 1$ . To illustrate our results in two dimensions, we employ the two-dimensional *nonlocal* generalization of

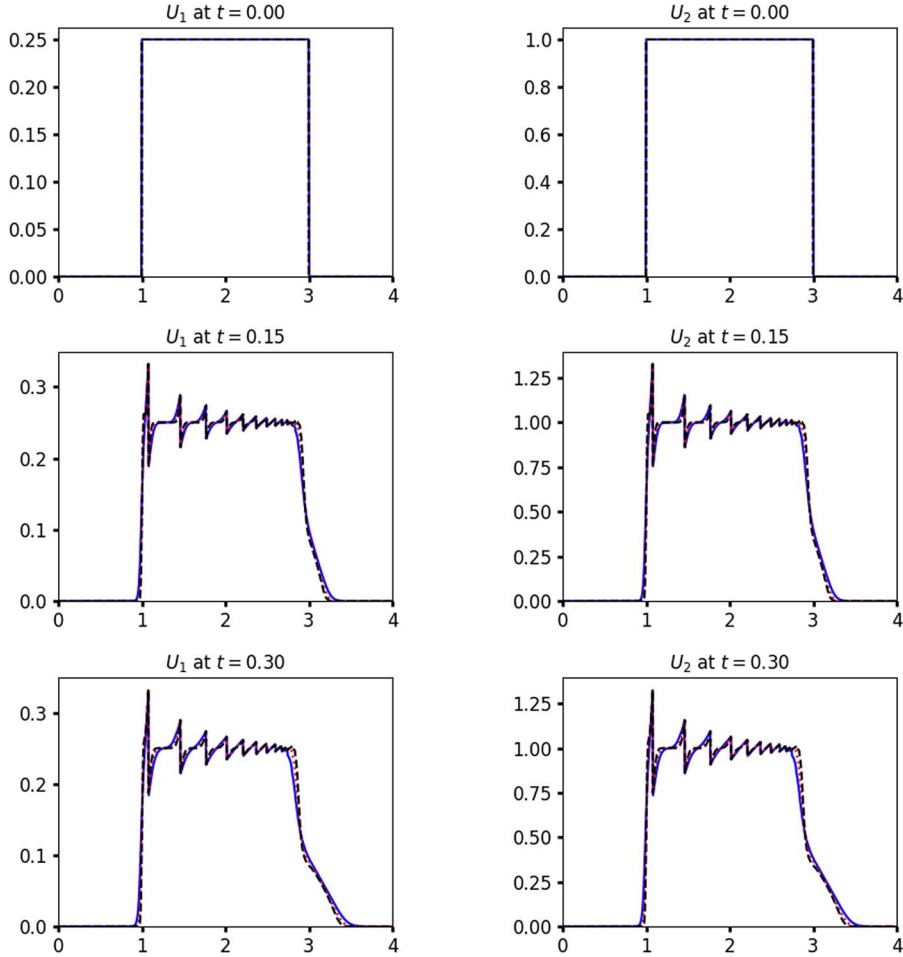


FIG. 1. Solution to (6.1) and (6.3) on the domain  $[0, 4]$  at times  $t = 0.00, 0.15, 0.3$  with decreasing mesh size  $\Delta x = 0.00625$  (—),  $\Delta x = 0.00625/2$  (---) and  $\Delta x = 0.00625/4$  (- - -).

Keyfitz–Kranzer system, introduced in Aggarwal *et al.* (2015), given by

$$\begin{cases} \partial_t U^1 + \partial_x (U^1 \phi_1(U^1 * \mu, U^2 * \mu)) + \partial_y (U^1 \phi_2(U^1 * \mu, U^2 * \mu)) = 0, \\ \partial_t U^2 + \partial_x (U^2 \phi_1(U^1 * \mu, U^2 * \mu)) + \partial_y (U^2 \phi_2(U^1 * \mu, U^2 * \mu)) = 0, \end{cases} \quad (6.4)$$

which is a system of two nonlocal conservation laws in two space dimensions. Here, we choose  $\phi_1(a, b) = \sin(a^2 + b^2)$ ,  $\phi_2(a, b) = \cos(a^2 + b^2)$  and

$$\mu(x, y) = L \left( \eta^2 - (x^2 + y^2) \right)^3 \mathbb{1}_{\{(x,y) : x^2 + y^2 \leq \eta^2\}}(x, y),$$

$\frac{\Delta x}{0.00625}$	$e_{\Delta x}(T)$	$\alpha$
1	$5.85e - 4$	1.32
1/2	$2.34e - 4$	1.31
1/4	$9.48e - 5$	1.18
1/8	$4.16e - 5$	

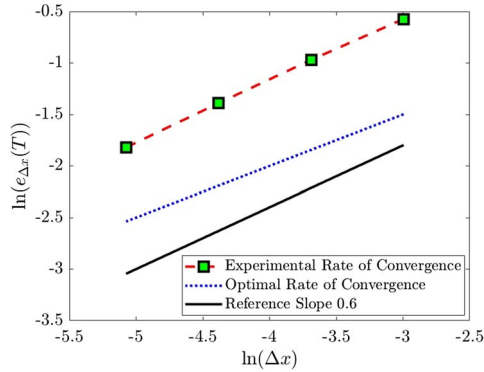


FIG. 2. Convergence rate  $\alpha$  for the numerical scheme (3.2) on the domain  $[0, 4]$  at time  $T=0.3$  for the approximate solutions to the problem (6.1), (6.3)

$\frac{\Delta x}{0.05}$	$e_{\Delta x}(T)$	$\alpha$
1	$5.64e - 1$	0.57
1/2	$3.79e - 1$	0.60
1/4	$2.49e - 1$	0.62
1/8	$1.62e - 1$	

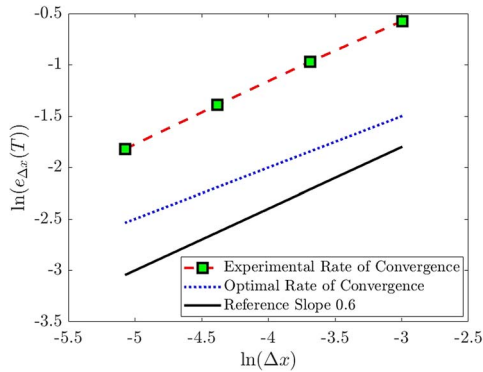


FIG. 3. Convergence rate  $\alpha$  for the numerical scheme (3.2) on the domain  $[-1.1, 1.1]^2$  at time  $T = 0.1$  for the approximate solutions to the problem (6.4)–(6.5).

where  $L$  is chosen so that  $\iint_{\mathbb{R}^2} \mu(x, y) \, dx \, dy = 1$ . The system fits into the setup of this article (cf. (1.1)) with  $N = 2, \mu = \bar{\mu} = \begin{bmatrix} \mu & \mu \\ \mu & \mu \end{bmatrix}, v^k = \phi_1, \bar{v}^k = \phi_2, f^k(u) = g^k(u) = u$  and  $\sigma^k = 1$ . We refer to Aggarwal et al. (2015) for the numerical integration of (6.4) on the domain the domain  $[-1.1, 1.1]^2$  and to the time interval  $[0, 0.1]$  with

$$U_o(x, y) = \begin{cases} (1, \sqrt{3}) & (x, y) \in ]0, 0.4] \times ]0, 0.4] \\ (\sqrt{2}, 1) & (x, y) \in ]-0.4, 0] \times ]0, 0.4] \\ (\frac{1}{2}, \frac{1}{3}) & (x, y) \in ]-0.4, 0] \times ]-0.4, 0] \\ (\sqrt{3}, \sqrt{2}) & (x, y) \in ]0, 0.4] \times ]-0.4, 0] \\ (0, 0) & \text{elsewhere} \end{cases} \quad (6.5)$$

and present the convergence rate analysis in this paper, with  $\eta = 0.4$ . Errors and the convergence rates obtained in the experiment are recorded in Fig. 3. The convergence rates thus obtained obey the claims of Theorem 5.2. The results recorded in Figs 2–3 show that the observed convergence rates exceed 0.5.

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## Conflict of interest

The authors have no conflicts of interest to declare.

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