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Erlend Bergtun

Algebraic Methods in Analysis and Topology

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of
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Engineering
Department of Mathematical Sciences

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To Oskar

Abstract

This thesis consists of two parts. Part I concerns the classification of unital minimal A_∞-structures on graded vector spaces concentrated in degree 0, 1 and 2 up to equivalence. Here we give a complete description of every possible unital A_∞-structure on such vector spaces, and partial results on the question of equivalence between such structures.

We find an invariant of equivalent minimal A_∞-structures, namely we prove that the first non-zero multiplications have to be equal up to automorphism. In particular, the integer where this happens is a discrete invariant of an A_∞-structure, and it can take any value from 2 to ∞. We also give an example where this invariant gives a complete answer to the question of equivalence.

In part II we show how to solve partial differential equations in the representation coefficients of a projective unitary representation of a Lie group. To do this we build up a theory which allows us to differentiate the representation itself. We then apply this to the case when the Lie group has a left-invariant complex structure to get holomorphic representation coefficients.

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Contents

Chapter 1

Introduction

This thesis consists of two parts, in two distinct fields, algebraic topology and abstract harmonic analysis. In both parts our goal is to introduce more structure to a given problem in the hope that it helps solve the problem.

In the first part we study A_∞-structures on graded algebras concentrated in degree 0, 1 and 2. Such algebras arises in topology as the cohomology of link complements. Our hope is therefore that the extra structure of A_∞-algebras can help classify links.

In the second part we show how to solve partial differential equations in the representation coefficients of a projective unitary representation of a Lie group. We then apply this to the case when the Lie group has a left-invariant complex structure to get holomorphic representation coefficients. This is useful as it allows us to transform the arbitrary Hilbert space of the representation into a Hilbert space of holomorphic functions on the group, which usually has more structure than the Hilbert space we started with.

1.1 Introduction to Part I

The goal of this work was to classify all minimal A_∞-structures on cube zero algebras over some field K. These are graded algebras which are concentrated in degree 0, 1 and 2, where the degree 0 part is 1-dimensional. Such algebras arise naturally as the cohomology rings of link complements, and so our hope is that further down the line this work can be applied to the study of link complements. However, it turned out that the classification of A_∞-structures is already relatively complicated, and so we do not go further into such topological applications here.

Let us quickly introduce the objects of study. By a cube zero algebra we mean a unital and associative graded algebra $A = \{A^n\}_{n\in\mathbb{Z}}$ with $A^n = 0$ for all $n \neq 0, 1, 2$ and $A^0 = K$. The underlying vector space of such an algebra we call connected and 3-coconnected. Let us also denote by $\mu\colon A \otimes A \to A$ the multiplication of A. A minimal A_∞-structure on (A, μ) is a sequence of maps $m_n\colon A^{\otimes n} \to A$, for $n \geq 2$, such that $m_2 = \mu$ and the so called Stasheff relations

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes m_r \otimes \mathrm{I}^{\otimes q}\big) = 0 \tag{1.1}$$

are satisfied.

The first problem we then want to answer is the following. What are all the possible A_∞-structures we can put on A? This is answered by theorem 4.14, which one also finds in the literature in [20, lemma 3.1]. It is the following statement.

Theorem. *The set of unital A_∞-structures on a connected and 3-coconnected vector space A is in one-to-one correspondence with the set $\mathrm{Hom}_K\big(\overline{\mathrm{T}}A^1, A^2\big)$.*

Here $\overline{\mathrm{T}}A^1$ is the vector space $A^1 \oplus (A^1)^{\otimes 2} \oplus (A^1)^{\otimes 3} \oplus \cdots$. Specializing this result to the minimal A_∞-structures on the cube zero algebra A we get the following. There is a one-to-one correspondence between the set of minimal A_∞-structures on A and the set $\mathrm{Hom}_K\big(\overline{\mathrm{T}}^{\geqslant 3}A^1, A^2\big)$, where $\overline{\mathrm{T}}^{\geqslant 3}A^1 = (A^1)^{\otimes 3} \oplus (A^1)^{\otimes 4} \oplus \cdots$.

Next, we need to introduce the morphisms of A_∞-algebras. These are called ∞-morphisms, and for two A_∞-algebras A and R they are defined as a sequence of maps $f_n\colon A^{\otimes n} \to R$ which satisfy the relations

$$\sum_{p+q+r=n} (-1)^{p+qr} f_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes m_r^A \otimes \mathrm{I}^{\otimes q}\big) = \sum_{i_1+\ldots+i_r=n} (-1)^{\varepsilon} m_r^R(f_{i_1} \otimes \cdots \otimes f_{i_r}) \tag{1.2}$$

where $\varepsilon = (r-1)(i_1-1) + (r-2)(i_2-1) + \ldots + (i_{r-1}-1)$. When f_1 is a quasi-isomorphism we call the ∞-morphism an equivalence.

The second problem we then want to answer is when are two minimal A_∞-structures on a cube zero algebra equivalent. Here our main result is an invariant of minimal A_∞-structures. To state it we need to introduce the following notation. For a minimal A_∞-algebra (A, m) we write

$$c(A, m) \coloneqq \min\{n \mid m_n \neq 0\} \tag{1.3}$$

for the smallest integer $n \geq 2$ such that the n-th multiplication is non-zero. If A is the trivial A_∞-algebra, i.e. if $m_n = 0$ for all $n \in \mathbb{N}$, we set $c(A) = \infty$. Our main result is then theorem 4.13, which is the following statement.

Theorem. *Let A be a graded vector space with m and m' two minimal A_∞-structures on A.*

(1) *If (A, m) and (A, m') are equivalent, then $c(A, m) = c(A, m')$.*

(2) *If* (A, m) *and* (A, m') *are strongly equivalent, then in addition to* (1) *we moreover have that* $m_c = m'_c$*, where* $c = c(A, m) = c(A, m')$*.*

This result shows that the integer $c(A, m)$ is an invariant of minimal A_∞-algebras. In particular it follows that there are infinitely many equivalence classes of A_∞-structures on a cube zero algebra A.

Part I of this thesis is divided into three chapters. The first chapter is chapter 2 were we review terminology and notation to be used throughout the remaining chapters. Next, in chapter 3 we define Hochschild cohomology of graded algebras. We then apply this to connected cube zero algebras and calculate an example. Finally, in chapter 4 we define A_∞-algebras and the ∞-morphisms between them. It is here that we find our main result on equivalence of A_∞-algebras. At the end of the chapter we specialize to case of a connected and 3-coconnected vector space.

1.2 Introduction to Part II

The idea for this work came from the following classical result by Bargmann [2]. The short-time Fourier transform (STFT) of a signal $f \in L^2(\mathbb{R})$ with respect to a window function $g \in L^2(\mathbb{R})$ is defined as

$$V_g f(x, y) := \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i y t}\, dt. \tag{1.4}$$

Bargmann then shows, among other things, that if we choose the window function to be the Gaussian

$$g_0(t) = \sqrt[4]{2}e^{-\pi t^2}, \tag{1.5}$$

then

$$F(x+iy) = e^{\frac{\pi}{2}|x+iy|^2 - \pi i x y} V_{g_0} f(x, -y) \tag{1.6}$$

is an entire function for every signal f.

This gave me the following question. Is it possible to find g_0 by solving an appropriate Cauchy–Riemann equation? As this work will show, the answer is yes. Moreover, this approach lead to a general theory applicable to many more problems than my original question.

The setting for the general theory is the following. Given a Lie group G and a smooth projective unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, we will show how to solve systems of left-invariant linear homogeneous partial differential equations for the representation coefficients of π. Chapter 5, 6 and 7 are devoted to this, with chapter 7 containing the main result of our text. After that we specialize this result to the Cauchy–Riemann equations in chapter 8.

To state our main result here in the introduction recall that the functions $g \mapsto \langle \pi(g)u, v\rangle$ as $u, v \in \mathcal{H}$ vary are called the representation coefficients of π. Given a set $D_1, \ldots, D_m$ of left-invariant differential operators on G, we will then show how to find $u \in \mathcal{H}$ and $a\colon G \to \mathbb{C}^\times$ which solves the system of equations

$$\begin{cases} D_1\big(a(g)\langle \pi(g)u, v\rangle\big) = 0 \\ \quad\vdots \\ D_m\big(a(g)\langle \pi(g)u, v\rangle\big) = 0 \end{cases} \tag{1.7}$$

for all $v \in \mathcal{H}$. We call the maximum of the orders of $D_1, \ldots, D_m$ for the order of the system. Our main result is then theorem 7.1, which is the following statement.

Theorem. *Let G be a Lie group, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a smooth projective unitary representation. Let $D_1, \ldots, D_m \in U\mathfrak{g}_{\mathbb{C}}$ be a system of order d of left-invariant differential operators. For all k write $D_k = p_k(\partial_1, \ldots, \partial_n)$ for some non-commutative polynomial p_k. If there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $u \in C^d(\pi)$ such that*

$$\begin{cases} p_1(d\pi(X_1) - \lambda_1, \ldots, d\pi(X_n) - \lambda_n)u = 0 \\ \quad\vdots \\ p_m(d\pi(X_1) - \lambda_1, \ldots, d\pi(X_n) - \lambda_n)u = 0, \end{cases} \tag{1.8}$$

and $a\colon G \to \mathbb{C}^\times$ solves the system

$$\begin{cases} \partial_1 a + b_1 a = -\lambda_1 a \\ \quad\vdots \\ \partial_n a + b_n a = -\lambda_n a \end{cases} \tag{1.9}$$

of first-order partial differential equations, where $b_j(g) = \partial_j c_g(e)$ and c is the cocycle map of π, then

$$\begin{cases} D_1\big(a(g)\langle \pi(g)u, v\rangle\big) = 0 \\ \quad\vdots \\ D_m\big(a(g)\langle \pi(g)u, v\rangle\big) = 0 \end{cases} \tag{1.10}$$

for all $v \in \mathcal{H}$.

Specializing to the case of the Cauchy–Riemann equations we need to assume G is even dimensional and with a left-invariant complex structure on it. The main result is then theorem 8.2, which is the following statement.

Theorem. *Let G be a $2n$-dimensional Lie group with a left-invariant complex structure, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a smooth projective unitary representation. For $u \in C^1(\pi)$ and $a\colon G \to \mathbb{C}^\times$ a differentiable function, the functions $g \mapsto a(g)\langle v, \pi(g)u\rangle$ are holomorphic for all $v \in \mathcal{H}$ if and only if u is an eigenvector of $d\pi(Z_k)$ for all $k = 1, \ldots, n$, and a solves the system*

$$\begin{cases} \bar{\partial}_1 a + \overline{b_1} a = -\overline{\lambda_1} a \\ \qquad\vdots \\ \bar{\partial}_n a + \overline{b_n} a = -\overline{\lambda_n} a \end{cases} \tag{1.11}$$

of first-order partial differential equations, where $b_k(g) = \partial_k c_g(e)$, c is the cocycle map of π, and λ_k is the eigenvalue of u with respect to $d\pi(Z_k)$ for each $k = 1, \ldots, n$.

In chapter 9 we define and study generalized Bargmann-Fock spaces. Finally, in the last chapter we calculate some specific examples of the theory developed in chapter 8.

Part I

On the Classification of A_∞-Algebras

Chapter 2

Preliminaries

Our main goal is to study A_∞-structures on the cohomology of link complements. Since the cohomology ring of a link complement has the known structure of being concentrated in degree 0, 1 and 2, we will study every A_∞-structure we can put on such a graded vector space. We will call such graded vector spaces connected and 3-coconnected.

In this chapter we mainly review terminology and notation for use in the rest of part I. We will also recall some result that are useful to us. Let us start by fixing a ground field K for the rest of part I. We write I or id for the identity mapping.

2.1 Graded Vector Spaces

By a *graded vector space* we shall mean a sequence of vector spaces over K indexed by $\mathbb{Z}$. For notation we will use cohomological grading conventions and so for a graded vector space V we write V^n for the degree n part. For elements $x \in V$ we use the notation $|x|$ for the degree of x.

By the category of graded vector spaces we shall mean the enriched category where the hom-sets are not just sets, but are themselves graded vector spaces. Specifically for two graded vector spaces V and W their graded hom-vector space is defined as

$$\operatorname{Hom}^n_{\mathrm{gr}}(V, W) \coloneqq \prod_{p \in \mathbb{Z}} \operatorname{Hom}_K(V^p, W^{n+p}), \quad n \in \mathbb{Z}. \tag{2.1}$$

In particular any linear map between graded vector spaces has a well-defined degree.

When we speak of K as a graded vector space we mean the graded vector space which has a copy of K in degree zero and is trivial in all other degrees. In general we can make any ungraded vector space a graded vector space by concentrating it in degree zero.

2.1.1 The Koszul Sign Convention

The Koszul sign convention [19, sec. 1.5.3] is the convention to introduce a sign in the tensor product of graded vector spaces. Specifically for linear maps $f\colon V \to V'$ and $g\colon W \to W'$ of graded vector spaces we define $f \otimes g\colon V \otimes W \to V' \otimes W'$ as the linear map

$$(f \otimes g)(x \otimes y) \coloneqq (-1)^{|g||x|} f(x) \otimes g(y). \tag{2.2}$$

Together with the switch-map $V \otimes W \to W \otimes V$, $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$, this makes the category of graded vector spaces a symmetric monoidal category.

Note that if $f'\colon V'' \to V$, $f\colon V \to V'$, $g'\colon W'' \to W$ and $g\colon W \to W'$ are pairwise composable maps, then

$$(f \otimes g)(f' \otimes g') = (-1)^{|g||f'|} ff' \otimes gg'. \tag{2.3}$$

This showcases the heuristic of the Koszul sign convention, namely that whenever two symbols are permuted there is also an introduction of a sign determined by the degrees of the symbols. By a symbol we mean any object which has a well-defined degree, like an element of a graded vector space or a linear map.

2.1.2 Connected and Coconnected Vector Spaces

Let A be a graded vector space. We call A *connected* if $A^0 = K$ and $A^n = 0$ for all $n < 0$. Dually, for an integer $n \in \mathbb{Z}$ we call A *n-coconnected* if $A^m = 0$ for all $m \geq n$.

2.1.3 Chain Complexes

Sticking to our cohomological grading conventions a *chain complex* is defined as a graded vector space A together with a differential d on A of degree 1. By a *differential* on A we mean a linear endomorphism $d \in \mathrm{Hom}_{\mathrm{gr}}(A, A)$ which satisfies $d^2 = 0$.

If A and B are chain complexes we define their tensor product as the chain complex $(A \otimes B, d_A \otimes \mathrm{I} + \mathrm{I} \otimes d_B)$. Note that the fact that $d_A \otimes \mathrm{I} + \mathrm{I} \otimes d_B$ is a differential follows from the Koszul sign convention.

The cohomology of a chain complex A is as usual defined as the cocycles modulo the coboundaries. This is itself a graded vector space and we use the notation $H(A)$ to denote it as a graded object.

A *chain map* between two chain complexes A and B is a linear map $f\colon A \to B$ of degree zero which commutes with the differentials of A and B, i.e. $d_B f = f d_A$. Given two chain maps $f, g\colon A \to B$ a *chain homotopy* from f to g is a linear map $h\colon A \to B$ of degree -1 such that $hd_A + d_B h = f - g$.

2.2 Associative Graded Algebras

An *associative graded algebra* over K is a graded vector space A together with a linear map $\mu\colon A\otimes A\to A$ of degree zero which is associative, i.e. μ satisfies $\mu(\mu\otimes \mathrm{I})=\mu(\mathrm{I}\otimes\mu)$. The map μ is called the multiplication of the algebra, and it is usually suppressed from the notation.

For two graded algebras A and B an *algebra homomorphism* $f\colon A\to B$ is a linear map of degree zero which commutes with the multiplications of A and B, i.e. $\mu_B(f\otimes f)=f\mu_A$. By definition the set of algebra homomorphisms between A and B is a subset of $\mathrm{Hom}^0_{\mathrm{gr}}(A,B)$, but it is in general not a linear subspace.

2.2.1 Differential Graded Algebras

A *differential graded algebra*, or dga for short, is an associative graded algebra (A,μ) together with a differential d on A of degree 1 which satisfies the Leibniz rule:

$$d\mu=\mu(d\otimes\mathrm{I}+\mathrm{I}\otimes d). \tag{2.4}$$

Note that because of the Koszul sign convention this becomes

$$d(ab)=(da)b+(-1)^{|a|}adb \tag{2.5}$$

on elements $a,b\in A$.

The cohomology of a dga is naturally an associative graded algebra. Furthermore, if A and B are dga's and $f\colon A\to B$ is an algebra homomorphism which is also a chain map, then $f^*\colon H(A)\to H(B)$ is an algebra homomorphism. Such morphisms are thus the natural morphisms of differential graded algebras.

2.2.2 Unital Algebras

An associative graded algebra (A,μ) is called *unital* if there is an identity element for μ in A^0. We denote this element by 1_A or just 1. Unless otherwise stated we shall use the shorthand *graded algebra* to mean a unital associative graded algebra. The smallest example a graded algebra is K itself. Lastly, an algebra homomorphism $f\colon A\to B$ between two graded algebras A and B is called unital if $f(1_A)=1_B$.

2.2.3 Augmented Algebras

An *augmented* algebra is a pair (A,ε) where A is a graded algebra and $\varepsilon\colon A\to K$ is a unital algebra homomorphism. The map ε is called the *augmentation map*. The *augmentation ideal* is defined as the kernel of ε

and denoted by $\overline{A}$. It is itself a non-unital associative graded algebra. By unitality the short exact sequence

$$0 \to \overline{A} \to A \xrightarrow{\varepsilon} K \to 0 \tag{2.6}$$

splits, which shows that $A = \overline{A} \oplus K1$ as a linear space.

A graded algebra whose underlying graded vector space is connected is called a *connected algebra*. Every connected algebra is in particular augmented, but the reverse is not true. For instance the polynomial ring $K[X]$ with $|X| = 0$ has an augmentation given by $X \mapsto 0$, but it is not connected.

Lemma 2.1. *A connected algebra is uniquely augmented.*

Proof. Let A be a connected algebra. The map from A to K which sends 1 to 1 and A^m to zero for all $m \geq 1$ is a unital algebra homomorphism. Hence it is an augmentation of A. Since this is the only unital algebra homomorphism from A to K it follows that A is uniquely augmented. □

2.3 Graded Lie Algebras

In this section we assume K has characteristic different from 2 and 3. Our main reference for graded (pre-)Lie algebras is Gerstenhaber [6]. By a *graded Lie algebra* we shall mean a graded vector space L together with a linear map $[-,-]\colon L \otimes L \to L$ of degree 0 which satisfies

(i) $[x, y] = -(-1)^{|x||y|}[y, x]$, (anti-symmetry)

(ii) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$, (Jacobi identity)

for all $x, y, z \in L$. Note that (i) is regular anti-symmetry with respect to the switch map of section 2.1.1. The linear map $[-,-]$ is called the *Lie bracket* of the Lie algebra.

2.3.1 Graded Pre-Lie Algebras

A *graded pre-Lie algebra* is a graded vector space L together with a linear map $L \otimes L \to L$ of degree zero which satisfies

$$(xy)z - x(yz) = (-1)^{|y||z|}\big((xz)y - x(zy)\big), \tag{2.7}$$

for all $x, y, z \in L$.

For any graded vector space V with a binary operation $(x, y) \mapsto xy$ we define the *graded commutator* as the binary operation given by

$$[x, y] := xy - (-1)^{|x||y|}yx \tag{2.8}$$

for all $x, y \in V$.

Lemma 2.2. [6, Thm. 1] *If L is a pre-Lie algebra, then the graded commutator is a Lie bracket on L.* □

Example 2.3. One can easily see that any associative graded algebra is a graded pre-Lie algebra. In particular, lemma 2.2 tells us that any associative graded algebra is a graded Lie algebra with respect to the graded commutator.

2.4 Graded Coalgebras

A *graded coalgebra* over K is a graded vector space C together with a linear map $\Delta\colon C \to C \otimes C$ of degree zero which is coassociative, i.e. which satisfies the identity

$$(\Delta \otimes \mathrm{I})\Delta = (\mathrm{I} \otimes \Delta)\Delta. \tag{2.9}$$

The map Δ is called the *comultiplication* or *coproduct* of the coalgebra.

Let C and D be two graded coalgebras. A *coalgebra morphism* from C to D is a linear map $f\colon C \to D$ of degree zero such that

$$(f \otimes f)\Delta_C = \Delta_D f. \tag{2.10}$$

We write $\mathrm{Hom}_{\mathrm{CoAlg}}(C, D)$ for the set of all coalgebra morphisms from C to D. By definition $\mathrm{Hom}_{\mathrm{CoAlg}}(C, D) \subseteq \mathrm{Hom}^0_{\mathrm{gr}}(C, D)$, but it is in general not a linear subspace.

One can also consider counital and coaugmented coalgebras. A *counital* coalgebra is a coalgebra C together with a counit map $\varepsilon\colon C \to K$ which makes the following diagram commute

$$\begin{array}{ccccc} & \swarrow^{\cong} & C & \searrow^{\cong} & \\ & & \downarrow \Delta & & \\ K \otimes C & \xleftarrow{\varepsilon \otimes \mathrm{I}} & C \otimes C & \xrightarrow{\mathrm{I} \otimes \varepsilon} & C \otimes K, \end{array} \tag{2.11}$$

and a *coaugmented* coalgebra is a counital coalgebra C together with a coaugmentation map $\eta\colon K \to C$ which is a counital coalgebra morphism. Here a coalgebra morphism $f\colon C \to D$ between two unital coalgebras is called *counital* if $\varepsilon_D f = \varepsilon_C$.

If C is a coaugmented coalgebra we can form the *reduced* coalgebra of C, which is a non-counital coalgebra which uniquely determines C. By definition it is the kernel $\overline{C} \coloneqq \ker \varepsilon$ with coproduct given by

$$\overline{\Delta}(x) \coloneqq \Delta(x) - \eta(1) \otimes x - x \otimes \eta(1). \tag{2.12}$$

This then determines C by the fact that $\overline{C}$ is a direct summand of C. Moreover, the set of unital coalgebra morphisms between coaugmented coalgebras is equal to the set of coalgebra morphisms between their reduced coalgebras. This is why we have chosen to work with non-unital coalgebras, because we could either have worked with coaugmented coalgebras and their reduced coalgebras or we can work directly with their reduced coalgebras as non-unital coalgebras.

2.4.1 Graded Comodules

Let C be a graded coalgebra. A *left* (resp. *right*) *comodule* over C is a graded vector space M together with a linear map $\lambda\colon M \to C \otimes M$ (resp. $\rho\colon M \to M \otimes C$) of degree 0 which satisfies

$$(\Delta \otimes \mathrm{I})\lambda = (\mathrm{I} \otimes \lambda)\lambda, \quad \big(\text{resp. } (\mathrm{I} \otimes \Delta)\rho = (\rho \otimes \mathrm{I})\rho\big). \tag{2.13}$$

If M is both a left and a right comodule and $(\mathrm{I} \otimes \rho)\lambda = (\lambda \otimes \mathrm{I})\rho$, then M is called a *bicomodule*.

Example 2.4. If C and D are coalgebras and $f\colon C \to D$ is a coalgebra morphism, then we can make C a bicomodule over D by $\lambda = (f \otimes \mathrm{I})\Delta_C$ and $\rho = (\mathrm{I} \otimes f)\Delta_C$.

2.4.2 Coderivations

Let C be a graded coalgebra. A *coderivation* on C is a linear map $d \in \mathrm{Hom}_{\mathrm{gr}}(C, C)$ which satisfies the co-Leibniz rule:

$$\Delta d = (d \otimes \mathrm{I} + \mathrm{I} \otimes d)\Delta. \tag{2.14}$$

Note that there are no restrictions on the degree of a coderivation. We write $\mathrm{Cod}_{\mathrm{gr}}(C)$ for the graded vector space of all coderivations on C.

By the Koszul sign convention the set of coderivations on C is closed under the graded commutator, i.e. if d_1 and d_2 are coderivations, then so is $[d_1, d_2] = d_1 d_2 - (-1)^{|d_1||d_2|} d_2 d_1$. This implies that $\mathrm{Cod}_{\mathrm{gr}}(C)$ is a graded Lie algebra under the graded commutator.

More generally, for a bicomodule M over C, a coderivation from M to C is a linear map $d \in \mathrm{Hom}_{\mathrm{gr}}(M, C)$ such that

$$\Delta d = (d \otimes \mathrm{I})\rho + (\mathrm{I} \otimes d)\lambda. \tag{2.15}$$

We write $\mathrm{Cod}_{\mathrm{gr}}(M, C)$ for the graded vector space of all coderivations from M to C.

Example 2.5. If C and D are coalgebras and $f\colon C \to D$ is a coalgebra morphism, then we saw in example 2.4 how to make C a comodule over D. Thus a coderivation from C to D with respect to f is a linear map $d \in \mathrm{Hom}_{\mathrm{gr}}(C, D)$ such that

$$\Delta_D d = (d \otimes f + f \otimes d)\Delta_C. \tag{2.16}$$

2.4.3 Differential Graded Coalgebras

A *differential graded coalgebra*, or dgc for short, is a graded coalgebra C together with a differential d on C of degree 1 which is also a coderivation on C, i.e. d should satisfy both $d^2 = 0$ and the co-Leibniz rule (2.14).

2.5 The Tensor Coalgebra

Given a graded vector space V over K, the *tensor coalgebra* of V is the graded vector space $\mathrm{T}V := K \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$ with coproduct

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \sum_{p=1}^{n-1} [x_1|\cdots|x_p] \otimes [x_{p+1}|\cdots|x_n] \tag{2.17}$$

for $x = [x_1|\cdots|x_n] \in V^{\otimes n} \subseteq \mathrm{T}V$. This is an example of a coaugmented coalgebra, and so we also have the *reduced tensor coalgebra* $\overline{\mathrm{T}}V = V \oplus V^{\otimes 2} \oplus \cdots$ with coproduct

$$\overline{\Delta}[x_1|\cdots|x_n] = \sum_{p=1}^{n-1} [x_1|\cdots|x_p] \otimes [x_{p+1}|\cdots|x_n]. \tag{2.18}$$

Note that the (reduced) tensor coalgebra is bigraded by the internal grading of V and the external grading given by the tensor power. We will call the external grading for *weight*.

2.5.1 The Gerstenhaber Product

Let V be a graded vector space. The *Gerstenhaber product* is a pre-Lie algebra structure on $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}V, V\big)$. For $f \in \mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}V, V\big)$ write f_n for the weight n part of f. The Gerstenhaber product of f and g is then defined as

$$(fg)_n := \sum_{p+q+r=n} f_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes g_r \otimes \mathrm{I}^{\otimes q}\big). \tag{2.19}$$

That this is a pre-Lie algebra product was shown in [6]. The resulting graded Lie bracket we get from taking the graded commutator of the Gerstenhaber product is called the *Gerstenhaber bracket*.

Proposition 2.6. [3, Lemma 2.5] *Let V and W be graded vector spaces.*

(a) *There is a linear isomorphism* $\mathrm{Hom}_{\mathrm{CoAlg}}\big(\overline{\mathrm{T}}V, \overline{\mathrm{T}}W\big) \cong \mathrm{Hom}^0_{\mathrm{gr}}\big(\overline{\mathrm{T}}V, W\big)$. *Moreover, each $f \in \mathrm{Hom}_{\mathrm{CoAlg}}\big(\overline{\mathrm{T}}V, \overline{\mathrm{T}}W\big)$ makes V a bicomodule over W and there is then a linear isomorphism* $\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{T}}V, \overline{\mathrm{T}}W\big) \cong \mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}V, W\big)$.

(b) *In the special case when $W = V$ the isomorphism* $\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{T}}V\big) \cong \mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}V, V\big)$ *is moreover a Lie algebra isomorphism.*

2.5.2 The Bar Construction

Let us first introduce the shift functor Σ. For a graded vector space V, the *shift* of V is the graded vector space ΣV given by $(\Sigma V)^n := V^{n+1}$. In general Σ shifts degrees downwards, so for instance something which lives in degree 0 lands in degree -1. On morphisms Σ does not change the degree, but it is defined as $\Sigma f := (-1)^{|f|} f$. An equivalent definition of Σ is to define it as tensoring from the left with K shifted one degree down.

The *bar construction* of a graded vector space V is defined as the tensor coalgebra of the shift of V, i.e. $\mathrm{B}V := \mathrm{T}\Sigma V$. The reason for having a shift in the bar construction has to do with getting the correct signs in applications. Similarly, the *reduced bar construction* of a graded vector space V is defined as the reduced tensor coalgebra of the shift of V, i.e. $\overline{\mathrm{B}}V := \overline{\mathrm{T}}\Sigma V$.

Chapter 3

Hochschild Cohomology

In the ungraded setting Hochschild cohomology over K is a cohomology theory taking K-algebras to K-vector spaces. Since we are working with graded objects we instead want a cohomology theory which takes graded algebras to graded vector spaces.

For an algebra A Hochschild cohomology takes an A-bimodule M as coefficients, and the resulting cohomology groups are written $HH^*(A, M)$. Since we do not need this generality we will only consider Hochschild cohomology with A itself as coefficients.

3.1 Definition and Relation to Coderivations

Let (A, μ) be an associative graded algebra. The *Hochschild cochain complex* of A, $C^\bullet(A, A)$, is a cochain complex of graded vector spaces. It is defined as $C^n(A, A) := \mathrm{Hom}_{\mathrm{gr}}(A^{\otimes n}, A)$, where by definition $A^{\otimes 0} = K$, and with differential given by

$$df = \mu(\mathrm{I} \otimes f) + \sum_{p=0}^{n-1} (-1)^{p+1} f\big(\mathrm{I}^{\otimes p} \otimes \mu \otimes \mathrm{I}^{\otimes n-p-1}\big) + (-1)^{n+1} \mu(f \otimes \mathrm{I}), \quad (3.1)$$

for $f \in C^n(A, A)$. Note that for $n = 0$ we have $df = \mu(\mathrm{I} \otimes f) - \mu(f \otimes \mathrm{I})$, and that by the Koszul sign convention there are some "hidden" signs in the notation.

Definition 3.1. The *Hochschild cohomology* of the associative graded algebra A, denoted by $HH^*(A, A)$, is the cohomology of the Hochschild cochain complex of A.

Since the Hochschild cochain complex of A is a cochain complex of graded vector spaces it is itself a bigraded complex. Let us represent it with internal grading going vertically and external grading horizontally, it then looks like

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
A^2 & \xrightarrow{d} & \mathrm{Hom}^2_{\mathrm{gr}}(A,A) & \xrightarrow{d} & \mathrm{Hom}^2_{\mathrm{gr}}(A\otimes A,A) & \xrightarrow{d} & \cdots \\
A^1 & \xrightarrow{d} & \mathrm{Hom}^1_{\mathrm{gr}}(A,A) & \xrightarrow{d} & \mathrm{Hom}^1_{\mathrm{gr}}(A\otimes A,A) & \xrightarrow{d} & \cdots \\
A^0 & \xrightarrow{d} & \mathrm{Hom}^0_{\mathrm{gr}}(A,A) & \xrightarrow{d} & \mathrm{Hom}^0_{\mathrm{gr}}(A\otimes A,A) & \xrightarrow{d} & \cdots \\
A^{-1} & \xrightarrow{d} & \mathrm{Hom}^{-1}_{\mathrm{gr}}(A,A) & \xrightarrow{d} & \mathrm{Hom}^{-1}_{\mathrm{gr}}(A\otimes A,A) & \xrightarrow{d} & \cdots \\
\vdots & & \vdots & & \vdots & &
\end{array}
\tag{3.2}
$$

We would like to relate this with $\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A\big)$ (where the shift in $\overline{\mathrm{B}}A$ is essential). Since we want to relate it to the reduced bar construction we will only look at $C^{\bullet\geqslant 1}(A,A)$. We then have the following lemma.

Lemma 3.2. *The shift of the totalization* $\Sigma\big(\mathrm{Tot}^{\Pi}\, C^{\bullet\geqslant 1}(A,A)\big)$ *is isomorphic to* $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}\Sigma A,\Sigma A\big)$ *as a graded vector space.*

Proof. The degree n part of $\Sigma\big(\mathrm{Tot}^{\Pi}\, C^{\bullet\geqslant 1}(A,A)\big)$ is

$$
\big(\Sigma\big(\mathrm{Tot}^{\Pi}\, C^{\bullet\geqslant 1}(A,A)\big)\big)^n = \prod_{\substack{p+q=n+1\\ p\geqslant 1}} \mathrm{Hom}^q_{\mathrm{gr}}(A^{\otimes p},A). \tag{3.3}
$$

Similarly, the degree n part of $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}\Sigma A,\Sigma A\big)$ is

$$
\begin{aligned}
\mathrm{Hom}^n_{\mathrm{gr}}\big(\overline{\mathrm{T}}\Sigma A,\Sigma A\big) &\cong \prod_{k\geq 1} \mathrm{Hom}^n_{\mathrm{gr}}\big((\Sigma A)^{\otimes k},\Sigma A\big) \\
&= \prod_{k\geq 1} \mathrm{Hom}^{n-k+1}_{\mathrm{gr}}\big(A^{\otimes k},A\big).
\end{aligned}
\tag{3.4}
$$

It is thus clear that we have a linear isomorphism of degree 0 between $\Sigma\big(\mathrm{Tot}^{\Pi}\, C^{\bullet\geqslant 1}(A,A)\big)$ and $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}\Sigma A,\Sigma A\big)$. □

By proposition 2.6 we know that $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}\Sigma A,\Sigma A\big)$ together with the Gerstenhaber bracket is isomorphic to $\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A\big)$ as a graded Lie algebra. Hence we would like to see if we can describe the differential of

$\Sigma\big(\mathrm{Tot}^{\Pi} C^{\bullet \geqslant 1}(A,A)\big)$ with respect to the Gerstenhaber bracket, and thus transfer it over to $\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A\big)$. To do this we need a description of the linear isomorphism in lemma 3.2.

Denote by $s\colon A \to \Sigma A$ the tautological map of degree -1. We might then describe the linear isomorphism in lemma 3.2 as follows. For $p+q=n+1$ an $f \in C^{p,q}(A,A) = \mathrm{Hom}^{q}_{\mathrm{gr}}(A^{\otimes p}, A)$ is mapped to $g = sf\big(s^{-1}\big)^{\otimes p}$. Note that g has the correct degree of $q+p-1=n$.

Now denote by $m_2 = s\mu\big(s^{-1}\big)^{\otimes 2} \in \mathrm{Hom}^{1}_{\mathrm{gr}}(\Sigma A \otimes \Sigma A, \Sigma A)$ and consider the Gerstenhaber bracket

$$[m_2, g] = m_2(g \otimes \mathrm{I} + \mathrm{I} \otimes g) - (-1)^n \sum_{k=0}^{p-1} g\big(\mathrm{I}^{\otimes k} \otimes m_2 \otimes \mathrm{I}^{\otimes p-k-1}\big). \tag{3.5}$$

A calculation then shows that $[m_2, g] = (-1)^q s(df)\big(s^{-1}\big)^{\otimes p+1}$. Hence up to a sign, which we can fix by tweaking the linear isomorphism, we have that the Hochschild differential is equal to $[m_2, -]$ in $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{T}}\Sigma A, \Sigma A\big)$. By proposition 2.6 we then get the following result.

Proposition 3.3. *Let (A,μ) be an associative graded algebra. There is an isomorphism of cochain complexes*

$$\Sigma\big(\mathrm{Tot}^{\Pi} C^{\bullet \geqslant 1}(A,A)\big) \cong \big(\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A\big), [m_2, -]\big), \tag{3.6}$$

where $m_2 = s\mu\big(s^{-1}\big)^{\otimes 2}$. □

For an associative graded algebra A we thus have a relation between the Hochschild cohomology of A and the coderivations of the reduced bar construction of A.

3.2 Augmented Algebras and Reduced Hochschild Cohomology

Let A be an augmented algebra with augmentation ideal $\overline{A}$. Then $\overline{A}$ is itself a non-unital associative graded algebra, and hence we can consider its Hochschild cohomology $HH\big(\overline{A}, \overline{A}\big)$. We call this cohomology for the *reduced Hochschild cohomology* of A.

3.3 Connected Cube Zero Algebras

Let us formally introduce the connected cube zero algebras which we saw in the introduction.

Definition 3.4. A *connected cube zero algebra* is a graded algebra whose underlying vector space is connected and 3-coconnected.

Note that by lemma 2.1 every connected cube zero algebra A is uniquely augmented. Thus A has a reduced Hochschild cochain complex which has the following shape:

$$
\begin{array}{llll}
A^2 & & & \\
A^1 \xrightarrow{d} \mathrm{Hom}^1_{\mathrm{gr}}(\overline{A}, \overline{A}) & & & \\
\mathrm{Hom}^0_{\mathrm{gr}}(\overline{A}, \overline{A}) \xrightarrow{d} \mathrm{Hom}^0_{\mathrm{gr}}(\overline{A} \otimes \overline{A}, \overline{A}) & & & \\
\mathrm{Hom}^{-1}_{\mathrm{gr}}(\overline{A}, \overline{A}) \xrightarrow{d} \mathrm{Hom}^{-1}_{\mathrm{gr}}(\overline{A} \otimes \overline{A}, \overline{A}) \xrightarrow{d} \mathrm{Hom}^{-1}_{\mathrm{gr}}(\overline{A}^{\otimes 3}, \overline{A}) & & & \\
\mathrm{Hom}^{-2}_{\mathrm{gr}}(\overline{A} \otimes \overline{A}, \overline{A}) \xrightarrow{d} \mathrm{Hom}^{-2}_{\mathrm{gr}}(\overline{A}^{\otimes 3}, \overline{A}) \xrightarrow{d} \cdots & & & \\
\ddots \qquad \ddots \qquad \ddots & & &
\end{array}
\tag{3.7}
$$

Example 3.5. Let A be a connected cube zero algebra with trivial multiplication. All differentials in the reduced Hochschild cochain complex of A are then zero, and so we can read off the reduced Hochschild cohomology of A directly from (3.7).

Example 3.6. A special case of example 3.5 is when $A^2 = 0$. The reduced Hochschild cochain complex of A degenerates even more in this case and we get that $HH^n(\overline{A}, \overline{A}) = \mathrm{Hom}_K\big((A^1)^{\otimes n}, A^1\big)$, or equivalently $HH^*(\overline{A}, \overline{A}) = \mathrm{Hom}_K\big(\mathrm{T}A^1, A^1\big)$.

Chapter 4

A_∞-Algebras

A_∞-algebras were introduced by James Stasheff in [29]. We follow Loday-Vallette [19, sec. 9.2.1] in our definitions.

Definition 4.1. Let A be a graded vector space. An *A_∞-structure* on A is a coderivation d of degree 1 on the reduced bar construction $\overline{\mathrm{B}}A$ which is also a differential, i.e. $d^2 = 0$. A graded vector space together with an A_∞-structure on it is called an *A_∞-algebra*.

Let A be an A_∞-algebra with A_∞-structure d on $\overline{\mathrm{B}}A$. Observe that definition 4.1 is exactly the same as saying $\big(\overline{\mathrm{B}}A, \overline{\Delta}, d\big)$ is a differential graded coalgebra, c.f. section 2.4.3.

By proposition 2.6 we know there is an isomorphism of Lie algebras $\mathrm{Cod}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A\big) \cong \mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A, \Sigma A\big)$. Hence we should translate what $d^2 = 0$ means in $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A, \Sigma A\big)$. First observe that since d is of degree 1, the graded commutator of d with itself is 2 times d^2, i.e. $\frac{1}{2}[d, d] = d^2$ when $\operatorname{char} K \neq 2$. Note that we can also directly show that $d^2 \in \mathrm{Cod}^2_{\mathrm{gr}}\big(\overline{\mathrm{B}}A\big)$ for we have

$$\Delta d^2 = (d \otimes \mathrm{I} + \mathrm{I} \otimes d)^2 \Delta = \big(d^2 \otimes \mathrm{I} + \mathrm{I} \otimes d^2\big)\Delta \tag{4.1}$$

by the Koszul sign convention.

Now, the commutator $[d, d]$ translated over to $\mathrm{Hom}_{\mathrm{gr}}\big(\overline{\mathrm{B}}A, \Sigma A\big)$ is given by the Gerstenhaber bracket. Let us write (d_n) for the image of d in $\mathrm{Hom}^1_{\mathrm{gr}}\big(\overline{\mathrm{B}}A, \Sigma A\big)$, where $d_n \colon (\Sigma A)^{\otimes n} \to \Sigma A$ is the weight n part. Then one can check that $d^2 = \frac{1}{2}[d, d] = 0$ becomes the relations

$$\sum_{p+q+r=n} d_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes d_r \otimes \mathrm{I}^{\otimes q}\big) = 0 \tag{4.2}$$

for $n \in \mathbb{N}$. In characteristic 2 we cannot directly use proposition 2.6, but it is still possible to get the relations (4.2).

Let $s\colon A \to \Sigma A$ still be the tautological map of degree -1. If we denote by $m_n := s^{-1} d_n s^{\otimes n}$, then $m_n\colon A^{\otimes n} \to A$ has degree $2-n$ and $d^2 = 0$ becomes the relations

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+q+1}\bigl(\mathrm{I}^{\otimes p} \otimes m_r \otimes \mathrm{I}^{\otimes q}\bigr) = 0 \tag{4.3}$$

for $n \in \mathbb{N}$. The sign $(-1)^{p+qr}$ comes from the Koszul sign convention. These are called the *Stasheff relations*, and they give us an equivalent definition of an A_∞-algebra.

Definition. An A_∞-algebra is a graded vector space A together with linear maps $m_n\colon A^{\otimes n} \to A$, for $n \geq 1$, of degree $2-n$ which satisfies the Stasheff relations (4.3).

We should examine the first few Stasheff relations to get a better understanding of m_1, m_2 and m_3. The first relation is

$$m_1^2 = 0, \tag{4.4}$$

which together with the fact that m_1 has degree 1 says that m_1 is a differential on A. Thus (A, m_1) is a chain complex. The next relation is

$$m_1 m_2 = m_2(m_1 \otimes \mathrm{I} + \mathrm{I} \otimes m_1), \tag{4.5}$$

which says that m_2 is a chain map with respect to m_1. Finally, the third relation is

$$m_2(\mathrm{I} \otimes m_2 - m_2 \otimes \mathrm{I}) = m_1 m_3 + m_3(m_1 \otimes \mathrm{I} \otimes \mathrm{I} + \mathrm{I} \otimes m_1 \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{I} \otimes m_1), \tag{4.6}$$

which says that m_2 is associative up to the chain homotopy m_3.

From the Stasheff relations on can also easily see that associative graded algebras and differential graded algebras are examples of A_∞-algebras if one sets m_2 equal to the multiplication map and m_1 equal to the differential.

4.1 Morphisms of A_∞-Algebras

Since A_∞-algebras are homotopical objects it is not surprising that there are several different strictness requirements for morphisms between them. In this section we define these.

4.1.1 1-Morphisms

Definition 4.2. A 1-*morphism* between two A_∞-algebras A and R is a linear map $f\colon A \to R$ of degree 0 such that

$$f m_n^A = m_n^R(f \otimes \cdots \otimes f) \tag{4.7}$$

for all $n \in \mathbb{N}$.

Example 4.3. If A and R are associative graded algebras, then one easily sees that a 1-morphism between A and R is the same as an algebra homomorphism between A and R. Similarly, if A and R are dga's, then a 1-morphism is the same as a morphism of dga's, i.e. an algebra homomorphism which is also a chain map.

4.1.2 ∞-Morphisms

Definition 4.4. An ∞-*morphism* between two A_∞-algebras A and R is a coalgebra morphism $F\colon \overline{\mathrm{B}}A \to \overline{\mathrm{B}}R$ which is also a chain map. In other words it must satisfy $(F \otimes F)\Delta_{\overline{\mathrm{B}}A} = \Delta_{\overline{\mathrm{B}}R}F$ and $Fd_A = d_R F$.

By proposition 2.6 we know that every unital coalgebra morphism $F\colon \overline{\mathrm{B}}A \to \overline{\mathrm{B}}R$ is given by a unique linear map in $\mathrm{Hom}^0_{\mathrm{gr}}\big(\overline{\mathrm{T}}A, \Sigma R\big)$. Also, given such a coalgebra morphism, $\overline{\mathrm{B}}A$ becomes a bicomodule over $\overline{\mathrm{B}}R$, and Fd_A and $d_R F$ becomes coderivations from $\overline{\mathrm{B}}A$ to $\overline{\mathrm{B}}R$. Hence by proposition 2.6 we can check the chain map condition in $\mathrm{Hom}^1_{\mathrm{gr}}\big(\overline{\mathrm{T}}A, \Sigma R\big)$. Denote by (F_n) the image of F in $\mathrm{Hom}^0_{\mathrm{gr}}\big(\overline{\mathrm{T}}A, \Sigma R\big)$, where $F_n\colon (\Sigma A)^{\otimes n} \to \Sigma R$. Then $Fd_A = d_R F$ is equivalent to

$$\sum_{p+q+r=n} F_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes d_r^A \otimes \mathrm{I}^{\otimes q}\big) = \sum_{i_1+\ldots+i_r=n} d_r^R(F_{i_1} \otimes \cdots \otimes F_{i_r}) \tag{4.8}$$

for all $n \in \mathbb{N}$.

We may replace F_n by $s^{-1}F_n s^{\otimes n}\colon A^{\otimes n} \to R$ and d_n by m_n to get similar relations as (4.8), but with signs coming from the Koszul sign convention. The resulting relations are

$$\sum_{p+q+r=n} (-1)^{p+qr} f_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes m_r^A \otimes \mathrm{I}^{\otimes q}\big) = \sum_{i_1+\ldots+i_r=n} (-1)^{\varepsilon} m_r^R(f_{i_1} \otimes \cdots \otimes f_{i_r}) \tag{4.9}$$

for all $n \in \mathbb{N}$, where $f_n = s^{-1}F_n s^{\otimes n}$ and

$$\varepsilon = (r-1)(i_1-1) + (r-2)(i_2-1) + \ldots + (i_{r-1}-1). \tag{4.10}$$

Note that the degree of f_n is $1-n$.

This gives us an equivalent definition of an ∞-morphism as a collection of maps $f_n\colon A^{\otimes n} \to R$, for $n \geq 1$, of degree $1-n$ which satisfies the relations (4.9). Let us also introduce the following notation for the resulting ∞-morphism, namely $f\colon A \rightsquigarrow R$, where we suppress the index n.

Let us examine the first few relations to get a better understanding of f_1 and f_2 in an ∞-morphism. The first relation is

$$f_1 m_1^A = m_1^R f_1, \tag{4.11}$$

which says that f_1 is a chain map. The second relation is

$$f_1 m_2^A - m_2^R(f_1 \otimes f_1) = m_1^R f_2 + f_2\big(m_1^A \otimes \mathrm{I} + \mathrm{I} \otimes m_1^A\big), \tag{4.12}$$

which says that f_1 preserves m_2 up to the chain homotopy f_2.

Lastly, let us show how 1-morphisms are a special case of ∞-morphisms.

Lemma 4.5. *Let A and R be A_∞-algebras. There is a one-to-one correspondence between* 1*-morphisms $f\colon A \to R$ and ∞-morphisms $g\colon A \rightsquigarrow R$ with $g_n = 0$ for all $n > 1$, given by $f = g_1$.*

Proof. If $f\colon A \to R$ is a 1-morphism, set $g_1 = f$ and $g_n = 0$ for all $n > 1$. The relation in (4.9) then degenerates to

$$g_1 m_n^A = m_n^R(g_1 \otimes \cdots \otimes g_1), \tag{4.13}$$

and thus we see that g is an ∞-morphism by the fact that f is a 1-morphism.

Conversely, if $g\colon A \rightsquigarrow R$ is an ∞-morphism with $g_n = 0$ for all $n > 1$, then (4.13) shows that g_1 is a 1-morphism. □

4.1.3 Isomorphism and Equivalence

Let A and R be two A_∞-algebras. We say that A and R are *isomorphic* if there exists a 1-morphism $f\colon A \to R$ which is bijective. We say that A and R are *equivalent* if there exists an ∞-morphism $f\colon A \rightsquigarrow R$ with f_1 a quasi-isomorphism. The following result was proved by Kadeishvili [16] and Prouté [25].

Theorem 4.6. *If $f\colon A \rightsquigarrow R$ is an equivalence of A_∞-algebras, then there exists an ∞-morphism $g\colon R \rightsquigarrow A$ such that g_1 is a quasi-inverse to f_1.* □

This results shows that equivalences of A_∞-algebras have inverses. This should be contrasted to other homotopical settings like weak equivalences between topological spaces, which may not have inverses.

4.2 Unital A_∞-Algebras

An A_∞-algebra A is called *unital* if there is an element $1 \in A^0$ which is a multiplicative identity for m_2, and $m_n(a_1, \ldots, a_n) = 0$ whenever some $a_i = 1$, for all $n \neq 2$. Observe that when we think of a graded algebra as an A_∞-algebra, then it is a unital A_∞-algebra. In particular K is a unital A_∞-algebra.

If A and R are unital A_∞-algebras, then an ∞-morphism $f\colon A \rightsquigarrow R$ is called *unital* if $f_1(1_A) = 1_R$, and $f_n(a_1, \ldots, a_n) = 0$ whenever some $a_i = 1_A$, for all $n > 1$. Note that by lemma 4.5 this terminology applies to 1-morphisms as well. In particular, when A and R are graded algebras, then a unital 1-morphism is the same as a unital algebra homomorphism, c.f. example 4.3.

4.2.1 Augmented A_∞-Algebras

A unital A_∞-algebra A together with a unital 1-morphism $\varepsilon\colon A \to K$ is called an *augmented* A_∞-algebra. The map ε is called the *augmentation map*. The *augmentation ideal* is defined as the kernel of ε and denoted by $\overline{A}$. By unitality the short exact sequence

$$0 \to \overline{A} \to A \xrightarrow{\varepsilon} K \to 0 \tag{4.14}$$

splits, which shows that $A = \overline{A} \oplus K1$ as a graded vector space.

The A_∞-structure d on $\overline{\mathrm{B}}A$ is then completely determined by its restriction to $\overline{\mathrm{B}}\overline{A}$. Furthermore, any unital ∞-morphism $F\colon \overline{\mathrm{B}}A \to \overline{\mathrm{B}}R$, to some unital A_∞-algebra R, factors through $\overline{\mathrm{B}}\overline{A}$.

4.2.2 Connected A_∞-Algebras

A unital A_∞-algebra whose underlying graded module is connected is called a *connected A_∞-algebra*. Note that we require connected A_∞-algebras to be unital.

Lemma 4.7. *A connected A_∞-algebra is uniquely augmented.*

Proof. Let A be a connected A_∞-algebra. The map from A to K which sends 1 to 1 and A^m to zero for all $m \geq 1$ is a unital 1-morphism. Hence it is an augmentation of A. Since this is the only unital 1-morphism from A to K it follows that A is uniquely augmented. $\square$

4.3 Minimal A_∞-Algebras

We are mostly interested in minimal A_∞-algebras, so let us define what these are.

Definition 4.8. An A_∞-algebra (A, m) is called *minimal* if $m_1 = 0$.

The first observation to make is that if (A, m) is a minimal A_∞-algebra, then by the Stasheff relations (4.3) it follows that (A, m_2) is an associative algebra. Furthermore, if A is augmented, then (A, m_2) is an augmented associative algebra. We call (A, m_2) the *underlying algebra* of the minimal A_∞-algebra A.

The following is a well-known corollary to theorem 4.6 applied to minimal A_∞-algebras.

Corollary 4.9. *Let A and B be minimal A_∞-algebras. An ∞-morphism $f\colon A \rightsquigarrow R$ is an equivalence if and only if f_1 is an isomorphism of the underlying algebras $\left(A, m_2^A\right)$ and $\left(R, m_2^R\right)$.*

Proof. The sufficiency is clear from the definition of an equivalence. Assume therefore that f is an equivalence. By theorem 4.6 we then know that f_1 has a quasi-inverse. However, since both A and R have trivial differential, f_1 must have an actual inverse, i.e. it is a bijection. That f_1 is an algebra homomorphism follows from the second relation in (4.9). ☐

4.3.1 Minimal A_∞-Algebras are Typical

In this section we want to show how abundant minimal A_∞-Algebras are. The first result we want to highlight is a famous result by Kadeishvili which shows how minimal A_∞-algebras arise naturally in the study of dga's.

Theorem 4.10 (Kadeishvili)**.** [15] *For a differential graded algebra A, there is an A_∞-structure on the cohomology $H(A)$ of A with m_2 given by the multiplication of A, and an equivalence $H(A) \rightsquigarrow A$.* ☐

A natural source of dga's is of course the singular cochain complex of a topological space, like a link complement. In this regard theorem 4.10 tells us that for any link complement there is a natural choice of A_∞-structure on the cohomology of the link complement which captures the topology.

Another source of dga's is the so-called cobar construction. This is a functor Ω which maps differential graded coalgebras to differential graded algebras. In this thesis we will treat this functor together with the next result as a black box to justify our focus on minimal A_∞-algebras.

Theorem 4.11. *For any A_∞-algebra A there is an equivalence $A \rightsquigarrow \Omega\overline{\mathrm{B}}A$.*

Proof. This is a special case of the very general theorem 11.4.4 in [19]. ☐

These two theorems shows that up to equivalence every A_∞-algebra is minimal. This will be important for the next section.

4.3.2 The Classification Problem

The classification problem is the problem of classifying all A_∞-algebras up to equivalence. By the previous section we know that in this regard it suffices to classify the minimal A_∞-algebras. Corollary 4.9 is then the first step, which tells us that two distinct graded vector spaces cannot be equipped with equivalent minimal A_∞-structures. Hence we can fix a graded vector space V and ask which A_∞-structures on V are equivalent.

The next step is then the following result, which I have not seen in the literature before.

Proposition 4.12. *Let A and B be minimal A_∞-algebras. Any equivalence $f\colon A \rightsquigarrow B$ factors as follows*

$$\begin{array}{ccc} A & \overset{f}{\rightsquigarrow} & B \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & \tilde{B} & \end{array} \tag{4.15}$$

where $\tilde{B}$ is an A_∞-structure on the underlying vector space of A, g is an equivalence with $g_1 = \mathrm{id}$, and h is an isomorphism (in particular a 1-morphism).

Proof. By corollary 4.9 we know that $f_1\colon A \to B$ is an isomorphism of the underlying algebras. Define the A_∞ algebra $\tilde{B}$ as follows. The underlying graded vector space of $\tilde{B}$ is the same as the underlying vector space of A. Next, set

$$m_r^{\tilde{B}} := f_1^{-1} m_r^B (f_1 \otimes \cdots \otimes f_1) \tag{4.16}$$

for all $n \geq 1$. This is then easily seen to satisfy the Stasheff relations (4.3), and so $\tilde{B}$ is an A_∞-algebra. Also note that by f_1 being an algebra isomorphism we have

$$m_2^{\tilde{B}} = f_1^{-1} m_2^B (f_1 \otimes f_1) = f_1^{-1} f_1 m_2^A = m_2^A. \tag{4.17}$$

Thus the underlying algebra of $\tilde{B}$ is equal to the underlying algebra of A.

Next, define the 1-morphism $h\colon \tilde{B} \to B$ by $h = f_1$. Then by the definition of the A_∞-structure on $\tilde{B}$, h is easily seen to be an isomorphism of A_∞-algebras.

Lastly, we want to define the equivalence $g\colon A \rightsquigarrow \tilde{C}$, which should have $g_1 = \mathrm{id}$. To see how to define g we consider the ∞-morphism relations (4.9) of f

$$\sum_{\substack{p+q+r=n\\ r\geqslant 2}} (-1)^{p+qr} f_{p+q+1}\big(\mathrm{I}^{\otimes p}\otimes m_r^A\otimes \mathrm{I}^{\otimes q}\big) = \\ = \sum_{\substack{i_1+\ldots+i_r=n\\ r\geqslant 2}} (-1)^{\varepsilon} m_r^B\big(f_{i_1}\otimes\cdots\otimes f_{i_r}\big). \tag{4.18}$$

Note that we have $r \geq 2$ in both sums by the fact that A and B are minimal. If we multiply this from the left by f_1^{-1} we get

$$\sum_{\substack{p+q+r=n\\ r\geqslant 2}} (-1)^{p+qr} f_1^{-1} f_{p+q+1}\big(\mathrm{I}^{\otimes p}\otimes m_r^A\otimes \mathrm{I}^{\otimes q}\big) = \\ = \sum_{\substack{i_1+\ldots+i_r=n\\ r\geqslant 2}} (-1)^{\varepsilon} f_1^{-1} m_r^B\big(f_{i_1}\otimes\cdots\otimes f_{i_r}\big). \tag{4.19}$$

And finally, if we use that $(f_1\otimes\cdots\otimes f_1)(f_1^{-1}\otimes\cdots\otimes f_1^{-1}) = \mathrm{id}$ we get

$$\sum_{\substack{p+q+r=n\\ r\geqslant 2}} (-1)^{p+qr} f_1^{-1} f_{p+q+1}\big(\mathrm{I}^{\otimes p}\otimes m_r^A\otimes \mathrm{I}^{\otimes q}\big) = \\ = \sum_{\substack{i_1+\ldots+i_r=n\\ r\geqslant 2}} (-1)^{\varepsilon} f_1^{-1} m_r^B\big(f_1\otimes\cdots\otimes f_1\big)\big(f_1^{-1}f_{i_1}\otimes\cdots\otimes f_1^{-1}f_{i_r}\big), \tag{4.20}$$

or equivalently

$$\sum_{\substack{p+q+r=n\\ r\geqslant 2}} (-1)^{p+qr} f_1^{-1} f_{p+q+1}\big(\mathrm{I}^{\otimes p}\otimes m_r^A\otimes \mathrm{I}^{\otimes q}\big) = \\ = \sum_{\substack{i_1+\ldots+i_r=n\\ r\geqslant 2}} (-1)^{\varepsilon} m_r^{\tilde{B}}\big(f_1^{-1}f_{i_1}\otimes\cdots\otimes f_1^{-1}f_{i_r}\big). \tag{4.21}$$

Thus we see that $g_n := f_1^{-1} f_n$ defines an ∞-morphism from A to $\tilde{B}$ with $g_1 = f_1^{-1} f_1 = \mathrm{id}$. Lastly, note that g is by definition an equivalence, as $g_1 = \mathrm{id}$ is certainly a quasi-isomorphism. □

What this result shows is that to solve the classification problem we can start by fixing a graded algebra (A,μ) and ask which A_∞-structures on A that extend μ (i.e. those with $m_2 = \mu$) are equivalent via an equivalence with $f_1 = \mathrm{id}$. In other words this is now the problem we want to solve:

1. Fix a graded algebra (A,μ).

2. Consider every minimal A_∞-structure on A with $m_2 = \mu$.

3. Given two such A_∞-structures, m and m', when does there exists an equivalence $f\colon (A, m) \rightsquigarrow (A, m')$ with $f_1 = \mathrm{id}$?

Let us call such morphisms described in 3. *strong equivalences*.

For strong equivalences there is also a link to Hochschild cohomology, that we now describe. Let (A, μ) be a fixed graded algebra and let d denote the Hochschild differential of A. Given two minimal A_∞-structures m and m' which extend μ, a strong equivalence between them is then a sequence of linear maps $f_n\colon A^{\otimes n} \to A$, for $n \geq 2$, of degree $1 - n$ which satisfies

$$df_{n-1} = m_n - m'_n + \sum_{\substack{p+q+r=n \\ 2<r<n}} (-1)^{p+qr} f_{p+q+1}\big(\mathrm{I}^{\otimes p} \otimes m_r \otimes \mathrm{I}^{\otimes q}\big) - \sum_{\substack{p+q+r=n \\ 2<r<n}} (-1)^{\varepsilon} m'_r(f_{i_1} \otimes \cdots \otimes f_{i_r}), \quad (4.22)$$

for all $n \geq 3$, where $f_1 = \mathrm{I}$ in the last sum and

$$\varepsilon = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \ldots + (i_{r-1} - 1). \quad (4.23)$$

This is just a rewriting of (4.9) to the special case of strong equivalences. Explicitly, the two first relations are

$$df_2 = m_3 - m'_3, \quad (4.24)$$

and

$$df_3 = m_4 - m'_4 + m'_2(f_2 \otimes f_2) - f_2\big(\mathrm{I} \otimes m_3\big) - m'_3(f_2 \otimes \mathrm{I} \otimes \mathrm{I}) + m'_3(\mathrm{I} \otimes f_2 \otimes \mathrm{I}) - m'_3(\mathrm{I} \otimes \mathrm{I} \otimes f_2) - f_2\big(m_3 \otimes \mathrm{I}\big). \quad (4.25)$$

Before stating our next result let us first introduce some notation. For a minimal A_∞-algebra (A, m) we write

$$c(A, m) \coloneqq \min\{n \mid m_n \neq 0\} \quad (4.26)$$

for the smallest integer $n \geq 2$ such that the n-th multiplication is non-zero. If A is the trivial A_∞-algebra, i.e. if $m_n = 0$ for all $n \in \mathbb{N}$, we set $c(A) = \infty$.

Theorem 4.13. *Let A be a graded vector space with m and m' two minimal A_∞-structures on A.*

(1) *If (A, m) and (A, m') are equivalent, then $c(A, m) = c(A, m')$.*

(2) *If* (A, m) *and* (A, m') *are strongly equivalent, then in addition to* (1) *we moreover have that* $m_c = m'_c$*, where* $c = c(A, m) = c(A, m')$*.*

Proof. It is clear that to prove (1) it is sufficient to prove the inequality $c(A, m) \leq c(A, m')$. This is the statement that if (A, m) and (A, m') are equivalent and $m_k = 0$ for all $k < n$, then $m'_k = 0$ for all $k < n$ as well. Let us prove this by induction on $n \geq 2$.

Denote by $f\colon (A, m) \rightsquigarrow (A, m')$ the given equivalence. The base case $n = 2$ follows by the assumption that both (A, m) and (A, m') are minimal.

Next, assume the induction hypothesis and let $m_k = 0$ for all $k \leq n$. By the induction hypothesis we then have that $m'_k = 0$ for all $k < n$. Hence if we look at the ∞-morphism relations (4.9) for f we see that the first non-trivial relation is the n-th relation, and that it degenerates to

$$f_1 m_n = m'_n(f_1 \otimes \cdots \otimes f_1). \tag{4.27}$$

By corollary 4.9 we know that f_1 is an automorphism of A, and hence we see that $m'_n = 0$ as well, which completes the proof of (1).

To prove (2) let $c = c(A, m) = c(A, m')$, and $f\colon (A, m) \rightsquigarrow (A, m')$ be the given strong equivalence. Looking at the ∞-morphism relations (4.9) for f we once again see that the first non-trivial relation is the c-th relation, and that it degenerates to

$$f_1 m_c = m'_c(f_1 \otimes \cdots \otimes f_1). \tag{4.28}$$

However, this time $f_1 = \mathrm{I}$ and hence $m_c = m'_c$, as we wanted to show. □

This result shows that for any given graded vector space A there are infinitely many equivalence classes of A_∞-structures on A. Moreover, it gives us a first invariant of minimal A_∞-structures up to strong equivalence, namely the first non-trivial operation m_n of A.

4.4 A_∞-Structures on Connected Cube Zero Algebras

In this section we want to study A_∞-structures on connected and 3-coconnected vector spaces. These are related to cube zero algebras in the following way. If (A, m) is a connected and 3-coconnected vector spaces with a minimal A_∞-structure m, then (A, m_2) is a connected cube zero algebra by the Stasheff relations (4.3).

4.4.1 Classification of A_∞-Structures

Firstly, we can give a complete description of every unital A_∞-structure on a connected and 3-coconnected vector space. This result is however not new; see lemma 3.1 in [20].

Theorem 4.14. *The set of unital A_∞-structures on a connected and 3-coconnected vector space A is in one-to-one correspondence with the set* $\mathrm{Hom}_K\big(\overline{\mathrm{T}}A^1, A^2\big)$.

Proof. Denote by $\overline{A}$ the graded vector space we get from A by deleting A^0, i.e. $\overline{A}^n = A^n$ for all $n \geq 1$ and $\overline{A}^n = 0$ for all $n < 1$. Since A is connected, the set of unital A_∞-structures on A is equal to the set of coderivations $d \in \mathrm{Cod}^1_{\mathrm{gr}}\big(\mathrm{B}\overline{A}\big)$ which are differentials. Now, $d^2 \in \mathrm{Cod}^2_{\mathrm{gr}}\big(\mathrm{B}\overline{A}\big)$ and by proposition 2.6 $\mathrm{Cod}^2_{\mathrm{gr}}\big(\mathrm{B}\overline{A}\big) \cong \mathrm{Hom}^2_{\mathrm{gr}}\big(\mathrm{B}\overline{A}, \Sigma\overline{A}\big) = 0$. Hence every degree 1 coderivation on $\mathrm{B}\overline{A}$ is also a differential, and so the set of unital A_∞-structures on A is equal to $\mathrm{Cod}^1_{\mathrm{gr}}\big(\mathrm{B}\overline{A}\big)$. Finally, this set is isomorphic to $\mathrm{Hom}_K\big(\overline{\mathrm{T}}A^1, A^2\big)$ by proposition 2.6. □

From this result we see that to choose a unital A_∞-structure on a connected and 3-coconnected vector space A is the same as choosing a collection of linear maps $m_n\colon \big(A^1\big)^{\otimes n} \to A^2$, one for each natural number $n \geq 1$, with no relation requirements. The minimal A_∞-structures are of course given by choosing $m_1 = 0$.

The next step in the classification problem is then to determine which minimal A_∞-structures are equivalent. For this we already have theorem 4.13, and so we should see if we can refine that result for connected and 3-coconnected vector spaces. In this regard, a natural question to ask is the following. Given a unital and minimal A_∞-algebra (A, m) with A connected and 3-coconnected, if we define

$$\overline{m}_n = \begin{cases} m_n & n = c(A, m) \\ 0 & \text{else} \end{cases} \tag{4.29}$$

is (A, m) equivalent to $(A, \overline{m})$? In general I do not know the answer to this question. However, there is an instance where the answer is yes, which we will show in the next section.

4.4.2 A Class of Examples

Fix an integer $n \geq 2$ and let V be an ungraded vector space. Define a connected and 3-coconnected vector space $A = A(V, n)$ by $A^1 = V$ and $A^2 = V^{\otimes n}$. Next, define an A_∞-structure m on A by $m_n = \mathrm{id}\colon V^{\otimes n} \to V^{\otimes n}$ and $m_k = 0$ for all $k \neq n$.

Theorem 4.15. *Let (A, m) be as above. If m' is another A_∞-structure on A with $c(A, m') = n$ and $m'_n = m_n$, then (A, m') is strongly equivalent to (A, m).*

Proof. We want to explicitly define a strong equivalence $f\colon (A, m) \rightsquigarrow (A, m')$. As such we need a good understanding of the components of such an ∞-morphism. What we need to define are a sequence of maps $f_k\colon A^{\otimes k} \to A$, for $k \geq 2$, of degree $1 - k$ which satisfies the relations in (4.9).

If we want f to be unital, which we want, then it suffices to define each f_k only on $\overline{A}^{\otimes k}$. Since f_k has degree $1 - k$ it then consists of two linear maps

$$\begin{aligned} \bigoplus^{k} A^2 \otimes \big(A^1\big)^{\otimes k-1} &\xrightarrow{f_{k,2}} A^2 \\ \big(A^1\big)^{\otimes k} \qquad &\xrightarrow{f_{k,1}} A^1 \end{aligned} \tag{4.30}$$

where $\bigoplus^{k} A^2 \otimes (A^1)^{\otimes k-1}$ denotes

$$\big(A^2 \otimes (A^1)^{\otimes k-1}\big) \oplus \big(A^1 \otimes A^2 \otimes (A^1)^{\otimes k-2}\big) \oplus \cdots \oplus \big((A^1)^{\otimes k-1} \otimes A^2\big). \tag{4.31}$$

We now define f_k as follows. Set $f_{k,1} = 0$, and $f_{k,2} = 0$ on each component in $\bigoplus\big(A^2 \otimes (A^1)^{\otimes k-1}\big)$ except the first component $A^2 \otimes (A^1)^{\otimes k-1}$ were we set $f_{k,2} = (-1)^{kn-n} m'_{n+k-1}$. Note that this makes sense because $A^2 = V^{\otimes n} = (A^1)^{\otimes n}$.

Now that we have defined f we need to check that it satisfies the relations in (4.9). By degree reasons it clear that each relation, say indexed by k, only needs to be checked on $(A^1)^{\otimes k}$.

Firstly, we see that for $k < n$, the relations are completely zero, and hence satisfied. Next, for $k = n$ we know and see that the relation degenerates to $m_n = m'_n$, which we have by assumption. Finally, for $k > n$ we see that (4.9) degenerates to

$$(-1)^{(k-n)n} f_{k-n+1,2}\big(m_n \otimes \mathrm{I}^{\otimes k-n}\big) = m'_k. \tag{4.32}$$

Since m_n is equal to the identity, this is also satisfied, and thus f defines a strong equivalence. $\square$

Part II

Partial Differential Equations and Holomorphic Representation Coefficients

Chapter 5

Preliminaries

In this section we recall some elementary theory of Lie groups and their continuous representation theory. We also fix some notation and conventions.

5.1 Lie Groups

A Lie group is a group object in the category of smooth manifolds and smooth maps, i.e. it is a smooth manifold which is also a group with smooth group operations. One of the most important objects for studying Lie groups are their Lie algebras. Recall that we may define the Lie algebra of a Lie group as either the tangent space at the identity element or as the space of left-invariant vector fields. By convention we let the Lie algebra $\mathfrak{g}$ of a Lie group G be the tangent space at the identity element. For an element $X \in \mathfrak{g}$ we then denote by ∂_X the left-invariant vector field defined by X.

We will also need the complexified Lie algebra of a Lie group. For a Lie algebra $\mathfrak{g}$ it is defined as $\mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. It is a vector space over $\mathbb{C}$ of complex dimension equal to the real dimension of $\mathfrak{g}$. Every element of $\mathfrak{g}_{\mathbb{C}}$ can be written on the form $X + iY$ with $X, Y \in \mathfrak{g}$. For such an element $X + iY$ in the complexified Lie algebra of a Lie group G we denote by ∂_{X+iY} the vector field $\partial_X + i\partial_Y$ in the complexified tangent bundle of G.

We denote by $U\mathfrak{g}$ the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Recall, e.g. from [31, Sec. 3.4], that when $\mathfrak{g}$ is the Lie algebra of a Lie group G, then we may identify $U\mathfrak{g}$ with the algebra of all left-invariant differential operators on G. Similarly, we may identify $U\mathfrak{g}_{\mathbb{C}}$ with the algebra of all left-invariant complex differential operators on G.

Lastly, we will need the exponential map of a Lie group G. We write $\exp\colon \mathfrak{g} \to G$ for the exponential map of G. It is defined by the following property. For each $X \in \mathfrak{g}$, the mapping $t \mapsto \exp tX$ is the unique one-parameter subgroup of G whose derivative at 0 is equal to X, see e.g. [31,

p. 84]. By a one-parameter subgroup we mean a continuous group homomorphism from $\mathbb{R}$ to G.

5.2 Projective Unitary Representations

Let G be a Lie group and $\mathcal{H}$ a complex Hilbert space. A unitary representation of G on $\mathcal{H}$ is a group homomorphism $\pi\colon G \to \mathrm{U}(\mathcal{H})$, where $\mathrm{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$. For us a unitary representation will also always be strongly continuous, i.e. that for each $v \in \mathcal{H}$ the map $g \mapsto \pi(g)v$ is continuous from G to $\mathcal{H}$.

The main reference for projective unitary representations of Lie groups is Bargmann [1] (note however that Bargmann uses the term "ray" instead of "projective"). Following Bargmann we define a projective unitary representation of G on $\mathcal{H}$ as a strongly continuous map $\pi\colon G \to \mathrm{U}(\mathcal{H})$ which satisfies $\pi(e) = I$ and

$$\pi(gh) = c(g,h)\pi(g)\pi(h) \tag{5.1}$$

for all $g, h \in G$, where $c\colon G \times G \to \mathbb{T}$ is some fixed unimodular map. It is an easy exercise to show that c is a cocycle in the continuous group cohomology of G with coefficients in $\mathbb{T}$. We will therefore call it the cocycle map of the projective representation π.

If the cocycle map of a projective representation π is smooth we will call π smooth. This will be our standing assumption throughout the text. Bargmann shows that this assumption is not particularly strict as it can always be assumed locally at the identity [1, Lemma 4.1], and everywhere when G is connected and simply connected [1, Thm. 5.1]. Note that any unitary representation is a smooth projective unitary representation.

5.2.1 The Associated Heisenberg Group

Given a smooth projective unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$, the associated (reduced) Heisenberg group is the Lie group H defined as the manifold $G \times \mathbb{T}$ with multiplication

$$(g,z)(h,w) = \big(gh, \overline{c(g,h)}zw\big), \tag{5.2}$$

where c is the cocycle map of π. This is clearly a smooth binary operation, and one might check that it is also associative. The identity element of H is $(e,1)$ and inverses are given by $(g,z)^{-1} = \big(g^{-1}, c\big(g^{-1}, g\big)\bar{z}\big)$. Thus H is a Lie group as claimed. Note that if π were not smooth, then H would not have been a Lie group as the multiplication would not have been a smooth map.

The natural projection from H to G is easily seen to be a group homomorphism, and thus H is a group extension of G. In fact one may check that H is a central extension of G with centre $\mathbb{T}$. There is also a lift of π to an ordinary (strongly continuous) unitary representation $\tilde{\pi}\colon H \to \mathrm{U}(\mathcal{H})$ by defining $\tilde{\pi}(g, z) = z\pi(g)$. This lift will be important for us later.

Chapter 6

Differentiating Representations

In this section we will show how to differentiate unitary and projective unitary representations. This theory is well established for unitary representations, see e.g. [7, 8, 9, 13, 17, 22, 26, 28]. We will add to this theory by also considering the weak case and projective unitary representations.

6.1 The Derivative of a Unitary Representation

Let G be a Lie group, $\mathcal{H}$ a Hilbert space, $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a non-projective unitary representation, and $X \in \mathfrak{g}$ a vector in the Lie algebra of G. By the definition of the exponential map of G, the mapping $t \mapsto \pi(\exp tX)$ is a strongly continuous one-parameter subgroup of $\mathrm{U}(\mathcal{H})$. Hence by Stone's theorem [30] there exists a unique unbounded skew-adjoint operator A on $\mathcal{H}$ such that $\pi(\exp tX) = e^{tA}$. The unbounded operator A is called the derivative of π with respect to X and we will denote it by $d\pi(X)$. From Stone's theorem we also get that $d\pi(X)$ is given by the limit

$$d\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp tX)v - v}{t}, \tag{6.1}$$

with domain $\mathcal{D}(d\pi(X))$ those $v \in \mathcal{H}$ for which this limit exists.

There is also another way in which we can differentiate π, namely by fixing a vector $v \in \mathcal{H}$ and looking at the maps $\tilde{v}\colon G \to \mathcal{H}$ given by $\tilde{v}(g) = \pi(g)v$. We then say that v is of class C^k, for $0 \le k \le \infty$, if $\tilde{v}$ is a map of class C^k. Recall that a map being of class C^k is defined inductively as being differentiable with partial derivatives of class C^{k-1}, and class C^0 means continuous. This then defines linear subspaces of $\mathcal{H}$ as follows

$$C^k(\pi) := \big\{ v \in \mathcal{H} \;\big|\; \tilde{v} \in C^k(G, \mathcal{H}) \big\}. \tag{6.2}$$

Finally, there is a third way to differentiate π, which is not mentioned in the literature before. Namely, for a fixed vector $v \in \mathcal{H}$ we can look at when $\tilde{v}$ is weakly of class C^k, i.e. when $\xi \circ \tilde{v}$ is of class C^k for all $\xi \in \mathcal{H}'$. We denote by $C^k_w(\pi)$ the resulting linear subspace of $\mathcal{H}$ defined similarly as (6.2). Let us mention that Grothendieck [11, note (1) p. 39] has shown that any map which is weakly of class C^k is (strongly) of class C^{k-1}, and hence we get $C^k_w(\pi) \subseteq C^{k-1}(\pi)$.

The next result compares these three ways to differentiate the representation π. Except for the inclusion of $C^k_w(\pi)$ this result is not new, see e.g. [8, Prop. 1.1].

Proposition 6.1. *Let G be a Lie group, $\mathcal{H}$ be a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a unitary representation. For any basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ we have that*

$$C^k(\pi) = C^k_w(\pi) = \bigcap_{1 \leqslant i_1, \dots, i_k \leqslant n} \mathcal{D}(d\pi(X_{i_1}) \cdots d\pi(X_{i_k})). \tag{6.3}$$

Moreover, for any $X \in \mathfrak{g}$ and $v \in C^1(\pi)$ we have that

$$\partial_X \tilde{v}(g) = \pi(g) d\pi(X) v. \tag{6.4}$$

Proof. For a proof of (6.4) and the equality

$$C^k(\pi) = \bigcap_{1 \leqslant i_1, \dots, i_k \leqslant n} \mathcal{D}(d\pi(X_{i_1}) \cdots d\pi(X_{i_k})) \tag{6.5}$$

see proposition 1.1 in [8]. Clearly we also have $C^k(\pi) \subseteq C^k_w(\pi)$, and so we only need to show that

$$C^k_w(\pi) \subseteq \bigcap_{1 \leqslant i_1, \dots, i_k \leqslant n} \mathcal{D}(d\pi(X_{i_1}) \cdots d\pi(X_{i_k})). \tag{6.6}$$

Let $v \in C^1_w(\pi)$ and $X \in \mathfrak{g}$. We will then show that $v \in \mathcal{D}(d\pi(X))$. For each $u \in \mathcal{H}$ we have that the Lie derivative of $g \mapsto \langle \pi(g)v, u \rangle$ with respect to X exists, i.e. that the following limit exists

$$\lim_{t \to 0} \frac{\langle \pi(\exp tX)v, u \rangle - \langle v, u \rangle}{t} = \lim_{t \to 0} \left\langle \frac{\pi(\exp tX)v - v}{t}, u \right\rangle. \tag{6.7}$$

It then follows by the Banach-Steinhaus theorem that the weak limit

$$\operatorname*{w-lim}_{t \to 0} \frac{\pi(\exp tX)v - v}{t} \tag{6.8}$$

exists in $\mathcal{H}$. Let us denote it by $w \in \mathcal{H}$.

Since $d\pi(X)$ is skew-adjoint we know that it is maximally skew-symmetric, i.e. that it has no proper skew-symmetric extension, for a proof see e.g. [27, p. 354]. We claim that this fact implies that $v \in \mathcal{D}(X)$ and $d\pi(X)v = w$. To prove this claim define another unbounded operator A on $\mathcal{H}$ by

$$Au := \operatorname*{w-lim}_{t\to 0} \frac{\pi(\exp tX)u - u}{t}, \tag{6.9}$$

with domain $\mathcal{D}(A)$ those $u \in \mathcal{H}$ for which this weak limit exists. Clearly A is an extension of $d\pi(X)$, and it is easy to check that A is also skew-symmetric. However, since $d\pi(X)$ was maximally skew-symmetric we must have $A = d\pi(X)$, which proves our claim.

We have thus shown that

$$C_w^1(\pi) \subseteq \bigcap_{X\in\mathfrak{g}} \mathcal{D}(d\pi(X)) \subseteq \bigcap_{i=1}^{n} \mathcal{D}(d\pi(X_i)), \tag{6.10}$$

and hence that $C_w^1(\pi) = C^1(\pi)$. Now assume by induction that we have shown that $C_w^{k-1}(\pi) = C^{k-1}(\pi)$. Letting $v \in C_w^k(\pi)$, then by induction $v \in C^{k-1}(\pi)$ and thus $\partial_{Y_1}\cdots\partial_{Y_{k-1}}\tilde{v}$ exists for all $Y_1,\dots,Y_{k-1} \in \mathfrak{g}$. Moreover, it is weakly of class C^1 and equal to $\pi(g)d\pi(Y_1)\cdots d\pi(Y_{k-1})v$ by (6.4). Hence $d\pi(Y_1)\cdots d\pi(Y_{k-1})v \in C^1(\pi)$, which shows that $v \in C^k(\pi)$. □

Let us denote by $\mathrm{Skew}(C^1(\pi),\mathcal{H})$ the vector space of all linear maps $C^1(\pi) \to \mathcal{H}$ which are skew-symmetric with respect to the inner product of $\mathcal{H}$. It is then clear that for each $X \in \mathfrak{g}$ we have $d\pi(X) \in \mathrm{Skew}(C^1(\pi),\mathcal{H})$.

Corollary 6.2. *Let G be a Lie group, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a unitary representation. The map $d\pi\colon \mathfrak{g} \to \mathrm{Skew}(C^1(\pi),\mathcal{H})$ is linear.*

Proof. It suffices to show that the map $\mathfrak{g} \times C^1(\pi) \to \mathcal{H}$ given by $(v, X) \mapsto d\pi(X)v$ is bilinear. This map is clearly linear in v and so we only need to show linearity in X. Fixing $v \in \mathfrak{g}$ we have that $d\pi(X)v = (D_e\tilde{v})X$, where $(D_e\tilde{v})$ is the total derivative of $\tilde{v}$ at the identity element of G. It is then clear from the linearity of $D_e\tilde{v}$ that the above map is also linear in X. □

Corollary 6.3. *Let G be a Lie group, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a unitary representation. If $v \in C^2(\pi)$ then for all $X, Y \in \mathfrak{g}$ we have*

$$d\pi([X,Y])v = d\pi(X)d\pi(Y)v - d\pi(Y)d\pi(X)v. \tag{6.11}$$

Proof. The Lie bracket on $\mathfrak{g}$ is defined such that $\partial_{[X,Y]} = \partial_X\partial_Y - \partial_Y\partial_X$. In particular $\partial_{[X,Y]}\tilde{v} = \partial_X\partial_Y\tilde{v} - \partial_Y\partial_X\tilde{v}$, and so the result follows from (6.4). □

We would like to define a map from the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ into some algebra of operators on $\mathcal{H}$. However, since the $d\pi(X)$'s are unbounded it is not so easy to get an algebra from the collection of all the $d\pi(X)$'s. The idea is to observe that if $v \in C^{\infty}(\pi)$ then $d\pi(X)v \in C^{\infty}(\pi)$ for all $X \in \mathfrak{g}$, and hence the algebra of operators we want is the endomorphism algebra $\mathrm{End}(C^{\infty}(\pi))$. Let us in this regard also note that several authors has shown that $C^{\infty}(\pi)$ is dense in $\mathcal{H}$ [4, 12, 13, 21].

Corollary 6.4. *Let G be a Lie group, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a unitary representation. The derivative $d\pi\colon \mathfrak{g} \to \mathrm{Skew}(C^{\infty}(\pi))$ induces a unique algebra homomorphism $d\pi\colon U\mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(C^{\infty}(\pi))$.*

Proof. Clearly $\mathrm{Skew}(C^{\infty}(\pi))$ is a Lie subalgebra of $\mathrm{End}(C^{\infty}(\pi))$ with the commutator as Lie bracket. Hence corollary 6.3 tells us that $d\pi\colon \mathfrak{g} \to \mathrm{Skew}(C^{\infty}(\pi))$ is a Lie algebra homomorphism. Complexifying both sides we get a Lie algebra homomorphism $d\pi\colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(C^{\infty}(\pi))$, which by the universal property of the universal enveloping algebra induces a unique algebra homomorphism $d\pi\colon U\mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(C^{\infty}(\pi))$. □

6.2 The Derivative of a Projective Unitary Representation

In the projective case we generally do not get a one-parameter subgroup from the mapping $t \mapsto \pi(\exp tX)$, and hence we cannot use Stone's theorem. However, we can use the limit in (6.1) to define the derivative of a projective unitary representation, and so this is what we will do.

Definition 6.5. Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a projective unitary representation, and $X \in \mathfrak{g}$ a vector in the Lie algebra of G. The derivative of π with respect to X is the unbounded operator $d\pi(X)$ on $\mathcal{H}$ defined by

$$d\pi(X)v \coloneqq \lim_{t\to 0} \frac{\pi(\exp tX)v - v}{t}, \tag{6.12}$$

with domain $\mathcal{D}(d\pi(X))$ those $v \in \mathcal{H}$ for which this limit exists.

Note that a priori we do not even know if $d\pi(X)$ is densely defined or skew-symmetric. The following result clears up our ignorance.

Lemma 6.6. *Let G be a Lie group, $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a smooth projective representation, and $X \in \mathfrak{g}$ a vector in the Lie algebra of G. The derivative of π with respect to X is an unbounded skew-adjoint operator on $\mathcal{H}$.*

Proof. Let H be the Associated Heisenberg group of π, and let $\tilde{\pi}$ the lift of π to H given in section 5.2.1. The idea is then to compare $d\pi$ to the non-projective derivative $d\tilde{\pi}$, which by Stone's theorem is skew-adjoint.

Let $\mathfrak{h}$ be the Lie algebra of H. The total derivative of the natural projection from H to G gives a surjective Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{g}$. Let $\tilde{X} \in \mathfrak{h}$ be any vector in the preimage of X, and let $\varphi\colon \mathbb{R} \to H$ be the one-parameter subgroup of $\tilde{X}$. When writing $\varphi = (\varphi_1, \varphi_2)$ it is clear that $\varphi_1(t) = \exp tX$. Hence

$$\begin{aligned} d\tilde{\pi}\big(\tilde{X}\big)v &= \lim_{t\to 0} \frac{\tilde{\pi}(\varphi(t))v - v}{t} \\ &= \lim_{t\to 0} \frac{\varphi_2(t)\pi(\exp tX)v - v}{t} \\ &= \lim_{t\to 0} \frac{\varphi_2(t) - 1}{t}\pi(\exp tX)v + \frac{\pi(\exp tX)v - v}{t}, \end{aligned} \tag{6.13}$$

and we see that $\mathcal{D}(d\pi(X)) = \mathcal{D}\big(d\tilde{\pi}\big(\tilde{X}\big)\big)$ and $d\pi(X)v = d\tilde{\pi}\big(\tilde{X}\big)v - \varphi_2'(0)v$. In particular if we choose $\tilde{X}$ such that $\varphi_2'(0) = 0$, then $d\pi(X) = d\tilde{\pi}\big(\tilde{X}\big)$ which proves the lemma. □

Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a smooth projective representation. Looking at the proof of lemma 6.6 let us see if we can learn more about $d\pi$ from this idea. As such let H be the associated Heisenberg group of π, and let $\tilde{\pi}$ the lift of π to H given in section 5.2.1.

Now, define $C^k(\pi)$ and $C^k_w(\pi)$ exactly the same way as we did in the previous section. Since $\tilde{\pi}(g,z)v = z\pi(g)v$ for $v \in \mathcal{H}$ we immediately see that

Lemma 6.7. *Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ and $\tilde{\pi}\colon H \to \mathrm{U}(\mathcal{H})$ be as above. We then have $C^k(\pi) = C^k(\tilde{\pi})$ and $C^k_w(\pi) = C^k_w(\tilde{\pi})$.* □

As we will see in corollary 6.9 the derivative $d\pi(X)$ is linear in $X \in \mathfrak{g}$. Hence we should investigate if it preserves the Lie bracket as well. As such choose for each $X \in \mathfrak{g}$ an $\tilde{X} \in \mathfrak{h}$ such that $d\pi(X) = d\tilde{\pi}\big(\tilde{X}\big)$. Then for a pair $X, Y \in \mathfrak{g}$ we have that $\big[\tilde{X}, \tilde{Y}\big]$ is in the preimage of $[X, Y]$ and hence if $v \in C^2(\pi)$ we get by corollary 6.3 that

$$d\pi(X)d\pi(Y)v - d\pi(Y)d\pi(X)v = d\pi([X,Y])v + \varphi_2'(0)v, \tag{6.14}$$

where $\varphi = (\varphi_1, \varphi_2)$ is the one-parameter subgroup of $\big[\tilde{X}, \tilde{Y}\big]$.

Note however that we do not necessarily have $\varphi_2'(0) = 0$, or in other words that $\big[\tilde{X}, \tilde{Y}\big]$ need not be equal to the chosen lift of $[X, Y]$. Thus $d\pi$ does not preserve the Lie bracket, and if we want $d\pi$ to be a Lie algebra homomorphism into $\mathrm{Skew}(C^\infty(\pi))$ we need to mod out by $\mathbb{R}i$. Here it is important to observe that $\mathbb{R}i$ is a Lie algebra ideal of $\mathrm{Skew}(C^\infty(\pi))$. We may also complexify and get a complex Lie algebra homomorphism $d\pi\colon \mathfrak{g}_\mathbb{C} \to \mathrm{End}(C^\infty(\pi))/\mathbb{C}$.

Let us summarize what we now know about $d\pi$. For each $X \in \mathfrak{g}$ we have that $d\pi(X)$ is an unbounded skew-adjoint operator on $\mathcal{H}$, but $d\pi(X)$ is not a Lie algebra homomorphism in X unless we mod out by the multiplication operators. In other words $d\pi(X)$ can be thought of as an unbounded operator on $\mathcal{H}$ defined up to shifting by a scalar.

Now that we have a better understanding of $d\pi$ let us end by generalizing proposition 6.1 to the projective setting. As far as I can tell this result is new to the literature.

Proposition 6.8. *Let G be a Lie group, $\mathcal{H}$ be a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a smooth projective unitary representation. For any basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ we have that*

$$C^k(\pi) = C^k_w(\pi) = \bigcap_{1 \leqslant i_1, \dots, i_k \leqslant n} \mathcal{D}(d\pi(X_{i_1}) \cdots d\pi(X_{i_k})). \tag{6.15}$$

Moreover, for any $X \in \mathfrak{g}$ and $v \in C^1(\pi)$ we have that

$$\partial_X \tilde{v}(g) = \partial_X c_g(e)\pi(g)v + \pi(g)d\pi(X)v \tag{6.16}$$

where $c_g(h) = c(g, h)$.

Proof. Using the idea of the proof of lemma 6.6 we see that (6.15) follows from proposition 6.1 and lemma 6.7.

To prove (6.16) let $X \in \mathfrak{g}$ and assume $v \in C^1(\pi)$. By (6.15) we then know that $v \in \mathcal{D}(d\pi(X))$. We also know that $\partial_X \tilde{v}$ is equal to the Lie derivative of $\tilde{v}$ with respect to ∂_X, i.e. that it is equal to the following limit

$$\begin{aligned} \partial_X \tilde{v}(g) &= \lim_{t\to 0} \frac{\pi(g \exp tX)v - \pi(g)v}{t} \\ &= \lim_{t\to 0} \frac{c(g, \exp tX)\pi(g)\pi(\exp tX)v - \pi(g)v}{t} \\ &= \pi(g)\left(\lim_{t\to 0} \frac{c(g, \exp tX)\pi(\exp tX)v - v}{t} \right). \end{aligned} \tag{6.17}$$

By the product rule we have that

$$\begin{aligned} \lim_{t\to 0} \frac{c(g, \exp tX)\pi(\exp tX)v - v}{t} &= \partial_X c_g(e)\pi(e)v + c(g, e)d\pi(X)v \\ &= \partial_X c_g(e)v + d\pi(X)v, \end{aligned} \tag{6.18}$$

and hence (6.16) follows. □

Corollary 6.9. *Let G be a Lie group, $\mathcal{H}$ be a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a smooth projective unitary representation. The derivative $d\pi\colon \mathfrak{g} \to \mathrm{Skew}(C^1(\pi), \mathcal{H})$ is linear.*

Proof. It suffices to show that the map $\mathfrak{g} \times C^1(\pi) \to \mathcal{H}$ given by $(v, X) \mapsto d\pi(X)v$ is bilinear. This map is clearly linear in v and so we only need to show linearity in X.

Let c be the cocycle map of π and note that $\pi(e) = I$ implies that $c(e, g) = 1$ for all $g \in G$. Hence if we fix $v \in C^1(\pi)$, then it follows from (6.16) in proposition (6.8) that $\partial_X \tilde{v}(e) = d\pi(X)v$ for all $X \in \mathfrak{g}$. Since $\tilde{v}$ is of class C^1 we also have that $\partial_X \tilde{v}(e) = (D_e\tilde{v})X$, where $(D_e\tilde{v})$ is the total derivative of $\tilde{v}$ at the e. It is thus clear from the linearity of $D_e\tilde{v}$ that $(v, X) \mapsto d\pi(X)v$ is linear in X, as we wanted to show. □

Chapter 7

Partial Differential Equations

Throughout this chapter fix a Lie group G and a smooth projective unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ of G on some fixed Hilbert space $\mathcal{H}$. Our goal is to find solutions among the representation coefficients of G to systems of left-invariant linear homogeneous partial differential equations. As such we need to define what we mean by a partial differential in this setting. The following seems to us to be the "right" definition.

Firstly, we need to make a choice of basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$. Then the partial derivatives on G will by definition be the left-invariant vector fields $\partial_{X_1}, \dots, \partial_{X_n}$ (which we will denote by $\partial_1, \dots, \partial_n$). Finally a left-invariant linear homogeneous partial differential equation on G is an element of $U\mathfrak{g}_{\mathbb{C}}$. This element is of course independent of our choice of basis for $\mathfrak{g}$, but we need the basis to make it a PDE. It becomes a PDE since every element of $U\mathfrak{g}_{\mathbb{C}}$ can be written as a non-commutative polynomial in $\partial_1, \dots, \partial_n$.

7.1 First Observations

Recall that for $v \in \mathcal{H}$ we denote by $\tilde{v}\colon G \to \mathcal{H}$ the map $\tilde{v}(g) = \pi(g)v$. Say we want to solve the simple equation $\partial_k \tilde{v} = 0$. By proposition 6.8 we then have

$$\partial_k c_g(e)\pi(g)v + \pi(g)d\pi(X_k)v = 0, \tag{7.1}$$

which implies $d\pi(X_k)v = -\partial_k c_g(e)v$. This however is a problem as the left hand side do not depend on g, while the right hand side does. Since $\partial_k c_g(e)$ is in general non-zero, we see that in that case the only solution is $v = 0$.

To solve this problem we will instead of looking for solutions in $v \in \mathcal{H}$ look for solutions $a(g)\tilde{v}(g)$ where $a\colon G \to \mathbb{C}^\times$ is a non-zero scalar-valued function. Conceptually this is not surprising that we have to do when

$\pi\colon G \to \mathrm{U}(\mathcal{H})$ is a projective representation. The reason being that π more naturally acts on the projective Hilbert space $\mathrm{P}\mathcal{H}$, where the elements are the rays $\mathbb{C}^\times v$ for $v \in \mathcal{H} \setminus \{0\}$.

With this observation let us give a precise description of the problem to be solved. Given a system $D_1, \ldots, D_m \in U\mathfrak{g}_\mathbb{C}$ of left-invariant differential operators we want to find $a\colon G \to \mathbb{C}^\times$ and $u \in \mathcal{H}$ such that

$$\begin{cases} D_1\big(a(g)\langle \pi(g)u, v\rangle\big) = 0 \\ \qquad \vdots \\ D_m\big(a(g)\langle \pi(g)u, v\rangle\big) = 0 \end{cases} \tag{7.2}$$

for all $v \in \mathcal{H}$.

Our approach to solving this problem is then the following. If d is the maximum of the orders of $D_1, \ldots, D_m$, let us call this the order of the system, then we want to find $a\colon G \to \mathbb{C}^\times$ of class C^d and $u \in C^d(\pi)$ such that

$$\begin{cases} D_1(a\tilde{u}) = 0 \\ \quad \vdots \\ D_m(a\tilde{u}) = 0. \end{cases} \tag{7.3}$$

Since (7.2) are the weak derivatives of $a\tilde{u}$, it is clear that (7.3) will imply (7.2). It might still happen that there exists solutions to (7.2) which are not strongly differentiable, but in that case our approach will not find them.

7.2 Main Result

Let $D_1, \ldots, D_m \in U\mathfrak{g}_\mathbb{C}$ be a system of order d of left-invariant differential operators. For all k write $D_k = p_k(\partial_1, \ldots, \partial_n)$ for some non-commutative polynomial p_k. Assume $a\colon G \to \mathbb{C}^\times$ is of class C^d and $u \in C^d(\pi)$. Let c be the cocycle map of π and denote by $b_j(g) \coloneqq \partial_j c_g(e)$, where $c_g(h) = c(g,h)$, for $j = 1, \ldots, n$. Observe that by proposition 6.8 we have

$$\begin{aligned} \partial_j(a\tilde{u})(g) &= \partial_j a(g)\tilde{u}(g) + a(g)\partial_j\tilde{u}(g) \\ &= \partial_j a(g)\tilde{u}(g) + a(g)\big(b_j(g)\tilde{u}(g) + \pi(g)d\pi(X_j)u\big), \end{aligned} \tag{7.4}$$

and so if we multiply this with $\pi\big(g^{-1}\big)$ we get

$$\big(\partial_j a(g) + a(g)b_j(g)\big)u + a(g)d\pi(X_j)u. \tag{7.5}$$

Thus $\pi(g^{-1})D_k(a\tilde{u})(g)$, for a fixed k, is some linear combination of the vectors u and $d\pi(X_{j_1})\cdots d\pi(X_{j_r})u$ for $1 \le j_1, \ldots, j_r \le n$ and $r \le d$.

In order to solve (7.3) we therefore need to find $u \in C^d(\pi)$ such that u and the $d\pi(X_{j_1})\cdots d\pi(X_{j_r})u$'s are linearly dependent, and solve partial differential equations in a such that the linear dependence is realized in $\pi(g^{-1})D_k(a\tilde{u})(g)$. Our main result is a sufficient condition to guarantee this.

Theorem 7.1. *Let G be a Lie group, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a smooth projective unitary representation. Let $D_1, \ldots, D_m \in U\mathfrak{g}_{\mathbb{C}}$ be a system of order d of left-invariant differential operators. For all k write $D_k = p_k(\partial_1, \ldots, \partial_n)$ for some non-commutative polynomial p_k. If there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $u \in C^d(\pi)$ such that*

$$\begin{cases} p_1(d\pi(X_1) - \lambda_1, \ldots, d\pi(X_n) - \lambda_n)u = 0 \\ \qquad\qquad\vdots \\ p_m(d\pi(X_1) - \lambda_1, \ldots, d\pi(X_n) - \lambda_n)u = 0, \end{cases} \tag{7.6}$$

and $a\colon G \to \mathbb{C}^\times$ solves the system

$$\begin{cases} \partial_1 a + b_1 a = -\lambda_1 a \\ \qquad\vdots \\ \partial_n a + b_n a = -\lambda_n a \end{cases} \tag{7.7}$$

of first-order partial differential equations, where $b_j(g) = \partial_j c_g(e)$ and c is the cocycle map of π, then

$$\begin{cases} D_1(a\tilde{u}) = 0 \\ \qquad\vdots \\ D_m(a\tilde{u}) = 0. \end{cases} \tag{7.8}$$

The idea of this theorem is to use the conceptual idea discussed in the previous chapter that $d\pi(X_j)$ is best understood only up to shifting by a scalar. This is what we allow for in (7.6), and then we need (7.7) to get the right linear combination to solve (7.8).

Proof. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $u \in C^d(\pi)$ be such that (7.6) is satisfied, and let $a\colon G \to \mathbb{C}^\times$ solve (7.7). For some $v \in C^1(\pi)$ and each $j = 1, \ldots, n$ we then have by proposition 6.8 that

$$\begin{aligned} \partial_j(a\tilde{v})(g) &= \big(\partial_j a(g) + a(g)b_j(g)\big)\tilde{v}(g) + a(g)\pi(g)d\pi(X_j)v. \\ &= -\lambda_j a(g)\tilde{v}(g) + a(g)\pi(g)d\pi(X_j)v \\ &= a(g)\pi(g)(d\pi(X_j) - \lambda_j)v. \end{aligned} \tag{7.9}$$

Now the last line is also on the form $a\tilde{w}$ for $w = (d\pi(X_j) - \lambda_j)v$. Hence for each $k = 1, \ldots, m$ we get

$$D_k(a\tilde{u})(g) = a(g)\pi(g)p_k(d\pi(X_1) - \lambda_1, \ldots, d\pi(X_n) - \lambda_n)u, \tag{7.10}$$

and thus $D_k(a\tilde{u}) = 0$ by (7.6). $\square$

Observe that the system (7.7) of first-order partial differential equations is independent of the system $D_1, \ldots, D_m$. Also, it is usually solvable for every choice of $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and so the main problem is then to find a vector $u \in C^d(\pi)$ which solves (7.6) for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

Chapter 8

Application: Holomorphic Representation Coefficients

In this chapter we want to show how to apply the previous chapter to find holomorphic representation coefficients. As such our setup will be a Lie group G with a left-invariant complex structure and a smooth projective unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ of G on some Hilbert space $\mathcal{H}$.

8.1 Left-Invariant Complex Structures

For an introduction to complex manifold theory consult e.g. [14]. In order to have complex structures on our Lie group we need it to be even dimensional (over the reals). So let G be an even dimensional Lie group with Lie algebra $\mathfrak{g}$. We say that an almost complex structure on G is left-invariant if it is left-invariant as a tensor field. The left invariant almost complex structures on G are thus exactly the linear complex structures on $\mathfrak{g}$.

By a left-invariant complex structure on G we shall mean a left invariant almost complex structure which is integrable. Equivalently, a left-invariant complex structure is a complex structure such that left multiplication is holomorphic.

By the Newlander-Nirenberg theorem [23, 24] we can detect on the Lie algebra $\mathfrak{g}$ whether or not a left-invariant almost complex structure will be integrable. Namely, if J is a linear complex structure on $\mathfrak{g}$, then the associated left-invariant almost complex structure will be integrable if and only if J satisfies the Nijenhuis condition:

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY] \tag{8.1}$$

for all $X, Y \in \mathfrak{g}$. By an abuse of language we say that J is integrable if it satisfies the Nijenhuis condition.

8.2 The Cauchy–Riemann Equations

Let G be a $2n$-dimensional Lie group with an integrable linear complex structure J on its Lie algebra $\mathfrak{g}$. Through J, $\mathfrak{g}$ becomes a complex vector space of dimension n. In addition we have the complexification $\mathfrak{g}_{\mathbb{C}}$ of the real vector space $\mathfrak{g}$, which is $2n$-dimensional over $\mathbb{C}$.

As in the previous chapter our partial differential equations depend on a choice of basis. So choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ over $\mathbb{C}$. Denote by $Y_k = JX_k$ for all k. Then $\{X_1, Y_1, \dots, X_n, Y_n\}$ is a basis of $\mathfrak{g}$ over $\mathbb{R}$. Finally, in $\mathfrak{g}_{\mathbb{C}}$ let

$$Z_k = \frac{1}{2}(X_k - iY_k), \qquad \overline{Z}_k = \frac{1}{2}(X_k + iY_k) \tag{8.2}$$

for all k. Then $\{Z_1, \overline{Z}_1, \dots, Z_n, \overline{Z}_n\}$ is a basis of $\mathfrak{g}_{\mathbb{C}}$, and we denote the corresponding left-invariant vector fields by $\partial_k = \partial_{Z_k}$ and $\overline{\partial}_k = \partial_{\overline{Z}_k}$ for $k = 1, \dots, n$. The Cauchy–Riemann equations, or $\overline{\partial}$-equations, for a function f on G are then

$$\begin{cases} \overline{\partial}_1 f = 0 \\ \quad\vdots \\ \overline{\partial}_n f = 0, \end{cases} \tag{8.3}$$

and f is holomorphic if and only if it solves the Cauchy–Riemann equations.

Similarly, the ∂-equations characterizes antiholomorphic functions. For a function f on G they are

$$\begin{cases} \partial_1 f = 0 \\ \quad\vdots \\ \partial_n f = 0. \end{cases} \tag{8.4}$$

8.3 Holomorphic Representation Coefficients

Let G be an even dimensional Lie group with a left-invariant complex structure and let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a smooth projective unitary representation of G on a Hilbert Space $\mathcal{H}$. Fix some complex basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$. The problem we want to solve in this chapter is to find $u \in \mathcal{H}$ and $a\colon G \to \mathbb{C}^\times$ such that $g \mapsto a(g)\langle v, \pi(g)u\rangle$ is holomorphic on G for all $v \in \mathcal{H}$. Note that in this chapter we are considering conjugate representation coefficients. We do this to get something which is linear in v.

Now we can also describe our problem as finding u and a such that $\overline{a(g)}\tilde{u}(g)$ is weakly antiholomorphic. The reason being that a function is holomorphic if and only if its complex conjugate is antiholomorphic. In the previous chapter we said that we had no method to solve the weak problem and would instead focus on the strong case. However, in this case we can solve the weak case by a result of Grothendieck, which says that any weakly holomorphic function into a quasi-complete locally convex space is necessarily holomorphic [11]. Using this result we get the following.

Lemma 8.1. *Let G and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be as above. For $u \in \mathcal{H}$ and $a\colon G \to \mathbb{C}^\times$ the functions $g \mapsto a(g)\langle v, \pi(g)u\rangle$ are holomorphic for all $v \in \mathcal{H}$ if and only if $\overline{a(g)}\tilde{u}(g)$ is antiholomorphic.*

Proof. Recall that the complex conjugate Hilbert space $\overline{\mathcal{H}}$ of the Hilbert space $\mathcal{H}$ is defined as the same abelian group as $\mathcal{H}$ but with scalar multiplication given by $z \cdot v = \bar{z}v$ and inner product given by $\overline{\langle -, - \rangle}$. It is then clear that $g \mapsto a(g)\langle v, \pi(g)u\rangle$ being holomorphic for all $v \in \mathcal{H}$ is the same as saying that $a(g) \cdot \tilde{u}(g)$ is weakly holomorphic into $\overline{\mathcal{H}}$.

By Grothendieck's result [11] we get that $a(g) \cdot \tilde{u}(g)$ is weakly holomorphic into $\overline{\mathcal{H}}$ if and only if it is holomorphic into $\overline{\mathcal{H}}$. Thus what we need to show is that $a(g) \cdot \tilde{u}(g)$ is holomorphic into $\overline{\mathcal{H}}$ if and only if it is antiholomorphic into $\mathcal{H}$.

Firstly, in $\mathcal{H}$ we have by definition that $a(g) \cdot \tilde{u}(g)$ is equal to $\overline{a(g)}\tilde{u}(g)$. Secondly, $a(g) \cdot \tilde{u}(g)$ is holomorphic into $\overline{\mathcal{H}}$ if and only if it satisfies the Cauchy–Riemann equations (8.3). Then since real derivatives are equal in $\mathcal{H}$ and $\overline{\mathcal{H}}$ we see that the $\bar{\partial}$-equations in $\overline{\mathcal{H}}$ are the same as the ∂-equations in $\mathcal{H}$. We thus get the desired result. □

We have thus reformulated our problem to the following. Find $a\colon G \to \mathbb{C}$ and $u \in \mathcal{H}$ solving the ∂-equations:

$$\begin{cases} \partial_1(\bar{a}\tilde{u}) = 0 \\ \quad\vdots \\ \partial_n(\bar{a}\tilde{u}) = 0. \end{cases} \tag{8.5}$$

If we assume a is of class C^1 and $u \in C^1(\pi)$, then this is solved by theorem 7.1.

Is this a reasonable assumption? Using the product rule and proposition 6.8 we can easily show that if $\bar{a}\tilde{u}$ is antiholomorphic and a is differentiable in at least one point, then $u \in C^1(\pi)$. We would therefore argue that it is reasonable to assume a is differentiable and $u \in C^1(\pi)$. Our main result is then the following, which shows that in this specific case theorem 7.1 is not only sufficient but also necessary.

Theorem 8.2. *Let G be a $2n$-dimensional Lie group with a left-invariant complex structure, $\mathcal{H}$ a Hilbert space, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ a smooth projective unitary representation. For $u \in C^1(\pi)$ and $a\colon G \to \mathbb{C}^\times$ a differentiable* *function, the function* $\overline{a(g)}\tilde{u}(g)$ *is antiholomorphic if and only if* u *is an eigenvector of* $d\pi(Z_k)$ *for all* $k = 1, \ldots, n$*, and* a *solves the system*

$$\begin{cases} \bar{\partial}_1 a + \overline{b_1} a = -\overline{\lambda_1} a \\ \vdots \\ \bar{\partial}_n a + \overline{b_n} a = -\overline{\lambda_n} a \end{cases} \tag{8.6}$$

of first-order partial differential equations, where $b_k(g) = \partial_k c_g(e)$*,* c *is the cocycle map of* π*, and* λ_k *is the eigenvalue of* u *with respect to* $d\pi(Z_k)$ *for each* $k = 1, \ldots, n$*.*

Proof. The sufficiency comes from theorem 7.1, and so we only need to show the necessity. Assume therefore that $\bar{a}\tilde{u}$ is antiholomorphic. Then $\bar{a}\tilde{u}$ satisfies the ∂-equations (8.5) and so for each $k = 1, \ldots, n$ we have by proposition 6.8 that

$$0 = \partial_k(\bar{a}\tilde{u})(g) = \big(\partial_k \overline{a(g)} + \overline{a(g)} b_k(g)\big)\tilde{u}(g) + \overline{a(g)}\pi(g) d\pi(Z_k)u, \tag{8.7}$$

where $b_k(g) = \partial_k c_g(e)$ (as above). Multiplying this with $1/\overline{a(g)}\,\pi(g^{-1})$ we get

$$d\pi(Z_k)u = -\frac{\partial_k \overline{a(g)} + \overline{a(g)} b_k(g)}{\overline{a(g)}} u. \tag{8.8}$$

Hence u must be an eigenvector of $d\pi(Z_k)$, say with eigenvalue λ_k, and since the left hand side does not depend on g we get

$$-\frac{\partial_k \overline{a(g)} + \overline{a(g)} b_k(g)}{\overline{a(g)}} = \lambda_k. \tag{8.9}$$

Taking the complex conjugate of this we see that a solves the system (8.6) of PDE's. □

Chapter 9

Bargmann–Fock Spaces

This chapter is a continuation of the previous chapter. Hopefully this chapter will make it clear why in the previous chapter we shifted our focus to conjugate representation coefficients.

Fix a $2n$-dimensional Lie group G with a left-invariant complex structure J, and a smooth projective unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$. Denote by $A(G)$ the linear space of holomorphic functions on G.

Assuming we have a differentiable function $a\colon G \to \mathbb{C}^\times$ and a vector $u \in C^1(\pi)$ such that $g \mapsto a(g)\langle v, \pi(g)u\rangle$ is holomorphic for all $v \in \mathcal{H}$, then it is clear that we get a linear map $T\colon \mathcal{H} \to A(G)$ given by $(Tv)(g) = a(g)\langle v, \pi(g)u\rangle$. Our goal in this chapter is to understand this map. Our first result is the following.

Lemma 9.1. *The linear map T defined above is injective if and only if u is cyclic.*

Proof. Since $a(g) \neq 0$ for all $g \in G$ it clear that $\ker T = (\pi(G)u)^\perp$. Thus T is injective if and only if $(\pi(G)u)^\perp = 0$, and recalling that u being cyclic is defined as the closed linear span of $\pi(G)u$ being equal to $\mathcal{H}$, it is clear that T is injective if and only if u is cyclic. $\square$

Since $\bar{\partial}$ is left-invariant we have on $A(G)$ the left-regular representation $L\colon G \to \mathrm{GL}(A(G))$ given by $L_g f(x) = f(g^{-1}x)$. It is then natural to wonder if T is a transfer map, i.e. if the following diagram commutes for all $g \in G$

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\pi(g)} & \mathcal{H} \\ \Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T} \\ A(G) & \xrightarrow{L_g} & A(G). \end{array} \tag{9.1}$$

This can however not be the case when π has a non-trivial cocycle map c since L is non-projective. So what can we say? A calculation shows that

$$\begin{aligned}(T\pi(g)v)(x) &= \overline{c(g,g^{-1}x)}a(x)\langle v, \pi(g^{-1}x)u\rangle \\ &= \overline{c(g,g^{-1}x)}\frac{a(x)}{a(g^{-1}x)}(L_gTv)(x).\end{aligned} \tag{9.2}$$

We claim that the functions $W_g(x) := \overline{c(g,g^{-1}x)}a(x)/a(g^{-1}x)$ are holomorphic.

Proof. Fix a $k \in \{1, \ldots, n\}$. Using the left-invariance of $\bar{\partial}_k$ we have

$$\begin{aligned}\bar{\partial}_k W_g(x) = \overline{\partial_k c(g,g^{-1}x)}&\frac{a(x)}{a(g^{-1}x)} \\ &+ \overline{c(g,g^{-1}x)}\frac{\bar{\partial}_k a(x)a(g^{-1}x) - a(x)\bar{\partial}_k a(g^{-1}x)}{a(g^{-1}x)^2}\end{aligned} \tag{9.3}$$

Next, by theorem 8.2 there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that a solves the system of partial differential equations in (8.6). We thus have that

$$\bar{\partial}_k a(x)a(g^{-1}x) - a(x)\bar{\partial}_k a(g^{-1}x) = -a(x)a(g^{-1}x)\big(\overline{b_k(x)} - \overline{b_k(g^{-1}x)}\big), \tag{9.4}$$

where we recall that $b_k(x) = \partial_k c_x(e)$. Plugging this into (9.3) we get

$$\bar{\partial}_k W_g(x) = \frac{a(x)}{a(g^{-1}x)}\overline{\big(\partial_k c_g(g^{-1}x) - c(g,g^{-1}x)\big(b_k(x) - b_k(g^{-1}x)\big)\big)}. \tag{9.5}$$

Lastly, we then need to show that

$$\partial_k c_g(g^{-1}x) - c(g,g^{-1}x)\big(b_k(x) - b_k(g^{-1}x)\big) = 0 \tag{9.6}$$

to conclude that W_g is holomorphic. Since c is a cocycle map we have that

$$c(x,y)c(g,g^{-1}x) = c(g,g^{-1}xy)c(g^{-1}x,y). \tag{9.7}$$

Differentiating this in the y variable with respect to ∂_k and setting $y = e$ we get

$$\begin{aligned}\partial_k c_x(e)c(g,g^{-1}x) &= \partial_k c_g(g^{-1}x)c(g^{-1}x,e) + c(g,g^{-1}x)\partial_k c_{g^{-1}x}(e) \\ &= \partial_k c_g(g^{-1}x)\cdot 1 + c(g,g^{-1}x)\partial_k c_{g^{-1}x}(e).\end{aligned} \tag{9.8}$$

which is what we wanted to show. □

We have thus shown that there exists a family of holomorphic functions $\{W_g\}_{g\in G} \subseteq A(G)$ such that

$$T\pi(g) = W_g L_g T \tag{9.9}$$

for all $g \in G$.

Lastly, let us study the range of T in $A(G)$. By general linear algebra $T(\mathcal{H})$ is a linear subspace of $A(G)$ which is linearly isomorphic to $\mathcal{H}/\ker T$. Moreover we saw in the proof of lemma 9.1 that $\ker T = (\pi(G)u)^{\perp}$ is closed in $\mathcal{H}$, and hence $\mathcal{H}/\ker T$ is naturally a Hilbert space. Transferring the inner product of $\mathcal{H}/\ker T$ over to $T(\mathcal{H})$ we get the following tautological result

Proposition 9.2. *Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ and $T\colon \mathcal{H} \to A(G)$ be as above. With the Hilbert space structure on $T(\mathcal{H})$ defined above, $T\colon \mathcal{H} \to T(\mathcal{H})$ is a partial isometry of $\overline{\mathrm{span}}(\pi(G)u)$ and $T(\mathcal{H})$.*

Proof. This is completely tautological as the Hilbert space structure on $T(\mathcal{H})$ defined above simply identifies $T(\mathcal{H})$ with $\mathcal{H}/\ker T$ via T. Then by general Hilbert space theory we know that the projection map $\mathcal{H} \to \mathcal{H}/\ker T$ is a partial isometry of $(\ker T)^{\perp} = \overline{\mathrm{span}}(\pi(G)u)$ and $\mathcal{H}/\ker T$, and so the result follows. □

In the next section we will investigate when we can give a description of the inner product on $T(\mathcal{H})$ internal to $A(G)$. What we need for this are orthogonality relations.

9.1 Orthogonality Relations

In order to be able to talk about orthogonality relations we need a square-integrable representation. Recall, see e.g. [5, Chap. VII, Sec. 1], that a representation is called square-integrable if its representation coefficients are square-integrable with respect to Haar measure.

Definition 9.3. Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a square-integrable projective unitary representation. A pair of vectors $u_1, u_2 \in \mathcal{H}$ is called an admissible pair if there exists a constant $C_{u_1,u_2} \in \mathbb{C}$ such that

$$\int_G \langle v_1, \pi(g)u_1\rangle \overline{\langle v_2, \pi(g)u_2\rangle}\, dg = C_{u_1,u_2}\langle v_1, v_2\rangle, \tag{9.10}$$

for all $v_1, v_2 \in \mathcal{H}$, where dg denotes Haar measure. The identity (9.10) is then called an orthogonality relation. If $u \in \mathcal{H}$ is an admissible pair with itself we simply call u admissible.

Lemma 9.4. *Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a square-integrable projective unitary representation. If $u \in \mathcal{H}$ is admissible, then u is cyclic.*

Proof. Assume $u \in \mathcal{H}$ is admissible, and let $v \in (\pi(G)u)^{\perp}$. Then by assumption $\langle v, \pi(g)u\rangle = 0$ for all $g \in G$, and thus

$$0 = \int_G |\langle v, \pi(g)u\rangle|^2 \, dg = \int_G \langle v, \pi(g)u\rangle \overline{\langle v, \pi(g)u\rangle} \, dg = \langle v, v\rangle = \|v\|^2 \quad (9.11)$$

by the orthogonality relation. Hence $v = 0$, and thus we see that u is cyclic as $\overline{\operatorname{span}}(\pi(G)u) = (\pi(G)u)^{\perp\perp} = \mathcal{H}$. □

Before we show how to apply orthogonality relations to the study of the linear map T defined above, let us first introduce the generalized Bargmann–Fock spaces.

Definition 9.5. Let G be a Lie group with a left-invariant complex structure and $a\colon G \to \mathbb{C}^{\times}$ be a continuous function. The generalized Bargmann–Fock space with weight a is defined as the intersection $\mathcal{F}_a := A(G) \cap L^2(dg/|a|^2)$ in $L^2(dg/|a|^2)$.

We can now state our main result of this chapter. This can be seen as a refinement of proposition 9.2 in the presence of an orthogonality relation.

Theorem 9.6. *Let G be a Lie group with a left-invariant complex structure, and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a square-integrable smooth projective unitary representation. Assume we have a differentiable function $a\colon G \to \mathbb{C}^{\times}$ and a vector $u \in C^1(\pi)$ such that $g \mapsto a(g)\langle v, \pi(g)u\rangle$ is holomorphic for all $v \in \mathcal{H}$. If u is admissible with constant $C_{u,u} = 1$, then the linear map $T\colon \mathcal{H} \to A(G)$ (defined above) has range contained in the generalized Bargmann-Fock space $\mathcal{F}_a$ and $T\colon \mathcal{H} \to \mathcal{F}_a$ is an isometry.*

Proof. Let $v \in \mathcal{H}$ be a vector. Using the orthogonality relation we then calculate

$$\begin{aligned} \int_G |Tv|^2 \frac{dg}{|a|^2} &= \int_G |a(g)\langle v, \pi(g)u\rangle|^2 \frac{dg}{|a(g)|^2} \\ &= \int_G \langle v, \pi(g)u\rangle \overline{\langle v, \pi(g)u\rangle} \, dg = \langle v, v\rangle = \|v\|^2. \end{aligned} \quad (9.12)$$

Hence it is clear that $Tv \in \mathcal{F}_a$, and that $T\colon \mathcal{H} \to \mathcal{F}_a$ is an isometry. □

Chapter 10

Examples

In this section we will show examples of how to find holomorphic representation coefficients. Since non-projective representations are a special case of projective representations we will treat every representation as a projective one and use theorem 8.2 to solve the problem.

10.1 The Affine Group

Let Aff be the affine group (also called the $ax+b$ group), i.e. $\mathrm{Aff} = \mathbb{R} \times \mathbb{R}_+$ with multiplication

$$(x, s)(y, r) = (x + sy, sr), \tag{10.1}$$

and let ρ be the unitary representation of Aff on $L^2(\mathbb{R})$ given by

$$\rho(x, s)u(t) = \frac{1}{\sqrt{s}} u\left(\frac{t - x}{s}\right). \tag{10.2}$$

We note that the representation coefficients of ρ are the Wavelet transforms.

Let $\mathfrak{aff}$ be the Lie algebra of Aff. For the generators $X = (1, 0)$ and $S = (0, 1)$ of $\mathfrak{aff}$ we have that $[X, S] = -X$. Hence by (8.1) any linear complex structure on $\mathfrak{aff}$ is integrable. So let

$$J = \begin{pmatrix} p & -\frac{1+p^2}{q} \\ q & -p \end{pmatrix} \tag{10.3}$$

with $p, q \in \mathbb{R}$, $q \neq 0$, be a general linear complex structure on $\mathfrak{aff}$.

Theorem 10.1. *Give* Aff *the left invariant complex structure coming from* J *defined above, and let* $\lambda = \xi + i\gamma$ *be a complex number. If* $\frac{2\gamma}{q} > 0$, $u \in L^2(\mathbb{R})$ *is equal to*

$$u(t) = (qt + p + i)^{\frac{2\lambda i}{q} - \frac{1}{2}}, \tag{10.4}$$

and $a\colon \mathrm{Aff} \to \mathbb{C}^\times$ *has the form*

$$a(x,s) = A(x,s) s^{\frac{2\bar{\lambda} i}{q}}, \tag{10.5}$$

where $A\colon \mathrm{Aff} \to \mathbb{C}^\times$ *is any holomorphic function, then the functions* $(x,s) \mapsto a(x,s)\langle v, \rho(x,s)u\rangle$ *are holomorphic for all* $v \in L^2(\mathbb{R})$*.*

Proof. Let $X, S \in \mathfrak{aff}$ be as above. Choosing X as our complex generator of $\mathfrak{aff}$ we get $Y = JX = pX + qS$ and $Z = \frac{1}{2}(X - iY)$ with the notation of section 8.2.

In order to calculate $d\rho(Z)$ we need the one-parameter subgroups of X and S. One finds that they are given by

$$\exp tX = (t,1), \qquad \exp tS = \bigl(0, e^t\bigr), \tag{10.6}$$

and we can then calculate

$$d\rho(X)u(t) = -u'(t), \qquad d\rho(S)u(t) = -\frac{1}{2}u(t) - tu'(t). \tag{10.7}$$

By corollary 6.2 we know that $d\rho$ is linear, and so we get that

$$\begin{aligned} d\rho(Z) &= \frac{1}{2}\bigl(d\rho(X) - ipd\rho(X) - iqd\rho(S)\bigr) \\ &= \frac{i}{2}\left(\frac{q}{2} + (p+i)\frac{d}{dt} + qt\frac{d}{dt}\right). \end{aligned} \tag{10.8}$$

Now by theorem 8.2 the first problem we need to solve is to find the eigenvectors of $d\rho(Z)$. In other words we need to find $u \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C}$ such that

$$\frac{q}{2}u + (p+i)u' + qtu' = 2\lambda i u. \tag{10.9}$$

Solving this differential equation we find that

$$u(t) = C(qt + p + i)^{\frac{2\lambda i}{q} - \frac{1}{2}}, \tag{10.10}$$

for C some constant. However, we also need to have $u \in L^2(\mathbb{R})$, and so we need to determine when

$$\int_{\mathbb{R}} \bigl((qt+p)^2 + 1\bigr)^{-\frac{2\gamma}{q} - \frac{1}{2}}\, dt < \infty, \tag{10.11}$$

where $\lambda = \xi + i\gamma$. One thus finds that $u \in L^2(\mathbb{R})$ if and only if $\frac{2\gamma}{q} > 0$.

Next, we then need to solve the differential equation

$$\bar{\partial} a = -\bar{\lambda} a \tag{10.12}$$

for $a\colon \mathrm{Aff} \to \mathbb{C}^\times$ and $\lambda = \xi + i\gamma$ with $\frac{2\gamma}{q} > 0$. It is clear that if we can find a single solution to this equation, then we can get every other solution by multiplication with a holomorphic function.

Now, in the standard coordinates of $\mathbb{R} \times \mathbb{R}_+$ (10.12) becomes the partial differential equation

$$\frac{\partial a}{\partial x} + i\left(p\frac{\partial a}{\partial x} + q\frac{\partial a}{\partial s}\right) = -\frac{2\bar{\lambda}}{s}a. \tag{10.13}$$

We may solve this by doing the following change of variables. Let $\tilde{x} = x - \frac{1}{q}(p-i)s$ and $\tilde{s} = x - \frac{1}{q}(p+i)s$, then (10.13) becomes

$$\frac{\partial a}{\partial \tilde{s}} = -\frac{2\bar{\lambda}i}{q(\tilde{x}-\tilde{s})}a, \tag{10.14}$$

which is solved by

$$a(\tilde{x},\tilde{s}) = B(\tilde{x})(q(\tilde{x}-\tilde{s}))^{\frac{2\bar{\lambda}i}{q}} \tag{10.15}$$

for B any function independent of $\tilde{s}$. In (x,s)-coordinates this becomes

$$a(x,s) = A(x,s)s^{\frac{2\bar{\lambda}i}{q}} \tag{10.16}$$

where A is any holomorphic function. This completes the proof of the result. □

10.2 The Heisenberg Representation

The Heisenberg representation is the projective unitary representation of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ given by

$$\rho(x,y)f(t) = e^{2\pi i y\cdot t}f(t-x), \tag{10.17}$$

where $y \cdot t$ denotes the standard inner product on $\mathbb{R}^n$. We note that the representation coefficients of ρ are the short time Fourier transforms.

The Lie algebra of $\mathbb{R}^{2n}$ is trivial, and hence by (8.1) any linear complex structure on it is integrable. However, we need some control over the complex structure in order to solve the given problem. We will therefore only consider those linear complex structures J which are adapted to the symplectic form $\Omega\big((x_1,y_1),(x_2,y_2)\big) = x_2\cdot y_1 - x_1\cdot y_2$ on $\mathbb{R}^{2n}$. By adapted we mean that $G(x,y) := \Omega(x,Jy)$ is an inner product. For further definitions and terminology of symplectic geometry see [18, Ch. I]. If we write J as a 2×2-block matrix

$$J = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{10.18}$$

then this assumption on J is equivalent to $-R$ and Q being symmetric positive-definite, and $S = -P^\intercal$.

Theorem 10.2. *Give $\mathbb{R}^{2n}$ the left invariant complex structure coming from J defined above, and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. If $u \in L^2(\mathbb{R}^n)$ is equal to*

$$u(t) = e^{-\pi t M(t-\Lambda)}, \tag{10.19}$$

with $M = (I + iS)Q^{-1}$ and $\Lambda = \frac{2}{\pi} R^{-1}\lambda$, and $a \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^\times$ has the form

$$a(x, y) = A(x, y) e^{-\frac{\pi}{2}(xRx + 2xSy - yQy) - \bar{\lambda}\cdot x + \pi i x\cdot y + \bar{\lambda}(P+iI)R^{-1}y}, \tag{10.20}$$

where $A \colon \mathbb{R}^{2n} \to \mathbb{C}^\times$ is any holomorphic function, then the functions $(x, y) \mapsto a(x, y)\langle v, \rho(x, y)u\rangle$ are holomorphic for all $v \in L^2(\mathbb{R}^n)$.

Proof. Let $V = \mathbb{R}^n \times \{0\}$, and note that this is a Lagrangian subspace of $(\mathbb{R}^{2n}, \Omega)$. Hence it follows that JV is orthogonal to V with respect to G. In particular any real basis of V is a complex basis of $\mathbb{R}^{2n}$ with respect to the complex structure J. In particular if $\{X_1, \ldots, X_{2n}\}$ is the standard basis of $\mathbb{R}^{2n}$, then $\{X_1, \ldots, X_n\}$ is a basis of V. Following the notation of section 8.2 we then have

$$Y_k = JX_k = \sum_{j=1}^{n} p_{jk} X_j + r_{jk} X_{n+j}, \tag{10.21}$$

and

$$Z_k = \frac{1}{2}(X_k - iY_k) = \frac{1}{2}\Big(X_k - i\sum_{j=1}^{n} p_{jk} X_j + r_{jk} X_{n+j}\Big). \tag{10.22}$$

It will be easier for us to work with an equivalent representation to ρ, and so we set

$$\tilde{\rho}(x, y) \coloneqq e^{-\pi i x\cdot y}\rho(x, y). \tag{10.23}$$

We then need to calculate $d\tilde{\rho}(Z_k)$ for all $k = 1, \ldots, n$. The one-parameter subgroup of X_k is just $t \mapsto tX_k$ and hence we find that

$$d\tilde{\rho}(X_k)f(t) = -\frac{\partial f}{\partial t_k}(t), \qquad d\tilde{\rho}(X_{n+k})f(t) = 2\pi i t_k f(t). \tag{10.24}$$

Using the fact that $d\tilde{\rho}$ is linear, c.f. corollary 6.9, we then get that

$$\begin{aligned} d\tilde{\rho}(Z_k) &= \frac{1}{2}\Big(-\frac{\partial}{\partial t_k} + \sum_{j=1}^{n} i p_{jk}\frac{\partial}{\partial t_j} + 2\pi r_{jk} t_j\Big) \\ &= \frac{1}{2}\big((iP^\intercal - I)\nabla + 2\pi R^\intercal t\big)_k. \end{aligned} \tag{10.25}$$

We are now ready to solve the problem of finding all holomorphic representation coefficients of $\tilde{\rho}$. By theorem 8.2 the first step is to find the simultaneous eigenvectors of $d\tilde{\rho}(Z_k)$ for $k = 1, \dots, n$. In other words we need to find $u \in L^2(\mathbb{R}^n)$ such that

$$\big(iP^\intercal - I\big)\nabla u = 2u\big(\lambda - \pi R^\intercal t\big). \tag{10.26}$$

Recall that $P^\intercal = -S$, and $-R$ is symmetric. Furthermore, we claim that $iS - I$ is invertible. For we have $(iS - I)(iS + I) = -S^2 - I$ and $S^2 + RQ = -I$, and hence $(iS+I)Q^{-1}R^{-1}$ is the inverse of $iS - I$. Thus if we let $\Lambda = \frac{2}{\pi}R^{-1}\lambda$ and $M = (iS + I)Q^{-1}$, then (10.26) is equivalent to

$$\nabla u = -\pi u M(2t - \Lambda). \tag{10.27}$$

Next, in order to solve this we claim that M is symmetric. For we have that $M^\intercal = \big(Q^\intercal\big)^{-1}\big(I^\intercal + iS^\intercal\big) = Q^{-1}(I - iP)$, and hence

$$Q\big(M - M^\intercal\big)Q = Q(I + iS) - (I - iP)Q = i(PQ + QS) = 0. \tag{10.28}$$

This shows that $M - M^\intercal = 0$, or in other words that M is symmetric as claimed. This then implies that (10.27) is solved by

$$u(t) = Ce^{-\pi t M(t-\Lambda)}, \tag{10.29}$$

with C any constant.

Now that we have a solution to (10.27) we need to determine if it lives in $L^2(\mathbb{R}^n)$. Write $\Lambda = \xi + i\gamma$, we then have

$$|u(t)|^2 = |C|^2 e^{-2\pi t Q^{-1}(t-\xi) + 2\pi t S Q^{-1}\gamma}, \tag{10.30}$$

and hence we see that u is always in $L^2(\mathbb{R}^n)$ by the fact that Q^{-1} is positive-definite.

Next, we then need to solve the first-order partial differential equations in (8.6). As such we need to determine b_k. The cocycle map of $\tilde{\rho}$ can be seen to equal

$$c\big((x, y), (z, w)\big) = e^{\pi i (x\cdot w - z\cdot y)}, \tag{10.31}$$

One then finds for $k = 1, \dots, n$ that

$$\partial_{X_k} c_{(x,y)}(0, 0) = -\pi i y_k, \qquad \partial_{X_{n+k}} c_{(x,y)}(0, 0) = \pi i x_k, \tag{10.32}$$

and hence

$$b_k(x, y) = -\frac{\pi}{2}\left(iy_k + \sum_{j=1}^{n} p_{jk} y_j - r_{jk} x_j\right). \tag{10.33}$$

Also, in the standard coordinates of $\mathbb{R}^n \times \mathbb{R}^n$ we have that

$$\bar{\partial}_k a = \frac{1}{2}\bigg(\frac{\partial a}{\partial x_k} + i\sum_{j=1}^{n} p_{jk}\frac{\partial a}{\partial x_j} + r_{jk}\frac{\partial a}{\partial y_j}\bigg). \tag{10.34}$$

Thus if we write $\nabla_x = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$ and $\nabla_y = \big(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\big)$, then we need to solve the partial differential equation

$$(I - iS)\nabla_x a + iR\nabla_y a = -\pi a(Rx + Sy + iy) - 2a\bar{\lambda}. \tag{10.35}$$

We may solve this by doing the following change of variables. Let $\tilde{x} = x - (P - iI)R^{-1}y$ and $\tilde{y} = x - (P + iI)R^{-1}y$, then (10.35) becomes

$$\nabla_{\tilde{y}} a = -a\Big(\frac{\pi}{2}R\tilde{x} + \bar{\lambda}\Big). \tag{10.36}$$

Which is solved by $\tilde{a}(\tilde{x}, \tilde{y}) = A(\tilde{x})e^{-\frac{\pi}{2}\tilde{y}R\tilde{x} - \bar{\lambda}\cdot\tilde{y}}$, for any $A\colon \mathbb{R}^{2n} \to \mathbb{C}$ which is independent of $\tilde{y}$. In (x, y)-coordinates this becomes

$$\tilde{a}(x, y) = A(x, y)e^{-\frac{\pi}{2}(xRx + 2xSy - yQy) - \bar{\lambda}\cdot x + \bar{\lambda}(P+iI)R^{-1}y}, \tag{10.37}$$

with A holomorphic. Lastly, since we worked with $\tilde{\rho}$ instead of ρ we need to set $a(x, y) = e^{\pi i x\cdot y}\tilde{a}(x, y)$, and we get the desired result. □

Before moving on to the next example we want to comment on how theorem 10.2 fits in with Bargmann's result, which we revealed in the introduction was what led to this research. Recall from the intro that Bargmann showed in [2], among other things, that for every $f \in L^2(\mathbb{R})$ the functions

$$F(x + iy) = e^{\frac{\pi}{2}|x+iy|^2 - \pi i x y}V_{g_0}f(x, -y) \tag{10.38}$$

are entire, when $g_0(t) = \sqrt[4]{2}e^{-\pi t^2}$ and

$$V_g f(x, y) = \int_{\mathbb{R}} f(t)\overline{g(t - x)}e^{-2\pi i y t}\, dt. \tag{10.39}$$

This is related to the Heisenberg representation of $\mathbb{R}^2$ by the fact that $V_g f(x, y) = \langle f, \rho(x, y)g\rangle$. Thus we should be able to use theorem 10.2 to find (10.38)

Step one is then to fix a complex structure on $\mathbb{R}^2$. Since we have the complex conjugate inside $V_{g_0}f$ in (10.38) the correct choice of complex structure will be the opposite of the usual complex structure, i.e.

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10.40}$$

It is then easy to check that J is adopted to Ω and so we may apply theorem 10.2.

Observe that $M = I$ and $\Lambda = -\frac{2}{\pi}\lambda$ for $\lambda \in \mathbb{C}$ any complex number. Thus theorem 10.2 tells us that

$$(x, y) \mapsto A(x, y)e^{\frac{\pi}{2}(x^2+y^2)+\pi ixy-\bar{\lambda}(x+iy)}\left\langle f, \rho(x, y)e^{-\pi t^2-2\lambda t}\right\rangle \tag{10.41}$$

is holomorphic, for any holomorphic function A. Note that we are here talking about being holomorphic with respect to J. If we want to consider the usual complex structure on $\mathbb{R}^2$ with $(x, y) = x + iy$ we need to precompose with complex conjugation. In other words theorem 10.2 tells us that

$$z \mapsto A(z)e^{\frac{\pi}{2}|z|^2-\pi ixy-\bar{\lambda}\bar{z}}\left\langle f, \rho(x, -y)e^{-\pi t^2-2\lambda t}\right\rangle \tag{10.42}$$

is entire for any complex number $\lambda \in \mathbb{C}$ and entire function A.

As expected we thus get Bargmann's result from theorem 10.2 by setting $\lambda = 0$ and A equal to the constant $\sqrt[4]{2}$. Moreover, it appears that we have found more choices of g_0 which lead to holomorphic STFT's. However this is only an illusion. For we have

$$\rho\left(\overline{z - \tfrac{\bar{\lambda}}{\pi}}\right)e^{-\pi t^2} = e^{-2ix-\pi^{-1}\alpha^2}\rho(\bar{z})e^{-\pi t^2-2\lambda t}, \tag{10.43}$$

when $\lambda = \alpha + i\beta$. Hence if we let F be as in (10.38) we may calculate that

$$F\left(z - \tfrac{\bar{\lambda}}{\pi}\right) = e^{\pi^{-1}\left(\frac{1}{2}|\lambda|^2+i\alpha\beta-\alpha^2\right)}e^{\frac{\pi}{2}|z|^2-\pi ixy-\bar{\lambda}\bar{z}}\left\langle f, \rho(\bar{z})e^{-\pi t^2-2\lambda t}\right\rangle, \tag{10.44}$$

and thus up to a constant, i.e. an appropriate choice of A, (10.42) is just a linear shift of F.

10.3 Second Order Heisenberg Representation

This representation is taken from [10, Example 3.3]. Let H be the unreduced Heisenberg group and let $G = \mathbb{R}\times H$. Explicitly $G = \mathbb{R}^4$ with multiplication given by

$$xy = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + x_2y_3). \tag{10.45}$$

There is a projective unitary representation $\rho\colon G \to \mathrm{U}\big(L^2(\mathbb{R}^2)\big)$, called the second order Heisenberg representation, given by

$$\rho(x)f(s, t) = e^{\pi iP(s,t,x)}f(s - x_1, t - x_2), \tag{10.46}$$

where P is the polynomial

$$P(s,t,x) = 2x_3 s - 2x_4 t + x_3 t^2 - x_1 x_3 + x_2 x_4 - \frac{1}{6} x_2^2 x_3. \tag{10.47}$$

The cocycle map of ρ is equal to

$$c(x,y) = e^{\pi i \left(x_1 y_3 - y_1 x_3 - x_2 y_4 + y_2 x_4 - \frac{1}{3} x_2 x_3 y_2 + \frac{2}{3} x_2 y_2 y_3 - \frac{1}{6} x_2^2 y_3 - \frac{1}{6} y_2^2 x_3 \right)}, \tag{10.48}$$

which is smooth and invariant on all one-parameter subgroups of G.

The Lie algebra of G is equal to $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H. Hence if we identify $\mathfrak{g}$ with $\mathbb{R}^4$ and write $\{X_1, X_2, X_3, X_4\}$ for the standard basis, then the only non-zero bracket is given by

$$[X_2, X_3] = X_4. \tag{10.49}$$

Using (8.1) one then finds that the left invariant complex structures on G are given by the linear complex structures J on $\mathfrak{g}$ of the form

$$J = \begin{pmatrix} p & r & \frac{1+p^2}{qv} w - \frac{p+u}{v} r & -\frac{1+p^2}{q} \\ 0 & u & -\frac{1+u^2}{v} & 0 \\ 0 & v & -u & 0 \\ q & w & \frac{pw - qr - uw}{v} & -p \end{pmatrix} \tag{10.50}$$

with $p, r, u, w \in \mathbb{R}$ and $q, v \in \mathbb{R} \setminus \{0\}$. Fix one such choice of J.

Theorem 10.3. *Give G the left invariant complex structure coming from J defined above. There is no non-trivial choice of $g \in L^2(\mathbb{R}^2)$ and $a \colon G \to \mathbb{C}^\times$ such that $x \mapsto a(x)\langle f, \rho(x) g\rangle$ is holomorphic for all $f \in L^2(\mathbb{R}^2)$.*

Proof. One can easily check that X_1 and X_2 are linearly independent over $\mathbb{C}$ with respect to J. We therefore choose $\{X_1, X_2\}$ as our complex basis, and we get

$$\begin{aligned} Z_1 &= \frac{1}{2}\big(X_1 - i(pX_1 + qX_4)\big), \\ Z_2 &= \frac{1}{2}\big(X_2 - i(rX_1 + uX_2 + vX_3 + wX_4)\big) \end{aligned} \tag{10.51}$$

with the notation of section 8.2.

Next we want to calculate $d\rho(Z_1)$ and $d\rho(Z_2)$. The one-parameter subgroup of X_k is $t \mapsto tX_k$, and so we find that

$$\begin{aligned} d\rho(X_1) f(s,t) &= -\frac{\partial f}{\partial s}(s,t) & d\rho(X_2) f(s,t) &= -\frac{\partial f}{\partial t}(s,t), \\ d\rho(X_3) f(s,t) &= \pi i \big(2s + t^2\big) f(s,t), & d\rho(X_4) f(s,t) &= -2\pi i t f(s,t). \end{aligned} \tag{10.52}$$

Using the fact that $d\rho$ is linear, c.f. corollary 6.9, we then get

$$\begin{aligned} d\rho(Z_1)f &= \frac{ip-1}{2}\frac{\partial f}{\partial s} - \pi q t f, \\ d\rho(Z_2)f &= \frac{ir}{2}\frac{\partial f}{\partial s} + \frac{iu-1}{2}\frac{\partial f}{\partial t} + \pi\Big(\frac{v}{2}t^2 - wt + vs\Big)f. \end{aligned} \tag{10.53}$$

By theorem 8.2 we now need to find simultaneous eigenvectors of $d\rho(Z_1)$ and $d\rho(Z_2)$. So let us start with $d\rho(Z_1)$. One finds that

$$\begin{aligned} d\rho(Z_1)f &= \lambda f \\ \frac{ip-1}{2}\frac{\partial f}{\partial s} - q\pi t f &= \lambda f \end{aligned} \tag{10.54}$$

is solved by

$$f(s,t) = g(t)\exp\Big(\frac{\pi q t + \lambda}{ip-1}2s\Big), \tag{10.55}$$

for g any function depending only on t. Plugging this f into $d\rho(Z_2)f = \mu f$ we get the ordinary differential equation

$$\frac{q\pi t + \lambda}{ip-1}irg + \frac{iu-1}{2}\frac{dg}{dt} + \frac{iu-1}{ip-1}\pi q s g + \pi\Big(\frac{v}{2}t^2 - wt + vs\Big)g = \mu g. \tag{10.56}$$

In order to solve this we need it to be independent of s. In other words we need

$$\frac{iu-1}{ip-1}\pi q s + \pi v s = 0, \tag{10.57}$$

which is equivalent to $v = -q$ and $u = p$. Assuming this we get the equation

$$\frac{q\pi t + \lambda}{ip-1}irg + \frac{ip-1}{2}\frac{dg}{dt} - \pi\Big(\frac{q}{2}t^2 + wt\Big)g = \mu g, \tag{10.58}$$

which is solved by

$$g(t) = C\exp\Big(\frac{\pi q}{3ip-3}t^3 - \frac{w + i(qr - pw)}{(ip-1)^2}\pi t^2 - \frac{\mu - ip\mu + ir\lambda}{(ip-1)^2}2t\Big), \tag{10.59}$$

for C any constant. Plugging this into (10.55) we arrive at

$$\begin{aligned} f(s,t) = C\exp\Big(&\frac{\pi q t^3 + 6\pi q s t + 6\lambda s}{3ip-3} - \frac{w + i(qr - pw)}{(ip-1)^2}\pi t^2 \\ &- \frac{\mu - ip\mu + ir\lambda}{(ip-1)^2}2t\Big). \end{aligned} \tag{10.60}$$

The last step we then need to do is to check that $f \in L^2(\mathbb{R}^2)$. Writing $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$ one might check that

$$\begin{aligned} |f(s,t)|^2 = |C|^2 \exp\bigg(&- \frac{2\pi q t^3 + 12\pi q s t + 12\lambda s}{3p^2+3} - \frac{w + p^2 w - 2pqr}{(p^2+1)^2} 2\pi t^2 \\ &+ \frac{r\lambda_2 + 2pr\lambda_1 - p^2 r\lambda_2 - (p^2+1)(\mu_1 - p\mu_2)}{(p^2+1)^2} 4t \bigg), \end{aligned} \tag{10.61}$$

and since q must be non-zero we cannot choose p, q, r, w, λ, μ such that $f \in L^2(\mathbb{R}^2)$. Hence there is no simultaneous eigenvectors of $d\rho(Z_1)$ and $d\rho(Z_2)$, and the result follows. □

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