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Optimal stability results and nonlinear duality for L^∞ entropy and L^1 viscosity solutions

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ABSTRACT

We give a new and rigorous duality relation between two central notions of weak solutions of nonlinear PDEs: entropy and viscosity solutions. It takes the form of the *nonlinear dual inequality:*

$$\int |S_t u_0 - S_t v_0|\varphi_0 \, \mathrm{d}x \le \int |u_0 - v_0| G_t \varphi_0 \, \mathrm{d}x, \quad \forall \varphi_0 \ge 0, \forall u_0, \forall v_0, \tag{$\star$}$$

where S_t is the entropy solution semigroup of the anisotropic degenerate parabolic equation

$$\partial_t u + \mathrm{div} F(u) = \mathrm{div}(A(u)Du),$$

and where we look for the smallest semigroup G_t satisfying ($\star$). This amounts to finding an optimal weighted L^1 contraction estimate for S_t. Our main result is that G_t is the viscosity solution semigroup of the Hamilton-Jacobi-Bellman equation

$$\partial_t \varphi = \sup_\xi \{F'(\xi) \cdot D\varphi + \mathrm{tr}(A(\xi)D^2\varphi)\}.$$

Since weighted L^1 contraction results are mainly used for possibly nonintegrable L^∞ solutions u, the natural spaces behind this duality are L^∞ for S_t and L^1 for G_t. We therefore develop a corresponding L^1 theory for viscosity solutions φ. But

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L^1 itself is too large for well-posedness, and we rigorously identify the weakest L^1 type Banach setting where we can have it – a subspace of L^1 called L^∞_{int}. A consequence of our results is a new domain of dependence like estimate for second order anisotropic degenerate parabolic PDEs. It is given in terms of a stochastic target problem and extends in a natural way recent results for first order hyperbolic PDEs by [N. Pogodaev, *J. Differ. Equ.*, 2018].

R É S U M É

Nous donnons une nouvelle relation de dualité entre deux notions de solutions faibles qui jouent un rôle central pour les EDPs non-linéaires. Il s'agit des solutions entropiques et des solutions de viscosité. Cette relation prend la forme de l'*inégalité de dualité non-linéaire* suivante :

$$\int |S_t u_0 - S_t v_0| \varphi_0 \,\mathrm{d}x \leq \int |u_0 - v_0| G_t \varphi_0 \,\mathrm{d}x, \quad \forall \varphi_0 \geq 0, \forall u_0, \forall v_0, \qquad (\star)$$

où S_t est le semi-groupe associé à l'équation parabolique, dégénérée et anisotropique

$$\partial_t u + \operatorname{div} F(u) = \operatorname{div}(A(u)Du),$$

et où nous cherchons le plus petit semi-groupe G_t satisfaisant ($\star$). Ceci revient à établir un principe de contraction L^1 à poids optimal pour S_t. Notre résultat principal est que G_t est le semi-groupe associé à l'équation de Hamilton-Jacobi-Bellman

$$\partial_t \varphi = \sup_\xi \{F'(\xi) \cdot D\varphi + \operatorname{tr}(A(\xi)D^2\varphi)\}.$$

Puisque de telles estimations à poids sont essentiellement utilisées pour les solutions u bornées et non-nécessairement intégrables, les espaces naturels dans ($\star$) sont L^∞ pour S_t et L^1 pour G_t. Ceci nous amène à développer une théorie L^1 pour les solutions de viscosité φ. Mais le problème dual est mal posé dans cet espace et nous identifions donc rigoureusement l'*espace de Banach de type* L^1 le plus faible dans lequel ce problème est bien posé. Ceci nous conduit à un espace appelé L^∞_{int}. Nos résultats généralisent en particulier les estimations récentes de [N. Pogodaev, *J. Differ. Equ.*, 2018] sur les domaines de dépendance des équations hyperboliques du premier ordre. Notre estimation est formulée en termes de problèmes de cibles et conserve un sens pour les équations paraboliques, dégénérées et anisotropiques, du second-ordre, pour lesquelles ces problèmes deviennent stochastiques.

Contents

1. Introduction

In this paper we study two central notions of weak solutions of nonlinear PDEs and their interplay – entropy solutions and viscosity solutions. Originally introduced for first order scalar conservation laws [39] and Hamilton-Jacobi equations [25] respectively, both solution concepts have later been extended to second order PDEs [37,36,18,22]. Conservation laws are divergence form equations arising in continuum physics [27], while Hamilton-Jacobi equations are nondivergence form equations from e.g. differential geometry and optimal control theory [31,5,4]. The well-posedness of these equations is an important topic and requires the entropy and viscosity solution theories in general. The literature is by now very large and includes lots of applications. See [31,28,5,4,48,27,24] for the state-of-the-art.

Here we develop a new connection between these solution concepts. It is already well-known that viscosity solutions are integrated entropy solutions in space dimension one [20,38,23]. Our connection is valid in any dimension and is expressed through weighted L^1 contraction results for entropy solutions: The optimal weight is the viscosity solution of a well-determined *dual equation*. Since L^∞ is a natural space for such weighted estimates, we need and do develop an L^1 theory for viscosity solutions of the dual equation. Consequences are a new domain of dependence like result for second order PDEs in terms of a stochastic target problem, a new rigorous form of duality between L^∞ entropy and L^1 viscosity solutions in terms of nonlinear semigroups, and a new characterization of viscosity supersolutions; see (8), (10) and (11) respectively.

The idea of using viscosity solutions to get estimates for entropy solutions was from [29]. The corresponding results were rather accurate but not optimal yet. In this paper we prove optimal estimates for entropy solutions – and – that viscosity solutions are in fact needed to prove this optimality. This is exactly what leads to rigorous duality results. Also note that we consider nonlinear anisotropic diffusions as opposed to [29]. For an early discussion and open questions about "duality between nonlinear semigroups," see [14, pp. 28–29]. We also mention the recent papers [19,45] which study transport equations with linear diffusion through viscosity solutions of their dual equations.

To be more precise, we consider the following two Cauchy problems: For the anisotropic degenerate parabolic convection-diffusion equation

$$\begin{aligned} \partial_t u + \operatorname{div} F(u) &= \operatorname{div}\left(A(u) Du\right) \qquad x \in \mathbb{R}^d, t > 0, \\ u(x,0) &= u_0(x) \qquad x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

and for the Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\varphi + \operatorname{tr}\left(a(\xi) D^2 \varphi\right) \right\} \qquad x \in \mathbb{R}^d, t > 0, \tag{2a}$$

$$\varphi(x,0) = \varphi_0(x) \qquad x \in \mathbb{R}^d, \tag{2b}$$

where "D," "D^2" and "div" respectively denote the gradient, the Hessian and the divergence in x, and "tr" is the trace. We assume that

$$F \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d) \quad \text{and} \quad A = \sigma^A \left(\sigma^A\right)^{\mathrm{T}} \quad \text{for} \quad \sigma^A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{d \times K}), \tag{H1}$$

as well as

$$\begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \to \mathbb{R}^d \text{ a bounded function,} \\ a = \sigma^a \left(\sigma^a\right)^{\mathrm{T}} \text{ for some bounded } \sigma^a : \mathcal{E} \to \mathbb{R}^{d\times K}, \end{cases} \tag{H2}$$

where K is the maximal rank of $A(u)$ and $a(\xi)$. The entropy solution theory for first order PDEs [39] was extended in [18,22] to show well-posedness of (1) in $L^1 \cap L^\infty$ or L^1. Well-posedness in L^∞ is less standard for second order PDEs, but results exist in [21,3,29,42]; see [32] for anisotropic diffusions. Our main objective is to derive an optimal weighted L^1 contraction result for L^∞ entropy solutions of (1). This then will require the development of a corresponding L^1 theory for a dual equation of the form (2), a nonstandard generalization of classical viscosity solution theory [25,37,36,24,31,5,4].

Contraction type estimates are quantitative continuous dependence results on the initial data. A simple example is the L^1 contraction principle [39,18,22]:

$$\|(u-v)(t)\|_{L^1} \le \|u_0 - v_0\|_{L^1}. \tag{3}$$

For possibly nonintegrable L^∞ solutions, we need weighted estimates. An important result is the finite speed of propagation property for first order PDEs [39]:

$$\int\limits_{|x-x_0|<R} |u(x,t)-v(x,t)| \,\mathrm{d}x \le \int\limits_{|x-x_0|<R+Ct} |u_0(x)-v_0(x)| \,\mathrm{d}x; \tag{4}$$

see [43] for more precise estimates. For second order PDEs, a standard example is given in [13,21,49,32]:

$$\int |u(x,t)-v(x,t)| \mathrm{e}^{-\sqrt{1+|x|^2}} \,\mathrm{d}x \le \mathrm{e}^{Ct} \int |u_0(x)-v_0(x)| \mathrm{e}^{-\sqrt{1+|x|^2}} \,\mathrm{d}x. \tag{5}$$

Note that (5) does not imply (3) and (4). A finer result that is closer to (4) is given in [29] but it still does not imply (3), see [29, Rem. 2.7(b)].

We continue with a formal presentation of our main results. We first give a very accurate weighted L^1 contraction estimate for (1). We need to be precise about the dependence of the estimates in u_0 and v_0. Note that C in (4) and (5) actually depends on L^∞ bounds on these initial data. These bounds will determine $\mathcal{E}$ in the dual equation of the form (2). For $m < M$, our new estimate for (1) is

$$\int |u(x,t)-v(x,t)| \varphi_0(x) \,\mathrm{d}x \le \int |u_0(x)-v_0(x)| \varphi(x,t) \,\mathrm{d}x, \tag{6}$$

where $\varphi_0 \ge 0$ is arbitrary, the weight φ is the viscosity solution of (2) with

$$b = F', \quad a = A, \quad \mathcal{E} = [m, M] \cap \left\{\text{Lebesgue points of } (F', A)\right\}, \tag{7}$$

and u_0 and v_0 are arbitrary with values in $[m, M]$. For a precise statement, see Theorem 19. Note that we also use another equivalent formulation of (2) in terms of $\operatorname{ess\,sup}$, see (21). The standard HJB form is, however, used especially for results specific to viscosity solutions.

Equation (2) is also related to stochastic control theory [31]. If we assume F' and A are continuous so $\mathcal{E}$ becomes compact, then the solution φ of (2) is the value function of a stochastic target problem and (6) can be rewritten as

$$\int_U |u(x,t)-v(x,t)|\,\mathrm{d}x \le \int |u_0(x)-v_0(x)| \sup_{\boldsymbol{\xi}_\cdot\in\Xi} \mathbb{P}\left(\boldsymbol{X}_t^x \in U\right)\mathrm{d}x, \tag{8}$$

where $U \subseteq \mathbb{R}^d$ is arbitrary, $\mathbb{P}$ is the probability, Ξ is the set of $[m, M]$-valued processes, $\boldsymbol{B}_s$ a Brownian motion, and $\boldsymbol{X}_s^x$ an Ito process satisfying the stochastic differential equation (SDE)

$$\begin{aligned} \mathrm{d}\boldsymbol{X}_s^x &= F'(\boldsymbol{\xi}_s)\,\mathrm{d}s + \sqrt{2}\,\sigma^A(\boldsymbol{\xi}_s)\,\mathrm{d}\boldsymbol{B}_s \qquad s>0,\\ \boldsymbol{X}_0^x &= x. \end{aligned} \tag{9}$$

For the precise statement, see Corollary 22. The control $\boldsymbol{\xi}_s$ is determined to maximize the probability for the controlled process $\boldsymbol{X}_s^x$ starting from x at time 0 to reach U at time t, and Equation (2a) with data (7) is the dynamic programming equation for this control problem. Interestingly (8) can be interpreted as a domain of dependence estimate for (1). Indeed if we consider the deterministic case $A \equiv 0$, then formally (9) becomes the characteristic equation of (1), $\frac{\mathrm{d}\boldsymbol{X}_s^x}{\mathrm{d}s} = F'(u(\boldsymbol{X}_s^x, s))$, if we take $\boldsymbol{\xi}_s = u(\boldsymbol{X}_s^x, s)$. In fact Estimate (8) reduces to the domain of dependence estimate of [43] for scalar conservation laws, see Corollary 21. This suggests that (8) is a natural extension of such estimates to the degenerate parabolic equation (1), where the second order term in (1) is taken into account via the Brownian part (the Ito integral) in (9).

Note that (6) and (8) imply (3), (4), (5), the related results in [29,43], and as we will see, they are optimal in a rigorous sense. To discuss the optimality of (6), we fix φ_0 and try to identify the minimal φ satisfying (6) for any u_0, v_0. The key result (Theorem 23) is a characterization of viscosity supersolutions of (2) in terms of contraction estimates for (1):

A nonnegative function φ is a viscosity supersolution of (2a) with data (7) if and only if

$$\int |u(x,t)-v(x,t)|\varphi(x,s)\,\mathrm{d}x \le \int |u_0(x)-v_0(x)|\varphi(x,t+s)\,\mathrm{d}x, \tag{10}$$

for all $t, s \ge 0$ and u_0, v_0 with values in $[m, M]$ with associated entropy solutions u, v of (1).

Roughly speaking this result implies that if we restrict to weights satisfying a natural semigroup property, then the best weight in (6) is the viscosity solution of (2) since by comparison solutions are always smaller than supersolutions. This then leads to our most original result (Corollary 37):

If S_t and G_t are the solution semigroups of (1) and (2), with data (7), then G_t is the smallest semigroup satisfying

$$\int |S_t u_0 - S_t v_0|\varphi_0\,\mathrm{d}x \le \int |u_0 - v_0| G_t\varphi_0\,\mathrm{d}x, \tag{11}$$

for all u_0, v_0 with values in $[m, M]$ and nonnegative φ_0.

We can interpret (11) as a *nonlinear dual inequality* and G_t as a *dual semigroup* of S_t, because G_t is entirely determined by (11) and knowledge of S_t. The duality in the other direction is open (Remark 39). Since S_t is taken on L^∞ from the beginning, it remains to properly define G_t on L^1.

Classical viscosity solution theory starting from [25,37,36] and summarized in e.g. [24,31,5,4], typically considers bounded continuous C_b solutions. For solutions in L^1 or L^p (in space) there are fewer results, see e.g. [16] for nondegenerate PDEs and [41,2,1,6,15,17,29] for various other PDEs. Here we show that (2) can be ill-posed in L^1 in general. We then consider stronger norm topologies and identify the weakest one for which (2) is well-posed in general: It is generated by the norm

$$\varphi_0 \mapsto \int \sup_{x+[-1,1]^d} |\varphi_0| \, \mathrm{d}x$$

which is the norm of the space L^∞_{int} as defined in [2,1]. Since $L^\infty_{\text{int}} \subset L^1 \cap L^\infty$, it follows that $C_b \cap L^\infty_{\text{int}}$ is a natural L^1 type Banach space for the dual equation (2) and its solution semigroup G_t in (11); see Theorem 33 and Corollary 36.

Our results on L^1 viscosity solutions are of independent interest, see in particular Theorem 35. Let us comment them further. The estimates of [2] are not in L^∞_{int} but in its predual $L^1_{\text{unif}} \not\subset L^1$, while [1] gives weighted L^∞_{int} estimates for unbounded solutions with linear diffusions. In [29] there are L^1 estimates for fully nonlinear degenerate PDEs with isotropic diffusions and exponentially decaying initial data. Equation (2) is fully nonlinear, degenerate, possibly anisotropic, and we consider general L^∞_{int} data while identifying L^∞_{int} as the most natural L^1 viscosity solution setting.

The rest of this paper is organized as follows. We recall basic facts in Section 2, we state our main results in Section 3, and prove them in Section 4. For completeness, some results for minimal discontinuous viscosity solutions are proved in Appendix A, a complete proof of well-posedness for L^∞ entropy solutions is given in Appendix B, and further comments on our duality results are postponed to Appendix C and Appendix D.

2. Preliminaries

This section recalls basic facts on C_b viscosity and L^∞ entropy solutions; for proofs, see e.g. [24,31,5,4] and [22,11,27] respectively. We also define the space L^∞_{int}.

2.1. Notation

Throughout $\mathbb{R}^+ := [0,\infty)$, balls and cubes of $\mathbb{R}^d$ with center x and radius $r > 0$ are $B_r(x) := \{y : |y-x| < r\}$ and $Q_r(x) := x + (-r,r)^d$, the symbol "co" denotes the convex hull of sets, "(ess) Im" the (essential) image of (measurable) functions, "Sp" the spectrum of matrices, and $\mathbf{1}_U$ the indicator function of a set U.

We follow standard notation for function spaces, e.g. C_c denotes continuous functions with compact support, $BLSC$ (resp. $BUSC$) bounded lower (resp. upper) semicontinuous functions, L^p stands for Lebesgue spaces, etc. For two normed spaces $X \subseteq Y$, we say that X is continuously embedded into Y if the canonical injection is continuous, and the completion of X is denoted by $\overline{X}^{\|\cdot\|_X} \subseteq Y$.

Concerning operations on functions, "$*$" is the convolution which is mostly taken in $x \in \mathbb{R}^d$, and we use "$*_{x,t}$" if it is taken in $(x,t) \in \mathbb{R}^{d+1}$, etc. To regularize functions of x, we use convolution with an approximate unit ρ_ν of the form

$$\rho_\nu(x) := \frac{1}{\nu^d} \rho\left(\frac{x}{\nu}\right), \tag{12}$$

where $0 \leq \rho \in C_c^\infty(\mathbb{R}^d)$ and $\int \rho = 1$, while for functions of t, we convolve with

$$\theta_\nu(t) := \frac{1}{\nu} \theta\left(\frac{t}{\nu}\right), \tag{13}$$

where $0 \leq \theta \in C_c^\infty((-\infty,0))$ and $\int \theta = 1$. If needed, we extend functions of $t \in \mathbb{R}^+$ by zero to all $t \in \mathbb{R}$ to give a meaning to the convolution. For locally bounded everywhere defined $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ or $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, we define the infconvolution [24,31,5,4] of e.g. φ_0 for all x by

$$(\varphi_0)_\varepsilon(x) := \inf_{y \in \mathbb{R}^d} \left\{ \varphi_0(y) + \frac{|x-y|^2}{2\varepsilon^2} \right\}. \tag{14}$$

Here the inf is pointwise and, to avoid confusion, we will use distinct notation for ess inf, etc. The upper φ^* (lower φ_*) semicontinuous envelope of φ is defined as

$$\varphi^*(x,t) := \limsup_{(y,s)\to(x,t)} \varphi(y,s) \quad \left(\varphi_*(x,t) := \liminf_{(y,s)\to(x,t)} \varphi(y,s)\right).$$

For a family $(\varphi_\varepsilon = \varphi_\varepsilon(x,t))_{\varepsilon>0}$, the upper and lower relaxed limits as $\varepsilon \to 0^+$ are, using standard notation [24,5,4],

$$\limsup{}^* \varphi_\varepsilon(x,t) := \limsup_{\substack{(y,s)\to(x,t)\\ \varepsilon\to 0^+}} \varphi_\varepsilon(y,s) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \tag{15}$$

and $\liminf_* \varphi_\varepsilon := -\limsup^* (-\varphi_\varepsilon)$. As is customary, we use the same notation $\limsup^*$ and $\liminf_*$ also when the limits are taken in another variable than $\varepsilon \to 0^+$, e.g. $R \to \infty$. We write $\lim_{\varepsilon\downarrow 0} \uparrow \varphi_\varepsilon$ for the limit if $\varphi_\varepsilon(x,t) \nearrow \sup_{\varepsilon>0} \varphi_\varepsilon(x,t)$ as $\varepsilon \searrow 0$. We use similar notation for $\varphi_0 = \varphi_0(x)$ as e.g. $(\varphi_0)^*(x) := \limsup_{y\to x} \varphi_0(y)$, etc.

As concerning stochastic processes, we fix

$$\begin{cases} \text{a complete filtered probability space } (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), \text{ and} \\ \text{a standard } d\text{-dimensional Brownian } \boldsymbol{B}_t \text{ on this filtration.} \end{cases} \tag{16}$$

The associated expectation w.r.t. $\mathbb{P}$ is denoted by $\mathbb{E}$. We will assume possibly without mentioning that all stochastic processes in this paper are defined on this filtered probability space, and that whenever we need a Brownian motion, then we take the above Brownian motion.

Less standard notation

Following [2,1],

$$L^\infty_{\text{int}}(\mathbb{R}^d) := \left\{\varphi_0 \in L^1_{\text{loc}}(\mathbb{R}^d) \,:\, \|\varphi_0\|_{L^\infty_{\text{int}}} < \infty\right\}, \tag{17}$$

where $\|\varphi_0\|_{L^\infty_{\text{int}}} := \int \operatorname{ess\,sup}_{\overline{Q}_1(x)} |\varphi_0| \, \mathrm{d}x$. For the pointwise sup, we use $\|\varphi_0\|_{\text{int}} := \int \sup_{\overline{Q}_1(x)} |\varphi_0| \, \mathrm{d}x$. Note that $\|\varphi_0\|_{\text{int}} = \|\varphi_0\|_{L^\infty_{\text{int}}}$ if φ_0 is continuous. For more details about L^∞_{int}, see Section 2.4.

For any $\varphi \in BLSC(\mathbb{R} \times \mathbb{R}^+)$, we associate a particular envelope defined as

$$\varphi_\#(x,t) := \liminf_{\substack{r\to 0^+\\ y\to x}} \frac{1}{\operatorname{meas}(B_r(y))} \int_{B_r(y)} \varphi(z,t) \, \mathrm{d}z. \tag{18}$$

This envelope will appear naturally in Theorem 23 and more properties will be given in Section 4.4.

2.2. Viscosity solutions of (2)

We begin by introducing the correct notion of solutions for HJB equations [24,31,5,4].

Definition 1 *(Viscosity solutions).* Assume (H2) and $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ is bounded.

(a) A locally bounded function $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (2) if

(i) for every $\phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ and local maximum $(x,t) \in \mathbb{R}^d \times (0,\infty)$ of $\varphi^* - \phi$ (resp. minimum of $\varphi_* - \phi$),

$$\partial_t \phi(x,t) \leq \sup_{\mathcal{E}} \left\{ b \cdot D\phi(x,t) + \operatorname{tr}\left(aD^2\phi(x,t)\right) \right\} \quad \text{(resp. } \geq \text{)},$$

(ii) and for every $x \in \mathbb{R}^d$,

$$\varphi^*(x,0) \leq (\varphi_0)^*(x) \quad \text{(resp. } \varphi_*(x,0) \geq (\varphi_0)_*(x)\text{)}.$$

(b) A function φ is a viscosity solution if it is both a sub and supersolution.

Remark 2. We say that φ is a viscosity subsolution (resp. supersolution) of (2a) if (ai) holds.

We recall the well-known comparison and the well-posedness for (2) [24,31].

Theorem 3 *(Comparison principle). Assume* (H2)*. If φ and ψ are bounded sub and supersolutions of* (2a)*, and*

$$\varphi^*(x,0) \leq \psi_*(x,0) \quad \forall x \in \mathbb{R}^d,$$

then $\varphi^* \leq \psi_*$ on $\mathbb{R}^d \times \mathbb{R}^+$.

Theorem 4 *(Existence and uniqueness). Assume* (H2) *and $\varphi_0 \in C_b(\mathbb{R}^d)$. Then there exists a unique viscosity solution $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}^+)$ of* (2).

Remark 5. By the comparison principle, $\inf \varphi_0 \leq \varphi \leq \sup \varphi_0$ and we have the following contraction property: $\|\varphi - \psi\|_\infty \leq \|\varphi_0 - \psi_0\|_\infty$ for every pair of solutions φ and ψ with initial data φ_0 and ψ_0.

We may take φ_0 to be discontinuous as in (8). In that case, we lose uniqueness and we have to work with minimal and maximal solutions [26,10,33] (see also [4] for bilateral solutions). For our considerations, we only need minimal solutions.

Theorem 6 *(Minimal solutions). Assume* (H2) *and $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ bounded. Then there exists a minimal viscosity solution $\underline{\varphi} \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ of* (2)*, in the sense that $\underline{\varphi} \leq \varphi$ for any bounded viscosity solution φ of* (2)*. Moreover $\underline{\varphi}(x, t = 0) = (\varphi_0)_*(x)$ for any $x \in \mathbb{R}^d$.*

Note that $\underline{\varphi}$ is unique by definition. Actually, it is more precisely the minimal supersolution.

Proposition 7. *Assume* (H2) *and $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ is bounded. Then any bounded supersolution φ of* (2) *is such that $\underline{\varphi} \leq \varphi_*$.*

Remark 8. In particular, we have the following comparison principle: $\underline{\varphi} \leq \underline{\psi}$ for any bounded $\varphi_0 \leq \psi_0$.

For completeness, the proofs of Theorem 6 and Proposition 7 are given in Appendix A.1 because [26,10, 33,4] consider slightly different problems. Let us continue with representation formulas for the solution $\underline{\varphi}$ from control theory [31,4,34,35].

Proposition 9 *(First order). Assume* (H2)*, $a \equiv 0$, and $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ bounded. Then the minimal viscosity solution of* (2) *is given by*

$$\underline{\varphi}(x,t) = \sup_{x+t\mathcal{C}} (\varphi_0)_* \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where $\mathcal{C} = \overline{\operatorname{co}\{\operatorname{Im}(b)\}}$.

In the second order case, we need a probabilistic framework.

Proposition 10 *(Second order). Assume* (H2), (16), *and*

$$\text{the set } \mathcal{E} \text{ is compact and the functions } b(\cdot) \text{ and } \sigma^a(\cdot) \text{ are continuous.} \tag{19}$$

Then the minimal viscosity solution of (2) *is given by*

$$\underline{\varphi}(x,t) = \sup_{\boldsymbol{\xi}_\cdot \in \Xi} \mathbb{E}\left\{ (\varphi_0)_*(\boldsymbol{X}_t^x) \right\},$$

where Ξ *is the set of progressively measurable* $\mathcal{E}$*-valued processes and* $\boldsymbol{X}_s^x$ *an Ito process satisfying the SDE*

$$\begin{cases} \mathrm{d}\boldsymbol{X}_s^x = b(\boldsymbol{\xi}_s)\,\mathrm{d}s + \sqrt{2}\,\sigma^a(\boldsymbol{\xi}_s)\,\mathrm{d}\boldsymbol{B}_s, & s > 0, \\ \boldsymbol{X}_{s=0}^x = x. \end{cases}$$

These results are standard for continuous viscosity solutions [31,4], see also [4,34,35] for maximal solutions. For minimal solutions, we did not find any reference so we provide the proofs in Appendix A.2.

2.3. Entropy solutions of (1)

Well-posedness of (1) in L^∞ is essentially established in [32] for smooth fluxes, see [22,11] for previous results in $L^\infty \cap L^1$ or L^1. Let us now recall these results in the form needed here and provide complementary proofs in Appendix B for completeness.

Definition 11 *(Entropy-entropy flux triple).* We say that (η, q, r) is an entropy-entropy flux triple if $\eta \in C^2(\mathbb{R})$ is convex, $q' = \eta' F'$ and $r' = \eta' A$.

Given $\beta \in C(\mathbb{R})$, we also need the notation

$$\zeta_{ik}(u) := \int_0^u \sigma_{ik}^A(\xi)\,\mathrm{d}\xi \quad \text{and} \quad \zeta_{ik}^\beta(u) := \int_0^u \sigma_{ik}^A(\xi)\beta(\xi)\,\mathrm{d}\xi.$$

Definition 12 *(Entropy solutions).* Assume (H1) and $u_0 \in L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ is an entropy solution of (1) if

(a) $\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L^2_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^+)$ for any $k = 1, \ldots, K$,
(b) for any $k = 1, \ldots, K$ and any $\beta \in C(\mathbb{R})$

$$\sum_{i=1}^d \partial_{x_i} \zeta_{ik}^\beta(u) = \beta(u) \sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u) \in L^2_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^+),$$

(c) and for all entropy-entropy flux triples (η, q, r) and $0 \leq \phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$,

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \left(\eta(u)\partial_t\phi + \sum_{i=1}^d q_i(u)\partial_{x_i}\phi + \sum_{i,j=1}^d r_{ij}(u)\partial^2_{x_ix_j}\phi \right) \mathrm{d}x\,\mathrm{d}t$$

$$+ \int\limits_{\mathbb{R}^d} \eta(u_0(x))\phi(x,0)\,\mathrm{d}x \geq \iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \eta''(u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}(u) \right)^2 \phi\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 13 *(Existence and uniqueness). Assume* (H1) *and* $u_0 \in L^\infty(\mathbb{R}^d)$. *Then there exists a unique entropy solution* $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ *of* (1).

See [32, Theorem 1.1] or Appendix B for the proof.

Remark 14.

(a) In the L^1 settings of [22,11], the following contraction principle holds: For solutions u and v of (1) with initial data u_0 and v_0,

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \quad \forall t \geq 0.$$

(b) In the L^∞ setting of [32], uniqueness is based on the weighted L^1 contraction principle (5), see also Lemma 63 in Appendix B.
(c) In all cases, we have comparison and maximum principles as stated in Lemma 65 in Appendix B.

In L^∞, uniqueness is based on a doubling of variables arguments developed in [39,18,11]. This argument leads to (20) below, and this inequality will be the starting point of our analysis.

Lemma 15 *(Kato inequality). Assume* (H1) *and* u, v *are entropy solutions of* (1) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. *Then for all* $T \geq 0$ *and nonnegative test functions* $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T])$,

$$\int\limits_{\mathbb{R}^d} |u-v|(x,T)\phi(x,T)\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^d} |u_0 - v_0|(x)\phi(x,0)\,\mathrm{d}x$$

$$+ \iint\limits_{\mathbb{R}^d\times(0,T)} \left(|u-v|\partial_t\phi + \sum_{i=1}^d q_i(u,v)\partial_{x_i}\phi + \sum_{i,j=1}^d r_{ij}(u,v)\partial^2_{x_ix_j}\phi \right) \mathrm{d}x\,\mathrm{d}t, \quad (20)$$

where

$$q_i(u,v) = \mathrm{sign}(u-v) \int\limits_v^u F_i'(\xi)\,\mathrm{d}\xi, \quad r_{ij}(u,v) = \mathrm{sign}(u-v) \int\limits_v^u A_{ij}(\xi)\,\mathrm{d}\xi.$$

See Appendix B for precise references to the computations in [11] on how to show this lemma in our setting.

2.4. *The function space* L^∞_{int}

Let us now give some basic properties on the space which was defined in (17).

Theorem 16. *The space $L^\infty_{\text{int}}(\mathbb{R}^d)$ is a Banach space, and it is continuously embedded into $L^1 \cap L^\infty(\mathbb{R}^d)$.*

See [2,1] for the proof and choice of the above notation. We also need the following result:

Lemma 17. *For any $r > 0$ and $\varepsilon \geq 0$, there is a constant $C_{r,\varepsilon} \geq 0$ such that*

$$\int \sup_{\overline{Q}_{r+\varepsilon}(x)} |\varphi_0| \, \mathrm{d}x \leq C_{r,\varepsilon} \int \sup_{\overline{Q}_r(x)} |\varphi_0| \, \mathrm{d}x \quad \forall \varphi_0 : \mathbb{R}^d \to \mathbb{R}.$$

Remark 18. This result will be used with the pointwise sup for discontinuous φ_0, typically lower or upper semicontinuous.

The proof can be found in [2,1], see e.g. [1, Lemma 2.5.1].

3. Main results

In this section we precisely state our results: the weighted L^1 contraction estimate for (1) in Section 3.1, the optimality of the weight in Section 3.2, and the interpretation in terms of dual nonlinear semigroup in Section 3.3. Section 3.3 contains the L^1 theory for (2), and the long proofs are postponed to Section 4.

3.1. Weighted L^1 contraction for entropy solutions

The weight φ of our new estimate for (1) is the viscosity solution of (2) with data (7), a problem which we rewrite in the more convenient form[4]

$$\partial_t \varphi = \operatorname*{ess\,sup}_{m \leq \xi \leq M} \left\{ F'(\xi) \cdot D\varphi + \operatorname{tr}\left(A(\xi) D^2 \varphi\right) \right\} \qquad x \in \mathbb{R}^d, t > 0, \tag{21a}$$

$$\varphi(x,0) = \varphi_0(x) \qquad x \in \mathbb{R}^d, \tag{21b}$$

for given $m < M$ and φ_0.

Theorem 19 *(Weighted L^1 contraction). Assume (H1), $m < M$, $u_0 = u_0(x)$ and $v_0 = v_0(x)$ are measurable with values in $[m, M]$, and $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$. Then the corresponding entropy solutions u and v of (1) and minimal viscosity solution $\underline{\varphi}$ of (21) satisfy*

$$\int_{\mathbb{R}^d} |u - v|(x,t) \varphi_0(x) \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \underline{\varphi}(x,t) \, \mathrm{d}x \quad \forall t \geq 0. \tag{22}$$

Remark 20.

(a) The right-hand side of (22) can be infinite. To get finite integrals, it suffices to take $u_0 - v_0 \in L^1$. We shall see later that another sufficient condition is that $\varphi_0 \in L^\infty_{\text{int}}$, since $\underline{\varphi}$ will then be L^1 in space by Theorem 35.

(b) The same result holds when $\underline{\varphi}$ is replaced by any measurable supersolution of (21), since it is greater than $\underline{\varphi}$.

[4] Viscosity solutions are understood as in Definition 1 via Problem (2) with data (7). But we let the reader check that we can equivalently redefine this notion via (21). More precisely φ is a viscosity supersolution of (2) with data (7) if and only if for every $\phi \in C^\infty$ and local max (x,t) of $\varphi^* - \phi$, $\partial_t \phi(x,t) \leq \operatorname{ess\,sup}_{m \leq \xi \leq M} \left\{ F'(\xi) \cdot D\phi(x,t) + \operatorname{tr}\left(A(\xi) D^2 \phi(x,t)\right) \right\}$, etc.

(c) We also point the interested reader to Lemma 54. There we prove that a consequence of the above result is that $u_0 - v_0 \in L^1$ implies $u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$.

From control theory there exist representation formulas for φ in the first and second order cases, see Propositions 9 and 10. Combining the above result with these representation formulas give us very precise domain of dependence results. In the first order case, we recover the precise results of [43], while in the second order case the result is new.

Corollary 21 *(First order equations). Assume (H1) with $A \equiv 0$, $m < M$, u_0 and v_0 are measurable functions with values in $[m, M]$, and u and v are entropy solutions of (1) with initial data u_0 and v_0. Then*

$$\int_B |u - v|(x,t)\,\mathrm{d}x \leq \int_{B - t\mathcal{C}} |u_0 - v_0|(x)\,\mathrm{d}x$$

for any Borel set $B \subseteq \mathbb{R}^d$ and $t \geq 0$, where

$$\mathcal{C} = \overline{\operatorname{co}\left\{\operatorname{ess\,Im}\left((F')|_{[m,M]}\right)\right\}}.$$

Proof. Let $U \supseteq B$ be an open set and take $\varphi_0 = \mathbf{1}_U$. By Proposition 9, the minimal solution of (21) is $\underline{\varphi}(x,t) = \mathbf{1}_{U - t\mathcal{C}}(x)$. Apply then Theorem 19 and take the infimum over all open $U \supseteq B$. □

Corollary 22 *(Second order equations). Assume (H1), (16), $F'(\cdot)$ and $\sigma^A(\cdot)$ continuous, $m < M$, u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, and u and v entropy solutions of (1) with u_0 and v_0 as initial data. Then for any open $U \subseteq \mathbb{R}^d$ and $t \geq 0$,*

$$\int_U |u - v|(x,t)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \sup_{\boldsymbol{\xi}_\cdot \in \Xi} \mathbb{P}\left(\boldsymbol{X}_t^x \in U\right)\mathrm{d}x,$$

where Ξ is the set of progressively measurable $[m, M]$-valued processes and $\boldsymbol{X}_s^x$ is an Ito process satisfying the SDE (9).

Proof. Take $\varphi_0 = \mathbf{1}_U$ and apply Proposition 10 to compute $\underline{\varphi}$ in Theorem 19. □

The proof of Theorem 19 is given in Section 4.3.

3.2. Optimality of the weight

Let us now discuss the optimality of the weight $\underline{\varphi}$ in a weighted L^1 contraction estimate for (1) such as (22). The first step is a reformulation of the definition of viscosity supersolutions of (21a) in terms of weights in L^1 contraction estimates for (1).

Theorem 23 *(Weights and supersolutions). Assume (H1), $m < M$, and $0 \leq \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$. Then the statements below are equivalent.*

(I) *For any measurable functions u_0 and v_0 with values in $[m, M]$ and entropy solutions u and v of (1) with initial data u_0 and v_0,*

$$\int_{\mathbb{R}^d} |u - v|(x,t)\varphi(x,s)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\varphi(x, t+s)\,\mathrm{d}x \quad \forall t, s \geq 0.$$

(II) *The function* $\varphi_{\#}$ *(cf. (18)) is a viscosity supersolution of (21a).*

Remark 24.

(a) We will see in Lemma 56(ii) that $\varphi_{\#}(\cdot,t)=\varphi(\cdot,t)$ a.e. in $\mathbb{R}^d$, for any t. Hence $\varphi_{\#}$ satisfies (I) if and only if φ does.
(b) For a fixed t, the classical precise representative [30,44] of $\varphi(\cdot,t)$, is defined over Lebesgue points (in space) as

$$\hat{\varphi}(x,t):=\lim_{r\to 0^+}\frac{1}{\mathrm{meas}(B_r(x))}\int_{B_r(x)}\varphi(y,t)\,\mathrm{d}y.$$

Assigning the value $\sup\varphi$ at all other points, and taking the lower semicontinuous envelope (in x), will exactly give $\varphi_{\#}(\cdot,t)$.
(c) Although $\varphi\in BLSC$ makes sense everywhere, we need to consider another precise representative in x for the viscosity inequalities to hold. This is because these inequalities are pointwise while (I) does not depend on the choice of such representatives. If e.g. modifying φ only at some (x_0,t_0) such that

$$\varphi(x_0,t_0)<\liminf_{(x_0,t_0)\neq(x,t)\to(x_0,t_0)}\varphi(x,t),$$

we would preserve (I) while losing the viscosity inequalities.[5]
(d) We do not need to change the precise representative in t, roughly speaking because we consider $BLSC$ weights satisfying (I) for all times.
(e) For simplicity, we restrict to $BLSC$ weights since this regularity is shared by $\underline{\varphi}$ from Theorem 19 and most of the weights from the literature. But we have a similar result for merely measurable weights in (x,t); see Appendix C for completeness.

We will therefore roughly speaking deduce from the comparison principle that our weight is optimal in the class of weights

$$\mathscr{W}_{m,M,\varphi_0}:=\left\{0\leq\varphi\in BLSC(\mathbb{R}^d\times\mathbb{R}^+)\text{ satisfying (I) and }\varphi(t=0)\geq\varphi_0\right\}.$$

Corollary 25 *(Optimality of the weight). Assume (H1), $m<M$, and $0\leq\varphi_0\in BLSC(\mathbb{R}^d)$. Then the weight $\underline{\varphi}$ from Theorem 19 belongs to the class $\mathscr{W}_{m,M,\varphi_0}$ and satisfies*

$$(\underline{\varphi})_{\#}(x,t)=\inf\left\{\varphi_{\#}(x,t):\varphi\in\mathscr{W}_{m,M,\varphi_0}\right\}\quad\forall(x,t)\in\mathbb{R}^d\times\mathbb{R}^+.$$

Remark 26.

(a) Property (I) is stronger than (22) since it holds for any $s\geq 0$. This may be interpreted as a certain semigroup property.
(b) Property (I) is satisfied by most of the weights from the literature, as e.g. for

$$\varphi\equiv 1,\quad\varphi(x,t)=\mathbf{1}_{|x-x_0|<R+Ct}\quad\text{and}\quad\varphi(x,t)=\mathrm{e}^{Ct}\mathrm{e}^{-\sqrt{1+|x|^2}},$$

in respectively (3), (4) and (5); see also the stability results from [39,13,21,49,22,29,32,43].

[5] Indeed $\varphi-\phi$ would achieve a local min in (x_0,t_0), for all $\phi\in C^\infty$.

The proofs of Theorem 23 and Corollary 25 are given in Section 4.4.

3.3. L^∞_{int}, semigroup formulation, and a new form of duality

We now interpret our results in terms of semigroups. This will reflect some form of duality for nonlinear semigroups, which will reduce to standard duality in the linear case. We first need to make the functional framework precise. Recall that L^1 might seem natural for the dual semigroup which will correspond to the weights in (22), but it is too weak for HJB equations and we will precisely explain why $L^\infty_{\text{int}} \subset L^1$ is a better and very natural setting. This preliminary study has also its own interest in viscosity solution theory, and is written for HJB equations in the usual form (2).

Preliminaries: $C_b \cap L^\infty_{\text{int}}$ as a natural L^1 setting for (2)

We first explain why the pure L^1 setting is too weak to develop a general well-posedness theory for (2). Consider a solution of the eikonal type equation[6]

$$\partial_t \varphi = \sum_{i=1}^{d} |\partial_{x_i} \varphi|. \tag{23}$$

Under which condition is it integrable?

Proposition 27 *(Necessary and sufficient integrability condition). Let φ be the viscosity solution of (23) with initial data $\varphi_0 \in C_b(\mathbb{R}^d)$. We then have*

$$\left[\varphi(\cdot,t) \in L^1(\mathbb{R}^d) \quad \forall t \geq 0\right] \quad \Longleftrightarrow \quad \left[\varphi_0^- \in L^1(\mathbb{R}^d) \text{ and } \varphi_0^+ \in L^\infty_{\text{int}}(\mathbb{R}^d)\right].$$

Proof. Since $\varphi(x,t) = \sup_{\overline{Q}_t(x)} \varphi_0$ by Proposition 9, we conclude by Lemma 17. □

We continue by showing that the L^1 topology is too weak to get the continuous dependence on the initial data, even for solutions which remain integrable.

Proposition 28 *(Failure of the L^1 continuous dependence). For all $n \geq 1$, let $\varphi_0^n(x) := (1 - n|x|)^+$, and φ_n be the solution of (23) with initial data φ_0^n. Then $\varphi_0^n \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ and*

$$\lim_{n\to\infty} \varphi_0^n = 0 \quad \text{in } L^1(\mathbb{R}^d),$$

but

$$\lim_{n\to\infty} \varphi_n(\cdot,t) = \mathbf{1}_{\overline{Q}_t}(\cdot) \neq 0 \quad \text{in } L^1(\mathbb{R}^d), \quad \forall t > 0.$$

Proof. Use again that $\varphi_n(x,t) = \sup_{\overline{Q}_t(x)} \varphi_0^n$. □

Interestingly a similar analysis works also for purely diffusive HJB equations. Consider e.g. an equation in one space dimension[7]

$$\partial_t \varphi = \left(\partial^2_{xx} \varphi\right)^+. \tag{24}$$

To have L^1 solutions, we need again that $\varphi_0^+ \in L^\infty_{\text{int}}$.

[6] Equation (23) is of the form (2) with $\mathcal{E} = \overline{Q}_1(0)$, $b(\xi) = \xi$, and $a \equiv 0$.
[7] Equation (24) is of the form (2) with $\mathcal{E} = [0,1]$, $b \equiv 0$, and $a(\xi) = \xi$.

Proposition 29 *(L^∞_{int} and nonlinear diffusions). Let $\varphi_0 \in C_b(\mathbb{R})$ be nonnegative and φ be the viscosity solution of (24) with φ_0 as initial data. Then,*

$$\left[\varphi(\cdot,t) \in L^1(\mathbb{R}) \quad \forall t \geq 0\right] \quad \Longleftrightarrow \quad \varphi_0 \in L^\infty_{\text{int}}(\mathbb{R}).$$

See Section 4.5 for the proof. We now use the lack of a fundamental solution of (24) to show that there is no continuous dependence on the initial data in L^1.

Proposition 30 *(Blow-up everywhere). For all $n \geq 1$, let φ_n be the viscosity solution of (24) with an approximate delta-function as initial data:*

$$\varphi_n(x, t = 0) = n\rho(nx), \tag{25}$$

where $0 \leq \rho \in C_c(\mathbb{R})$ is nontrivial. Then $\lim_{n\to\infty} \varphi_n(x,t) = \infty$, $\forall x \in \mathbb{R}, \forall t > 0$.

See Section 4.5 for the proof.

Remark 31. A counterexample to the L^1 continuous dependence for (24) is then given by the sequence of solutions

$$\psi_n(x,t) := \varphi_n(x,t)/\sqrt{\|\varphi_n(\cdot,t_0)\|_{L^1}} \quad \text{for a fixed } t_0 > 0,$$

since $\|\psi_n(t=0)\|_{L^1} \to 0$ while $\|\psi_n(\cdot,t)\|_{L^1} \geq \|\psi_n(\cdot,t_0)\|_{L^1} \to \infty$ for any $t \geq t_0$.

In view of the previous results, we now look for a Banach space $X \subset L^1$ that is strong enough to get well-posedness for (2) in general. We are mainly interested in properly defining an associated semigroup; see e.g. [14,12] for a general presentation of nonlinear semigroups.

Definition 32. Let E be a normed space.

(a) A family of maps $G_t : E \to E$ parametrized by $t \geq 0$ is a semigroup on E if

$$\begin{cases} G_{t=0} = \text{id (the identity), and} \\ G_{t+s} = G_t G_s \text{ (meaning the composition) for any } t, s \geq 0. \end{cases}$$

(b) It is a semigroup of continuous operators if in addition $G_t : E \to E$ is continuous for each $t \geq 0$.
(c) And it is strongly continuous if for each $\varphi_0 \in E$, $t \geq 0 \mapsto G_t\varphi_0 \in E$ is strongly continuous (i.e. continuous in norm).

Let φ be the unique viscosity solution of (2) and define

$$G_t : \varphi_0 \in C_b(\mathbb{R}^d) \mapsto \varphi(\cdot,t) \in C_b(\mathbb{R}^d). \tag{26}$$

Then G_t is a semigroup of Lipschitz continuous (in C_b) operators by Remark 5. A natural construction is to define X as the completion of some $E \subseteq C_b \cap L^1$, such that $X \subseteq L^1$ and G_t can be extended from E onto X. More precisely we require that

$$\begin{cases} E \text{ is a vector subspace of } C_b \cap L^1(\mathbb{R}^d), \\ E \text{ is a normed space,} \\ E \text{ is continuously embedded into } L^1(\mathbb{R}^d), \end{cases} \tag{27}$$

and for any data $(\mathcal{E}, b, a)$ satisfying (H2), the semigroup (26) satisfies:

$$\forall t \geq 0, \quad \begin{cases} G_t(E) \subseteq X := \overline{E}^{\|\cdot\|_E}, \, G_t : E \to X \text{ is continuous, and} \\ G_t \text{ admits an extension onto } X \text{ as a continuous operator.} \end{cases} \tag{28}$$

Here $\overline{E}^{\|\cdot\|_E} \subseteq L^1(\mathbb{R}^d)$ is the completion, see Section 2.1.

The best E is given below.

Theorem 33 *(A natural L^1 setting for (2)). The space $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ is a Banach space satisfying the properties (27)–(28). Moreover, any other space E satisfying (27)–(28) is continuously embedded into $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$.*

Remark 34. Since the best $E = X$ is a Banach space by Theorem 16, it is a posteriori not necessary to extend G_t outside C_b. The classical notion of viscosity solutions is then already satisfactory to study L^1 solutions of fully nonlinear degenerate PDEs.

Theorem 33 relies on the following estimate:

Theorem 35 *(General L^∞_{int} stability). Assume (H2) and $T \geq 0$. For any bounded subsolution φ and supersolution ψ of (2a),*

$$\int \sup_{\overline{Q}_1(x) \times [0,T]} (\varphi^* - \psi_*)^+ \, \mathrm{d}x \leq C \int \sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+ (\cdot, 0) \, \mathrm{d}x, \tag{29}$$

for some constant $C = C(d, \|a\|_\infty, \|b\|_\infty, T) \geq 0$.

As a consequence we have the following result:

Corollary 36 *(L^∞_{int} well-posedness of (2)). Assume (H2) and G_t is the solution semigroup defined in (26). Then its restriction to $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ is a strongly continuous semigroup of Lipschitz continuous operators.*

The proofs of Theorem 35 and Corollary 36 are given in Section 4.2, while Theorem 33 is proved in Section 4.5.

A certain duality between nonlinear semigroups

For each $t \geq 0$, let

$$S_t : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d)$$

where u is the entropy solution of (1), and let

$$G_t : \varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \mapsto \varphi(\cdot, t) \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$$

where φ is the viscosity solution of (21). Note that $G_t = G_t^{m,M}$ depends on the parameters m and M through Equation (21a).

Corollary 37 *(A form of duality). Assume (H1), $m < M$, and consider the semigroups S_t and G_t defined as above. Then G_t is the smallest strongly continuous semigroup of continuous operators on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t \varphi_0 \, \mathrm{d}x, \tag{30}$$

for every u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$, and $t \geq 0$.

The proof of Corollary 37 is given in Section 4.4.

Remark 38. Here "smallest" means that any other semigroup H_t satisfying the same properties is such that

$$G_t \varphi_0 \leq H_t \varphi_0 \quad \forall \varphi_0 \geq 0, \forall t \geq 0.$$

Remark 39.

(a) Inequality (30) can be seen as a nonlinear dual inequality between S_t and G_t, and G_t as a dual semigroup of S_t whose restriction over the cone $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d, \mathbb{R}^+)$ is entirely determined by S_t through (30).

(b) The question of duality in the other direction is open. Let us formulate it precisely. Consider S_t and the whole family $\{G_t^{m,M} : m < M\}$ defined just before Corollary 37.

Open question. *Is S_t the* **unique** *weakly-$\star$ continuous semigroup on $L^\infty(\mathbb{R}^d)$ such that for all $m < M$, $G_t^{m,M}$ is the smallest strongly continuous semigroup of continuous operators on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_t^{m,M} \varphi_0 \, \mathrm{d}x, \tag{31}$$

for all u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$, and $t \geq 0$?

A positive answer would mean that S_t is conversely entirely determined by the family $\{G_t^{m,M} : m < M\}$ through (31).

(c) Following part (a), we might be tempted to define a notion of dual for more general nonlinear semigroups. It is not our aim to explore such a direction, but note however that it would make sense only if
  (i) we have a reciprocal duality as discussed in part (b), and
  (ii) we can recover standard duality notions in the linear case.
  We can say more about (ii), and in Appendix D we give a sample result for slightly more abstract semigroups, for which we would not a priori know the associated equations.

4. Proofs

This section is devoted to the proofs of the results of Section 3. We will prove them in a certain order to arrive at Corollaries 25 and 37, thus concluding by the optimality of the weight and the interpretation in terms of dual nonlinear semigroup. The proofs of Propositions 29 and 30 and Theorem 33 are independent of this development and given at the end of the section.

4.1. More on viscosity solutions of (2)

We need further classical results that can be found in [24,31,5,4].

Proposition 40 *(Stability w.r.t. sup). Assume (H2) and $\mathcal{F} \neq \emptyset$ is a uniformly locally bounded family of viscosity subsolutions of (2a). Then, the function*

$$(x,t) \mapsto \sup\{\varphi(x,t) : \varphi \in \mathcal{F}\}$$

is a viscosity subsolution of (2a).

The next results concern relaxed limits; cf. (15).

Proposition 41 *(Stability w.r.t. relaxed limits). Assume* (H2) *and let* $(\varphi_\varepsilon)_{\varepsilon>0}$ *be a family of uniformly locally bounded viscosity subsolutions (resp. supersolutions) of* (2a)*. Then* $\limsup^* \varphi_\varepsilon$ *(resp.* $\liminf_* \varphi_\varepsilon$*) is a subsolution of* (2a) *(resp. supersolution).*

Remark 42. The notion of solution (or semisolution) is thus stable under local uniform convergence (equivalent to $\limsup^* \varphi_\varepsilon = \liminf_* \varphi_\varepsilon$).

Proposition 43 *(Limiting initial data). Assume* (H2) *and* $(\varphi_\varepsilon)_{\varepsilon>0}$ *is a uniformly locally bounded family of viscosity subsolutions (resp. supersolutions) of* (2a)*. Then* $\limsup^* \varphi_\varepsilon$ *(resp.* $\liminf_* \varphi_\varepsilon$*) satisfies*

$$\limsup^* \varphi_\varepsilon(x,0) = \limsup^* \big[(\varphi_\varepsilon)^*(\cdot,0)\big](x) \quad \forall x \in \mathbb{R}^d$$

(resp. $\liminf_* \varphi_\varepsilon(x,0) = \liminf_* \big[(\varphi_\varepsilon)_*(\cdot,0)\big](x)$*).*

Remark 44. For subsolutions this means that

$$\limsup_{\substack{\mathbb{R}^d\times\mathbb{R}^+\ni(y,s)\to(x,0)\\ \varepsilon\to 0^+}} \varphi_\varepsilon(y,s) = \limsup_{\substack{\mathbb{R}^d\ni y\to x\\ \varepsilon\to 0^+}} (\varphi_\varepsilon)^*(y,0),$$

where $(\varphi_\varepsilon)^*$ is the upper semicontinuous envelope computed in (x,t). The proof can be found in [9] and [5, Theorem 4.7]. The idea is to first consider $\varphi := \limsup^* \varphi_\varepsilon$, $\varphi_0(x) := \limsup^* \big[(\varphi_\varepsilon)^*(\cdot,0)\big](x)$, and show that $\min\{\partial_t\varphi - H(D\varphi, D^2\varphi), \varphi - \varphi_0\} \leq 0$ at $t = 0$ in the viscosity sense. Fix then some x and use the viscosity inequalities at a max $(\overline{y}, \overline{t})$ of the function $\varphi(y,t) - |y-x|^2/\tilde{\varepsilon} - Ct$ with C large enough such that $\overline{t} = 0$. We get $\varphi(x,0) \leq \varphi_0(\overline{y})$ and conclude as $\tilde{\varepsilon} \to 0^+$.

Here is the stability for minimal solutions, see Appendix A.1 for the proof.

Proposition 45 *(Stability of minimal solutions). Assume* (H2) *and* $(\varphi_0^n)_n$ *is a nondecreasing uniformly globally bounded sequence. If* $\underline{\varphi}_n$ *is the minimal solution of* (2) *with* φ_0^n *as initial data, then* $\sup_n \underline{\varphi}_n$ *is the minimal solution of* (2) *with initial data* $\sup_n (\varphi_0^n)_*$*.*

Let us continue with regularization procedures. Usually we consider inf and supconvolutions, but for convex Hamiltonians we can use the classical convolution for supersolutions, see [7,8] (the ideas were introduced in [40]).

Lemma 46. *Assume* (H2)*,* $\varphi \in BLSC(\mathbb{R}^d \times (0,\infty))$ *is a supersolution of* (2a)*, and* $0 \leq f \in L^1(\mathbb{R}^d \times (-\infty,0))$*. Then* $\varphi *_{x,t} f$ *is a supersolution of* (2a)*.*

Below is another version that will be needed.

Lemma 47. *Assume* (H2)*,* $\varphi \in C_b(\mathbb{R}^d \times (0,\infty))$ *is a supersolution of* (2a)*, and* $0 \leq g \in L^1(\mathbb{R}^d)$*. Then* $\varphi *_x g$ *remains a supersolution.*

The latter lemma is not proven in [7,8], but can be obtained via a standard approximation procedure. Let us give it for completeness.

Proof of Lemma 47. By Lemma 46, $\varphi_\nu := \varphi *_x g *_x \rho_\nu *_t \theta_\nu$ is a supersolution of (2a). It remains to pass to the limit as $\nu \to 0^+$. We will show that the convergence is local uniform towards $\varphi *_x g$, which will be sufficient by stability of the equation. We only need a local uniform convergence for $t > 0$ because the conclusion concerns the PDE only. With the assumed regularity on φ,

$$\lim_{\nu\to 0^+} \varphi *_x \rho_\nu *_t \theta_\nu = \varphi \quad \text{locally uniformly},$$

and $\|\varphi *_x \rho_\nu *_t \theta_\nu\|_\infty \le \|\varphi\|_\infty$. Moreover, for any $x \in \mathbb{R}^d$, $t > 0$ and $R \ge 0$,

$$\begin{aligned} |\varphi_\nu - \varphi *_x g|(x,t) &\le |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi| *_x g(x,t) \\ &\le \left(\sup_{|y|\le R} |\varphi *_x \rho_\nu *_t \theta_\nu - \varphi|(x-y,t) \right) \int\limits_{|y|\le R} g(y)\,\mathrm{d}y \\ &\quad + 2\|\varphi\|_\infty \int\limits_{|y|>R} g(y)\,\mathrm{d}y. \end{aligned}$$

This is enough to conclude since $\lim_{R\to\infty} \int_{|y|>R} g(y)\,\mathrm{d}y = 0$. □

4.2. L^∞_{int} well-posedness: proofs of Theorem 35 and Corollary 36

Let us now show that (2) is well-posed in L^∞_{int} as stated in Corollary 36. We first need to prove Theorem 35 for which we will use the lemmas below.

Lemma 48. *Assume (H2), and φ and ψ are sub and supersolutions of (2a). Then $(\varphi^* - \psi_*)^+$ remains a subsolution.*

Sketch of proof. First note that $\varphi - \psi$ is a subsolution since

$$\partial_t(\varphi - \psi) \le \sup_{\xi\in\mathcal{E}} H_\xi(\varphi) - \sup_{\xi\in\mathcal{E}} H_\xi(\psi) \le \sup_{\xi\in\mathcal{E}} \left(H_\xi(\varphi) - H_\xi(\psi)\right),$$

for $H_\xi(\varphi) := b(\xi)\cdot D\varphi + \operatorname{tr}\left(a(\xi)D^2\varphi\right)$. Since $(\varphi^* - \psi_*)^+ = \max\{\varphi^* - \psi_*, 0\}$, it is a subsolution by stability of viscosity subsolutions w.r.t. max, see Proposition 40. □

The rigorous justification of the above computations can be done by using a test function, Ishii lemma, and semijets [24, Theorem 8.3]. The details are standard and left to the reader. Here is a second lemma involving the profile

$$U : r \ge 0 \mapsto c_0 \int_r^\infty \mathrm{e}^{-\frac{s^2}{4}}\,\mathrm{d}s,$$

where $c_0 > 0$ is chosen such that $U(0) = 1$.

Lemma 49. *Let $L_b \geq 0$ and $L_a > 0$. For any $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$, define*

$$\Psi(x,t) := \begin{cases} U\left((|x| - 1 - L_b t)^+ / \sqrt{L_a t} \right) & \text{if } t > 0, \\ \mathbf{1}_{|x|<1} & \text{if } t = 0. \end{cases} \tag{32}$$

Then in the viscosity sense,

$$\partial_t \Psi \geq L_b |D\Psi| + L_a \sup_{\lambda \in \mathrm{Sp}(D^2\Psi)} \lambda^+ \quad \text{in } \mathbb{R}^d \times (0,\infty)$$

(it is in fact an equality). Moreover $\Psi \in C_b(\mathbb{R}^d \times (0,\infty)) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ where the latter time continuity holds up to $t = 0$.

Remark 50. Roughly speaking, we will use Ψ as a fundamental solution to construct L^1 supersolutions of (2), but we cannot take it as a Dirac mass at $t = 0$ because of Proposition 30.

Proof. The desired PDE holds if $|x| < 1 + L_b t$ since Ψ is constant in that region. It is also satisfied if $|x| = 1 + L_b t$ because the subjets are empty. Now if $|x| > 1 + L_b t$, then

$$\partial_t \Psi = -L_a \frac{|x| - 1 - L_b t}{2(L_a t)^{\frac{3}{2}}} U' - \frac{L_b}{\sqrt{L_a t}} U', \quad D\Psi = \frac{x}{|x|} \frac{U'}{\sqrt{L_a t}},$$

and

$$\partial^2_{x_i x_j} \Psi = \left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right) \frac{U'}{\sqrt{L_a t}} + \frac{x_i x_j}{|x|^2} \frac{U''}{L_a t}.$$

Since $U' \leq 0$ and $U'' \geq 0$, we have $\sum_{i,j=1}^d \partial^2_{x_i x_j} \Psi h_i h_j \leq \frac{U''}{L_a t}$ for any $h = (h_i)$ with $|h| = 1$. Hence $\sup_{\lambda \in \mathrm{Sp}(D^2\Psi)} \lambda^+ \leq \frac{U''}{L_a t}$ and

$$\partial_t \Psi - L_b |D\Psi| - L_a \sup_{\lambda \in \mathrm{Sp}(D^2\Psi)} \lambda^+ \geq -\frac{rU'(r)/2 + U''(r)}{t}$$

with $r = (|x| - 1 - L_b t)/\sqrt{L_a t}$. The right-hand side is zero by definition of U, and we obtain the desired equation for positive times. Now the detailed verification that $\Psi \in C_b(\mathbb{R}^d \times (0,\infty)) \cap C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ does not contain any particular difficulty and is left to the reader. The proof is complete. □

Proof of Theorem 35. Let $L_b := \|b\|_\infty$ and $L_a := \|\operatorname{tr}(a)\|_\infty$ and assume $L_a > 0$. We will use the following Ky Fan inequality [46]:

$$\operatorname{tr}(XY) \leq \sum_{i=1}^d \lambda_i(X) \lambda_i(Y) \quad \forall X, Y \text{ real } d \times d \text{ symmetric matrices}, \tag{33}$$

with the ordered eigenvalues $\lambda_1 \leq \cdots \leq \lambda_d$. It implies that any subsolution of (2a) is a subsolution of the equation

$$\partial_t \varphi = L_b |D\varphi| + L_a \sup_{\lambda \in \mathrm{Sp}(D^2\varphi)} \lambda^+. \tag{34}$$

Consider now arbitrary bounded sub and supersolutions φ and ψ of (2a). By Lemma 48, $(\varphi^* - \psi_*)^+$ is a subsolution of (2a) thus of (34). To prove Estimate (29), we will construct an integrable supersolution of (34). We will take it of the form

$$\psi := \Psi *_x \sup_{\overline{Q}_1(\cdot)} \phi_0,$$

where $\phi_0(x) := (\varphi^* - \psi_*)^+ (x, t = 0)$ and Ψ is defined in Lemma 49. Let us use Lemma 47 to show that ψ is a supersolution of (34). We need $\sup_{\overline{Q}_1(\cdot)} \phi_0$ to be integrable, and this can be assumed without loss of generality since (29) trivially holds if not. Now recalling that $\Psi \in C_b(\mathbb{R}^d \times (0, \infty))$ is a supersolution of (34), Lemma 47 applies and ψ remains a supersolution. Since moreover $\Psi \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ and $\sup_{\overline{Q}_1(\cdot)} \phi_0 \in L^\infty(\mathbb{R}^d)$, this supersolution is continuous up to $t = 0$ and satisfies

$$\psi(x, 0) = \int \mathbf{1}_{|y|<1} \sup_{\overline{Q}_1(x-y)} \underbrace{\phi_0}_{=(\varphi^*-\psi_*)^+(t=0)} \mathrm{d}y \geq (\varphi^* - \psi_*)^+ (x, 0).$$

Hence $(\varphi^* - \psi_*)^+ \leq \psi$ everywhere by the comparison principle, and

$$\begin{aligned}\int \sup_{\overline{Q}_1(x)\times[0,T]} (\varphi^* - \psi_*)^+ \,\mathrm{d}x &\leq \int \sup_{\overline{Q}_1(x)\times[0,T]} \psi \,\mathrm{d}x \\ &\leq \int \sup_{t\in[0,T]} \Psi(y, t) \,\mathrm{d}y \int \sup_{\overline{Q}_2(x)} \phi_0 \,\mathrm{d}x,\end{aligned}$$

by the Fubini theorem, etc. The first integral satisfies

$$\int \sup_{t\in[0,T]} \Psi(y, t) \,\mathrm{d}y \leq \int U \left((|y| - 1 - L_b T)^+ / \sqrt{L_a T} \right) \mathrm{d}y < \infty,$$

by (32) and since U is nondecreasing and integrable. For the second integral, Lemma 17 implies that

$$\int \sup_{\overline{Q}_2(x)} \phi_0 \,\mathrm{d}x \leq C \int \sup_{\overline{Q}_1(x)} (\varphi^* - \psi_*)^+ (\cdot, 0) \,\mathrm{d}x,$$

for a constant C which only depends on d. Combining the three inequalities above completes the proof of (29) when $L_a = \|\operatorname{tr}(a)\|_\infty > 0$. If $L_a = 0$, there is no diffusive part in (2a), and (29) follows from Proposition 9 and Lemma 17. □

We are ready to prove Corollary 36. We need the result below.

Lemma 51. *Assume (H2) and φ and ψ are continuous viscosity solutions of (2a). Then $|\varphi - \psi|$ is a subsolution of the same PDE.*

Proof. Use that $|\varphi - \psi| = \max\{(\varphi - \psi)^+, (\psi - \varphi)^+\}$ and Lemma 48. □

Proof of Corollary 36. The fact that G_t maps $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ into itself follows from Theorem 35. Indeed, if $\varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$, then the function $(x, t) \mapsto |G_t \varphi_0(x)|$ is a bounded subsolution of (2a), by Lemma 51 with $\psi \equiv 0$. Estimate (29) then implies that for any $t \geq 0$,

$$\|G_t \varphi_0\|_{\text{int}} = \int \sup_{\overline{Q}_1(x)} |G_t \varphi_0| \,\mathrm{d}x \leq C \int \sup_{\overline{Q}_1(x)} |\varphi_0| \,\mathrm{d}x,$$

for some constant $C = C(d, \|a\|_\infty, \|b\|_\infty, t)$. Let us now prove that

$$G_t : C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \to C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$$

is Lipschitz continuous for any $t \geq 0$. Let us apply again (29) to

$$(x,t) \mapsto |G_t\varphi_0(x) - G_t\psi_0(x)|,$$

which is a subsolution of (2a) by Lemma 51. As above we get that

$$\|G_t\varphi_0 - G_t\psi_0\|_{\text{int}} \leq C \int \sup_{\overline{Q}_1(x)} |\varphi_0 - \psi_0| \,\mathrm{d}x,$$

and deduce the desired continuity because C does not depend on the initial data. Hence G_t is a semigroup of Lipschitz continuous operators on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ and it remains to prove the time continuity. Fix $t_0 \geq 0$ and let us show that

$$\int \sup_{\overline{Q}_1(x)} |G_t\varphi_0 - G_{t_0}\varphi_0| \,\mathrm{d}x \to 0 \quad \text{as } t \to t_0.$$

The pointwise convergence follows from the continuity of $(x,t) \mapsto G_t\varphi_0(x)$ (as continuous solution of (2)), and a dominating function is given by

$$x \mapsto \sup_{(y,s) \in \overline{Q}_1(x) \times [0,t_0+1]} 2|G_s\varphi_0(y)|$$

which is integrable by Theorem 35. □

4.3. Weighted L^1 contraction: proof of Theorem 19

We continue with the general weighted L^1 contraction principle for (1).

Proof of Theorem 19. We have to show that

$$\int_{\mathbb{R}^d} |u-v|(x,T)\varphi_0(x) \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\underline{\varphi}(x,T) \,\mathrm{d}x \quad \forall T \geq 0. \tag{35}$$

Let us use the Kato inequality (20) with $0 \leq \phi \in C_c^\infty(\mathbb{R}^d \times [0,T])$. We then obtain, for a.e. $x \in \mathbb{R}^d$ and $t \geq 0$,

$$\begin{aligned} &\left\{ \sum_{i=1}^d q_i(u,v)\partial_{x_i}\phi + \sum_{i,j=1}^d r_{ij}(u,v)\partial^2_{x_i x_j}\phi \right\}(x,t) \\ &= \operatorname{sign}(u(x,t) - v(x,t)) \int_{v(x,t)}^{u(x,t)} \left\{ F'(\xi) \cdot D\phi(x,t) + \operatorname{tr}\left(A(\xi)D^2\phi(x,t)\right) \right\} \mathrm{d}\xi \\ &\leq |u(x,t) - v(x,t)| \operatorname*{ess\,sup}_{m \leq \xi \leq M} \left\{ F'(\xi) \cdot D\phi(x,t) + \operatorname{tr}\left(A(\xi)D^2\phi(x,t)\right) \right\}, \end{aligned} \tag{36}$$

where we have taken the sup over $[m,M]$ because of the maximum principle Lemma 65. Injecting into (20), we get that

$$\int_{\mathbb{R}^d} |u-v|(x,T)\phi(x,T)\,\mathrm{d}x \le \int_{\mathbb{R}^d} |u_0-v_0|(x)\phi(x,0)\,\mathrm{d}x + \iint_{\mathbb{R}^d\times(0,T)} |u-v| \left(\partial_t\phi + \operatorname*{ess\,sup}_{m\le\xi\le M}\left\{F'(\xi)\cdot D\phi + \operatorname{tr}\left(A(\xi)D^2\phi\right)\right\}\right)\mathrm{d}x\,\mathrm{d}t. \quad (37)$$

In the third integral, we recognize the backward in time version of (21a). The proof of (35) then consists in taking $\phi(x,t) = \underline{\varphi}(x,T-t)$.

Simplified case: $0 \le \varphi_0 \in C_c(\mathbb{R}^d)$.
Now (21) has a unique viscosity solution φ which coincides with $\underline{\varphi}$. It belongs to $C_b(\mathbb{R}^d\times\mathbb{R}^+)\cap C(\mathbb{R}^+;L^1(\mathbb{R}^d))$ by Corollary 36 and Theorem 16. Let us regularize it by convolution

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu\theta_\nu),$$

with the mollifiers (12) and (13). It follows that

$$\varphi_\nu \in C^\infty(\mathbb{R}^d\times\mathbb{R}^+)\cap C(\mathbb{R}^+;L^1(\mathbb{R}^d))$$

along with all its derivatives. This is enough to take $\phi_\nu(x,t) := \varphi_\nu(x,T-t)$ as a test function in (37) by approximation. Note that ϕ_ν is a supersolution of the backward version of (21a) by Lemma 46, i.e.

$$\partial_t\phi_\nu + \operatorname*{ess\,sup}_{m\le\xi\le M}\left\{F'(\xi)\cdot D\phi_\nu + \operatorname{tr}\left(A(\xi)D^2\phi_\nu\right)\right\} \le 0 \quad \text{for any } t<T.$$

Inequality (37) with the test function ϕ_ν then implies that

$$\int_{\mathbb{R}^d} |u-v|(x,T)\varphi_\nu(x,0)\,\mathrm{d}x \le \int_{\mathbb{R}^d} |u_0-v_0|(x)\varphi_\nu(x,T)\,\mathrm{d}x,$$

for any $T\ge 0$ and $\nu>0$. By the $C(\mathbb{R}^+;L^1(\mathbb{R}^d))$ regularity of φ, the convolution $\varphi_\nu = \varphi *_{x,t}(\rho_\nu\theta_\nu)$ converges to φ in $C([0,T];L^1(\mathbb{R}^d))$ as $\nu\to 0^+$. Passing to the limit as $\nu\to 0^+$ then yields (35).

General case: $0\le\varphi_0\in BLSC(\mathbb{R}^d)$.
We would like to pointwise approximate φ_0 by a monotone sequence $\varphi_0^n \uparrow \varphi_0$ such that $0\le\varphi_0^n\in C_c(\mathbb{R}^d)$. Take

$$\varphi_0^n(x) := \inf_{y\in\mathbb{R}^d}\left\{\varphi_0(y)\mathbf{1}_{|y|<n} + n|x-y|^2\right\} \ge 0.$$

Then φ_0^n is continuous as an infconvolution, see e.g. [24,31,5,4]. Also,

$$\varphi_0^n(x) \le \varphi_0(x)\mathbf{1}_{|x|<n} \quad \forall x\in\mathbb{R}^d,$$

which implies that $\varphi_0^n\in C_c(\mathbb{R}^d)$. In the limit $n\to\infty$, we have $\varphi_0^n\uparrow(\varphi_0)_* = \varphi_0$. Let φ_n be the solution of (21) with initial data φ_0^n, then by the previous step,

$$\int_{\mathbb{R}^d} |u-v|(x,T)\varphi_0^n(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} |u_0-v_0|(x)\varphi_n(x,T)\,\mathrm{d}x,$$

for any $T\ge 0$ and n. By the stability of minimal solutions (see Proposition 45), these solutions satisfy $\varphi_n\uparrow\underline{\varphi}$ pointwise. So we conclude the proof of (35) by passing to the limit as $n\to\infty$ using the monotone convergence theorem. □

Remark 52. Going back to (20) and (36), we might think about the kinetic setting for (1) since

$$\operatorname{sign}(u(x,t)-v(x,t))\int_{v(x,t)}^{u(x,t)}\left\{\partial_t\phi(x,t)+F'(\xi)\cdot D\phi(x,t)+\operatorname{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\}\mathrm{d}\xi$$

$$=\int_{\mathbb{R}}|\chi(\xi;u)-\chi(\xi;v)|\left\{\partial_t\phi(x,t)+F'(\xi)\cdot D\phi(x,t)+\operatorname{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\}\mathrm{d}\xi,$$

with the usual kinetic function χ; cf. [22]. However, we did not explore this. For L^1 kinetic solutions of (1), u and v would take values outside any bounded interval, so there would be further terms for large $|\xi|$ and we do not have any idea of what might then be a reasonable version of (6).

4.4. Duality: proofs of Theorem 23 and Corollaries 25 and 37

Let us now establish the new characterization of viscosity supersolutions (Theorem 23). We need several technical lemmas.

Here is a first classical result on entropy solutions.

Lemma 53. *Assume (H1) and $u_0\in L^\infty(\mathbb{R}^d)$. Then, the entropy solution of (1) is a distributional solution of (1),*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^+}\left(u\partial_t\phi+\sum_{i=1}^d F_i(u)\partial_{x_i}\phi+\sum_{i,j=1}^d\mathcal{A}_{ij}(u)\partial^2_{x_ix_j}\phi\right)\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^d}u_0(x)\phi(x,0)\,\mathrm{d}x=0\quad\forall\phi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^+),$$

where $\mathcal{A}_{ij}(u)=\int_0^u A_{ij}(\xi)\,\mathrm{d}\xi$.

Proof. Take $\eta(u)=\pm u$ successively in the entropy inequalities, Definition 12(c). □

Here is another result on the continuity in time.

Lemma 54. *Assume (H1), $u_0,v_0\in L^\infty(\mathbb{R}^d)$ with $u_0-v_0\in L^1(\mathbb{R}^d)$, u and v entropy solutions of (1) with initial data u_0 and v_0. Then $u-v\in C(\mathbb{R}^+;L^1(\mathbb{R}^d))$.*

Proof of Lemma 54. By Theorem 19 with $\varphi_0\equiv 1$, we have

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1}\le\|u_0-v_0\|_{L^1}\quad\forall t\ge 0.$$

Since the left-hand side is finite, $u-v\in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d))$. By the continuity in time with values in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ of these functions, it remains to prove that

$$\lim_{R\to\infty}\sup_{t\in[0,T]}\int_{|x|\ge R}|u(x,t)-v(x,t)|\,\mathrm{d}x=0\quad\forall T\ge 0.\tag{38}$$

To do so, we will use again Theorem 19.

Fix $m<M$ such that u_0 and v_0 take their values in $[m,M]$, and consider

$$\varphi_0^R(x):=\varphi_0\left(\frac{x}{R}\right),\quad R>0,$$

where $\varphi_0 = \varphi_0(x)$ is some kernel such that

$$\begin{cases} 0 \le \varphi_0 \in C_b(\mathbb{R}^d), \\ \varphi_0(x) = 0 \text{ for } |x| \le 1/2, \\ \text{and } \varphi_0(x) = 1 \text{ for } |x| \ge 1. \end{cases}$$

With that choice, $\varphi_0^R \to 0$ as $R \to \infty$ locally uniformly in $\mathbb{R}^d$. We then claim that the solutions φ_R of (21) with initial data φ_0^R converge locally uniformly in $\mathbb{R}^d \times \mathbb{R}^+$ to zero too. This is a consequence of the method of relaxed semilimits [9]. Let us give details for completeness. By the maximum principle,

$$\|\varphi_R\|_\infty \le \|\varphi_0^R\|_\infty = \|\varphi_0\|_\infty \quad \forall R > 0.$$

We can then apply Propositions 41 and 43 to $\limsup^* \varphi_R$ as $R \to \infty$ and get that it is a subsolution of (21a) satisfying

$$\limsup{}^* \varphi_R(x, 0) = \limsup{}^* \varphi_0^R(x) = 0 \quad \forall x \in \mathbb{R}^d.$$

Let us recall that the above $\limsup^* \varphi_0^R$ as $R \to \infty$ is only taken in space; cf. (15) and Remark 44. Similarly $\liminf_* \varphi_R$ as $R \to \infty$ is a supersolution of (2) with zero as initial data. The comparison principle then implies that

$$\limsup{}^* \varphi_R \le \liminf{}_* \varphi_R.$$

Hence φ_R converges locally uniformly in $\mathbb{R}^d \times \mathbb{R}^+$, as $R \to \infty$, to the unique solution of (21) with zero initial data, that is zero itself.

Now we can show (38). By Theorem 19 with the previous m, M, and φ_0^R,

$$\begin{aligned} \int_{|x|\ge R} |u(x,t) - v(x,t)| \,\mathrm{d}x &\le \int_{\mathbb{R}^d} |u(x,t) - v(x,t)| \varphi_0^R(x) \,\mathrm{d}x \\ &\le \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \varphi_R(x,t) \,\mathrm{d}x \le \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \sup_{s \in [0,T]} \varphi_R(x,s) \,\mathrm{d}x, \end{aligned}$$

for any $T \ge t \ge 0$. The right-hand side vanishes as $R \to \infty$ by the discussion above and the dominated convergence theorem. The proof of (38) is complete. □

Here is a regularization procedure for the weights.

Lemma 55. *Assume* (H1)*,* $m < M$*,* ρ_ν *and* θ_ν *are defined in* (12) *and* (13)*, and* $0 \le \varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ *satisfies* (I) *in Theorem 23. Then for any* $\nu > 0$*, the convolution*

$$\varphi_\nu := \varphi *_{x,t} (\rho_\nu \theta_\nu) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$$

also satisfies (I) *in Theorem 23.*

Proof. By assumption,

$$\int_{\mathbb{R}^d} |u - v|(x,t) \varphi(x,s) \,\mathrm{d}x \le \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t+s) \,\mathrm{d}x, \tag{39}$$

for any $t, s \geq 0$, u_0 and v_0 with values in $[m, M]$, and entropy solutions u and v of (1) with u_0 and v_0 as initial data. Our aim is to get the same result for φ_ν. Let us use (39) not for u_0 and v_0, but their translations $u_0(\cdot + y)$ and $v_0(\cdot + y)$ for some fixed $y \in \mathbb{R}^d$. Since the PDE part of (1) is invariant w.r.t. translation, the corresponding solutions are $u(x + y, t)$ and $v(x + y, t)$. Hence,

$$\int_{\mathbb{R}^d} |u - v|(x + y, t)\varphi(x, s)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x + y)\varphi(x, t + s)\,\mathrm{d}x$$

for any $t, s \geq 0$. By changing the variable of integration, we obtain that

$$\int_{\mathbb{R}^d} |u - v|(x, t)\varphi(x - y, s)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\varphi(x - y, t + s)\,\mathrm{d}x.$$

Now we fix $\tau \leq 0$ and apply this formula, not for s but $s - \tau$. We deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t)\varphi(x - y, s - \tau)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\varphi(x - y, t + s - \tau)\,\mathrm{d}x. \tag{40}$$

Multiply then by $\rho_\nu(y)\theta_\nu(\tau)$ and integrate over $(y, \tau) \in \mathbb{R}^d \times \mathbb{R}^-$ to conclude. □

Later we will pass to the limit[8]

$$\varphi_\flat := \liminf{}_* \varphi_\nu \quad \text{as } \nu \to 0^+, \tag{41}$$

and compare $\varphi_\flat$ with the function $\varphi_\#$ defined in (18). To compare the two limits, we will assume in addition that

$$\operatorname{supp}(\rho_\nu) \subset B_\nu(0) \quad \text{and} \quad \operatorname{supp}(\theta_\nu) \subset (-\nu, 0). \tag{42}$$

Here are fundamental properties of $\varphi_\flat$ and $\varphi_\#$ that will be needed.

Lemma 56. *Assume $\varphi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$, $\varphi_\flat$ and $\varphi_\#$ are as above, and (42) holds. Then:*

(i) *The limit $\varphi_\flat$ is the pointwise largest function in $BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ that is less than or equal φ a.e. in $\mathbb{R}^d \times \mathbb{R}^+$. Moreover $\varphi_\flat = \varphi$ a.e. in $\mathbb{R}^d \times \mathbb{R}^+$.*

(ii) *For any $t \geq 0$, $\varphi_\#(\cdot, t)$ is the pointwise largest function in $BLSC(\mathbb{R}^d)$ less than or equal $\varphi(\cdot, t)$ a.e. in $\mathbb{R}^d$. Moreover $\varphi_\#(\cdot, t) = \varphi(\cdot, t)$ a.e. in $\mathbb{R}^d$.*

Remark 57.

(a) Above "pointwise largest function" means, e.g. for the item (i), that if any other $\psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ is such that $\psi \leq \varphi$ a.e. in $\mathbb{R}^d \times \mathbb{R}^+$, then necessarily

$$\psi(x, t) \leq \varphi_\flat(x, t) \quad \text{for all } (x, t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

The second item has to be understood similarly.

[8] This is the relaxed limit in (15) with the parameter ν instead of ε.

(b) In the sequel, it is understood that "a.e." holds in (x,t) in (i) and x in (ii), without possibly recalling it.

Proof. Let us prove (i). Note first that $\varphi_\flat$ is lower semicontinuous as a lower relaxed limit. To prove that $\varphi_\flat \leq \varphi$ a.e., it suffices to do it for the Lebesgue points of φ. Such points $(x,t) \in \mathbb{R}^d \times (0,\infty)$ satisfy

$$\lim_{\nu \to 0^+} \frac{1}{\nu^{d+1}} \iint\limits_{B_\nu(x)\times(t-\nu,t+\nu)} |\varphi(y,s) - \varphi(x,t)| \, \mathrm{d}y \, \mathrm{d}s = 0,$$

so by the assumptions on the mollifiers, see (12), (13) and (42), we find that

$$|\varphi_\nu(x,t) - \varphi(x,t)| \leq \frac{1}{\nu^{d+1}} \iint\limits_{B_\nu(x)\times(t,t+\nu)} |\varphi(y,s) - \varphi(x,t)| \, \rho\left(\frac{x-y}{\nu}\right) \theta\left(\frac{t-s}{\nu}\right) \mathrm{d}y \, \mathrm{d}s \to 0 \quad \text{as } \nu \to 0^+.$$

It follows that

$$\varphi_\flat(x,t) \leq \lim_{\nu \to 0^+} \varphi_\nu(x,t) = \varphi(x,t),$$

at any Lebesgue point. Moreover, for any fixed (x,t), lower semicontinuity of φ implies that

$$\varphi_\nu(y,s) = \iint\limits_{B_\nu(y)\times(s,s+\nu)} \underbrace{\varphi(z,\tau)}_{\geq \varphi(x,t)+o(1)} \, \rho\left(\frac{y-z}{\nu}\right) \theta\left(\frac{s-\tau}{\nu}\right) \mathrm{d}z \, \mathrm{d}\tau \geq \varphi(x,t) + o(1)$$

as $(y,s,\nu) \to (x,t,0^+)$, and we get that

$$\varphi_\flat(x,t) = \liminf{}_* \varphi_\nu(x,t) \geq \varphi(x,t).$$

We conclude that $\varphi_\flat = \varphi$ a.e.

Now, to complete the proof of (i), it remains to prove that $\varphi_\flat \geq \psi$ pointwise for any other $\psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ such that $\psi \leq \varphi$ a.e. Given such a function, let

$$\psi_\flat := \liminf{}_* \psi *_{x,t} (\rho_\nu \theta_\nu).$$

As above, $\psi \leq \psi_\flat$ pointwise; but also $\psi_\flat \leq \varphi_\flat$ pointwise since

$$\psi *_{x,t} (\rho_\nu \theta_\nu) \leq \varphi *_{x,t} (\rho_\nu \theta_\nu).$$

This proves (i) and the arguments for (ii) are similar. □

Here is also a general inequality between $\varphi_\flat$ and $\varphi_\#$ that will be needed.

Lemma 58. *Under the hypotheses of the previous lemma, $(\varphi_\#)_* \leq \varphi_\flat$ pointwise in $\mathbb{R}^d \times \mathbb{R}^+$.*

Proof. Let us first prove that $\varphi_\#$ is measurable in (x,t). We have

$$\varphi_\#(x,t) = \sup_{n\geq 1} \overbrace{\inf_{m\geq n} \underbrace{\inf_{\substack{\frac{1}{m}\leq r\leq \frac{1}{n} \\ |y|\leq \frac{1}{n}}} \frac{1}{\mathrm{meas}(B_r(y))} \int\limits_{B_r(y)} \varphi(x+z,t) \, \mathrm{d}z}_{=:\varphi_{n,m}(x,t)}}^{=:\varphi_n(x,t)},$$

where n and m are integers. For each $\frac{1}{m} \le r \le \frac{1}{n}$ and $|y| \le \frac{1}{n}$, the function

$$(x,t) \mapsto \frac{1}{\operatorname{meas}(B_r(y))} \int\limits_{B_r(y)} \varphi(x+z,t)\,\mathrm{d}z$$

is lower semicontinuous by Fatou's lemma and $\varphi \in BLSC$ (assumption in the previous lemma). The infimum $\varphi_{n,m}$ remains lower semicontinuous, because r and y live in compact sets. Hence, $\varphi_n = \inf_{m \ge n} \varphi_{n,m}$ is measurable in (x,t) and so is $\varphi_\# = \sup_{n \ge 1} \varphi_n$.

We can now prove the lemma. For any $t \ge 0$, the measurable functions $\varphi, \varphi_\#$ satisfy $\varphi_\#(\cdot,t) = \varphi(\cdot,t)$ a.e., hence we may use the Fubini theorem to conclude that

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_\# = \varphi\}}\,\mathrm{d}x\,\mathrm{d}t = \int\limits_{\mathbb{R}^+} \left(\int\limits_{\mathbb{R}^d} \mathbf{1}_{\{\varphi_\#(x,t) = \varphi(x,t)\}}\,\mathrm{d}x \right) \mathrm{d}t = 0.$$

This proves that $\varphi_\# = \varphi$ a.e. in (x,t), so that $(\varphi_\#)_* \le \varphi$ a.e. in (x,t). Hence $(\varphi_\#)_* \le \varphi_\flat$ pointwise by Lemma 56(i). □

Here are further properties that we will need.

Lemma 59. *Let $\varphi, \psi \in BLSC(\mathbb{R}^d \times \mathbb{R}^+)$ and $\varphi_\#, \psi_\#$ as in (18). Then*

(i) *$\varphi \le (\varphi_\#)_*$ pointwise, and*
(ii) *if $\varphi \le \psi_\#$ pointwise, then $\varphi_\# \le \psi_\#$ pointwise.*

Proof. We can show that $\varphi \le \varphi_\#$ from the definition of $\varphi_\#$ and the lower semicontinuity of φ, exactly as we showed that $\varphi \le \varphi_\flat$ in the proof of Lemma 56. In particular, $\varphi \le (\varphi_\#)_*$ which is part (i). For part (ii), use Lemma 56(ii). It says that $\psi_\#(\cdot,t) = \psi(\cdot,t)$ a.e. in x, for each fixed $t \ge 0$. Hence, $\varphi(\cdot,t) \le \psi(\cdot,t)$ a.e. and the desired inequality follows again from the definitions of $\varphi_\#$ and $\psi_\#$. □

We are now in position to prove Theorem 23.

Proof of Theorem 23. Let us proceed in several steps.

1) (II) $\Longrightarrow$ (I).

By (II), $(\varphi_\#)_*$ is a $BLSC$ supersolution of (21a). In particular, for any fixed $s \ge 0$, the function

$$(x,t) \mapsto (\varphi_\#)_*(x,t+s)$$

is also a supersolution of (21a). By Remark 20(b), we can apply Theorem 19 to this supersolution with the $BLSC$ initial weight $(\varphi_\#)_*(\cdot,s)$. The result is that

$$\int\limits_{\mathbb{R}^d} |u-v|(x,t)(\varphi_\#)_*(x,s)\,\mathrm{d}x \le \int\limits_{\mathbb{R}^d} |u_0 - v_0|(x)(\varphi_\#)_*(x,t+s)\,\mathrm{d}x,$$

for any $u_0 = u_0(x)$ and $v_0 = v_0(x)$ with values in $[m,M]$, u and v entropy solutions of (1) with u_0 and v_0 as initial data, and $t,s \ge 0$. This is exactly (I) but with $(\varphi_\#)_*$ instead of φ. To replace $(\varphi_\#)_*$ by φ, we use Lemma 59(i) for the left-hand side. For the right-hand side, we use that $(\varphi_\#)_* \le \varphi_\#$ pointwise and the fact that $\varphi_\#(x,t+s) = \varphi(x,t+s)$ for a.e. x, see Lemma 56(ii). This implies (I) with φ, as desired.

2) (I) $\Longrightarrow$ (II) *for smooth weights* φ.

Let us prove the reverse implication when $0 \leq \varphi \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^+)$. We will appropriately choose u_0 and v_0 later. For the moment, we assume that

$$m \leq v_0 \leq u_0 \leq M \quad \text{and} \quad u_0 - v_0 \in L^1(\mathbb{R}^d).$$

By Lemmas 65 and 54, $0 \leq u - v \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$, and then we can use (I) to get

$$\int_{\mathbb{R}^d} (u-v)(x,T)\varphi(x,s)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, T+s)\,\mathrm{d}x, \tag{43}$$

for any $T, s \geq 0$. Let us fix $s > 0$ and determine what PDE φ satisfies. This will be done by injecting the weak formulation of (1) into (43) and then pass to the limit as $T \to 0^+$. By Lemma 53,

$$\begin{aligned}\int_{\mathbb{R}^d} (u-v)(x,T)\phi(x,T)\,\mathrm{d}x = &\iint_{\mathbb{R}^d\times(0,T)} \Bigg((u-v)\partial_t \phi + \sum_{i=1}^d (F_i(u) - F_i(v))\partial_{x_i}\phi \\ &+ \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v))\partial^2_{x_i x_j}\phi \Bigg)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathbb{R}^d} (u_0 - v_0)(x)\phi(x,0)\,\mathrm{d}x,\end{aligned}$$

for any $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T])$ and $\mathcal{A}'_{ij} = A_{ij}$. Note that we have rewritten the equation given by Lemma 53 with integrals in $t < T$ and an additional final term at $t = T$. This follows from standard arguments using the L^1_{loc} continuity in time of u and v. Since $\varphi \in C_b^\infty$, $u - v \in C_t(L^1_x)$ and $u, v \in L^\infty$, a standard approximation argument shows that we can take ϕ to be

$$\phi(x,t) = \varphi(x, t+s-T),$$

and get that

$$\begin{aligned}\int_{\mathbb{R}^d} (u-v)(x,T)\varphi(x,s)\,\mathrm{d}x = &\iint_{\mathbb{R}^d\times(0,T)} \Bigg((u-v)\partial_t \varphi(t+s-T) \\ &+ \sum_{i=1}^d (F_i(u) - F_i(v))\partial_{x_i}\varphi(t+s-T) \\ &+ \sum_{i,j=1}^d (\mathcal{A}_{ij}(u) - \mathcal{A}_{ij}(v))\partial^2_{x_i x_j}\varphi(t+s-T) \Bigg)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, s-T)\,\mathrm{d}x.\end{aligned} \tag{44}$$

Here we assume that $s > 0$ and T is so small that $s - T > 0$. Inserting (44) into (43), we get

$$\int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, s+T)\,\mathrm{d}x - \int_{\mathbb{R}^d} (u_0 - v_0)(x)\varphi(x, s-T)\,\mathrm{d}x \geq \iint_{\mathbb{R}^d\times(0,T)} \Big(\dots \Big)\,\mathrm{d}x\,\mathrm{d}t.$$

We now would like to divide by $2T$ and pass to the limit as $T \to 0^+$. All the computations are justified, again because $\varphi \in C_b^\infty$, the solutions u and v are bounded, and $u - v \in C_t(L_x^1)$. We get that

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u_0(x) - v_0(x))\partial_s\varphi(x,s)\,\mathrm{d}x \\
&\geq \frac{1}{2}\int_{\mathbb{R}^d} \Bigg((u_0 - v_0)\partial_s\varphi(s) + \sum_{i=1}^{d}(F_i(u_0) - F_i(v_0))\partial_{x_i}\varphi(s) \\
&\quad + \sum_{i,j=1}^{d}(\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0))\partial^2_{x_i x_j}\varphi(s) \Bigg)\,\mathrm{d}x.
\end{aligned}$$

Subtracting the term $\int (u_0 - v_0)\partial_s\varphi(s)\,\mathrm{d}x/2$ of the right-hand side implies that

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u_0(x) - v_0(x))\partial_s\varphi(x,s)\,\mathrm{d}x \\
&\geq \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d}(F_i(u_0) - F_i(v_0))\partial_{x_i}\varphi(s) + \sum_{i,j=1}^{d}(\mathcal{A}_{ij}(u_0) - \mathcal{A}_{ij}(v_0))\partial^2_{x_i x_j}\varphi(s) \right)\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\int_{v_0(x)}^{u_0(x)} \left\{ F'(\xi)\cdot D\varphi(x,s) + \operatorname{tr}\left(A_{ij}(\xi)D^2\varphi(x,s)\right)\right\}\mathrm{d}\xi\,\mathrm{d}x,
\end{aligned} \tag{45}$$

for any $s > 0$ and $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ such that both u_0 and v_0 take their values in the interval $[m, M]$. It remains to choose $u_0 - v_0$ as an approximate unit, up to some multiplicative constant.

Let us introduce new parameters: $x_0 \in \mathbb{R}^d$, $\varepsilon > 0$ and $m \leq a < b \leq M$. We would like to choose

$$u_0 - v_0 = (b - a)\mathbf{1}_{x_0 + (-\varepsilon,\varepsilon)^d}, \tag{46}$$

with the constraint that both u_0 and v_0 only take the two values a and b. Writing $x = (x_i)$, take e.g.

$$u_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon, \\ b & \text{if not,} \end{cases}$$

and

$$v_0(x) := \begin{cases} a & \text{if } x_1 > (x_0)_1 + \varepsilon \text{ or } x \in x_0 + (-\varepsilon,\varepsilon)^d, \\ b & \text{if not.} \end{cases}$$

Then $m \leq v_0 \leq u_0 \leq M$ and $u_0 - v_0 \in L^1(\mathbb{R}^d)$ as required. Inserting our choice into (45) and dividing by $(b - a)\varepsilon^d$, we deduce that

$$\begin{aligned}
&\frac{1}{\varepsilon^d}\int_{x_0 + (-\varepsilon,\varepsilon)^d} \partial_s\varphi(x,s)\,\mathrm{d}x \\
&\geq \frac{1}{\varepsilon^d}\int_{x_0 + (-\varepsilon,\varepsilon)^d} \frac{1}{b - a}\int_a^b \left\{ F'(\xi)\cdot D\varphi(x,s) + \operatorname{tr}\left(A_{ij}(\xi)D^2\varphi(x,s)\right)\right\}\mathrm{d}\xi\,\mathrm{d}x.
\end{aligned}$$

Let now $\xi \in (m, M)$ be any Lebesgue point of any arbitrarily chosen a.e. representative of (F', A). Take first the limit as $a, b \to \xi$ such that ξ is the center of each $[a, b]$ in order to use the Lebesgue point property; take next the limit as $\varepsilon \to 0^+$. This gives us that

$$\partial_s \varphi(x_0, s) \geq F'(\xi) \cdot D\varphi(x_0, s) + \operatorname{tr}\left(A_{ij}(\xi) D^2 \varphi(x_0, s)\right),$$

for any $x_0 \in \mathbb{R}^d$, $s > 0$, and Lebesgue point ξ. That is φ is a supersolution of (21). This completes the proof of the remaining implication in the case where φ is C_b^∞ (and then $\varphi_\# = \varphi$).

3) (I) $\Longrightarrow$ (II) *for nonnegative BLSC weights* φ.

In this case we use the regularization procedure of Lemma 55. By this lemma

$$\varphi_\nu = \varphi *_{x,t} (\rho_\nu \theta_\nu)$$

satisfies (I) since φ does by assumption. By the previous step we deduce that φ_ν is a supersolution of (21a). Hence

$$\varphi_\flat = \liminf{}_* \varphi_\nu$$

is also a supersolution by stability (cf. Proposition 41). But to prove (II), we need to show that $\varphi_\#$ is a supersolution. We will do this by showing that $\varphi_\flat = (\varphi_\#)_*$ pointwise (at least for positive times). To prove that $\varphi_\flat \leq (\varphi_\#)_*$, we need to use (I). By (I),

$$\int_{\mathbb{R}^d} |u - v|(x, t) \varphi(x, s) \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, t + s) \, \mathrm{d}x,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and corresponding solutions u and v of (1) and $t, s \geq 0$. By Lemma 56(i), we also have that $\varphi_\flat = \varphi$ a.e. In particular, there is a null set $N \subset \mathbb{R}^+$ such that $\varphi(\cdot, s) = \varphi_\flat(\cdot, s)$ a.e., for any $s \notin N$.[9] Fixing $T > 0$, there thus exists a sequence $s_n \to T^-$ such that $s_n \notin N$, for any n. Choosing moreover $t_n := T - s_n$, we deduce that

$$\int_{\mathbb{R}^d} |u - v|(x, t_n) \varphi_\flat(x, s_n) \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, \mathrm{d}x.$$

Let us pass to the limit as $n \to \infty$ in the left-hand side. To do so, we use Fatou's lemma, which is possible because of the lower semicontinuity of $\varphi_\flat$ and the continuity of entropy solutions with values in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ which implies that

$$|u - v|(x, t_n) \to |u_0 - v_0|(x) \quad \text{for a.e. } x$$

(along a subsequence). In the limit, it then follows that

$$\int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi_\flat(x, T) \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x) \varphi(x, T) \, \mathrm{d}x$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$ and $T > 0$. To continue, we argue as in the previous step where we chose $0 \leq u_0 - v_0 \in L^1(\mathbb{R}^d)$ to be an approximate unit up to a multiplicative constant, cf. (46). The same arguments imply that for any $T > 0$,

[9] To find N use that $\iint_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{\{\varphi_\flat = \varphi\}} \, \mathrm{d}x \, \mathrm{d}s = 0 = \int_{\mathbb{R}^+} \operatorname{meas}\{\varphi(\cdot, s) = \varphi_\flat(\cdot, s)\} \, \mathrm{d}s$ by Fubini.

$$\varphi_\flat(\cdot, T) \leq \varphi(\cdot, T) \quad \text{a.e.}$$

By Lemma 56(ii), we conclude that $\varphi_\flat \leq \varphi_\#$ pointwise (for positive times). Hence, $\varphi_\flat \leq (\varphi_\#)_*$ and then $\varphi_\flat = (\varphi_\#)_*$ pointwise (for positive times) by Lemma 58. This implies that $(\varphi_\#)_* = \varphi_\flat$ is a supersolution of (21a). The proof of Theorem 23 is complete. □

We have now established all preliminary results and are ready to prove our duality results (Corollaries 25 and 37).

Proof of Corollary 25. We already know that $\underline{\varphi} \in \mathscr{W}_{m,M,\varphi_0}$ by Theorem 19. Let us prove the formula with the inf. Take $\varphi \in \mathscr{W}_{m,M,\varphi_0}$, which means that $\varphi \in BLSC$ and satisfies Theorem 23(I) with $\varphi(t=0) \geq \varphi_0$. By this theorem, φ satisfies (II) as well, that is $\varphi_\#$ is a supersolution of (21a). Recall that $\varphi \leq (\varphi_\#)_*$ pointwise by Lemma 59(i). In particular

$$(\varphi_\#)_*(t=0) \geq \varphi(t=0) \geq \varphi_0.$$

Thus $\varphi_\#$ is a supersolution of the Cauchy problem (21), and $\underline{\varphi} \leq \varphi_\#$ by Proposition 7. Then Lemma 59(ii) implies that $(\underline{\varphi})_\# \leq \varphi_\#$ pointwise, and we conclude that

$$(\underline{\varphi})_\#(x,t) = \inf \{\varphi_\#(x,t) : \varphi \in \mathscr{W}_{m,M,\varphi_0}\} \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+$$

(with an equality because $\underline{\varphi} \in \mathscr{W}_{m,M,\varphi_0}$). The proof is complete. □

Proof of Corollary 37. Fix $m < M$. By what precedes, the solution semigroup G_t of (21) is a strongly continuous semigroup of continuous operators on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ and satisfies (30). Let now H_t be another arbitrary such semigroup satisfying (30), i.e. such that

$$\int_{\mathbb{R}^d} |S_t u_0 - S_t v_0| \varphi_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| H_t \varphi_0 \, \mathrm{d}x,$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d, [m, M])$, $0 \leq \varphi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$, and $t \geq 0$. We have to prove that for any such φ_0 and t,

$$G_t \varphi_0 \leq H_t \varphi_0.$$

First the minimal solution of (21) is the unique continuous solution, that is

$$\underline{\varphi}(x,t) = G_t \varphi_0(x) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+.$$

Moreover, the above assumption on H_t implies that

$$H_t \varphi_0(x) \in \mathscr{W}_{m,M,\varphi_0}.$$

By Corollary 25 we deduce that for any $x \in \mathbb{R}^d$ and $t \geq 0$,

$$(G_t \varphi_0)_\# (x) \leq (H_t \varphi_0)_\# (x),$$

where we recall that

$$(G_t\varphi_0)_\#(x) = \liminf_{\substack{r\to 0^+\\ y\to x}} \frac{1}{\text{meas}(B_r(y))} \int_{B_r(y)} G_t\varphi_0(z)\,\mathrm{d}z$$

(and similarly for H). Since both $G_t\varphi_0(x)$ and $H_t\varphi_0(x)$ are continuous in x, we have $(G_t\varphi_0)_\# = G_t\varphi_0$ and $(H_t\varphi_0)_\# = H_t\varphi_0$ pointwise and the proof is complete. □

4.5. L^∞_{int} versus L^1: proofs of Propositions 29, 30, and Theorem 33

Recall that these results justify the use of L^∞_{int} for (2), instead of L^1. We need a result on the profile $U(r) = c_0 \int_r^\infty \mathrm{e}^{-\frac{s^2}{4}}\,\mathrm{d}s$ with c_0 such that $U(0) = 1$.

Lemma 60. *For any $(x,t) \in \mathbb{R}\times\mathbb{R}^+$, let*

$$\psi(x,t) := \begin{cases} U\left(|x|/\sqrt{t}\right) & \text{if } t>0,\\ \mathbf{1}_{\{0\}}(x) & \text{if } t=0.\end{cases}$$

Then $\psi \in BUSC(\mathbb{R}\times\mathbb{R}^+)$ and is a subsolution of (24).

Proof. Let us prove that ψ is a subsolution of (24). In the domain $\{x\neq 0, t>0\}$, we find as in the proof of Lemma 49 that

$$\partial_t\psi = \partial^2_{xx}\psi = (\partial^2_{xx}\psi)^+$$

in the classical sense. If now $x = 0$, we have

$$\partial_t\psi(0,\cdot) = 0 \le (\partial^2_{xx}\psi(0,\cdot))^+$$

since $\psi(0,\cdot)$ is constant in time. Let us now show that ψ is $BUSC$. It is clearly continuous for positive t and it only remains to prove that

$$\mathbf{1}_{x=0} \ge \limsup_{\mathbb{R}\times\mathbb{R}^+\ni(y,t)\to(x,0)} U\left(|y|/\sqrt{t}\right),$$

for any $x\in\mathbb{R}$. If $x = 0$, the result follows since $U(r)\le U(0)=1$ for any $r\ge 0$. If $x\neq 0$, then we use that

$$|y|/\sqrt{t}\to\infty \quad \text{as } (y,t)\to(x,0^+)$$

together with the fact that $\lim_{r\to\infty} U(r) = 0$. The proof of Lemma 60 is now complete. □

Proof of Proposition 29. Theorem 35 implies the if-part. Let us prove the only-if-part. It is based on the following pointwise lower bound:

$$\varphi(x,t) \ge U\left(1/\sqrt{t}\right) \sup_{x+[-1,1]} \varphi_0 \quad \forall x\in\mathbb{R}, \forall t>0, \tag{47}$$

where U is the profile from the previous lemma, $0\le\varphi_0\in C_b(\mathbb{R})$ and φ is the solution of (24) with φ_0 as initial data. Let us prove (47). Fix x and t. The sup on the right-hand side is attained at some $x_0\in x+[-1,1]$. By the previous lemma,

$$(y,s)\mapsto \varphi_0(x_0)U\left(|y-x_0|/\sqrt{s}\right)$$

is a $BUSC$ subsolution of (24). At $s = 0$, it equals the function

$$y \mapsto \varphi_0(x_0)\mathbf{1}_{\{x_0\}}(y)$$

which is less or equal to $\varphi_0 = \varphi_0(y)$. By the comparison principle (Theorem 3),

$$\varphi(y, s) \geq \varphi_0(x_0)U\left(|y - x_0|/\sqrt{s}\right) \quad \forall y \in \mathbb{R}, \forall s > 0.$$

Taking $(y, s) = (x, t)$, we then get that

$$\varphi(x, t) \geq \underbrace{\varphi_0(x_0)}_{=\sup_{x+[-1,1]}\varphi_0} \underbrace{U\left(|x - x_0|/\sqrt{t}\right)}_{\geq U(1/\sqrt{t})}.$$

This completes the proof of (47). From that bound the only-if-part of Proposition 29 is obvious since $U(1/\sqrt{t})$ is positive for $t > 0$. $\square$

Proof of Proposition 30. Let $x_0 \in \mathbb{R}$ and $c > 0$ be such that

$$\rho \geq c\mathbf{1}_{\{x_0\}},$$

where ρ is defined in (25), and define

$$\psi_n(x, t) := nc\psi\left(nx - x_0, n^2 t\right),$$

where ψ is given by Lemma 60. It is easy to see that ψ_n remains a subsolution of (24). Moreover, it is $BUSC$ with

$$\varphi_n(x, 0) \geq \psi_n(x, 0) \quad \forall x \in \mathbb{R},$$

by (25). Hence $\varphi_n \geq \psi_n$ by the comparison principle and it suffices to show that

$$\lim_{n\to\infty} \psi_n(x, t) = \infty \quad \forall x \in \mathbb{R}, \forall t > 0.$$

But this is quite easy because

$$\psi_n(x, t) = ncU\left(\left|x - \frac{x_0}{n}\right|/\sqrt{t}\right),$$

for any $x \in \mathbb{R}$ and $t > 0$, and both the constant c and the profile $U(\cdot)$ are positive. The proof of Proposition 30 is complete. $\square$

To show Theorem 33, we need the following lemma whose proof is elementary and left to the reader.

Lemma 61. *For any* $\varphi_0 : \mathbb{R}^d \to \mathbb{R}^d$, $\sup|\varphi_0| \leq |\sup\varphi_0| + |\inf\varphi_0|$.

Proof of Theorem 33. The fact that $E = C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ satisfies (27)–(28) follows from Theorem 16 and Corollary 36. Let now E be another normed space satisfying such properties and let us prove that it is continuously embedded into $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$. Recall that (28) is required to hold for any data $b = b(\xi)$ and $a = a(\xi)$ satisfying (H2). Choose e.g. the eikonal equation

$$\partial_t \varphi = \sum_{i=1}^{d} |\partial_{x_i} \varphi|$$

and denote by G_t^e its semigroup. By the representation Proposition 9,

$$G_t^e \varphi_0(x) = \sup_{x+t[-1,1]^d} \varphi_0.$$

Since $G_{t=1}^e$ maps $E \subseteq C_b \cap L^1(\mathbb{R}^d)$ into $X = \overline{E}^{\|\cdot\|_E} \subseteq L^1(\mathbb{R}^d)$ by assumption, the function

$$x \mapsto \sup_{x+[-1,1]^d} \varphi_0$$

belongs to $L^1(\mathbb{R}^d)$ for any $\varphi_0 \in E$. Using that E is a vector space, $-\varphi_0 \in E$, and the function

$$x \mapsto \inf_{x+[-1,1]^d} \varphi_0$$

also belongs to $L^1(\mathbb{R}^d)$. By Lemma 61, we conclude that $E \subseteq C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$. Finally we use that $G_{t=1}^e : E \to X$ is continuous at $\varphi_0 \equiv 0$ to obtain that for any $\|\varphi_0^n\|_E \to 0$, as $n \to \infty$, we have $G_{t=1}^e \varphi_0^n \to 0$ in X. Combining this with the continuity of the inclusion $X \subseteq L^1(\mathbb{R}^d)$, we obtain that

$$\left\| \sup_{x+[-1,1]^d} \varphi_0^n \right\|_{L^1_x} \to 0.$$

Using once again that E is a normed space, the same holds with $-\varphi_0$, that is

$$\left\| \inf_{x+[-1,1]^d} \varphi_0^n \right\|_{L^1_x} \to 0.$$

By Lemma 61, we conclude that $\|\varphi_0^n\|_{\text{int}} \to 0$ which completes the proof. □

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Appendix A. Complementary proofs for viscosity solutions

A.1. Minimal viscosity solutions

Here are the proofs of Theorem 6 and Propositions 7 and 45; the ideas are inspired by [26,10,33] and the details are given for completeness.

Proof of Theorem 6. Consider the infconvolution $(\varphi_0)_\varepsilon$ as in (14), which is at least C_b with $\inf \varphi_0 \leq (\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \sup \varphi_0$, and

$$\lim_{\varepsilon \downarrow 0} \uparrow (\varphi_0)_\varepsilon = \sup_{\varepsilon > 0} (\varphi_0)_\varepsilon = (\varphi_0)_*,$$

see e.g. [24,31,5,4]. Let φ_ε be the viscosity solution of (2a) with initial data $(\varphi_0)_\varepsilon$, whose well-posedness is ensured by Theorem 4. By the maximum principle, see Remark 5, we have the bounds

$$\inf \varphi_0 \leq \varphi_\varepsilon \leq \sup \varphi_0.$$

We can then define the real-valued and bounded function

$$\underline{\varphi} := \sup_{\varepsilon>0} \varphi_\varepsilon.$$

We will see that this is our desired minimal solution.

The key step is to prove that

$$\underline{\varphi} = \sup_{\varepsilon>0} \varphi_\varepsilon = \liminf{}_* \varphi_\varepsilon \tag{A.1}$$

where the relaxed limit is taken as $\varepsilon \to 0^+$. This follows by elementary arguments (see e.g. [5,4]) since φ^ε is at least lower semicontinuous and nondecreases as $\varepsilon \downarrow 0$, which follows by comparison since $(\varphi_0)^\varepsilon$ nondecreases as $\varepsilon \downarrow 0$. Let us give details for the reader's convenience. For any fixed (x,t),

$$\liminf{}_* \varphi_\varepsilon(x,t) \leq \lim_{\varepsilon\to 0^+} \varphi_\varepsilon(x,t) = \underline{\varphi}(x,t).$$

Moreover, for any sequence $(x_n, t_n, \varepsilon_n) \to (x, t, 0^+)$ such that $\varepsilon_n \leq \varepsilon_m$ for any $n \geq m$, we have $\varphi_{\varepsilon_n}(x_n,t_n) \geq \varphi_{\varepsilon_m}(x_n,t_n)$. Fixing m and taking the limit in n,

$$\liminf_{n\to\infty} \varphi_{\varepsilon_n}(x_n,t_n) \geq \liminf_{n\to\infty} \varphi_{\varepsilon_m}(x_n,t_n) \geq \varphi_{\varepsilon_m}(x,t)$$

by lower semicontinuity of φ_{ε_m}. Taking the limit in m,

$$\liminf_{n\to\infty} \varphi_{\varepsilon_n}(x_n,t_n) \geq \lim_{m\to\infty} \varphi_{\varepsilon_m}(x,t) = \underline{\varphi}(x,t).$$

This proves (A.1).

By stability by sup (Proposition 40), $\underline{\varphi}$ is a subsolution of (2a), and by stability by relaxed limit (Proposition 41), $\underline{\varphi}$ is a supersolution of (2a). To pass to the limit in the initial data, use Proposition 43 to infer that

$$(\underline{\varphi})^*(x, t=0) \leq \limsup{}^* \varphi_\varepsilon(x,0) = \limsup{}^* \left[\varphi_\varepsilon(\cdot,0)\right](x) \leq (\varphi_0)^*(x)$$

(the first relaxed limit as $\varepsilon \to 0^+$ is in (x,t) and the second in x). This gives the inequality of subsolution as in Definition 1(aii). For the other inequality, use that $\underline{\varphi}$ is lower semicontinuous, as a sup of continuous functions, with

$$\underline{\varphi}(x, t=0) = \sup_{\varepsilon>0} \varphi_\varepsilon(x,0) = (\varphi_0)_*(x).$$

This proves that $\underline{\varphi}$ is a solution of (2). It only remains to prove that it is minimal. Let φ be another bounded discontinuous solution. Noting that

$$(\varphi_0)_\varepsilon \leq (\varphi_0)_* \leq \varphi_*(t=0),$$

we use once more the comparison principle to deduce that $\varphi_\varepsilon \leq \varphi$, for any $\varepsilon > 0$, so $\underline{\varphi} \leq \varphi$ as $\varepsilon \to 0^+$. □

Proof of Proposition 7. We argue as in the end of the proof of Theorem 6: Assume φ is a bounded supersolution of (2), then $(\varphi_0)_\varepsilon \le (\varphi_0)_* \le \varphi_*(t=0)$ and, by comparison, $\varphi_\varepsilon \le \varphi$, etc. □

Proof of Proposition 45. Let $\underline{\varphi}$ denote the minimal solution of (2) with initial data $\varphi_0 := \sup_n (\varphi_0^n)_*$. We have to prove that $\underline{\varphi} = \sup_n \underline{\varphi}_n$, where $\underline{\varphi}_n$ is the minimal solution of (2) with initial data φ_0^n. By Proposition 7, we have $\underline{\varphi}_n \le \underline{\varphi}$ for any integer n. We thus already know that $\underline{\varphi} \ge \sup_n \underline{\varphi}_n$ and it only remains to prove the other inequality. To do so, it suffices to show that $\sup_n \underline{\varphi}_n$ is a supersolution of (2) (with initial data φ_0). Indeed, by Proposition 7, we then get $\underline{\varphi} \le \sup_n \underline{\varphi}_n$. It is at this stage that we need to use monotonicity. Recall that $n \mapsto \varphi_0^n(x)$ is nondecreasing for any x. By the comparison principle, cf. Remark 8, the same monotonicity holds for the minimal solutions which means that $n \mapsto \underline{\varphi}_n(x,t)$ is nondecreasing for any fixed x and t. Since $\underline{\varphi}_n$ is lower semicontinuous, we can argue as for (A.1) and get that

$$\sup_n \underline{\varphi}_n = \liminf{}_* \underline{\varphi}_n,$$

where the above relaxed limit is taken as $n \to \infty$. By stability, see Propositions 41 and 43, we deduce that $\liminf_* \underline{\varphi}_n$ is a supersolution of (2a) with initial data

$$\liminf{}_* \underline{\varphi}_n(t=0) = \liminf{}_* (\varphi_0^n)_*.$$

But this initial data is precisely

$$\liminf{}_* (\varphi_0^n)_* = \sup_n (\varphi_0^n)_* = \varphi_0,$$

again by similar arguments than for (A.1). This completes the proof. □

A.2. Representation formulas

Let us prove Propositions 9 and 10. These results are classical in control theory, but usually written for continuous or maximal solutions, see [31,4,34,35]. Here we give the proofs for minimal solutions.

Proof of Proposition 9. By the assumption that $a \equiv 0$, (2a) is now

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \{ b(\xi) \cdot D\varphi \} = \sup_{q \in \mathcal{C}} \{ q \cdot D\varphi \},$$

where $\mathcal{C} = \overline{\mathrm{co}\,\{\mathrm{Im}(b)\}}$ is compact. By control theory [5,4] the viscosity solutions of (2) are given by

$$\varphi(x,t) = \sup_{x+t\mathcal{C}} \varphi_0$$

if φ_0 is bounded and uniformly continuous. In the general case, consider the infconvolution (14). Recall that $(\varphi_0)_\varepsilon$ is at least bounded and uniformly continuous, and $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ pointwise as $\varepsilon \downarrow 0$. It follows that the solution of (2a) with $(\varphi_0)_\varepsilon$ as initial data is

$$\varphi_\varepsilon(x,t) = \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon.$$

By Proposition 45, the minimal solution of (2) is thus

$$\underline{\varphi}(x,t) = \sup_{\varepsilon>0} \varphi_\varepsilon(x,t) = \sup_{\varepsilon>0} \sup_{x+t\mathcal{C}} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} \sup_{\varepsilon>0} (\varphi_0)_\varepsilon = \sup_{x+t\mathcal{C}} (\varphi_0)_*.$$

Rigorously speaking, Proposition 45 implies that this is the minimal solution with initial data $(\varphi_0)_*$, but it coincides with the minimal solution associated to φ_0 by Proposition 7. $\square$

Proof of Proposition 10. Equation (2a) is given by

$$\partial_t \varphi = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\varphi + \operatorname{tr}\left(\sigma^a(\xi)(\sigma^a)^{\mathrm{T}}(\xi) D^2\varphi\right) \right\},$$

where $\mathcal{E}$ is compact and the coefficients b and σ^a are continuous by (19). By stochastic control theory [31], the viscosity solution of (2) is given by

$$\varphi(x,t) = \sup_{\boldsymbol{\xi}_\cdot \in \Xi} \mathbb{E}\left\{\varphi_0(\boldsymbol{X}_t^x)\right\}$$

if φ_0 is bounded and uniformly continuous, where Ξ and $\boldsymbol{X}_s^x$ are defined in Proposition 10. Let us now repeat the argument of the proof of Proposition 9 considering the infconvolution $(\varphi_0)_\varepsilon$ and the corresponding solution of (2a)

$$\varphi_\varepsilon(x,t) = \sup_{\boldsymbol{\xi}_\cdot \in \Xi} \mathbb{E}\left\{(\varphi_0)_\varepsilon(\boldsymbol{X}_t^x)\right\}.$$

We find that the minimal solution of (2) is

$$\underline{\varphi}(x,t) = \sup_{\varepsilon>0} \varphi_\varepsilon(x,t) = \sup_{\boldsymbol{\xi}_\cdot \in \Xi} \sup_{\varepsilon>0} \mathbb{E}\left\{(\varphi_0)_\varepsilon(\boldsymbol{X}_t^x)\right\}.$$

Since $(\varphi_0)_\varepsilon \uparrow (\varphi_0)_*$ as $\varepsilon \downarrow 0$, we conclude the proof using the monotone convergence theorem:

$$\begin{aligned}&\sup_{\varepsilon>0} \mathbb{E}\left\{(\varphi_0)_\varepsilon(\boldsymbol{X}_t^x)\right\}\\ &= \lim_{\varepsilon\downarrow 0}\uparrow \mathbb{E}\left\{(\varphi_0)_\varepsilon(\boldsymbol{X}_t^x)\right\} = \mathbb{E}\left\{\lim_{\varepsilon\downarrow 0}\uparrow(\varphi_0)_\varepsilon(\boldsymbol{X}_t^x)\right\} = \mathbb{E}\left\{(\varphi_0)_*(\boldsymbol{X}_t^x)\right\}. \quad \square\end{aligned}$$

Appendix B. Complementary proofs for entropy solutions

For completeness, we recall the proof of Theorem 13 which is Theorem 1.1 in [32] under (H1). We will take the opportunity to give details, but we will not perform the doubling of variables to show Lemma 15 for which we will refer to [11].

Recall that [22,11] proved the well-posedness of L^1 kinetic or renormalized solutions which are equivalent to entropy solutions in $L^1 \cap L^\infty$. The definition of entropy solutions in $L^1 \cap L^\infty$ uses the energy estimate (2.8) of [22],

$$\iint_{\mathbb{R}^d \times \mathbb{R}^+} \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} \zeta_{ik}(u)\right)^2 \mathrm{d}x\, \mathrm{d}t \le \frac{1}{2}\|u_0\|_{L^2} < \infty \quad \text{if } u_0 \in L^1 \cap L^\infty,$$

where $\zeta_{ik}(u) = \int_0^u \sigma_{ik}^A(\xi)\,\mathrm{d}\xi$. As a consequence "$L^2$" was used e.g. in [11, Definition 2.2] instead of "L^2_{loc}" in Definition 12. But we have the following result:

Lemma 62 *(Local energy estimate). Assume (H1), $u_0 \in L^\infty(\mathbb{R}^d)$, $0 \le \phi \in C_c^\infty(\mathbb{R}^d)$, and $T \ge 0$. If u is an entropy solution of (1) in the sense of Definition 12 and*

$$\|u_0\|_{L^\infty} + \|u\|_{L^\infty} + \|\phi\|_{W^{2,1}} \le M,$$

then there is a constant C only depending on T, M, F and A such that

$$\iint\limits_{\mathbb{R}^d\times(0,T)} \sum_{k=1}^{K}\left(\sum_{i=1}^{d}\partial_{x_i}\zeta_{ik}(u(x,t))\right)^2 \phi(x)\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Proof. We use Definition 12(c) with the entropy $\eta(u)=|u|^2$ and the corresponding fluxes

$$q(u)=2\int_0^u \xi F'(\xi)\,\mathrm{d}\xi \quad \text{and} \quad r(u)=2\int_0^u \xi A(\xi)\,\mathrm{d}\xi.$$

We also take a test function $\phi(x)\mathbf{1}_{[0,T]}(t)$ where $0\le\phi\in C_c^\infty(\mathbb{R}^d)$. It is not smooth in time but a standard approximation argument shows that it can be used in Definition 12(c) if we add also a final value term at $t=T$. Here we need the L^1_{loc} continuity in time of entropy solutions. The result is

$$\overbrace{\int\limits_{\mathbb{R}^d} u^2(x,T)\phi(x)\,\mathrm{d}x}^{\ge 0} + 2\iint\limits_{\mathbb{R}^d\times(0,T)} \sum_{k=1}^{K}\left(\sum_{i=1}^{d}\partial_{x_i}\zeta_{ik}(u)\right)^2\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int\limits_{\mathbb{R}^d} u_0^2(x)\phi(x)\,\mathrm{d}x + \iint\limits_{\mathbb{R}^d\times(0,T)}\left(\sum_{i=1}^{d} q_i(u)\partial_{x_i}\phi + \sum_{i,j=1}^{d} r_{ij}(u)\partial^2_{x_ix_j}\phi\right)\mathrm{d}x\,\mathrm{d}t.$$

By assumption $\|u_0\|_{L^\infty}+\|u\|_{L^\infty}+\|\phi\|_{W^{2,1}}\le M$, so it follows that

$$\begin{cases}\|q(u)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^+,\mathbb{R}^d)} \le 2M^2 \operatorname{ess\,sup}_{-M\le\xi\le M}|F'(\xi)|, \text{ and}\\ \|r(u)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^+,\mathbb{R}^{d\times d})} \le 2M^2 \operatorname{ess\,sup}_{-M\le\xi\le M}|A(\xi)|.\end{cases}$$

With all these estimates, the conclusion readily follows. □

Let us now give precise references on how to show the Kato inequality.

Sketch of the proof of Lemma 15. Copy the proof of Theorem 3.1 of [11] with $l=\infty$ and zero renormalization measures $\mu_l^u\equiv 0\equiv\mu_l^v$. With the aid of the previous local energy estimate, check that every computation holds until (3.19) – even if u and v satisfy (a)–(b) of Definition 12 with L^2_{loc} and not L^2 as in [11]. This gives (20) with $\phi\in C_c^\infty(\mathbb{R}^d\times(0,\infty))$. Use an approximation argument for $\phi(x,t)\mathbf{1}_{[0,T]}(t)$ and the continuity in time with values in L^1_{loc} to get initial and final terms. □

To show the uniqueness of entropy solutions, it suffices to find a *good* ϕ in (20), e.g. an exponential as in [21,32]. This gives the result below.

Lemma 63. *Assume (H1) and u,v are L^∞ entropy solutions of (1) with initial data $u_0,v_0\in L^\infty(\mathbb{R}^d)$. Then for any $t\ge 0$ and $m<M$ such that u and v take their values in $[m,M]$,*

$$\int\limits_{\mathbb{R}^d}|u-v|(x,t)\mathrm{e}^{-|x|}\,\mathrm{d}x \le \mathrm{e}^{(L_F+L_A)t}\int\limits_{\mathbb{R}^d}|u_0-v_0|(x)\mathrm{e}^{-|x|}\,\mathrm{d}x,$$

where $L_F=\operatorname{ess\,sup}_{[m,M]}|F'|$ and $L_A=\operatorname{ess\,sup}_{[m,M]}\operatorname{tr}(A)$.

Remark 64. By the maximum principle, the result remains true for any $[m, M]$ containing the values u_0 and v_0. But at this stage of this appendix, this principle is only known in $L^1 \cap L^\infty$ (or L^1) by [22,11] and it will follow later in L^∞.

Sketch of the proof. The proof is inspired by [21,32]. Consider

$$\phi_\varepsilon(x,t) := \mathrm{e}^{(L_F+L_A)(T-t)-\sqrt{\varepsilon^2+|x|^2}},$$

for some arbitrary $\varepsilon > 0$, and check that

$$|u-v|\partial_t\phi_\varepsilon + \sum_{i=1}^d q_i(u,v)\partial_{x_i}\phi_\varepsilon + \sum_{i,j=1}^d r_{ij}(u,v)\partial^2_{x_ix_j}\phi_\varepsilon \le |u-v|\left\{\partial_t\phi_\varepsilon + L_F|D\phi_\varepsilon| + L_A \sup_{\lambda\in \mathrm{Sp}(D^2\phi_\varepsilon)} \lambda^+\right\} \le 0$$

by the Ky Fan inequality (33). Then by the Kato inequality (20) with ϕ_ε,

$$\int_{\mathbb{R}^d} |u-v|(x,T)\mathrm{e}^{-\sqrt{\varepsilon^2+|x|^2}}\,\mathrm{d}x \le \mathrm{e}^{(L_F+L_A)T}\int_{\mathbb{R}^d} |u_0-v_0|(x)\mathrm{e}^{-\sqrt{\varepsilon^2+|x|^2}}\,\mathrm{d}x$$

and the result follows in the limit $\varepsilon \to 0^+$. □

Proof of Theorem 13. By Lemma 63, it remains to show the existence. The proof is inspired by [22,11]. Given $u_0 \in L^\infty(\mathbb{R}^d)$, take $(u_0^n)_n$ in $L^1 \cap L^\infty(\mathbb{R}^d)$ such that

$$-\operatorname{ess\,sup} u_0^- \le u_0^n \le \operatorname{ess\,sup} u_0^+ \quad \text{and} \quad u_0^n \to u_0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d). \tag{B.1}$$

Let u_n be the entropy solution of (1) with initial data u_0^n. By the maximum principle (in $L^1 \cap L^\infty$), we know that

$$-\operatorname{ess\,sup} u_0^- \le u_n \le \operatorname{ess\,sup} u_0^+. \tag{B.2}$$

Moreover, by Lemma 63, we have for any $R \ge 0$, $T \ge 0$, and integers n, m,

$$\begin{aligned}
&\|u_m - u_n\|_{C([0,T];L^1(\{|x|<R\}))} \\
&= \sup_{t\in[0,T]} \int_{|x|<R} |u_m(x,t) - u_n(x,t)|\,\mathrm{d}x \\
&\le \mathrm{e}^R \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |u_m(x,t)-u_n(x,t)|\mathrm{e}^{-|x|}\,\mathrm{d}x \\
&\le \mathrm{e}^R\mathrm{e}^{(L_F+L_A)T} \int_{\mathbb{R}^d} |u_0^m(x) - u_0^n(x)|\mathrm{e}^{-|x|}\,\mathrm{d}x,
\end{aligned}$$

where the latter integral tends to zero as $n, m \to \infty$ by (B.1). Hence there exists some $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+) \cap C(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ such that

$$\lim_{n\to\infty} u_n = u \quad \text{in } C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d)), \quad \forall T \ge 0. \tag{B.3}$$

It remains to show that u is an entropy solution with initial data u_0.

We have to derive the L^2_{loc} energy estimate of Definition 12(a), and check that it is enough to pass to the limit in the equation as in [22,11]. By Lemma 62 and the L^∞ bounds in (B.2), the sequence

$$\left\{\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u_n)\right\} \subset L^2(\mathbb{R}^d \times \mathbb{R}^+)$$

is uniformly bounded in $L^2(\mathcal{K})$, for any $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$. It then weakly converges in $L^2(\mathcal{K})$ to $\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u)$. We can identify the limit because $\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u_n)$ also converges to $\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u)$ in the distribution sense. Indeed

$$\zeta_{ik}(\cdot) = \int_0^{\cdot} \sigma^A_{ik}(\xi)\,\mathrm{d}\xi$$

is locally Lipschitz continuous since $\sigma^A_{ik}(\cdot)$ is locally bounded, and (B.2) and (B.3) imply that $\zeta_{ik}(u_n) \to \zeta_{ik}(u)$ in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$ for all $T \geq 0$. And as claimed previously, all corresponding derivatives necessarily converge in the distribution sense. The proof of part (a) in Definition 12 is complete. Moreover we have found that

$$\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u_n) \rightharpoonup \sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u) \quad \text{in } L^2(\mathcal{K}),$$

for any $k = 1, \dots, K$ and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

To show the chain rule in part (b) of Definition 12, we start from the chain rule for u_n,

$$\sum_{i=1}^{d} \partial_{x_i}\zeta^{\beta}_{ik}(u_n) = \beta(u_n)\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u_n) \in L^2(\mathbb{R}^d \times \mathbb{R}^+), \tag{B.4}$$

valid for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and integer n. Recall also that

$$\zeta^{\beta}_{ik}(u_n) = \int_0^{u_n} \sigma^A_{ik}(\xi)\beta(\xi)\,\mathrm{d}\xi.$$

By the previous convergence results and bounds, the right-hand side of (B.4) converges weakly in $L^2(\mathcal{K})$ to $\beta(u)\sum_{i=1}^{d} \partial_{x_i}\zeta_{ik}(u)$. We can argue as before to show that the left-hand side converges weakly in $L^2(\mathcal{K})$ to $\sum_{i=1}^{d} \partial_{x_i}\zeta^{\beta}_{ik}(u)$. We thus get part (b) of Definition 12 in the limit. Moreover,

$$\sum_{i=1}^{d} \partial_{x_i}\zeta^{\beta}_{ik}(u_n) \rightharpoonup \sum_{i=1}^{d} \partial_{x_i}\zeta^{\beta}_{ik}(u) \quad \text{in } L^2(\mathcal{K}), \tag{B.5}$$

for any $\beta \in C(\mathbb{R})$, $k = 1, \dots, K$, and compact $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}^+$.

Now, it remains to prove part (c) of Definition 12. The only difference with [22,11] is that the previous convergences hold locally in L^2 and not globally. But since we use test functions, the reasoning is the same. Let us recall it for completeness. We focus on the quadratic term. Take $\beta = \sqrt{\eta''}$ and apply the chain rule Definition 12(b),

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \eta''(u_n)\sum_{k=1}^K\left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}(u_n)\right)^2\phi\,\mathrm{d}x\,\mathrm{d}t\\
&=\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \eta''(u_n)\sum_{k=1}^K\left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}(u_n)\right)\left(\sum_{j=1}^d \partial_{x_j}\zeta_{jk}(u_n)\right)\phi\,\mathrm{d}x\,\mathrm{d}t\\
&=\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \sum_{k=1}^K\left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u_n)\right)\left(\sum_{j=1}^d \partial_{x_j}\zeta_{jk}^{\sqrt{\eta''}}(u_n)\right)\phi\,\mathrm{d}x\,\mathrm{d}t\\
&=\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \sum_{k=1}^K\left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u_n)\sqrt{\phi}\right)^2\mathrm{d}x\,\mathrm{d}t\\
&=\sum_{k=1}^K\left\|\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u_n)\sqrt{\phi}\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}.
\end{aligned}$$

But, by (B.5), we have for any $k=1,\dots,K$,

$$\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u_n)\sqrt{\phi} \rightharpoonup \sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u)\sqrt{\phi} \quad \text{in } L^2(\mathbb{R}^d\times\mathbb{R}^+).$$

It follows that

$$\left\|\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u)\sqrt{\phi}\right\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \le \liminf_{n\to\infty}\left\|\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u_n)\sqrt{\phi}\right\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)},$$

that is

$$\begin{aligned}
&\liminf_{n\to\infty}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \eta''(u_n)\sum_{k=1}^K\left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}(u_n)\right)^2\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\ge\sum_{k=1}^K\left\|\sum_{i=1}^d \partial_{x_i}\zeta_{ik}^{\sqrt{\eta''}}(u)\sqrt{\phi}\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}\\
&=\iint\limits_{\mathbb{R}^d\times\mathbb{R}^+} \eta''(u)\sum_{k=1}^K\left(\sum_{i=1}^d \partial_{x_i}\zeta_{ik}(u)\right)^2\phi\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where similar chain rule computations have been used for u. This is enough to pass to the limit in the entropy inequalities of Definition 12(b) and the proof is complete. □

As a byproduct of the previous proof, we get the lemma below.

Lemma 65. *Assume* (H1), *$u_0 \in L^\infty(\mathbb{R}^d)$, and u is the entropy solution of* (1). *Then* $\operatorname{ess\,inf} u_0 \le u \le \operatorname{ess\,sup} u_0$. *Moreover, if v is the entropy solution with initial data v_0, then $u_0 \ge v_0$ implies $u \ge v$.*

Proof. For the comparison principle, define $u_0^n(x) := u_0(x)\mathbf{1}_{|x|<n}$ and v_0^n similarly. As previously, the associated entropy solutions u_n and v_n respectively converge towards u and v in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^d))$, $T\ge 0$, and thus a.e. up to taking a (common) subsequence. If $u_0 \ge v_0$, then $u_0^n \ge v_0^n$ for all n, so $u_n \ge v_n$ by the

comparison principle in $L^1 \cap L^\infty$, and $u \geq v$ at the limit. For the maximum principle, apply the comparison principle to $v_0 := \operatorname{ess\,inf} u_0$ and $\operatorname{ess\,sup} u_0$ successively. □

Appendix C. Measurable weights and viscosity supersolutions

Let us provide for completeness a version of Theorem 23 for measurable and essentially bounded weights $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$. The result will involve a version of $\varphi_\#$ from (18) in both space and time. It is defined as

$$\varphi_{\#\#}(x,t) := \liminf_{\substack{r\to 0^+\\ y\to x\\ \mathbb{R}^+\ni s\to t}} \frac{1}{r\,\mathrm{meas}(B_r(y))} \iint\limits_{B_r(y)\times(s,s+r)} \varphi(z,\tau)\,\mathrm{d}z\,\mathrm{d}\tau. \tag{C.1}$$

Notably $\varphi_{\#\#} \in BLSC$ with $\varphi_{\#\#} \leq \varphi$ a.e., but we may not have $\varphi_{\#\#} = \varphi$ a.e. when $\varphi \notin BLSC$; cf. Remark 67.

Theorem 66 *(Measurable weights and supersolutions). Assume (H1), $m < M$, and $\varphi : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ is measurable, nonnegative, essentially bounded, and such that*

$$\int\limits_{\mathbb{R}^d} |u-v|(x,t)\varphi(x,s)\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^d} |u_0-v_0|(x)\varphi(x,t+s)\,\mathrm{d}x \quad \text{a.e. } t,s\geq 0, \tag{C.2}$$

for any u_0 and v_0 in $L^\infty(\mathbb{R}^d,[m,M])$ with respective associated entropy solutions u and v of (1). Then $\varphi_{\#\#}$ in (C.1) is a viscosity supersolution of (21a).

Remark 67. The reciprocal assertion may fail. A one dimensional example is $\varphi(x,t) = \mathbf{1}_E(x)$ with a fat Cantor set E (a closed nowhere dense set of positive measure). Indeed $\varphi_{\#\#} \equiv 0$[10] is always a solution of (21a), but choosing (1) as the heat equation $\partial_t u = \partial^2_{xx} u$, $u_0 = \mathbf{1}_{\mathbb{R}\setminus E}$, and $v_0 \equiv 0$, we cannot have (C.2) because the right-hand side is zero and the left-hand side is positive.

Remark 68 *(The optimal measurable weight is $\underline{\varphi}$).* Given in addition $0 \leq \varphi_0 \in BLSC(\mathbb{R}^d)$ such that

$$\varphi_0(x) \leq \varphi_{\#\#}(x,t=0) \quad \text{for all } x\in\mathbb{R}^d, \tag{C.3}$$

we have

$$\int\limits_{\mathbb{R}^d} |u-v|(x,t)\varphi_0(x)\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^d} |u_0-v_0|(x)\varphi(x,t)\,\mathrm{d}x \quad \text{a.e. } t\geq 0.$$

This is (22) with the merely measurable weight φ. Notably the minimal viscosity solution $\underline{\varphi}$ of (21) remains optimal within this class of weights satisfying (C.2) and (C.3), because $\underline{\varphi} \leq \varphi_{\#\#} \leq \varphi$ where the last inequality holds a.e.

Remark 69.

(a) Going back to $\varphi \in BLSC$ satisfying Theorem 23(I), and applying Theorems 23 and 66, we get two viscosity supersolutions $\varphi_\#$ and $\varphi_{\#\#}$ of (21a). This is however coherent because they actually represent the same supersolution. Indeed $\varphi_{\#\#}$ is nothing else than $\varphi_\flat$ in (41), and we have seen during the proof of (I) $\Longrightarrow$ (II) that $(\varphi_\#)_* = \varphi_\flat = \varphi_{\#\#}$ pointwise.

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[10] Use that $\varphi_{\#\#}$ is LSC, nonnegative, and equals zero in the dense open set $(\mathbb{R}\setminus E)\times\mathbb{R}^+$.

(b) For general $\varphi \in BLSC$, $(\varphi_\#)_* \leq \varphi_{\#\#}$ pointwise by Lemma 58 but the reverse inequality may fail. An example is $\varphi(x,t) = \mathbf{1}_{t \neq t_0}$ with some fixed t_0, which gives $(\varphi_\#)_* = \varphi$ and $\varphi_{\#\#} \equiv 1$.

We actually already proved the above theorem since $\varphi_{\#\#} = \varphi_\flat$ from (41). But let us give details for the reader's convenience.

Proof of Theorem 66. By "a.e." in (C.2), we assume having a null set $N \subset \mathbb{R}^+$ such that (C.2) holds for all $t, s \geq 0$ such that $s \notin N$ and $t + s \notin N$. Fix $r > 0$ and define

$$\varphi_r(x,t) := \frac{1}{r\,\mathrm{meas}(B_r(x))} \iint\limits_{B_r(x)\times(t,t+r)} \varphi(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

As for (40), it is easy to deduce from (C.2) that

$$\int\limits_{\mathbb{R}^d} |u-v|(x,t)\varphi(x-y,s-\tau)\,\mathrm{d}x \leq \int\limits_{\mathbb{R}^d} |u_0-v_0|(x)\varphi(x-y,t+s-\tau)\,\mathrm{d}x,$$

for all $y \in \mathbb{R}^d$, $t \geq 0$, and $s - \tau \geq 0$, such that $s - \tau \notin N$ and $t + s - \tau \notin N$. Fix $t, s \geq 0$, multiply by $\frac{1}{r\,\mathrm{meas}(B_r(x))}$ and integrate over $(y,\tau) \in B_r(0)\times(-r,0)$, which we can do excepted for $\tau \in (s-N)\cup(t+s-N)$. But the latter set is a null set, and this shows that φ_r satisfies (C.2) for all $t, s \geq 0$. Since moreover φ_r is continuous in (x,t), it is a viscosity supersolution of (21a) by Theorem 23 and so is $\varphi_{\#\#} = \liminf_* \varphi_r$ as $r \to 0^+$. □

Appendix D. Nonlinear to linear semigroups

In this section we give a sample result on how we from nonlinear duality can recover standard duality notions in the linear case. It contains the discussion and results mentioned in Remark 39(c) and the notation and setting is taken from Section 3.3. First note that $X = C_b \cap L^\infty_{\mathrm{int}}$ was a natural space for the weight semigroup G_t, but other X could be more appropriate if we consider other semigroups than S_t. Here are some reasonable assumptions which we will need:

$$X \neq \emptyset \text{ is a Banach space continuously embedded and dense in } L^1, \tag{D.1}$$

such that

$$\forall \varphi_0 \in L^1, \forall \psi_0 \in X, \quad |\varphi_0| \leq |\psi_0| \Rightarrow \big[\varphi_0 \in X \text{ and } \|\varphi_0\|_X \leq \|\psi_0\|_X\big], \tag{D.2}$$

(i.e. X is a Banach lattice) and for any mollifier ρ_ν (cf. (12)) and $\varphi_0 \in X$,

$$X \ni \rho_\nu * \varphi_0 \to \varphi_0 \text{ strongly in } X \text{ with } \|\rho_\nu * \varphi_0\|_X \leq \|\rho_\nu\|_X \|\varphi_0\|_{L^1}. \tag{D.3}$$

Note that hereafter $L^1 = L^1(\mathbb{R}^d)$ etc.

Proposition 70 *(Relation with standard duality). Take a weakly-⋆ continuous semigroup T_t of weakly-⋆ continuous linear operators on $L^\infty = (L^1)^\star$ such that each T_t is positive and commutes with translations.*[11] *Let $(T_t)_\star$ be its predual semigroup on L^1 defined by*

[11] That is $T\varphi_0 \geq 0$ if $\varphi_0 \geq 0$, and $T(\varphi_0(\cdot + h)) = (T\varphi_0)(\cdot + h)$ for all $\varphi_0 \in L^\infty$ and $h \in \mathbb{R}^d$.

$$\int_{\mathbb{R}^d} \varphi_0 T_t u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} u_0 (T_t)_\star \varphi_0 \, \mathrm{d}x, \quad \forall u_0 \in L^\infty, \forall \varphi_0 \in L^1, \forall t \geq 0. \tag{D.4}$$

Assume also that there exist X satisfying (D.1)–(D.2)–(D.3), a strongly continuous semigroup H_t of continuous operators on $X^+ := \{\varphi_0 \in X : \varphi_0 \geq 0\}$ satisfying

$$\int_{\mathbb{R}^d} |T_t u_0 - T_t v_0| \varphi_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| H_t \varphi_0 \, \mathrm{d}x, \quad \forall u_0, v_0 \in L^\infty, \forall \varphi_0 \in X^+, \forall t \geq 0, \tag{D.5}$$

and that H_t is the minimal such semigroup. Then $(T_t)_\star$ is necessarily the unique extension of H_t from X^+ onto L^1 as a semigroup of bounded linear operators.

Remark 71. For a general duality theory for linear semigroups, see [47]. Let us recall that (D.4) defines a strongly continuous semigroup $(T_t)_\star$ of bounded linear operators on L^1.[12] The semigroup H_t would be the new predual defined as in Remark 39(a), which would thus coincide with $(T_t)_\star$ in the linear case.

Proof. Take $u_0 \geq 0$, $v_0 \equiv 0$ and $\varphi_0 \geq 0$ in (D.5), to get

$$\int_{\mathbb{R}^d} u_0 (T_t)_\star \varphi_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \varphi_0 T_t u_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} u_0 H_t \varphi_0 \, \mathrm{d}x.$$

This shows that

$$(T_t)_\star \leq H_t \quad \text{on } X^+. \tag{D.6}$$

To continue, we claim that $(T_t)_\star$ is a strongly continuous semigroup of continuous operators on X^+ satisfying (D.5). Let us verify this claim. Let us prove that $(T_t)_\star$ satisfies (D.5), as H_t does. Since $T_t \geq 0$ and is linear,

$$|T_t u_0 - T_t v_0| = |T_t (u_0 - v_0)^+ - T_t (u_0 - v_0)^-| \leq T_t (u_0 - v_0)^+ + T_t (u_0 - v_0)^-$$

for any u_0 and v_0 in L^∞. Hence

$$\begin{aligned} \int_{\mathbb{R}^d} |T_t u_0 - T_t v_0| \varphi_0 \, \mathrm{d}x &\leq \int_{\mathbb{R}^d} \left(T_t (u_0 - v_0)^+ + T_t (u_0 - v_0)^- \right) \varphi_0 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \left((u_0 - v_0)^+ + (u_0 - v_0)^- \right) (T_t)_\star \varphi_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} |u_0 - v_0| (T_t)_\star \varphi_0 \, \mathrm{d}x, \end{aligned}$$

for any $\varphi_0 \in X^+$. To show next that $(T_t)_\star$ is bounded for $\| \cdot \|_X$, we use that $(T_t)_\star \geq 0$, the previous bound (D.6), the assumption (D.2), and the continuity of H_t for this norm. For the time continuity of $(T_t)_\star$, we regularize any $\varphi_0 \in X$ by convolution thanks to (D.3). Take $\varphi_0^\nu := \rho_\nu * \varphi_0 \to \varphi_0$ in X as $\nu \to 0^+$, and note that

$$\|\varphi_0 - (T_t)_\star \varphi_0\|_X \leq \|\varphi_0 - \varphi_0^\nu\|_X + \|\varphi_0^\nu - (T_t)_\star \varphi_0^\nu\|_X + \underbrace{\|(T_t)_\star (\varphi_0^\nu - \varphi_0)\|_X}_{\leq \sum_\pm \|H_t (\varphi_0^\nu - \varphi_0)^\pm\|_X \text{ by (D.6)}}. \tag{D.7}$$

[12] Indeed, given $\varphi_0 \in L^1$, $u_0 \in L^\infty \mapsto \int \varphi_0 T_t u_0$ is weakly-$\star$ continuous thus corresponding to a unique element $(T_t)_\star \varphi_0 \in L^1 \subset (L^\infty)^\star$. This operator $(T_t)_\star$ is bounded in L^1 since T_t is weakly-$\star$ continuous in L^∞, thus bounded. The semigroup $(T_t)_\star$ is weakly continuous thus strongly continuous.

Note also that $(T_t)_\star(\varphi_0 * \rho_\nu) = \rho_\nu * (T_t)_\star\varphi_0$ since $(T_t)_\star : L^1 \to L^1$ is linear, bounded, and commutes with translations. Hence

$$\|\varphi_0^\nu - (T_t)_\star\varphi_0^\nu\|_X = \|\rho_\nu * (\varphi_0 - (T_t)_\star\varphi_0)\|_X \leq \|\rho_\nu\|_X\|\varphi_0 - (T_t)_\star\varphi_0\|_{L^1}$$

by (D.3), and letting $t \to 0^+$ before $\nu \to 0^+$ in (D.7) implies that

$$\lim_{t\to 0^+} \|\varphi_0 - (T_t)_\star\varphi_0\|_X = 0$$

by the (time) strong continuity of $(T_t)_\star$ on L^1. This proves our claim, and we infer that $(T_t)_\star = H_t$ on X^+. The result follows by density. □

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