PRECISE ERROR BOUNDS FOR NUMERICAL APPROXIMATIONS OF FRACTIONAL HJB EQUATIONS

INDRANIL CHOWDHURY AND ESPEN R. JAKOBSEN

Abstract. We prove precise rates of convergence for monotone approximation schemes of fractional and nonlocal Hamilton-Jacobi-Bellman (HJB) equations. We consider diffusion corrected difference-quadrature schemes from the literature and new approximations based on powers of discrete Laplacians, approximations which are (formally) fractional order and 2nd order methods. It is well-known in numerical analysis that convergence rates depend on the regularity of solutions, and here we consider cases with varying solution regularity: (i) Strongly degenerate problems with Lipschitz solutions, and (ii) weakly non-degenerate problems where we show that solutions have bounded fractional derivatives of order $\sigma \in (1, 2)$. Our main results are optimal error estimates with convergence rates that capture precisely both the fractional order of the schemes and the fractional regularity of the solutions. For strongly degenerate equations, these rates improve earlier results. For weakly nondegenerate problems of order greater than one, the results are new. Here we show improved rates compared to the strongly degenerate case, rates that are always better than $\mathcal{O}(h^{\frac{1}{2}})$.

CONTENTS

2020 Mathematics Subject Classification. 49L25, 35J60, 34K37, 35R11, 35J70, 45K05, 49L25, 49M25, 93E20, 65N06, 65R20, 65N15, 65N12.

Key words and phrases. Fractional and nonlocal equations, fully nonlinear equation, HJB equations, degenerate equation, weakly non-degenerate equation, stochastic control, Lévy processes, error estimate, rate of convergence, viscosity solution, numerical method, monotone scheme, powers of discrete Laplacians.

I.C. was supported by ERCIM "Alain Bensoussan" Fellowship programme and INSPIRE faculty fellowship (IFA22-MA187).

E.R.J. received funding from the Research Council of Norway under Grant Agreement No. 325114 "IMod. Partial differential equations, statistics and data: An interdisciplinary approach to data-based modelling".

1. Introduction

In this paper we prove precise rates of convergence for monotone approximation schemes of fractional and nonlocal Hamilton-Jacobi-Bellman (HJB) equations. Weakly non-degenerate problems are studied, and we give error bounds with convergence rates capturing both the fractional orders of accuracy of schemes and regularity of solutions.

HJB equations are fully nonlinear possibly degenerate PDEs from optimal control theory with a large number of applications in engineering, science, economics etc. [\[7,](#page-29-0) [37,](#page-30-0) [55,](#page-30-1) [40\]](#page-30-2). In this paper we focus on the following nonlocal version:

$$
\sup_{\alpha \in \mathcal{A}} \{ f^{\alpha}(x) + c^{\alpha}(x)u(x) - \mathcal{I}^{\alpha}[u](x) \} = 0 \quad \text{in} \quad \mathbb{R}^{N}, \tag{1.1}
$$

where $N \in \mathbb{N}$, A is a compact metric space, the integral operator

$$
\mathcal{I}^{\alpha}[\phi](x) := \int_{\mathbb{R}^N \setminus \{0\}} \left(\phi(x + \eta^{\alpha}(z)) - \phi(x) - \eta^{\alpha}(z) \cdot \nabla_x \phi(x) 1_{|z| < 1} \right) \nu_{\alpha}(dz), \tag{1.2}
$$

and the Lévy measure ν_{α} is nonnegative with $\int |z|^2 \wedge 1 \nu_{\alpha}(dz) < \infty$. The operator \mathcal{I}^{α} is a fractional (convection-)diffusion operator of maximal order $\sigma \in (0, 2)$. Our assumptions encompass fractional Laplacians, tempered operators from finance, and any other generator of a pure jump Lévy process. The coefficients are bounded, continuous, and x-Lipschitz uniformly in α , see Section [2](#page-4-0) for all assumptions.

Equation (1.1) is the dynamic programming equation for the value function of an infinite horizon optimal stochastic control problem [\[55,](#page-30-1) [40\]](#page-30-2):

$$
u(x) = \inf_{\alpha \in \mathcal{A}_{ad}} \int_0^\infty e^{-\int_0^s e^{\alpha r} (X_r) dr} f^{\alpha_s}(X_s) ds,
$$

where \mathcal{A}_{ad} is the set of admissible controls and the (controlled) process X_s is given by a Lévy driven SDE $[1, 27]$ $[1, 27]$ of the form

$$
dX_s = x + b^{\alpha_s} ds + \int_{|z| < 1} \eta^{\alpha_s}(z) \tilde{N}(\alpha_s, dz, ds) + \int_{|z| \ge 1} \eta^{\alpha_s}(z) N(\alpha_s, dz, ds). \tag{1.3}
$$

The (compensated) Poisson random measure $N(\tilde{N})$ of the driving Lévy process has an intensity/Lévy measure ν_a such that $\mathbb{E}[N(a, B, (0,t))] = \nu_a(B)t^1$ $\mathbb{E}[N(a, B, (0,t))] = \nu_a(B)t^1$ for Borel sets $B \not\supseteq 0$ and $t > 0$. For simplicity we focus on pure (jump) diffusion processes with $b = 0$ and HJB equations of the type (1.1) , but at the end of the paper we give results for more general equations with $b \neq 0$.

The operator \mathcal{I}^{α} will always be at least degenerate elliptic under our assumptions. When we also assume (loosely speaking) that

$$
\left(\frac{d\nu_{\alpha}}{dz}, \eta^{\alpha}(z)\right) \to \left(\frac{c_1}{|z|^{N+\sigma}}, c_2 z\right) \quad \text{as} \quad z \to 0,
$$

¹This is the expected number of jumps $z \in B$ of the Lévy process up to time t [\[1,](#page-28-1) [55\]](#page-30-1).

 \mathcal{I}^{α} will be non-degenerate and uniformly elliptic. We refer to $(\mathbf{B}.1)$ $(\mathbf{B}.1)$ and $(\mathbf{A}.6)$ $(\mathbf{A}.6)$ for precise assumptions. The HJB equation (1.1) is *strongly degenerate* if the operators \mathcal{I}^{α} are degenerate for every α ,^{[2](#page-2-0)} and it is *weakly non-degenerate* if there is at least one α for which \mathcal{I}^{α} is elliptic/non-degenerate. Obstacle problems for elliptic operators are examples of weakly non-degenerate problems [\(1.1\)](#page-1-1), and they are known to have non-smooth solutions (at the contact set). The correct (weak) solution concept for this type of problems is viscosity solutions [\[46,](#page-30-3) [47,](#page-30-4) [3\]](#page-28-2). Wellposedness, regularity, asymptotics, approximations, and other properties of viscosity solutions for nonlocal PDEs has been intensely studied in recent years. Regularity in the strongly degenerate case comes from comparison type of arguments and typically gives preservation of the regularity of the data $[47]$. Solutions can then be at most Lipschitz continuous. In non-degenerate cases there is a regularizing effect. The regularity theory has mostly been developed for uniformly elliptic/parabolic problems, and the huge literature includes seminal works of Caffarelli and Silvestre [\[18,](#page-29-2) [19\]](#page-29-3). In the weakly non-degenerate case there are few results, and most relevant for us (our inspiration) is $[32]$ for local problems. We show here that weakly non-degenerate problems of order $\sigma \in (1,2)$ have solutions with bounded fractional derivatives of order σ ^{[3](#page-2-1)}. Hence solutions are more smooth than in the strongly degenerate case. Independently, similar type of regularity results have been obtained in the very recent preprint [\[59\]](#page-31-0) on nonlocal obstacle problems.

There is a huge literature on numerical methods for local HJB equations including finite differences, semi-Lagrangian, finite elements, spectral, Monte Carlo, and many more, see e.g. [\[29,](#page-29-4) [53,](#page-30-6) [36,](#page-30-7) [6,](#page-29-5) [54,](#page-30-8) [17,](#page-29-6) [14,](#page-29-7) [30,](#page-29-8) [61,](#page-31-1) [16,](#page-29-9) [39\]](#page-30-9). For fractional and nonlocal problems, there is the added difficulty of discretizing the fractional and nonlocal operators in a monotone, stable, and consistent way. These operators are singular integral operators, and can be discretized by quadrature after truncating the singular part and correct with a suitable second derivative term. This diffusion corrected approximation was introduced on the level of processes in [\[2\]](#page-28-3) and then for linear PDEs e.g. in [\[28\]](#page-29-10) in connection with difference-quadrature schemes, see also $[48, 11, 41]$ $[48, 11, 41]$ $[48, 11, 41]$ $[48, 11, 41]$. In the setting of HJB equations, it was introduced in $[48, 22, 11]$ $[48, 22, 11]$ $[48, 22, 11]$ $[48, 22, 11]$ with further developments in e.g. $[8, 26, 56, 34]$ $[8, 26, 56, 34]$ $[8, 26, 56, 34]$ $[8, 26, 56, 34]$ $[8, 26, 56, 34]$ $[8, 26, 56, 34]$. We will give new results for this approximation here, and focus on a version based on semi-Lagrangian type approximations $[21, 30]$ $[21, 30]$ of the nonlocal operators $[22]$. Another way of discretizing certain fractional operators, is via subordination: When the operator is a fractional Laplacian, it can be discretized by a (fractional) power of the discrete (FDM) Laplacians which can be seen as a quadrature rule with explicit weights [\[25\]](#page-29-16). While the diffusion corrected approximation has fractional order accuracy, the power of discrete Laplacian approximation is always of second order and faster when the order of the equation is close to 2. This last approximation has previously been used to solve linear and porous medium equations [\[35,](#page-30-14) [13\]](#page-29-17). In this paper, we will explain how it can be used to solve HJB equations and provide error bounds.

The main focus of the paper is on precise error bounds for the schemes and regularity settings mentioned above, especially the weakly non-degenerate case. In numerical analysis it is well-known that such bounds must depend on both the accuracy of the method and the regularity of solutions. In our fractional setting,

 ${}^{2}E.g.$ there could be no diffusion in some directions, or the operator could be a 0 order operator with bounded Lévy measure. There could be different degeneracies for different α 's.

³We assume that the data is semiconcave to achieve this.

both of these may be fractional, and previous results are either not optimal or lacking. While linear, local, and smooth problems can analyzed in a rather simple and classical way [\[57\]](#page-31-2), error analysis is more complicated in our fully nonlinear and non-smooth setting. There are two main approaches:^{[4](#page-3-0)} (i) The 'doubling of variables' technique for fully nonlinear equations of 1st order $[23, 29, 62]$ $[23, 29, 62]$ $[23, 29, 62]$ $[23, 29, 62]$ or fractional order less than 2×26 ; and (ii) the 'shaking of coefficients' method for convex HJB equations of 2nd order [\[4,](#page-28-4) [5,](#page-29-19) [32,](#page-30-5) [49,](#page-30-15) [50,](#page-30-16) [51\]](#page-30-17) or fractional order [\[10,](#page-29-20) [11,](#page-29-11) [48\]](#page-30-10).

The 'shaking of coefficients' method, originally introduced by Krylov, relies on constructing smooth subsolutions of both the equation and the scheme which can then be used to get one-sided error estimates via the comparison principle and local consistency bounds. If precise regularity results for both the scheme and the equation are known, along with sharp consistency bounds, the method produces optimal rates. We refer to [\[32,](#page-30-5) [43,](#page-30-18) [51\]](#page-30-17) for local 2nd order problems and [\[10,](#page-29-20) [22,](#page-29-12) [48\]](#page-30-10) for nonlocal problems. If regularity of the scheme is not known (this is difficult in general), sub-optimal rates can still be proved $[4, 5, 11]$ $[4, 5, 11]$ $[4, 5, 11]$ $[4, 5, 11]$, and these latter bounds holds for a very large class of monotone schemes. Note that the 'shaking of coefficients' method has the advantage that it can handle arbitrary high order error equations and therefore also higher order methods, while the 'doubling of variables' method only work optimally for schemes with (at most) 2nd order truncation errors. For nonlocal HJB equations, most of the progress on optimal error bounds for monotone schemes have addressed bounded (non-singular) integral operators [\[10,](#page-29-20) [48\]](#page-30-10). Non-optimal bounds for problems with singular operators can then be obtained after first approximation by bounded operators. Without this approximation step, sub-optimal rates have been obtained in [\[11\]](#page-29-11) for singular integral operators.

Our main contributions:

(a) A rigorous error analysis for monotone approximations of weakly non-degenerate problems is developed in Section [5.](#page-15-0) This is new and based on the "method of shaking the coefficients". The proof amounts to extending the analysis of [\[32\]](#page-30-5) to nonlocal/fractional equations and schemes. Our setting is more involved and technical. The main challenges are related to the fractional approximation, regularization, and regularity results needed – both for the equation and the scheme. As opposed to previous nonlocal results, we cannot use standard mollifiers for regularization but crucially need fractional heat kernels. For the schemes, the results are discrete and contain error terms, and a very careful analysis is needed to get optimal results.

(b) $C^{1,\sigma-1}$ -regularity results for weakly non-degenerate HJB-equations of order $\sigma \in$ $(1, 2)$ given in Theorem [2.7.](#page-9-0) These are natural extensions to nonlocal/fractional problems of the $W^{2,\infty}$ results of [\[32\]](#page-30-5). They seem to be new for equations of fractional order (but see also [\[59\]](#page-31-0)) and are of independent interest. Our proof is based on uniform estimation of approximate fractional derivatives based on semi-concavity estimates and exploitation of weak non-degeneracy followed by an application of regularity results for linear problems in [\[58\]](#page-31-4). We also need and prove discrete versions of such results.

(c) Precise error bounds for diffusion corrected difference-quadrature schemes in Section [3.](#page-10-0) Under various assumptions, we roughly speaking show that if σ is the

 4 In the uniformly elliptic case, there are other methods $[20, 52, 63]$ $[20, 52, 63]$ $[20, 52, 63]$ $[20, 52, 63]$. These results are not explicit nor optimal, but they apply also to nonconvex problems. See also [\[44,](#page-30-20) [15,](#page-29-22) [45\]](#page-30-21).

order of equation [\(1.1\)](#page-1-1), u its solution, and u_h the solution of scheme, then

$$
||u - u_h||_{L^{\infty}} \leq \begin{cases} C h^{\frac{1}{2}(4-\sigma)} & \text{when solutions are smooth } (C_b^4), \\ C h^{\frac{\sigma}{4+\sigma}(4-\sigma)} & \text{in the weakly non-degenerate case and } \sigma > 1, \\ C h^{\frac{1}{4+\sigma}(4-\sigma)} & \text{in the strongly degenerate case or when } \sigma \leq 1. \end{cases}
$$

Here the accuracy is a decreasing function of σ , which is reflected in decreasing rates in σ when the regularity is fixed (strongly degenerate and smooth cases). In the weakly non-degenerate case, regularity is increasing with σ and so are the rates despite decreasing accuracy. Rates are higher when solutions are more regular and maximal in the smooth case. These results are sharper than previous results [\[10,](#page-29-20) [11,](#page-29-11) [8\]](#page-29-13) in the strongly degenerate case, and new in the weakly non-degenerate case where the rate increases from $\frac{3}{5}$ at $\sigma = 1$ to $\frac{2}{3}$ in the limit as $\sigma \to 2$.

(d) New approximations based on powers of discrete Laplacians are introduced in Section [4](#page-14-0) for HJB equations with fractional Laplacians, $\mathcal{I}^{\alpha}[\phi] = -a^{\alpha}(-\Delta)^{\frac{\sigma}{2}}\phi$. These problems are always weakly non-degenerate, and we prove precise error bounds,

$$
||u - u_h||_{L^{\infty}} \le \begin{cases} Ch^{\frac{1}{2}} & \text{for} \quad 0 < \sigma \le 1, \\ Ch^{\frac{\sigma}{2}} & \text{for} \quad 1 < \sigma < 2. \end{cases}
$$
 (1.4)

Under our assumptions these rates are optimal, and as $\sigma \to 2$, the error bounds approach the $\mathcal{O}(h)$ bound in the local 2nd order case [\[32\]](#page-30-5).^{[5](#page-4-1)}

Outline. The remaining part of this paper is organized as follows: In Section [2](#page-4-0) we introduce the notation and assumptions for the strongly degenerate and weakly non-degenerate problems, and give wellposedness and regularity results for equation [\(1.1\)](#page-1-1) in both cases. In Section [3](#page-10-0) we consider the diffusion corrected differencequadrature approximations of [\(1.1\)](#page-1-1) for general nonlocal operators and state our main error bounds. In Section [4](#page-14-0) we give the results for approximation based on powers of discrete Laplacians. The proofs of these results are given in Sections [5](#page-15-0) and [6.](#page-24-0) In Section [7](#page-26-0) we discuss extensions to problems with non-zero drift and more non-symmetric diffusions.

2. Strongly and weakly non-degenerate fractional HJB equations

In this section we present the assumptions on nonlocal HJB equations and give wellposedness and regularity results. We start by introducing some notation. By C, K etc. we mean various constants which may change from line to line, | | is the euclidean norm, and the norms $||u||_0 = \sup_x |u(x)|$ and $||u||_1 =$ $|u|_0 + \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|}$ $\frac{x)-u(y)}{|x-y|}$. $C_b(Q)$ is the space of bounded continuous functions on $Q \subset \mathbb{R}^N$, while $C^n(Q)$ and $C^{n,\gamma}(Q)$ for $n \in \mathbb{N}$ and $\gamma \in (0,1]$, denote the spaces of n-th time continuously differentiable functions on Q with finite norms

$$
||u||_n = \sum_{j=0}^n ||D^j u||_0
$$
 and $||u||_{n,\gamma} = ||u||_n + \sup_{x \neq y} \frac{|D^n u(x) - D^n u(y)|}{|x - y|^{\gamma}},$

where $D^n u$ is the (*n*-form of) *n*-th order derivatives of *u*.

⁵When $\sigma \to 2$, problem [\(1.1\)](#page-1-1) converges by [\[24\]](#page-29-23) to local the 2nd order problem of [\[32\]](#page-30-5).

2.1. Assumptions and wellposedness of (1.1) . First we list assumptions needed for wellposedness and Lipschitz regularity of viscosity solutions of (1.1) .

- (A.1) A is a separable metric spaces, $c^{\alpha}(x) \geq \lambda > 0$, and $c^{\alpha}(x)$, $f^{\alpha}(x)$, and $\eta^{\alpha}(z)$ are continuous in α , x, and z.
- $(A.2)$ There is a $K > 0$ such that

$$
||f^{\alpha}||_1 + ||c^{\alpha}||_1 + ||\eta^{\alpha}||_0 \leq K \quad \text{for} \quad \alpha \in \mathcal{A}.
$$

 $(A.3)$ There is a $K > 0$ such that

$$
|\eta^{\alpha}(z)| \le K|z| \quad \text{for} \quad |z| < 1, \quad \alpha \in \mathcal{A}.
$$

(A.4) ν_{α} is a nonnegative Radon measures on \mathbb{R}^{N} and there is $K > 0$ such that

$$
\int_{|z|\leq 1}|z|^2\nu_{\alpha}(dz)+\int_{|z|> 1}\nu_{\alpha}(\,dz)\leq K.
$$

In some results we also need symmetry assumptions on the nonlocal terms and upper bounds on the density of the Lévy measure.

- (**A**.5) $\nu_{\alpha}(dz) 1_{|z|<1}$ is symmetric for $\alpha \in \mathcal{A}$.
- (A.6) ν_{α} is absolutely continuous on $|z| < 1$, and there are $\sigma \in (0, 2)$, $M \in \mathbb{N}$, and $C > 0$ such that

$$
0 \le \frac{d\nu_{\alpha}}{dz} \le \frac{C}{|z|^{M+\sigma}} \quad \text{for} \quad |z| < 1, \quad \alpha \in \mathcal{A}.
$$

$$
(\mathbf{A}.7) \ \eta^{\alpha}(-z) = -\eta^{\alpha}(z) \ \text{ for } \ |z| < 1 \text{ and } \alpha \in \mathcal{A}.
$$

Remark 2.1. (a) Under $(A.3)$ $(A.3)$ and $(A.4)$, any pure jump Lévy process is allowed as a driver for the SDE [\(1.3\)](#page-1-3). This includes stable, processes, tempered processes, spectrally one-sided process, compound Poisson processes, and most jump processes considered in finance [\[1,](#page-28-1) [27\]](#page-29-1). The generators of these processes are \mathcal{I}^{α} .

(b) Assumption $(A.6)$ $(A.6)$ is a restriction implying that \mathcal{I}^{α} (which may be degenerate) contains fractional derivatives of orders at most σ . It can be replaced by a more general integral condition to also cover non-absolutely continuous L´evy measures,

$$
r^{-2+\sigma}\int_{|z|
$$

for some $C > 0$ independent of α and $r \in (0,1)$. This condition is satisfied e.g. sums of one-dimensional operators (possibly of different orders) satisfying $(A.6)$.

(c) By symmetry (**A**[.5\)](#page-5-4) and (**A**[.7\)](#page-5-5) it is clear that $\int_{\delta < |z| < 1} \eta^{\alpha}(z) \nu_{\alpha}(dz) = 0$. Hence we can also define \mathcal{I}^{α} in [\(1.2\)](#page-1-4) using principal values and dropping the gradient (compensator) term.

(d) Note that $(A.3)$ – $(A.7)$ $(A.7)$ give no restrictions on the tails of the Lévy measures and the nonsingular part of the nonlocal operators. This possibly non-symmetric part could be the generator of any compound Poisson process.

(e) The fractional Laplacian $-(-\Delta)^{\frac{\sigma}{2}}$, where $\eta^{\alpha} = z$ and $\nu(dz) = \frac{c_{\alpha,N}}{|z|^{N+\sigma}}dz$, is a special case satisfying all assumptions $(A.3)$ – $(A.7)$, see also section [4.](#page-14-0)

A definition and general theory of viscosity solution for the nonlocal equations like (1.1) can be found e.g. in [\[46,](#page-30-3) [3\]](#page-28-2), but we do not need this generality here. In particular since there is no local diffusion, we could follow the simpler (comparison) arguments of [\[24\]](#page-29-23). Wellposedness and Lipschitz regularity for solutions of equation [\(1.1\)](#page-1-1) are given in the next result.

Proposition 2.2. Assume $(A.1)$ $(A.4)$.

(a) If u and v are bounded upper semicontinuous viscosity subsolution and bounded lower semicontinuous supersolution of (1.1) , then

$$
u \le v \quad in \quad \mathbb{R}^N.
$$

- (b) There exists a unique viscosity solution $u \in C_b(\mathbb{R}^N)$ of equation [\(1.1\)](#page-1-1).
- (c) The viscosity solution u of (1.1) is Lipschitz continuous,

$$
||u||_0 \leq \frac{1}{\lambda} \sup_{\alpha \in \mathcal{A}} ||f^{\alpha}||_0
$$
, $||Du||_0 \leq \frac{1}{\lambda} \sup_{\alpha \in \mathcal{A}} (||Df^{\alpha}||_0 + ||Dc^{\alpha}||_0 ||u||_0)$.

Proof. We refer to $[24]$ Theorems 2.1, 2.3, and Corollary 2.3 for the proof (see also $[42]$) of parts (a), (b), and the first part of (c). The second estimate in (c) follows by the comparison principle in a standard way. \Box

2.2. Extra regularity for weakly non-degenerate equations. A weakly nondegenerate version of (1.1) is

$$
\lambda u(x) + \sup_{\alpha \in \mathcal{A}} \{ f^{\alpha}(x) - \mathcal{I}^{\alpha}[u](x) \} = 0, \tag{2.1}
$$

where to simplify we have set $c^{\alpha}(x) \equiv \lambda > 0$. We assume slightly more regularity of f and weak degeneracy in the following sense:

(B.1) Weak-degeneracy: There are $\alpha_0 \in \mathcal{A}$, $c_{\alpha_0} > 0$, and $K \geq 0$, such that

(i)
$$
\frac{d\nu_{\alpha_0}}{dz} \ge \frac{c_{\alpha_0}}{|z|^{N+\sigma}}
$$
 for $|z| < 1$,
\n(ii) $|\eta^{\alpha_0}(z) - \eta^{\alpha_0}(0) - z| \le K|z|^2$ for $|z| < 1$.

(**B**.2) There is $\beta > (\sigma - 1)^+$ and $K > 0$ such that $||f^{\alpha}||_{1, \beta} \leq K$ for every $\alpha \in \mathcal{A}$.

Remark 2.3. (a) Assumption (B[.1\)](#page-6-1) is a lower bound on the order of differentiability of \mathcal{I}^{α_0} and implies that it is elliptic/non-degenerate. The lower bounds behaves as $z \to 0$ as the $\frac{\sigma}{2}$ -fractional Laplacian.

(b) weakly non-degenerate in $(\mathbf{B}.1)$ $(\mathbf{B}.1)$ means that there is at least one α_0 such that \mathcal{I}^{α_0} is non-degenerate. If \mathcal{I}^{α} is non-degenerate for all α , with uniform bounds in $(B.1)$, then equation (1.1) is (uniformly/strongly) non-degenerate and have classical solutions.

We prove our regularity results via an approximate problem where the Lévy measure is truncated near origin:

$$
\lambda u(x) + \sup_{\alpha \in \mathcal{A}} \{ f^{\alpha}(x) - \mathcal{I}^{\alpha, r}[u](x) \} = 0 \quad \text{in} \quad \mathbb{R}^{N}, \tag{2.2}
$$

where $\mathcal{I}^{\alpha,r}$ is defined by

$$
\mathcal{I}^{\alpha,r}\phi(x) := \int_{|z|>r} \left(\phi(x + \eta^{\alpha}(z)) - \phi(x) \right) \nu_{\alpha}(dz).
$$

Note that $\mathcal{I}^{\alpha,r}$ is a bounded operator, well-defined for bounded functions, and then that viscosity solutions of equation [\(2.2\)](#page-6-2) also will be pointwise/classical solutions. This problem is well-posed by Proposition [2.2,](#page-6-3) and we have the following stability and approximation results:

Lemma 2.4. Assume $(A.1)-(A.4)$ $(A.1)-(A.4)$ $(A.1)-(A.4)$, $(A.6)$, u_r and u are the unique bounded solutions of (2.2) and (2.1) . Then there is a $C > 0$ independent of r such that

$$
||u_r||_{0,1} \leq \frac{1}{\lambda} \sup_{\alpha \in \mathcal{A}} ||f^{\alpha}||_{0,1}
$$
 and $||u - u_r||_0 \leq C r^{1-\frac{\sigma}{2}}$.

Proof. The first part follows from Proposition [2.2](#page-6-3) (c). By a continuous dependence result,

$$
||u - u_r||_0 \le K \sup_{\alpha \in \mathcal{A}} \left(\int_{|z| < r} |z|^2 \, \nu_\alpha(dz) \right)^{\frac{1}{2}}
$$

for some $K > 0$ independent of r. Since $\int_{|z| < r} |z|^2 \nu_{\alpha}(dz) = C r^{2-\sigma}$ by $(\mathbf{A}.6)$, the second part follows. The continuous dependence result is the stationary version of Theorem 4.1 in [\[46\]](#page-30-3) and can be proved in a similar way. We omit the proof here. \Box

We introduce a truncated fractional Laplacian,

$$
\Delta^{\sigma,r}[\phi](x) = \int_{|z|>r} (\phi(x+z) - \phi(x)) \frac{dz}{|z|^{N+\sigma}}.
$$

Theorem 2.5. Assume $(A.1)-(A.7), (B.1)-(B.2),$ $(A.1)-(A.7), (B.1)-(B.2),$ $(A.1)-(A.7), (B.1)-(B.2),$ $(A.1)-(A.7), (B.1)-(B.2),$ $(A.1)-(A.7), (B.1)-(B.2),$ $(A.1)-(A.7), (B.1)-(B.2),$ and u_r is the unique viscosity solution of [\(2.2\)](#page-6-2). Then for any $r > 0$ there is a $K > 0$ independent of r such that

$$
\|\Delta^{\sigma,r}[u_r]\|_0 \le \frac{K}{c_{\alpha_0}}.\tag{2.3}
$$

Proof. Let us define the bounded auxiliary operator

$$
\mathcal{J}^r[\phi](x) = \int_{|z|>r} (\phi(x+\eta^{\alpha_0}(z)) - \phi(x)) \frac{c_{\alpha_0}dz}{|z|^{N+\sigma}}.
$$

1) A uniform bound on $w_r := -\mathcal{J}^r[u_r]$. Fix $x \in \mathbb{R}^N$. By [\(2.2\)](#page-6-2) and properties of suprema, for any $\epsilon > 0$ there exists $\bar{\alpha} \in \mathcal{A}$ such that

$$
\lambda u_r(x) + f^{\bar{\alpha}}(x) - \mathcal{I}^{\bar{\alpha},r} u_r(x) \ge -\epsilon,
$$
\n(2.4)

and (trivially) for any $y \in \mathbb{R}^N$,

$$
\lambda u_r(x+y) + f^{\bar{\alpha}}(x+y) - \mathcal{I}^{\bar{\alpha},r}u_r(x+y) \le 0. \tag{2.5}
$$

Take $y = \eta^{\alpha_0}(z)$, subtract equations [\(2.4\)](#page-7-0) and [\(2.5\)](#page-7-1), multiply by $\frac{c_{\alpha_0}}{|z|^{N+\sigma}}$, and integrate over $|z| > r$. The result is then

$$
\lambda\left(-\mathcal{J}^r[u_r](x)\right)-\mathcal{J}^r[f^{\bar{\alpha}}](x)-\mathcal{J}^r\big[-\mathcal{I}^{\bar{\alpha},r}[u_r]\big](x)\geq-\epsilon.
$$

This inequality holds for $\bar{\alpha}$ and then also holds for the supremum over all $\alpha \in \mathcal{A}$. Since $\epsilon > 0$ and $x \in \mathbb{R}^N$ are arbitrary, \mathcal{J}^r and $\mathcal{I}^{\alpha,r}$ are linear operators, and by Fubini $\mathcal{J}^r[\mathcal{I}^{\bar{\alpha},r}[u_r]] = \mathcal{I}^{\bar{\alpha},r}[\mathcal{J}^r[u_r]]$, by the definition of w_r we have

$$
\lambda w_r(x) + \sup_{\alpha \in \mathcal{A}} \left\{ -\mathcal{J}^r[f^\alpha](x) - \mathcal{I}^{\alpha, r}[w_r](x) \right\} \ge 0 \quad \text{in} \quad \mathbb{R}^N. \tag{2.6}
$$

By assumption (B[.2\),](#page-6-5) $C := \sup_{\alpha \in \mathcal{A}} ||\mathcal{J}^r[f^\alpha]||_0 < \infty$, so $-\frac{C}{\lambda}$ is a subsolution of (2.6) .^{[6](#page-8-0)} Then by comparison, Proposition [2.2](#page-6-3) (a),^{[7](#page-8-1)}

$$
-\mathcal{J}^r[u_r] = w_r \ge -\frac{C}{\lambda} \quad \text{in} \quad \mathbb{R}^N. \tag{2.7}
$$

To get a lower bound on $\mathcal{J}^r[u_r]$, we use the upper bound and weak degeneracy: $\tilde{\nu}_{\alpha_0}(z) - \frac{c_{\alpha_0}}{|z|^{N+\sigma}} \ge 0$ for $|z| < 1$. Let $y = \eta^{\alpha_0}(z)$, subtract (2.5) and (2.4) , multiply by $(\tilde{\nu}_{\alpha_0}(z) - \frac{c_{\alpha_0}}{|z|^{N+\sigma}})$, and integrate over $r < |z| < 1$. The result is

$$
\lambda \left(-(\mathcal{I}_1^{\alpha_0,r} - \mathcal{J}_1^r)[u_r](x) \right) - (\mathcal{I}_1^{\alpha_0,r} - \mathcal{J}_1^r)[f^{\bar{\alpha}}](x) - \mathcal{I}^{\bar{\alpha},r} \left[-(\mathcal{I}_1^{\alpha_0,r} - \mathcal{J}_1^r)[u_r] \right](x) \ge -\epsilon.
$$

where $\mathcal{J}_1^r[\phi](x) = \int_{r<|z|<1} (\phi(x + \eta^{\alpha_0}(z)) - \phi(x)) \frac{c_{\alpha_0}dz}{|z|^{N+\sigma}}$. Then arguing as for the upper bound we have

$$
-(\mathcal{I}_1^{\alpha_0,r} - \mathcal{J}_1^r)[u_r] \ge -\frac{C}{\lambda} \quad \text{in} \quad \mathbb{R}^N. \tag{2.8}
$$

The above estimate implies $-\mathcal{J}_1^r[u_r](x) \leq \frac{C}{\lambda} + \sup_{\alpha \in \mathcal{A}} \left\{-\mathcal{I}_1^{\alpha_0,r}[u_r](x)\right\}$, and therefore since u_r solves (2.2) , that

$$
-\mathcal{J}_1^r[u_r](x)
$$

\n
$$
\leq \frac{C}{\lambda} + \sup_{\alpha \in \mathcal{A}} \{-\mathcal{I}^{\alpha,r}[u_r] + f^{\alpha}(x)\} + \lambda u_r(x)
$$

\n
$$
+ \sup_{\alpha \in \mathcal{A}} ||f^{\alpha}||_0 + \lambda ||u_r||_0 + \sup_{\alpha \in \mathcal{A}} \left| \int_{|z| > 1} \left(u_r(x + \eta^{\alpha}(z)) - u_r(x) \right) \nu_{\alpha}(dz) \right|
$$

\n
$$
\leq \frac{C}{\lambda} + 0 + \sup_{\alpha \in \mathcal{A}} ||f^{\alpha}||_0 + \left(\lambda + 2 \sup_{\alpha \in \mathcal{A}} \int_{|z| > 1} \nu_{\alpha}(dz) \right) ||u_r||_0.
$$
 (2.9)

Let $\mathcal{J}^r = \mathcal{J}_1^r + \mathcal{J}^{1,r}$ where $\mathcal{J}^{1,r} = \int_{|z|>1} (\cdots) \frac{c_{\alpha_0} dz}{|z|^{N+\sigma}}$. By (A[.2\),](#page-5-7) (A[.4\),](#page-5-3) and Lemma [2.4,](#page-7-3) both the right hand side of (2.9) and $\mathcal{J}^{1,r}[u_r]$ are bounded, and hence

$$
-\mathcal{J}^r[u_r] \le C \qquad \text{in} \qquad \mathbb{R}^N, \tag{2.10}
$$

for some constant $C > 0$ independent of r. By (2.7) and (2.10) we conclude that $|w_r| = |\mathcal{J}^r[u_r]| \leq C_1$ for some other $C_1 > 0$ independent of r.

2) The bound on $\Delta^{\sigma,r}[u_r]$. Since $c_{\alpha_0} > 0$ by $(\mathbf{B}.1)$, from step 1) it follows that

$$
I := \Big| \int_{|z|>r} \left(u_r(x + \eta^{\alpha_0}(z)) - u_r(x) \right) \frac{dz}{|z|^{N+\sigma}} \Big| \leq \frac{C_1}{c_{\alpha_0}}.
$$

From this estimate, the bound $||u_r||_{0,1} \leq K$, and $(\mathbf{B}.1)(ii)$ $(\mathbf{B}.1)(ii)$ $(\mathbf{B}.1)(ii)$ and $(\mathbf{A}.3)$ $(\mathbf{A}.3)$ (implying $\eta^{\alpha}(0) = 0$, we see that

$$
|\Delta^{\sigma,r}[u_r](x)| \leq I + \int_{|z|>r} |u_r(x + \eta^{\alpha_0}(z)) - u_r(x + z)| \frac{dz}{|z|^{N+\sigma}}
$$

$$
\leq \frac{C_1}{c_{\alpha_0}} + ||Du_r||_0 \int_{|r| \leq |z| \leq 1} |z|^2 \frac{dz}{|z|^{N+\sigma}} + 2||u_r||_0 \int_{|z|>1} \frac{dz}{|z|^{N+\sigma}}.
$$

The right hand side is uniformly bounded so the proof is complete. \Box

 6 Replace \geq by $=$ in [\(2.6\)](#page-7-2).

⁷Equation [\(2.6\)](#page-7-2) (replace \geq by =) is of same form as in [\(1.1\)](#page-1-1).

Sending $r \to 0$ in the above result, we get a key result for this paper.

Corollary 2.6. Assume $(A.1)-(A.7)$ $(A.1)-(A.7)$ $(A.1)-(A.7)$, $(B.1)-(B.2)$, and u it the unique viscosity solution of [\(2.1\)](#page-6-4). Then $(-\Delta)^{\frac{\sigma}{2}}[u] \in L^{\infty}(\mathbb{R}^{N}).$

Proof. Note that since u is bounded, $(-\Delta)^{\frac{\sigma}{2}}[u]$ defines a distribution by

$$
((-\Delta)^{\frac{\sigma}{2}}[u], \phi) = \int_{\mathbb{R}^N} u(x) \, (-\Delta)^{\frac{\sigma}{2}}[\phi](x) \, dx \quad \text{for any} \quad \phi \in C_c^{\infty}(\mathbb{R}^N).
$$

To complete the proof we must show that this distribution can be represented by a function in $L^{\infty}(\mathbb{R}^N)$. Let u_r be the bounded solution of (2.2) , and note that

$$
\left| \int_{\mathbb{R}^N} u(x) \left(-\Delta \right)^{\frac{\sigma}{2}} [\phi](x) \, dx - \int_{\mathbb{R}^N} u_r(x) \left(-\Delta^{\sigma,r} [\phi](x) \right) dx \right|
$$
\n
$$
\leq \left| \int_{\mathbb{R}^N} (u - u_r)(x) (-\Delta)^{\frac{\sigma}{2}} [\phi](x) \, dx \right| + \|u_r\|_0 I,\tag{2.11}
$$

where $(-\Delta)^{\frac{\sigma}{2}}[\phi] \in L^1(\mathbb{R}^N)^8$ $(-\Delta)^{\frac{\sigma}{2}}[\phi] \in L^1(\mathbb{R}^N)^8$ and by Taylor,

$$
I = \int_{\mathbb{R}^N} \left| \left(-\Delta^{\sigma,r}[\phi] - (-\Delta)^{\frac{\sigma}{2}}[\phi] \right)(x) \right| dx
$$

\n
$$
= \int_{\mathbb{R}^N} \left| \int_{|z| < r} \left(\phi(x+z) - \phi(x) - z \cdot \nabla \phi(x) \right) \frac{dz}{|z|^{N+\sigma}} \right| dx
$$

\n
$$
\leq \| D^2 \phi \|_{L^1(\mathbb{R}^N)} \int_{|z| < r} |z|^2 \frac{dz}{|z|^{N+\sigma}} \leq C \| D^2 \phi \|_{L^1(\mathbb{R}^N)} r^{2-\sigma}.
$$

By Lemma [2.4,](#page-7-3) $||u_r||_0$ is bounded independently of r and $u_r \to u$ in L^{∞} , hence since $\Delta^{\sigma,r}$ is self-adjoint, it follows from (2.11) that

$$
\int_{\mathbb{R}^N} u(x) \left(-\Delta \right)^{\frac{\sigma}{2}} [\phi](x) dx = \lim_{r \to 0} \int_{\mathbb{R}^N} u_r(x) \left(-\Delta^{\sigma,r} [\phi] \right)(x) dx
$$

$$
= \lim_{r \to 0} \int_{\mathbb{R}^N} \left(-\Delta^{\sigma,r} [u_r] \right)(x) \phi(x) dx. \tag{2.12}
$$

By Theorem [2.5,](#page-7-4) $\|\Delta^{\sigma,r}[u_r]\|_0 \leq K$ for some $K > 0$ independent of r. By weak star compactness (Alaoglou/Helly) there is an $f \in L^{\infty}(\mathbb{R}^N)$ and a subsequence $\{r_n\}_n$ such that $r_n \to 0$ and $(-\Delta^{\sigma,r_n}[u_{r_n}]) \stackrel{*}{\rightharpoonup} f$ in L^{∞} . Passing to the limit in (2.12) ,

$$
\int_{\mathbb{R}^N} u(x) \, (-\Delta)^{\frac{\sigma}{2}} [\phi](x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} (-\Delta^{\sigma, r_n}[u_{r_n}]) (x) \, \phi(x) \, dx = \int_{\mathbb{R}^N} f(x) \, \phi(x) \, dx.
$$
\nThe proof is complete.

We immediately observe an improvement of regularity for the viscosity solution of (2.1) in the case that $\sigma > 1$ (compare with Proposition [2.2\)](#page-6-3).

Theorem 2.7. Assume $\sigma > 1$, $(A.1)-(A.7)$ $(A.1)-(A.7)$ $(A.1)-(A.7)$, $(B.1)-(B.2)$, and u is the unique viscosity solution of [\(2.1\)](#page-6-4). Then $u \in C^{1,\sigma-1}(\mathbb{R}^N)$ and

$$
||u||_{1,\sigma-1} \leq K(||u||_0 + ||(-\Delta)^{\frac{\sigma}{2}}[u]||_0).
$$

Proof. By the Corollary [2.6,](#page-9-4) $(-\Delta)^{\frac{\sigma}{2}}[u] \in L^{\infty}(\mathbb{R}^{N})$, and from the definition of viscosity solution $u \in L^{\infty}(\mathbb{R}^{N})$. Therefore the result follows from Theorem 1.1(a) of the article $[58]$ by Ros-Oton and Serra.

⁸A Taylor expansion shows that $\|(-\Delta)^{\frac{\sigma}{2}}[\phi]\|_{L^1} \leq c \|\phi\|_{W^{2,1}}$, and $\|\phi\|_{W^{2,1}} < \infty$ for $\phi \in C_c^{\infty}$.

Remark 2.8. When $\sigma < 1$ we get no improvement in regularity from Lipschitz (Proposition $2.2(c)$ $2.2(c)$). But here Lipschitz regularity is sufficient for solutions to be point-wise classical solutions of (2.1) .

3. Diffusion corrected difference-quadrature scheme

In this section we construct monotone discretizations for equation (1.1) (and [\(2.1\)](#page-6-4)), and give precise results on their convergence rates. There are two main steps to construct the schemes: (i) approximate the singular part of the nonlocal operator by a local diffusion, and (ii) discretize the resulting equations using semi-Lagrangian type of difference quadrature schemes.

By symmetry (**A**[.5\)](#page-5-4) and (**A**[.7\),](#page-5-5) $\left(\int_{\delta < |z| < 1} \eta^{\alpha}(z) \nu_{\alpha}(dz)\right) \cdot \nabla \phi(x) = 0$. For $\delta \in$ $(0, 1)$, we then write the nonlocal operator \mathcal{I}^{α} as

$$
\mathcal{I}^{\alpha}[\phi](x) = \left(\int_{|z| < \delta} + \int_{|z| > \delta}\right) \left(\phi(x + \eta^{\alpha}(z)) - \phi(t, x) - \eta^{\alpha}(z) \cdot \nabla \phi(x)\right) \nu_{\alpha}(dz)
$$

\n
$$
= \int_{|z| < \delta} \left(\phi(x + \eta^{\alpha}(z)) - \phi(t, x) - \eta^{\alpha}(z) \cdot \nabla \phi(x)\right) \nu_{\alpha}(dz)
$$

\n
$$
+ \int_{|z| > \delta} \left(\phi(x + \eta^{\alpha}(z)) - \phi(t, x)\right) \nu_{\alpha}(dz)
$$

\n
$$
:= \mathcal{I}_{\delta}^{\alpha}[\phi](x) + \mathcal{I}^{\alpha, \delta}[\phi](x).
$$
 (3.1)

The δ will be chosen later. We say that $\mathcal{I}^{\alpha}_{\delta}$ is the singular part^{[9](#page-10-2)} of \mathcal{I}^{α} , while $\mathcal{I}^{\alpha,\delta}$ is always a bounded operator.

3.1. Approximation of the singular part of the nonlocal operator. The simplest (but not very accurate) discretization of $\mathcal{I}_{\delta}^{\alpha}[\phi]$ is to replace it by 0. Better approximations can be obtained using local diffusion terms [\[27,](#page-29-1) [48\]](#page-30-10). This corresponds to approximating the small jumps in the SDE (1.3) by an appropriate Brownian motion [\[2\]](#page-28-3). We define

$$
a_{\delta}^{\alpha} = \frac{1}{2} \int_{|z| < \delta} \eta^{\alpha}(z) \eta^{\alpha}(z)^{T} \nu_{\alpha}(dz) \quad \text{and} \quad \mathcal{L}_{\delta}^{\alpha}[\phi](x) := tr[a_{\delta}^{\alpha} D^{2} \phi],
$$

where a_{δ}^{α} is a constant non-negative matrix and $\phi \in C_b^2(\mathbb{R}^N)$. We approximate equation [\(1.1\)](#page-1-1) by replacing $\mathcal{I}_{\delta}^{\alpha}[\phi]$ with $\mathcal{L}_{\delta}^{\alpha}[\phi](x)$:

$$
\sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + c^{\alpha}(x)u(x) - \mathcal{L}_{\delta}^{\alpha}[\phi](x) - \mathcal{I}^{\alpha,\delta}[u](x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^{N}.
$$
 (3.2)

Lemma 3.1. Assume $(A.1)$ - $(A.7)$ $(A.7)$ and $\delta \in (0,1)$. Then there are $C, K > 0$ independent of δ, α, ϕ such that

$$
(i) \quad |\mathcal{I}_{\delta}^{\alpha}[\phi] - \mathcal{L}_{\delta}^{\alpha}[\phi]| \le C\delta^{4-\sigma} \|D^4\phi\|_0, \tag{3.3}
$$

$$
(ii) \quad |a_{\delta}^{\alpha}| \le \int_{|z| \le \delta} |\eta^{\alpha}(z)|^2 \, \nu_{\alpha}(dz) \le K\delta^{2-\sigma}.
$$

⁹When ν has a singularity at the origin, this is a singular integral operator. If the singularity is strong enough, the operator will be a fractional differential operator of positive order.

Proof. By Taylor's expansion theorem and smooth ϕ ,

$$
\int_{|z|<\delta} \left(\phi(x+\eta^{\alpha}(z)) + \phi(x) - \eta^{\alpha}(z) \cdot \nabla \phi \right) \nu_{\alpha}(dz)
$$
\n
$$
= \int_{|z|<\delta} \left(\eta^{\alpha}(z) \cdot D^2 \phi(x) \cdot \eta^{\alpha}(z)^T + \sum_{|\beta|=3} \frac{1}{\beta!} [\eta^{\alpha}(z)]^{\beta} D^{\beta} \phi(x) \right) \nu_{\alpha}(dz) + Err_{\delta},
$$

where $Err_{\delta} = \frac{|\beta|}{\beta!}$ $\frac{|\beta|}{\beta!} \sum_{|\beta|=4} \left[\int_{|z| < \delta} \int_0^1 (1-s)^{|\beta-1|} [\eta^{\alpha}(z)]^{\beta} D^{\beta} \phi(x+sz) ds \nu_{\alpha}(dz) \right].$ By the assumptions $(A.5)$ $(A.5)$ and $(A.7)$ $(A.7)$ and then by $(A.6)$ $(A.6)$ we have

$$
\sum_{|\beta|=3} \int_{|z|\leq \delta} \left[\eta^\alpha(z) \right]^\beta D^\beta \phi(x) \, \nu_\alpha(dz) = 0 \quad \text{and} \quad |Err_\delta| \leq C\delta^{4-\sigma} \|D^4 \phi\|_0.
$$

That proves part (*i*). Part (*ii*) follows by $(A.3)$ $(A.3)$ and $(A.4)$.

3.2. Consistent monotone discretization of the approximate equation. We now approximate the local and nonlocal part of equation [\(3.2\)](#page-10-3) separately.

(i) Discretization of the local term: Since a_{δ}^{α} is symmetric and nonnegative $(\xi^T a_{\delta}^{\alpha})$ (*i*) Discretization of the local term: Since a_{δ}^{α} is symmetric and nonnegative $(\xi^T a_{\delta}^{\alpha} \xi = \int_{|z| < \delta} (\eta^{\alpha}(z) \cdot \xi)^2 \nu_{\alpha}(dz) \ge 0)$, it has a square root with columns $(\sqrt{a_{\delta}^{\alpha}})_i$. We then introduce the semi Lagrangian (SL) approximation (inspired by [\[21,](#page-29-15) [30\]](#page-29-8))

$$
\mathcal{L}_{\delta}^{\alpha}[\phi] = tr[a_{\delta}^{\alpha} D^2 \phi]
$$

$$
\approx \sum_{i=1}^{N} \frac{\phi(x + k(\sqrt{a_{\delta}^{\alpha}})_{i}) + \phi(x - k(\sqrt{a_{\delta}^{\alpha}})_{i}) - 2\phi(x)}{2 k^2} \equiv \mathcal{D}_{\delta,k}^{\alpha}[\phi](x).
$$
 (3.5)

This approximation is monotone by construction, and by Taylor expansions,

$$
|\mathcal{L}_{\delta}^{\alpha}[\phi] - \mathcal{D}_{\delta,k}^{\alpha}[\phi]| \le K|a_{\delta}^{\alpha}|^{2}k^{2}||D^{4}\phi||_{0} \le K\delta^{2(2-\sigma)}k^{2}||D^{4}\phi||_{0}.
$$
 (3.6)

Since $x_j \pm k(\sqrt{a_0^{\alpha}})_i$ may not be on the grid, we interpolate to get a full discretiza- $\sum_{\mathbf{j}\in\mathbb{Z}^N} \phi(x_j)\omega_{\mathbf{j}}(x)$ where the basis functions $\omega_{\mathbf{j}} \geq 0$ and $\sum_{\mathbf{j}\in\mathbb{Z}^N} \omega_{\mathbf{j}} = 1$. Let tion. To preserve monotonicity, we use linear/multilinear interpolation $i_h(\phi)(x) =$

$$
\mathcal{L}_{\delta,k,h}^{\alpha}[\phi](x) = \sum_{i=1}^{N} \frac{i_h \big[\phi(x + k(\sqrt{a_{\delta}^{\alpha}})_{i})\big] + i_h \big[\phi(x - k(\sqrt{a_{\delta}^{\alpha}})_{i})\big] - 2\phi(x)}{2k^2}.
$$
 (3.7)

By the property of multilinear interpolation, this approximation is monotone with

$$
|\mathcal{L}_{\delta,k,h}^{\alpha}[\phi] - \mathcal{D}_{\delta,k}^{\alpha}[\phi]| \le C \frac{h^2}{k^2} ||D^2 \phi||_0.
$$
 (3.8)

By [\(3.6\)](#page-11-1) and [\(3.8\)](#page-11-2) we have a truncation error bound for the local approximate term.

Lemma 3.2. Assume $(A.3)-(A.7)$ $(A.3)-(A.7)$ $(A.3)-(A.7)$. Then there is $K > 0$ independent of h, δ, α, ϕ such that

$$
\left| \mathcal{L}_{\delta,k,h}^{\alpha} [\phi](x) - \mathcal{L}_{\delta}^{\alpha} [\phi](x) \right| \le K \left(\delta^{2(2-\sigma)} k^2 \| D^4 \phi \|_0 + \frac{h^2}{k^2} \| D^2 \phi \|_0 \right). \tag{3.9}
$$

(ii) Discretization of the nonlocal term: We follow $[8, \text{Section 3}]$ and approximate $\mathcal{I}^{\alpha,\delta}$ by the quadrature

$$
\mathcal{I}_h^{\alpha,\delta}[\phi] = \sum_{\mathbf{j}\in\mathbb{Z}^N} \left(\phi(x+x_{\mathbf{j}}) - \phi(x) \right) \kappa_{h,\mathbf{j}}^{\alpha,\delta}; \quad \kappa_{h,\mathbf{j}}^{\alpha,\delta} = \int_{|z|>\delta} \omega_{\mathbf{j}}(\eta^{\alpha}(z);h) \nu_{\alpha}(dz), \tag{3.10}
$$

where $\{\omega_j\}_j$ is the basis for multilinear interpolation defined above. Since $\omega_j \geq 0$, $\kappa_{h,\mathbf{j}}^{\alpha,\delta} \geq 0$, and the approximation $\mathcal{I}_h^{\alpha,\delta}$ is monotone. A Taylor expansion gives an estimate on the local truncation error, c.f. [\[8\]](#page-29-13):

Lemma 3.3. Assume $(A.3)-(A.4)$ $(A.3)-(A.4)$ $(A.3)-(A.4)$ $(A.3)-(A.4)$ and $(A.6)$. Then there is $K > 0$ independent of h, δ , α , ϕ such that

$$
\left|\mathcal{I}^{\alpha,\delta}[\phi](x) - \mathcal{I}_h^{\alpha,\delta}[\phi](x)\right| \le K \frac{h^2}{\delta^{\sigma}} \|D^2 \phi\|_0.
$$
 (3.11)

(iii) Discretization of the nonlocal equation (1.1) :

$$
\sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + c^{\alpha}(x)u(x) - \mathcal{L}^{\alpha}_{\delta,k,h}[u](x) - \mathcal{I}^{\alpha,\delta}_{h}[u](x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^{N},\qquad(3.12)
$$

or in weakly non-degenerate case [\(2.1\)](#page-6-4) where $c^{\alpha}(x) = \lambda$,

$$
\lambda v(x) + \sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) - \mathcal{L}^{\alpha}_{\delta,k,h}[v](x) - \mathcal{I}^{\alpha,\delta}_{h}[v](x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^{N}.
$$
 (3.13)

3.3. Properties and convergence analysis for the schemes. We state wellposedness, comparison, L^{∞} -stability, and L^{∞} -convergence results for the schemes in different settings.

Theorem 3.4 (wellposedness, stability). Assume $(A.1)$ - $(A.4)$.

- (a) There exists a unique solution $u_h \in C_b(\mathbb{R}^N)$ of (3.12) .
- (b) If $u_h, v_h \in C_b(\mathbb{R}^N)$ are sub and supersolutions of (3.12) , then $u_h \leq v_h$.
- (c) If u_h is the unique solution of [\(3.12\)](#page-12-1), then $|u_h|_0 \leq C \sup_{\alpha \in A} |f^{\alpha}|_0$.

Proof. Part (a) can be proved using Banach fixed point arguments, we refer to $[9, 6]$ Lemma 3.1] for details. Part (b) is a consequence of the scheme having positive coefficients. Finally, part (c) follows from (b) by taking $\pm \frac{1}{\lambda} \sup_{\alpha \in A} |f^{\alpha}|_0$ as super and sub-solution of the scheme (3.12) respectively.

If the solutions of (1.1) are very smooth (C_b^4) , then we get the best possible convergence rate for our scheme – what some would call the accuracy of the method:

Proposition 3.5 (Smooth solutions). Assume $(A.4)$ – $(A.7)$, $\sigma \in (0, 2)$, $u \in C_b^4(\mathbb{R}^N)$ solves [\(1.1\)](#page-1-1), and u_h solves [\(3.12\)](#page-12-1) with $k = O(h^{\frac{\sigma}{4}})$ and $\delta = O(h^{\frac{1}{2}})$. Then there is $C > 0$ such that

$$
|u - u_h| \leq C h^{2 - \frac{\sigma}{2}}.
$$

This rate is always better than 1, and approaches 1 as $\sigma \to 2^-$. We will not discuss assumptions to have so smooth solutions, but below we will give results that holds for the solutions that exist under the assumptions of this paper.

Proof. By equation (1.1) and the errors bounds (3.3), (3.9), (3.11), for any
$$
\alpha \in \mathcal{A}
$$
,
\n
$$
f^{\alpha}(x) + c^{\alpha}(x)u(x) - \mathcal{L}_{\delta,k,h}^{\alpha}[u](x) - \mathcal{I}_{h}^{\alpha,\delta}[u](x) \leq \mathcal{I}^{\alpha}[u](x) - \mathcal{L}_{\delta,k,h}^{\alpha}[u](x) - \mathcal{I}_{h}^{\alpha,\delta}[u](x)
$$
\n
$$
\leq C\Big(\delta^{4-\sigma} \|D^4u\|_0 + \frac{h^2}{k^2} \|D^2u\|_0 + \delta^{2(2-\sigma)}k^2 \|D^4u\|_0 + \frac{h^2}{\delta^{\sigma}} \|D^2u\|_0\Big) := B_{h,\delta}.
$$

This implies $u(x) - \frac{B_{h,\delta}}{\lambda}$ is a subsolution of [\(3.12\)](#page-12-1), and by Theorem [3.4](#page-12-3) (b) that

$$
u-u_h\leq \frac{B_{h,\delta}}{\lambda}.
$$

Again by [\(1.1\)](#page-1-1), the definition of the sup, and the errors bounds, for any $x \in \mathbb{R}^N$ and $\epsilon > 0$, there is a $\alpha_{\epsilon} \in \mathcal{A}$ such that

$$
f^{\alpha_{\epsilon}}(x) + c^{\alpha_{\epsilon}}(x)u(x) - \mathcal{L}_{k,h}^{\alpha_{\epsilon},\delta}[u](x) - \mathcal{I}_{h}^{\alpha_{\epsilon},\delta}[u](x)
$$

\n
$$
\geq -\epsilon + \mathcal{I}^{\alpha_{\epsilon}}[u](x) - \mathcal{L}_{k,h}^{\alpha_{\epsilon},\delta}[u](x) - \mathcal{I}_{h}^{\alpha_{\epsilon},\delta}[u](x) \geq -\epsilon - B_{h,\delta}.
$$

Let $\tilde{u} = u + \frac{B_{h,\delta}}{\lambda}$, and note that

$$
\sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + c^{\alpha}(x)\tilde{u}(x) - \mathcal{L}^{\alpha}_{\delta,k,h}[\tilde{u}](x) - \mathcal{I}^{\alpha,\delta}_h[\tilde{u}](x) \right\} \ge -\epsilon.
$$

Since ϵ and x are arbitrary, \tilde{u} is a supersolution of [\(3.12\)](#page-12-1), and then $u_h - u \leq \frac{B_{h,\delta}}{\lambda}$ λ by Theorem [3.4](#page-12-3) (b). Since $u \in C_b^4(\mathbb{R}^N)$, we have shown that

$$
|u - u_h| \leq \frac{C}{\lambda} \left(\delta^{4-\sigma} + \frac{h^2}{k^2} + \delta^{2(2-\sigma)} k^2 + \frac{h^2}{\delta^{\sigma}} \right).
$$

We conclude by taking the optimal choices $k^2 = O(\frac{h}{\delta^{2-\sigma}})$ and then $\delta = O(h^{\frac{1}{2}})$. \Box

The next two results form the main contribution of this paper along with the result of section [4.](#page-14-0) These results give very precise rates of convergence for our monotone numerical approximations in cases of strongly and weakly non-degenerate equations respectively. Note that in these results the solutions u of (1.1) and (2.1) will not be smooth. The proofs of these results are given in Section [5.](#page-15-0)

Theorem 3.6 (Strongly degenerate equations). Assume $\sigma \in (0, 2)$, $h \in (0, 1)$, $(A.1)-(A.7)$ $(A.1)-(A.7)$ $(A.1)-(A.7)$, u and u_h are solutions of [\(1.1\)](#page-1-1) and [\(3.12\)](#page-12-1) for $k = O(h^{\frac{2\sigma}{4+\sigma}})$ and $\delta = O(h^{\frac{4}{4+\sigma}})$. Then there is a $C > 0$ such that

$$
|u - u_h| \le C h^{\frac{4-\sigma}{4+\sigma}}.\tag{3.14}
$$

Remark 3.7. (a) The rate $\frac{4-\sigma}{4+\sigma}$ is decreasing in σ . It equals $\frac{3}{5}$ at $\sigma = 1$, approaches 1 as $\sigma \to 0^+$, and $\frac{1}{3}$ as $\sigma \to 2^-$.

(b) The "CFL" conditions $k = O(h^{\frac{2\sigma}{4+\sigma}})$ and $\delta = O(h^{\frac{4}{4+\sigma}})$ imply that $\frac{h}{k} \to 0$ and $\frac{h}{\delta} \to 0$ as $h \to 0$.

(c) Conditions $(A.5)$ $(A.5)$ and $(A.7)$ $(A.7)$ are symmetry assumptions on the singular part of \mathcal{I}^{α} which lead to best possible rates. We refer to Section [7](#page-26-0) for extensions to nonsymmetric nonlocal operators and the corresponding (slightly) lower rates.

In the weakly non-degenerate case we get an improvement in the rate due to the better regularity of solutions both for the equation and the numerical scheme:

Theorem 3.8 (weakly non-degenerate equations). Assume $\sigma \in (0, 2)$, $h \in (0, 1)$, $(A.1)$ - $(A.7)$, $(B.1)$ - $(B.2)$, u and u_h are the solutions of [\(2.1\)](#page-6-4) and [\(3.12\)](#page-12-1) for $k =$ $O(h^{\frac{2\sigma}{4+\sigma}})$ and $\delta = O(h^{\frac{4}{4+\sigma}})$. Then there is $C > 0$ independent of h such that

$$
|u - u_h| \le \begin{cases} C h^{\frac{4-\sigma}{4+\sigma}} & \text{for} \quad 0 < \sigma \le 1, \\ C h^{\frac{\sigma(4-\sigma)}{4+\sigma}} & \text{for} \quad 1 < \sigma < 2. \end{cases}
$$
 (3.15)

Remark 3.9. For $\sigma \leq 1$, the results are the same as in Theorem [3.6.](#page-13-0) For $\sigma > 1$, the rate of convergence is always more than $\mathcal{O}(h^{\frac{1}{2}})$, and the rate approaches $\mathcal{O}(h^{\frac{2}{3}})$ when $\sigma \to 2$. The "CFL" conditions are the same as in Theorem [3.6.](#page-13-0)

4. Powers of discrete Laplacian

In this section we consider versions of equation [\(1.1\)](#page-1-1) where the nonlocal operator is the fractional Laplacian,

$$
\lambda u(x) + \sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + a^{\alpha} \left(-\Delta \right)^{\frac{\sigma}{2}} u(x) \right\} = 0. \tag{4.1}
$$

In other words, $\mathcal{I}^{\alpha} = -a^{\alpha} \left(-\Delta\right)^{\frac{\sigma}{2}}, \nu_{\alpha}(dz) = a^{\alpha} \frac{c_{N,\sigma}}{|z|^{N+2\sigma}} dz$, and $\eta^{\alpha}(z) = z$ in [\(1.2\)](#page-1-4). Here $(A.3)$ – $(A.7)$ $(A.7)$ trivially holds. We assume $(B.1)$, the equation is weakly nondegenerate (otherwise the equation is purely algebraic), which here is equivalent to

there is
$$
\alpha_0 \in \mathcal{A}
$$
 such that $a^{\alpha_0} > 0$. (4.2)

Under assumptions $(A.1)$, $(A.2)$, $(B.1)$, and $(B.2)$, we can use Proposition [2.2,](#page-6-3) Lemma [2.4,](#page-7-3) and Theorem [2.7](#page-9-0) to conclude wellposedness, stability, approximation, and regularity results for (4.1) . Here we introduce and analyse a discretization

$$
\lambda u_h(x) + \sup_{\alpha \in \mathcal{A}} \{ f^{\alpha}(x) + a^{\alpha} (-\Delta_h)^{\frac{\sigma}{2}} [u_h](x) \} = 0, \tag{4.3}
$$

based on powers of the discrete Laplacian $(-\Delta_h)^{\frac{\sigma}{2}}$, see [\[25,](#page-29-16) [35\]](#page-30-14) and also [\[13\]](#page-29-17). As far as we know, this is the first time this type of discretization has been considered for HJB equations. It is a very good approximation in the sense that it is a monotone method of second order accuracy. This is better than the diffusion corrected discretization of Section [3.](#page-10-0)

Let $\Delta_h \phi(x) = \sum_{k=1}^N \frac{1}{h^2} (\phi(x+he_k) - 2\phi(x) - \phi(x-he_k))$ be the 2nd order central finite difference approximation of the Laplacian $\Delta \phi$, then

$$
(-\Delta_h)^{\frac{\sigma}{2}}\phi(x) := \frac{1}{\Gamma(-\frac{\sigma}{2})} \int_0^\infty \left(e^{t\Delta_h} \phi(x) - \phi(x) \right) \frac{dt}{t^{1+\frac{\sigma}{2}}},\tag{4.4}
$$

where $U(t) = e^{t\Delta_h}\psi$ is the solution of semi-discrete heat equation

$$
\partial_t U(x,t) = \Delta_h U(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R}^N \times (0,\infty),
$$

$$
U(x,0) = \psi(x) \quad \text{for} \quad x \in \mathbb{R}^N.
$$

An explicit formula for $e^{t\Delta_h}\phi$ and details related to this approximation can be found in Section 4.5 of $[35]$. We can write approximation (4.4) as a quadrature,

$$
-(-\Delta_h)^{\frac{\sigma}{2}}\phi(x) = \sum_{\mathbf{j}\in\mathbb{Z}^N\backslash\{0\}} \left(\phi(x+x_{\mathbf{j}}) - \phi(x)\right)\kappa_{h,\mathbf{j}} \quad \text{with} \quad \kappa_{h,\mathbf{j}} \ge 0.
$$

This is obviously a monotone approximation of the fractional Laplacian, and by Lemma 4.22 in [\[35\]](#page-30-14), it has the following local truncation error:

Lemma 4.1. Assume $\sigma \in (0, 2)$. Then for any smooth bounded function ϕ ,

$$
\left| \left(-\Delta_h \right)^{\frac{\sigma}{2}} \phi(x) - \left(-\Delta \right)^{\frac{\sigma}{2}} \phi(x) \right| \le C h^2 \left(\| D^4 \phi \|_0 + \| \phi \|_0 \right). \tag{4.5}
$$

We note that Theorem 3.4 (wellposedness and stability) also holds for (4.3) . We now state an error bound for this scheme. The proof is given in Section [6.](#page-24-0)

Theorem 4.2. Assume $h \in (0,1)$, $(A.1)$, $(A.2)$, $(B.1)$, $(B.2)$, u and u_h are solutions of equation [\(4.1\)](#page-14-1) and the scheme [\(4.3\)](#page-14-3). Then there is $C > 0$ such that

$$
||u - u_h||_0 \le \begin{cases} Ch^{\frac{1}{2}} & \text{for} \quad 0 < \sigma \le 1, \\ Ch^{\frac{\sigma}{2}} & \text{for} \quad 1 < \sigma < 2. \end{cases}
$$
 (4.6)

Remark 4.3. The problem is weakly non-degenerate and the regularity of the solution can be seen in the rate for $\sigma > 1$, cf. Theorem [2.7.](#page-9-0) This σ dependence seems to be optimal, and is consistent as $\sigma \to 2$ with the $\mathcal{O}(h)$ bound obtained in the 2nd order case in [\[32\]](#page-30-5). For $\sigma \in (\frac{4}{3}, 2)$, the rate is better than for the diffusion corrected discretization in Theorem [3.8.](#page-13-1)

5. Proofs of the error bounds for monotone quadrature schemes

Here, we give proof of the convergence rates discussed in Section [3.](#page-10-0)

5.1. **Strongly-degenerate equations** – **the proof of Theorem [3.6.](#page-13-0)** Let $(\rho_{\epsilon})_{\epsilon>0}$ be the standard mollifier on \mathbb{R}^{N} and define $u_{\epsilon,h} = u_h * \rho_{\epsilon}$. By [\(3.12\)](#page-12-1),

$$
f^{\alpha}(x) + c^{\alpha}(x)u_h(x) - \mathcal{L}^{\alpha}_{\delta,k,h}u_h(x) - \sum_{\mathbf{j}\in\mathbb{Z}^N} \left(u_h(x+x_{\mathbf{j}}) - u_h(x)\right)\kappa_{h,\mathbf{j}}^{\alpha,\delta} \le 0
$$

for any $\alpha \in \mathcal{A}$. Let $f_{\epsilon}^{\alpha} = f^{\alpha} * \rho_{\epsilon}$, convolve by ρ_{ϵ} , to get

$$
f_{\epsilon}^{\alpha}(x) + (c^{\alpha}u_{h,\epsilon}) * \rho_{\epsilon}(x) - \mathcal{L}_{\delta,k,h}^{\alpha} u_{h,\epsilon}(x) - \sum_{\mathbf{j} \in \mathbb{Z}^N} \left(u_{h,\epsilon}(x+x_{\mathbf{j}}) - u_{h,\epsilon}(x) \right) \kappa_{h,\mathbf{j}}^{\alpha,\delta} \leq 0.
$$

Since $||f^{\alpha} * \rho_{\epsilon} - f^{\alpha}||_0 \leq K\epsilon$ and $||(c^{\alpha}u_h) * \rho_{\epsilon} - c^{\alpha}u_{h,\epsilon}||_0 \leq \sup_{\alpha} ||Dc^{\alpha}||_0 ||u_h||_0 \epsilon \leq$ $CK^2\epsilon$, we then find that

$$
f^{\alpha}(x) + c^{\alpha}(x)u_{\epsilon,h}(x) - \mathcal{I}^{\alpha}[u_{\epsilon,h}](x)
$$

\n
$$
\leq \left\| \mathcal{I}^{\alpha}[u_{\epsilon,h}] - \left(\mathcal{L}^{\alpha}_{\delta,k,h} u_{\epsilon,h} + \mathcal{I}^{\alpha,\delta}_{h}[u_{\epsilon,h}] \right) \right\|_{0} + (CK^{2} + K)\epsilon.
$$
 (5.1)

By Lemmas [3.1,](#page-10-5) [3.2,](#page-11-4) [3.3,](#page-12-4) and $|D^k u_{\epsilon,h}|_0 \leq \frac{C \|u_h\|_{0,1}}{\epsilon^{k-1}}$ $\frac{u_h\|0,1}{\epsilon^{k-1}},$ it follows that

$$
\left| \mathcal{I}^{\alpha}[u_{\epsilon,h}] - \left(\mathcal{L}^{\alpha}_{\delta,k,h} u_{\epsilon,h} + \mathcal{I}^{\alpha,\delta}_h[u_{\epsilon,h}] \right) \right|_0
$$

$$
\leq M_{\epsilon,\delta} := C \left(\delta^{4-\sigma} \frac{1}{\epsilon^3} + k^2 \delta^{2(2-\sigma)} \frac{1}{\epsilon^3} + \frac{h^2}{k^2} \frac{1}{\epsilon} + \frac{h^2}{\delta^{\sigma}} \frac{1}{\epsilon} \right).
$$
 (5.2)

Therefore $u_{\epsilon,h} - \frac{C}{\lambda} \tilde{M}_{\epsilon,\delta}$, for $\tilde{M}_{\epsilon,\delta} = M_{\epsilon,\delta} + (CK^2 + K)\epsilon$, is a classical (and hence also viscosity) subsolution of equation (1.1) . By comparison for equation (1.1) (Proposition [2.2](#page-6-3) (a)), $u_{\epsilon,h} - \frac{C}{\lambda} \tilde{M}_{\epsilon,\delta} \leq u$. Since $||u_h - u_{\epsilon,h}||_0 \leq \epsilon ||Du_h||_0$, we get

$$
u_h - u \le K\left(\epsilon + M_{\epsilon,\delta}\right). \tag{5.3}
$$

The bound on $u - u_h$ can be proved in similar way. Let $u_{\epsilon} = u * \rho_{\epsilon}$. Arguing as above, using Lemmas [3.1,](#page-10-5) [3.2,](#page-11-4) [3.3,](#page-12-4) and $||D^k u_{\epsilon}||_0 \leq \frac{C ||u||_{0,1}}{\epsilon^{k-1}}$ $\frac{\|u\|_{0,1}}{\epsilon^{k-1}},$ we have

$$
f^{\alpha}(x) + c^{\alpha}(x) u_{\epsilon}(x) - \mathcal{L}^{\alpha}_{\delta,k,h} u_{\epsilon}(x) - \mathcal{I}^{\alpha,\delta}_{h}[u_{\epsilon}](x)
$$

\$\leq \left\| \mathcal{I}^{\alpha}[u_{\epsilon}] - \left(\mathcal{L}^{\alpha}_{\delta,k,h} u_{\epsilon} + \mathcal{I}^{\alpha,\delta}_{h}[u_{\epsilon}]\right) \right\|_{0} + (CK^{2} + K)\epsilon \leq M_{\epsilon,\delta} + (CK^{2} + K)\epsilon.\$

This implies $u_{\epsilon} - \frac{C}{\lambda} \tilde{M}_{\epsilon,\delta}$ is a subsolution of the numerical scheme [\(3.12\)](#page-12-1). Compar-ison for the scheme [\(3.12\)](#page-12-1) (Theorem [3.4\(](#page-12-3)b)) and $||u - u_{\epsilon}||_0 < \epsilon ||Du||_0$ lead to

$$
u_h - u \ge -C(\epsilon + M_{\epsilon,\delta}).\tag{5.4}
$$

By [\(5.3\)](#page-15-2) and [\(5.4\)](#page-15-3) we get $|u - u_h| \leq C(\epsilon + M_{\epsilon,\delta})$, and then we optimize with respect to k, δ , and ϵ . The optimal choices $k^2 = O\left(\frac{h\epsilon}{\delta^{2-\sigma}}\right)$ and $\epsilon = O\left(\frac{h}{\delta^2}\right)$ $\frac{h}{\delta^{\frac{\sigma}{2}}}$) lead to

$$
|u - u_h| \le C \left(\delta^{4 + \frac{\sigma}{2}} h^{-3} + \delta^2 h^{-1} + \frac{h}{\delta^{\frac{\sigma}{2}}} \right),\tag{5.5}
$$

and the result follows by choosing $\delta = O(h^{\frac{4}{4+\sigma}})$

5.2. Intermezzo on regularisations. In the remaining proofs we need high order estimates for two different regularisation procedures: (i) Convolution with standard mollifiers and (ii) convolution with fractional heat kernels. These estimates are proved in this section.

Let $\rho_{\varepsilon}(x) = \frac{1}{\varepsilon^{N}} \rho\left(\frac{x}{\varepsilon}\right)$ for some $\rho \in C_c^{\infty}(\mathbb{R}^N)$ with support in $B(0, 1)$ and $\int_{\mathbb{R}^N} \rho \, dx =$ 1. Hence supp $\rho_{\epsilon} = \overline{B(0, \epsilon)}$ and $\int_{\mathbb{R}^N} \rho_{\epsilon} dx = 1$. We define

$$
v^{(\epsilon)} = v * \rho_{\epsilon} \tag{5.6}
$$

 \Box

for bounded continuous functions v. It then easily follows that $v^{(\epsilon)} \in C_b^{\infty}$.

Lemma 5.1. If $v \in C^{1,\beta}(\mathbb{R}^N)$ for $\beta \in (0,1]$ and ρ is a radial function, then

$$
||v^{(\epsilon)} - v||_0 \leq C\epsilon^{1+\beta} ||v||_{1,\beta} \qquad and \qquad ||D^m v^{(\epsilon)}||_0 \leq \frac{K}{\epsilon^{m-1-\beta}} ||v||_{1,\beta}
$$

for any $m \geq 2$, where C and K are independent of ϵ .

Proof. The first inequality follows since $\int_{\mathbb{R}^N} y \rho_{\epsilon}(y) dy = 0$ and then

$$
|v^{(\epsilon)}(x) - v(x)| = \Big| \int_{\mathbb{R}^N} (v(x - y) - v(x) - y \cdot \nabla v(x)) \rho_{\epsilon}(y) dy \Big|
$$

$$
\leq C \|v\|_{1, \beta} \int_{\mathbb{R}^N} |y|^{1 + \beta} \rho_{\epsilon}(y) dy \leq C \|v\|_{1, \beta} \epsilon^{1 + \beta}.
$$

Since $\int_{\mathbb{R}^N} D^{m-1} \rho_{\epsilon}(y) dy = 0$ by the divergence theorem, the second inequality follows since $Dv \in C^{\beta}$ and

$$
D^{m}v^{(\epsilon)} = Dv * D^{m-1}\rho_{\epsilon} = \int_{\mathbb{R}^{N}} [Dv(x - y) - Dv(x)]D^{m-1}\rho_{\epsilon}(y)dy.
$$

Let $\tilde{K}^{\sigma}(t,x) := \mathcal{F}^{-1}(e^{-t|\cdot|^{\sigma}})(x)$ be the fractional heat kernel, the fundamental solution of the fractional heat equation $u_t + (-\Delta)^{\frac{\sigma}{2}}u = 0$. Convolution with \tilde{K}^{σ} defines a smooth approximation of a bounded continuous function v ,

$$
v^{[\epsilon]}(x) := v(\cdot) * \tilde{K}^{\sigma}(\epsilon^{\sigma}, \cdot)(x). \tag{5.7}
$$

Let $K^{\sigma}(x) = \tilde{K}^{\sigma}(1,x)$. To prove estimates on $v^{[\epsilon]}$, we need some well-known properties of \tilde{K}^{σ} :

(i)
$$
\tilde{K}^{\sigma} \in C^{\infty}((0,\infty) \times \mathbb{R}^{N}), \tilde{K}^{\sigma} \ge 0
$$
, and $\int_{\mathbb{R}^{N}} \tilde{K}^{\sigma}(t,x) dx = 1$ for $t > 0$.

(ii) $\tilde{K}^{\sigma}(t+s,x) = \tilde{K}^{\sigma}(t,x) * \tilde{K}^{\sigma}(s,x)$ for $t,s \geq 0$ (convolution semigroup).

(iii) For
$$
t > 0
$$
 and $x \in \mathbb{R}^N$, $\tilde{K}^\sigma(t, x) = t^{-\frac{N}{\sigma}} K^\sigma\left(\frac{x}{t^{\frac{1}{\sigma}}}\right)$ where

$$
\frac{c_1t}{\left(t^{\frac{2}{\sigma}}+|x|^2\right)^{\frac{N+\sigma}{2}}} \leq \tilde{K}^{\sigma}(t,x) \leq \frac{C_2t}{\left(t^{\frac{2}{\sigma}}+|x|^2\right)^{\frac{N+\sigma}{2}}}.
$$

(iv) (Theorem 1.1(c) in [\[38\]](#page-30-23)) For any $m > 0$ and multi-index β with $|\beta| = m$,

$$
|D^{\beta} K^{\sigma}(x)| \le \frac{B_m}{1+|x|^{N+\sigma}} \quad \text{for} \quad x \in \mathbb{R}^N.
$$

We refer to $[12, 33, 38]$ $[12, 33, 38]$ $[12, 33, 38]$ $[12, 33, 38]$ for the proofs.

Lemma 5.2. Assume $\epsilon > 0$, $\sigma > 1$, $\beta \in (\sigma - 1, 1)$, and $v \in C^{1,\beta}(\mathbb{R}^N)$. Then there is $C > 0$ independent of ϵ , such that

$$
||v^{[\epsilon]} - v||_0 \leq C\epsilon^{\sigma}.
$$

Proof. Let S_t be the fractional heat semigroup, i.e. $v^{[\epsilon]} = S_{\epsilon^{\sigma}}(v)$. Since $\int_{\mathbb{R}^N} \tilde{K}^{\sigma}(r, y) dy =$ 1, by Fubini's Theorem and property (i) above,

$$
\begin{aligned} |(-\Delta)^{\frac{\sigma}{2}}[S_r(v)](x)| &= \Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v(x+z-y) - v(x-y)}{|z|^{N+\sigma}} \tilde{K}^\sigma(r,y) dx dy \Big| \\ &\leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\sigma}{2}}[v](x-y) | \tilde{K}^\sigma(r,y) dy \leq \|(-\Delta)^{\frac{\sigma}{2}}[v] \|_0. \end{aligned}
$$

Since $\|(-\Delta)^{\frac{\sigma}{2}}[v]\|_0 \leq K\|v\|_{1,\beta}$, for any $t \geq s > 0$,

$$
|S_t(v) - S_s(v)| = \Big| \int_s^t \partial_r [S_r(v)] dr \Big| = \Big| \int_s^t (-\Delta)^{\frac{\sigma}{2}} [S_r(v)] dr \Big| \le K(t-s) \|v\|_{1,\beta}.
$$

The lemma then follows by taking $t = e^{\sigma}$ and using that $S_s(v) \to v$ pointwise as $s \to 0$.

Lemma 5.3. Assume $\epsilon > 0$, $\sigma > 1$, $m \geq 2$, $v \in C^{0,1}(\mathbb{R}^N)$, and define $\epsilon_1 = \frac{\epsilon}{\epsilon_1^2}$ $rac{\epsilon}{2^{\frac{1}{\sigma}}}$. Then there exists $C > 0$ independent of ϵ such that

$$
||D^m v^{[\epsilon]}||_0 \le \frac{C}{\epsilon^{m-1}} ||v||_{0,1} \quad \text{and} \quad ||D^m v^{[\epsilon]}||_0 \le \frac{C}{\epsilon^{m-\sigma}} ||v^{[\epsilon_1]}||_{1,\sigma-1}.
$$

Proof. The first estimate is classical and follows from differentiating \tilde{K}^{σ} (m – 1) times and v once (c.f. Lemma [5.1\)](#page-16-1) and noting that $|x||D_x^m \tilde{K}^\sigma(t,x)| \in L^1(\mathbb{R}^N)$ for $\sigma > 1$ and $t > 0$ by property (iv) above.

For the second estimate, we must estimate $\partial_{x_i} D^{\alpha} v^{[\epsilon]}$ for any multiindex α with $|\alpha| = m - 1$. Rewriting $v^{[\epsilon]}$ as

$$
v^{[\epsilon]}=v\ast \tilde{K}^{\sigma}(\epsilon^{\sigma},\cdot)=v\ast \tilde{K}^{\sigma}\big(\frac{\epsilon^{\sigma}}{2},\cdot\big)\ast \tilde{K}^{\sigma}\big(\frac{\epsilon^{\sigma}}{2},\cdot\big)=v^{[\epsilon_{1}]} \ast \tilde{K}^{\sigma}\big(\frac{\epsilon^{\sigma}}{2},x\big),
$$

we find that

$$
\partial_{x_i}D^{\alpha}v^{[\epsilon]}=\partial_{x_i}v^{[\epsilon_1]}*\,D^{\alpha}\tilde{K}^{\sigma}\big(\frac{\epsilon^{\sigma}}{2},\cdot\big).
$$

First, by the divergence theorem and the decay at infinity (property (iv) above), $\int_{\mathbb{R}^N} D^{\alpha} \tilde{K}^{\sigma} \left(\frac{\epsilon^{\sigma}}{2} \right)$ $(\frac{x^2}{2}, y)dy = 0$. Then, by self-similarity (property (iii)) and $y = \frac{e}{2\pi\sqrt{3}}$ $rac{\epsilon}{2\frac{1}{\sigma}}z,$

$$
(D_y^{\alpha}\tilde{K})\left(\frac{\epsilon^{\sigma}}{2},y\right) = 2^{\frac{N}{\sigma}}\frac{1}{\epsilon^{N+(m-1)}}(D_z^{\alpha}K)(z).
$$

Combining these facts with the change of variables $y = \epsilon z$, we see that

$$
\begin{split} |\partial_{x_i} D^{\alpha} v^{[\epsilon]}| &= \Big| \int_{\mathbb{R}^N} \left(\partial_{x_i} v^{[\epsilon_1]}(x - y) - \partial_{x_i} v^{[\epsilon_1]}(x) \right) D_y^{\alpha} \tilde{K}^{\sigma} \left(\frac{\epsilon^{\sigma}}{2}, y \right) dy \Big| \\ &\leq \frac{2^{\frac{N}{\sigma}}}{\epsilon^{m-1}} \int_{\mathbb{R}^N} |\partial_{x_i} v^{[\epsilon_1]}(x - \epsilon z) - \partial_{x_i} v^{[\epsilon_1]}(x) | |D_z^{\alpha} K^{\sigma}(z)| dz \\ &\leq \frac{K}{\epsilon^{m-\sigma}} \|v^{[\epsilon_1]} \|_{1, \sigma^{-1}} \int_{\mathbb{R}^N} |z|^{\sigma-1} |D_z^{\alpha} K^{\sigma}(z)| dz. \end{split}
$$

The proof is complete since $|x|^{\sigma-1} |D^{\alpha} K^{\sigma}(x)| \in L^1$ by property (iv) above. \square

5.3. Weakly-degenerate equations $-$ the proof of Theorem [3.8.](#page-13-1) We first prove a discrete version of the bound on nonlocal operator in Theorem [2.5.](#page-7-4) Then we show that these bounds leads to regularity of the numerical solution. From regularity, approximation, and comparison arguments the error bounds follows. Regularization arguments and the results of the previous section are used throughout. For $h, k, \epsilon > 0$, and $\delta \in (0, 1)$, we define

$$
\label{eq:decomp} \begin{aligned} \hat{\mathcal{I}}^{\alpha}_{\delta,k,h}[\phi] &:= \mathcal{L}^{\alpha}_{\delta,k,h}[\phi] + \mathcal{I}^{\alpha,\delta}_h[\phi],\\ \mathcal{J}^{\alpha,\delta}_h[\phi] &:= \sum_{\mathbf{j}\in\mathbb{Z}^N} \left(\phi(x+x_\mathbf{j}) - \phi(x)\right) \int_{|z|>\delta} \omega_\mathbf{j}(\eta^\alpha(z)) \frac{dz}{|z|^{N+\sigma}}, \end{aligned}
$$

where $\mathcal{L}_{\delta,k,h}^{\alpha}, \mathcal{I}_{h}^{\alpha,\delta}$, and the weight function ω_{j} are defined in section [3.2.](#page-11-0) By definition $\mathcal{J}_h^{\alpha,\delta}$ is a monotone approximation of the non-singular part of the operator

$$
\mathcal{J}^{\alpha}[\phi] := \int_{\mathbb{R}^N} \left(\phi(x + \eta^{\alpha}(z)) - \phi(x) - \nabla \phi(x) \cdot \eta^{\alpha}(z) 1_{|z| < \delta} \right) \frac{dz}{|z|^{N+\sigma}}
$$
(5.8)

with local truncation error (Taylor expand, see e.g. [\[8,](#page-29-13) Section 3])

$$
|\mathcal{J}_h^{\alpha,\delta}[\phi](x) - \mathcal{J}^{\alpha}[\phi](x)| \le C(||\phi||_0 + ||D^2\phi||_0) \left(\delta^{2-\sigma} + h^2 \delta^{-\sigma}\right). \tag{5.9}
$$

The discrete version of Theorem [2.5](#page-7-4) is the following result.

Theorem 5.4. Assume $(A.1)-(A.5)$ $(A.1)-(A.5)$ $(A.1)-(A.5)$, $(B.1)-(B.2)$, and u_h solves (3.13) . Then for $\delta \in (0,1), \ \delta \geq h$, and $k \geq \delta^{\frac{\sigma}{2}}$, there is a $K > 0$ independent of h, k, δ such that

$$
\|\hat{\mathcal{I}}_{\delta,k,h}^{\alpha_0}[u_h]\|_0 \le K,\tag{5.10}
$$

$$
\|\mathcal{J}_h^{\alpha_0,\delta}[u_h]\|_0 \le \frac{K}{c_{\alpha_0}}.\tag{5.11}
$$

The proof relies on the following technical lemma.

Lemma 5.5. Assume $(A.1)-(A.6)$ $(A.1)-(A.6)$ $(A.1)-(A.6)$, $(B.1)-(B.2)$, and α_0 is defined in $(B.1)$. For $\sigma \in (0,1)$, there is a $K > 0$ independent of δ, h, k such that

$$
\|\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}]\|_0 \leq K\Big[\frac{h^{\sigma}}{k^2} + k^{\sigma-2}\delta^{\frac{\sigma(2-\sigma)}{2}}\Big] \|f^{\alpha}\|_{1,\sigma-1}.
$$

Proof. Let $f^{\alpha}_{(\gamma)} := f^{\alpha} * \rho_{\gamma} \in C_b^{\infty}(\mathbb{R}^N)$. By Lemma [5.1](#page-16-1) and the fact that $f^{\alpha} \in$ $C^{1,\sigma-1}(\mathbb{R}^N)$ by $(\mathbf{B}.2)$,

$$
||D^m f^{\alpha}_{(\gamma)}||_0 \le \frac{C||f^{\alpha}||_{1,\sigma-1}}{\gamma^{m-\sigma}} \quad \text{and} \quad ||f^{\alpha} - f^{\alpha}_{(\gamma)}||_0 \le C\gamma^{\sigma}||f^{\alpha}||_{1,\sigma-1}.
$$
 (5.12)

Then by [\(3.8\)](#page-11-2), [\(3.5\)](#page-11-5), the bound on a_{δ}^{α} in [\(3.4\)](#page-10-6), and first part of [\(5.12\)](#page-18-1),

$$
\left| \mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}_{(\gamma)}] \right| \leq \left| \mathcal{D}_{\delta,k}^{\alpha_0}[f^{\alpha}_{(\gamma)}] \right| + \frac{Ch^2}{k^2} \| D^2 f^{\alpha}_{(\gamma)} \|_0
$$

$$
\leq K |(\sqrt{a^{\alpha}_{\delta}})_i)|^2 \| D^2 f^{\alpha}_{(\gamma)} \|_0 + C \frac{h^2}{k^2} \| D^2 f^{\alpha}_{(\gamma)} \|_0
$$

$$
\leq \frac{K}{\gamma^{2-\sigma}} \left(\delta^{2-\sigma} + \frac{h^2}{k^2} \right) \| f^{\alpha} \|_{1,\sigma^{-1}}.
$$

By the second part of [\(5.12\)](#page-18-1) and the definition of $\mathcal{L}_{\delta,k,h}^{\alpha_0}$ in [\(3.7\)](#page-11-6),

$$
\left|\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}_{(\gamma)}]-\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}]\right|=\left|\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}_{(\gamma)}-f^{\alpha}]\right|\leq K\frac{\gamma^{\sigma}}{k^2}\|f^{\alpha}\|_{1,\sigma-1},
$$

and then

$$
\|\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}]\|_{0} \leq \|\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}_{(\gamma)}]\|_{0} + \|\mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}_{(\gamma)}] - \mathcal{L}_{\delta,k,h}^{\alpha_0}[f^{\alpha}]\|_{0}
$$

$$
\leq K \Big[\frac{1}{\gamma^{2-\sigma}} \Big(\frac{h^2}{k^2} + \delta^{2-\sigma}\Big) + \frac{\gamma^{\sigma}}{k^2}\Big] \|f^{\alpha}\|_{1,\sigma-1}.
$$
 (5.13)

The result follows by taking $\gamma = \max\{h, k \, \delta^{\frac{2-\sigma}{2}}\}.$

Proof of Theorem [5.4.](#page-18-2) (i) Since u_h solves (3.13) , we find as in the proof of Theo-rem [2.5,](#page-7-4) that $-\mathcal{I}_h^{\alpha_0,\delta}[u_h]$ is a supersolution of

$$
\lambda v(x) + \sup_{\alpha \in \mathcal{A}} \left\{ -\mathcal{L}_{\delta,k,h}^{\alpha}[v] - \mathcal{I}_h^{\alpha,\delta}[v] - \mathcal{I}_h^{\alpha_0,\delta}[f^{\alpha}](x) \right\} = 0. \tag{5.14}
$$

By assumptions $(**B**.2)$ $(**B**.2)$ and $(**A**.3)$,

$$
\|\mathcal{I}_h^{\alpha_0,\delta}[f^{\alpha}]\|_0 \leq C_1 := \|f^{\alpha}\|_{1,\beta-1} \int_{|z|<1} |z|^{\beta} \nu_{\alpha}(dz) + 2\|f^{\alpha}\|_0 \int_{|z|\geq 1} \nu_{\alpha}(dz),
$$

where the constant $C_1 \geq 0$ is independent of α , δ , and h . Since $-\frac{C_1}{\lambda}$ is a subsolution of [\(5.14\)](#page-19-0), the comparison principle yields that $\mathcal{I}_h^{\alpha_0,\delta}[u_h](x) \leq \frac{C_1}{\lambda}$. Arguing in the same way for the operator $\mathcal{L}_{\delta,k,h}^{\alpha_0}$ and using Lemma [5.5,](#page-18-3) we get that

$$
\mathcal{L}_{\delta,k,h}^{\alpha_0}[u_h] \le \frac{K}{\lambda} \left[\frac{h^{\sigma}}{k^2} + k^{\sigma-2} \delta^{\frac{\sigma(2-\sigma)}{2}} \right] ||f^{\alpha}||_{1,\sigma-1}.
$$

Taking $k \geq C \max\{\delta^{\frac{\sigma}{2}},h^{\frac{\sigma}{2}}\} = C\delta^{\frac{\sigma}{2}}$ (assuming $\delta \geq h$) we find a constant $C_2 \geq$ 0 independent of α , k, h, and δ such that $\mathcal{L}_{\delta,k,h}^{\alpha_0}[u_h] \leq \frac{C_2}{\lambda}$. Combining the two estimates then gives

$$
\hat{\mathcal{I}}_{\delta,k,h}^{\alpha_0}[u_h](x) \leq \frac{C_1+C_2}{\lambda}.
$$

To get the lower bound, we use the definition of $\hat{\mathcal{I}}^{\alpha_0}_{\delta,k,h}[u_h]$ and the fact that u_h is a subsolution of [\(3.13\)](#page-12-5), to see that

$$
- \hat{\mathcal{I}}_{\delta,k,h}^{\alpha_0}[u_h](x) \le \sup_{\alpha \in \mathcal{A}} \{-\hat{\mathcal{I}}_{\delta,k,h}^{\alpha}[u_h](x)\}
$$

\n
$$
\le \lambda u_h(x) + \sup_{\alpha \in \mathcal{A}} \{-\mathcal{L}_{\delta,k,h}^{\alpha}[u_h] - \mathcal{I}_h^{\alpha,\delta}[u_h](x) + f^{\alpha}(x)\} + \left(\lambda \|u_h\|_0 + \|f^{\alpha}\|_0\right)
$$

\n
$$
\le \left(\lambda \|u_h\|_0 + \|f^{\alpha}\|_0\right).
$$

In view of $(A.2)$ $(A.2)$ and Theorem [3.4](#page-12-3) this completes the proof of (5.10) .

(ii) The upper bound for $-\mathcal{J}_h^{\alpha_0,\delta}[u_h]$ follows from the same reasoning that led to the upper bound in part (i). To prove the lower bound, we first note that $\int_{\delta<|z|<1}\omega_{\mathbf{j}}(\eta^{\alpha_0}(z))\left(\frac{d\nu_{\alpha_0}}{dz}(z)-\frac{c_{\alpha_0}}{|z|^{N+\sigma}}\right)$ $\left(dz \geq 0 \right)$ by $(\mathbf{B}.1)(i)$ $(\mathbf{B}.1)(i)$ $(\mathbf{B}.1)(i)$ and the fact $\omega_j \geq 0$. By arguments similar to those that led to estimate (2.8) , we then find that

$$
\sum_{\mathbf{j}\in\mathbb{Z}^N}\left(u_h(x+x_{\mathbf{j}})-u_h(x)\right)\int_{\delta<|z|<1}\omega_{\mathbf{j}}(\eta^{\alpha_0}(z))\left(\frac{d\nu_{\alpha_0}}{dz}(z)-\frac{c_{\alpha_0}}{|z|^{N+\sigma}}\right)dz\leq\frac{K}{\lambda}.
$$

Then by [\(5.10\)](#page-18-4) (this bound also holds for $\mathcal{I}_h^{\alpha_0,\delta}[u_h]$, see the proof), $\sum_{\mathbf{j}\in\mathbb{Z}^N}\omega_{\mathbf{j}}(\eta^{\alpha_0}(z))=$ 1, and $(A.4)$, we have

$$
-\mathcal{J}_h^{\alpha_0,\delta}[u_h] \leq \frac{K}{\lambda} - \mathcal{I}_h^{\alpha_0,\delta}[u_h]
$$

+
$$
\sum_{\mathbf{j}\in\mathbb{Z}^N} (u_h(x+x_\mathbf{j}) - u_h(x)) \int_{|z|>1} \omega_{\mathbf{j}}(\eta^{\alpha_0}(z)) \left(\frac{d\nu_{\alpha_0}}{dz}(z) - \frac{c_{\alpha_0}}{|z|^{N+\sigma}}\right) dz
$$

$$
\leq K + C \|u_h\|_0 \Big(\int_{|z|>1} \nu_{\alpha_0}(dz) + \int_{|z|>1} \frac{c_{\alpha_0}dz}{|z|^{N+\sigma}}\Big) \leq K + C \|u_h\|_0.
$$

This completes the proof.

By Theorem [2.7](#page-9-0) the solution u of (2.1) and its regularization $u^{(\epsilon)}$ satisfy the bounds of Lemma [5.1](#page-16-1) with $\beta = \sigma - 1$. We now show similar bounds for the solution u_h of the scheme [\(3.13\)](#page-12-5) and regularizations of u_h . The results will incorporate error terms due to truncation bounds for approximate operators.

Lemma 5.6. Assume $(A.1)-(A.7), (B.1)-(B.2), \delta \in (0,1), \delta \geq h$ $(A.1)-(A.7), (B.1)-(B.2), \delta \in (0,1), \delta \geq h$, u_h solves [\(3.13\)](#page-12-5), and $\tilde{u}_h = u_h * \phi$ for $0 \leq \phi \in C^{\infty}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \phi \, dx = 1$. Then there are $K_1, K_2 > 0$ independent of δ, k, h and ϕ such that

$$
(i) \quad ||(-\Delta)^{\frac{\sigma}{2}}[\tilde{u}_h]||_0 \leq K_1 \Big(||u_h||_{0,1} + \delta^{2-\sigma} (||u_h||_0 + ||D^2 \tilde{u}_h||_0) \Big),
$$

\n
$$
(ii) \quad ||\tilde{u}_h||_{1,\sigma-1} \leq K_2 \Big(1 + ||u_h||_{0,1} + \delta^{4-\sigma} ||D^4 \tilde{u}_h||_0 + \delta^{2(2-\sigma)} k^2 ||D^4 \tilde{u}_h||_0
$$

\n
$$
+ \frac{h^2}{k^2} ||D^2 \tilde{u}_h||_0 + h^2 \delta^{-\sigma} ||D^2 \tilde{u}_h||_0 \Big).
$$

Note that a bound like (ii) follows from (i) by elliptic regularity, but bound (ii) is an improvement on any bound coming from (i).

Proof. (i) Note that $\eta^{\alpha_0}(0) = 0$ by $(\mathbf{A}.3)$, $z - \eta_0^{\alpha}(z) = \mathcal{O}(|z|^2)$ by $(\mathbf{B}.1)(ii)$ $(\mathbf{B}.1)(ii)$ $(\mathbf{B}.1)(ii)$, and $\|\tilde{u}_h\|_{0,1} \leq \|u_h\|_{0,1}$ by properties of convolutions. By the definition of \mathcal{J}^{α_0} [\(5.8\)](#page-18-5), assumptions $(A.3)$, $(A.5)$ – $(A.7)$, $(B.1)$, and the truncation error bound (5.9) ,

$$
\begin{split}\n|(-\Delta)^{\frac{\sigma}{2}}[\tilde{u}_h](x)| \\
&\leq |\mathcal{J}^{\alpha_0}[\tilde{u}_h](x)| + \Big| \int_{|z| < 1} (z - \eta^{\alpha_0}(z)) \cdot \nabla \tilde{u}_h(x) \frac{dz}{|z|^{N+\sigma}} \Big| \\
&+ \Big| \int_{|z| > 1} \tilde{u}_h(x+z) - \tilde{u}_h(x + \eta^{\alpha_0}(z)) \frac{dz}{|z|^{N+\sigma}} \Big| \\
&\leq \|\mathcal{J}^{\alpha_0}[\tilde{u}_h]\|_0 + \|\nabla \tilde{u}_h\|_0 \int_{|z| < 1} \frac{K|z|^2 \, dz}{|z|^{N+\sigma}} + 2 \|\tilde{u}_h\|_0 \int_{|z| > 1} \frac{dz}{|z|^{N+\sigma}} \\
&\leq \|\mathcal{J}_h^{\alpha_0, \delta}[\tilde{u}_h]\|_0 + C(\|u_h\|_0 + \|D^2 \tilde{u}_h\|_0) \Big(\delta^{2-\sigma} + h^2 \delta^{-\sigma}\Big) + c_{\sigma} \|u_h\|_{0, 1}.\n\end{split}
$$

The proof is complete since by Theorem [5.4](#page-18-2) and properties of convolutions,

$$
\|\mathcal{J}_h^{\alpha_0,\delta}[\tilde{u}_h]\|_0 \leq C \|\mathcal{J}_h^{\alpha_0,\delta}[u_h]\|_0 \leq K.
$$

(ii) By Theorem [5.4](#page-18-2) and properties of convolutions, $\|\hat{\mathcal{I}}_{\delta,k,h}^{\alpha_0}[\tilde{u}_h]\|_0 \leq C \|\hat{\mathcal{I}}_{\delta,k,h}^{\alpha_0}[u_h]\|_0 \leq$ K. From the error bounds (3.3) , (3.9) , and (3.11) , it then follows that

$$
\begin{split} \|\mathcal{I}^{\alpha_{0}}[\tilde{u}_{h}]\|_{0} &\leq K_{1}\Big(K+\delta^{4-\sigma}\|D^{4}\tilde{u}_{h}\|_{0}+\delta^{2(2-\sigma)}k^{2}\|D^{4}\tilde{u}_{h}\|_{0} \\ &+\frac{h^{2}}{k^{2}}\|D^{2}\tilde{u}_{h}\|_{0}+\frac{h^{2}}{\delta^{\sigma}}\|D^{2}\tilde{u}_{h}\|_{0}\Big). \end{split} \tag{5.15}
$$

We define the operator

$$
\widetilde{\mathcal{J}}[\phi](x) := \int_{|z| < 1} \left(\phi(x+z) - \phi(x) - z \cdot \nabla \phi(x) \right) \nu^{\alpha_0}(dz) + \int_{|z| > 1} \left(\phi(x+z) - \phi(x) - z \cdot \nabla \phi(x) \right) \frac{c_{\alpha_0}}{|z|^{N+\sigma}}.
$$

Since $z - \eta_0^{\alpha}(z) = \mathcal{O}(|z|^2)$ by $(B.1)(ii)$ $(B.1)(ii)$ $(B.1)(ii)$ and $\|\tilde{u}_h\|_{0,1} \le \|u_h\|_{0,1}$, by (5.15) we have

$$
\begin{split}\n|\tilde{\mathcal{J}}[\tilde{u}_h](x)| &\leq |\mathcal{I}^{\alpha_0}[\tilde{u}_h](x)| \\
&+ \Big| \int_{|z| < 1} \left(\tilde{u}_h(x+z) - \tilde{u}_h(x+\eta^{\alpha_0}(z)) - (z-\eta^{\alpha_0}(z)) \cdot \nabla \tilde{u}_h(x) \right) \nu^{\alpha_0}(dz) \Big| \\
&+ \Big| \int_{|z| > 1} \left(\tilde{u}_h(x+z) - \tilde{u}_h(x) \right) \frac{c_{\alpha_0} dz}{|z|^{N+\sigma}} - \int_{|z| > 1} \left(\tilde{u}_h(x+j^{\alpha}(z)) - \tilde{u}_h(x) \right) \nu^{\alpha_0}(dz) \Big| \\
&\leq |\mathcal{I}^{\alpha_0}[\tilde{u}_h](x)| + 2 \|\nabla \tilde{u}_h\|_0 \int_{|z| < 1} |z - \eta^{\alpha_0}(z)| \nu^{\alpha_0}(dz) + 2 \|\tilde{u}_h\|_0 \int_{|z| > 1} \frac{(c_{\alpha_0} + C)}{|z|^{N+\sigma}} \\
&\leq C \Big(K + \delta^{4-\sigma} \| D^4 \tilde{u}_h \|_0 + \delta^{2(2-\sigma)} k^2 \| D^4 \tilde{u}_h \|_0 \\
&\quad + \frac{h^2}{k^2} \| D^2 \tilde{u}_h \|_0 + \frac{h^2}{\delta^{\sigma}} \| D^2 \tilde{u}_h \|_0 + \|u_h \|_{0,1} \Big). \tag{5.16}\n\end{split}
$$

Hence $\widetilde{\mathcal{J}}[\tilde{u}_h] \in L^{\infty}(\mathbb{R}^N)$ for fixed δ and h . By $(\mathbf{B}.1)(i)$ $(\mathbf{B}.1)(i)$ $(\mathbf{B}.1)(i)$ and $(\mathbf{A}.6)$, the assumptions of the regularity result [\[31,](#page-30-25) Theorem 3.8] are satisfied, and we conclude that

$$
\|\tilde{u}_h\|_{1,\sigma-1}\leq K\Big(\|\tilde{u}_h\|_0+\|\widetilde{\mathcal{J}}[\tilde{u}_h]\|_0\Big).
$$

The result then follows from (5.16) .

We now give results approximation and derivative bounds mollifications of u_h by the fractional heat kernel. These are discrete versions of Lemmas [5.2](#page-17-0) and [5.3.](#page-17-1)

Lemma 5.7. Assume $\delta \in (0,1)$, $h \leq \delta$, $\epsilon > 0$, $(A.1)$ - $(A.5)$, $(B.1)$ - $(B.2)$, u_h solves [\(3.13\)](#page-12-5), and its mollification $u_h^{[\varepsilon]}$ $h^{\text{[s]}}$ is defined in [\(5.7\)](#page-16-2). Then for $m \geq 2$,

$$
\|D^m u^{[\varepsilon]}_h\|_0 \leq K \frac{\|u_h\|_{0,1}}{\varepsilon^{m-\sigma}} \Big(1+\big(\delta^{4-\sigma}+\delta^{2(2-\sigma)}k^2\big)\frac{1}{\varepsilon^3}+\big(h^2k^{-2}+h^2\delta^{-\sigma}\big)\frac{1}{\varepsilon}\Big).
$$

$$
\sqcup
$$

Proof. By Lemma [5.3](#page-17-1) and Lemma [5.6](#page-20-0) (*ii*) with $\phi(x) = \tilde{K}^{\sigma}(\varepsilon^{\sigma}, x)$,

$$
\|D^m u_h^{[\varepsilon]}\|_0 \le \frac{K}{\varepsilon^{m-\sigma}} \|u_h^{[\varepsilon_1]}\|_{1,\sigma-1}
$$

$$
\le \frac{K}{\varepsilon^{m-\sigma}} \Big(1 + \|u_h\|_{0,1} + \delta^{4-\sigma} \|D^4 u_h^{[\varepsilon_1]}\|_0 + \delta^{2(2-\sigma)} k^2 \|D^4 u_h^{[\varepsilon_1]}\|_0
$$

$$
+ \frac{h^2}{k^2} \|D^2 u_h^{[\varepsilon_1]}\|_0 + \frac{h^2}{\delta^{\sigma}} \|D^2 u_h^{[\varepsilon_1]}\|_0 \Big),
$$

where $\varepsilon_1 = \frac{\varepsilon}{\sqrt{2}}$ $\frac{\varepsilon}{2^{\frac{1}{\sigma}}}$. The result then follows from the first part of Lemma [5.3.](#page-17-1) \Box

Lemma 5.8. Assume $0 < h \le \delta \le \epsilon$, $\delta \in (0,1)$, $(A.1)-(A.5)$ $(A.1)-(A.5)$ $(A.1)-(A.5)$, $(B.1)-(B.2)$, u_h solves [\(3.13\)](#page-12-5), and its mollification $u_h^{[\varepsilon]}$ $\binom{[c]}{h}$ is defined in [\(5.7\)](#page-16-2). Then

$$
||u_h^{[\varepsilon]} - u_h||_0 \leq C(\delta + \varepsilon^{\sigma} + \delta^{2-\sigma} \varepsilon^{2(\sigma-1)} + k^2 \delta^{1-\sigma} + \frac{h^2}{k^2} \delta^{\sigma-1}).
$$

Proof. Let S_t be the fractional heat semigroup (c.f. the proof of Lemma [5.2\)](#page-17-0) so that $u_h^{[\varepsilon]} = S_{\varepsilon}(\varepsilon u_h)$. By properties of S_t and Lemmas [5.6](#page-20-0) (i) and [5.7,](#page-21-2) we have

$$
|S_t[u_h] - S_s[u_h]| = \Big| \int_s^t (-\Delta)^{\frac{\sigma}{2}} \Big[S_r[u_h] \Big] dr \Big| \leq C \int_s^t \left(1 + \delta^{2-\delta} (1 + \|D^2 u_h^{[r^{\frac{1}{\sigma}}]} \|_0) \right) dr
$$

\n
$$
\leq C \int_s^t \left(1 + \frac{\delta^{2-\sigma}}{r^{\frac{2-\sigma}{\sigma}}} \Big(1 + \frac{\delta^{4-\sigma} + \delta^{2(2-\sigma)} k^2}{r^{\frac{3}{\sigma}}} + \frac{h^2 k^{-2} + h^2 \delta^{-\sigma}}{r^{\frac{1}{\sigma}}} \Big) \right) dr
$$

\n
$$
\leq C \left(t + \delta^{2-\sigma} t^{\frac{2(\sigma-1)}{\sigma}} + \left(\delta^{6-2\sigma} + \delta^{6-3\sigma} k^2 \right) s^{\frac{2\sigma-5}{\sigma}} + \left(h^2 \delta^{2-2\sigma} + h^2 \delta^{2-\sigma} k^{-2} \right) s^{\frac{2\sigma-3}{\sigma}} \right).
$$

Since $||S_s[u_h] - [u_h]||_0 \leq Cs^{\frac{1}{\sigma}} ||u_h||_{0,1}$, we then find that

$$
||S_t[u_h] - [u_h]||_0 \le C \left(s^{\frac{1}{\sigma}} + t + \delta^{2-\sigma} t^{\frac{2(\sigma-1)}{\sigma}} + \left(\delta^{6-2\sigma} + \delta^{6-3\sigma} k^2\right) s^{\frac{2\sigma-5}{\sigma}} + \left(h^2 \delta^{2-2\sigma} + h^2 \delta^{2-\sigma} k^{-2}\right) s^{\frac{2\sigma-3}{\sigma}}\right).
$$

This estimate holds for any $s \in (0, t)$. Note that since $h \leq \delta$, the Take $t = \varepsilon^{\sigma}$ and $s=\delta^{\sigma}$ to find that

$$
||u_h^{[\varepsilon]} - u_h||_0 \le C\Big(\delta + \varepsilon^{\sigma} + \delta^{2-\sigma}\varepsilon^{2(\sigma-1)} + (\delta^{6-2\sigma} + \delta^{6-3\sigma}k^2)\delta^{2\sigma-5} + (h^2\delta^{2-2\sigma} + h^2\delta^{2-\sigma}k^{-2})\delta^{2\sigma-3}\Big).
$$

$$
\le C(\delta + \varepsilon^{\sigma} + \delta^{2-\sigma}\varepsilon^{2(\sigma-1)} + \delta + k^2\delta^{1-\sigma} + \delta + \frac{h^2}{k^2}\delta^{\sigma-1}).
$$
This completes the proof.

In the last proof the dependence on the parameters are only partially optimized, but the result is still good enough for our purposes – the optimal error bound that we will prove next.

Proof of Theorem [3.8.](#page-13-1) The proof is similar to the proof of Theorem [3.6,](#page-13-0) and only the case $\sigma > 1$ is new. Let $(\rho_{\epsilon})_{\epsilon > 0}$ be the standard mollifier on \mathbb{R}^{N} and define $u^{(\epsilon)} = u * \rho_{\epsilon}$. Since u is the viscosity solution of (2.1) , $u^{(\epsilon)}$ is a smooth solution of

$$
\lambda\,u^{(\epsilon)}+\sup_{\alpha\in\mathcal{A}}\Big\{(f^{\alpha})^{(\epsilon)}(x)-\mathcal{I}^{\alpha}[u^{(\epsilon)}]\Big\}\leq 0.
$$

By Theorem [2.7,](#page-9-0) $u \in C^{1,\sigma-1}(\mathbb{R}^N)$, and by $(B.2)$ $(B.2)$ and Lemma [5.1,](#page-16-1) $||f^{\alpha}-(f^{\alpha})^{(\varepsilon)}||_0 \le$ $K\varepsilon^{\sigma}$. Therefore, from the truncation error bounds (3.3) , (3.9) and (3.11) we get

$$
\lambda u^{(\epsilon)} + \sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) - \mathcal{I}_{h}^{\alpha} u^{(\epsilon)} \right\} \le \sup_{\alpha \in \mathcal{A}} \left[\| f^{\alpha} - (f^{\alpha})^{(\epsilon)} \|_{0} + \| \mathcal{I}_{h}^{\alpha} [u^{(\epsilon)}] - \mathcal{I}^{\alpha} [u^{(\epsilon)}] \|_{0} \right]
$$

\n
$$
\le C\epsilon^{\sigma} + C\left(\delta^{4-\sigma} \| D^{4} u^{(\epsilon)} \|_{0} + \delta^{2(2-\sigma)} k^{2} \| D^{4} u^{(\epsilon)} \|_{0} + \frac{h^{2}}{k^{2}} \| D^{2} u^{(\epsilon)} \|_{0} \right)
$$

\n
$$
\le C\left(\epsilon^{\sigma} + \delta^{4-\sigma} \frac{1}{\epsilon^{4-\sigma}} + \delta^{2(2-\sigma)} k^{2} \frac{1}{\epsilon^{4-\sigma}} + \frac{h^{2}}{k^{2}} \frac{1}{\epsilon^{2-\sigma}} + \frac{h^{2}}{\delta^{\sigma}} \frac{1}{\epsilon^{2-\sigma}} \right) := A_{\epsilon}.
$$

Hence $u^{(\epsilon)} - \frac{C}{\lambda}A_{\epsilon}$ is a subsolution of the equation [\(3.13\)](#page-12-5), and the comparison principle for [\(3.13\)](#page-12-5) then implies that $u^{(\epsilon)} - \frac{C}{\lambda} A_{\epsilon} \leq u_h$. By Theorem [2.7](#page-9-0) and Lemma [5.1,](#page-16-1) $||u^{(\epsilon)} - u|| \leq K\epsilon^{\sigma}$, and we conclude that

$$
u(x) - u_h(x) \leq C\epsilon^{\sigma} + \frac{C}{\lambda}A_{\varepsilon}.
$$

Minimizing by taking $k^2 = O(\frac{h\varepsilon}{\delta^{2-\sigma}})$, $\delta = O(h^{\frac{1}{2}}\varepsilon^{\frac{1}{2}})$, and $\varepsilon = O(h^{\frac{4-\sigma}{4+\sigma}})$, leads to

$$
u(x) - u_h(x) \le Kh^{\frac{\sigma(4-\sigma)}{4+\sigma}}.
$$

The lower bound on $u-u_h$ follows from a similar argument based on the solution u_h of the scheme [\(3.13\)](#page-12-5). For technical reasons, we need to work with a different regularisation $u_h^{[\epsilon]}$ $h^{\text{[c]}}_h$ based on the fractional heat kernel, see the definition in [\(5.7\)](#page-16-2). Since u_h solves (3.13) , we have

$$
\lambda\, u_h^{[\epsilon]} + \sup_{\alpha\in\mathcal{A}} \left\{ (f^\alpha)^{[\epsilon]}(x) - \mathcal{I}_h^\alpha[u_h^{[\epsilon]}]\right\} \leq 0.
$$

By (**B**[.2\)](#page-6-5) and Lemma [5.2,](#page-17-0) $||f^{\alpha} - (f^{\alpha})^{[\epsilon]}|| \leq C\epsilon^{\sigma}$, and then by Lemmas [3.1,](#page-10-5) [3.2](#page-11-4) and [3.3,](#page-12-4)

$$
\lambda u_h^{[\epsilon]} + \sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) - \mathcal{I}^{\alpha}[u_h^{[\epsilon]}] \right\} \leq K \epsilon^{\sigma} + ||\mathcal{I}^{\alpha}[u_h^{[\epsilon]}] - \mathcal{I}_h^{\alpha}[u_h^{[\epsilon]}]||_0
$$

$$
\leq K \epsilon^{\sigma} + C \Big(\Big(\delta^{4-\sigma} + \delta^{2(2-\sigma)} k^2 \Big) ||D^4 u_h^{[\epsilon]}||_0 + \Big(h^2 k^{-2} + h^2 \delta^{-\sigma} \Big) ||D^2 u_h^{[\epsilon]}||_0 \Big) := B_{\epsilon}.
$$

Hence $u_h^{[\epsilon]} - \frac{C}{\lambda} B_{\epsilon}$ is a subsolution of equation [\(2.1\)](#page-6-4), and the comparison principle for [\(2.1\)](#page-6-4) then implies that $u_h^{[\epsilon]} - u \leq \frac{C}{\lambda} B_{\epsilon}$. Therefore by Lemma [5.8](#page-22-0) and the bounds on $\|D^4u_h^{[\epsilon]}$ $_{h}^{[\epsilon]} \|_{0}$ and $\|D^{2}u_{h}^{[\epsilon]}$ $\frac{e}{h}$ of from Lemma [5.7,](#page-21-2) we get

$$
u_h - u \leq C \left(\delta + \varepsilon^{\sigma} + \delta^{2-\sigma} \varepsilon^{2(\sigma-1)} + k^2 \delta^{1-\sigma} + \frac{h^2}{k^2} \delta^{\sigma-1} \right)
$$

+
$$
C \left(\delta^{4-\sigma} + \delta^{2(2-\sigma)} k^2 \right) \frac{1 + (\delta^{4-\sigma} + \delta^{2(2-\sigma)} k^2) \frac{1}{\varepsilon^3} + (h^2 k^{-2} + h^2 \delta^{-\sigma}) \frac{1}{\varepsilon}}{\varepsilon^{4-\sigma}}
$$

+
$$
C \left(h^2 k^{-2} + h^2 \delta^{-\sigma} \right) \frac{1 + (\delta^{4-\sigma} + \delta^{2(2-\sigma)} k^2) \frac{1}{\varepsilon^3} + (h^2 k^{-2} + h^2 \delta^{-\sigma}) \frac{1}{\varepsilon}}{\varepsilon^{2-\sigma}}.
$$

As in the proof of the upper bound, we now take $k^2 = O(\delta^{\sigma})$ so that

$$
u_h - u \le K \left(\varepsilon^{\sigma} + \delta^{2-\sigma} \varepsilon^{2(\sigma-1)} + \delta + h^2 \delta^{-1} \right)
$$

+
$$
C \left(\frac{h^2 \delta^{-\sigma}}{\varepsilon^{2-\sigma}} + \frac{h^4 \delta^{-2\sigma}}{\varepsilon^{3-\sigma}} + \frac{\delta^{4-\sigma}}{\varepsilon^{4-\sigma}} + 2 \frac{h^2 \delta^{4-2\sigma}}{\varepsilon^{5-\sigma}} + \frac{\delta^{8-2\sigma}}{\varepsilon^{7-\sigma}} \right)
$$

= $A_1 + A_2$.

To continue note we can factor the second term,

$$
A_2 = C \frac{1}{\varepsilon^{1-\sigma}} \Big(\frac{h^2}{\varepsilon \delta^{\sigma}} + \frac{\delta^{4-\sigma}}{\varepsilon^3} \Big) \Big(1 + \frac{h^2}{\varepsilon \delta^{\sigma}} + \frac{\delta^{4-\sigma}}{\varepsilon^3} \Big).
$$

Taking $\delta = O(h^{\frac{1}{2}} \varepsilon^{\frac{1}{2}})$ as in the upper bound, we balance terms in A_2 , and $A_2 =$ $\frac{1}{\varepsilon^{1-\sigma}}a(1+a)$ for $a^2 = O(\frac{h^{4-\sigma}}{\varepsilon^{2+\sigma}})$ $\frac{h^{1-\sigma}}{\varepsilon^{2+\sigma}}$). Finally (as for the upper bound) we take $\varepsilon =$ $O(h^{\frac{4-\sigma}{4+\sigma}})$. Then it is easy to check (for $h < 1$) that $a = O(\varepsilon)$ and

$$
A_2 \le O\left(\frac{a}{\varepsilon^{1-\sigma}}\right) = O(\varepsilon^{\sigma}) = O(h^{\frac{\sigma(4-\sigma)}{4+\sigma}}).
$$

In the remaining A_1 term, using $h \leq \delta \leq \varepsilon$ to estimate the 2nd and 4th terms, and a direct computation for the δ -term, we find that the 2nd and 4th terms are $O(\varepsilon^{\sigma})$ and $O(h)$, while $\delta = O(h^{\frac{4}{4+\sigma}})$. Since $\frac{\sigma(4-\sigma)}{4+\sigma} \leq \frac{4}{4+\sigma} \leq 1$, we conclude that

$$
u_h - u \leq Ch^{\frac{\sigma(4-\sigma)}{4+\sigma}}.
$$

This completes the proof of the theorem.

6. Proof of error bound for powers of discrete Laplacian

We start with an analogous (uniform in h) bound as in Theorem [5.4.](#page-18-2)

Theorem 6.1. Assume $(A.1)-(A.5)$ $(A.1)-(A.5)$ $(A.1)-(A.5)$, $(B.1)-(B.2)$, and u_h solves (4.3) . Then there is $K > 0$ independent of h such that

$$
\|(-\Delta_h)^{\frac{\sigma}{2}}[u_h]\|_0 \leq K.
$$

We omit the proof which is similar to the proof of Theorem [5.4,](#page-18-2) but simpler since we have no diffusion correction term in the approximation this time. Next we state the analogous results to Lemmas [5.6](#page-20-0) and [5.7](#page-21-2) for regularisations $u_h^{[\epsilon]}$ $h^{\left[\epsilon\right]}(x)$ by the fractional heat semigroup defined in [\(5.7\)](#page-16-2).

Lemma 6.2. Assume $\sigma > 1$, (**A**[.1\)-](#page-6-1)(**A**[.7\),](#page-5-5) (**B**.1)-(**B**[.2\),](#page-6-5) u_h solves [\(4.3\)](#page-14-3), and $u_h^{[\varepsilon]}$ h is defined in [\(4.3\)](#page-14-3). Then there is $K > 0$ independent of h and ε such that

$$
||u_h^{[\varepsilon]}||_{1,\sigma-1} \le K\Big(1 + \frac{h^2}{\varepsilon^3}\Big),\tag{6.1}
$$

.

and for $m \geq 2$,

$$
||D^m u_h^{[\varepsilon]}||_0\leq \frac{K}{\varepsilon^{m-\sigma}}\Big(1+\frac{h^2}{\varepsilon^3}\Big)
$$

Proof. By Theorem [6.1](#page-24-1) and properties of \tilde{K}^{σ} , $\|(-\Delta_h)^{\frac{\sigma}{2}}[u_h^{[\varepsilon]}]$ $\|h_{h}^{\left[\varepsilon\right]}\|_{0} \leq C \|(-\Delta_{h})^{\frac{\sigma}{2}}[u_{h}]\|_{0} \leq$ CK , and we conclude from the truncation error bound (4.5) that

$$
\|(-\Delta)^{\frac{\sigma}{2}}u_h^{[\varepsilon]}\|_0 \le K_1\Big(\|(-\Delta_h)^{\frac{\sigma}{2}}[u_h]\|_0 + h^2(\|D^4u_h^{[\varepsilon]}\|_0 + \|u_h^{[\varepsilon]}\|_0)\Big). \tag{6.2}
$$

Since $\|D^m u_h^{(\varepsilon)}\|$ $\|h_k^{(\varepsilon)}\|_0 \leq \frac{C}{\varepsilon^{m-1}} \|u_h\|_{0,1}$ by Lemma [5.3,](#page-17-1) estimate (6.1) follows from the regularity estimate $[58,$ Theorem 1.1(a)] by Ros-Oton and Serra for fractional Laplace operators. The second part follows from (6.1) and Lemma [5.3.](#page-17-1)

We give a version of Lemma 5.8 for powers of the discrete fractional Laplacian.

Lemma 6.3. Assume $\sigma > 1$, $0 < h \leq \epsilon^{\frac{4-\sigma}{2}}$, $(A.1)$ - $(A.5)$, $(B.1)$ - $(B.2)$, u_h solves (4.3) , and $u_h^{[\varepsilon]}$ $\binom{[e]}{h}$ is defined in [\(5.7\)](#page-16-2). Then

$$
||u_h^{[\varepsilon]} - u_h||_0 \le K \Big(\varepsilon^{\sigma} ||(-\Delta_h)^{\frac{\sigma}{2}} [u_h]||_0 + h^{\frac{2}{4-\sigma}} ||u_h||_{0,1}\Big). \tag{6.3}
$$

Proof. The proof is similar to the proof of Lemma [5.8.](#page-22-0) By definition [\(5.7\)](#page-16-2), $u_h^{[\varepsilon]} =$ $S_r(u_h)$ where $\varepsilon = r^{\frac{1}{\sigma}}$ and S_t is the fractional heat semigroup. Therefore using properties of heat kernels, estimate [\(6.2\)](#page-24-3), and the first part of Lemma [5.3,](#page-17-1) we have

$$
|S_t(u_h) - S_s(u_h)| \leq \int_s^t K\Big(\|(-\Delta_h)^{\frac{\sigma}{2}} u_h\|_0 + \frac{h^2}{r^{\frac{3}{\sigma}}} \|u_h\|_{0,1} \Big) dr
$$

\n
$$
\leq K(t-s) \|(-\Delta_h)^{\frac{\sigma}{2}} u_h\|_0 + Kh^2 \|u_h\|_{0,1} \Big(\frac{1}{s^{\frac{3-\sigma}{\sigma}}} - \frac{1}{t^{\frac{3-\sigma}{\sigma}}} \Big),
$$

\n
$$
|S_s(u_h) - u_h| = \Big| \int_{\mathbb{R}^N} \Big(u_h(x - s^{\frac{1}{\sigma}} y) - u_h(x) \Big) K^{\sigma}(y) dy \Big| \leq K s^{\frac{1}{\sigma}} \|u_h\|_{0,1}.
$$

Moreover,

$$
|S_t(u_h) - u_h| \le K\Big(t\|(-\Delta_h)^{\frac{\sigma}{2}}u_h\|_0 + \Big(\frac{h^2}{s^{\frac{3-\sigma}{\sigma}}} + s^{\frac{1}{\sigma}}\Big)\|u_h\|_{0,1}\Big).
$$

The result now follows by taking $t = \varepsilon^{\sigma}$ and $s = h^{\frac{2\sigma}{4-\sigma}}$, noting that $s \leq t$ by the assumption that $h \leq \varepsilon^{\frac{4-\sigma}{2}}$ $\frac{-\sigma}{2}$.

Proof of Theorem [4.2.](#page-15-4) The case $\sigma < 1$ uses no more than Lipschitz continuity of solutions and follows in straight forward way from the local truncation error bound in Lemma [4.1](#page-14-5) and the regularisation/comparison arguments in the proof of Theorem [3.6.](#page-13-0) Therefore we focus on the case $\sigma > 1$. The arguments are same as in the proof of Theorem [3.8.](#page-13-1) To prove the upper bound on $u - u_h$, we regularize equation (4.1) and use the truncation error bound (4.5) to find that

$$
\lambda u^{(\epsilon)} + \sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + a^{\alpha} (-\Delta_h)^{\frac{\sigma}{2}} u^{(\epsilon)} \right\}
$$

\n
$$
\leq \sup_{\alpha \in \mathcal{A}} \| f^{\alpha} - (f^{\alpha})^{(\epsilon)} \|_{0} + \sup_{\alpha \in \mathcal{A}} a^{\alpha} \| (-\Delta_h)^{\frac{\sigma}{2}} u^{(\epsilon)} - (-\Delta)^{\frac{\sigma}{2}} u^{(\epsilon)} \|_{0}
$$

\n
$$
\leq K \epsilon^{\sigma} + C h^{2} \Big(\| D^{4} u^{(\epsilon)} \|_{0} + \| u^{(\epsilon)} \|_{0} \Big).
$$

Hence $u^{(\epsilon)} - \frac{C}{\lambda} (\epsilon^{\sigma} + h^2(||D^4 u^{(\epsilon)}||_0 + ||u^{(\epsilon)}||_0))$ is a subsolution of equation [\(4.3\)](#page-14-3). By the comparison principle for (4.3) , regularity of u given by Theorem [2.7,](#page-9-0) and the bounds given by Lemma [5.1,](#page-16-1) we have

$$
u(x) - u_h(x) \le K\Big(\epsilon^{\sigma} + \frac{h^2}{\epsilon^{4-\sigma}}\Big).
$$

We optimize the right hand side by choosing $\epsilon = O(h^{\frac{1}{2}})$ and get

$$
u(x) - u_h(x) \le Kh^{\frac{\sigma}{2}}.
$$

To prove the lower bound we mollify/regularize the scheme (4.3) using the frac-tional heat semigroup. Then by Lemma [5.2](#page-17-0) for f^{α} and the truncation error [\(4.5\)](#page-14-4),

$$
\lambda u_h^{[\epsilon]} + \sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + a^{\alpha} (-\Delta)^{\frac{\sigma}{2}} u_h^{[\epsilon]} \right\} \leq C\epsilon^{\sigma} + Ch^2 \Big(\| D^4 u_h^{[\epsilon]} \|_0 + \| u_h^{[\epsilon]} \|_0 \Big).
$$

Therefore $u_h^{[\epsilon]} - \frac{C}{\lambda} (\epsilon^{\sigma} + h^2 ||D^4 u^{[\epsilon]}||_0 + h^2 ||u^{[\epsilon]}||_0)$ is a subsolution of equation [\(4.1\)](#page-14-1), and comparison for [\(4.1\)](#page-14-1) then yields

$$
u_h^{[\epsilon]} - u \le \frac{C}{\lambda} \Big(\epsilon^{\sigma} + h^2 \| D^4 u_h^{[\epsilon]} \|_0 + h^2 \| u_h^{[\epsilon]} \|_0 \Big).
$$

Then by Lemma [6.3](#page-25-0) (needs $h \leq \varepsilon^{\frac{4-\sigma}{2}}$) and the $||D^4u_h^{[\epsilon]}||$ $\mathbb{R}^{\lfloor \epsilon \rfloor}$ | 0-bound of Lemma [6.2,](#page-24-4)

$$
u_h - u \le C \Big(\epsilon^{\sigma} + h^{\frac{2}{4-\sigma}} + \frac{h^2}{\epsilon^{4-\sigma}} + \frac{h^4}{\epsilon^{7-\sigma}} \Big).
$$

Optimizing in ϵ by choosing $\epsilon = O(h^{\frac{1}{2}})$, we get the final estimate

$$
u_h - u \le K\left(h^{\frac{\sigma}{2}} + h^{\frac{2}{4-\sigma}}\right).
$$

The result now follows since $\frac{2}{4-\sigma} > \frac{\sigma}{2}$ for $\sigma > 1$ and $h = \varepsilon^2 \leq \varepsilon^{\frac{2}{4-\sigma}}$ (for $h < 1$). \Box

7. Extensions

In this section we discuss two related extensions of our previous results: (i) to nonlocal HJB equations with drift/advection terms, and (ii) to jump diffusions with nonsymmetric singular parts in the sense that we drop condition (A[.7\).](#page-5-5) Consider

$$
\sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + c^{\alpha}(x)u(x) - b^{\alpha} \cdot \nabla u(t, x) - \mathcal{I}^{\alpha}[u](x) \right\} = 0, \quad \text{in } \mathbb{R}^{N},\tag{7.1}
$$

where $b^{\alpha} \in \mathbb{R}^{N}$ and a modified version of $(\mathbf{A}.2)$ $(\mathbf{A}.2)$ holds:

 $(A.2')$ There is a $K > 0$ such that

$$
||f^{\alpha}||_1 + ||c^{\alpha}||_1 + |b^{\alpha}| + ||\eta^{\alpha}||_0 \le K \quad \text{for} \quad \alpha \in \mathcal{A}.
$$

Under assumptions $(A.1)$, $(A.2')$, $(A.3)$, $(A.4)$ $(A.4)$ equation (7.1) is well-posed, comparison holds, and the C_b and Lipschitz bounds of Proposition [2.2](#page-6-3) hold. The proof is the same as for Proposition [2.2.](#page-6-3) Note that x-independent b^{α} is consistent with x-independent η^{α} in [\(1.2\)](#page-1-4) and simplifies the presentation below.

Dropping $(A.7)$ $(A.7)$ means that

$$
\tilde{b}^{\alpha,\delta} := \int_{\delta < |z| < 1} \eta^\alpha(z) \, \nu_\alpha(dz) \neq 0,
$$

and there is a new drift term in our equation. We can write the nonlocal term as

$$
\mathcal{I}^{\alpha}[\phi](x) = \mathcal{I}^{\alpha}_{\delta}[\phi](x) + \mathcal{I}^{\alpha,\delta}[\phi](x) - \tilde{b}^{\alpha,\delta} \cdot \nabla \phi(x),
$$

where $\mathcal{I}^{\alpha}_{\delta}$, $\mathcal{I}^{\alpha,\delta}$ are defined in Section [3.](#page-10-0) The term $\tilde{b}^{\alpha,\delta}$ is bounded under a $C^{1,1}$ condition for η^{α} at $z = 0$, a uniform in α version of assumption (**B**[.1\)](#page-6-1) (ii):

(A.8) There is $K > 0$ such that

$$
|\eta^{\alpha}(z) - 2\eta^{\alpha}(0) - \eta^{\alpha}(-z)| \le K|z|^2 \quad \text{for} \quad |z| < 1, \quad \alpha \in \mathcal{A}.
$$

This assumption is satisfied in most applications. The next result is a version of Lemma [3.1](#page-10-5) without $(A.7)$.

Lemma 7.1. Assume $(A.1)$, $(A.2')$, $(A.3)$ $(A.3)$ - $(A.6)$ $(A.6)$ and $\delta \in (0.1)$.

(i) There is $K > 0$ independent of δ, α, ϕ such that

$$
|\mathcal{I}_{\delta}^{\alpha}[\phi] - tr[a_{\delta}^{\alpha} D^2 \phi]| \leq K \delta^{3-\sigma} ||D^3 \phi||_0. \tag{7.2}
$$

(ii) If also $(A.\delta)$ holds, there is $C > 0$ independent of δ, α such that $|\tilde{b}^{\alpha,\delta}| \leq C$.

Proof. (i) The proof is similar to the proof of Lemma [3.1.](#page-10-5) After a Taylor expansion of ϕ , we find that

$$
\mathcal{I}_{\delta}^{\alpha}[\phi](x) = tr[a_{\delta}^{\alpha} D^2 \phi] + Err_{1,\delta},
$$

where $Err_{\delta} = \frac{|\beta|}{\beta!}$ $\frac{|\beta|}{\beta!}\sum_{|\beta|=3}\big[\int_{|z|<\delta}\int_0^1(1-s)^{|\beta-1|}D^{\beta}\phi(x+s\eta^{\alpha}(z))\eta^{\alpha}(z)^{\beta}\,ds\,\nu_{\alpha}(dz)\big]$ and a_{δ}^{α} is defined in Lemma [3.1.](#page-10-5) By $(A.6)$ $(A.6)$ we have $|Err_{1,\delta}| \leq C\delta^{3-\sigma} ||D^3\phi||_0$. (ii) Since $\eta^{\alpha}(0) = 0$ by $(A.3)$, assumptions $(A.5)$ $(A.5)$ and $(A.8)$ $(A.8)$ lead to

$$
\left|\tilde{b}^{\alpha,\delta}\right| = \frac{1}{2} \Big| \int_{\delta < |z| < 1} \left(\eta^\alpha(z) + \eta^\alpha(-z) \right) \nu_\alpha(dz) \Big| \leq K \int_{\delta < |z| < 1} |z|^2 \, \nu_\alpha(dz).
$$

By $(A.4)$, this completes the proof.

Following the approach of Section 3 , to discretize (7.1) we first approximate small jumps by a diffusion. This leads to equation [\(3.2\)](#page-10-3) with a redefined operator $\mathcal{L}_{\delta}^{\alpha}$ to account for the drift:

$$
\mathcal{L}_{\delta}^{\alpha}[\phi](x) := tr[a_{\delta}^{\alpha} D^2 \phi](x) + b_{\delta}^{\alpha} \cdot \nabla \phi(x), \qquad b_{\delta}^{\alpha} = b^{\alpha} - \tilde{b}^{\alpha, \delta}, \tag{7.3}
$$

where b^{α}_{δ} is bounded under $(A.2')$ $(A.2')$ and $(A.8)$. Then we approximate $\mathcal{L}^{\alpha}_{\delta}$ by

$$
\bar{\mathcal{L}}^{\alpha}_{\delta,k,h}\phi = \mathcal{L}^{\alpha}_{\delta,k,h}[\phi] + b^{\alpha,+}_k \delta_{h,e_k}\phi + b^{\alpha,-}_k \delta_{h,-e_k}\phi, \tag{7.4}
$$

where $\mathcal{L}_{\delta,k,h}^{\alpha}$ is defined in [\(3.7\)](#page-11-6), e_k are basis vectors in \mathbb{R}^N , $b_{\delta}^{\alpha} = (b_1^{\alpha}, \cdots, b_N^{\alpha})$, and

$$
\delta_{h,l}u(x) = \frac{u(x+hl) - u(x)}{h} \quad \text{for} \quad l \in \mathbb{R}^N, \neq 0.
$$

Here the drift term is discretized by an upwind finite difference method^{[10](#page-27-0)}, and the total discretization is still monotone. We estimate the truncation error next.

Lemma 7.2. Assume $(A.1)$, $(A.2')$, $(A.3)$ - $(A.6)$, $(A.8)$, $\phi \in C^4(\mathbb{R}^N)$, and $\mathcal{L}_{\delta}^{\alpha}$ and $\bar{\mathcal{L}}_{\delta,k,h}^{\alpha}$ are defined by [\(7.3\)](#page-27-1) and [\(7.4\)](#page-27-2). Then there is K independent of h, k, δ such that

$$
\left| \bar{\mathcal{L}}_{\delta,k,h}^{\alpha}[\phi] - \mathcal{L}_{\delta}^{\alpha}[\phi] \right| \le K \left(h \| D^2 \phi \|_0 + \delta^{2(2-\sigma)} k^2 \| D^4 \phi \|_0 + \frac{h^2}{k^2} \| D^2 \phi \|_0 \right). \tag{7.5}
$$

Proof. The first term on the right hand side of (7.5) is classical and due to the approximation of the drift. The remaining terms come from Lemma [3.2.](#page-11-4) \Box

The numerical scheme for equation (7.1) is defined by

$$
\sup_{\alpha \in \mathcal{A}} \left\{ f^{\alpha}(x) + c^{\alpha}(x)u(x) - \bar{\mathcal{L}}^{\alpha}_{\delta,k,h}[u](x) - \mathcal{I}^{\alpha,\delta}_{h}[u](x) \right\} = 0 \quad \text{in} \quad \mathbb{R}^{N},\qquad(7.6)
$$

where $\bar{\mathcal{L}}_{\delta,k,h}^{\alpha}$ and $\mathcal{I}_h^{\alpha,\delta}$ are given by [\(7.4\)](#page-27-2) and [\(3.10\)](#page-11-7). This is a consistent, monotone, and L^{∞} -stable scheme. In the strongly degenerate case, an error estimate given by the next result.

 10 This is just an example, many other monotone discretizations would also work here, also SL schemes.

Theorem 7.3. Assume $\sigma \in (0, 2)$, $h, k \in (0, 1)$, $\delta \geq h$, $(\mathbf{A}.1)$, $(\mathbf{A}.2)$, $(\mathbf{A}.3)$ - $(\mathbf{A}.6)$, $(A.8)$, u and u_h solves (7.1) and (7.6) .

(a) If
$$
k^2 = O(\frac{h^2}{\delta^{2-\frac{\sigma}{2}}})
$$
, then there is $C > 0$ such that
\n
$$
|u - u_h| \leq \begin{cases} C h^{\frac{1}{2}} & \text{for} \quad 0 < \sigma \leq \frac{6}{5} \quad \text{and} \quad \delta = O(h^{\frac{1}{\sigma}}) \\ C h^{\frac{2(3-\sigma)}{6+\sigma}} & \text{for} \quad \frac{6}{5} < \sigma < 2 \quad \text{and} \quad \delta = O(h^{\frac{6}{6+\sigma}}). \end{cases}
$$
\n(7.7)

(b) When $(A.\gamma)$ holds and $k^2 = O(\frac{h^2}{s^2})$ $\frac{h^2}{\delta^{2-\frac{\sigma}{2}}}$, then there is $C>0$ such that

$$
|u - u_h| \leq \begin{cases} Ch^{\frac{1}{2}} & \text{for} \quad 0 < \sigma \leq \frac{4}{3} \quad \text{and} \quad \delta = O\left(h^{\frac{1}{\sigma}}\right) \\ Ch^{\frac{4-\sigma}{4+\sigma}} & \text{for} \quad \frac{4}{3} < \sigma < 2 \quad \text{and} \quad \delta = O\left(h^{\frac{4}{4+\sigma}}\right). \end{cases} (7.8)
$$

Remark 7.4. (a) When $\sigma \leq 1$, the error can can not be better than $\mathcal{O}(h^{\frac{1}{2}})$ because of the (local) drift term in (7.1) . In this case the diffusion correction does not improve the rate as it did in Section [3.](#page-10-0)

(b) Under assumption $(**A**.8)$, we get improved convergence rates for any $\sigma > 1$, see Theorem [7.3](#page-28-5) (b). The rate approaches $\frac{1}{3}$ as $\sigma \to 2$, compared to $\frac{1}{4}$ in part (a).

Sketch of proof: The proof is similar to the proof of Theorem 3.6 , we only explain the main differences. In view of Lemmas [3.1](#page-10-5) and [7.1,](#page-26-4) replacing Lemma [3.2](#page-11-4) by Lemma [7.2](#page-27-5) when estimating [\(5.1\)](#page-15-5), the constant $M_{\epsilon,\delta}$ in [\(5.2\)](#page-15-6) gets a $O(\frac{h}{\epsilon})$ contribution from the drift and becomes

$$
M_{\epsilon,\delta} = \begin{cases} \delta^{3-\sigma} \frac{1}{\epsilon^2} + h \frac{1}{\epsilon} + k^2 \delta^{2(2-\sigma)} \frac{1}{\epsilon^3} + \frac{h^2}{k^2} \frac{1}{\epsilon} + \frac{h^2}{\delta^{\sigma}} \frac{1}{\epsilon} & \text{for part (a)},\\ \delta^{4-\sigma} \frac{1}{\epsilon^3} + h \frac{1}{\epsilon} + k^2 \delta^{2(2-\sigma)} \frac{1}{\epsilon^3} + \frac{h^2}{k^2} \frac{1}{\epsilon} + \frac{h^2}{\delta^{\sigma}} \frac{1}{\epsilon} & \text{for part (b)}. \end{cases}
$$
(7.9)

In case (a) the nonlocal operator is not symmetric, so we have used Lemma [7.1](#page-26-4) (i) to get the first term. By [\(5.3\)](#page-15-2) and [\(5.4\)](#page-15-3) we get $|u-u_h| \leq C(\epsilon + M_{\epsilon,\delta})$ and optimize with respect to k, δ , and ϵ . First we take $k^2 = O(\frac{h\epsilon}{\delta^{2-\sigma}})$, then using $h \leq \delta$, we take $\delta = O(h^{\frac{2}{3}}\epsilon^{\frac{1}{3}})$ for part (a) and $\delta = O(h^{\frac{1}{2}}\epsilon^{\frac{1}{2}})$ for part (b) to get

$$
|u - u_h| \leq \begin{cases} C\left(h^{\frac{2}{3}(3-\sigma)}\epsilon^{-\frac{1}{3}(3+\sigma)} + h\frac{1}{\epsilon} + \epsilon\right) & \text{for part (a)},\\ C\left(h^{\frac{1}{2}(4-\sigma)}\epsilon^{-\frac{1}{2}(2+\sigma)} + h\frac{1}{\epsilon} + \epsilon\right) & \text{for part (b)}. \end{cases}
$$
(7.10)

For part (a), the rate [\(7.7\)](#page-28-6) follows by choosing $\epsilon = O(\max\{h^{\frac{1}{2}}, h^{\frac{2(3-\sigma)}{6+\sigma}}\})$, i.e. $\epsilon = O\left(h^{\frac{1}{2}}\right)$ for $0 < \sigma \leq \frac{5}{6}$, and $\epsilon = O\left(h^{\frac{2(3-\sigma)}{6+\sigma}}\right)$ for $\frac{5}{6} \leq \sigma < 2$. For part (b), the con-vergence rate [\(7.8\)](#page-28-7) is observed by choosing ϵ optimally as $\epsilon = O\left(\max\left\{h^{\frac{1}{2}}, h^{\frac{4-\sigma}{4+\sigma}}\right\}\right),$ i.e. $\epsilon = O(h^{\frac{1}{2}})$ for $0 < \sigma \leq \frac{4}{3}$ and $\epsilon = O(h^{\frac{4-\sigma}{4+\sigma}})$ for $\frac{4}{3} \leq \sigma < 2$.

REFERENCES

[1] D. Applebaum. Lévy processes and stochastic calculus. Cambridge University Press, Cambridge, 116 (2) (2009).

^[2] S. Asmussen and J. Rosiński. Approximations of Small Jumps of Lévy Processes with a View Towards Simulation. Journal of Applied Probability, 38(2) (2001), 482–493.

^[3] G. Barles and C. Imbert. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. Ann. Inst. H. Poincaré C Anal. Non Linéaire, 25 (3) (2008), 567–585.

^[4] G. Barles and E. R. Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. M2AN Math. Model. Numer. Anal., 36 (1) (2002), 33–54.

- [5] G. Barles and E. R. Jakobsen. Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations. SIAM J. Numer. Anal., 43 (2) (2005), 540–558.
- [6] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. Asymptotic Anal., 4 (1991), 271–283.
- [7] R. Bellman. Dynamic programming. Reprint of the 1957 edition. Princeton, 2010.
- [8] I. H. Biswas, I. Chowdhury and E. R. Jakobsen. On the rate of convergence for monotone numerical schemes for nonlocal Isaacs' equations. SIAM J. Numer. Anal., 57 (2) (2019), 799– 827.
- [9] I. H. Biswas, E. R. Jakobsen and K. H. Karlsen. Error estimates for finite difference quadrature schemes for a class of nonlocal Bellman equations with variable diffusion. In Stochastic Analysis and Partial Differential Equations, Contemp. Math., 429 (2007), 19–31.
- [10] I.H. Biswas, E.R. Jakobsen and K.H. Karlsen. Error estimates for a class of finite differencequadrature schemes for fully nonlinear degenerate parabolic integro-PDEs. J. Hyperbolic Differ. $Equ., 5 (1) (2008), 187-219.$
- [11] I.H. Biswas, E.R. Jakobsen and K.H. Karlsen. Difference-quadrature schemes for nonlinear degenerate parabolic integro-PDE. SIAM J. Numer. Anal., 48 (3) (2010), 1110–1135.
- [12] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song and Z. Vondraček. Potential analysis of stable processes and its extensions. Lecture Notes in Mathematics: Springer-Verlag, Berlin, 1980 (2009).
- [13] A. Bonito and J. E. Pasciak. Numerical Approximation of fractional powers of elliptic operators. Math. Comp., 84 (295) (2015), 2083–2110.
- [14] J. F. Bonnans and H. Zidani. Consistency of generalized finite difference schemes for the stochastic HJB equation. SIAM J. Numer. Anal. 41(3) (2003), 1008–1021.
- [15] J. F. Bonnans, S. Maroso, and H. Zidani. Error estimates for stochastic differential games: The adverse stopping case. IMA J. Numer. Anal., 26 (2006), 188–212.
- [16] B. Bouchard, and N. Touzi. Discrete-time approximation and monte-carlo simulation of backward stochastic differential equations. Stoch. Process Their Appl. 111 (2) (2004), 175–206.
- [17] M. Boulbrachene and M. Haiour. The finite element approximation of Hamilton-Jacobi-Bellman equations. Comput. Math. Appl. 41 (7-8) (2001), 993–1007.
- [18] L. Caffarelli, L. Silvestre. Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math., 62 (5) (2009), 597–638.
- [19] L. Caffarelli, L. Silvestre. The Evans–Krylov theorem for nonlocal fully nonlinear equations, Ann. of Math., 174 (2) (2011), 1163–1187.
- [20] L. Caffarelli, P. E. Souganidis. A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs, Comm. Pure Appl. Math., 61 (2008), 1–17.
- [21] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes, RAIRO Modél. Math. Anal. Numér, 29 (1) (1995), 97-122.
- [22] F. Camilli and E. R. Jakobsen. A finite element like scheme for integro-partial differential Hamilton-Jacobi-Bellman equations, SIAM J. Numer. Anal., 47 (2009), 2407–2431.
- [23] I. Capuzzo-Dolcetta and M. Falcone. Discrete dynamic programming and viscosity solutions of the Bellman equation, Ann. Inst. H. Poincaré Anal. Non Linéarie, 6 (1989), 161–183.
- [24] E. Chasseigne and E. R. Jakobsen. On nonlocal quasilinear equations and their local limits, J. Differential Equations, 262 (6) (2017), 3759–3804.
- [25] O. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea, and J. L. Varona. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. Adv. Math., 330 (2018), 688–738.
- [26] G. M. Coclite, O. Reichmann, and N. H. Risebro. A convergent difference scheme for a class of partial integro-differential equations modeling pricing under uncertainty, SIAM J. Numer. Anal., 54 (2016), 588–605.
- [27] R. Cont and P. Tankov. Financial modelling with jump processes. Chapman $\mathcal B$ Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, (2004), xvi+535 pp.
- [28] R. Cont and E. Voltchkova. A Finite Difference Scheme for Option Pricing in Jump Diffusion and Exponential Lévy Models. $SIAM$ J. Numer. Anal. 43 (4) (2005), 1596–1626.
- [29] M. G. Crandall and P. L. Lions. Two approximations of solutions of Hamilton-Jacobi equations, Math. Comp. , 43 (1984), 1–19.
- [30] K. Debrabant and E.R. Jakobsen. Semi-Lagrangian schemes for linear and fully non-linear diffusion equations. Math. Comp., 82 (283)(2013), 1433–1462.
- [31] S. Dipierro, X. Ros-Oton, J. Serra and E. Valdinoci. Non-symmetric stable operators: regularity theory and integration by parts. Adv. Math., 401 (2022), 100pp.
- [32] H. Dong and N.V. Krylov. On the rate of convergence of finite-difference approximations for Bellman equations with constant coefficients. Algebra i Analiz., 17 (2) (2005), 108–132;
- [33] J. Droniou, T. Gallouët and J. Vovelle. Global solution and smoothing effect for a nonlocal regularization of a hyperbolic equation. J. of Evolution Eqn. , 499 (3) (2003), 499–521.
- [34] R. Dumitrescu, C. Reisinger, Y. Zhang. Approximation schemes for mixed optimal stopping and control problems with nonlinear expectations and jumps. Appl. Math. Optim. 83 (3) (2021), 1387–1429.
- [35] J. Endal, E.R. Jakobsen and F. D. Teso. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: Schemes and experiments. SIAM J. Numer. Anal., 56 (6) (2018), 3611–3647.
- [36] M. Falcone and R. Ferretti. Semi-Lagrangian approximation schemes for linear and Hamilton-Jacobi equations. Society for Industrial and Applied Mathematics (SIAM), 2014.
- [37] W. H. Fleming and H. M. Soner. Controlled Markov processes and viscosity solutions, Springer, New York, 2006.
- [38] T. Grzywny and K. Szczypkowski. Estimates of heat kernels of non-symmetric Lévy processes. Forum Math. 23 (5)(2021), 1207–1236.
- [39] W. E, J. Han, and A. Jentzen. Solving high-dimensional partial differential equations using deep learning. Proc. Natl. Acad. Sci. USA 115(34) (2018), 8505–8510.
- [40] F. L. Hanson. Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation. Society for Industrial and Applied Mathematics (SIAM), 2007.
- [41] Y. Huang, A. Oberman. Numerical methods for the fractional Laplacian: a finite differencequadrature approach. SIAM J. Numer. Anal. 52 (6) (2014), 3056–3084.
- [42] H. Ishii and A Roch. Existence and uniqueness of viscosity solutions of an integro-differential equation arising in option pricing. $SIAM$ J. Financial Math. 12 (2)(2021), 604–640.
- [43] E. R. Jakobsen. On the rate of convergence of approximation schemes for Bellman equations associated with optimal stopping time problems. Math. Models Methods Appl. Sci. , 13 (2003), 613–644.
- [44] E. R. Jakobsen. On error bounds for approximation schemes for non-convex degenerate elliptic equations. BIT , 44 (2004), 269–285.
- [45] E. R. Jakobsen. On error bounds for monotone approximation schemes for multi-dimensional Isaacs equations. Asymptot. Anal. 49 (2006), 249–273.
- [46] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of integro-PDEs. J. Differential Equations, 212 (2) (2005), 278–318.
- [47] E. R. Jakobsen and K. H. Karlsen. A "maximum principle for semicontinuous functions" applicable to integro-partial differential equations. NoDEA Nonlinear Differential Equations Appl., 13 (2006), 137–165.
- [48] E. R. Jakobsen, K. H. Karlsen and C. La Chioma. Error estimates for approximate solutions to Bellman equations associated with controlled jump-diffusions. Numer. Math., 110 (2) (2008), 221–255.
- [49] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations, $Algebra\ i\ Analiz$, 9 (1997), 245-256.
- [50] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients, Probab. Theory Related Fields, 117 (2000), 1–16.
- [51] N. V. Krylov. The rate of convergence of finite-difference approximations for Bellman equations with Lipschitz coefficients, Appl. Math. Optim., 52 (2005), 365–399.
- [52] N. V. Krylov. On the rate of convergence of finite-difference approximations for elliptic Isaacs equations in smooth domains, $Appl. Math. Optim.$, 40 (2015), 1393– 1407.
- [53] H. J. Kushner and P. Dupuis. Numerical methods for stochastic control problems in continuous time. Springer, 2001.
- [54] O. Lepsky. Spectral Viscosity Approximations to Hamilton–Jacobi Solutions. SIAM J. Numer. Anal. 38 (5) (2000), 1439–1453.
- [55] B. Øksendal and A. Sulem. Applied stochastic control of jump diffusions. Springer, Cham, 2019.
- [56] C. Reisinger, Y. Zhang. A penalty scheme and policy iteration for nonlocal HJB variational inequalities with monotone nonlinearities. Comput. Math. Appl. 93 (2021), 199–213.

[57] R. D. Richtmyer and K. W. Morton. Difference methods for initial-value problems. John Wiley & Sons, 1967.

[58] X. Ros-Oton and J. Serra. Regularity theory for general stable operators. *J. Differential* Equations, 260 (12) (2016), 8675–8715.

[59] X. Ros-Oton and M. Weidner. Obstacle problems for nonlocal operators with singular kernels. Preprint arXiv:2308.01695.

[60] L. Silvestre, On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion. Adv. Math., 226 (2011), 2020–2039.

[61] I. Smears and E. Süli. Discontinuous Galerkin finite element approximation of nondivergence form elliptic equations with Cordès coefficients. SIAM J. Numer. Anal. 51 (4) (2013), 2088–2106.

[62] P. E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations, J. Differential Equations, 59 (1) (1985), 1–43.

[63] O. Turanova. Error estimates for approximations of nonlinear uniformly parabolic equations, NoDEA Nonlinear Differential Equations Appl., 22 (2015), 345–389.

Indian Institute of Technology - Kanpur, India Email address: indranil@iitk.ac.in

Norwegian University of Science and Technology, Norway Email address: espen.jakobsen@ntnu.no

