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William Proctor Hornslien

Topics in motivic and toric homotopy theory

Doctoral thesis

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Norwegian University of Science and Technology
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Contents

Chapter 1

Introduction

This thesis is about problems in motivic and toric homotopy and consists of three papers. In the first two papers of this thesis, we will do various computations in motivic homotopy theory using different methods. In the third paper, we will study a variation of a construction used in the second paper.

Paper I is about $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, the group of $\mathbb{A}^1$-homotopy classes of endomorphisms of the projective line. The computation of the group $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a motivic homotopy theoretic analogue the computation of the fundamental group $\pi_1(S^1)$ of the circle in classical topology. Computing $\pi_1(S^1)$ is something a student might learn in their first algebraic topology class. Computing $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, on the other hand, requires highly technical abstract mathematical machinery. In Paper I, titled *Making the motivic group structure on the endomorphisms of the projective line explicit*, we describe the group $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ using elementary algebraic geometry.

Another way of making it easier to compute things is by decomposing whatever you want to study into pieces you already understand. There is a family of topological spaces called *polyhedral products* that allow us to do this. Roughly speaking, a polyhedral product is a collection topological spaces glued together according to some combinatorial data. Thus, it is not strange to expect that one can express information about a polyhedral product in terms of its combinatorial data and collection of topological spaces.

In Paper II we generalize polyhedral products such that they can be considered in motivic homotopy theory. We focus our attention to a certain family of motivic polyhedral products and use the good properties of polyhedral products to do various computations that would otherwise be difficult. The paper is titled *Polyhedral products in abstract and motivic homotopy theory*.

In Paper III we dualize the polyhedral product construction and define *polyhedral coproducts*. We then explore if and how classical theorems about polyhedral products dualize. The paper is titled *Polyhedral coproducts*.

In the upcoming sections we will introduce and discuss some mathematics that play a central role in the thesis. We also aim to prepare the reader for the more technical summaries of the papers in Section 1.3. In Section 1.1 we discuss polyhedral products and discuss other relevant information for Paper II and III.

In Section 1.2 motivic homotopy theory is introduced to prepare the reader for Paper I and II.

1.1 Polyhedral products

In this section we will present several results from toric topology. Toric topology is a field centered around studying topological spaces with torus actions and started with seminal work by Davis and Januskiewicz [11]. Polyhedral products, which are the main focus of the upcoming sections, are generalizations of certain objects called moment-angle complexes (see Section 1.1.3) from toric topology. By toric homotopy theory, we mean the use of homotopy theory to study objects that appear in toric topology.

1.1.1 The polyhedral product

Let K be a simplicial complex with m vertices and let

$$(\underline{X},\underline{A}) = ((X_1,A_1),\ldots,(X_m,A_m))$$

be a sequence of m pairs of pointed topological spaces $A_i \subseteq X_i$. We call $(\underline{X},\underline{A})$ a *family of pairs*. We will now define the polyhedral product. The following definition is due to Bahri, Bedersky, Cohen, and Gitler [5].

Definition 1.1. Let $(\underline{X},\underline{A})$ be a family of pairs and let K be a simplicial complex. We define the polyhedral product $(\underline{X},\underline{A})^K$ as the union

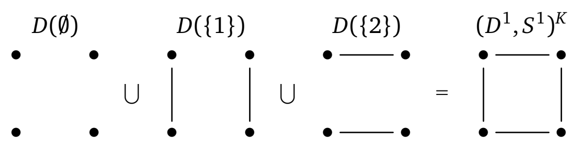
$$(\underline{X},\underline{A})^K = \bigcup_{\sigma\in K} D(\sigma) \subseteq \prod_{i=1}^{m} X_i,$$

where

$$D(\sigma) = \prod_{i=1}^{m} Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Remark 1.2. When all pairs (X_i,A_i) are the same pair, we will write $(X,A)^K$ for the polyhedral product $(\underline{X},\underline{A})^K$.

Example 1.3. Let K be two disjoint points. We wish to compute the polyhedral product $(D^1,S^0)^K$. Then $D(\emptyset) = S^0 \times S^0$, $D(\{1\}) = D^1 \times S^0$, and $D(\{2\}) = S^0 \times D^1$. The illustration below shows process of taking the union of the three pieces and that $(D^1,S^0)^K \simeq S^1$.



Example 1.4. Let $K = \partial\Delta^{m-1}$, then $(D^1, S^0)^K \simeq S^{m-1}$.

Example 1.5.
1. When K is a full $(m-1)$-simplex, the polyhedral product is the product $\prod_{i=1}^m X_i$.
2. Let K be m disjoint points, and let A_i be the basepoint of X_i for all $1 \le i \le m$. Then $(\underline{X}, \underline{A})^K = \bigvee_{i=1}^m X_i$.
3. Let $K = \partial\Delta^{m-1}$, and let A_i be the basepoint of X_i for all $1 \le i \le m$. Then $(\underline{X}, \underline{A})^K$ is the fat wedge of the spaces X_i. That is, the subset of $\prod_{i=1}^m X_i$. where at least one coordinate is the basepoint.

The previous example shows that the polyhedral product interpolates between the wedge $\bigvee_{i=1}^m X_i$ and the product $\prod_{i=1}^m X_i$ as K ranges from m disjoint points to a full $(m-1)$-simplex.

Example 1.6 ([9, Proposition 4.2.5])**.** Let K and K' be two simplicial complexes and let $K \star K'$ denote their join. Then $(X, A)^K \times (X, A)^{K'} = (X, A)^{K \star K'}$.

Even though the definition of a polyhedral product is fairly recent, it is evident that several relevant constructions in topology and homotopy theory can be expressed as polyhedral products. As we have just seen, this includes spheres, wedges of spaces, and wedge filtrations. Some further examples also include joins and half-smash products of spaces. Surveys about polyhedral products and their connections to other fields can be found in [4, 9].

1.1.2 The Stanley–Reisner ring

The following ring associated to a simplical complex plays an important role in toric topology.

Definition 1.7. Let $\mathbf{k}$ be a ring. For a simplicial complex K, we define the *Stanley–Reisner ideal* I_K as the square-free monomial ideal corresponding to non-faces of K, i.e.

$$I_K = (x_{i_1} \dots x_{i_r} | \{i_1, \dots, i_r\} \notin K).$$

We define the *Stanley–Reisner ring* as the quotient

$$\mathbf{k}[K] := \mathbf{k}[x_1, \dots, x_m]/I_K.$$

Example 1.8. Let $K = \partial\Delta^{m-1}$ and $\mathbf{k}$ be a ring. The simplicial complex K only has one missing face and its Stanley–Reisner ring is

$$\mathbf{k}[K] = \frac{\mathbf{k}[x_1, \dots, x_m]}{(x_1 x_2 \cdots x_m)}.$$

The Stanley–Reisner ring shows up in areas such as toric geometry, polytopes, and splines [21, Chapter III].

1.1.3 Moment-angle complexes

We will now present a certain family of polyhedral products named moment-angle complexes.

Definition 1.9. Let K be a simplicial complex. The *moment-angle complex* Z_K is the polyhedral product

$$Z_K := (D^2, S^1)^K.$$

Example 1.10. Let K be two disjoint points. Then

$$Z_K = D^2 \times S^1 \bigcup_{S^1 \times S^1} S^1 \times D^2 \simeq \partial(D^2 \times D^2) \simeq S^3.$$

Example 1.11. Let $K = \partial\Delta^{m-1}$, then $Z_K \simeq S^{2m-1}$.

Moment-angle complexes in the way we present them here is due to Buchstaber and Panov [8]. Moment-angle complexes naturally arise as complements of coordinate subspace arrangements, intersections of quadrics, level sets of moment maps in symplectic topology, and as complex points of toric varieties (see [4, 9]). Thus the study of moment-angle complexes is highly interdisciplinary and can be viewed from many angles. In this thesis we are mostly concerned about moment-angle complexes from the view of topology, homotopy theory, and combinatorics. For example, the following theorem gives a nice presentation of the cohomology ring of a moment-angle complex.

Theorem 1.12 ([6, Theorem 1],[7, Theorem 1])**.** *Let K be a simplicial complex on the vertex set $[m]$. Let $\mathbf{k}$ be a field or $\mathbb{Z}$. There is an isomorphism of groups*

$$H^i(Z_K;\mathbf{k}) \cong \begin{cases} \mathbf{k} & i = 0, \\ \bigoplus_{I \notin K} \widetilde{H}^{i-|I|-1}(K_I;\mathbf{k}) & i > 0. \end{cases}$$

In particular, there is an isomorphism of algebras

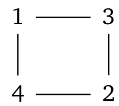
$$H^*(Z_K;\mathbf{k}) \cong \mathbf{k} \oplus \bigoplus_{I \notin K} \widetilde{H}^*(K_I;\mathbf{k}).$$

The products in the sum on the right are given as follows: for $I, J \notin K$, with $I \cap J = \emptyset$, let $\alpha \in \widetilde{H}^p(K_I;\mathbf{k})$ and $\beta \in \widetilde{H}^q(K_J;\mathbf{k})$ be nontrivial cohomology classes. Then there exists a nontrivial cohomology class $\gamma \in \widetilde{H}^{p+q}(K_{I\cup J};\mathbf{k})$ such that $\alpha \smile \beta = \gamma$. All products of cohomology classes in $H^*(Z_K;\mathbf{k})$ arise in this way.

Example 1.13. Let L and L' be two disjoint points. Then $K = L \star L'$ is a square and we get

$$Z_K \simeq S^3 \times S^3.$$

The following is an illustration of the simplicial complex K with labeled vertices.



The cohomology classes of Z_K are generated by non-contractible full subcomplexes of K. The full subcomplex $K_{\{1,2\}} = L$ gives rise to a class α in degree 3, and similarly for $K_{\{3,4\}} = L'$ we get another degree 3 class α'. Lastly, since the complex $K_{\{1,2,3,4\}} = L \star L'$ is not contractible and is homotopic to S^1 there is a class β in degree 6. By Theorem 1.12, since $\{1,2\} \cup \{3,4\} = \{1,2,3,4\}$, we get the relation $\alpha\alpha' = \beta \in H^6(Z_K, \mathbf{k})$.

1.1.4 Colimit and homotopy colimit descriptions

Our current definition of polyhedral products is as a union of certain subspaces of $\prod_{i=1}^m X_i$. We will now describe polyhedral products as colimits of diagrams indexed by a category related to a simplicial complex.

Definition 1.14. Let K be a simplicial complex. Let $\mathrm{cat}(K)$ be the *face poset category* of K. The objects of $\mathrm{cat}(K)$ are given by the simplices of K, including an initial object $\emptyset$ which corresponds to the empty simplex. Let $\sigma, \tau \in K$ be two simplices of K. If σ is a subface of τ, then there is a unique morphism $f_{\sigma \leq \tau} \colon \sigma \to \tau$.

An alternate description of the polyhedral product is as the colimit of a $\mathrm{cat}(K)$-shaped diagram

$$(\underline{X}, \underline{A})^K = \operatorname*{colim}_{\sigma \in K} D(\sigma),$$

where $D(\sigma)$ is defined as before. However, colimits are not homotopy invariant, which is not good if we want to do homotopy theory. Homotopy colimits, on the other hand, are homotopy invariant. For simplices $\sigma \leq \tau$, the maps $D(\sigma) \to D(\tau)$ are cofibrations, thus by [23, Lemma 3.1] there is a homotopy equivalence

$$(\underline{X}, \underline{A})^K = \operatorname*{colim}_{\sigma \in K} D(\sigma) \simeq \operatorname*{hocolim}_{\sigma \in K} D(\sigma).$$

Going forward, we will always think of a polyhedral product as a homotopy colimit, since it will allow us to use techniques from homotopy theory to speak about polyhedral products.

1.1.5 The polyhedral smash product

A variation of the polyhedral product is the polyhedral smash product. It was first defined in [5].

Definition 1.15. Let $(\underline{X},\underline{A})$ be a family of pairs and let K be a simplicial complex on the vertex set $[m]$. Let $\mathcal{K}$ be the face poset category of K ordered by inclusions. We define the *polyhedral smash product* $\widehat{(\underline{X},\underline{A})}^K$ as

$$\widehat{(\underline{X},\underline{A})}^K := \operatorname*{hocolim}_{\sigma\in K} \widehat{D}(\sigma),$$

with $\widehat{D}(\sigma)$ defined as follows:

$$\widehat{D}(\sigma) = \bigwedge_{i=1}^{m} Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

For any pair of simplices $\sigma \subset \tau \in K$ the map from $\widehat{D}(\sigma)$ to $\widehat{D}(\tau)$ is induced by the maps ι_i and the identity.

Example 1.16. Let K be two disjoint points. Then the polyhedral smash product $\widehat{(D^2,S^1)}^K$ is the given by the homotopy pushout

$$\begin{array}{ccc} S^1 \wedge S^1 & \longrightarrow & D^2 \wedge S^1 \\ \downarrow & \ulcorner & \downarrow \\ S^1 \wedge D^2 & \longrightarrow & \widehat{(D^2,S^1)}^K. \end{array}$$

Since $D^2 \wedge S^1$ is contractible, there is an equivalence

$$\widehat{(D^2,S^1)}^K \simeq \Sigma(S^1 \wedge S^1) \simeq S^3.$$

1.1.6 Stable splittings

In [5] suspensions of polyhedral products are studied. The following theorem gives a relation between the suspension of a polyhedral product and the polyhedral smash product.

Theorem 1.17 ([5, Theorem 2.10])**.** *Let K be a simplicial complex. There is a homotopy equivalence*

$$\Sigma(\underline{X},\underline{A})^K \simeq \Sigma \bigvee_{I\notin K} \widehat{(\underline{X},\underline{A})}^{K_I}.$$

When all the spaces X_i are contractible, it is possible to say even more. Given m spaces $X_1,\dots,X_m$ and $I = \{i_1,\dots,i_k\} \subset [m]$, we write $X^{\wedge I} := X_{i_1} \wedge \dots \wedge X_{i_k}$.

Theorem 1.18 ([5, Theorem 2.21])**.** *Let K be a simplicial complex. Assume that X_i is contractible for all i. There is a homotopy equivalence*

$$\Sigma(\underline{X},\underline{A})^K \simeq \Sigma^2 \bigvee_{I\notin K} |K_I| \wedge A^{\wedge I}.$$

For the moment-angle complex, this results in the following equivalence

$$\Sigma Z_K \simeq \Sigma^2 \bigvee_{I \notin K} |K_I| \wedge S^{|I|} \simeq \bigvee_{I \notin K} \Sigma^{|I|+2}|K_I|.$$

Example 1.19. Let K be the boundary of a square as in Example 1.13. Then the full subcomplexes for both $I = \{1,2\}$ and $I = \{3,4\}$ yield $|K_I| \simeq S^0$. The full subcomplex $|K_I|$ is contractible whenever $|I| = 3$. When $I = \{1,2,3,4\}$, the full subcomplex is the whole simplicial complex and we get $|K_I| = |K| \simeq S^1$. Consequently, we have

$$\Sigma Z_K \simeq \Sigma^2 \big((S^0 \wedge S^2) \vee (S^0 \wedge S^2) \vee (S^1 \wedge S^4)\big) \simeq S^4 \vee S^4 \vee S^7 \simeq \Sigma(S^3 \times S^3).$$

1.1.7 Porter's decomposition and the Hilton–Milnor theorem

We have just seen how certain polyhedral products split into a wedge after a suspension. There are cases where the suspension is not needed for the polyhedral product to have the homotopy type of a wedge. A result of Porter [20, Theorem 1] identifies the homotopy type of $(\underline{C\Omega X}, \underline{\Omega X})^K$ in the case that each X_i is simply connected and K is a disjoint union of m points. For a space X and $k \geq 1$, let $X^{\vee k}$ be the k-fold wedge of X.

Theorem 1.20. *Let $X_1, \ldots, X_m$ be pointed simply connected CW-complexes, and K be m disjoint points. There is a homotopy equivalence*

$$(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \bigvee_{k=2}^{m} \bigvee_{1 \leq i_1 < \ldots < i_k \leq m} (\Sigma\Omega X_{i_1} \wedge \ldots \wedge \Omega X_{i_k})^{\vee(k-1)} \simeq \bigvee_{I \in [m], |I| \geq 2} \Sigma((\Omega X)^{\wedge I})^{\vee |I|-1}.$$

Moreover, this homotopy equivalence is natural for maps $X_i \to Y_i$.

Example 1.21. Recall that moment-angle complexes are defined as

$$Z_K := (D^2, S^1)^K = (C\Omega\mathbb{CP}^\infty, \Omega\mathbb{CP}^\infty)^K.$$

Let K be three disjoint points. Then there is an equivalence

$$Z_K \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4.$$

Porter's decomposition allows us to say something about the loop space of a wedge $X_1 \vee \ldots \vee X_m$ of simply connected spaces X_i. Let K be m disjoint points. By [12], there is a homotopy fibration

$$(\underline{C\Omega X}, \underline{\Omega X})^K \to \bigvee_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i.$$

The inclusion of the wedge into the product has a right homotopy inverse after looping. Hence, the whole sequence splits after looping, which yields the following theorem.

Theorem 1.22. *Let $X_1,\dots,X_m$ be pointed simply connected spaces. There is a homotopy equivalence*

$$\Omega(X_1 \vee \dots \vee X_m) \simeq \prod_{i=1}^{m} \Omega X_i \times \Omega\left(\bigvee_{I\in[m],|I|\geq 2} \Sigma((\Omega X)^{\wedge I})^{\vee |I|-1}\right).$$

We are once again left with a loop space of a wedge of spaces, but this time its a wedge of suspension. The Hilton–Milnor theorem, which we will now recall, allows us to say something more in this situation. Let L be the free (ungraded) Lie algebra over $\mathbb{Z}$ on the elements $x_1,\dots,x_m$, and let B be a Hall basis of L. For a bracket $b \in B$, let $k_i(b)$ be the number of instances of x_i in b. For a space X and $k \geq 0$, denote by $X^{\wedge k}$ to be the k-fold smash of X. We will define the 0-fold smash of X to be omission of the corresponding term, rather than a trivial space.

Theorem 1.23. *(Hilton–Milnor theorem [13, 17]) Let $X_1,\dots,X_m$ be connected topological spaces. Then there is a homotopy equivalence*

$$\Omega\left(\bigvee_{i=1}^{m} \Sigma X_i\right) \simeq \prod_{b\in B} \Omega\Sigma(X_1^{\wedge k_1(b)} \wedge \dots \wedge X_m^{\wedge k_m(b)}).$$

Moreover, this homotopy equivalence is natural for maps $X_i \to Y_i$.

The Hilton–Milnor theorem can be used to further decompose the right hand side of Theorem 1.22. This will be of use in Paper III.

1.2 Motivic homotopy theory

In this section, we will introduce motivic homotopy theory. Motivic homotopy theory will play a central role of Papers I and II. For the rest of this section, whenever we speak of a field k we will assume k to have characteristic different from 2. We define the affine line as the affine variety $\mathbb{A}^1 := \mathrm{Spec}(k[T])$.

1.2.1 The category of motivic spaces

In this section, we will briefly introduce the motivic homotopy category over a field. We refer the reader to [1, 14, 18, 24] for further details.

Let Sm_k be the category of smooth k-schemes of finite type. Morel and Voevodsky defined the motivic homotopy category over a perfect field [19]. To do this, they constructed a model structure that contained Sm_k and then considered the associated homotopy category. The category Sm_k does not have all small limits and colimts. To fix this, one first considers simplicial presheaves on Sm_k, which we will denote by $\mathrm{PShv}(\mathrm{Sm}_k)$. We say that a simplicial presheaf $F \in \mathrm{PShv}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-*invariant*, if the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence of simplicial sets $F(X \times \mathbb{A}^1) \to F(X)$ for any $X \in \mathrm{Sm}_k$. We will write $\mathrm{PShv}_{\mathbb{A}^1}(\mathrm{Sm}_k)$ for the full subcategory of $\mathrm{PShv}(\mathrm{Sm}_k)$ consisting of $\mathbb{A}^1$-invariant presheaves. The inclusion $\mathrm{PShv}_{\mathbb{A}^1}(\mathrm{Sm}_k) \hookrightarrow \mathrm{PShv}(\mathrm{Sm}_k)$ admits a left adjoint

$$\mathrm{L}_{\mathbb{A}^1} : \mathrm{PShv}(\mathrm{Sm}_k) \to \mathrm{PShv}_{\mathbb{A}^1}(\mathrm{Sm}_k),$$

which we will call $\mathbb{A}^1$-localization. With $\mathbb{A}^1$-localization, we make $\mathbb{A}^1$ weakly equivalent to the point. We now need to define a suitable topology on Sm_k. In this case, we choose the Nisnevich topology, which is a Grothendieck topology on Sm_k generated by the following squares

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow p \\ U & \xrightarrow{\;i\;} & X. \end{array} \in \mathrm{Sm}_k$$

where $p \colon Y \to X$ is étale, $i \colon U \to X$ is an open immersion, and $p^{-1}(X \setminus U) \to X \setminus U$ is an isomorphism of reduced induced schemes. One can consider the full subcategory $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ of Nisnevich sheaves in $\mathrm{PShv}(\mathrm{Sm}_k)$. The category of motivic spaces Spc_k is the full subcategory of $\mathrm{PShv}(\mathrm{Sm}_k)$ spanned by $\mathbb{A}^1$-invariant Nisnevich sheaves. There is a localization functor

$$\mathrm{L}_{\mathrm{Mot}} \colon \mathrm{PShv}(\mathrm{Sm}_k) \to \mathrm{Spc}_k,$$

which we will refer to as *motivic localization*. The objects of Spc_k are called motivic spaces. We view an object of $X \in \mathrm{Sm}_k$ as a motivic space by considering taking motivic localization of the presheaf $\mathrm{Hom}_{\mathrm{Sm}_k}(-, X)$. A simplicial set S can be viewed as a motivic space by taking the motivic localization of the constant presheaf with value S. By inverting weak equivalences, we end up with the motivic homotopy category $\mathcal{H}(k)$. There is also a motivic homotopy category of pointed spaces, which we will denote by $\mathcal{H}_*(k)$. The terminal and initial object $\mathcal{H}_*(k)$ is $\mathrm{Spec}(k)$. For two motivic spaces X, Y, we will write $[X, Y]^{\mathbb{A}^1}$ for the set of pointed morphism $X \to Y$ in $\mathcal{H}(k)$, that is $[X, Y]^{\mathbb{A}^1} := \mathrm{Hom}_{\mathcal{H}_*(k)}(X, Y)$.

In Paper II we consider the category of motivic spaces as n ∞-category. This can be done by taking the nerve embedding of Spc_k, or by doing $\mathbb{A}^1$-localization and Nisnevich localization on the ∞-category of simplicial presheaves over Sm_k equipped with the Nisnevich topology.

1.2.2 Naive motivic homotopy theory

For topological spaces X and Y and morphisms $f : X \to Y$ and $g : X \to Y$, a homotopy is a morphism $H : X \times [0, 1] \to Y$, such that $H(0) = f$ and $H(1) = g$. $\mathbb{A}^1$ plays the role of the interval in motivic homotopy theory. The following definition is a motivic analogue of the notion of homotopy in classical topology.

Definition 1.24. Let $X, Y \in \mathrm{Sm}_k$ be pointed at k-points x and y respectively. A *pointed elementary homotopy* between two pointed scheme morphisms $f : X \to Y$ and $g : X \to Y$ is given by a morphism $H(T) \colon X \times \mathbb{A}^1 \to Y$ with the additional properties that $\{x\} \times \mathbb{A}^1$ maps to y, and $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. When a pointed elementary homotopy between f and g exists, we say f and g are *pointed elementary homotopic* and write $f \sim g$.

This relation on morphism X to Y is symmetric and reflexive, but not transitive. Consequently, there exists f, g, h such that $f \sim g$ and $g \sim h$, but $f \not\sim h$. By taking the transitive closure of $\sim$, we are left with an equivalence relation $\simeq$ on the set $\mathrm{Sm}_k(X, Y)_*$.

Definition 1.25. Let $X, Y \in \mathrm{Sm}_k$ be pointed at k-points x and y respectively. A *pointed naive homotopy* between two pointed morphisms $f : X \to Y$ and $g : X \to Y$ is given by a chain of pointed elementary homotopies such that

$$f \sim h_1 \sim h_2 \sim \ldots \sim h_n \sim g.$$

If such a chain of pointed elementary homotopies exists, we say that f and g are *pointed naively homotopic* and write $f \simeq g$.

For $X, Y \in \mathrm{Sm}_k$, we write $[X, Y]^N$ for the set of pointed naive homotopy classes of morphisms.

1.2.3 Comparison of homotopy classes

We have just described two different notions of pointed homotopy classes of pointed smooth k-schemes X, Y: the *true* homotopy classes $[X, Y]^{\mathbb{A}^1}$ and the *naive* homotopy classes $[X, Y]^N$. There is a canonical map

$$\phi : [X, Y]^N \to [X, Y]^{\mathbb{A}^1},$$

but this rarely a bijection. Asok, Hoyois, and Wendt [2, 3], studied certain schemes Y, such that ϕ is a bijection for all affine X. They call Y $\mathbb{A}^1$*-naive* if the map ϕ is a bijection for all affine X. Examples of $\mathbb{A}^1$-naive spaces are $\mathbb{P}^n$ and SL_2. The results on $\mathbb{A}^1$-naive spaces were originally proven for unpointed homotopy classes. In appendix A of Paper I we verify that these results hold in the category of pointed motivic spaces.

1.2.4 Motivic spheres

In the motivic homotopy category there are two circles. There is the geometric circle $S^{1,1} := \mathbb{G}_\mathrm{m} = \mathbb{A}^1 \setminus 0$, also known as the Tate circle, and the simplicial circle $S^{1,0}$ represented by the motivic localization of the constant simplicial presheaf with value S^1. We may smash these spheres together just like in classical homotopy theory yielding $S^{a,b} \simeq S^{a-b,0} \wedge S^{b,b} \simeq (S^1)^{\wedge(a-b)} \wedge (\mathbb{G}_\mathrm{m})^{\wedge b}$. The question concerning which motivic spheres have smooth scheme representatives is an open question. However, there are some examples such as $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_\mathrm{m} \simeq S^{2,1}$ and $\mathbb{A}^n \setminus 0 \simeq S^{2n-1,n}$.

1.2.5 The Grothendieck–Witt ring of quadratic forms

The Grothendieck–Witt ring is an important object in motivic homotopy theory. Let $\mathrm{MW}(k)$ be the semiring of non-degenerate quadratic forms of k-vector spaces. The addition of $\mathrm{MW}(k)$ is given by direct sum $\oplus$ and the product is given by tensor product $\otimes$. We call $\mathrm{MW}(k)$ the *Witt monoid*. There is a map $\mathrm{MW}(k) \to \mathbb{N}$ given by the rank of the quadratic form and there is a map $\mathrm{MW}(k) \to k^\times/k^{\times 2}$ given by its discriminant. For any $u \in k^\times$, let $\langle u \rangle \in \mathrm{MW}(k)$ denote the rank one quadratic form $x \mapsto ux^2$. Since we have assumed k to have characteristic 2, any quadratic form can be diagonalized. This means that $\mathrm{MW}(k)$ is generated by the rank 1 quadratic forms $\langle u \rangle$ under addition. The Grothendieck–Witt ring $\mathrm{GW}(k)$ is the group completion of $\mathrm{MW}(k)$ with respect to $\oplus$. The following lemma describes the relations of the Witt monoid and the Grothendieck–Witt ring.

Lemma 1.26 ([16, Theorem II.4.1])**.** *The Grothendieck–Witt ring (resp. Witt monoid) of a field k is generated as a group (resp. monoid) by generators $\langle u \rangle$, where $u \in k^\times$, and the following relations*

1. $\langle uv^2 \rangle = \langle u \rangle$, *for* $u, v \in k^\times$.
2. $\langle u \rangle + \langle v \rangle = \langle u + v \rangle + \langle uv(u+v) \rangle$, *for* $u, v, (u+v) \in k^\times$.
3. $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle$, *for* $u \in k^\times$.

Example 1.27. When k is an algebraically closed field, any unit is a square. Hence $\langle u \rangle = \langle 1 \rangle$, for all $u \in k^\times$. Thus $\mathrm{MW}(k) = \mathbb{N}$ and $\mathrm{GW}(k) \cong \mathbb{Z}$.

Example 1.28. Let k be the real numbers. In this case we have the two generators $\langle 1 \rangle$ and $\langle -1 \rangle$. Thus $\mathrm{MW}(\mathbb{R}) = \mathbb{N} \times \mathbb{N}$ and $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$.

Example 1.29. Let k be a finite field and let u be a nonsquare unit. The generators are $\langle 1 \rangle$ and $\langle u \rangle$. Thus $\mathrm{MW}(k) \cong \mathbb{N} \times \mathbb{Z}_2$ and $\mathrm{GW}(k) \cong \mathbb{Z} \times \mathbb{Z}_2$.

The following result shows why the Grothendieck–Witt ring is important in motivic homotopy theory.

Theorem 1.30 ([18, Corollary 6.43])**.** *Let a, b be integers such that $a - b \geq 2$, then the set $[S^{a,b}, S^{a,b}]^{\mathbb{A}^1}$ can be equipped with a group operation $\oplus^{\mathbb{A}^1}$ and there is an isomorphism*

$$([S^{a,b}, S^{a,b}]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}) \cong \mathrm{GW}(k).$$

The theorem above shows that the zeroth stable homotopy group of the motivic spheres is $\mathrm{GW}(k)$. The following result, due to Morel, describes the motivic analogue of the fundamental group of the circle.

Theorem 1.31 ([18, Theorem 7.36])**.** *There is an isomorphism of groups*

$$([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}) \cong \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times.$$

The map from $GW(k)$ to $k^\times/k^{\times 2}$ in the theorem above is given by the discriminant map.

1.2.6 Milnor–Witt K-theory

Another crucial ring that is closely related to $\mathrm{GW}(k)$ is Milnor–Witt K-theory. The following definition is due to Hopkins and Morel.

Definition 1.32 ([18, Definition 3.1]). The Milnor–Witt K-theory of the field k, denoted $K_*^{\mathrm{MW}}(k)$, is the graded associative ring generated by symbols $[u]$ in degree 1 for $u \in k^\times$ and the symbol η in degree -1 subject to the following relations:

1. For each $u \in k^\times \setminus \{1\}$, $[u].[1-u] = 0$.
2. For each pair $u, v \in (k^\times)^2$, $[uv] = [u] + [v] + \eta.[u].[v]$.
3. For each $u \in k^\times$, $\eta.[u] = [u].\eta$.
4. Let $h := \eta.[-1] + 2$. Then $\eta.h = 0$.

There is an isomorphism $K_0^{\mathrm{MW}}(k) \cong \mathrm{GW}(k)$. The following proposition is an extension of Theorem 1.30

Theorem 1.33 ([18, Corollary 6.43]). *Let a, i and j be non-negative integers such that $a - i \geq 2$. There is an isomorphism*

$$([S^{a,i}, S^{a,j}], \oplus^{\mathbb{A}^1}) \cong K_{j-i}^{\mathrm{MW}}(k).$$

Morel also proved the following.

Theorem 1.34 ([18, §7.3]). *There is an isomorphism of groups*

$$([\mathbb{P}^1, \mathbb{A}^2 \setminus 0]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}) \cong K_1^{\mathrm{MW}}(k).$$

Note that $\mathbb{A}^2 \setminus 0 \simeq S^{3,2}$, which is outside the range of Theorem 1.33. The result still holds, but follows from different computations.

Example 1.35.
1. $K_1^{\mathrm{MW}}(\mathbb{C}) \cong \mathbb{C}^\times$
2. $K_1^{\mathrm{MW}}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{R}_+^\times$
3. $K_1^{\mathrm{MW}}(\mathbb{F}_p) \cong \mathbb{Z}_{p-1}$

In Paper I Milnor–Witt K-theory in degree 1 will play an important role. Milnor–Witt K-theory can be extended to a definition of sheaves of abelian groups on Sm_k. Morel does this in [18, §3.2] and defines the unramified Milnor–Witt K-theory sheaves $\mathbf{K}_i^{\mathrm{MW}}$.

1.2.7 The Jouanolou–Thompson homotopy lemma

The following result is due to Jouanolou.

Lemma 1.36 ([15, Lemma 1.5]). *Let X be a quasi-projective scheme over a field k. There exists a pair $(\widetilde{X}, \pi)$, where $\widetilde{X}$ is affine, and the morphism $\pi\colon \widetilde{X} \to X$ is a Zariski locally trivial smooth morphism with fibers isomorphic to affine spaces.*

This lemma is particularly useful in motivic homotopy theory. Since the bundle is Zariski locally trivial with contractible fibers, the map $\pi\colon \widetilde{X} \to X$ is a homotopy equivalence. The quasi-projective hypothesis of Lemma 1.36 can be weakened to any smooth scheme due to Thomason [22, Proposition 4.4].

Theorem 1.37. *(Jouanolou–Thomason homotopy lemma) Let $X \in \mathrm{Sm}_k$, then there exists and affine scheme $\widetilde{X} \in \mathrm{Sm}_k$ such that $\widetilde{X} \simeq X$ in $\mathcal{H}(k)$.*

We will say that $\widetilde{X}$ is a *Jouanolou device* for X. We will now look at some example Jouanolou devices.

Example 1.38. A Jouanolou device for $\mathbb{A}^2 \setminus 0$ is given by the affine variety

$$\mathrm{SL}_2 := \mathrm{Spec}\left(\frac{k[a,b,c,d]}{(ad-bc-1)}\right).$$

Example 1.39. A Jouanolou device for $\mathbb{P}^1$ is given by the affine variety

$$\mathcal{J} := \mathrm{Spec}\left(\frac{k[x,y,z,w]}{(x+w-1, xw-yz)}\right).$$

These two Jouanolou devices and their coordinate rings play a crucial part of Paper I.

1.2.8 Naive homotopy classes of endomorphisms of the projective line

In [10], Cazanave computes the pointed naive homotopy classes of endomorphisms of the projective line. In other words, Cazanave studies the set $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$. Cazanave shows that the set $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ can be equipped with a monoid structure, which we will denote $([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}})$.

Theorem 1.40 ([10, Corollary 3.10])**.** *There is a canonical isomorphism of graded monoids:*

$$[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}} \cong \mathrm{MW}(k) \times_{k^\times/k^{\times 2}} k^\times.$$

The surprising thing about Cazanave's monoid structure is that it is independent of motivic homotopy thoery, and relies only on basic algebraic geometry and knowledge of quadratic forms. As mentioned in Section 1.2.3, there is a map

$$\phi\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$$

from naive to true homotopy classes. Since $\mathbb{P}^1$ is not affine, we can not expect this to be a bijection. Furthermore, $([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}})$ can only be made into a monoid, but the set right-hand side can be given the structure of a group by Theorem 1.31. However, Cazanave shows that ϕ is the best thing it can be in this circumstance: a group completion.

Theorem 1.41 ([10, Theorem 1.2])**.** *The canonical map*

$$\nu_{\mathbb{P}^1}\colon ([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}) \to ([\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$$

induced by ϕ, is a group completion.

1.3 Paper summaries

In this section, a short summary of each paper is presented as well as highlighting the main results.

1.3.1 Summary of Paper I

In this paper, we construct an explicit group structure on the set of naive homotopy classes of maps from the Jouanolou device of the projective line to the projective line. By utilizing the fact that $\mathbb{P}^1$ is an $\mathbb{A}^1$-naive space and that $\mathcal{J} \simeq \mathbb{P}^1$, we have a bijection of sets

$$\phi : [\mathcal{J}, \mathbb{P}^1]^N \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}.$$

We construct an explicit group structure on the set pointed naive homotopy classes of maps $[\mathcal{J}, \mathbb{P}^1]^N$ and denote this group by $([\mathcal{J}, \mathbb{P}^1]^N, \oplus)$. The group structure uses basic elementary algebraic geometry and is independent of the construction of the motivic homotopy category. We prove the following relation between Cazanave's monoid structure on $[\mathbb{P}^1, \mathbb{P}^1]^N$ and our group structure on $[\mathcal{J}, \mathbb{P}^1]^N$.

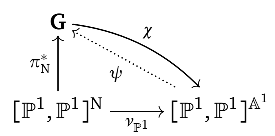
Theorem 1.42 (Theorem 95 in Paper I)**.** *The map*

$$\pi_{\mathrm{N}}^* : ([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^N) \to ([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus)$$

induced by the map $\pi : \mathcal{J} \to \mathbb{P}^1$ is a morphism of monoids.

We then study the image of π_{N}^* in $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$. In particular, we find that it lies within a subgroup $\mathbf{G} \subseteq ([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus)$ which is generated by an easy to describe set of explicit morphisms and inherits the group operation $\oplus$. In the case where k is a finite field, we show that $\mathbf{G} = [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$.

Theorem 1.43 (Theorem 8 in Paper I)**.** *The monoid morphism $\pi_{\mathrm{N}}^* : [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to \mathbf{G}$ is a group completion. There is a unique isomorphism $\chi : \mathbf{G} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ such that the diagram below commutes. Moreover, χ and ψ are mutual inverses to each other.*



The group $\mathbf{G}$ is a geometric model of the group $([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1})$, which is the motivic analogue of the fundamental group of the circle.

1.3.2 Summary of Paper II

In this paper, we generalize polyhedral products to an ∞-categorical setting. In particular, we generalize the stable splitting result from [5] to cartesian closed ∞-categories with small colimits. Let $\mathcal{C}$ be a cartesian closed ∞-categories with small colimits and K a simplicial complex. we write $|K|$ for the geometric realization of K in $\mathcal{C}$. The following is a generalization of Theorem 1.18.

Theorem 1.44. *Let $\mathcal{C}$ be a cartesian closed ∞-category and fix a morphism $i: A \to X$ of pointed objects where X is contractible. Let K be a simplicial complex. Then there is an equivalence*

$$\Sigma(X,A)^K \simeq \Sigma^2 \bigvee_{I \notin K} |K_I| \wedge A^{\wedge |I|}.$$

We then study polyhedral products in the motivic homotopy category, specifically a motivic refinement of moment-angle complexes.

Definition 1.45. Let K be a simplicial complex, we define the *motivic moment-angle complex* $Z_K^{\mathbb{A}^1}$ to be the polyhedral product

$$Z_K^{\mathbb{A}^1} := (\mathbb{A}^1, \mathbb{G}_{\mathrm{m}})^K$$

in the ∞-category $\mathcal{H}(k)$.

Application of Theorem 1.44 yields the following theorem.

Theorem 1.46. *Let K be a simplicial complex. Then there is an equivalence in $\mathcal{H}(k)$*

$$\Sigma Z_K^{\mathbb{A}^1} \simeq \Sigma \left(\bigvee_{I \notin K} |K_I| \star \mathbb{G}_{\mathrm{m}}^{\wedge |I|} \right) \simeq \bigvee_{I \notin K} |K_I| \wedge S^{|I|+2,|I|}.$$

The splitting result is used to compute various invariants of the motivic moment-angle complexes, such as cellular $\mathbb{A}^1$-homology and $\mathbb{A}^1$-Euler characteristic.

1.3.3 Summary of Paper III

In this paper, we introduce and study an Eckmann–Hilton dual of polyhedral products, named polyhedral coproducts. Polyhedral products are (homotopy) colimits of products, and we define polyhedral coproducts to be a homotopy limit of coproducts.

Definition 1.47. Let $\underline{f} = (f_1, \ldots, f_m)$ be an m-tuple of maps $f_i : X_i \to A_i$ of pointed spaces. We define the *polyhedral coproduct* associated to $\underline{f}$ and a simplicial complex K as the homotopy limit

$$\underline{f}_{\mathrm{co}}^K = \operatorname*{holim}_{\sigma \in K} \bigvee_{i=1}^{m} Y_i(\sigma), \quad \text{where} \quad Y_i(\sigma) = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The paper acts as a survey, we present several classical result about polyhedral products and study their Eckmann–Hilton duals. In particular, we prove dual versions of Theorem 1.17 and 1.18. In the following theorem $B_{\Delta^{m-1}}$ is a certain Hall basis. For $b \in B_{\Delta^{m-1}}$ there is an associated subset $I_b \subset [m]$ and for $1 \leq i \leq m$ an integer $l_i(b) \geq 0$. See page 14 of Paper III for further details. The following theorem is the Eckmann–Hilton dual of Theorem 1.18.

Theorem 1.48 (Theorem 4.5 in Paper I)**.** *Let K be a simplicial complex on $[m]$ and $f_i : X_i \to A_i$ where X_i is contractible and A_i is a pointed, simply connected CW-complex for $1 \leq i \leq m$. Then there is a homotopy equivalence*

$$\Omega \underline{f}^K_{\mathrm{co}} \simeq \prod_{b \in B_{\triangle^{m-1}}, I_b \notin K} \Omega \mathrm{Map}_*(\Sigma |K_{I_b}|, \Sigma \Omega A_1^{l_1(b)} \wedge \ldots \wedge \Omega A_m^{l_m(b)}).$$

The theorem describes the loop space of a polyhedral coproduct in terms of mapping spaces of full subcomplexes of K. We also study how joins and pushouts of simplicial complexes affect polyhedral coproducts.

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Paper I

Making the motivic group structure on the endomorphisms of the projective line explicit

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We construct a group structure on the set of pointed naive homotopy classes of scheme morphisms from the Jouanolou device to the projective line. The group operation is defined via matrix multiplication on generating sections of line bundles and only requires basic algebraic geometry. In particular, it is completely independent of the construction of the motivic homotopy category. We show that a particular scheme morphism, which exhibits the Jouanolou device as an affine torsor bundle over the projective line, induces a monoid morphism from Cazanave's monoid to this group. Moreover, we show that this monoid morphism is a group completion to a subgroup of the group of scheme morphisms from the Jouanolou device to the projective line. This subgroup is generated by a set of morphisms that are simple to describe.

Contents

1 Introduction

The $\mathbb{A}^1$-homotopy category over a field k introduced by Morel and Voevodsky [23] makes it possible to define a group structure on the $\mathbb{A}^1$-homotopy classes of pointed maps $\mathbb{P}^1 \to \mathbb{P}^1$. This construction plays the role of the fundamental group of the circle in motivic homotopy theory. Although the group operation mimics the usual construction in algebraic topology, the set of $\mathbb{A}^1$-homotopy classes of maps $\mathbb{P}^1 \to \mathbb{P}^1$ is not simply the set of morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ modulo an equivalence relation. It is unsettling that such an important group does not arise easily in some elementary way as a set of morphisms up to a homotopy relation with some geometrically defined group operation. Thankfully, the work of Asok, Hoyois, and Wendt in [6], along with Cazanave's investigation in [13] lays a foundation to build off of.

In [13] Cazanave defines an operation $\oplus^{\mathrm{N}}$ which turns the set $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ of pointed naive homotopy classes into a monoid and shows that the canonical map $\nu_{\mathbb{P}^1} : [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group completion. However, this approach cannot yield candidates for scheme morphisms which represent inverses of $\mathbb{A}^1$-homotopy classes.

In [6] Asok, Hoyois, and Wendt show that the set $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ of motivic homotopy classes is in bijection to an explicit set of maps modulo the naive homotopy relation by using the larger set of maps $\mathrm{Sm}_k(\mathcal{J}, \mathbb{P}^1)$ where $\mathcal{J}$ denotes the Jouanolou device of $\mathbb{P}^1$, which we consider equipped with a morphism $\pi : \mathcal{J} \to \mathbb{P}^1$ that exhibits $\mathcal{J}$ as an affine torsor bundle. This resolves the problem of a lack of candidates of morphisms which may represent inverses in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, but it is not clear at all how the group operation on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ or the operation of [13] may be lifted.

In the present paper we define an explicit group structure on the set $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ of pointed naive homotopy classes. The construction of this group operation is independent of the general machinery of motivic homotopy theory and only uses basic algebraic geometry. We then show that the induced map $\pi_{\mathrm{N}}^* : [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ is a morphism of monoids where $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ has the monoid structure of [13, §3]. Moreover, we show that π_{N}^* has image in a concrete subgroup $\mathbf{G}$ and that the map $\pi_{\mathrm{N}}^* : [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to \mathbf{G}$ is a group completion. Hence there are canonical isomorphisms between $\mathbf{G}$ and $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ which are compatible with π_{N}^* and $\nu_{\mathbb{P}^1}$. A key feature of the group $\mathbf{G}$ is that is defined by explicit generating scheme morphisms $\mathcal{J} \to \mathbb{P}^1$ that are defined in terms of very simple (2×2)-matrices.

We will now describe our results in more detail. First we recall the conventional group operation $\oplus^{\mathbb{A}^1}$ on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, the set of maps in the pointed $\mathbb{A}^1$-homotopy category over a field k. It is a simple exercise to produce an $\mathbb{A}^1$-weak equivalence $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ using the standard covering of $\mathbb{P}^1$ by two affine lines with intersection $\mathbb{G}_m$. The simplicial circle S^1 (or some suitable homotopy equivalent model of it, like $\partial\Delta^2$) admits the structure of an h-cogroup, or just a cogroup in the homotopy category. Explicitly, the pointed simplicial set $\partial\Delta^2 \simeq S^1$ admits two maps: a pinch map $\mu : S^1 \to S^1 \vee S^1$ and an inverse map $S^1 \to S^1$. These operations fit into homotopy commutative diagrams that give the expected algebraic properties, like associativity and the definition of the inverse [24, Chapter 2]. These two observations together allow us to define a group operation on $[S^1 \wedge \mathbb{G}_m, \mathbb{P}^1]^{\mathbb{A}^1}$ as follows. Given two maps $f, g : S^1 \wedge \mathbb{G}_m \to \mathbb{P}^1$ in the $\mathbb{A}^1$-homotopy category, the composition below represents the sum $f \oplus^{\mathbb{A}^1} g$ of the maps

f and g.

$$(1) \qquad S^1 \wedge \mathbb{G}_m \xrightarrow{\mu\wedge 1} (S^1 \vee S^1) \wedge \mathbb{G}_m \xrightarrow{\cong} (S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) \xrightarrow{f\vee g} \mathbb{P}^1$$

Note that the morphism $f \vee g$ exists by the universal property of wedge sums. One must take the time to verify that the operation defined above does indeed make $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ into a group, but the pleasant properties of the $\mathbb{A}^1$-homotopy category make this doable. This operation seems quite explicit, however, it is not evident how to describe maps $S^1 \wedge \mathbb{G}_m \to \mathbb{P}^1$ as the domain is only defined as a simplicial presheaf on Sm/k. This is where the work of Asok, Hoyois, and Wendt [5–7] can be used to get a geometric description of the set of maps $\mathbb{P}^1 \to \mathbb{P}^1$.

The Jouanolou device of $\mathbb{P}^1$ over a field k is the smooth affine scheme $\mathcal{J} = \mathrm{Spec}(R)$ where

$$R = \frac{k[x,y,z,w]}{(x+w-1, xw-yz)}.$$

The ring R is used to represent (2×2)-matrices with trace 1 and determinant 0. Namely, a ring homomorphism $R \to S$ is equivalent to a (2×2)-matrix over S with trace 1 and determinant 0. The Jouanolou device of $\mathbb{P}^1$ is $\mathbb{A}^1$-homotopy equivalent to $\mathbb{P}^1$ and also an affine scheme. These good properties are exploited by Asok, Hoyois, and Wendt [6]. Briefly, their work can be used to show that there is a bijection $\xi\colon [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, where $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ is the set of naive homotopy classes of pointed maps $\mathcal{J} \to \mathbb{P}^1$. The map ξ is the composite of the canonical map $\nu\colon [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ and the inverse of the map $\pi^*_{\mathbb{A}^1}\colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \to [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ induced by a scheme morphism $\pi\colon \mathcal{J} \to \mathbb{P}^1$. The set $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ is concrete in the following sense: it is the set of pointed scheme morphisms $\mathcal{J} \to \mathbb{P}^1$ modulo an equivalence relation generated by naive homotopies. A naive homotopy between two pointed maps $f, g : \mathcal{J} \to \mathbb{P}^1$ is given by a map $H : \mathcal{J} \times \mathbb{A}^1 \to \mathbb{P}^1$ satisfying the evident restrictions $H_0 = f$ and $H_1 = g$. We note that H must be pointed in the sense that $* \times \mathbb{A}^1$ maps to the basepoint of $\mathbb{P}^1$. We explain in appendix A how the unpointed results of [5] and [6] imply that the canonical map $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ is a bijection.

The bijection $\xi : [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ gets us started on our quest for an elementary description of the group structure. There are at least two possible approaches: We first describe the one we did not take in this paper. One can hope to construct a cogroup structure on $\mathcal{J}$. This is not so easy. However, Asok and Fasel have done much of the work to make this possible. In [3], Asok and Fasel give an explicit construction of a smooth scheme $\widetilde{\mathcal{J} \vee \mathcal{J}}$ that is $\mathbb{A}^1$-weak equivalent to the wedge sum $\mathcal{J} \vee \mathcal{J}$. We have constructed an explicit map $\mu : \mathcal{J} \to \widetilde{\mathcal{J} \vee \mathcal{J}}$ that conjecturally represents the pinch map $\mathbb{P}^1 \to \mathbb{P}^1 \vee \mathbb{P}^1$. We also have a candidate for a map representing the inverse map

$\mathcal{J} \to \mathbb{P}^1$, but unfortunately both of these claims have proven too difficult to verify. We therefore decided not to include this construction in this paper.

The approach we take is more explicit. Recall that a morphism $f\colon \mathcal{J} \to \mathbb{P}^1$ is determined by an invertible sheaf $\mathcal{L}$ over $\mathcal{J}$ and a choice of two generating sections $s_0, s_1 \in \Gamma(\mathcal{L}, \mathcal{J})$. The invertible sheaf $\mathcal{L}$ is the pullback $f^*\mathcal{O}(1)$. We say that a morphism $f\colon \mathcal{J} \to \mathbb{P}^1$ has degree 0 if $f^*\mathcal{O}(1)$ is the structure sheaf on $\mathcal{J}$. As we will show in section 2.5, the maps $\mathcal{J} \to \mathbb{P}^1$ of degree 0 are exactly the maps which factor through the Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$. A little work revealed that the set of maps $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ has an apparent group structure. Seen one way, a map $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ is given by a unimodular row (A, B) in R^2, i.e., there exist $U, V \in R$ for which $AU + BV = 1$. Any such unimodular row can be completed to a (2×2)-matrix over R, and the product of these matrices defines a group operation on $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$. Seen another way, the punctured plane $\mathbb{A}^2 \setminus \{0\}$ is $\mathbb{A}^1$-weak equivalent to SL_2, the scheme representing (2×2)-matrices with determinant 1. The scheme SL_2 is a group scheme, and so has the structure of a group object in the $\mathbb{A}^1$-homotopy category. It is standard algebraic topology then to use the structure of SL_2 to make the set of maps $[\mathcal{J}, \mathrm{SL}_2]^{\mathrm{N}}$ into a group [24]. Passing to the $\mathbb{A}^1$-homotopy classes of maps, it follows that this is the correct group structure, because $[\mathcal{J}, \mathrm{SL}_2]^{\mathrm{N}} \cong [S^1 \wedge \mathbb{G}_m, \mathrm{SL}_2]^{\mathbb{A}^1}$ has the domain space a cogroup object and the codomain space a group object, and the Eckmann–Hilton argument implies that these two structures define isomorphic group operations on $[S^1 \wedge \mathbb{G}_m, \mathrm{SL}_2]^{\mathbb{A}^1}$.

What remains then? The subgroup of degree 0 maps $\mathcal{J} \to \mathbb{P}^1$ is quite large. Once we have verified that $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ is a group, we will show that the quotient group $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} / [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} \cong \mathbb{Z}$ is the group of integers. This isomorphism serves as our guiding principle to determine the group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$. More precisely, the bijection ξ fits into the commutative diagram below in which the bottom row is an exact sequence of groups by the work of Morel [22].

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} & \longrightarrow & [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} & \xrightarrow{\deg} & \mathrm{Pic}(\mathcal{J}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \xi_0} & & {\scriptstyle \phi}\downarrow\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle q} & & \\
1 & \longrightarrow & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & \xrightarrow{\deg} & \mathrm{Pic}(\mathbb{P}^1) & \longrightarrow & 1
\end{array} \tag{2}
$$

Since we have an explicit group structure on $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$, it then suffices to pick representative lifts for each integer $n \in \mathbb{Z} \cong \mathrm{Pic}(\mathcal{J})$ and to understand how the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ acts on them. More concretely, for any integer n, let $n\pi$ denote a morphism $n\pi\colon \mathcal{J} \to \mathbb{P}^1$ which represents the $\mathbb{A}^1$-homotopy class $n[\mathrm{id} : \mathbb{P}^1 \to \mathbb{P}^1]$ under the bijection $\xi\colon [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. Let $f\colon \mathcal{J} \to \mathbb{P}^1$ and $g\colon \mathcal{J} \to \mathbb{P}^1$ be morphisms of degrees n and m respectively. We will show that there are degree 0 maps $f_0 : \mathcal{J} \to \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ and $g_0 : \mathcal{J} \to \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ for which $f \simeq f_0 \oplus n\pi$ and $g \simeq g_0 \oplus m\pi$. We then define the sum of $[f]$ and $[g]$ to be

$$\begin{aligned}[f] \oplus [g] &:= ([f_0] \oplus [n\pi]) \oplus ([g_0] \oplus [m\pi]) \\ &= ([f_0] \oplus [g_0]) \oplus [(n+m)\pi].\end{aligned}$$

The term $[f_0] \oplus [g_0]$ is calculated by matrix multiplication in the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$. The key idea for how $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ acts on the set $\{[n\pi] : n \in \mathbb{Z}\}$ is that any map $n\pi$ is given by the choice of a line bundle together with two generating sections. We can then let a morphism $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ given by a (2×2)-matrix act on the sections via matrix multiplication. We explain the details of this operation in section 3.2.

In section 4 we prove the following result, see theorem 80:

Theorem 3 The operation $\oplus$ of definition 73, which is described above, makes $\left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus\right)$ an abelian group. There is a group isomorphism

$$\phi\colon \left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus\right) \xrightarrow{\cong} \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right)$$

which restricts to ξ_0 on the subgroup $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$.

The step we are not able to prove yet is that the bijection $\xi\colon [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is compatible with $\oplus$ on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ and the conventional group structure $\oplus^{\mathbb{A}^1}$ on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. Nevertheless, we hope to ultimately prove the following conjecture for which we report on further evidence in appendix C.

Conjecture 4 The bijection $\xi\colon [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group isomorphism and equals ϕ.

The isomorphism ϕ is produced by rather formal arguments and does not yet provide a concrete description of the group $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. We will therefore now explain how we do obtain a very explicit expression of the latter group in terms of concrete scheme morphisms $\mathcal{J} \to \mathbb{P}^1$.

Recall that Cazanave shows in [13, Theorem 3.22] that there is an operation $\oplus^{\mathrm{N}}$ which provides $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ with the structure of a commutative monoid and that the canonical map $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group completion. We show that the morphism $\pi\colon \mathcal{J} \to \mathbb{P}^1$ also induces a group completion in the following way. First we prove in section 5 that the operation $\oplus$ is compatible with $\oplus^{\mathrm{N}}$, see theorem 95:

Theorem 5 The morphism $\pi\colon \mathcal{J} \to \mathbb{P}^1$ induces a morphism of commutative monoids

$$\pi_{\mathrm{N}}^*\colon \left([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}}\right) \to \left([\mathcal{J},\mathbb{P}^1]^{\mathrm{N}},\oplus\right)$$

where the left-hand side denotes the monoid of [13, § 3].

The proof of theorem 5 is based on the following observation. Let $u \in k^\times$. As in [13] we identify a rational function X/u in the indeterminate X with the morphism $\mathbb{P}^1 \to \mathbb{P}^1$ defined by $[x_0 : x_1] \mapsto [x_0 : ux_1]$. For $u, v \in k^\times$, we let $g_{u,v}\colon \mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ denote the morphism given by the unimodular row $\left(x + \frac{v}{u}w, (u-v)y\right)$ in R^2. For the rational functions X/u and X/v we then have the identity

$$g_{u,v} \oplus \pi_{\mathrm{N}}^*\left(X/v\right) = \pi_{\mathrm{N}}^*\left(X/u\right) \tag{6}$$

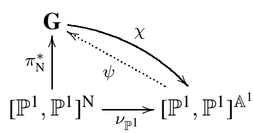
which we emphasize is an identity of morphisms not just homotopy classes. In particular, for $v = 1$ and $\pi_{\mathrm{N}}^*\left(X/1\right) = \pi$, formula (6) reads $g_{u,1} \oplus \pi = \pi_{\mathrm{N}}^*\left(X/u\right)$ and reduces computations for $\pi_{\mathrm{N}}^*\left(X/u\right)$ to computations for $g_{u,1}$ and π. The key technical result needed to prove theorem 5 is that, for every pointed morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, we have an explicit naive homotopy

$$\pi_{\mathrm{N}}^*\left(X/u \oplus^{\mathrm{N}} f\right) \simeq g_{u,1} \oplus \left(\pi_{\mathrm{N}}^*\left(X/1 \oplus^{\mathrm{N}} f\right)\right). \tag{7}$$

The construction of the concrete homotopy in formula (7) is based on computations of the resultants of certain morphisms which we provide in section 5.1 and appendix B. Theorem 5 then follows from the fact that the set of homotopy classes $[X/u]$ for all $u \in k^\times$ generates $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ and a successive application of formula (7).

Identity (6) also implies that the image of π_{N}^* is contained in the subgroup $\mathbf{G} \subseteq [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ generated by the homotopy classes $[g_{u,v}]$ for all $u, v \in k^\times$ and $[\pi]$. Theorem 5 and the work of Cazanave [13, Theorem 3.22] then imply that there is a unique group homomorphism $\psi\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1} \to \mathbf{G}$ such that $\psi \circ \nu_{\mathbb{P}^1} = \pi_{\mathrm{N}}^*$. In section 6 we show the following key result, see theorem 111:

Theorem 8 The monoid morphism $\pi_{\mathrm{N}}^*\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}} \to \mathbf{G}$ is a group completion. There is a unique isomorphism $\chi\colon \mathbf{G} \to [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$ such that the diagram below commutes. Moreover, χ and ψ are mutual inverses to each other.



Theorem 8 gives a very concrete description of all pointed endomorphisms of $\mathbb{P}^1$ in the unstable $\mathbb{A}^1$-homotopy category in the following sense: the group $\mathbf{G}$ is given by a simple set of generating morphisms, and the group operation $\oplus$ in $\mathbf{G}$ inherited from $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ is defined in basic algebro-geometric terms. The only thing we are missing is a concrete morphism $\mathcal{J}\to\mathbb{P}^1$ that is sent to the homotopy class $-[\mathrm{id}\colon\mathbb{P}^1\to\mathbb{P}^1]$. We speculate that the map $\tilde{\pi}$ defined in the beginning of section 2 corresponds with $-[\mathrm{id}]$. In theorem 103 we prove that the image of the naive homotopy class of $\tilde{\pi}$ under the motivic Brouwer degree is the class $-\langle 1\rangle$ in $\mathrm{GW}(k)$. This shows that the image of the class of $\tilde{\pi}$ represents $-[\mathrm{id}]$ in the stable motivic homotopy category. However, this is not sufficient to determine the class of $\tilde{\pi}$ unstably. We elaborate on this claim in section 4 and report on our evidence in section 6.1 and appendix C.

Finally, we note that the isomorphisms $\mathbf{G}\xrightarrow{\chi}[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}\xleftarrow{\phi}[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ do not imply that $\mathbf{G}$ equals $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$. However, we conjecture that the inclusion $\mathbf{G}\subseteq[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ is an equality and we show in section 6.3 that this is true for all finite fields by computing the first Milnor–Witt K-theory $K_1^{MW}(\mathbb{F}_q)$, which is isomorphic to $[\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ and to the subgroup generated by all classes $[g_{u,v}]$ in $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$.

To conclude this discussion we remark that Cazanave speculates in his thesis [12, page 31] whether $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$ may be used to give a concrete model to study the motivic homotopy group $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$. Our cursory attempts at equipping this set with a group operation did not succeed. The obvious choices for an operation on matrices do not yield the conventional group structure, and we were unable to find a suitable candidate for an operation. We did not include our findings in this paper. We refer to remark 84 for some additional comments.

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2 The Jouanolou device and morphisms to $\mathbb{P}^1$

In this section we work out the details needed about the Jouanolou device $\mathcal{J}$, morphisms $\mathcal{J}\to\mathbb{P}^1$, and the pointed naive homotopy relation. We keep the terminology as elementary as possible and hope that the details provided help making our approach accessible.

Throughout this paper k will always denote a field. All schemes are schemes over $\operatorname{Spec} k$. The letter R will always denote the following ring.

Definition 9 We let R denote the ring

$$R = \frac{k[x,y,z,w]}{(x+w-1, xw-yz)}.$$

We will frequently identify R with the ring $k[x,y,z]/(x(1-x)-yz)$ where it is convenient. The Jouanolou device of $\mathbb{P}^1$ is the smooth affine k-scheme $\mathcal{J} = \operatorname{Spec} R$. We consider $\mathcal{J}$ to be pointed at $\mathbf{j} = (x-1, y, z, w)$.

The Jouanolou device may also be considered as the ring representing (2×2)-matrices with trace 1 and determinant 0.

While we will discuss morphisms $\mathcal{J} \to \mathbb{P}^1$ in more detail later, we point out that there are two evident morphisms that exhibit $\mathcal{J}$ as an affine torsor bundle over $\mathbb{P}^1$. The matrices over R

$$p_1 = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \text{ and } q_1 = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$$

are idempotent matrices which thus define projective modules $\mathcal{P}_1 = \operatorname{Im}(p_1)$ and $\mathcal{Q}_1 = \operatorname{Im}(q_1)$ as the image of the associated map of R-modules. Both $\mathcal{P}_1$ and $\mathcal{Q}_1$ have rank 1, and so they yield invertible sheaves on $\mathcal{J}$. We obtain a map $\pi\colon \mathcal{J} \to \mathbb{P}^1$ by selecting the invertible sheaf associated to $\mathcal{P}_1$ and the generating sections

$$s_0 = \begin{pmatrix} x \\ z \end{pmatrix} \text{ and } s_1 = \begin{pmatrix} y \\ w \end{pmatrix}.$$

We intuitively understand this map as sending a point in $\mathcal{J}$ corresponding to a matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ to either the point with homogeneous coordinates $[x : y]$ or $[z : w]$, depending on which is defined. When both points make sense in $\mathbb{P}^1$, they agree, so the map is well-defined. Similarly, we obtain a map $\tilde{\pi}\colon \mathcal{J} \to \mathbb{P}^1$ by using $\mathcal{Q}_1$ and the choice of generating sections

$$s_0 = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } s_1 = -\begin{pmatrix} z \\ w \end{pmatrix}.$$

Both π and $\tilde{\pi}$ exhibit $\mathcal{J}$ as an affine torsor bundle over $\mathbb{P}^1$, hence they are $\mathbb{A}^1$-homotopy equivalences. It follows that

$$\pi^*_{\mathbb{A}^1}\colon \left[\mathbb{P}^1, \mathbb{P}^1\right]^{\mathbb{A}^1} \to \left[\mathcal{J}, \mathbb{P}^1\right]^{\mathbb{A}^1} \tag{10}$$

is a bijection. We show in proposition 128 in appendix A that the canonical map $\nu\colon [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}} \to [\mathcal{J},\mathbb{P}^1]^{\mathbb{A}^1}$ is a bijection because $\mathcal{J}$ is affine and $\mathbb{P}^1$ is $\mathbb{A}^1$-naive. Thus, the composition of the bijection ν and the inverse of π^* is a bijection

$$\xi\colon \left[\mathcal{J},\mathbb{P}^1\right]^{\mathrm{N}} \to \left[\mathbb{P}^1,\mathbb{P}^1\right]^{\mathbb{A}^1}. \tag{11}$$

This bijection may be described as follows. A naive pointed homotopy class of maps $[f]$ represented by the pointed scheme morphism $f\colon \mathcal{J} \to \mathbb{P}^1$ is sent to $\xi([f]) = [f\circ\pi^{-1}]^{\mathbb{A}^1}$, the pointed $\mathbb{A}^1$-homotopy class of the zig-zag $\mathbb{P}^1 \xleftarrow{\pi} \mathcal{J} \xrightarrow{f} \mathbb{P}^1$.

In the following sections we will investigate the domain of ξ, i.e., the set $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ of pointed naive homotopy classes of pointed morphisms $\mathcal{J} \to \mathbb{P}^1$.

2.1 Convenient coordinates for $\mathcal{J}$

The map $\pi\colon \mathcal{J} \to \mathbb{P}^1$ encourages the choice of a convenient set of coordinate charts for $\mathcal{J}$. For $\mathbb{P}^1$, we use the standard notation $U_0 = \mathbb{P}^1 \setminus \{[0:1]\}$ and $U_1 = \mathbb{P}^1 \setminus \{[1:0]\}$. It is straightforward to verify that the preimages under π of U_0 and U_1 are $\pi^{-1}(U_0) = D(x) \cup D(z)$ and $\pi^{-1}(U_1) = D(y) \cup D(w)$. Both of these open sets are isomorphic to $\mathbb{A}^2$ under the following maps.

Lemma 12 The open set $D(x) \cup D(z) \subseteq \mathcal{J}$ is isomorphic to $\mathrm{Spec}(k[a,b])$ under the map $\phi_0\colon \mathbb{A}^2 \to \mathcal{J}$ given by $x \mapsto 1-ab$, $y \mapsto a(1-ab)$, $z \mapsto b$, and $w \mapsto ab$.

Similarly, the open set $D(y) \cup D(w) \subseteq \mathcal{J}$ is isomorphic to $\mathrm{Spec}(k[s,t])$ under the map $\phi_1\colon \mathbb{A}^2 \to \mathcal{J}$ given by $x \mapsto st$, $y \mapsto t$, $z \mapsto s(1-st)$, and $w \mapsto 1-st$.

Proof The proof proceeds by studying the map locally. For instance, ϕ_0 induces an isomorphism of rings $k[a,b][(1-ab)^{-1}] \to R[x^{-1}]$ and also of $k[a,b][b^{-1}] \to R[z^{-1}]$. The open sets $D(1-ab)$ and $D(b)$ cover $\mathbb{A}^2$, so it follows that ϕ_0 maps surjectively onto $D(x) \cup D(z)$. The inverse map is obtained by gluing the maps that are defined on $D(x)$ and $D(z)$, giving the result. A similar argument works for ϕ_1. □

Remark 13 The open affine subschemes $D(x) \cup D(z)$ and $D(y) \cup D(w)$ of $\mathrm{Spec}(R)$ have the odd property that their ring of global sections is not a localization of R.

2.2 Invertible sheaves on $\mathcal{J}$

By [17, Theorem II.7.1], a morphism $\mathcal{J} \to \mathbb{P}^1$ is determined by an invertible sheaf $\mathcal{L}$ on $\mathcal{J}$ and two generating global sections of $\mathcal{L}$. We now take the time to study the invertible sheaves on $\mathcal{J}$ to enable our study of the morphisms $\mathcal{J} \to \mathbb{P}^1$. We will assume familiarity with the basic terminology presented in for example [25, Chapter 1] and [8].

Since $\mathcal{J} = \mathrm{Spec}(R)$ is an irreducible affine scheme, the invertible sheaves on $\mathcal{J}$ correspond to projective R-modules of rank 1. We have already seen the projective modules $\mathcal{P}_1$ and $\mathcal{Q}_1$ used to define π and $\tilde{\pi}$ above. Since the map $\pi\colon \mathcal{J} \to \mathbb{P}^1$ is an $\mathbb{A}^1$-weak equivalence and the Picard group functor is homotopy invariant, the induced map on Picard groups is an isomorphism $\pi^*\colon \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(\mathcal{J})$. Since $\pi^*(\mathcal{O}(1)) = \mathcal{P}_1$ and $\mathrm{Pic}(\mathcal{J}) \cong \mathbb{Z}$, it follows that $\mathcal{P}_1$ generates the Picard group of $\mathcal{J}$. For future reference, we state this as a lemma.

Lemma 14 The Picard group of $\mathcal{J}$ is isomorphic to $\mathbb{Z}$ and $\mathcal{P}_1$ is a generator.

Furthermore, $\mathcal{Q}_1 = -\mathcal{P}_1$ in $\mathrm{Pic}(\mathcal{J})$ as the following proposition shows.

Proposition 15 There is an isomorphism $\mathcal{P}_1 \otimes \mathcal{Q}_1 \cong R$.

Proof The R-module $\mathcal{P}_1 \otimes \mathcal{Q}_1$ is generated by

$$\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \otimes \begin{bmatrix} z \\ w \end{bmatrix} \right\}.$$

Consider the module homomorphism $\mu\colon R^2 \otimes R^2 \longrightarrow R^2$ induced by component-wise multiplication $\mu\left(\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} ac \\ bd \end{bmatrix}$. We restrict μ to $\mathcal{P}_1 \otimes \mathcal{Q}_1$ and observe that the image of $\mathcal{P}_1 \otimes \mathcal{Q}_1$ under μ is the submodule $\left\langle \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle \subseteq R^2$ (use $x + w = 1$). This is a free R-module of rank 1. As $\mu\colon \mathcal{P}_1 \otimes \mathcal{Q}_1 \to \left\langle \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle$ is surjective, it follows that it is locally an isomorphism at all maximal ideals $\mathfrak{m} \subseteq R$. Hence the map μ itself restricted to $\mathcal{P}_1 \otimes \mathcal{Q}_1$ is an isomorphism onto its image. $\square$

We would like to understand the tensor powers of $\mathcal{P}_1$ and $\mathcal{Q}_1$. It turns out they can be described by the following R-modules that are constructed by open patching data. We will investigate these modules before returning to the claim that they are indeed tensor powers of $\mathcal{P}_1$ and $\mathcal{Q}_1$.

Definition 16 As in [25, 2.5 on page 14], we define projective R-modules of rank 1 by open patching data for the open sets $D(x)$, $D(w)$ as $x + w = 1$. For $n \geq 0$, we define the R-modules

$$\mathcal{O}\mathcal{P}_n = \left\{(f_x, f_w) \in R[x^{-1}] \times R[w^{-1}] \,|\, f_w = \left(\frac{z}{x}\right)^n f_x\right\},$$
$$\mathcal{O}\mathcal{Q}_n = \left\{(f_x, f_w) \in R[x^{-1}] \times R[w^{-1}] \,|\, f_w = \left(\frac{y}{x}\right)^n f_x\right\}.$$

Definition 17 We denote by $\mathcal{P}_n$ and $\mathcal{Q}_n$ the submodules of R^2 generated as follows

$$\mathcal{P}_n = \left\langle \begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} x^{n-1}y \\ z^{n-1}w \end{bmatrix}, \dots, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right\rangle, \text{ and } \mathcal{Q}_n = \left\langle \begin{bmatrix} x^n \\ y^n \end{bmatrix}, \begin{bmatrix} x^{n-1}z \\ y^{n-1}w \end{bmatrix}, \dots, \begin{bmatrix} z^n \\ w^n \end{bmatrix} \right\rangle.$$

The following lemma is useful for simplifying proofs. It shows that what we prove about $\mathcal{P}_n$ by symmetry holds for $\mathcal{Q}_n$.

Lemma 18 We define the involutive automorphism $\tau\colon R \to R$ by

$$\tau : x \mapsto x \qquad y \mapsto z$$
$$z \mapsto y \qquad w \mapsto w.$$

Pulling back along τ gives R-module isomorphisms $\tau^*\mathcal{P}_n \cong \mathcal{Q}_n$ and $\tau^*\mathcal{Q}_n \cong \mathcal{P}_n$.

Proof To more easily distinguish between them, we give the domain and codomain of τ different names and write $\tau\colon R \to R'$. Pulling back the R'-module $\mathcal{P}_n$, we get the R-module $\tau^*\mathcal{P}_n$, where the multiplication is defined by $r \cdot_R p = \tau(r) \cdot_{R'} p$. The map $\tau^*\mathcal{P}_n \to \mathcal{Q}_n$ is defined on basis elements by

$$\begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} \mapsto \begin{bmatrix} x^{n-i}z^i \\ y^{n-i}w^i \end{bmatrix}.$$

It is easily checked that f is bijective and R-linear and hence an R-module isomorphism.

To see that $\tau^*\mathcal{Q}_n \cong \mathcal{P}_n$, we pull back the isomorphism $\mathcal{Q}_n \cong \tau^*\mathcal{P}_n$, which we just proved, along τ on both sides. Since $\tau \circ \tau = \mathrm{id}$, this simplifies to $\tau^*\mathcal{Q}_n \cong \tau^*\tau^*\mathcal{P}_n = \mathcal{P}_n$. ◻

Proposition 19 The R-modules $\mathcal{P}_n$ and $\mathcal{Q}_n$ are also generated in the following way

$$\mathcal{P}_n = \left\langle \begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right\rangle \quad \text{and} \quad \mathcal{Q}_n = \left\langle \begin{bmatrix} x^n \\ y^n \end{bmatrix}, \begin{bmatrix} z^n \\ w^n \end{bmatrix} \right\rangle.$$

Proof We only prove the claim for $\mathcal{P}_n$, as the proof for $\mathcal{Q}_n$ is analogous by lemma 18. Containment in one direction is clear by definition of $\mathcal{P}_n$. Now fix n and pick a number $0 \le i \le n$. We then have

$$\begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = (x+w)^n \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = \sum_{d=0}^{n} \binom{n}{d} x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix}.$$

For each d, one of the following hold, which completes the proof

$$x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = x^{n-i-d}y^i w^d \begin{bmatrix} x^n \\ z^n \end{bmatrix} \qquad \text{if } i+d \leq n,$$

$$x^{n-d}w^d \begin{bmatrix} x^{n-i}y^i \\ z^{n-i}w^i \end{bmatrix} = x^{n-d}z^{n-i}w^{d+i-n} \begin{bmatrix} y^n \\ w^n \end{bmatrix} \quad \text{if } i+d > n.$$

□

Proposition 20 There are isomorphisms $\mathcal{P}_n \cong \mathcal{O}\mathcal{P}_n$ and $\mathcal{Q}_n \cong \mathcal{O}\mathcal{Q}_n$ for all natural numbers n. Hence $\mathcal{P}_n$ and $\mathcal{Q}_n$ are algebraic line bundles over R (that is, finitely generated R-modules of constant rank 1) and determine invertible sheaves on $\mathcal{J}$.

Proof The canonical projections $p_1 \colon \mathcal{P}_n[x^{-1}] \to R[x^{-1}]$ and $p_2 \colon \mathcal{P}_n[w^{-1}] \to R[w^{-1}]$ are isomorphisms. For $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{P}_n[x^{-1}, w^{-1}]$, one checks that $(z/x)^n f = g$, which is the same open patching data for $\mathcal{O}\mathcal{P}_n$. The evident map $\mathcal{P}_n \to \mathcal{O}\mathcal{P}_n$ is thus an isomorphism. In the same way, the evident map $\mathcal{Q}_n \to \mathcal{O}\mathcal{Q}_n$ is an isomorphism. □

Remark 21 Proposition 20 shows us how to interpret an element

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \text{ with } a_0, a_1 \in R,$$

which is a global section of the invertible sheaf associated to $\mathcal{P}_n$. Namely, the global section s_0 restricted to $D(x)$ is described by $a_0 x^n + a_1 y^n$, while on $D(w)$ the section is $a_0 z^n + a_1 w^n$. On the overlap, the two sections agree when compared using the appropriate transition functions.

Note that proposition 20 also gives a canonical form for the elements $(f_x, f_w) \in \mathcal{O}\mathcal{P}_n$. Any such element is described as

$$\begin{bmatrix} f_x \\ f_w \end{bmatrix} = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \text{ for some } a_0, a_1 \in R.$$

The algebraic line bundles $\mathcal{P}_n$ and $\mathcal{Q}_n$ may also be described as the image of an idempotent (2×2)-matrix of rank 1. For $n \geq 1$, let $A = \sum_{k=0}^{n-1} \binom{2n-1}{k} x^{n-1-k} w^k$ and $B = \sum_{k=n}^{2n-1} \binom{2n-1}{k} x^{2n-1-k} w^{k-n}$. Then $x^n A + w^n B = (x+w)^{2n-1} = 1$. Define

$$M_n = \begin{pmatrix} x^n A & y^n B \\ z^n A & w^n B \end{pmatrix} \text{ and } M_n' = \begin{pmatrix} x^n A & z^n B \\ y^n A & w^n B \end{pmatrix}. \tag{22}$$

Proposition 23 For every $n \geq 1$, the matrices M_n and M_n' are idempotent of rank 1 and have image $\mathcal{P}_n$ and $\mathcal{Q}_n$ respectively.

Proof It is straightforward to verify that M_n is idempotent using the relation $1 = x^nA + w^nB$ and that $\mathrm{Im}(M_n) \subset \mathcal{P}_n$. Note that

$$x^n \begin{bmatrix} x^nA \\ z^nA \end{bmatrix} + z^n \begin{bmatrix} y^nB \\ w^nB \end{bmatrix} = (x^nA + w^nB) \begin{bmatrix} x^n \\ z^n \end{bmatrix} = \begin{bmatrix} x^n \\ z^n \end{bmatrix}$$

and similarly,

$$y^n \begin{bmatrix} x^nA \\ z^nA \end{bmatrix} + w^n \begin{bmatrix} y^nB \\ w^nB \end{bmatrix} = (x^nA + w^nB) \begin{bmatrix} y^n \\ w^n \end{bmatrix} = \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

So $\mathcal{P}_n \subset \mathrm{Im}(M_n)$, and the image is equal to $\mathcal{P}_n$. The argument for M'_n and $\mathcal{Q}_n$ is similar. □

Proposition 24 The morphisms $\mu\colon \mathcal{P}_1^{\otimes n} \to \mathcal{P}_n$ and $\mu'\colon \mathcal{Q}_1^{\otimes n} \to \mathcal{Q}_n$ obtained from component-wise multiplication are isomorphisms.

Proof Consider the R-module map $\mu\colon R^2 \otimes R^2 \to R^2$ induced by component-wise multiplication. By the description of the generators of the modules $\mathcal{P}_n$, it is clear that μ restricts to a map $\mu\colon \mathcal{P}_n \otimes \mathcal{P}_1 \to \mathcal{P}_{n+1}$ and this map is surjective. As both $\mathcal{P}_n \otimes \mathcal{P}_1$ and $\mathcal{P}_{n+1}$ are algebraic line bundles, the map μ is surjective locally at every maximal ideal $\mathfrak{m} \subseteq R$ and hence an isomorphism. Thus $\mu\colon \mathcal{P}_n \otimes \mathcal{P}_1 \to \mathcal{P}_{n+1}$ is itself an isomorphism by [8, Proposition 3.9]. The claim now follows by induction. □

We now have a complete description of the isomorphism $\mathbb{Z} \cong \mathrm{Pic}(\mathcal{J})$.

Proposition 25 Under the isomorphism $\mathrm{Pic}(\mathcal{J}) \cong \mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ arising from the $\mathbb{A}^1$-homotopy equivalence $\pi : \mathcal{J} \to \mathbb{P}^1$, the modules $\mathcal{P}_n$ and $\mathcal{Q}_n$ correspond to n and $-n$, respectively, while the trivial invertible sheaf $\mathcal{O}$ corresponds to 0.

Proof By lemma 14, $\mathrm{Pic}(\mathcal{J}) = \mathbb{Z}$, and $\mathcal{P}_1$ generates the Picard group. By proposition 15, the inverse of $\mathcal{P}_1$ is $\mathcal{Q}_1$. By proposition 24, the modules $\mathcal{P}_n$ and $\mathcal{Q}_n$ correspond to n and $-n$ in the Picard group. □

2.3 Pointed morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ and $\mathcal{J} \to \mathbb{P}^1$

We will now study morphisms to $\mathbb{P}^1$ in more detail. By [17, Theorem II.7.1], for a smooth k-scheme X, the data needed to give a morphism $f : X \to \mathbb{P}^1$ are an invertible sheaf $\mathcal{L}$ over X and the choice of two global sections $s_0, s_1 \in \Gamma(X, \mathcal{L})$ that generate the invertible sheaf $\mathcal{L}$. That is, at every point $p \in X$, the stalks of the sections $(s_0)_p$

and $(s_1)_p$ generate the local ring $\mathcal{L}_p$. We then write $[s_0, s_1]$ for the map $X \to \mathbb{P}^1$ given by the data above, where we usually omit the invertible sheaf $\mathcal{L}$ from the notation.

The scheme $\mathbb{P}^1$ is pointed at $\infty = [1 : 0]$. A pointed map $f : \mathbb{P}^1 \to \mathbb{P}^1$ by definition is a map satisfying $f(\infty) = \infty$. A pointed morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by the invertible sheaf $\mathcal{O}(n)$ on $\mathbb{P}^1$ with two generating sections $\sigma_0, \sigma_1 \in k[x_0, x_1]_n$ has the following special form by work of Cazanave [13].

Proposition 26 [13, Proposition 2.3] A pointed k-scheme morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ corresponds uniquely to the data of a natural number n and a choice of two polynomials, $A = \sum_{i=0}^{n} a_i X^i$ and $B = \sum_{i=0}^{n-1} b_i X^i$ in $k[X]$ for which $a_n = 1$ and the resultant $\mathrm{res}_{n,n}(A, B)$ is non-zero. The integer n is called the degree of f and is denoted $\deg(f)$; the scalar $\mathrm{res}(f) = \mathrm{res}_{n,n}(A, B) \in k^\times$ is called the resultant of f.

One easily translates from the morphism given by n, A, and B in proposition 26 to the morphism given by the invertible sheaf $\mathcal{O}(n)$ and the choice of global sections $\sigma_0 = \sum_{i=0}^{n} a_i x_0^i x_1^{n-i}$ and $\sigma_1 = \sum_{i=0}^{n} b_i x_0^i x_1^{n-i}$ where we understand $b_n = 0$. The resultant condition guarantees that these global sections generate $\mathcal{O}(n)$. The condition $a_n = 1$ is a normalizing condition to give a bijective correspondence between morphisms and the data n, A, and B. We will find it more convenient to use the data $[\sigma_0, \sigma_1]\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mathcal{O}(n)$ to describe a pointed map in what follows.

Proposition 27 Consider a pointed map $[\sigma_0, \sigma_1]\colon \mathbb{P}^1 \to \mathbb{P}^1$ with invertible sheaf $\mathcal{O}(n)$, $\sigma_0 = \sum_{i=0}^{n} a_i x_0^i x_1^{n-i}$ and $\sigma_1 = \sum_{i=0}^{n} b_i x_0^i x_1^{n-i}$. The composition $[\sigma_0, \sigma_1] \circ \pi$ is the map $[s_0, s_1]\colon \mathcal{J} \to \mathbb{P}^1$ with invertible sheaf $\mathcal{P}_n$ and global sections

$$s_0 = \sum_{i=0}^{n} a_i \begin{bmatrix} x^i y^{n-i} \\ z^i w^{n-i} \end{bmatrix} \text{ and } s_1 = \sum_{i=0}^{n} b_i \begin{bmatrix} x^n y^{n-i} \\ z^i w^{n-i} \end{bmatrix}. \tag{28}$$

Proof This is a straightforward calculation. The resultant condition ensures that the sections s_0 and s_1 generate $\mathcal{P}_n$. ◻

Remark 29 We note that the difference between a general map $[s_0, s_1]\colon \mathcal{J} \to \mathbb{P}^1$ with invertible sheaf $\mathcal{P}_n$ and a map $\mathcal{J} \to \mathbb{P}^1$ which factors as $f \circ \pi$ with $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is that the coefficients a_i and b_i in the expressions of the sections in equation (28) are in the field k when the map factors, but in general the coefficients are in R.

For later purposes, we extend the definition of the resultant to homogeneous polynomials in two variables. We collect some further facts about resultants in appendix B.

Definition 30 Let $R[\alpha,\beta]$ be a polynomial ring over R in two variables and let $R[\alpha,\beta]_n$ denote the subgroup of homogeneous polynomials of degree n. For every $n\geq 1$, the map $\sigma\colon R[\alpha,\beta]_n\to\mathcal{P}_n$, defined by $\sigma(\alpha^i\beta^{n-i})=\begin{bmatrix}x^iy^{n-i}\\ z^iw^{n-i}\end{bmatrix}$ for all $0\leq i\leq n$ is a surjective morphism of R-modules. For a pair of homogeneous polynomials (S_0,S_1), we write $\sigma((S_0,S_1))$ for the pair $(\sigma(S_0),\sigma(S_1))$ in $(\mathcal{P}_n)^2$ by slight abuse of notation. For a pair of homogeneous polynomials $(S_0,S_1)=(a_n\alpha^n+\ldots+a_0\beta^n, b_n\alpha^n+\ldots+b_0\beta^n)\in(R[\alpha,\beta]_n)^2$, we divide by β^n to form univariate polynomials $(\widehat{S}_0,\widehat{S}_1)=(\sum_{i=0}^n a_i\mathscr{X}^i,\sum_{i=0}^n b_i\mathscr{X}^i)$ in $\mathscr{X}:=\alpha/\beta$. We define the resultant $\mathrm{res}(S_0,S_1)$ of the pair (S_0,S_1) to be

$$\mathrm{res}(S_0,S_1):=\mathrm{res}\left(\sum_{i=0}^n a_i\mathscr{X}^i,\sum_{i=0}^n b_i\mathscr{X}^i\right)=\det\left(\mathrm{Syl}(\widehat{S}_0,\widehat{S}_1)\right),$$

where $\mathrm{Syl}(\widehat{S}_0,\widehat{S}_1)$ is the Sylvester matrix of the pair of polynomials $(\widehat{S}_0,\widehat{S}_1)$ in $R[\mathscr{X}]$.

Lemma 31 Consider a pair of homogeneous polynomials of degree $n\geq 1$ in $R[\alpha,\beta]_n$, denoted by $S_0=\sum_{i=0}^n a_i\alpha^{n-i}\beta^i$ and $S_1=\sum_{i=0}^n b_i\alpha^{n-i}\beta^i$. If $\mathrm{res}(S_0,S_1)$ is a unit, then the pair of sections $(\sigma(S_0),\sigma(S_1))$ generates $\mathcal{P}_n$ and defines a morphism $[\sigma(S_0),\sigma(S_1)]:\mathcal{J}\to\mathbb{P}^1$.

Proof It suffices to show that (s_0,s_1) generate $\mathcal{P}_n$ on the open patches $D(x)$ and $D(w)$. On $D(x)$, this requires showing that the ideal $(a_nx^n+\ldots a_0y^n, b_nx^n+\ldots+b_0y^n)$ is the unit ideal in $R[x^{-1}]$. The ideal is the same as the ideal $(a_n+\ldots a_0\frac{y^n}{x^n}, b_n+\ldots+b_0\frac{y^n}{x^n})$ which corresponds to a pair of polynomials of degree n in the variable $\frac{y}{x}$. By lemma 130, this pair of polynomials has unit resultant. Since the resultant is a unit, there exists $U_x,V_x\in R[x^{-1}]$ by lemma 129 giving a Bézout relation $U_x(a_n+\ldots a_0\frac{y^n}{x^n})+V_x(b_n+\ldots+b_0\frac{y^n}{x^n})=1$. On $D(w)$ we need to prove that the ideal $(a_nz^n+\ldots a_0w^n, b_nz^n+\ldots+b_0w^n)$ is the unit ideal in $R[w^{-1}]$. The ideal is equal to the ideal $(a_n\frac{z^n}{w^n}+\ldots a_0, b_n\frac{z^n}{w^n}+\ldots+b_0)$. This pair of polynomials has unit resultant by assumption. By lemma 129, unit resultant implies existence of a Bézout relation $U_w(a_n\frac{z^n}{w^n}+\ldots a_0)+V_x(b_n\frac{z^n}{w^n}+\ldots+b_0)=1$ in $R[w^{-1}]$. This proves that $[s_0,s_1]$ defines a morphism $\mathcal{J}\to\mathbb{P}^1$. □

Remark 32 Let $[\sigma_0,\sigma_1]\colon\mathbb{P}^1\to\mathbb{P}^1$ be a pointed map given by invertible sheaf $\mathcal{O}(n)$, and sections $\sigma_0=\sum_{i=0}^n a_ix_0^ix_1^{n-i}$ and $\sigma_1=\sum_{i=0}^n b_ix_0^ix_1^{n-i}$. Then the pair of homogeneous polynomials $S_0=\sum_{i=0}^n a_i\alpha^{n-i}\beta^i$ and $S_1=\sum_{i=0}^n b_i\alpha^{n-i}\beta^i$ in $R[\alpha,\beta]_n$ has unit resultant. By lemma 31, the pair $(\sigma(S_0),\sigma(S_1))$ defines a morphism $[\sigma(S_0),\sigma(S_1)]:\mathcal{J}\to\mathbb{P}^1$. This morphism is equal to the morphism $[s_0,s_1]$, constructed from $[\sigma_0,\sigma_1]$ in proposition 27.

Remark 33 We note that there exist pairs of polynomials $(S_0, S_1), (S'_0, S'_1)$ such that $\sigma((S_0, S_1)) = \sigma((S'_0, S'_1))$, while $\operatorname{res}(S_0, S_1) \neq \operatorname{res}(S'_0, S'_1)$. An example is given by $(x\alpha + z\beta, \beta)$ and (α, β). We calculate

$$\sigma((x\alpha + z\beta, \beta)) = \left(x \begin{bmatrix} x \\ z \end{bmatrix} + z \begin{bmatrix} y \\ w \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right) = \left(\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right) = \sigma((\alpha, \beta)).$$

Their resultants are

$$\operatorname{res}(x\alpha + z\beta, \beta) = x \neq 1 = \operatorname{res}(\alpha, \beta).$$

We now look at the data needed to describe a general morphism $\mathcal{J} \to \mathbb{P}^1$ and also see what condition pointedness imposes.

Construction 34 A morphism $f\colon \mathcal{J} \to \mathbb{P}^1$ is determined by the following data: an invertible sheaf $\mathcal{L}$ on $\mathcal{J}$ and the choice of two global sections $s_0, s_1 \in \Gamma(\mathcal{J}, \mathcal{L})$ that generate $\mathcal{L}$ [17, Theorem II.7.1]. Since $\operatorname{Pic}(\mathcal{J}) \cong \mathbb{Z}$, the invertible sheaf $\mathcal{L}$ may be chosen to be either $\mathcal{P}_n$, $\mathcal{Q}_n$, or $\mathcal{O}$. We call the integer corresponding to the class of $\mathcal{L}$ in $\operatorname{Pic}(\mathcal{J}) \cong \mathbb{Z}$ the degree of f.

We will now make the assignment $(\mathcal{L}, s_0, s_1) \mapsto f$ explicit. We will study only the case of $\mathcal{P}_n$, as $\mathcal{Q}_n$ is handled in the same way by proposition 19, and the case of $\mathcal{O}$ is discussed later in section 2.5.

In the case of $\mathcal{P}_n$, two generating sections $s_0, s_1 \in \Gamma(\mathcal{J}, \mathcal{P}_n)$ may be chosen to be of the form

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} \qquad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

Define $D(s_i) = \{p \in \mathcal{J} \,|\, (s_i)_p \notin \mathfrak{m}_p(\mathcal{P}_n)_p\}$. The map $[s_0, s_1]$ is defined on the open set $D(s_i)$ to map into $U_i = \{[x_0, x_1] \,|\, x_i \neq 0\}$. Here $U_0 \cong \operatorname{Spec}(k[y_1])$ and $U_1 \cong \operatorname{Spec}(k[y_0])$, where $y_0 = x_0/x_1$ and $y_1 = x_1/x_0$. The map $D(s_i) \to U_i$ is given by the corresponding map of rings $k[y_j] \to \mathcal{P}_n[s_i^{-1}]$ determined by $y_j \mapsto s_j/s_i$. This requires some explanation due to the description of the sheaf $\mathcal{P}_n$. Proposition 20 shows that the components of each section s_i describe the section on the open sets $D(x)$ and $D(w)$. Hence there are four cases to consider to get a description of the map in concrete terms of affine open sets.

(1) $D(x) \cap D(s_0)$: Here s_0 is described by $a_0x^n + a_1y^n$ in the ring $R[x^{-1}]$ and s_1 is given by $b_0x^n + b_1y^n$ in the ring $R[x^{-1}]$. Hence on $D(s_0)$ the corresponding ring map $k[y_1] \to R[x^{-1}, (a_0x^n + a_1y^n)^{-1}]$ is given by $y_1 \mapsto \frac{b_0x^n + b_1y^n}{a_0x^n + a_1y^n}$.

(2) $D(x) \cap D(s_1)$: Here s_0 is described by $a_0x^n + a_1y^n$ in the ring $R[x^{-1}]$ and s_1 is given by $b_0x^n + b_1y^n$ in the ring $R[x^{-1}]$. Hence on $D(s_1)$ the corresponding ring map $k[y_0] \to R[x^{-1}, (b_0x^n + b_1y^n)^{-1}]$ is given by $y_0 \mapsto \frac{a_0x^n + a_1y^n}{b_0x^n + b_1y^n}$.

(3) $D(w) \cap D(s_0)$: Here s_0 is described by $a_0 z^n + a_1 w^n$ in the ring $R[w^{-1}]$ and s_1 is given by $b_0 z^n + b_1 w^n$ in the ring $R[w^{-1}]$. Hence on $D(s_0)$ the corresponding ring map $k[y_1] \to R[w^{-1}, (a_0 z^n + a_1 w^n)^{-1}]$ is given by $y_1 \mapsto \frac{b_0 z^n + b_1 w^n}{a_0 z^n + a_1 w^n}$.

(4) $D(w) \cap D(s_1)$: Here s_0 is described by $a_0 z^n + a_1 w^n$ in the ring $R[w^{-1}]$ and s_1 is given by $b_0 z^n + b_1 w^n$ in the ring $R[w^{-1}]$. Hence on $D(s_1)$ the corresponding ring map $k[y_0] \to R[w^{-1}, (b_0 z^n + b_1 w^n)^{-1}]$ is given by $y_0 \mapsto \frac{a_0 z^n + a_1 w^n}{b_0 z^n + b_1 w^n}$.

This information can be consolidated into the two maps $D(x) \to \mathbb{P}^1$ and $D(w) \to \mathbb{P}^1$ given in terms of the pair of sections $[a_0 x^n + a_1 y^n, b_0 x^n + b_1 y^n]$ and $[a_0 z^n + a_1 w^n, b_0 z^n + b_1 w^n]$ respectively. Written in this form, we see that a map $\mathcal{J} \to \mathbb{P}^1$ given by the invertible sheaf $\mathcal{P}_n$ with two generating sections s_0, s_1 should be interpreted as giving a map to $\mathbb{P}^1$ on the open sets $D(x)$ and $D(w)$ according to the first component of the sections s_0, s_1 on $D(x)$ and according to the second component of the sections s_0, s_1 on $D(w)$.

Remark 35 Recall that $\mathcal{J}$ is pointed at $\mathbf{j} = (x-1, y, z, w)$ and $\mathbb{P}^1$ is pointed at $\infty = [1:0]$. A map $f : \mathcal{J} \to \mathbb{P}^1$ is pointed if $f(\mathbf{j}) = \infty$. If $f = [s_0, s_1]$ with line bundle $\mathcal{L}$ and generating sections s_0, s_1, pointedness can be verified by checking that the stalk $s_1(\mathbf{j})$ satisfies $s_1(\mathbf{j}) = 0$ in the local ring $\mathcal{L}_{\mathbf{j}}$. For us, it suffices to work on $D(x)$ where our line bundles are trivial, and verify that modulo $\mathbf{j}$ the section s_1 vanishes.

We give a concrete criterion for checking pointedness of a map $f : \mathcal{J} \to \mathbb{P}^1$ with line bundle $\mathcal{P}_n$. The case of $\mathcal{Q}_n$ is similar.

Proposition 36 A map $[s_0, s_1] : \mathcal{J} \to \mathbb{P}^1$ with invertible sheaf $\mathcal{P}_n$ and generating sections

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \qquad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}$$

is pointed if and only if $b_0 \in \mathbf{j}$, i.e., $b_0(\mathbf{j}) = 0$.

Proof First, assume the map $[s_0, s_1]$ is pointed. Construction 34 gives a description of the map in local coordinates. Note that for $\mathbf{j}$ to map to $\infty \in U_0$, it is necessary that $\mathbf{j} \in D(s_0)$. Since $\mathbf{j} \in D(x) \cap D(s_0)$, the map in local coordinates is obtained by taking Spec of the ring map $\sigma \colon k[y_1] \to R[x^{-1}, (a_0 x^n + a_1 y^n)^{-1}]$ which is given by $\sigma(y_1) = \frac{b_0 x^n + b_1 y^n}{a_0 x^n + a_1 y^n}$. The condition for pointedness is then that the preimage of $\mathbf{j}$ under σ is the maximal ideal (y_1). This is equivalent to the condition that y_1 maps into the ideal $(x-1, y, z, w) \subseteq R[x^{-1}, (a_0 x^n + a_1 y^n)^{-1}]$. By the definition of σ, the requirement is that $\frac{b_0 x^n + b_1 y^n}{a_0 x^n + a_1 y^n} \in (x-1, y, z, w)$, which is equivalent to $b_0 x^n + b_1 y^n \in (x-1, y, z, w)$.

Since $y \in (x-1, y, z, w)$ and x is invertible, this condition is met when $b_0 \in \mathbf{j}$. Thus when $[s_0, s_1]$ is pointed, $\mathbf{j} \in D(s_0)$ and $b_0 \in \mathbf{j}$.

Now assume that $b_0 \in \mathbf{j}$. This implies $\mathbf{j} \in D(s_0)$, since the sections s_0, s_1 generate $(\mathcal{P}_n)_\mathbf{j}$, and $b_0 \in \mathbf{j}$ implies $s_1(\mathbf{j}) = 0$. Here we can use the same construction above, since $\mathbf{j} \in D(x) \cap D(s_0)$. The algebra above shows that when $b_0 \in \mathbf{j}$ the preimage of $\mathbf{j}$ under σ is (y_1), i.e., the map $[s_0, s_1]$ is pointed. □

Proposition 37 Let $f = [s_0, s_1] \colon \mathcal{J} \to \mathbb{P}^1$ be a pointed map with invertible sheaf $\mathcal{P}_n$. If $\alpha = s_0(\mathbf{j})$, then $\left[\frac{s_0}{\alpha}, \frac{s_1}{\alpha}\right] : \mathcal{J} \to \mathbb{P}^1$ is a pointed map that is equal to f. Thus any pointed map with line bundle $\mathcal{P}_n$ may be represented by a pair of generating global sections $[s_0, s_1]$ where $s_0(\mathbf{j}) = 1$ and $s_1(\mathbf{j}) = 0$.

Proof Proposition 36 has established that $s_1(\mathbf{j}) = 0$ and $s_0(\mathbf{j}) = \alpha$ is a unit. We verify that the maps $[s_0, s_1]$ and $\left[\frac{s_0}{\alpha}, \frac{s_1}{\alpha}\right]$ are equal in local coordinates by construction 34, where the constants $\frac{1}{\alpha}$ cancel out in every local coordinate chart. □

Proposition 38 Let s_0 and s_1 be the following sections in $\mathcal{P}_n$

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}.$$

The sections s_0, s_1 generate $\mathcal{P}_n$ if and only if there exist $U_x, V_x, U_w, V_w \in R$ such that

$$U_x(x^n a_0 + y^n a_1) + V_x(x^n b_0 + y^n b_1) + U_w(z^n a_0 + w^n a_1) + V_w(z^n b_0 + w^n b_1) = 1.$$

Employing similar notation, sections s_0, s_1 generate $\mathcal{Q}_n$ if and only if there exist $U_x, V_x, U_w, V_w \in R$ such that

$$U_x(x^n a_0 + z^n a_1) + V_x(x^n b_0 + z^n b_1) + U_w(y^n a_0 + w^n a_1) + V_w(y^n b_0 + w^n b_1) = 1.$$

Proof By lemma 18, it suffices to prove this for $\mathcal{P}_n$. Assume s_0, s_1 generate $\mathcal{P}_n$. Then there exist U, V such that $Us_0 + Vs_1 = \begin{bmatrix} x^n \\ z^n \end{bmatrix}$. The first component of this identity gives

$$\big(U(x^n a_0 + y^n a_1) + V(x^n b_0 + y^n b_1)\big) = x^n.$$

Similarly, there exist U', V' such that $U's_0 + V's_1 = \begin{bmatrix} y^n \\ w^n \end{bmatrix}$. This gives the relation

$$\big(U'(z^n a_0 + w^n a_1) + V'(z^n b_0 + w^n b_1)\big) = w^n.$$

Since the equation $1 = (x+w)^{2n}$ demonstrates that 1 can be expressed as a linear combination of x^n and w^n, we are done. Now we assume that there exist elements

$U_x, V_x, U_w, V_w \in R$ such that

$$U_x(x^n a_0 + y^n a_1) + V_x(x^n b_0 + y^n b_1) + U_w(z^n a_0 + w^n a_1) + V_w(z^n b_0 + w^n b_1) = 1.$$

A straight forward computation yields $(U_x x^n + U_w z^n)s_0 + (V_x x^n + V_w z^n)s_1 = \begin{bmatrix} x^n \\ z^n \end{bmatrix}$, and $(U_x y^n + U_w w^n)s_0 + (V_x y^n + V_w w^n)s_1 = \begin{bmatrix} y^n \\ w^n \end{bmatrix}$. These two elements generate $\mathcal{P}_n$, thus $[s_0, s_1]$ do as well. □

For brevity, we write maps $\mathcal{J} \to \mathbb{P}^1$ of nonzero degree using the following notation.

Definition 39 Let n be a positive integer. We write $(a_0, a_1; b_0, b_1)_n$, respectively $(a_0, a_1; b_0, b_1)_{-n}$, for the map $\mathcal{J} \to \mathbb{P}^1$ with invertible sheaf $\mathcal{P}_n$, respectively $\mathcal{Q}_n$, and generating sections

$$s_0 = a_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + a_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ z^n \end{bmatrix} + b_1 \begin{bmatrix} y^n \\ w^n \end{bmatrix} \text{ with } a_0, a_1, b_0, b_1 \in R,$$

respectively with invertible sheaf $\mathcal{Q}_n$ and generating sections

$$s_0 = a_0 \begin{bmatrix} x^n \\ y^n \end{bmatrix} + a_1 \begin{bmatrix} z^n \\ w^n \end{bmatrix}, \quad s_1 = b_0 \begin{bmatrix} x^n \\ y^n \end{bmatrix} + b_1 \begin{bmatrix} z^n \\ w^n \end{bmatrix} \text{ with } a_0, a_1, b_0, b_1 \in R.$$

2.4 The pointed naive homotopy relation

Naive homotopy theory for schemes is a generalization of the homotopy theory of rings in classical algebra, see [16] for a definition. Naive homotopy classes of maps between schemes do not generally have the good properties one expects from a homotopy theory, but in our case, thanks to work of Asok, Hoyois, and Wendt, it is sufficiently good [6].

Definition 40 Let X and Y be smooth schemes over the field k. For $a \in k$, let $i_a = \mathrm{id}_X \times a$ be the map obtained by taking the Cartesian product of id_X and the inclusion map $a\colon \operatorname{Spec} k \to \mathbb{A}^1$ given by the ring map $k[t] \to k$ sending t to a. An elementary homotopy between two morphisms $f\colon X \to Y$ and $g\colon X \to Y$ is given by a morphism $H(T)\colon X \times \mathbb{A}^1 \to Y$ satisfying $H(0) = f$ and $H(1) = g$, i.e., $H(0) = H(T) \circ i_0$ and $H(1) = H(T) \circ i_1$. We say that f and g are elementarily homotopic and write $f \sim g$.

The relation of morphisms being elementarily homotopic is symmetric and reflexive, but not transitive. To obtain an equivalence relation on the set of morphisms $\mathrm{Sm}_k(X, Y)$, we take the transitive closure of $\sim$. That is, we define two morphisms $f, g \in \mathrm{Sm}_k(X, Y)$ to be naively homotopic if there is a finite sequence of elementary homotopies $H_i(T)\colon X\times \mathbb{A}^1 \to Y$, for $0 \le i \le n$ with $H_0(0) = f$, $H_n(1) = g$, and for all $0 \le i < n$ $H_i(1) = H_{i-1}(0)$. We write $f \simeq g$ in this case. The relation $\simeq$ is now an equivalence relation on $\mathrm{Sm}_k(X,Y)$, so we can study the set of naive homotopy classes of morphisms from X to Y.

For our constructions, we will work with pointed maps and pointed naive homotopies.

Definition 41 If X and Y are smooth k-schemes, pointed at k-points x and y respectively, we say that an elementary homotopy $H(T)\colon X\times\mathbb{A}^1 \to Y$ is pointed if the generic point of $\{x\}\times\mathbb{A}^1$ maps to y. Said another way, the points x and y correspond to morphisms $x\colon \mathrm{Spec}(k) \to X$ and $y\colon \mathrm{Spec}(k)\to Y$, and we require that $H(T)\circ(x\times \mathrm{id}_{\mathbb{A}^1}) = y\circ p_1$ where $p_1\colon \mathrm{Spec}(k)\times\mathbb{A}^1 \to \mathrm{Spec}(k)$ is the projection onto the first factor.

As in the unpointed case, the relation on the set of pointed morphisms $\mathrm{Sm}_k(X,Y)_*$ given by pointed elementary homotopies is not an equivalence relation. We say that pointed morphisms $f,g\in \mathrm{Sm}_k(X,Y)_*$ are naively homotopic, and write $f\simeq g$, if there is a chain of pointed elementary homotopies from f to g. The naive homotopy relation is an equivalence relation on pointed morphisms. We write $[X,Y]^{\mathrm{N}} = \mathrm{Sm}_k(X,Y)_*/\simeq$ for the set of equivalence classes.

For us, the most important case is when $X = \mathcal{J} = \mathrm{Spec}(R)$ with basepoint $\mathbf{j} = (x-1,y,z,w)$. This ideal extends to $\mathbf{j}' = (x-1,y,z,w)\subseteq R[T]$. The condition that a homotopy $H(T)\colon \mathcal{J}\times\mathbb{A}^1\to Y$ be pointed is simply that $H(T)(\mathbf{j}') = y$, where y is the basepoint of Y.

2.5 Morphisms $\mathcal{J}\to\mathbb{A}^2\setminus\{0\}$

We write $\deg\colon [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}} \to \mathrm{Pic}(\mathcal{J})\cong\mathbb{Z}$ for the map that sends a map f to $f^*\mathcal{O}(1)$. Our choices thus far set $\deg(\pi) = 1$ and $\deg(\tilde{\pi}) = -1$. Write $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}_n$ for the set of naive homotopy classes of maps $\mathcal{J}\to\mathbb{P}^1$ with degree n. Our goal for this section is to describe the maps $\mathcal{J}\to\mathbb{P}^1$ of degree 0. We consider the scheme $\mathbb{A}^2\setminus\{0\} = \mathrm{Spec}\,(k[s,t])\setminus\{(s,t)\}$ to be pointed at $(s-1,t)$ and write $\eta\colon \mathbb{A}^2\setminus\{0\}\to\mathbb{P}^1$ for the Hopf map given by the trivial algebraic line bundle $\mathcal{O}_{\mathbb{A}^2\setminus\{0\}}$ with the choice of sections $\eta_0 = s$, $\eta_1 = t$. Let SL_2 denote the affine scheme $\mathrm{Spec}\,k[a,b,c,d]/(ad-bc-1)$ pointed at the ideal $(a-1,b,c,d-1)$. Intuitively, this is the scheme of (2×2)-matrices with determinant 1, pointed at the identity matrix.

Proposition 42 Consider a map $f\colon \mathcal{J} \to \mathbb{P}^1$. Then we have $\deg(f) = 0$ if and only if f factors through the Hopf map $\eta\colon \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$.

Proof If $f^*\mathcal{O}(1) = \mathcal{O}_R$, the map is given by sections $s_0, s_1 \in R$ that generate $\mathcal{O}_R$, i.e., $(s_0, s_1) = R$. Hence there are $U, V \in R$ for which $s_0U + s_1V = 1$. This is exactly the data needed to construct a scheme morphism $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$, as such a morphism is given by a morphism $\mathcal{J} \to \mathbb{A}^2$ that does not have $\{0\}$ in the image. Since $\mathrm{Pic}(\mathbb{A}^2 \setminus \{0\}) = 0$, it follows that any map that factors as $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\} \xrightarrow{\eta} \mathbb{P}^1$ has degree 0. □

Corollary 43 Let $f\colon \mathcal{J} \to \mathbb{P}^1$ be a pointed map of degree 0. Then there exists a unique pointed map $f'\colon \mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ such that $f = f' \circ \eta$.

Proof Let $(s_0, s_1)\colon \mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ be a factorization of f through the Hopf map. Note that $\alpha = s_0(\mathbf{j})$ need not be 1, although α is a unit of k. Instead, the map $f' = \left(\frac{1}{\alpha}s_0, \frac{1}{\alpha}s_1\right)$ is pointed and satisfies $f = f' \circ \eta$.

To show uniqueness, let $(s_0', s_1')\colon \mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ be another pointed map that factors f through η. That is, we assume $[s_0, s_1] = [s_0', s_1']$. Note that in this case, $D(s_0) = D(s_0') = D(f^*x_0)$ and $D(s_1) = D(s_1') = D(f^*x_1)$. Working locally in $D(s_1) = D(s_1')$, we have $s_0/s_1 = s_0'/s_1'$ in $R[s_1^{-1}]$ by construction. We may write $s_0' = c_0s_0$ for $c_0 = s_1'/s_1 \in R[s_1^{-1}]$. Similarly, in $D(s_0) = D(s_0')$, we obtain $s_1' = c_1s_1$ for $c_1 = s_0'/s_0$. In the intersection $D(s_0) \cap D(s_1)$ we have $s_0'/s_1' = s_0/s_1$, which implies $s_0'/s_0 = s_1'/s_1$. This is exactly the equation $c_0 = c_1$. The elements $c_1 \in R[s_0^{-1}]$ and $c_0 \in R[s_1^{-1}]$ therefore glue together to an element $c \in R$. Hence c satisfies $s_0' = cs_0$ and $s_1' = cs_1$. Observe that $c(s_0u' + s_1v') = 1$, that is, $c \in R^\times = k^\times$. The pointedness assumption forces $c(\mathbf{j}) = 1$, hence, $c = 1$ with which we conclude $(s_0, s_1) = (s_0', s_1')$. □

Remark 44 The previous proposition says, in other words, that a map $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ is equivalent to a unimodular row (A, B) of length two in R. Furthermore, a pointed map $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ is equivalent to a unimodular row (A, B) of length two in R that also satisfies $A(\mathbf{j}) = 1$ and $B(\mathbf{j}) = 0$.

Pointed elementary homotopies between maps of degree 0 can also be lifted to a pointed elementary homotopy of maps $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$.

Proposition 45 Let $H(T) = [s_0(T), s_1(T)]\colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{P}^1$ be a pointed elementary homotopy between maps $H(0)$ and $H(1)$ which have degree 0. There is a pointed elementary homotopy $H'(T)\colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{A}^2 \setminus \{0\}$ between the lifts $H'(0)$ and $H'(1)$.

Proof Since $H(0)$ and $H(1)$ have degree 0, the homotopy $H(T)$ is degree 0 too, that is, the line bundle it determines is the trivial one $\mathcal{O}_{\mathcal{J}\times\mathbb{A}^1}$. We can use the two generating global sections $s_0(T), s_1(T) \in R[T]$ to build a map $(s_0(T), s_1(T))\colon \mathcal{J}\times\mathbb{A}^1 \to \mathbb{A}^2\setminus\{0\}$. Note that since $s_0(T)$ and $s_1(T)$ generate $R[T]$, there are $u(T), v(T) \in R[T]$ for which $s_0(T)u(T) + s_1(T)v(T) = 1$. Since $H(T)$ is pointed, $s_1(T)(\mathbf{j}') = 0$ in $R[T]/\mathbf{j}'$. This implies that $s_0(T)(\mathbf{j}')u(T)(\mathbf{j}') = 1$ in $R[T]/\mathbf{j}'$. The ring $R[T]/\mathbf{j}'$ is easily seen to be isomorphic to $k[T]$. Hence $\alpha = s_0(T)(\mathbf{j}')$ is a unit of $k[T]$, and the units of $k[T]$ are exactly the units of k. With this, the map $\left(\frac{1}{\alpha}s_0(T), \frac{1}{\alpha}s_1(T)\right) : \mathcal{J}\times\mathbb{A}^1 \to \mathbb{A}^2\setminus\{0\}$ is a pointed homotopy between $H'(0)$ and $H'(1)$. □

Let (A, B) be a unimodular row in R. That is, there exist $U, V \in R$ for which $AU + BV = 1$. Thus the data of a map $\mathcal{J} \to \mathbb{A}^2\setminus\{0\}$ can be used to produce a matrix $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R)$, in other words, a map $\mathcal{J} \to \mathrm{SL}_2$.

Lemma 46 A pointed map $(A, B)\colon \mathcal{J} \to \mathbb{A}^2\setminus\{0\}$ can be lifted to a pointed map $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} : \mathcal{J} \to \mathrm{SL}_2$.

Proof Let $\begin{pmatrix} A & -V_1 \\ B & U_1 \end{pmatrix}$ be an arbitrary lift of (A, B). Note that $A(\mathbf{j}) = 1$, $B(\mathbf{j}) = 0$, and $U_1(\mathbf{j}) = 1$, but $V_1(\mathbf{j}) = v$ for some $v \in k$. For any $d \in R$, we can construct a different lift by setting $U_2 = U_1 + Bd$ and $V_2 = V_1 - Ad$. Set $d = v$. Then $U_2(\mathbf{j}) = 1$, and $V_2(\mathbf{j}) = 0$, so $\begin{pmatrix} A & -V_2 \\ B & U_2 \end{pmatrix}$ is pointed. □

Construction 47 The pointed lift is not unique in general. For example, the unimodular row $(1, 0)$ lifts to the pointed maps

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Generally, any two pointed lifts of a pointed unimodular row (A, B) are naively homotopic. Let

$$\tilde{f}_i = \begin{pmatrix} A & -V_i \\ B & U_i \end{pmatrix}$$

for $i \in \{1, 2\}$ be two pointed lifts of (A, B). A pointed elementary homotopy between $\tilde{f}_1$ and $\tilde{f}_2$ is given by

$$\tilde{f}_t = \begin{pmatrix} A & -(TV_1 + (1-T)V_2) \\ B & TU_1 + (1-T)U_2 \end{pmatrix}. \tag{48}$$

Proposition 49 Every pointed elementary homotopy

$$H(T) = (s_0(T), s_1(T)) \colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{A}^2 \setminus \{0\}$$

can be lifted to a pointed elementary homotopy

$$\begin{pmatrix} s_0(T) & -V(T) \\ s_1(T) & U(T) \end{pmatrix} : \mathcal{J} \times \mathbb{A}^1 \to \mathrm{SL}_2.$$

Proof Recall $\mathbf{j}' = (x-1, y, z, w)$ must map to the basepoint for the homotopy $H(T)$ to be pointed. The sections $s_0(T)$ and $s_1(T)$ generate the unit ideal, hence there exist $u(T)$ and $v(T)$ in $R[T]$ for which $s_0(T)u(T) + s_1(T)v(T) = 1$. The pointedness assumption gives the relation among ideals $(s_0(T) - 1, s_1(T)) \subseteq \mathbf{j}' \subseteq R[T]$.

With these data, we construct the matrix

$$\begin{pmatrix} s_0(T) & -v(T) \\ s_1(T) & u(T) \end{pmatrix} \in \mathrm{SL}_2(R[T]).$$

This matrix determines a map $\mathcal{J} \times \mathbb{A}^1 \to \mathrm{SL}_2$ that lifts the unimodular row $(s_0(T), s_1(T))$ we started with. This homotopy need not be a pointed homotopy. However, for any choice of $d(T) \in R[T]$, the matrix

$$\begin{pmatrix} s_0(T) & -v(T) + s_0(T)d(T) \\ s_1(T) & u(T) + s_1(T)d(T) \end{pmatrix}$$

is also a lift of $(s_0(T), s_1(T))$. We will now show that, for $d(T) = v(T)$, the map this matrix determines is a pointed homotopy. Write $u_2(T) = u(T) + s_1(T)v(T)$ and $v_2(T) = v(T) - s_0(T)v(T)$. Our assumption that $(s_0(T), s_1(T))$ is pointed gives us $(s_0(T) - 1, s_1(T)) \subseteq \mathbf{j}' \subseteq R[T]$. We must show that $(v_2(T), u_2(T) - 1) \subseteq \mathbf{j}'$ too. Since $(s_0(T) - 1) \in \mathbf{j}'$, we have $v_2(T) = -v(T)(s_0(T) - 1) \in \mathbf{j}'$. Observe that $u_2(T) - 1 \in \mathbf{j}'$ if $u(T) - 1 \in \mathbf{j}'$ since $s_1(T) \in \mathbf{j}'$. Since $s_0(T)u(T) + s_1(T)v(T) = 1$, it follows that $s_0(T)u(T) - 1 \in \mathbf{j}'$. This can be rewritten as $s_0(T)u(T) - 1 = (s_0(T) - 1)u(T) + u(T) - 1$. Since $s_0(T) - 1 \in \mathbf{j}'$ it follows that $u(T) - 1 \in \mathbf{j}'$ too. □

Definition 50 Let $\phi\colon \mathrm{SL}_2 \to \mathbb{A}^2 \setminus \{0\}$ be the morphism determined by the ring map $f\colon k[s,t] \to k[a,b,c,d]/(ad - bc - 1)$ given by $f(s) = a$ and $f(t) = c$. Intuitively, this is the morphism that extracts the first column from a matrix in SL_2. As given, this map has codomain $\mathbb{A}^2$, but it is clear from the relation $ad - bc = 1$ that ϕ maps into $\mathbb{A}^2 \setminus \{0\}$.

Proposition 51 The maps $\phi\colon \mathrm{SL}_2 \to \mathbb{A}^2 \setminus \{0\}$ and $\eta\colon \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ induce bijections of naive homotopy classes of pointed maps

$$[\mathcal{J}, \mathrm{SL}_2]^{\mathrm{N}} \xrightarrow{\phi_*} [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} \xrightarrow{\eta_*} [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}_0.$$

Proof The map ϕ_* is surjective by lemma 46. Construction 47 shows that ϕ_* is injective. Corollary 43 shows that η_* is bijective on the level of pointed morphisms. This shows that η_* is surjective. To show that η_* is injective, it suffices to show that a pointed elementary homotopy $H(T)\colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{P}^1$ between degree 0 maps lifts to a pointed elementary homotopy $H'(T)\colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{A}^2 \setminus \{0\}$, which is done in proposition 45. □

3 Operations on naive homotopy classes of morphisms

3.1 Group structure on maps of degree 0

We may now define a binary operation on naive homotopy classes of morphisms $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$. This is analogous to Cazanave's naive sum of pointed rational functions. The group structure is obtained by lifting maps $f, g\colon \mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ to $\tilde{f}, \tilde{g}\colon \mathcal{J} \to \mathrm{SL}_2$, multiplying the two resulting maps using the group structure on SL_2, then mapping back down to $\mathbb{A}^2 \setminus \{0\}$ via the map $\phi\colon \mathrm{SL}_2 \to \mathbb{A}^2 \setminus \{0\}$.

Definition 52 A morphism $f\colon \mathcal{J} \to \mathrm{SL}_2$ is equivalent to the data of a matrix $M \in \mathrm{SL}_2(R)$. A matrix $M \in \mathrm{SL}_2(R)$ corresponds to a pointed morphism if upon evaluation at $\mathbf{j}$, the resulting matrix is the identity matrix. The set of pointed maps corresponds to a subgroup of $\mathrm{SL}_2(R)$. The operation of matrix multiplication respects the naive homotopy relation for pointed maps and therefore defines a group operation on $[\mathcal{J}, \mathrm{SL}_2]^{\mathrm{N}}$, the set of pointed naive homotopy classes of morphisms. It suffices to prove the following proposition, given that the naive homotopy relation is the transitive closure of pointed elementary homotopies.

Proposition 53 Let $M(T) \in \mathrm{SL}_2(R[T])$ be a pointed elementary homotopy between the matrices $M_0 = M(0) \in \mathrm{SL}_2(R)$ and $M_1 = M(1) \in \mathrm{SL}_2(R)$ corresponding to pointed morphisms. Let $N(T) \in \mathrm{SL}_2(R[T])$ be another elementary homotopy where similar notation is employed. The pointed morphisms corresponding to $M_0 \cdot N_0$ and $M_1 \cdot N_1$ are elementarily homotopic.

Proof All that needs to be verified is that the morphism corresponding to the matrix product $M(T) \cdot N(T)$ is pointed. Since both $M(T)$ and $N(T)$ are pointed, evaluation at $\mathbf{j}'$ gives the identity matrix in $\mathrm{SL}_2(k[T])$. It's clear then that the product $M(T) \cdot N(T)$ will evaluate to the identity matrix at $\mathbf{j}'$ too. □

Definition 54 Consider two pointed naive homotopy classes $[(A_i,B_i)\colon \mathcal{J}\to \mathbb{A}^2\setminus\{0\}]$ for $i=1,2$ represented by the unimodular rows $(A_i,B_i)\in R^2$. Pick completions of the unimodular rows to matrices corresponding to pointed maps, as guaranteed by lemma 46:

$$\begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix}, \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix} \in \mathrm{SL}_2(R).$$

We define $[(A_1,B_1)]\oplus[(A_2,B_2)]$ to be the naive homotopy class $[(A_3,B_3)]$ where (A_3,B_3) is the unimodular row obtained from the matrix product

$$\begin{pmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{pmatrix} = \begin{pmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{pmatrix}.$$

Proposition 55 The operation $\oplus$ of definition 54 is well-defined and gives the set $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ the structure of a group.

Proof We first show that the operation does not depend on the particular completion to a matrix in $\mathrm{SL}_2(R)$. Let M_1 and M_1' be two pointed completions of (A_1,B_1), and similarly let M_2 and M_2' be two pointed completions of (A_2,B_2). There are two representatives for the product $[(A_1,B_1)]\oplus[(A_2,B_2)]$ from these choices. They are (A_3,B_3), taken from the first column of $M_1\cdot M_2$ and (A_3',B_3'), the first column of $M_1'\cdot M_2'$. Any two completions of a unimodular row to a matrix in $\mathrm{SL}_2(R)$ are homotopic by construction 47, hence $[M_1]=[M_1']$ and $[M_2]=[M_2']$. By the proof of proposition 49 there is an elementary pointed naive homotopy between $M_1\cdot M_2$ and $M_1'\cdot M_2'$. Extracting the first column of this homotopy gives a homotopy between the resulting unimodular rows defining the resulting map.

We now show that the operation does not depend on the representative of the naive homotopy class chosen. Let $(A_1(T),B_1(T))$ and $(A_2(T),B_2(T))$ be pointed elementary homotopies. These can be completed to matrices $M_1(T)\in\mathrm{SL}_2(R[T])$ and $M_2(T)\in\mathrm{SL}_2(R[T])$ by proposition 49. The first column of the product $M_1(T)\cdot M_2(T)$ provides the homotopy between the two possible representations of the product. We conclude that the operation is well-defined on the set $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ of pointed naive homotopy classes.

The identity for the operation is given by the unimodular row $(1,0)\colon \mathcal{J}\to\mathbb{A}^2\setminus\{0\}$. Associativity of $\oplus$ follows from the associativity of matrix multiplication. Finally, let $(A,B)\colon \mathcal{J}\to\mathbb{A}^2\setminus\{0\}$ be given by the unimodular row $(A,B)\in R^2$ and complete it to a matrix $\begin{pmatrix} A & -V \\ B & U \end{pmatrix}\in\mathrm{SL}_2(R)$ giving a pointed map. The inverse of this matrix in $\mathrm{SL}_2(R)$ is the matrix $\begin{pmatrix} U & V \\ -B & A \end{pmatrix}$, and the first column of this matrix represents the inverse of (A,B) in $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ for $\oplus$. That is, $-[(A,B)] = [(U,-B)]$. □

Lemma 56 The map $\phi\colon \mathrm{SL}_2 \to \mathbb{A}^2\setminus\{0\}$ induces an isomorphism of groups

$$\phi_*\colon [\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \xrightarrow{\cong} [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}.$$

Proof The map ϕ_* is a group homomorphism by our definition of $\oplus$ on $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ in terms of matrix multiplication. We have shown in proposition 51 that ϕ_* is bijective, hence the result. □

For the next result, we recall that, as described before equation (1), the cogroup structure of $\mathbb{P}^1 \cong S^1\wedge\mathbb{G}_m$ in the pointed $\mathbb{A}^1$-homotopy category endows $[\mathbb{P}^1,X]^{\mathbb{A}^1}$ with a group operation for any motivic space X. We refer to this structure as the *conventional group structure*.

Definition 57 We let $\xi_0\colon [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}} \to [\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ denote the composition of the natural map $\nu\colon [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}} \to [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ and the bijection $(\pi^*_{\mathbb{A}^1})^{-1}\colon [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1} \to [\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ that is given by the inverse of the bijection $\pi^*_{\mathbb{A}^1}$. We note that ν is a bijection by proposition 125 since $\mathbb{A}^2\setminus\{0\}$ is $\mathbb{A}^1$-naive.

Theorem 58 The map ξ_0 is an isomorphism of groups between the group $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ with operation $\oplus$ and the group $[\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ with the conventional group operation.

Proof Let $\xi_0'\colon [\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \to [\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$ denote the composition of the canonical map $[\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \to [\mathcal{J},\mathrm{SL}_2]^{\mathbb{A}^1}$ and the bijection $[\mathcal{J},\mathrm{SL}_2]^{\mathbb{A}^1} \to [\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$ which is given by the inverse of the bijection $\pi^*_{\mathbb{A}^1}$. Since both groups $[\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$ and $[\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ inherit their operation from the cogroup structure of $\mathbb{P}^1$, the $\mathbb{A}^1$-weak equivalence $\mathrm{SL}_2 \xrightarrow{\phi} \mathbb{A}^2\setminus\{0\}$ induces an isomorphism of groups $\phi_*\colon [\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1} \to [\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$. By lemma 56 the map $\phi_*\colon [\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \to [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ is a group isomorphism. We then have the following commutative diagram.

$$\begin{array}{ccc}
[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}} & \xrightarrow{\xi_0} & [\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1} \\
{\scriptstyle\phi_*}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\phi_*} \\
[\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} & \xrightarrow[\xi_0']{} & [\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}
\end{array}$$

Hence, in order to establish that ξ_0 is a group isomorphism, it suffices to show that ξ_0' is a group isomorphism. Since $\mathcal{J}$ is affine and SL_2 is $\mathbb{A}^1$-naive by [6, Theorem 4.2.1], we know that ξ_0' is a bijection by proposition 125. Hence it suffices to show that ξ_0' is a group homomorphism.

Again, because SL_2 is $\mathbb{A}^1$-naive, the canonical map $[\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \to [\mathcal{J},\mathrm{SL}_2]^{\mathbb{A}^1}$ is a bijection by proposition 125. This bijection is a group isomorphism because the operation on both sets is defined using the same construction, that is, the sum of two maps is given by

$$\mathcal{J} \xrightarrow{\Delta} \mathcal{J}\times\mathcal{J} \xrightarrow{f\times g} \mathrm{SL}_2\times\mathrm{SL}_2 \xrightarrow{m} \mathrm{SL}_2,$$

where $m\colon \mathrm{SL}_2\times\mathrm{SL}_2\to\mathrm{SL}_2$ is the multiplication on SL_2. In other words, the group structure is induced by the group object structure on SL_2.

Similarly, the set $[\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$ also obtains the structure of a group using that SL_2 is a group object in the pointed $\mathbb{A}^1$-homotopy category. The Eckmann–Hilton argument given in [24, Proposition 2.25] can be applied in this scenario to show that this group structure coincides with the conventional group structure, see also [2, Proposition 2.2.12]. Hence we may assume that the group operation on $[\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$ is induced by the group object structure on SL_2. Combining these observations shows that the composition

$$[\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \to [\mathcal{J},\mathrm{SL}_2]^{\mathbb{A}^1} \to [\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$$

is a group homomorphism. This is the map ξ_0' which proves the assertion. □

Corollary 59 The group $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ is abelian.

Proof Since $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ is isomorphic to $[\mathbb{P}^1,\mathrm{SL}_2]^{\mathbb{A}^1}$, the Eckmann–Hilton argument shows that this group is abelian. □

Remark 60 Morel shows in [22, §7.3] that the group $[\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$ is isomorphic to $K_1^{MW}(k)$, the first Milnor–Witt K-theory group of the field k. In short, the computation [22, Theorem 7.13] and the $\mathbb{A}^1$-weak equivalence between SL_2 and $\mathbb{A}^2\setminus\{0\}$ gives $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2\setminus\{0\}) \cong \underline{\mathbf{K}}_2^{MW}$. The contraction of this sheaf evaluated at $\mathrm{Spec}(k)$ then computes $[\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1}$.

$$[\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1} \cong \pi_1^{\mathbb{A}^1}(\mathbb{A}^2\setminus\{0\})_{-1}(\mathrm{Spec}(k)) \cong (\underline{\mathbf{K}}_2^{MW})_{-1}(\mathrm{Spec}(k)) \cong K_1^{MW}(k)$$

Remark 61 For any two pointed matrices $M,N\in\mathrm{SL}_2(R)$, which represent pointed morphisms $\mathcal{J}\to\mathrm{SL}_2$, there is a chain of elementary homotopies connecting $M\cdot N$ and $N\cdot M$. We do not know of a general algorithm to construct this chain of homotopies explicitly.

The following explicit naive homotopies will be of use in the later sections.

Lemma 62 Consider a matrix $M = \begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R)$. Then M and $(M^{-1})^T$ are naively homotopic. Thus, the unimodular rows (A,B) and (U,V) are naively homotopic.

Proof Consider the matrix $H = \begin{pmatrix} 1-T^2 & -T \\ T(2-T^2) & 1-T^2 \end{pmatrix} \in \mathrm{SL}_2(R[T])$. The matrix H defines an unpointed homotopy from the identity matrix to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is straightforward to verify that the product HMH^{-1} is a pointed homotopy between M and $(M^{-1})^T$ as claimed. □

Lemma 63 Consider a matrix $\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in SL_2(R)$ and let $u \in k^\times$. Then there is an elementary homotopy

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \simeq \begin{pmatrix} A & -\frac{1}{u^2}V \\ u^2B & U \end{pmatrix}. \tag{64}$$

Thus, the unimodular row (A,B) is naively homotopic to the unimodular row (A,u^2B).

Proof The matrix on the right-hand side of (64) can be written as the following product

$$\begin{pmatrix} A & -\frac{1}{u^2}V \\ u^2B & U \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} A & -V \\ B & U \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}.$$

The diagonal matrices can be decomposed to a product of elementary matrices, which are all homotopic to the identity. □

3.2 Action of degree 0 maps on degree n maps

Recall that we write $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}_n$ for the set of naive homotopy classes of maps $\mathcal{J} \to \mathbb{P}^1$ with degree n. We define a group action of $[\mathcal{J},\mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} \cong [\mathcal{J},\mathrm{SL}_2]^{\mathrm{N}} \cong [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}_0$ on $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}_n$ for all $n \neq 0$. We start by first defining an operation on actual morphisms, and then show that the operation respects the naive homotopy equivalence relation.

Definition 65 Let $M\colon \mathcal{J} \to \mathrm{SL}_2$ be a morphism with corresponding matrix

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix}$$

and consider a map $[s_0, s_1]\colon \mathcal{J} \to \mathbb{P}^1$ determined by $n \in \mathbb{N}$, the algebraic line bundle $\mathcal{P}_n$ or $\mathcal{Q}_n$, and generating global sections s_0, s_1.

We define $M \oplus [s_0, s_1]\colon \mathcal{J} \to \mathbb{P}^1$ to be the morphism determined by the same algebraic line bundle with the generating global sections $M \oplus [s_0, s_1] = [As_0 - Vs_1, Bs_0 + Us_1]$ which are obtained from the following matrix multiplication

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} As_0 - Vs_1 \\ Bs_0 + Us_1 \end{pmatrix}.$$

Proposition 66 Given a map $[s_0, s_1]\colon \mathcal{J} \to \mathbb{P}^1$ with algebraic line bundle $\mathcal{L}$ (either $\mathcal{P}_n$ or $\mathcal{Q}_n$) and a map $M\colon \mathcal{J} \to \mathrm{SL}_2$, the construction $M \oplus [s_0, s_1]$ is a morphism from $\mathcal{J}$ to $\mathbb{P}^1$. If both maps are pointed, the result is also pointed. Furthermore, the operation is a left group action.

Proof The morphism $M\colon \mathcal{J} \to \mathrm{SL}_2$ is described by a matrix

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \in \mathrm{SL}_2(R).$$

We observe that $U(As_0 - Vs_1) + V(Bs_0 + Us_1) = s_0$, and $-B(As_0 - Vs_1) + A(Bs_0 + Us_1) = s_1$. By assumption, the sections s_0, s_1 generate the algebraic line bundle $\mathcal{L}$. Hence the pair of sections $As_0 - Vs_1$, $Bs_0 + Us_1$ generate $\mathcal{L}$ as well. This proves the first assertion.

That the map $[s_0, s_1]$ is pointed means that $s_1 \in \mathbf{j} \subseteq R$, or equivalently, $s_1(\mathbf{j}) = 0$ in $R/\mathbf{j}$. That M is pointed means $M(\mathbf{j})$ is the identity matrix. To verify $M \oplus [s_0, s_1]$ is pointed, we must check that $B(\mathbf{j})s_0(\mathbf{j}) + U(\mathbf{j})s_1(\mathbf{j}) = 0$, but this is clear as $B(\mathbf{j}) = 0$ and $s_1(\mathbf{j}) = 0$ from our assumptions.

The fact that the operation is a left group action follows from the associativity of matrix multiplication and the definition of the group structure on maps $\mathcal{J} \to \mathrm{SL}_2$. □

The next theorem employs the notation of definition 39 for morphisms $\mathcal{J} \to \mathbb{P}^1$.

Theorem 67 Let $f\colon \mathcal{J} \to \mathbb{P}^1$ be a map of degree n. Then there exists a matrix $M \in \mathrm{SL}_2(R)$ such that $f = M \oplus (1, 0 : 0, 1)_n$.

Proof Let $f = (a_0, a_1 : b_0, b_1)_n$. For $c, c', d, d' \in R$, we consider the matrix

$$M = \begin{pmatrix} a_0 + y^n c + w^n c' & a_1 - x^n c - z^n c' \\ b_0 - y^n d - w^n d' & b_1 + x^n d + z^n d' \end{pmatrix}. \tag{68}$$

Note that for any choice of $c, c', d, d' \in R$ we have

$$M \oplus \left[\begin{bmatrix} x^n \\ z^n \end{bmatrix}, \begin{bmatrix} y^n \\ w^n \end{bmatrix} \right] = (a_0, a_1 : b_0, b_1)_n.$$

We now show that there always exist c, c', d, d' such that $M \in SL_2(R)$. The determinant of M is given by the formula

$$\det(M) = a_0b_1 - a_1b_0 + c(x^n b_0 + y^n b_1) + c'(z^n b_0 + w^n b_1) + d(x^n a_0 + y^n a_1) + d'(z^n a_0 + w^n a_1).$$

Since $(a_0, a_1 : b_0, b_1)_n$ determines a morphism of schemes, it follows from proposition 38 that the ideal $I := (x^n a_0 + y^n a_1, z^n a_0 + w^n a_1, x^n b_0 + y^n b_1, z^n b_0 + w^n b_1)$ is the unit ideal. Thus $1 - a_0b_1 - a_1b_0$ is in I, and there exist elements c, c', d, d' such that $\det(M) = 1$. This shows $M \in SL_2(R)$ and proves the assertion by definition of the operation $\oplus$. □

Corollary 69 Let $f, g \colon \mathcal{J} \to \mathbb{P}^1$ be two morphisms of degree n. Then there exists a matrix $M \in \mathrm{SL}_2(R)$ such that $M \oplus f = g$.

Proof By theorem 67, there exist M' and M'' such that $M' \oplus (1, 0 : 0, 1)_n = f$ and $M'' \oplus (1, 0 : 0, 1)_n = g$. The desired matrix is now given by $M = M'' \cdot (M')^{-1}$. □

Remark 70 The $\mathrm{SL}_2(R)$-matrix M constructed in the proof of theorem 67 is not always pointed, even if the map f we started with is pointed. For example, following the construction for the map $f = (1, 1 : 0, 1)_1$ yields the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which is not pointed, since $M(\mathbf{j})$ is not the identity matrix.

Remark 70 shows that we have to improve our argument in order to get an action on pointed homotopy classes. We will now prove the necessary adjustments.

Proposition 71 Let $f \colon \mathcal{J} \to \mathbb{P}^1$ be a pointed map of degree $n \neq 0$. Then there is a pointed naive homotopy between f and a map of the form $M_f \oplus (1, 0 : 0, 1)_n$ for some pointed matrix $M_f \in \mathrm{SL}_2(R)$.

Proof Let $f = (a_0, a_1 : b_0, b_1)_n$, where we may assume $a_0(\mathbf{j}) = 1$ by proposition 37. By theorem 67 we can find a matrix $M' \in \mathrm{SL}_2(R)$ such that $M' \oplus (1, 0 : 0, 1)_n = f$. However, M' may not be pointed. We can replace M' with a pointed map M_f as follows. Assuming M' is of the form (68) we get $b_1(\mathbf{j}) + d(\mathbf{j}) = 1$. Moreover, this implies that there is an element $e \in k$ such that $M'(\mathbf{j}) = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ and $e = a_1(\mathbf{j}) - c(\mathbf{j})$. Define M_f to be $M_f = \begin{pmatrix} 1 & -e \\ 0 & 1 \end{pmatrix} M'$. We compute $M_f \oplus (1,0:0,1)_n = (a_0 - eb_0, a_1 - eb_1 : b_0, b_1)_n$. The assertion now follows from the fact that the morphism $(a_0 - Teb_0, a_1 - Teb_1 : b_0, b_1)_n$ is a pointed homotopy between $M_f \oplus (1,0:0,1)_n$ and f. ◻

Corollary 72 Let $f, g \colon \mathcal{J} \to \mathbb{P}^1$ be two pointed morphisms of degree n. There exists a pointed map $M \colon \mathcal{J} \to \mathrm{SL}_2$ such that $M \oplus f$ is pointed naively homotopic to g.

Definition 73 Let $[(A,B)] \in [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} \cong [\mathcal{J}, \mathbb{P}^1]_0^{\mathrm{N}}$ be a pointed naive homotopy class represented by the map with unimodular row (A,B) in R. Let $[f] \in [\mathcal{J}, \mathbb{P}^1]_n^{\mathrm{N}}$ be a pointed naive homotopy class of degree n with $n \neq 0$ represented by a pointed morphism $f \colon \mathcal{J} \to \mathbb{P}^1$. We define $[(A,B)] \oplus [f] := [M \oplus f]$ where M is a completion of (A,B) to a matrix in $\mathrm{SL}_2(R)$ corresponding to a pointed map.

Theorem 74 The operation of definition 73 is well-defined and for each $n \in \mathbb{Z}$ provides the set $[\mathcal{J}, \mathbb{P}^1]_n^{\mathrm{N}}$ with a left-action by the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$.

Proof First, consider a pointed map $f = [s_0, s_1]$. We show that $[(A,B)] \oplus [f]$ is independent of the choice of completion of (A,B) to a matrix in $\mathrm{SL}_2(R)$. So let

$$M = \begin{pmatrix} A & -V \\ B & U \end{pmatrix}, \; M' = \begin{pmatrix} A & -V' \\ B & U' \end{pmatrix}$$

be two completions to matrices in $\mathrm{SL}_2(R)$ which correspond to pointed maps. Then the naive homotopy $H(T) = \begin{pmatrix} A & -(TV + (1-T)V' \\ B & TU + (1-T)U' \end{pmatrix}$ is pointed independently of T, and $H(T) \oplus f$ is a pointed homotopy between $M \oplus f$ and $M' \oplus f$.

Now we show that $[(A,B)] \oplus [f]$ is independent of the choice of the representing unimodular row (A,B). Suppose we have a pointed elementary homotopy between two unimodular rows, $(A(T), B(T))$. Proposition 49 shows that we can lift it to a pointed elementary homotopy $M(T) = \begin{pmatrix} A(T) & -V(T) \\ B(T) & U(T) \end{pmatrix} \in \mathrm{SL}_2(R[T])$. Then $M(T) \oplus f$ is a pointed homotopy between $(A,B) \oplus f$ and $(A', B') \oplus f$. Now we consider a unimodular row (A,B) and let $M = \begin{pmatrix} A & -V \\ B & U \end{pmatrix}$ be a lift to a matrix in $\mathrm{SL}_2(R)$. Let $f_0, f_1 \colon \mathcal{J} \to \mathbb{P}^1$ be two pointed morphisms which are homotopic via a pointed naive homotopy. Let $f(T) \colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{P}^1$ be a pointed naive homotopy. We define the map $H(T) := M \oplus f(T) \colon \mathcal{J} \times \mathbb{A}^1 \to \mathbb{P}^1$ with the same algebraic line bundle $\mathcal{L}'$ on $\mathcal{J} \times \mathbb{A}^1$ and global sections

$$\begin{pmatrix} A & -V \\ B & U \end{pmatrix} \cdot \begin{pmatrix} s_0(T) \\ s_1(T) \end{pmatrix} = \begin{pmatrix} As_0(T) - Vs_1(T) \\ Bs_0(T) + Us_1(T) \end{pmatrix}.$$

We note that $H(T)$ thus defined is in fact a morphism $\mathcal{J} \times \mathbb{A}^1 \to \mathbb{P}^1$, since we have $U(As_0(T) - Vs_1(T)) + V(Bs_0(T) + Us_1(T)) = s_0(T)$, and $-B(As_0(T) - Vs_1(T)) + A(Bs_0(T) + Us_1(T)) = s_1(T)$. By assumption, the sections $s_0(T)$, $s_1(T)$ generate the line bundle $\mathcal{L}'$. Hence $(As_0(T) - Vs_1(T), Bs_0(T) + Us_1(T))$ generate $\mathcal{L}'$ as well. This shows that $H(T)$ defines a morphism. We now verify that $H(T)$ is pointed by showing $Bs_0(T) + Us_1(T) \in \mathbf{j}'$. Pointedness of $[s_0(T), s_1(T)]$ means that $s_1(T)(\mathbf{j}') = 0$ in $R[T]/\mathbf{j}'$. Pointedness of (A, B) means $M(\mathbf{j})$ is the identity matrix. We calculate

$$B(\mathbf{j}')s_0(T)(\mathbf{j}') + U(\mathbf{j}')s_1(T)(\mathbf{j}') = 0 \cdot s_0(T)(\mathbf{j}') + 1 \cdot 0 = 0$$

which completes the verification. This shows that $\oplus$ is independent of the choice of representatives in both naive homotopy classes and completes the proof of the first assertion. The second assertion then follows from proposition 66. □

Remark 75 There are several variations to the operation given in definition 73 that produce valid group actions. For $M \in \mathrm{SL}_2(R)$, the operation in definition 65 is given by the matrix multiplication $M \cdot (s_0\ s_1)^T$. We could have taken equally well either $M^T \cdot (s_0\ s_1)^T$ or $M^{-1} \cdot (s_0\ s_1)^T$, although this would give a right-action rather than a left-action on maps. Up to homotopy, the latter two choices in fact agree, since M^T is homotopic to M^{-1} by lemma 62. Thus there are two natural choices for this action, one of which applies the inverse operation to the morphism in $\mathrm{SL}_2(R)$ before acting. In appendix C we will use real realization to check which of these operations can represent the group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ induced from Morel's group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ via $\pi^*_{\mathbb{A}^1}$. In fact, in example 145 we show that only the choice of definitions 65 and 73 can be compatible.

4 The group structure on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$

We now state an explicit group structure on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$.

Definition 76 Let $-[\mathrm{id}]$ denote the additive inverse of $[\mathrm{id} : \mathbb{P}^1 \to \mathbb{P}^1]$ under the conventional group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. Define $-\pi : \mathcal{J} \to \mathbb{P}^1$ to be a morphism which represents the $\mathbb{A}^1$-homotopy class $-[\mathrm{id} : \mathbb{P}^1 \to \mathbb{P}^1] \in [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ under the bijection $\xi : [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ of equation (11). More generally, for any integer n, let $n\pi$ denote a morphism $n\pi : \mathcal{J} \to \mathbb{P}^1$ which represents the $\mathbb{A}^1$-homotopy class $n[\mathrm{id} : \mathbb{P}^1 \to \mathbb{P}^1]$ under the bijection $\xi : [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$.

We are now ready to define a group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$.

Definition 77 Let $f\colon \mathcal{J} \to \mathbb{P}^1$ and $g\colon \mathcal{J} \to \mathbb{P}^1$ be morphisms of degrees n and m respectively. By corollary 72 there are degree 0 maps $f_0\colon \mathcal{J} \to \mathbb{P}^1$ and $g_0\colon \mathcal{J} \to \mathbb{P}^1$ for which $f \simeq f_0 \oplus n\pi$ and $g \simeq g_0 \oplus m\pi$. We define the sum of $[f]$ and $[g]$ to be

$$\begin{aligned}[f] \oplus [g] &= ([f_0] \oplus [n\pi]) \oplus ([g_0] \oplus [m\pi]) \\ &= ([f_0] \oplus [g_0]) \oplus [(n+m)\pi].\end{aligned}$$

The term $[f_0] \oplus [g_0]$ is calculated by matrix multiplication via definition 54. The group action of definition 73 is used to compute $(f_0 \oplus g_0) \oplus (n+m)\pi$.

Remark 78 It follows from theorem 74 that the operation $\oplus$ of definition 77 is well-defined. We also note that, for $n > 0$, the proofs of theorem 67 and proposition 71 may be used to write down a concrete algorithm to find a map f_0 such that $f \simeq f_0 \oplus n\pi$ for any degree n map f.

Remark 79 For $n > 0$, we may construct morphisms $n\pi$ by using Cazanave's group operation on morphisms $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ and lift it to an element in $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$. A recursive description of the maps $n\pi$ for $n > 0$ can be given as follows: We set $F_0 = 1$ and $F_1 = \begin{bmatrix} x \\ z \end{bmatrix}$. For $n > 0$, we define F_{n+1} recursively by setting

$$F_{n+1} = \begin{bmatrix} x \\ z \end{bmatrix} \cdot F_n - \begin{bmatrix} y^2 \\ w^2 \end{bmatrix} \cdot F_{n-1}.$$

For $n > 0$, the morphism $n\pi$ is given by sections

$$\left[F_n, \begin{bmatrix} y \\ w \end{bmatrix} \cdot F_{n-1} \right].$$

Note that $[(1, 0 : 0, 1)_n]$ is in general not equal to $[n\pi]$ for $n > 1$.

We are now ready to prove one of our main results.

Theorem 80 The operation $\oplus$ turns the set $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ into an abelian group. Moreover, there is an isomorphism of groups $\phi\colon \left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus\right) \xrightarrow{\cong} \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right)$.

Proof We observe that the set $\{[n\pi] : n \in \mathbb{Z}\}$ inherits the structure of an abelian group from $\mathbb{Z}$. By definition, $\left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus\right)$ is isomorphic to the direct product of the two groups $\{[n\pi] : n \in \mathbb{Z}\}$ and $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$. Both are abelian by corollary 59. This implies the first assertion.

By definition of the operation $\oplus$, the group $\left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus\right)$ fits into the short exact sequence displayed in the top row of diagram (81) below. By the work of Morel in [22, §7.3], the group $\left([\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1},\oplus^{\mathbb{A}^1}\right)$ fits into the short exact sequence displayed in the bottom row.

(81)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}} & \longrightarrow & [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}} & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 1\\
 & & \xi_0\downarrow\cong & & \phi\,\vdots & & \cong\downarrow q & & \\
1 & \longrightarrow & [\mathbb{P}^1,\mathbb{A}^2\setminus\{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1} & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

By theorem 58, the vertical map ξ_0 on the left-hand side is an isomorphism. The vertical map q on the right-hand side is an isomorphism as well. We define ϕ to be the unique group homomorphism satisfying $\phi([\pi]) = [\mathrm{id}]$ and $\phi([f_0]) = \xi_0([f_0])$ for all $[f_0] \in [\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$. The diagram commutes by our definition of ϕ. Since ξ_0 and q are isomorphisms, ϕ is an isomorphism of groups as well by the five-lemma. □

Unfortunately, theorem 80 does not give a satisfactory way to describe an $\mathbb{A}^1$-homotopy class $\gamma \in [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$ as a concrete homotopy class of a scheme morphism $g\colon \mathcal{J} \to \mathbb{P}^1$. However, since ξ restricts to an isomorphism on the subgroups $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ and $\{[n\pi]\,|\,n\in\mathbb{Z}\}$, we do believe that the bijection ξ is a group isomorphism, which we state as a conjecture below.

Conjecture 82 The bijection $\xi\colon [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$ is a group isomorphism and equals ϕ.

One obstacle to prove conjecture 82 is that, for $n < 0$, we do not know which morphism $\mathcal{J} \to \mathbb{P}^1$ is sent to $n[\mathrm{id}]$ under ξ. In particular, we do not know which morphism $\mathcal{J} \to \mathbb{P}^1$ is mapped to the motivic homotopy class $-[\mathrm{id} : \mathbb{P}^1 \to \mathbb{P}^1]$. A potential candidate for $-\pi$ may be the map $\tilde{\pi} = (1,0:0,-1)_{-1}$ determined by the line bundle $\mathcal{Q}_1$ and generating sections

$$s_0 = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } s_1 = -\begin{pmatrix} z \\ w \end{pmatrix}.$$

Question 83 Is $\tilde{\pi}$ the inverse of π for $\oplus$, i.e., is $\tilde{\pi}$ naively homotopic to $-\pi$?

In appendix C we present further evidence for conjecture 82. We use the real realization functor for fields $k \subset \mathbb{R}$ and Morel's theorem which states that the signature of the motivic Brouwer degree equals the topological Brouwer degree under real realization. This provides a potential obstruction to the compatibility of $\oplus^{\mathbb{A}^1}$ and the action of $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ on $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$. We then compute concrete examples and show that other choices for the action of $[\mathcal{J},\mathbb{A}^2\setminus\{0\}]^{\mathrm{N}}$ on $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ are not compatible with

$\oplus^{\mathbb{A}^1}$, while our choice of operation in definition 73 is compatible with $\oplus^{\mathbb{A}^1}$ after real realization in the chosen examples.

In proposition 102 we show that the naive homotopy class of π is mapped to the class $(\langle 1\rangle, 1)$ in $\mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$ as expected if ξ is a group homomorphism. Based on the computations in appendix C we prove in theorem 103 that the image of $[\tilde{\pi}]$ under the motivic Brouwer degree is the class $-\langle 1\rangle$ in $\mathrm{GW}(k)$. This brings us very close to a positive answer to question 83. We are, however, not able to compute the image of $\tilde{\pi}$ in $k^\times$.

We end this section with a comment on a question by Cazanave in [12]:

Remark 84 Since π is an $\mathbb{A}^1$-weak equivalence, it induces a bijection $\pi_*\colon [\mathcal{J},\mathcal{J}]^{\mathrm{N}} \to [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$. Hence there is a bijection between $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$ and the set of pointed naive homotopy classes $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$. In [12, page 31] Cazanave speculates whether $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$ can be used to study the group structure on $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$. A morphism $\mathcal{J}\to\mathcal{J}$ corresponds to a ring homomorphism $R\to R$, or equivalently, the data of a (2×2)-matrix with entries in R and with trace 1 and determinant 0. For every map $f\colon \mathcal{J}\to\mathbb{P}^1$ we can find a map $F\colon \mathcal{J}\to\mathcal{J}$ such that $f=\pi\circ F$. We will refer to such a map F as a lift of f. There is a particularly nice way to construct a lift in the case $f\colon \mathcal{J}\to\mathbb{P}^1$ has degree 0. Assume that f is given by a unimodular row (A,B). Let $U,V\in R$ be such that $\begin{pmatrix} A & -V \\ B & U \end{pmatrix}$ has determinant 1. Then F is given by the matrix $\begin{pmatrix} AU & BU \\ AV & BV \end{pmatrix}$ which has trace 1 and determinant 0. Composing the map with π yields the $\mathcal{J}\to\mathbb{P}^1$ map given by either $[AU:BU]$ or $[AV:BV]$, whenever they are defined, which coincides with the map corresponding to the unimodular row (A,B). If f has non-zero degree, there is also concrete procedure to find a lift of f, which we leave to the reader.

Since morphisms $\mathcal{J}\to\mathcal{J}$ can be represented by matrices, it may seem plausible that one can find a suitable operation on $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$ which may help to describe the group $([\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1},\oplus^{\mathbb{A}^1})$. However, neither addition nor multiplication of matrices equip the set $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$ with an operation which is compatible with the conventional group structure on $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$. We have verified in examples that composition of maps in $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$ descends to the operation $\circ$ on $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ of [13, Definition 4.5]. As pointed out in [13, Remark 4.7] the latter does not distribute over the conventional group structure on $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$.

We were not able to make a reasonable guess which other operation on $[\mathcal{J},\mathcal{J}]^{\mathrm{N}}$ might work. We have therefore not pursued this path further.

5 Compatibility with Cazanave's monoid structure

The goal of this section is to show that the map

$$\pi_{\mathrm{N}}^{*}\colon \left([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}}\right) \to \left([\mathcal{J},\mathbb{P}^1]^{\mathrm{N}},\oplus\right)$$

is a morphism of monoids, where $\oplus^{\mathrm{N}}$ denotes the monoid operation defined by Cazanave in [13]. We will achieve this goal in theorem 95.

5.1 Compatibility with certain degree 0 maps

We first study an important family of degree 0 morphisms and their compatibility with $\oplus$, $\oplus^{\mathrm{N}}$, and π_{N}^{*}.

Definition 85 For $u, v \in k^{\times}$, we write $g_{u,v}$ for the pointed morphism $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$ given by the unimodular row $\left(x + \frac{v}{u}w, (u-v)y\right)$ in R. This unimodular row can be completed to the $\mathrm{SL}_2(R)$-matrix

$$m_{u,v} = \begin{pmatrix} x + \frac{v}{u}w & \frac{u-v}{uv}z \\ (u-v)y & x + \frac{u}{v}w \end{pmatrix}.$$

We now develop some basic properties of the maps $g_{u,v}$ which will be necessary to prove that π_{N}^{*} is a monoid morphism.

Lemma 86 For all $u, v, s \in k^{\times}$, we have the identity $g_{u,v} \oplus g_{v,s} = g_{u,s}$. In particular, we have $g_{u,v} \oplus g_{v,u} = (1,0)$ and $g_{u,v} \oplus g_{v,1} = g_{u,1}$.

Proof A direct computation, using $xw = yz$, shows

$$m_{u,v} \cdot m_{v,s} = \begin{pmatrix} x^2 + \frac{uv+vs}{uv}xw + \frac{s}{u}w^2 & \frac{uv-sv}{usv}xz + \frac{-vs+uv}{usv}zw \\ (u-s)xy + (u-s)wy & x^2 + \frac{uv+sv}{sv}xw + \frac{u}{s}w^2 \end{pmatrix},$$

and since $x + w = 1$, this simplifies to the matrix $m_{u,s}$. Then $g_{u,v} \oplus g_{v,u} = g_{u,u} = (1,0)$ and $g_{u,v} \oplus g_{v,1} = g_{u,1}$ are special cases for respectively $s = u$ and $s = 1$. □

Lemma 87 Let $u, v, c \in k^{\times}$. Then $[g_{u,v}] = [g_{c^2u,c^2v}]$.

Proof By lemma 63, we have

$$g_{u,v} = \left(x + \frac{v}{u}w, (u-v)y\right) \simeq \left(x + \frac{v}{u}w, c^2(u-v)y\right) = g_{c^2u,c^2v}.$$

□

Lemma 88 Let $u, v \in k^\times$ and let v be a square. Then we have $[g_{u,1}] \oplus [g_{v,1}] = [g_{uv,1}]$.

Proof Using lemma 87 and lemma 86 shows

$$g_{u,1} \oplus g_{v,1} \simeq g_{u,1} \oplus g_{1,1/v} = g_{u,1/v} \simeq g_{uv,1},$$

and hence the claim. □

Next we study the relationship of the maps $g_{u,v}$ with $\oplus$ and π_{N}^*.

Lemma 89 For every $u \in k^\times$, we have $g_{u,1} \oplus \pi = \pi_{\mathrm{N}}^* \left(\frac{X}{u}\right)$.

Proof A direct computation using the facts that $x \begin{bmatrix} y \\ w \end{bmatrix} = y \begin{bmatrix} x \\ z \end{bmatrix}$ and $z \begin{bmatrix} y \\ w \end{bmatrix} = w \begin{bmatrix} x \\ z \end{bmatrix}$ shows

$$\begin{pmatrix} x + \frac{1}{u} w & \frac{u-1}{u} z \\ (u-1)y & x + uw \end{pmatrix} \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ \begin{bmatrix} y \\ w \end{bmatrix} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ u \begin{bmatrix} y \\ w \end{bmatrix} \end{pmatrix}$$

and hence the result by definition of the maps involved. □

Lemma 90 For all $u, v \in k^\times$, we have the identity

$$g_{u,v} \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{v}\right) = \pi_{\mathrm{N}}^* \left(\frac{X}{u}\right).$$

Proof Using lemmas 86 and 89 and definition 77 we get

$$g_{u,v} \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{v}\right) = (g_{u,v} \oplus g_{v,1}) \oplus \pi = g_{u,1} \oplus \pi = \pi_{\mathrm{N}}^* \left(\frac{X}{u}\right)$$

and hence the result. □

We are now ready to prove a key result for the compatibility of π_{N}^* with the monoid operations.

Proposition 91 Let $u \in k^\times$ and $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a pointed morphism. Then we have

$$\pi_{\mathrm{N}}^* \left(\frac{X}{u} \oplus^{\mathrm{N}} f\right) \simeq g_{u,1} \oplus \left(\pi_{\mathrm{N}}^* \left(\frac{X}{1} \oplus^{\mathrm{N}} f\right)\right).$$

Proof Let $\pi_{\mathrm{N}}^*(f) = [f_1, f_2]$, and let n denote its degree. The pair of generating sections of $\mathcal{P}_{n+1}$ which determines the morphism $\pi_{\mathrm{N}}^*\left(\frac{X}{u} \oplus^{\mathrm{N}} f\right)$ is given by the matrix product

$$\pi_{\mathrm{N}}^*\left(\frac{X}{u} \oplus^{\mathrm{N}} f\right) = \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} & -\frac{1}{u}\begin{bmatrix} y \\ w \end{bmatrix} \\ u\begin{bmatrix} y \\ w \end{bmatrix} & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Note that here we use the isomorphism given in Proposition 24 to identify the product of a pair of column vectors with its image in $\mathcal{P}_{n+1}$. The pair of generating sections which determines the morphism $g_{u,1} \oplus \left(\pi^*\left(\frac{X}{1} \oplus^{\mathrm{N}} f\right)\right)$ is given by the matrix product

$$g_{u,1} \oplus \left(\pi_{\mathrm{N}}^*\left(\frac{X}{1} \oplus^{\mathrm{N}} f\right)\right) = \begin{pmatrix} x + \frac{1}{u}w & \frac{u-1}{u}z \\ (u-1)y & x + uw \end{pmatrix} \cdot \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} & -\begin{bmatrix} y \\ w \end{bmatrix} \\ \begin{bmatrix} y \\ w \end{bmatrix} & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Note that we have the following equality of matrix products

$$\begin{aligned} \begin{pmatrix} x + \frac{1}{u}w & \frac{u-1}{u}z \\ (u-1)y & x + uw \end{pmatrix} \cdot \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} & -\begin{bmatrix} y \\ w \end{bmatrix} \\ \begin{bmatrix} y \\ w \end{bmatrix} & 0 \end{pmatrix} &= \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} & -(x + \frac{1}{u})\begin{bmatrix} y \\ w \end{bmatrix} \\ u\begin{bmatrix} y \\ w \end{bmatrix} & (1-u)y\begin{bmatrix} y \\ w \end{bmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} & -\frac{1}{u}\begin{bmatrix} y \\ w \end{bmatrix} \\ u\begin{bmatrix} y \\ w \end{bmatrix} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{u-1}{u}y \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We claim that the rows of the matrix $h(T)$ defined by the following product

$$\begin{aligned} h(T) &= \begin{pmatrix} \begin{bmatrix} x \\ z \end{bmatrix} & -\frac{1}{u}\begin{bmatrix} y \\ w \end{bmatrix} \\ u\begin{bmatrix} y \\ w \end{bmatrix} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{u-1}{u}yT \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \begin{pmatrix} f_1\begin{bmatrix} x \\ z \end{bmatrix} + \frac{u-1}{u}yTf_2\begin{bmatrix} x \\ z \end{bmatrix} - \frac{1}{u}f_2\begin{bmatrix} y \\ w \end{bmatrix} \\ uf_1\begin{bmatrix} y \\ w \end{bmatrix} + (u-1)yTf_2\begin{bmatrix} y \\ w \end{bmatrix} \end{pmatrix} \end{aligned}$$

provide generating sections of $\mathcal{P}_{n+1}$ and define the desired pointed homotopy from $h(0) = \pi_{\mathrm{N}}^*\left(\frac{X}{u} \oplus^{\mathrm{N}} f\right)$ to $h(1) = g_{u,1} \oplus \left(\pi_{\mathrm{N}}^*\left(\frac{X}{1} \oplus^{\mathrm{N}} f\right)\right)$.

Now we prove the remaining claim. Since f is a morphism $\mathbb{P}^1 \to \mathbb{P}^1$, we can write it as a rational function $\frac{X^n+a_{n-1}X^{n-1}+\ldots a_0}{b_{n-1}X^{n-1}+\ldots+b_0}$. We then define

$$F=(F_0,F_1)=(\alpha^n+\ldots+a_0\beta^n, b_{n-1}\alpha^{n-1}\beta+\ldots+b_0\beta^n)\in(R[\alpha,\beta]_n)^2.$$

Using the construction of definition 30, we observe that $\sigma((F_0,F_1))=[f_0,f_1]=\pi^*(f)$. Let H be the following product of $R[\alpha,\beta,T]$-matrices

$$\begin{aligned}H(T)&=\begin{pmatrix}\alpha & -\frac{1}{u}\beta\\ u\beta & 0\end{pmatrix}\cdot\begin{pmatrix}1 & \frac{u-1}{u}yT\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}F_1\\ F_2\end{pmatrix}\\ &=\begin{pmatrix}F_1\alpha+\frac{u-1}{u}yTF_2\alpha-\frac{1}{u}F_2\alpha\\ uF_1\beta+(u-1)yTF_2\beta\end{pmatrix}.\end{aligned}$$

We have $\sigma(H)=h$. Since F has unit resultant, lemma 131 implies that the pair $(F_1+\frac{u-1}{u}yTF_2,F_2)$ also has unit resultant. By lemma 132 the resultant of $H(T)$ is a unit as well. By lemma 31 this implies that h consists of a pair of generating sections and finishes the proof. □

To give a concrete example of the homotopy constructed in the proof of proposition 91, we look at the special case $f=X/1$:

Example 92 For every $u\in k^\times$, the morphism H defined by

$$H=\left[\begin{bmatrix}x^2\\ z^2\end{bmatrix}+T\frac{u-1}{u}y\begin{bmatrix}xy\\ zw\end{bmatrix}-\left(x+\frac{1}{u}w\right)\begin{bmatrix}y^2\\ w^2\end{bmatrix}, u\begin{bmatrix}xy\\ zw\end{bmatrix}+(T(u-1)-(u-1)y)\begin{bmatrix}y^2\\ w^2\end{bmatrix}\right]$$

is a homotopy between $H(0)=g_{u,1}\oplus 2\pi$ and $H(1)=\pi_{\mathrm{N}}^*\left(\frac{X}{u}\oplus^{\mathrm{N}}\frac{X}{1}\right)$.

5.2 The map π_{N}^* is a monoid morphism

We will now prove that π_{N}^* is a morphism of monoids.

Lemma 93 We have

$$\pi_{\mathrm{N}}^*\left(\frac{X}{1}\oplus^{\mathrm{N}}\cdots\oplus^{\mathrm{N}}\frac{X}{1}\right)\simeq\pi\oplus\cdots\oplus\pi,$$

where there are n summands on both sides.

Proof Since $\nu_{\mathbb{P}^1}\left(\frac{X}{1}\right)=\mathrm{id}_{\mathbb{P}^1}$ and $\nu_{\mathbb{P}^1}$ is a morphism of monoids by [13, Proposition 3.23], we have the equality $n[\mathrm{id}]=\left[\frac{X}{1}\oplus^{\mathrm{N}}\cdots\oplus^{\mathrm{N}}\frac{X}{1}\right]$ in $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$. Thus $\pi_{\mathrm{N}}^*\left(\frac{X}{1}\oplus^{\mathrm{N}}\cdots\oplus^{\mathrm{N}}\frac{X}{1}\right)$ is naively homotopic to $n\pi$. By definition, $[\pi]\oplus\cdots\oplus[\pi]=[n\pi]$, hence the result follows. □

Proposition 94 For $u_1, \dots, u_n \in k^\times$ we have

$$\pi_{\mathrm{N}}^* \left(\frac{X}{u_1} \oplus^{\mathrm{N}} \frac{X}{u_2} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{X}{u_n} \right) \simeq \pi_{\mathrm{N}}^* \left(\frac{X}{u_1} \right) \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{u_2} \right) \oplus \cdots \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{u_n} \right).$$

Proof We use that $\oplus^{\mathrm{N}}$ and $\oplus$ are commutative and that $\oplus$ is associative and apply proposition 91 and lemma 93 to get

$$\begin{aligned}
\pi_{\mathrm{N}}^* \left(\frac{X}{u_1} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{X}{u_n} \right) &\simeq g_{u_1,1} \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{1} \oplus^{\mathrm{N}} \frac{X}{u_2} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{X}{u_n} \right) \\
&\simeq g_{u_1,1} \oplus g_{u_2,1} \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{1} \oplus^{\mathrm{N}} \frac{X}{1} \oplus^{\mathrm{N}} \frac{X}{u_3} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{X}{u_n} \right) \\
&\simeq g_{u_1,1} \oplus \cdots \oplus g_{u_n,1} \oplus n\pi \\
&\simeq (g_{u_1,1} \oplus \pi) \oplus (g_{u_2,1} \oplus \pi) \oplus \cdots \oplus (g_{u_n,1} \oplus \pi) \\
&\simeq \pi_{\mathrm{N}}^* \left(\frac{X}{u_1} \right) \oplus \cdots \oplus \pi_{\mathrm{N}}^* \left(\frac{X}{u_n} \right).
\end{aligned}$$

For the final step we used lemma 89. ◻

Theorem 95 The map $\pi_{\mathrm{N}}^* \colon \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}} \right) \to \left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus \right)$ induced by π is a morphism of monoids.

Proof Let $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ be two pointed morphisms. By [13, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ is generated by elements in degree 1. Hence we can assume $f \simeq \frac{X}{u_1} \oplus^{\mathrm{N}} \frac{X}{u_2} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{X}{u_n}$ and $g \simeq \frac{X}{v_1} \oplus^{\mathrm{N}} \frac{X}{v_2} \oplus^{\mathrm{N}} \cdots \oplus^{\mathrm{N}} \frac{X}{v_m}$ for some $u_1, \dots, u_n, v_1, \dots, v_m \in k^\times$. Then proposition 94 implies the identity

$$\pi_{\mathrm{N}}^* \left([f] \oplus^{\mathrm{N}} [g] \right) = \left[\pi_{\mathrm{N}}^*(f) \right] \oplus \left[\pi_{\mathrm{N}}^*(g) \right]$$

and hence the result. ◻

6 Group completion

The morphism $\pi \colon \mathcal{J} \to \mathbb{P}^1$ induces the following commutative diagram of solid arrows.

(96)
$$\begin{array}{ccc}
[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} & \xrightarrow{\nu_{\mathcal{J}}} & [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1} \\
\uparrow \pi_{\mathrm{N}}^* & \overset{\psi}{\nwarrow} & \pi_{\mathbb{A}^1}^* \uparrow \downarrow (\pi_{\mathbb{A}^1}^*)^{-1} \\
[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} & \xrightarrow[\nu_{\mathbb{P}^1}]{} & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}
\end{array}$$

Lemma 97 We have the identity of morphisms $\xi \circ \pi_{\mathrm{N}}^* = \nu_{\mathbb{P}^1}$.

Proof Since the outer diagram in (96) commutes, we have $(\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu_{\mathcal{J}} \circ \pi_{\mathrm{N}}^* = \nu_{\mathbb{P}^1}$. Since $\xi = (\pi_{\mathbb{A}^1}^*)^{-1} \circ \nu_{\mathcal{J}}$ by definition, this shows

$$\xi \circ \pi_{\mathrm{N}}^* = \nu_{\mathbb{P}^1}$$

as desired. □

In [13, Theorem 3.22] Cazanave proves that the canonical map $\nu_{\mathbb{P}^1} \colon \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right) \to \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right)$ is a group completion. Hence there exists a unique group homomorphism

$$\psi\colon \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right) \to \left([\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}, \oplus\right)$$

making the lower triangle in diagram (96) commute.

We will show in this section that π_{N}^* has image in a certain subgroup and induces a group completion. Together with Cazanave's result this implies that we have a canonical isomorphism between the two group completions induced by π_{N}^* and $\nu_{\mathbb{P}^1}$, respectively. The main result is proven in theorem 111.

6.1 Motivic Brouwer degree

In [21] Morel describes the analog of the topological Brouwer degree map in $\mathbb{A}^1$-homotopy theory. For pointed endomorphisms of $\mathbb{P}^1$ it defines a homomorphism

$$\deg^{\mathbb{A}^1}\colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \to \mathrm{GW}(k).$$

We recall that by the work of Cazanave [13, Corollary 3.10] and Morel [22, Theorem 7.36] the map given by

$$f \mapsto \left(\deg^{\mathbb{A}^1}(f), \mathrm{res}(f)\right),$$

where $\mathrm{res}(f)$ denotes Cazanave's resultant of [13] described in proposition 26, induces an isomorphism of groups

$$\rho\colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\cong} \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times. \tag{98}$$

Since our definition of deg is compatible with the notion of degree of a rational function used by Cazanave in [13], the work of Cazanave and Morel implies that, for every pointed morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ we have

$$\deg([f]) = \mathrm{rank}\left(\deg^{\mathbb{A}^1}([f])\right),$$

where rank denotes the homomorphism $\mathrm{GW}(k) \to \mathbb{Z}$ induced by the rank of a quadratic form. For a pointed morphism $g\colon \mathcal{J} \to \mathbb{P}^1$ with $\xi([g]) = [f]$, we have

$$\deg([g]) = \mathrm{rank}\left(\deg^{\mathbb{A}^1}(\xi[g])\right).$$

Hence we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} & \longrightarrow & [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} & \xrightarrow{\deg} & \mathrm{Pic}(\mathcal{J}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \xi_0}\ \cong & & \downarrow{\scriptstyle \xi}\ \text{bijection} & & \downarrow \cong & & \\ 1 & \longrightarrow & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & \xrightarrow{\deg} & \mathrm{Pic}(\mathbb{P}^1) & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow{\scriptstyle \rho}\ \cong & & \downarrow \cong & & \\ 1 & \longrightarrow & \mathrm{GW}(k)_0 \times_{k^\times/k^{\times 2}} k^\times & \longrightarrow & \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times & \xrightarrow{\mathrm{rank}} & \mathbb{Z} & \longrightarrow & 1 \end{array} \tag{99}$$

where $\mathrm{GW}(k)_0 \times_{k^\times/k^{\times 2}} k^\times$ denotes the kernel of the rank homomorphism.

Proposition 100 The map π_{N}^* is injective.

Proof By lemma 97 we know $\xi \circ \pi_{\mathrm{N}}^* = \nu_{\mathbb{P}^1}$. Since ξ is a bijection, it suffices to show that $\nu_{\mathbb{P}^1}$ is injective. The isomorphism ρ fits into the commutative diagram

$$\begin{array}{ccc} [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} & \xrightarrow{\nu_{\mathbb{P}^1}} & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \\ \downarrow \cong & & \cong \downarrow{\scriptstyle \rho} \\ \mathrm{MW}^s(k) \times_{k^\times/k^{\times 2}} k^\times & \xrightarrow[\hat{w}]{} & \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times \end{array}$$

where $\mathrm{MW}^s(k)$ denotes the stable monoid of symmetric bilinear forms as in [13, Definition 3.8] and $\hat{w}$ is the group completion induced by the group completion $w\colon \mathrm{MW}^s(k) \to \mathrm{GW}(k)$. Since the vertical maps are isomorphisms by [13, Corollary 3.10] and [22, §7.3], $\nu_{\mathbb{P}^1}$ is injective if and only if $\hat{w}$ is injective. To show that $\hat{w}$ is injective, it suffices to show that the group completion $w\colon \mathrm{MW}^s(k) \to \mathrm{GW}(k)$ is injective. Since $\mathrm{MW}^s(k)$ satisfies the cancellation property by the definition of $\mathrm{MW}^s(k)$ in [13, Definition 3.8], respectively by Witt's cancellation theorem, the map w is indeed injective. This proves the assertion. □

Proposition 101 Let $u, v \in k^\times$. If $u \neq v$, then $[g_{u,1}] \neq [g_{v,1}]$.

Proof Assume that $u \neq v \in k^\times$. By proposition 100 this implies $\pi_{\mathrm{N}}^*([X/u]) \neq \pi_{\mathrm{N}}^*([X/v])$. By proposition 91 we have $\pi_{\mathrm{N}}^*([X/u]) = [g_{u,1}] \oplus \pi$ and $\pi_{\mathrm{N}}^*([X/v]) = [g_{v,1}] \oplus \pi$. Since $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ is a group, this implies $[g_{u,1}] \neq [g_{v,1}]$. □

Proposition 102 For every $u \in k^\times$ we have

$$\left(\deg^{\mathbb{A}^1}(\xi[\pi_{\mathrm{N}}^*(X/u)]), \mathrm{res}(\xi[\pi_{\mathrm{N}}^*(X/u)])\right) = \left(\langle u\rangle, u\right) \text{ in } \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times.$$

In particular, for $\pi_{\mathrm{N}}^*(X/1) = \pi$, we get

$$\left(\deg^{\mathbb{A}^1}(\xi[\pi]), \mathrm{res}(\xi[\pi])\right) = \left(\langle 1\rangle, 1\right) \text{ in } \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times.$$

Proof By lemma 97 we know $\xi[\pi_{\mathrm{N}}^*(X/u)] = \nu_{\mathbb{P}^1}([X/u])$. In [13, 3.4] Cazanave shows that the image of $\nu_{\mathbb{P}^1}([X/u])$ in $\mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$ is $\left(\langle u\rangle, u\right)$ by assigning it to the rank 1 symmetric matrix $[u]$, which has determinant u and corresponds to the quadratic form $\langle u\rangle$. □

In light of question 83 we would like to show that $\rho(\xi[\tilde{\pi}])$ is the class $(-\langle 1\rangle, 1)$ in $\mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$. We are not able to confirm this yet, since we do not know how to compute the resultant of $\xi[\tilde{\pi}]$. We can, however, make the following observation based on computations of topological degrees in appendix C. We thank Kirsten Wickelgren for mentioning to us the idea to use the arguments of [9] and [10] to reduce the computation to the Grothendieck–Witt group of the integers.

Theorem 103 For every field k, we have

$$\deg^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1\rangle \text{ in } \mathrm{GW}(k).$$

Proof First we assume $k = \mathbb{F}_2$. By [13, Lemma 3.13], $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ is generated by elements in degree 1, i.e., the class of $X/1$ generates $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$. Since $\nu_{\mathbb{P}^1} \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ is a group completion, this implies that $\deg_{\mathbb{F}_2}^{\mathbb{A}^1} \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\cong} \mathrm{GW}(\mathbb{F}_2) = \mathbb{Z}$ is an isomorphism, where we refer to [20] for the Grothendieck group $\mathrm{GW}(\mathbb{F}_2)$ of symmetric bilinear forms over $\mathbb{F}_2$ and the isomorphism $\mathrm{GW}(\mathbb{F}_2) = \mathbb{Z}$ (see [10, Lemma B.5], [20, III Remark (3.4)]). Since the right-hand side of diagram (99) commutes, the fact that we have $\deg([\tilde{\pi}]) = -1$ implies $\deg^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1\rangle$ in $\mathrm{GW}(k)$.

Next we let k be a field of characteristic 2. Then the canonical morphism $\mathrm{Spec}\, k \to \mathrm{Spec}\, \mathbb{F}_2$ induces a commutative diagram of group homomorphisms

$$\begin{array}{ccc} [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{F}_2}^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]_k^{\mathbb{A}^1} \\ {\scriptstyle \deg_{\mathbb{F}_2}^{\mathbb{A}^1}}\downarrow & & \downarrow{\scriptstyle \deg_k^{\mathbb{A}^1}} \\ \mathrm{GW}(\mathbb{F}_2) & \longrightarrow & \mathrm{GW}(k). \end{array}$$

Since $\tilde{\pi}$ is defined over $\mathbb{F}_2$ and $\deg_{\mathbb{F}_2}^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1\rangle$ by the first case, this implies $\deg_k^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1\rangle$ in $\mathrm{GW}(k)$.

Now we assume that k is a field of characteristic $\neq 2$. The proof for this case is also based on the fact that $\tilde{\pi}$ is already defined over $\mathbb{Z}$ and not just k. To make the argument work, however, requires a bit more effort. For a ring A, let $\mathcal{SH}(A)$ denote the stable motivic homotopy category over $\operatorname{Spec} A$. Let $KO_k \in \mathcal{SH}(k)$ denote the motivic spectrum over $\operatorname{Spec} k$ which represents Hermitian K-theory. It is equipped with a unit morphism $\varepsilon_k \colon \mathbb{1}_k \to KO_k$ in $\mathcal{SH}(k)$. Let $\mathfrak{s} \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \to \mathbb{1}_k^{0,0}(\operatorname{Spec} k)$ denote the homomorphism defined by stabilisation and note that there is a canonical isomorphism $KO_k^{0,0}(\operatorname{Spec} k) \cong \mathrm{GW}(k)$. We then define the homomorphism δ as the following composition.

$$\delta \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\mathfrak{s}} \mathbb{1}_k^{0,0}(\operatorname{Spec} k) \xrightarrow{\varepsilon_k} KO_k^{0,0}(\operatorname{Spec} k) \cong \mathrm{GW}(k)$$

We claim that the homomorphism δ can be identified with the motivic Brouwer degree $\deg_k^{\mathbb{A}^1}$ over k. To prove the claim we follow the argument of Levine and Raksit in [19, proof of Theorem 8.6, page 1845]. By Morel's computation [22, Theorem 6.40] the isomorphism $\mathrm{GW}(k) \cong \mathbb{1}_k^{0,0}(\operatorname{Spec} k)$ sends $\langle u \rangle \in \mathrm{GW}(k)$, for $u \in k^\times$, to $\mathfrak{s}(\nu_{\mathbb{P}^1}[X/u])$, the image of the class of $X/u \colon \mathbb{P}_k^1 \to \mathbb{P}_k^1$, $[x_0 : x_1] \mapsto [x_0 : ux_1]$, in $\mathbb{1}_k^{0,0}(\operatorname{Spec} k)$. Hence the classes $\mathfrak{s}(\nu_{\mathbb{P}^1}[X/u])$ for all $u \in k^\times$ generate $\mathbb{1}_k^{0,0}(\operatorname{Spec} k)$. Thus, in order to prove the claim it suffices to show that $\delta(\nu_{\mathbb{P}^1}[X/u]) = \langle u \rangle$ in $\mathrm{GW}(k)$, since $\deg_k^{\mathbb{A}^1}([X/u]) = \langle u \rangle \in \mathrm{GW}(k)$. That is we need to show $\varepsilon_k(\mathfrak{s}(\nu_{\mathbb{P}^1}[X/u])) = \langle u \rangle$. This follows from [1, Corollary 6.2] which proves the claim.

In [9, § 3.8.3] Bachmann and Hopkins construct a motivic spectrum $KO'_{\mathbb{Z}} \in \mathcal{SH}(\mathbb{Z})$ with a unit morphism $\varepsilon'_{\mathbb{Z}} \colon \mathbb{1}_{\mathbb{Z}} \to KO'_{\mathbb{Z}}$, and write KO'_k and ε'_k for the pullback of $KO'_{\mathbb{Z}}$ and $\varepsilon'_{\mathbb{Z}}$ to $\mathcal{SH}(k)$ along the canonical morphism $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}$. Since the characteristic of k is not 2, there is an equivalence of ring spectra $KO'_k \simeq KO_k$ by [9, Lemma 3.38 (3)], which induces an isomorphism $(KO'_k)^{0,0}(\operatorname{Spec} k) \cong KO_k^{0,0}(\operatorname{Spec} k)$. Thus, there is an isomorphism $(KO'_k)^{0,0}(\operatorname{Spec} k) \cong KO_k^{0,0}(\operatorname{Spec} k)$ which fits into the following commutative diagram.

$$\begin{array}{ccc} \mathbb{1}_k^{0,0}(\operatorname{Spec} k) & \xrightarrow{\varepsilon'_k} & (KO'_k)^{0,0}(\operatorname{Spec} k) \\ & \searrow^{\varepsilon_k} & \downarrow \cong \\ & & KO_k^{0,0}(\operatorname{Spec} k) \end{array}$$

By the above, we may therefore identify $\deg_k^{\mathbb{A}^1}$ over k with the composed homomorphism

$$[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\mathfrak{s}} \mathbb{1}_k^{0,0}(\operatorname{Spec} k) \xrightarrow{\varepsilon'_k} (KO'_k)^{0,0}(\operatorname{Spec} k) \cong KO_k^{0,0}(\operatorname{Spec} k) \cong \mathrm{GW}(k).$$

Furthermore, by [9, Lemma 3.38 (2)], there is an isomorphism $\pi_{0,0}(KO'_{\mathbb{Z}}) \cong \mathrm{GW}(\mathbb{Z})$, where $\mathrm{GW}(\mathbb{Z})$ denotes the Grothendieck–Witt group over $\mathbb{Z}$ defined in [20, Chapter II]. Let $[\mathbb{P}^1,\mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1}$ denote the set of endomorphisms of $\mathbb{P}^1$ in the pointed unstable $\mathbb{A}^1$-homotopy category over $\operatorname{Spec}\mathbb{Z}$. We now define the homomorphism $\deg_{\mathbb{Z}}^{\mathbb{A}^1}\colon [\mathbb{P}^1,\mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} \to \mathrm{GW}(\mathbb{Z})$ as the composition

$$\deg_{\mathbb{Z}}^{\mathbb{A}^1}\colon [\mathbb{P}^1,\mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} \xrightarrow{\mathfrak{s}_{\mathbb{Z}}} \mathbb{1}_{\mathbb{Z}}^{0,0}(\operatorname{Spec}\mathbb{Z}) \xrightarrow{\varepsilon'_{\mathbb{Z}}} (KO'_{\mathbb{Z}})^{0,0}(\operatorname{Spec}\mathbb{Z}) \cong \mathrm{GW}(\mathbb{Z}).$$

The canonical homomorphism $\mathbb{Z} \to k$ then induces the following commutative square

$$\begin{array}{ccc} [\mathbb{P}^1,\mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} & \xrightarrow{\deg_{\mathbb{Z}}^{\mathbb{A}^1}} & \mathrm{GW}(\mathbb{Z}) \\ \downarrow & & \downarrow \mathfrak{b}_k \\ [\mathbb{P}^1,\mathbb{P}^1]_{k}^{\mathbb{A}^1} & \xrightarrow[\deg_{k}^{\mathbb{A}^1}]{} & \mathrm{GW}(k) \end{array}$$

where $\mathfrak{b}_k\colon \mathrm{GW}(\mathbb{Z}) \to \mathrm{GW}(k)$ denotes the change of coefficients homomorphism. As a consequence we see that if $[\alpha] \in [\mathbb{P}^1,\mathbb{P}^1]_{k}^{\mathbb{A}^1}$ is in the image of the homomorphism $[\mathbb{P}^1,\mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1} \to [\mathbb{P}^1,\mathbb{P}^1]_{k}^{\mathbb{A}^1}$, then

$$\deg_{k}^{\mathbb{A}^1}([\alpha]) = \mathfrak{b}_k(\deg_{\mathbb{Z}}^{\mathbb{A}^1}([\alpha])). \tag{104}$$

By [10, Lemma 5.6] (see also [20, Theorem II.4.3]), $\mathrm{GW}(\mathbb{Z})$ is generated over $\mathbb{Z}$ by the classes $\langle 1\rangle$ and $\langle -1\rangle$. For a class $q \in \mathrm{GW}(\mathbb{Z})$, let $q_{\mathbb{C}}$ and $q_{\mathbb{R}}$ denote the images of q in $\mathrm{GW}(\mathbb{C})$ and $\mathrm{GW}(\mathbb{R})$, respectively. Then $q \in \mathrm{GW}(\mathbb{Z})$ is uniquely determined by the integers $r(q) := \operatorname{rank}(q_{\mathbb{C}})$ and $s(q) := \operatorname{sgn}(q_{\mathbb{R}})$, given by the rank and signature of $q_{\mathbb{C}}$ and $q_{\mathbb{R}}$, respectively, via the formula

$$q = \frac{r(q)+s(q)}{2}\langle 1\rangle + \frac{r(q)-s(q)}{2}\langle -1\rangle \in \mathrm{GW}(\mathbb{Z}). \tag{105}$$

Since $\mathcal{J}$ and both morphisms π and $\tilde{\pi}$ are defined over $\operatorname{Spec}\mathbb{Z}$, we can now apply the above observations to prove the assertion of the proposition. Since π is an $\mathbb{A}^1$-weak equivalence over $\operatorname{Spec}\mathbb{Z}$ as well, we can form the pointed $\mathbb{A}^1$-homotopy class $\xi_{\mathbb{Z}}([\tilde{\pi}]) := [\tilde{\pi}\circ\pi^{-1}]^{\mathbb{A}^1} \in [\mathbb{P}^1,\mathbb{P}^1]_{\mathbb{Z}}^{\mathbb{A}^1}$ defined by the zig-zag $\mathbb{P}^1_{\mathbb{Z}} \xleftarrow{\pi} \mathcal{J}_{\mathbb{Z}} \xrightarrow{\tilde{\pi}} \mathbb{P}^1_{\mathbb{Z}}$. The class $\xi_{\mathbb{Z}}([\tilde{\pi}])$ is sent to $\xi([\tilde{\pi}])$ under base change. Thus, by the above arguments, to determine $\deg_{k}^{\mathbb{A}^1}(\xi[\tilde{\pi}])$ in $\mathrm{GW}(k)$ it suffices to compute the rank and signature of $\deg_{\mathbb{Z}}^{\mathbb{A}^1}(\xi_{\mathbb{Z}}[\tilde{\pi}])$ after base change to $\mathbb{C}$ and $\mathbb{R}$, respectively. Since the right-hand side of diagram (99) commutes, the fact that we have $\deg([\tilde{\pi}]) = -1$ implies $\operatorname{rank}(\deg_{\mathbb{C}}^{\mathbb{A}^1}(\xi[\tilde{\pi}])) = -1$. In appendix C and example 144 we show that the signature of $\deg_{\mathbb{R}}^{\mathbb{A}^1}(\xi[\tilde{\pi}])$ over $\mathbb{R}$ is -1. Thus, by formula (105), we get $\deg_{\mathbb{Z}}^{\mathbb{A}^1}(\xi_{\mathbb{Z}}[\tilde{\pi}]) = -\langle 1\rangle$ in $\mathrm{GW}(\mathbb{Z})$. By equation (104) we can therefore conclude that $\deg_{k}^{\mathbb{A}^1}(\xi[\tilde{\pi}]) = -\langle 1\rangle$ in $\mathrm{GW}(k)$. ◻

6.2 Group completion of naive homotopy classes

We will now describe the homomorphism $\psi\colon \left([\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1},\oplus^{\mathbb{A}^1}\right) \to \left([\mathcal{J},\mathbb{P}^1]^{\mathrm{N}},\oplus\right)$ induced by the universal property of the group completion $\nu_{\mathbb{P}^1}$ in more detail. By [13, Lemma 3.13], $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ is generated by elements in degree 1, i.e., it is generated by the set of classes $[X/u]$ for all $u \in k^\times$. Hence, since $\nu_{\mathbb{P}^1}$ is a group completion, every element in $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$ of degree 0 can be written as a sum of the differences $\gamma_{u,v} := \nu_{\mathbb{P}^1}([X/u]) - \nu_{\mathbb{P}^1}([X/v])$ for suitable $u,v \in k^\times$. Thus the set of classes $\gamma_{u,v}$ for all $u,v \in k^\times$ generates the subgroup $[\mathbb{P}^1,\mathbb{P}^1]_0^{\mathbb{A}^1}$ of degree 0 elements. Because of this we would like to understand the image of the $\gamma_{u,v}$ under ψ. Since ψ is a group homomorphism, we know $\psi(\gamma_{u,v}) \oplus \psi(\nu_{\mathbb{P}^1}([X/v])) = \psi(\nu_{\mathbb{P}^1}([X/u]))$. Since $\psi \circ \nu_{\mathbb{P}^1} = \pi_{\mathrm{N}}^*$, this implies $\psi(\gamma_{u,v}) \oplus \pi_{\mathrm{N}}^*([X/v]) = \pi_{\mathrm{N}}^*([X/u])$. By lemma 90, the map $g_{u,v}$ satisfies $[g_{u,v}] \oplus \pi_{\mathrm{N}}^*([X/v])) = \pi_{\mathrm{N}}^*([X/u])$. Hence, since $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ is a group, we get

$$\psi(\gamma_{u,v}) = [g_{u,v}] \text{ in } [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}.$$

This motivates the following definition of the subgroup $\mathbf{G} \subseteq [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$.

Definition 106 Let $\mathbf{G}_0 := \langle [g_{u,v}] \mid u,v \in k^\times \rangle \subseteq [\mathcal{J},\mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ denote the subgroup generated by the homotopy classes of the maps $g_{u,v}$. Let $\mathbf{G} \subseteq [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ be the subgroup generated by $\mathbf{G}_0$ and $[\pi]$.

We note that by lemma 86, we have $-[g_{u,v}] = [g_{v,u}]$, while $[g_{u,u}]$ is the neutral element, and we therefore have $-([g_{u,v}] \oplus \pm n[\pi]) = [g_{v,u}] \oplus \mp n[\pi]$ in $\mathbf{G}$.

Remark 107 Let $\mathbf{M}_0$ denote the submonoid of $[\mathcal{J},\mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ generated by the set of homotopy classes $[g_{u,1}]$ for all $u \in k^\times$. By lemma 86, we have $[g_{u,v}] \oplus [g_{v,1}] = [g_{u,1}]$ for all $u,v \in k^\times$. Thus, every element in $\mathbf{G}_0$ is a sum of differences of elements in $\mathbf{M}_0$. The implies that the inclusion $\mathbf{M}_0 \subset \mathbf{G}_0$ is a group completion.

Lemma 108 The morphism of monoids $\pi_{\mathrm{N}}^*\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}} \to [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ has image in $\mathbf{G}$.

Proof By [13, Lemma 3.13], $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ is generated by elements in degree 1, i.e., it is generated by the set of classes $[X/u]$ for all $u \in k^\times$. Hence it suffices to show that $\pi_{\mathrm{N}}^*([X/u])$ is contained in $\mathbf{G}$. This follows from lemma 89. ◻

Proposition 109 The morphism of monoids $\pi_{\mathrm{N}}^*\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}} \to \mathbf{G}$ is a group completion.

Proof Let H be an abelian group and $\mu\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}} \to H$ be a morphism of monoids. We will show that there is a unique homomorphism of groups $\widetilde{\mu}\colon \mathbf{G} \to H$ such that $\widetilde{\mu} \circ \pi_{\mathrm{N}}^* = \mu$.

We set $\widetilde{\mu}([\pi]) := \mu([X/1])$. By lemma 90, we have $[g_{u,v}] \oplus \pi_{\mathrm{N}}^*([X/v]) = \pi_{\mathrm{N}}^*([X/u])$ in $\mathbf{G} \subseteq [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$. Hence compatibility with μ forces the definition

$$\widetilde{\mu}([g_{u,v}]) := \mu([X/u]) - \mu([X/v]).$$

By definition of $\mathbf{G}$ this induces a unique group homomorphism $\widetilde{\mu}$, once we have shown that it is well-defined.

Now we show that $\widetilde{\mu}$ is well-defined. Because $\mathbf{G} \cong \mathbf{G}_0 \oplus \mathbb{Z}$, all relations in $\mathbf{G}$ amongst the generators arise from relations of the classes $[g_{u,v}]$. Consider a relation of the form

$$[g_{u_1,v_1}] \oplus \ldots \oplus [g_{u_s,v_s}] = 0. \tag{110}$$

We must then show that $\sum_i \widetilde{\mu}\left([g_{u_i,v_i}]\right) = 0$ in H. Since $\mathbf{G}$ is a group and by lemma 90, we have

$$\sum_i g_{u_i,v_i} \oplus \pi_{\mathrm{N}}^*([X/v_i]) = \sum_i \pi_{\mathrm{N}}^*([X/u_i]).$$

Since $\mathbf{G}$ is abelian, this implies

$$\sum_i [g_{u_i,v_i}] = \sum_i \pi_{\mathrm{N}}^*([X/u_i]) - \pi_{\mathrm{N}}^*([X/v_i]) = \sum_i \pi_{\mathrm{N}}^*([X/u_i]) - \sum_i \pi_{\mathrm{N}}^*([X/v_i]) = 0.$$

Hence

$$\sum_i \pi_{\mathrm{N}}^*([X/u_i]) = \sum_i \pi_{\mathrm{N}}^*([X/v_i])$$

in $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$. Since π_{N}^* is an injective monoid morphism, in $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ we have the equation

$$\sum_i [X/u_i] = \sum_i [X/v_i].$$

It thus follows that

$$\mu\left(\sum_i [X/u_i]\right) = \mu\left(\sum_i [X/v_i]\right) \text{ in } H.$$

We calculate

$$\widetilde{\mu}\left(\sum_i [g_{u_i,v_i}]\right) = \sum_i \mu([X/u_i]) - \mu([X/v_i]) = 0,$$

as desired. This shows that $\widetilde{\mu}$ is well-defined.

It remains to show $\widetilde{\mu} \circ \pi_{\mathrm{N}}^* = \mu$. By [13, Lemma 3.13], $[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}$ is generated by the set of classes $[X/u]$ for all $u \in k^{\times}$. Hence μ is completely determined by the images of $[X/u]$ for all $u \in k^\times$. Thus, in order to show $\widetilde{\mu} \circ \pi_{\mathrm{N}}^* = \mu$, it suffices to show $\widetilde{\mu}\left(\pi_{\mathrm{N}}^*([X/u])\right) = \mu([X/u])$ for every $u \in k^\times$. This is now immediate from the definition of $\widetilde{\mu}$ and lemma 90:

$$\begin{aligned}\widetilde{\mu}\left(\pi_{\mathrm{N}}^*([X/u])\right) &= \widetilde{\mu}\left([g_{u,1}] \oplus \pi_{\mathrm{N}}^*([X/1])\right)\\ &= \widetilde{\mu}\left([g_{u,1}]\right) + \widetilde{\mu}([\pi])\\ &= \mu\left([X/u]\right) - \mu\left([X/1]\right) + \mu\left([X/1]\right)\\ &= \mu\left([X/u]\right).\end{aligned}$$

This shows that $\pi_{\mathrm{N}}^* \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}} \to \mathbf{G}$ has the universal property of a group completion and finishes the proof. □

Theorem 111 There is a unique isomorphism of groups

$$\chi \colon \mathbf{G} \to [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$$

such that $\chi \circ \pi_{\mathrm{N}}^* = \nu_{\mathbb{P}^1}$.

The homomorphism χ sends $[g_{u,v}]$ to the unique element $\gamma_{u,v}$ that satisfies $\nu_{\mathbb{P}^1}^*([X/u]) = \gamma_{u,v} \oplus^{\mathbb{A}^1} \nu_{\mathbb{P}^1}([X/v])$ and $[\pi]$ to $[\mathrm{id}]$. Moreover, χ and the homomorphism $\psi \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \to \mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ are mutual inverses to each other.

Proof The existence of χ and its definition is a consequence of proposition 109 and its proof. The assertion that χ is the inverse of ψ follows from the fact that $\nu_{\mathbb{P}^1}$ is a group completion proven by Cazanave in [13, Theorem 3.22] and the universal property of group completion. □

As a particular consequence of theorem 111 we get the following result.

Proposition 112 The restriction χ_0 of the homomorphism χ to $\mathbf{G}_0$ defines an isomorphism of groups

$$\chi_0 \colon \mathbf{G}_0 \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}.$$

Proof The elements $\gamma_{u,v}$ are of degree 0, and hence they lie in the subgroup $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$. As a consequence, the homomorphism χ and its restriction χ_0 fit in the following commutative diagram of abelian groups.

$$\begin{array}{ccccccccc}1 & \longrightarrow & \mathbf{G}_0 & \longrightarrow & \mathbf{G} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1\\ & & \downarrow{\scriptstyle \chi_0} & & \downarrow{\scriptstyle \chi}\,{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\ 1 & \longrightarrow & [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} & \longrightarrow & [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1\end{array}$$

Since the middle and right-most maps are isomorphisms, the assertion follows from the five-lemma. □

The existence of the isomorphisms $\mathbf{G} \xrightarrow{\chi} [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xleftarrow{\phi} [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ does not imply that $\mathbf{G}$ equals $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$. However, we make the following conjecture on the a prioiri subgroups $\mathbf{G}_0$ and $\mathbf{G}$.

Conjecture 113 The inclusions $\mathbf{G}_0 \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ and $\mathbf{G} \subseteq [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ are equalities.

We will show in theorem 120 in section 6.3 that conjecture 113 is true whenever $k = \mathbb{F}_q$ is a finite field. This follows from an explicit computation of $\mathbf{G}_0$ and $K_1^{MW}(\mathbb{F}_q)$, the first Milnor–Witt K-theory of $\mathbb{F}_q$.

Remark 114 It follows from the structure of the group $\mathbf{G}$ as a product of $\mathbf{G}_0$ and $\{n[\pi] \mid n \in \mathbb{Z}\}$ that in order to prove conjecture 113 it suffices to show that the inclusion $\mathbf{G}_0 \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ is an equality, i.e., that the set of homotopy classes $[g_{u,v}]$ for all $u, v \in k^\times$ generates the group $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$.

Remark 115 If conjecture 113 is true, then the group homomorphism $\psi \colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \to [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$, induced by the fact that $\nu_{\mathbb{P}^1}$ is a group completion, is an isomorphism and it agrees with ϕ^{-1}, the inverse of the isomorphism of theorem 80. We note, however, that this does not yet imply that the bijection ξ is a group homomorphism.

6.3 Milnor–Witt K-theory and morphisms in degree 0

Our final goal is to prove conjecture 113 for finite fields. For the proof we use the Milnor–Witt K-theory of a field which we now recall from [22, Definition 3.1].

Definition 116 The Milnor–Witt K-theory of the field k, denoted $K_*^{MW}(k)$, is the graded associative ring generated by symbols $[u]$ in degree 1 for $u \in k^\times$ and the symbol η in degree -1 subject to the following relations:

(1) For each $u \in k^\times \setminus \{1\}$, $[u].[1-u] = 0$.

(2) For each pair $u, v \in (k^\times)^2$, $[uv] = [u] + [v] + \eta.[u].[v]$.

(3) For each $u \in k^\times$, $\eta.[u] = [u].\eta$.

(4) Let $h := \eta.[-1] + 2$. Then $\eta.h = 0$.

Remark 117 It follows directly from the defining relations for $K_*^{MW}(k)$ that $[1] = 0$ and $\eta.[u^2] = 0$ for each $u \in k^\times$. See [22, § 3.1] for a proof and other basic properties of Milnor–Witt K-theory.

Recall that $\mathbf{G}_0 := \langle [g_{u,v}] \,|\, u, v \in k^\times \rangle \subseteq [\mathcal{J}, \mathbb{A}^2 \backslash \{0\}]^{\mathrm{N}}$ denotes the subgroup generated by the homotopy classes of the maps $g_{u,v}$. In this subsection we write $\mathbf{G}_0(k)$ and $\mathbf{G}(k)$ for the groups $\mathbf{G}_0$ and $\mathbf{G}$, respectively, to emphasize the dependency of the base field k.

Proposition 118 For every field k, there is an isomorphism $\mathbf{G}_0(k) \cong K_1^{MW}(k)$.

Proof By proposition 112, we have an isomorphism $\chi_0 \colon \mathbf{G}_0(k) \xrightarrow{\cong} [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$. As recalled in remark 60, the work of Morel in [22, § 7.3] implies that there is an isomorphism of groups $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1} \cong K_1^{MW}(k)$. The composition of these isomorphisms yields the assertion. □

Proposition 119 Let $k = \mathbb{F}_q$ be a finite field. Then $K_1^{MW}(\mathbb{F}_q)$ is a finite cyclic group of order $q - 1$.

Proof First we assume that q is even. Then the squaring homomorphism is surjective, and hence every unit is a square. Fix u to be a multiplicative generator of $\mathbb{F}_q^\times$. It follows from [22, Lemma 3.6 (1)] that $K_1^{MW}(\mathbb{F}_q)$ is generated by the elements $[v]$ for $v \in k^\times$, which are subject to the relation $[vv'] = [v] + [v']$ for all $v, v' \in \mathbb{F}_q^\times$. The fact that $u^{q-1} = 1$ yields the result that $K_1^{MW}(\mathbb{F}_q)$ is cyclic of order $q - 1$ generated by the symbol $[u]$.

Now we assume that q is odd. Then the kernel of the squaring homomorphism has two elements, -1 and 1, i.e., $\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$. Because 1 is a square, $\mathbb{F}_q \setminus \{0, 1\}$ contains more non-squares than squares. Construct pairs $(s, 1 - s)$ from elements $s \in \mathbb{F}_q \setminus \{0, 1\}$, and observe that there must exist at least one non-square s such that $1 - s$ is also a non-square. For the rest of the proof we fix s to be one such non-square and pick a multiplicative generator u of $\mathbb{F}_q^\times$.

By relation (1) in $K_1^{MW}(\mathbb{F}_q)$, we have $[s].[1 - s] = 0$. Let v_1, v_2 be non-squares in $\mathbb{F}_q^\times$. Then there exist units c_1 and c_2 such that $c_1^2 s = v_1$ and $c_2^2(1 - s) = v_2$ since $\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$. We can then compute

$$\begin{aligned}
[v_1 v_2] &= [v_1] + [v_2] + \eta.[v_1].[v_2] \\
&= [v_1] + [v_2] + \eta.[c_1^2 s].[c_2^2(1 - s)] \\
&= [v_1] + [v_2] + \eta.([c_1^2] + [s]).([c_2^2] + [1 - s]) \\
&= [v_1] + [v_2],
\end{aligned}$$

where the last equality follows from the fact $\eta.[c_i^2] = 0$ of remark 117 and the relation $[s].[1 - s] = 0$. Additionally, for every non-square v and every square c^2, we get

$[vc_2^2] = [v] + [c^2]$. Since $\mathbb{F}_q^\times$ is cyclic and $[u^n] = n[u]$, this shows that $K_1^{MW}(\mathbb{F}_q)$ is cyclic of order $q-1$ generated by the symbol $[u]$. □

Theorem 120 Let $k = \mathbb{F}_q$ be a finite field. Then conjecture 113 is true, i.e., the inclusions $\mathbf{G}_0(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ and $\mathbf{G}(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ are equalities.

Proof By remark 114 it suffices to prove the assertion for $\mathbf{G}_0(\mathbb{F}_q)$. By propositions 118 and 119, both $\mathbf{G}_0(\mathbb{F}_q)$ and $K_1^{MW}(\mathbb{F}_q)$ are finite groups of the same cardinality. Since $[\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ and $K_1^{MW}(\mathbb{F}_q)$ are isomorphic and since $\xi_0 \colon [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}} \to [\mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}]^{\mathbb{A}^1}$ is an isomorphism, $[\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ is a finite group of the same cardinality as $\mathbf{G}_0(\mathbb{F}_q)$ as well. Hence $\mathbf{G}_0(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ is an inclusion of finite groups of the same cardinality. This implies that the inclusion $\mathbf{G}_0(\mathbb{F}_q) \subseteq [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ is an equality. □

We conclude this section with the following observation. While proposition 118 shows that there is an isomorphism between $\mathbf{G}_0(k)$ and $K_1^{MW}(k)$, the proof of proposition 119 suggests that the following map provides a concrete isomorphism. We consider this an interesting observation about $K_1^{MW}(k)$ that arises from our work on maps $\mathcal{J} \to \mathbb{A}^2 \setminus \{0\}$.

Proposition 121 Let k be one of the following fields: a field in which every unit is a square, a finite field, or $\mathbb{R}$. Then the assignment $[u] \mapsto [g_{u,1}]$ defines an isomorphism $\kappa \colon K_1^{MW}(k) \to \mathbf{G}_0(k)$.

Proof We will show that $K_1^{MW}(k)$ and $\mathbf{G}_0(k)$ are generated by the classes $[u]$ and $[g_{u,1}]$, respectively, and that these generators satisfy exactly the same type of relations. This implies that κ is both a well-defined homomorphism and an isomorphism. We will prove the claim by looking at each type of field separately.

First we assume that k is a field where every unit is a square. The group $K_1^{MW}(k)$ is generated by the elements $[u]$ and the relation $[uv] = [u] + [v]$ for all $u, v \in k^\times$. To prove that κ is an isomorphism we need to show that $\mathbf{G}_0(k)$ is generated by the classes $[g_{u,1}]$ subject to the relation $[g_{uv,1}] = [g_{u,1}] \oplus [g_{v,1}]$. Since every unit in k is a square, we get $[g_{u,v}] = [g_{u/v,1}]$ by lemma 87. Hence $\mathbf{G}_0(k)$ is generated by elements $[g_{u,1}]$. By proposition 101 we know that $[g_{u,1}] \neq [g_{v,1}]$ for $u \neq v \in k^\times$. By lemma 88 we get the relation $[g_{u,1}] \oplus [g_{v,1}] = [g_{uv,1}]$. Thus, the map sending $[u]$ to $[g_{u,1}]$ induces a homomorphism which is surjective and injective. Hence κ is an isomorphism.

For $k = \mathbb{F}_q$, proposition 119 shows that $K_1^{MW}(\mathbb{F}_q)$ is generated by the symbol $[u]$ for a multiplicative generator $u \in \mathbb{F}_q^\times$. When q is even, every unit is a square, and in this case κ is an isomorphism. So we assume that q is odd, and will now show that every element in $\mathbf{G}_0(\mathbb{F}_q)$ can be written in the form $m[g_{u,1}] \oplus [g_{u^{2m'},1}]$ for some $m, m' \in \mathbb{Z}$. We will use that every square in $\mathbb{F}_q$ is equal to an even power of the generator $u \in \mathbb{F}_q^\times$, and distinguish three cases: Assume first $v_1, v_2 \in \mathbb{F}_q^\times$ are squares. By lemma 87 we then have $[g_{v_1,v_2}] = [g_{v_1/v_2,1}] = [g_{u^{2m}}, 1]$ for some m. Second, if v_1 is not a square and v_2 is a square, then $v_1/v_2 = u^{2m+1}$ for some $m \in \mathbb{Z}$. Then by lemma 87 and 88 we know $[g_{v_1,v_2}] = [g_{u^{2m+1},1}] = [g_{u,1}] \oplus [g_{u^{2m},1}]$. Note that $[g_{v_2,v_1}] = -[g_{v_1,v_2}] = -[g_{u,1}] \oplus [g_{u^{-2m},1}]$. Third, assume that both v_1 and v_2 are non-squares in $\mathbb{F}_q^\times$. Since $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$, we can find an $m \in \mathbb{Z}$ such that $v_1/v_2 = u^{2m}$. We have $[g_{v_1,v_2}] = [g_{u\cdot u^{2m},u}]$ by lemma 87 and scaling by the square u/v_2. We can now apply lemma 86 and then lemma 88 to get

$$[g_{u\cdot u^{2m},u}] = [g_{u\cdot u^{2m},1}] \oplus [g_{1,u}] = [g_{u,1}] \oplus [g_{u^{2m},1}] \oplus [g_{1,u}] = [g_{u^{2m},1}].$$

To conclude the argument we note that, for $v_1, v_2 \in \mathbb{F}_q^\times$ with $v_1 + v_2 \neq 1$, there is the relation $\langle v_1\rangle + \langle v_2\rangle = \langle v_1 + v_2\rangle + \langle (v_1 + v_2)v_1v_2\rangle$ in $\mathrm{GW}(\mathbb{F}_q)$. For s and $1 - s$ in $\mathbb{F}_q^\times$, this gives $\langle s\rangle + \langle 1 - s\rangle = \langle 1\rangle + \langle s(1 - s)\rangle = \langle 1, 1\rangle$. In particular, since u, s, and $1 - s$ all differ by squares and hence $\langle u\rangle = \langle s\rangle = \langle 1 - s\rangle$ in $\mathrm{GW}(\mathbb{F}_q)$, we have $\langle u\rangle + \langle u\rangle = \langle 1, 1\rangle = \langle u^2, 1\rangle$ in $\mathrm{GW}(\mathbb{F}_q)$. By [13, Corollary 3.10] this relation implies $[X/u] \oplus^{\mathrm{N}} [X/u] = [X/u^2] \oplus^{\mathrm{N}} [X/1]$ in $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$. By proposition 94 and lemma 90, this implies the equality $[g_{u,1}] \oplus [g_{u,1}] = [g_{u^2,1}]$ in $\mathbf{G}_0(\mathbb{F}_q)$. Iterating this argument, we get $(q-1)[g_{u,1}] = [g_{u^{q-1},1}] = [g_{1,1}]$. Since $[g_{v_1}, 1] \neq [g_{v_2}, 1]$ for $v_1 \neq v_2 \in \mathbb{F}_q^\times$ by proposition 101, this implies that $\mathbf{G}_0(\mathbb{F}_q)$ is cyclic of order $q - 1$ generated by $[g_{u,1}]$. Hence the map $[u] \mapsto [g_{u,1}]$ is a well-defined homomorphism which is surjective and injective. Thus κ is an isomorphism in this case as well.

Finally, we assume $k = \mathbb{R}$. First we determine the generators and relations for $K_1^{MW}(\mathbb{R})$. For $u > 0$, we have $[-u] = [-1] + [u] + \eta.[-1].[u] = [-1] + [u]$ and $-[u] = [1/u]$. Thus every element in $K_1^{MW}(\mathbb{R})$ can be written as $n[-1] + [u]$ with $u > 0$ for some $n \in \mathbb{Z}$ subject to the relation $(n[-1] + [u]) + (m[-1] + [v]) = (n + m)[-1] + [uv]$. Next we show that $\mathbf{G}_0(\mathbb{R})$ has analogous generators and relations. Assume $u, v > 0$. Since v is a square, lemmas 87 and 88 imply the identities

$$[g_{u,v}] = [g_{u/v,1}], \text{ and } [g_{-u,v}] = [g_{-u/v,1}] = [g_{-1,1}] \oplus [g_{u/v,1}].$$

Using that v is a square, lemma 87 yields the following identity

$$[g_{u,-v}] = [g_{u/v,-1}] = [g_{u/v,1}] \oplus [g_{1,-1}] = -[g_{-1,1}] \oplus [g_{u/v,1}]$$

where we have $[g_{1,-1}] = -[g_{-1,1}]$ by lemma 86. Finally, using lemma 87 and 88 we get

$$[g_{-u,-v}] = [g_{-u/v,-1}] = [g_{-u/v,1}] \oplus [g_{1,-1}] = [g_{-1,1}] \oplus [g_{u/v,1}] \oplus [g_{1,-1}] = [g_{u/v,1}].$$

This implies every element of $\mathbf{G}_0(\mathbb{R})$ can be written as a sum $n[g_{-1,1}] \oplus [g_{u,1}]$ with $u > 0$ and $n \in \mathbb{Z}$. By proposition 101 we know that $[g_{u,1}] \neq [g_{v,1}]$ for $u \neq v \in \mathbb{R}^\times$. By lemma 88 we get the relation $\big(n[g_{-1,1}] \oplus [g_{u,1}]\big) \oplus \big(m[g_{-1,1}] \oplus [g_{v,1}]\big) = (n+m)[g_{-1,1}] \oplus [g_{uv,1}]$ when $u, v > 0$. Hence the map $[u] \mapsto [g_{u,1}]$ is a well-defined homomorphism which is surjective and injective. Thus κ is an isomorphism in this case. This finishes the proof. □

A Affine representability for pointed spaces and homotopies

In this section, we discuss the results of Asok, Hoyois, and Wendt in [5], [6], and how we apply them. While the definition of our proposed group operation on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ in definition 77 is independent of motivic homotopy theory and the results of [6], we use the affine representability results of [6] to compare our group operation with the conventional group structure on $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. A minor technical point to overcome is that Asok, Hoyois, and Wendt work in the unpointed motivic homotopy category, whereas we need the analogous results in the pointed setting. The purpose of this appendix is to explain how the pointed analogs can be deduced. To keep the presentation brief, we use the conventions and notation of the papers [5] and [6]. We thank Marc Hoyois for helpful comments.

Let $\mathrm{Spc}(k)$ denote the category of simplicial presheaves on Sm_k. Let $\mathrm{Spc}_*(k)$ denote the category of pointed simplicial presheaves on Sm_k. We refer to an object in $\mathrm{Spc}(k)$ (respectively in $\mathrm{Spc}_*(k)$) as a (pointed) motivic space. For a motivic space $\mathcal{Y}$, let $\mathrm{Sing}^{\mathbb{A}^1}\mathcal{Y}$ denote the singular functor defined in [5, §4.1], see also [23, page 88]. If $\mathcal{Y}$ is pointed by a morphism $y\colon \operatorname{Spec} k \to \mathcal{Y}$, then the pointed singular functor $\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{Y}$ is defined as the fiber over y. More precisely, let X be a pointed smooth k-scheme pointed by the morphism $x\colon \operatorname{Spec} k \to X$, then $\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{Y}$ is determined by the pullback square of simplicial sets

$$\begin{array}{ccc} (\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{Y})(X) & \longrightarrow & (\mathrm{Sing}^{\mathbb{A}^1}\mathcal{Y})(X) \\ \downarrow & & \downarrow{\scriptstyle x^*} \\ \text{point} = (\mathrm{Sing}^{\mathbb{A}^1}\operatorname{Spec} k)(\operatorname{Spec} k) & \xrightarrow{y_*} & (\mathrm{Sing}^{\mathbb{A}^1}\mathcal{Y})(\operatorname{Spec} k). \end{array} \tag{122}$$

We note that, on 0-simplices, x induces induces a map of sets $x^*\colon \mathrm{Spc}(k)(X,\mathcal{Y}) \to \mathrm{Spc}(k)(\operatorname{Spec} k,\mathcal{Y})$. Hence, $(\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{Y})(X)_0$ is the set $\mathrm{Spc}_*(k)(X,\mathcal{Y})$ of pointed morphisms $X \to \mathcal{Y}$. On 1-simplices, x induces a map of sets $x^*\colon \mathrm{Spc}(k)(X \times \mathbb{A}^1,\mathcal{Y}) \to \mathrm{Spc}(k)(\mathbb{A}^1,\mathcal{Y})$. Hence, $(\mathrm{Sing}_*^{\mathbb{A}^1}\mathcal{Y})(X)_1$ is the set of pointed naive $\mathbb{A}^1$-homotopies of pointed morphisms $X \to \mathcal{Y}$.

Remark 123 In particular, if $\mathcal{Y} = Y$ is represented by a pointed smooth k-scheme Y, the set $\pi_0((\mathrm{Sing}_*^{\mathbb{A}^1} Y)(X))$ is the set of pointed naive homotopy classes of pointed morphisms $X \to Y$ described in section 2.4, that is,

$$\pi_0((\mathrm{Sing}_*^{\mathbb{A}^1} Y)(X)) = [X, Y]^{\mathrm{N}}.$$

We recall the following definition from [6]:

Definition 124 [6, Definition 2.1.1] Let $\mathcal{F} \in \mathrm{Spc}(k)$ and let $\mathcal{F} \to \widetilde{\mathcal{F}}$ be a fibrant replacement in the $\mathbb{A}^1$-model structure on $\mathrm{Spc}(k)$. There is a canonical map $\mathrm{Sing}^{\mathbb{A}^1} \mathcal{F} \to \widetilde{\mathcal{F}}$ that is well-defined up to simplicial homotopy equivalence. Then $\mathcal{F} \in \mathrm{Spc}(k)$ is called $\mathbb{A}^1$*-naive* if the map $(\mathrm{Sing}^{\mathbb{A}^1} \mathcal{F})(X) \to \widetilde{\mathcal{F}}(X)$ is a weak equivalence of simplicial sets for every affine smooth k-scheme X.

We will now show how the unpointed notion of $\mathbb{A}^1$-naivity of definition 124 translates to the pointed setting.

Proposition 125 Let $\mathcal{Y} \in \mathrm{Spc}_*(k)$ be a pointed motivic space. Assume that the underlying unpointed motivic space $\mathcal{Y}$ is $\mathbb{A}^1$-naive. Then, for every affine pointed smooth k-scheme X, the canonical map $\pi_0((\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)) \xrightarrow{\cong} [X, \mathcal{Y}]^{\mathbb{A}^1}$ is a bijection.

Proof Let (X, x) be a pointed smooth k-scheme, and let $p \colon X \to \mathrm{Spec}\, k$ denote the canonical morphism. Then p induces a map $p^* \colon (\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(\mathrm{Spec}\, k) \to (\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(X)$ of simplicial sets such that $x^* \circ p^*$ is the identity on $(\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(\mathrm{Spec}\, k)$. This shows that the map x^* is a Kan fibration. Since the Kan–Quillen model structure on simplicial sets is right proper, this implies that $(\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X)$ is the homotopy fiber of x^*.

Let $\mathcal{Y} \to R_{\mathbb{A}^1} \mathcal{Y}$ be a fibrant replacement of $\mathcal{Y}$ in the $\mathbb{A}^1$-model structure on $\mathrm{Spc}_*(k)$. After forgetting the basepoint, $R_{\mathbb{A}^1} \mathcal{Y}$ is fibrant in the $\mathbb{A}^1$-model structure on the category $\mathrm{Spc}(k)$ of unpointed motivic spaces. Since the singular functor preserves $\mathbb{A}^1$-fibrations, $\mathrm{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y}$ is fibrant and we may assume $\widetilde{\mathcal{Y}} = \mathrm{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y}$. Moreover, we get the following commutative diagram of simplicial sets which, by the above argument, is a morphism of homotopy fiber sequences for every pointed smooth k-scheme (X, x).

(126)
$$\begin{array}{ccc}
(\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{Y})(X) & \longrightarrow & (\mathrm{Sing}_*^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(X) \\
\downarrow & & \downarrow \\
(\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(X) & \longrightarrow & (\mathrm{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(X) \\
{\scriptstyle x^*}\downarrow & & \downarrow{\scriptstyle x^*} \\
(\mathrm{Sing}^{\mathbb{A}^1} \mathcal{Y})(\mathrm{Spec}\, k) & \longrightarrow & (\mathrm{Sing}^{\mathbb{A}^1} R_{\mathbb{A}^1} \mathcal{Y})(\mathrm{Spec}\, k)
\end{array}$$

Now we assume that the underlying simplicial presheaf of $\mathcal{Y}$ is $\mathbb{A}^1$-naive and that X is affine. Since $\mathcal{Y}$ is $\mathbb{A}^1$-naive and both X and $\operatorname{Spec} k$ are affine, the horizontal maps in the middle and at the bottom are weak equivalences of simplicial sets. Thus, since diagram (126) is a morphism of homotopy fiber sequences, the top horizontal map is a weak equivalence of simplicial sets. Hence it induces a bijection on π_0. Since $\pi_0((\operatorname{Sing}_*^{\mathbb{A}^1} R_{\mathbb{A}^1}\mathcal{Y})(X)) = [X, \mathcal{Y}]^{\mathbb{A}^1}$, this proves the assertion. □

Lemma 127 The smooth k-schemes $\mathcal{J}$ and $\mathbb{P}^1$ are $\mathbb{A}^1$-naive.

Proof Let Q_2 be the smooth affine quadric over $\mathbb{Z}$ defined by $xy = z(z+1)$. By [6, Theorem 4.2.2], Q_2 is $\mathbb{A}^1$-naive. The scheme endomorphism of $\operatorname{Spec} \mathbb{Z}[x,y,z]$ given by the ring homomorphism defined by sending $x \mapsto z$, $y \mapsto -y$, $z \mapsto -x$ induces an isomorphism $Q_2 \cong \mathcal{J}$. Hence $\mathcal{J}$ is $\mathbb{A}^1$-naive. By [6, Lemma 4.2.4] an affine torsor bundle over a base space is $\mathbb{A}^1$-naive if and only if the base space is $\mathbb{A}^1$-naive. Since $\mathcal{J}$ is $\mathbb{A}^1$-naive and an affine torsor bundle over $\mathbb{P}^1$, it follows that $\mathbb{P}^1$ is $\mathbb{A}^1$-naive. □

Proposition 128 The canonical map $\nu\colon [X, \mathbb{P}^1]^{\mathrm{N}} \xrightarrow{\cong} [X, \mathbb{P}^1]^{\mathbb{A}^1}$ is a bijection for every affine pointed smooth k-scheme X.

Proof The proposition follows from remark 123 and proposition 125, since lemma 127 shows $\mathbb{P}^1$ is $\mathbb{A}^1$-naive. □

For $X = \mathcal{J}$, the previous proposition yields the comparison of the sets $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ and $[\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$ of pointed homotopy classes that we wanted.

B Facts about the resultant

In sections 2.3 and 5.1 we used the following facts about the resultant for which we now provide references or proofs.

Throughout this section we let S be an integral domain and let $A = a_nX^n + a_{n-1}X^{n-1} + \ldots + a_0$ and $B = b_nX^n + b_{n-1}X^{n-1} + \ldots + b_0$ be polynomials over S in the indeterminate X.

Lemma 129 (Remark 4 on page IV.76 in [11]) Assume $\operatorname{res}(A,B) \in S^\times$. Then there exist polynomials $U, V \in S[X]$ such that $AU + BV = 1$.

Lemma 130 (Remark 1 on page IV.76 in [11]) Let $\tilde{A} = a_n + a_{n-1}X + \ldots + a_0X^n$ and $\tilde{B} = b_n + b_{n-1}X + \ldots + b_0X^n$ be the reversed polynomials of A and B. If $\mathrm{res}(A,B) \in S^\times$, then $\mathrm{res}(\tilde{A},\tilde{B}) = (-1)^n \mathrm{res}(A,B) \in S^\times$.

Lemma 131 (Remark 5 on page IV.77 in [11]) Assume $\mathrm{res}(A,B) \in S^\times$. Let $C \in S[X]$ be a polynomial such that $\deg(A) \geq \deg(BC)$. Then we have $\mathrm{res}(A+BC,B) = \mathrm{res}(A,B)$.

Lemma 132 Assume $\mathrm{res}(A,B) \in S^\times$ and that A is monic. Then we have

$$\mathrm{res}\left(AX - \frac{1}{u}B, uA\right) = -u\mathrm{res}(A,B) \text{ for all } u \in S^\times.$$

Proof The strategy of the proof is as follows: We determine the Sylvester matrix for the pair $\left(AX - \frac{1}{u}B, uA\right)$ and will then use elementary row and column operations to confirm that it has the determinant claimed.

First note that $\mathrm{res}\left(AX - \frac{1}{u}B, uA\right) = u^{n+1}\mathrm{res}\left(AX - \frac{1}{u}B, A\right)$. Let $A = \sum_{i=0}^n a_iX^i$, and $B = \sum_{i=0}^n b_iX^i$. Let $c_i = a_{i-1} - \frac{1}{u}b_i$ for $0 \leq i \leq n+1$ and set $a_{-1} = b_{n+1} = 0$. Then $AX - \frac{1}{u}B = \sum_{i=0}^{n+1} c_iX^i$. The Sylvester matrix for the pair $\left(AX - \frac{1}{u}B, A\right)$ is

$$\begin{pmatrix}
c_{n+1} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
c_n & c_{n+1} & & \vdots & a_n & 0 & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \\
c_1 & c_2 & \ldots & c_{n+1} & a_1 & \ldots & a_n & 0 \\
c_0 & c_1 & \ldots & c_n & a_0 & \ldots & a_{n-1} & a_n \\
0 & c_0 & \ldots & c_{n-1} & 0 & a_0 & \ldots & a_{n-1} \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \ldots & c_0 & 0 & 0 & \ldots & a_0
\end{pmatrix}.$$

Since $c_{n+1} = 1$, we can remove the first row and first column to obtain a submatrix with the same determinant, namely

$$\begin{pmatrix}
c_{n+1} & 0 & \dots & 0 & a_n & 0 & \dots & 0 & 0 \\
c_n & c_{n+1} & & \vdots & a_{n-1} & a_n & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \\
c_2 & c_3 & \dots & c_{n+1} & a_1 & a_2 & \dots & a_n & 0 \\
c_1 & c_2 & \dots & c_n & a_0 & a_1 & \dots & a_{n-1} & a_n \\
c_0 & c_1 & \dots & c_{n-1} & 0 & a_0 & \dots & a_{n-2} & a_{n-1} \\
\vdots & c_0 & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & c_1 & 0 & 0 & \dots & a_0 & a_1 \\
0 & 0 & \dots & c_0 & 0 & 0 & \dots & 0 & a_0
\end{pmatrix}.$$

Subtracting column 1 from column $n+1$ yields

$$\begin{pmatrix}
c_{n+1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
c_n & c_{n+1} & & \vdots & \frac{1}{u}b_n & a_n & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & & \\
c_2 & c_3 & \dots & c_{n+1} & \frac{1}{u}b_2 & a_2 & \dots & a_n & 0 \\
c_1 & c_2 & \dots & c_n & \frac{1}{u}b_1 & a_1 & \dots & a_{n-1} & a_n \\
c_0 & c_1 & \dots & c_{n-1} & \frac{1}{u}b_0 & a_0 & \dots & a_{n-2} & a_{n-1} \\
\vdots & c_0 & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & c_1 & 0 & 0 & \dots & a_0 & a_1 \\
0 & 0 & \dots & c_0 & 0 & 0 & \dots & 0 & a_0
\end{pmatrix}.$$

Once again, the determinant of this matrix is the same as that of the submatrix where the first row and first and column removed. We remove them and obtain

$$\begin{pmatrix}
c_{n+1} & 0 & \dots & 0 & \frac{1}{u}b_n & a_n & 0 & \dots & 0 & 0 \\
c_n & c_{n+1} & & \vdots & \frac{1}{u}b_{n-1} & a_{n-1} & a_n & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\
c_3 & c_4 & \dots & c_{n+1} & \frac{1}{u}b_2 & a_2 & a_3 & \dots & a_n & 0 \\
c_2 & c_3 & \dots & c_n & \frac{1}{u}b_1 & a_1 & a_2 & \dots & a_{n-1} & a_n \\
c_1 & c_2 & \dots & c_{n-1} & \frac{1}{u}b_0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\
c_0 & c_1 & \dots & c_{n-2} & 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} \\
\vdots & c_0 & \ddots & \vdots & \vdots & & & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & c_1 & 0 & 0 & 0 & \dots & a_0 & a_1 \\
0 & 0 & \dots & c_0 & 0 & 0 & 0 & \dots & 0 & a_0
\end{pmatrix}.$$

We subtract column $n+i$ from column i for each $i<n$, and the result is that at each entry c_i, we get instead $c_i - a_{i-1} = -\frac{1}{u}b_i$.

$$\begin{pmatrix}
-\frac{1}{u}b_{n+1} & 0 & \dots & 0 & \frac{1}{u}b_n & a_n & 0 & \dots & 0 & 0 \\
-\frac{1}{u}b_n & -\frac{1}{u}b_{n+1} & & \vdots & \frac{1}{u}b_{n-1} & a_{n-1} & a_n & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\
-\frac{1}{u}b_3 & -\frac{1}{u}b_4 & \dots & -\frac{1}{u}b_{n+1} & \frac{1}{u}b_2 & a_2 & a_3 & \dots & a_n & 0 \\
-\frac{1}{u}b_2 & -\frac{1}{u}b_3 & \dots & -\frac{1}{u}b_n & \frac{1}{u}b_1 & a_1 & a_2 & \dots & a_{n-1} & a_n \\
-\frac{1}{u}b_1 & -\frac{1}{u}b_2 & \dots & -\frac{1}{u}b_{n-1} & \frac{1}{u}b_0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\
-\frac{1}{u}b_0 & -\frac{1}{u}b_1 & \dots & -\frac{1}{u}b_{n-2} & 0 & 0 & a_0 & \dots & a_{n-3} & a_{n-2} \\
\vdots & -\frac{1}{u}b_0 & \ddots & \vdots & \vdots & & & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & -\frac{1}{u}b_1 & 0 & 0 & 0 & \dots & a_0 & a_1 \\
0 & 0 & \dots & -\frac{1}{u}b_0 & 0 & 0 & 0 & \dots & 0 & a_0
\end{pmatrix}$$

Then multiplying the first n columns by u and applying a cyclic permutation of the n first columns yields the Sylvester matrix of the pair (B,A). The sign of the permutation is $(-1)^{n-1}$. Interchanging column i with $n+i$ for all $i \leq n$ yields (A,B), and this needed another permutation of sign $(-1)^{n-1}$, so the signs cancel out. □

C Testing compatibility via real realization and signatures

Now we provide the additional evidence for conjecture 82 and a positive answer to question 83 referred to in section 4.

We assume that k is a subfield of $\mathbb{R}$. Let $\mathcal{H}_*(k)$ denote the homotopy category of pointed smooth k-schemes and let $\mathcal{H}_*$ be the homotopy category of pointed topological spaces. By [23] sending a smooth k-scheme X to the topological space $X(\mathbb{R})$ equipped with its usual structure of a real manifold extends to a functor $\Re\colon \mathcal{H}_*(k) \to \mathcal{H}_*$, see also [4, page 14] and [14, Section 5.3].

For a smooth map f between oriented compact smooth manifolds of the same dimension, let $\deg^{\mathrm{top}}(f) \in \mathbb{Z}$ denote the topological Brouwer degree of f. In [21] Morel describes the analog of the topological degree map in $\mathbb{A}^1$-homotopy theory. For endomorphisms of $\mathbb{P}^1$ it defines a homomorphism

$$\deg^{\mathbb{A}^1}\colon [\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1} \to \mathrm{GW}(k).$$

Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism. Since we assume that k is a subfield of $\mathbb{R}$, we can form the real realization $\Re(f)\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$. Following Morel, the signature, denoted sgn, of the quadratic form given by the $\mathbb{A}^1$-Brouwer degree of f equals the topological Brouwer degree of $\Re(f)$, i.e.,

$$\operatorname{sgn}\left(\deg^{\mathbb{A}^1}(f)\right) = \deg^{\mathrm{top}}(\Re(f)). \tag{133}$$

We note that, in some form, this was also shown by Eisenbud, Levine, and Teissier in [15, Theorem 1.2] for the local degree of maps between real affine spaces. The latter approach has been incorporated into the motivic theory by Kass and Wickelgren [18].

The motivic Brouwer degree map $\deg^{\mathbb{A}^1}$ is a homomorphism for the conventional group structure $\oplus^{\mathbb{A}^1}$ on $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$, and the signature is additive. Hence, for morphisms $f,g\colon \mathbb{P}^1 \to \mathbb{P}^1$ and their sum $f \oplus^{\mathbb{A}^1} g$ in $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$, (133) implies

$$\begin{aligned}\deg^{\mathrm{top}}\left(\Re\left(f \oplus^{\mathbb{A}^1} g\right)\right) &= \operatorname{sgn}\left(\deg^{\mathbb{A}^1}\left(f \oplus^{\mathbb{A}^1} g\right)\right) \\ &= \deg^{\mathrm{top}}(\Re(f)) + \deg^{\mathrm{top}}(\Re(g)).\end{aligned} \tag{134}$$

We will now use this fact to test the compatibility of the action of definition 73 and thereby of definition 77 with the conventional group structure in the following way.

The real points $\mathcal{J}(\mathbb{R})$ of $\mathcal{J}$ form a surface in $\mathbb{R}^3$ given by the equation $x(1-x)-yz = 0$. The intersection with the plane defined by $y = z$ is the circle given by the set of points satisfying $x(1-x)-y^2 = 0$. Its center is the point $(1/2,0,0) \in \mathbb{R}^3$. We parameterize this circle via the map $\gamma : \mathbb{S}^1 \to \mathcal{J}(\mathbb{R})$ given by

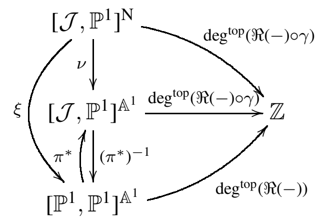
$$\gamma : \theta \mapsto \left(1/2 + \cos(\theta)/2, \sin(\theta)/2, \sin(\theta)/2\right).$$

The real realization of $\mathbb{P}^1$ is the topological real projective line $\mathbb{RP}^1$. Hence, for a morphism $f\colon \mathcal{J} \to \mathbb{P}^1$, we may form the composition $\Re(f)\circ\gamma$ which is a smooth map $\mathbb{S}^1 \to \mathbb{RP}^1$. We can then apply the topological Brouwer degree to the composition $\Re(f)\circ\gamma$. Since the real realization of a naive homotopy induces a homotopy of maps between topological spaces, this induces a well-defined map

$$\begin{aligned}\deg^{\mathrm{top}}(\Re(-)\circ\gamma)\colon [\mathcal{J},\mathbb{P}^1]^{\mathrm{N}} &\longrightarrow \mathbb{Z} \\ f &\longmapsto \deg^{\mathrm{top}}(\Re(f)\circ\gamma).\end{aligned}$$

Lemma 135 The following diagram commutes.

(136)



Proof To prove the assertion it suffices to show that both parts of the diagram commute. The functor $\Re$ commutes with the canonical map $\nu\colon [\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}} \to [\mathcal{J}, \mathbb{P}^1]^{\mathbb{A}^1}$. This implies that the upper part commutes. We verify in example 142 that for $\pi\colon \mathcal{J} \to \mathbb{P}^1$ the composite map $\Re(\pi) \circ \gamma\colon \mathbb{S}^1 \to \mathbb{RP}^1$ is an orientation preserving diffeomorphism. Now let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism. Since the composition with $\Re(\pi) \circ \gamma$ preserves degrees, we obtain the identity

$$\deg^{\mathrm{top}}(\Re(f \circ \pi) \circ \gamma) = \deg^{\mathrm{top}}(\Re(f) \circ \Re(\pi) \circ \gamma) = \deg^{\mathrm{top}}(\Re(f)).$$

This implies that the lower part of diagram (136) commutes and finishes the proof. □

This implies the following necessary condition for the compatibility of the operations $\oplus$ and $\oplus^{\mathbb{A}^1}$:

Proposition 137 Assume that ξ is a group homomorphism. Then we have

$$\deg^{\mathrm{top}}(\Re(\xi(f \oplus g))) = \deg^{\mathrm{top}}(\Re(f) \circ \gamma) + \deg^{\mathrm{top}}(\Re(g) \circ \gamma).$$

Proof The assumption that ξ is a group homomorphism implies

$$\deg^{\mathrm{top}}(\Re(\xi(f \oplus g))) = \deg^{\mathrm{top}}(\Re(\xi(f) \oplus^{\mathbb{A}^1} \xi(g))).$$

Identity (134) implies

$$\deg^{\mathrm{top}}(\Re(\xi(f) \oplus^{\mathbb{A}^1} \xi(g))) = \deg^{\mathrm{top}}(\Re(\xi(f))) + \deg^{\mathrm{top}}(\Re(\xi(g))).$$

Commutativity of diagram (136) implies

$$\deg^{\mathrm{top}}(\Re(\xi(f))) + \deg^{\mathrm{top}}(\Re(\xi(g))) = \deg^{\mathrm{top}}(\Re(f) \circ \gamma) + \deg^{\mathrm{top}}(\Re(g) \circ \gamma).$$

Putting these identities together yields the assertion. □

As a special case, we get the following necessary condition for the compatibility of $\oplus$ with the conventional group structure:

Corollary 138 Let $F\colon \mathcal{J} \to \mathbb{P}^1$ be a pointed morphism. Then, if ξ is a group homomorphism, we must have

$$\deg^{\mathrm{top}}(\Re(F \oplus \pi) \circ \gamma) = \deg^{\mathrm{top}}(\Re(F) \circ \gamma) + 1.$$

In the following section we will apply corollary 138 in a concrete case in example 143.

Moreover, we exclude a potential alternative to the operation $\oplus$ of definition 73 in example 146.

Remark 139 Let $F\colon \mathcal{J} \to \mathbb{P}^1$ again be a pointed morphism. Assume that question 83 has a positive answer, i.e., assume that $\tilde{\pi}$ is naively homotopic to $-\pi$. Then proposition 137 shows that, if ξ is a group homomorphism, then we must expect to get

$$\deg^{\mathrm{top}}(\Re(F \oplus \tilde{\pi}) \circ \gamma) = \deg^{\mathrm{top}}(\Re(F) \circ \gamma) - 1. \tag{140}$$

Note that, since we do not know whether $\tilde{\pi}$ is naively homotopic to $-\pi$, (140) may fail to hold for some F even though ξ is a group homomorphism.

However, we confirm formula (140) in a concrete case in example 145.

We will now compute the topological degrees and thereby the signatures of several maps and apply the previous observations.

Example 141 Consider the morphism $g_{1,-1} : \mathcal{J} \to \mathbb{P}^1$ defined by the unimodular row $(2x-1, 2y)$. Its real realization is the map $\Re(g_{1,-1}) : \Re(\mathcal{J}) \to \Re(\mathbb{P}^1)$ defined by

$$\Re(g_{1,-1}) : (x,y,z) \mapsto [2x-1 : 2y].$$

Precomposing with γ gives

$$\Re(g_{1,-1}) \circ \gamma : \theta \mapsto [\cos(\theta) : \sin(\theta)]\,,$$

which is the usual double cover of $\mathbb{RP}^1$ by $\mathbb{S}^1$ and has topological Brouwer degree 2.

As explained in section 2.3, a morphism $f : \mathcal{J} \to \mathbb{P}^1$ may be described by gluing together partially defined maps on open subsets. In the following examples we will define a morphism $\Re(f) \circ \gamma : \mathbb{S}^1 \to \mathbb{RP}^1$ by gluing $\Re(f|_{D(x)}) \circ \gamma \colon \gamma^{-1}(\Re(D(x))) \to \mathbb{RP}^1$ and $\Re(f|_{D(1-x)}) \circ \gamma \colon \gamma^{-1}(\Re(D(1-x))) \to \mathbb{RP}^1$ on their overlaps in the respective domains.

Example 142 The real realization of $\pi : \mathcal{J} \to \mathbb{P}^1$ is defined on $\Re(D(x))$ by

$$\Re(\pi|_{D(x)}) : (x,y,z) \mapsto [x : y],$$

and on $\Re(D(1-x))$ by

$$\Re(\pi|_{D(1-x)}) : (x, y, z) \mapsto [z : 1-x].$$

Precomposing with γ gives

$$\Re(\pi|_{D(x)}) \circ \gamma : \theta \mapsto [1/2 + \cos(\theta)/2 : \sin(\theta)/2],$$
$$\Re(\pi|_{D(1-x)}) \circ \gamma : \theta \mapsto [\sin(\theta)/2 : 1/2 - \cos(\theta)/2],$$

which glue together to give a map of degree 1:

$$(\Re(\pi) \circ \gamma)(\theta) = \begin{cases} [1 + \cos(\theta) : \sin(\theta)] & \theta \neq \pi, \\ [0 : 1] & \theta = \pi. \end{cases}$$

This shows that $\Re(\pi) \circ \gamma$ is an orientation preserving diffeomorphism and has topological Brouwer degree 1.

In the following example we test the necessary condition of corollary 138 in a concrete case.

Example 143 Recall that the unimodular row $g_{1,-1} = (2x-1, 2y)$ can be augmented to the following matrix with determinant 1:

$$m_{1,-1} = \begin{pmatrix} 2x-1 & -2z \\ 2y & 2x-1 \end{pmatrix}.$$

The group action of definition 73 yields the map $F := g_{1,-1} \oplus \pi = (2x-1, -2z : 2y, 2x-1)_1$.

Taking real realization and precomposing with γ yields the map the map $\Re(F) \circ \gamma \colon \mathbb{S}^1 \to \mathbb{RP}^1$ given by

$$(\Re(F) \circ \gamma)(\theta) = \begin{cases} [\cos(\theta) + \cos(2\theta) : \sin(\theta) + \sin(2\theta)] & \theta \neq \pi, \\ [0 : 1] & \theta = \pi. \end{cases}$$

The topological degree of this map is 3.

Hence our computation confirms that

$$\deg^{\text{top}}(\Re(F) \circ \gamma) = 3 = 2 + 1 = \deg^{\text{top}}(\Re(g_{-1,1}) \circ \gamma) + \deg^{\text{top}}(\Re(\pi) \circ \gamma),$$

as required for the compatibility of $\oplus$ with $\oplus^{\mathbb{A}^1}$.

The next example confirms that the signature of the motivic Brouwer degree of $\tilde{\pi}$ has the value -1 as expected if question 83 has a positive answer.

Example 144 The real realization of the morphism $\tilde{\pi} = (1, 0 : 0, -1)_{-1} : \mathcal{J} \to \mathbb{P}^1$ is defined on $\Re(D(x))$ and $\Re(D(1-x))$ respectively by $\Re(D(x))$ by

$$\Re(\tilde{\pi}|_{D(x)}) : (x, y, z) \mapsto [x : -z],$$
$$\Re(\tilde{\pi}|_{D(1-x)}) : (x, y, z) \mapsto [y : -1 + x].$$

Precomposing with γ gives

$$\Re(\tilde{\pi}|_{D(x)}) \circ \gamma : \theta \mapsto [1/2 + \cos(\theta)/2 : -\sin(\theta)/2],$$
$$\Re(\tilde{\pi}|_{D(1-x)}) \circ \gamma : \theta \mapsto [\sin(\theta)/2 : -1/2 + \cos(\theta)/2],$$

which glue together to give a map of topological Brouwer degree -1.

Now we confirm that identity (140) of remark 139 does hold in an example.

Example 145 Consider the unimodular row $F = (2x-1, 2z)$ which can be augmented to the following matrix

$$M = \begin{pmatrix} 2x-1 & -2y \\ 2z & 2x-1 \end{pmatrix}$$

with determinant 1. Note that F is homotopic to $g_{1,-1}$ by lemma 62. We let F act on $\tilde{\pi}$ via the action of definition 73. This yields the map $L := F \oplus \tilde{\pi} = (2x-1, 2y : 2z, -2x+1)_{-1}$. Precomposing its real realization with γ yields the same map as in example 142 where we showed it has topological Brouwer degree 1.

Hence our computation confirms

$$\deg^{\mathrm{top}}(\Re(L) \circ \gamma) = 1 = 2 - 1 = \deg^{\mathrm{top}}(\Re(F) \circ \gamma) + \deg^{\mathrm{top}}(\Re(\tilde{\pi}) \circ \gamma),$$

as required in remark 139.

Example 146 Consider now an alternative action $\boxplus$ of $[\mathcal{J}, \mathbb{P}^1]_0^{\mathrm{N}}$ on $[\mathcal{J}, \mathbb{P}^1]^{\mathrm{N}}$ defined as follows. For $[(A, B)] \in [\mathcal{J}, \mathbb{A}^2 \setminus \{0\}]^{\mathrm{N}}$ and $[s_0, s_1] \in [\mathcal{J}, \mathbb{P}^1]_n^{\mathrm{N}}$, extend the unimodular row (A, B) to a matrix M in $\mathrm{SL}_2(R)$ and define

$$[(A, B)] \boxplus [s_0, s_1] := [M^T \cdot (s_0, s_1)^T].$$

Again we look at the unimodular row $F = (A, B) = (2x-1, 2z)$ and the matrix M of example 145. The action $\boxplus$ of F on π yields the morphism $H := F \boxplus \pi = (2x-1, 2z : -2y, 2x-1)_1 = (1, 0 : 0, -1)_1$. Taking real realization and precomposing with γ yields the map $\Re(H) \circ \gamma \colon \mathbb{S}^1 \to \mathbb{RP}^1$ given by

$$(\Re(H) \circ \gamma)(\theta) = \begin{cases} [1 + \cos(\theta) : -\sin(\theta)] & \theta \neq \pi, \\ [0 : 1] & \theta = \pi. \end{cases}$$

This map has topological Brouwer degree -1. Hence our computation shows

$$\deg^{\text{top}}(\Re(F \boxplus \pi) \circ \gamma) = -1 \neq 3 = \deg^{\text{top}}(\Re(F) \circ \gamma) + \deg^{\text{top}}(\Re(\pi) \circ \gamma).$$

Thus, by the analogous statement of corollary 138 for $\boxplus$, we see that $\boxplus$ cannot be used to define an operation compatible with the conventional group structure.

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Paper II

Polyhedral products in abstract and motivic homotopy theory

William Hornslien *

Abstract

We introduce polyhedral products in an ∞-categorical setting. We generalize a splitting result by Bahri, Bendersky, Cohen, and Gitler that determines the stable homotopy type of the a polyhedral product. We also introduce a motivic refinement of moment-angle complexes and use the splitting result to compute cellular $\mathbb{A}^1$-homology, and $\mathbb{A}^1$-Euler characteristics.

1 Introduction

Toric geometry is an important part of algebraic geometry. Since its inception in the 70s, it has grown substantially and proven its usefulness in other fields such as combinatorics, commutative algebra, and algebraic statistics. Toric geometry is the study of a certain class of algebraic varieties called toric varieties. These algebraic varieties are defined particularly nice and combinatorially, which makes it easier to do computations and prove theorems. In algebraic geometry, toric varieties are great for testing theories before proving results for larger classes of algebraic varieties. In the past decades, ways of studying toric geometry through the lens of topology have been developed. One way is by using methods from the field of toric topology, which in short, looks at the real and complex points of toric varieties as manifolds and studies their topological properties. Toric topology only considers toric varieties over the real and complex numbers, but what about different bases? Morel and Voevodsky's motivic homotopy category has made it possible to do homotopy theory with smooth algebraic varieties over any base field. In this paper we unite the two topological viewpoints and use methods from toric topology in motivic homotopy theory to study toric geometry over any field.

The field of toric topology started with work by Davis and Januszkiewicz [15]. They wanted to study a family of manifolds called quasi-toric manifolds. The quasi-toric manifolds were defined by simple polytopes and were homotopic to the complex points of a smooth projective toric varieties. Along with each quasi-toric manifold, they defined an auxiliary space, which we will call Z_K, and showed that the quasi-toric manifold could be realized as orbit space of a torus (a product of n circles) acting on Z_K. This is a

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topological version of what is known as Cox's construction in algebraic geometry [13]. The space Z_K turned out to be an interesting object in its own right, and was named the moment-angle complex by Buchstaber and Panov [9]. They found out that Z_K could be described as a union of disks and circles according to the combinatorial data of a simplicial complex (see Section 2.2 and 2.3). The quasi-toric manifolds were quotients of moment-angle complexes associated to the case where the simplicial complex was a triangulation of a sphere. In [6] Bahri, Bendersky, Cohen, and Gitler introduce what they call polyhedral products $(X, A)^K$ for a pair of spaces $A \subset X$ (see Section 2.2 for a precise definition). Moment-angle complexes were unions of disks and circles and coincided with the polyhedral product $(D^2, S^1)^K$. There is also a closely related *real moment-angle complex* $\mathbb{R}Z_K$, which is the polyhedral product $(D^1, S^0)^K$. Results dating all the way back to the 60s, such as a paper by Porter [36] could be put into the framework of polyhedral products. Depending on the choice of pairs of spaces $A \subset X$, polyhedral products have connections to surprisingly many fields such as commutative algebra, geometric group theory, and robotics. A survey covering the vast applications and properties of polyhedral products can be found in [5].

Motivic homotopy theory takes place in Morel and Voevodsky's category of motivic spaces $\mathcal{H}(k)$ over a field k [35]. The category of motivic spaces is a homotopy theory for smooth schemes (algebraic varieties) over a field. See Section 5.3 for a more in-depth explanation. In motivic homotopy theory, the affine line $\mathbb{A}^1$ takes the role of the interval in classical homotopy theory, and is in particular contractible in $\mathcal{H}(k)$. Motivic homotopy theory allows for using methods from algebraic topology for algebraic varieties, but it usually comes with some complications. One disadvantage with motivic homotopy theory is that it is in general difficult to do explicit computations. For example, the motivic cohomology of a point is only known in special cases [38].

We introduce a motivic moment-angle complex $Z_K^{\mathbb{A}^1}$, which is a motivic refinement of its classical counterpart. The motivic moment-angle complex is roughly speaking a union of $\mathbb{A}^1$'s and $\mathbb{G}_{\mathrm{m}}$'s, which act like disks and circles in $\mathcal{H}(k)$. One could define polyhedral products directly in $\mathcal{H}(k)$, but we have chosen to generalize the construction to an abstract homotopical setting. That is, in Definition 3.7 we define polyhedral products in a cartesian closed ∞-category with small colimits in the sense of Lurie [30]. In the ∞-category of topological spaces, this definition recovers the original definition. We then prove an abstract homotopical version of splitting result of Bahri, Bendersky, Cohen, and Gitler [6] concerning how certain polyhedral products split into a wedge of simpler pieces after a suspension (see Theorem 4.5). Before stating our results, we will need to introduce some notation. Fix a cartesian closed ∞-category $\mathcal{C}$ with small colimits and let K be a simplicial complex. We write $|K|$ for the geometric realization K in $\mathcal{C}$ (see Definition 4.1 and 4.3). By $I \notin K$, we mean a set I of vertices in K such that they do not span a face in K. For a set of vertices I, we let K_I be the full subcomplex of K corresponding to the vertex set I. The cardinality of the set I is denoted by $|I|$. We denote suspension functor by Σ, and the smash product of two pointed objects X, Y in $\mathcal{C}$ by $X \wedge Y$. The following theorem is a special case of Theorem 4.5.

Theorem A. *Let $\mathcal{C}$ be a cartesian closed ∞-category with small colimits and fix a* *morphism $i\colon A \to X$ of pointed objects where X is contractible. Let K be a simplicial complex. Then there is an equivalence*

$$\Sigma(X,A)^K \simeq \Sigma^2 \bigvee_{I \notin K} |K_I| \wedge A^{\wedge |I|}.$$

The statement above for topological spaces was applied in [6] to describe the stable homotopy type of moment-angle complexes.

The motivic moment-angle complex is defined as the polyhedral product $(\mathbb{A}^1, \mathbb{G}_{\mathrm{m}})^K$, and since $\mathbb{A}^1$ is contractible in $\mathcal{H}(k)$, we can apply Theorem A to $Z_K^{\mathbb{A}^1}$.

Theorem B (Theorem 5.3)**.** *Let K be a simplicial complex. Then there is an equivalence in $\mathcal{H}(k)$*

$$\Sigma Z_K^{\mathbb{A}^1} \simeq \Sigma^2 \left(\bigvee_{I \notin K} |K_I| \wedge \mathbb{G}_{\mathrm{m}}^{\wedge |I|} \right) \simeq \bigvee_{I \notin K} |K_I| \wedge S^{|I|+2,|I|}.$$

We are now able to compute various motivic invariants of $Z_K^{\mathbb{A}^1}$. In [34] Morel and Sawant introduced cellular $\mathbb{A}^1$-homology which is valued in strictly $\mathbb{A}^1$-invariant sheaves of abelian groups. They compute the cellular $\mathbb{A}^1$-homology of punctured affine spaces, projective spaces, and also low dimensional homology groups of some flag varieties. We use the stable splitting to describe the cellular $\mathbb{A}^1$-homology of $Z_K^{\mathbb{A}^1}$ in terms of K. Let $\mathbf{K}_i^{\mathrm{MW}}$ denote the ith unramified Milnor–Witt K-theory sheaf. Let $\widetilde{\mathbf{H}}_i(|K|)$ be the ith reduced singular homology singular homology group of $|K|$ viewed as a sheaf of abelian groups.

Theorem C (Theorem 7.11)**.** *Let K be a simplicial complex. Then $\mathbf{H}_0^{\mathrm{cell}}(Z_K^{\mathbb{A}^1}) = \mathbb{Z}$ and for $i > 0$*

$$\mathbf{H}_i^{\mathrm{cell}}(Z_K^{\mathbb{A}^1}) \cong \bigoplus_{I \notin K} \widetilde{\mathbf{H}}_{i-1}(|K_I|) \otimes \mathbf{K}_{|I|}^{\mathrm{MW}}.$$

To prove the theorem above, one actually needs to apply Theorem A to the derived category of chain complexes of $\mathbb{A}^1$-invariant sheaves. Note that this theorem also gives examples of varieties with integral torsion in their cellular $\mathbb{A}^1$-homology groups, e.g. when K is a triangulation of $\mathbb{RP}^2$ (see Example 7.13).

There is a motivic version of the Euler characteristic called the $\mathbb{A}^1$-Euler characteristic. By using the stable splitting we compute the $\mathbb{A}^1$-Euler characteristic for $Z_K^{\mathbb{A}^1}$ in terms of K. The $\mathbb{A}^1$-Euler characteristic is valued in $\mathrm{GW}(k)$ i.e. the Grothendieck–Witt ring of quadratic forms over k.

Theorem D (Theorem 7.25)**.** *The $\mathbb{A}^1$-Euler characteristic of the motivic moment-angle complex is*

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \langle 1 \rangle - \sum_{I \notin K} (-1)^{|I|} (\chi(K_I) - 1) \cdot \langle -1 \rangle^{|I|}.$$

The theorem above allows us to describe the $\mathbb{A}^1$-Euler characteristic of $Z_K^{\mathbb{A}^1}$ in terms of the topological Euler characteristic of full subcomplexes of K. In classical topology, knowing the (co)homology of a space is sufficient for determining the Euler characteristic, but this is not the case in motivic homotopy theory. When we view $\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1})$ as a quadratic form over $\mathbb{R}$, the rank of $\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1})$ is $\chi(Z_K)$ and the signature of $\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1})$ is $\chi(\mathbb{R}Z_K)$. This result examplifies why $Z_K^{\mathbb{A}^1}$ is a motivic refinement of Z_K and $\mathbb{R}Z_K$.

The paper is structured as follows. In Section 2 the classical definition of a polyhedral product and various classical results are recalled. In Section 3, we define polyhedral products in ∞-categories. Theorem A is then proven in Section 4. In Section 5, we review polyhedral products in equivariant and motivic homotopy theory. We also define motivic moment-angle complexes. In Section 6, we give various smooth models of motivic moment-angle complexes, and briefly study their connection to toric varieties. Section 7 is dedicated to computing various invariants of the motivic moment-angle complexes.

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Conventions

By a topological space we mean a CW-complex. Starting in Section 2, we will use the language of ∞-categories as developed by Lurie in [30]. By ∞-category, we mean an $(\infty, 1)$-category. When necessary, we will implicitly view 1-categories as ∞-categories through the nerve embedding. From Section 5, we will assume k to be a perfect field of characteristic not equal to 2.

2 Polyhedral products

In this section we give a brief overview of some core properties of the polyhedral product functor and review some classical results.

2.1 Preliminaries on simplicial complexes

Definition 2.1. Let m be a positive integer. An abstract simplicial complex K is a family of subsets of $[m] := \{1, \ldots, m\}$ that is closed under taking subsets. In more geometric terms, K is a simplicial complex with m vertices labeled by the set $[m] = \{1, \ldots, m\}$. A $(n-1)$-face σ of K is given by a subset $\sigma = \{i_1, \ldots, i_n\}$ with $1 \leq i_1 < \ldots < i_n \leq m$. All subsets $\tau \subset \sigma$ define faces in K as well. In particular, K includes the empty face $\emptyset$.

Whenever we say simplicial complex, we mean an abstract simplicial complex. We will now define an important family of subcomplexes.

Definition 2.2. Let K be a simplicial complex and $I \subset [m]$. The full subcomplex K_I consists of all faces of K that have their vertex set as a subset of I, i.e.

$$K_I := \{\sigma \cap I | \sigma \in K\}.$$

Simplicial complexes can be seen as topological spaces, this is done by geometric realization.

Definition 2.3. Denote the geometric realization of K as a topological space by $|K|$.

A simplicial complex comes with the natural structure of a poset. That is, the face poset ordered by inclusion. Each face represents an object and if σ is a subface of τ, then $\sigma < \tau$ in the poset. To this poset is an associated category, which will be essential going forward.

Definition 2.4. Let K be a simplicial complex. The *face poset category* $\mathcal{K}$ is defined as follows. The objects of $\mathcal{K}$ are given by the simplices of K, including an initial object $\emptyset$ which corresponds to the empty face. Let $\sigma, \tau \in K$ be two simplices of K. If σ is a subface of τ, then there is a unique morphism $f_{\sigma \leq \tau} \colon \sigma \to \tau$. Let $I \subset [m]$, we denote the face poset category of K_I by $\mathcal{K}_I$.

The following two constructions are central in combinatorics.

Definition 2.5. Let $\mathbf{k}$ be a ring. For a simplicial complex K, we define the *Stanley–Reisner ideal* I_K as the square-free monomial ideal corresponding to non-faces of K, i.e.

$$I_K = (x_{i_1} \dots x_{i_r} | \{i_1, \dots, i_r\} \notin K).$$

We define the *Stanley–Reisner ring* as the quotient

$$\mathbf{k}[K] := \mathbf{k}[x_1, \dots, x_m]/I_K.$$

The Stanley–Reisner ring has connections to fields such as toric geometry, polytopes, and splines [37, Chapter III].

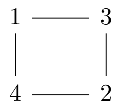
Definition 2.6. To a simplicial complex K, the *Alexander dual* $K^\vee$ is the simplicial complex whose faces are complements of non-faces of K, i.e.

$$K^\vee := \{\sigma \in [m] | [m] \setminus \sigma \notin K\}.$$

See [10, Example 2.26 and Corollary 2.28] for further results on Alexander duality of simplicial complexes.

Example 2.7. Let K be the boundary of a square, i.e.

$$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{2,3\}, \{2,4\}, \{1,4\}\}.$$

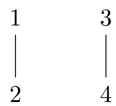


Then

$$\mathbf{k}[K] = \mathbf{k}[x_1, x_2, x_3, x_4]/(x_1x_2, x_3x_4)$$

and the Alexander dual is the simplicial complex

$$K^\vee = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{3,4\}\}.$$



Note the swap of position of vertices 2 and 4 in the picture.

2.2 The classical construction

Polyhedral products were first defined by Bahri, Bendersky, Cohen, and Gitler in [6]. A good survey of the work done and its connections to other fields can be found in [5]. Let

$$(\underline{X}, \underline{A}) = ((X_1, A_1), \ldots, (X_m, A_m))$$

be a sequence of m pairs of pointed topological spaces $A_i \subset X_i$. Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$. The polyhedral product $(\underline{X}, \underline{A})^K$ is defined as the following union

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} D(\sigma) \subset \prod_{i=1}^{m} X_i,$$

where

$$D(\sigma) = \prod_{i=1}^{m} Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Remark 2.8. When all pairs (X_i, A_i) are the same pair and K is a simplicial complex, we will write $(X, A)^K$ for the associated polyhedral product.

A wide array of spaces can be created as polyhedral products.

Example 2.9. Let K be a simplicial complex with two disjoint vertices.

1. $(D^1, S^0)^K = D^1 \times S^0 \cup S^0 \times D^1 \simeq S^1$

2. $(D^2, S^1)^K = D^2 \times S^1 \cup S^1 \times D^2 \simeq S^3$

3. $(S^1, *)^K = S^1 \times * \cup * \times S^1 \simeq S^1 \vee S^1$

Example 2.10. Let K be a disjoint union of m points. Then there is a homotopy equivalence $(\underline{X}, *)^K = X_1 \vee \ldots \vee X_m$.

In this case, one can observe how $(\underline{X}, *)^K$ interpolates between $X_1 \times \ldots \times X_m$ and $X_1 \vee \ldots \vee X_m$ when K ranges from a full $(m-1)$-simplex to m disjoint points. The following proposition tells us how certain operations with simplicial complexes affect the polyhedral products.

Proposition 2.11 ([11, Proposition 4.2.5])**.** *Let K and K' be two simplicial complexes and let $K \star K'$ denote their join. Then $(X, A)^K \times (X, A)^{K'} \simeq (X, A)^{K \star K'}$.*

Example 2.12. Let $K_1 = \{\emptyset, \{1\}, \{2\}\}$, $K_2 = \{\emptyset, \{3\}, \{4\}\}$ be simplicial complexes. In this case K_1 and K_2 are complexes consisting of two disjoint points. Let $K = K_1 \star K_2$, then K is a complex shaped like the boundary of a square as in Example 2.7.

1. $(D^1, S^0)^K \simeq S^1 \times S^1$

2. $(D^2, S^1)^K \simeq S^3 \times S^3$

Example 2.13. Let $K = \partial \Delta^n$.

1. $(D^1, S^0)^K \simeq S^n$

2. $(D^2, S^1)^K \simeq S^{2n+1}$

Example 2.14 ([20, Corollary 9.7])**.** Let K be the disjoint union of m points. Then there is a homotopy equivalence $(D^2, S^1)^K \simeq \bigvee_{l=2}^{m} (S^{l+1})^{\vee (l-1)\binom{m}{l}}$.

The following two results are due to Bahri, Bendersky, Cohen, and Gitler. In particular, they show how certain polyhedral products split into a wedge of nice pieces after suspending. Let Σ denote the reduced suspension functor. When $I = \{i_1, \ldots, i_n\} \subset [m]$ and $Y_1, \ldots, Y_m$ are pointed topological spaces, we will write $\widehat{Y}^I := Y_{i_1} \wedge \ldots \wedge Y_{i_k}$.

Theorem 2.15 ([6, Theorem 2.15])**.** *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be a family of pairs. If A_i is contractible for each $1 \leq i \leq m$, then*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \bigvee_{I \in K} \widehat{X}^I.$$

There is also a version of the splitting when all the X_i's are contractible.

Theorem 2.16 ([6, Theorem 2.21])**.** *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be a family of pairs. If X_i is contractible for each $1 \leq i \leq m$, then*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \bigvee_{I \notin K} |K_I| \star \widehat{A}^I.$$

In particular, the two splitting results above are special cases of a theorem that requires the map $A_i \to X_i$ to be null-homotopic for all i [6, Theorem 2.13].

In [14] Davis finds a general formula for the Euler characteristic of the polyhedral product.

Theorem 2.17 ([14]). *Let K be a simplicial complex on the vertex set $[m]$ and $A \subset X$ be two finite CW-complexes. Then*

$$\chi((X,A)^K) = \sum_{\sigma\in K} (\chi(X) - \chi(A))^{|\sigma|}\chi(A)^{m-|\sigma|}.$$

Example 2.18. Let K be a simplicial complex and X a finite CW-complex.

1. $\chi((D^2, S^1)^K) = 0$
2. $\chi((D^1, S^0)^K) = \Sigma_{\sigma\in K}(-1)^{|\sigma|} \cdot 2^{m-|\sigma|}$
3. $\chi((X, *)^K) = \Sigma_{\sigma\in K}(\chi(X) - 1)$

2.3 Moment-angle complexes

A particularly well studied family of polyhedral products are the *moment-angle complexes*. For a simplicial complex K, the associated moment-angle complex is defined as the polyhedral product $Z_K := (D^2, S^1)^K$. In fact, moment-angle complexes were studied long before polyhedral products were defined. The original construction goes back to Davis and Januskiewicz in [15, §4.1], but only for a family of moment-angle complexes known as moment-angle manifolds. However, the definition that inspired polyhedral products, i.e. moment-angle complexes viewed as a union of products, is due to Buchstaber and Panov [10, Definition 6.10]. Since Buchstaber and Panov's reinterpretation of moment-angle complexes, a lot of progress has been made. We will now present some results that both will be of use later, and emphasise how moment-angle complexes are studied.

Since D^2 is contractible, we can use Theorem 2.16 to describe ΣZ_K. There is a homotopy equivalence

$$\Sigma Z_K \simeq \bigvee_{I\notin K} \Sigma^{|I|+2}|K_I|.$$

This decomposition makes it possible to describe the homology of Z_K in terms of full subcomplexes of K for any homology theory. In particular, it describes the singular cohomology groups of Z_K. The cohomological ring structure can also be described. In [10], the cohomology of Z_K is computed using the Eilenberg–Moore spectral sequence. For a ring $\mathbf{k}$, recall that $\mathbf{k}[K]$ is the Stanley–Reisner ring as defined in Definition 2.5.

Theorem 2.19 ([10, Theorem 7.6]). *Let K be a simplicial complex on the vertex set $[m]$. Let $\mathbf{k}$ be a field or $\mathbb{Z}$. There is an isomorphism of algebras*

$$H^{2j-i}(Z_K; \mathbf{k}) \cong \mathrm{Tor}^{-i,2j}_{\mathbf{k}[x_1,\dots,x_m]}(\mathbf{k}, \mathbf{k}[K]).$$

The isomorphism of Theorem 2.19 gives a natural bigrading on the cohomology ring of Z_K. The wedge decomposition of ΣZ_K splits the cohomology of Z_K into nice subgroups given by the cohomology of full subcomplexes of K. A result by Baskakov makes it possible to describe the ring structure with respect to the full subcomplexes K_I.

Theorem 2.20 ([7, Theorem 1], [8, Theorem 1])**.** *Let K be a simplicial complex on m vertices. Let $\mathbf{k}$ be a field or $\mathbb{Z}$. There is an isomorphism of groups*

$$H^i(Z_K;\mathbf{k}) \cong \begin{cases} \mathbf{k} & i=0, \\ \bigoplus_{I \notin K} \widetilde{H}^{i-|I|-1}(K_I;\mathbf{k}) & i>0. \end{cases}$$

In particular, there is an isomorphism of algebras

$$H^*(Z_K;\mathbf{k}) \cong \mathbf{k} \oplus \bigoplus_{I \notin K} \widetilde{H}^*(K_I;\mathbf{k}).$$

The products in the sum on the right are given as follows: for $I, J \notin K$, with $I \cap J = \emptyset$, let $\alpha \in \widetilde{H}^p(K_I;\mathbf{k})$ and $\beta \in \widetilde{H}^q(K_J;\mathbf{k})$ be nontrivial cohomology classes. Then there exists a nontrivial cohomology class $\gamma \in \widetilde{H}^{p+q}(K_{I\cup J};\mathbf{k})$ such that $\alpha \smile \beta = \gamma$. All products of cohomology classes in $H^*(Z_K;\mathbf{k})$ arise in this way.

It is also possible to describe Massey products [21] and Steenrod operations [1] in terms of full subcomplexes of K.

Moment-angle complexes also share a deep connection to toric geometry, since the complex points of any smooth projective toric variety can be realized as the orbits of a torus acting on the polyhedral product $(\mathbb{C},\mathbb{C}^\times)^K$. Note that we have a homotopy equivalence $Z_K \simeq (\mathbb{C},\mathbb{C}^\times)^K$. This is a topological version of something known as Cox construction [13, Theorem 2.1] of smooth projective algebraic varieties in algebraic geometry. See [11, §5.4] for a description of the Cox construction from a toric topological viewpoint.

There is also related polyhedral product called the *real moment-angle complexes* defined as $\mathbb{R}Z_K := (D^1, S^0)^K$. The real moment-angle complex can be seen as the fixed points of a C_2-action on Z_K. Theorem 2.16 also makes it possible to describe $\mathbb{R}Z_K$,

$$\Sigma\mathbb{R}Z_K \simeq \bigvee_{I \notin K} \Sigma^2|K_I|.$$

This makes it straightforward to describe the cohomology groups of $\mathbb{R}Z_K$. The ring structure has been computed [12], but is far more complicated than the case of the moment-angle complex.

3 ∞-categorical setup

We will now consider an ∞-categorical version of polyhedral products. From now on we freely use the language of ∞-categories as developed by Lurie in [30]. Let $\mathcal{C}$ be a cartesian closed ∞-category with finite colimits. That is, for each $X \in \mathcal{C}$ the product functor $X \times -$ has a right adjoint, and thus preserves all colimits. Denote the terminal object of $\mathcal{C}$ by $*$. A pointed object in $\mathcal{C}$ is an object X together with a map $x\colon * \to X$. From now on, when we talk about colimits, we mean in the ∞-categorical sense. Whenever $\mathcal{C}$ is the nerve of a model category, computing the homotopy colimit in the underlying model category suffices.

Definition 3.1. The *suspension* of an object $X \in \mathcal{C}$ is the pushout

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \Sigma X. \end{array}$$

Definition 3.2. The *wedge* of two pointed objects $X, Y \in \mathcal{C}$ is the pushout

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & X \vee Y. \end{array}$$

Definition 3.3. The *smash product* of two pointed objects $X, Y \in \mathcal{C}$ is the pushout

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & X \wedge Y. \end{array}$$

Definition 3.4. The *join* of two pointed objects $X, Y \in \mathcal{C}$ is the pushout

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & X \star Y. \end{array}$$

Lemma 3.5 ([26, Lemma 3.5])**.** *Let $X, Y \in C$ be pointed. There is an equivalence*

$$X \star Y \simeq \Sigma X \wedge Y.$$

Definition 3.6. Let $\mathcal{C}$ be an ∞-category and let

$$(\underline{X}, \underline{A}) = ((X_1, A_1), \ldots, (X_m, A_m))$$

be a sequence of pairs of pointed objects in $\mathcal{C}$ equipped with a map $\iota_i\colon A_i \to X_i$. We call $(\underline{X}, \underline{A})$ a *family of pairs.*

Definition 3.7. Let $(\underline{X}, \underline{A})$ be a family of pairs and let K be a simplicial complex on the vertex set $[m]$. Let $\mathcal{K}$ be the face poset category of K ordered by inclusions, that is $\sigma > \tau$ if $\sigma \subsetneq \tau$. We define the polyhedral product $(\underline{X}, \underline{A})^K$ as

$$(\underline{X}, \underline{A})^K := \operatorname*{colim}_{\sigma \in K} D(\sigma),$$

with $D(\sigma)$ defined as follows:

$$D(\sigma) = \prod_{i=1}^{m} Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

For any pair of simplices $\sigma \subset \tau \in K$ the map from $D(\sigma)$ to $D(\tau)$ is induced by the products of the maps ι_i and the identity. In other words, $(\underline{X}, \underline{A})^K$ is the colimit of the diagram

$$D \colon \mathcal{K} \to \mathcal{C}.$$

Example 3.8.
1. Suppose that each A_i is the terminal object $*$. If K is the disjoint union of m points then $(\underline{X}, \underline{*})^K$ is $X_1 \vee X_2 \vee \ldots \vee X_m$.
2. Suppose that K is the $(m-1)$-simplex then $(\underline{X}, \underline{A})^K$ is $X_1 \times X_2 \times \ldots \times X_m$.
3. Suppose that K is the complex of two disjoint vertices and that each $X_i \simeq *$ then $(\underline{X}, \underline{A})^K \simeq \Sigma A_1 \wedge A_2 \simeq A_1 \star A_2$, the join of A_1 and A_2.

The following proposition is an ∞-version of Proposition 3.9 does.

Proposition 3.9. *Suppose K and K' are two simplicial complexes, and denote their join by $K \star K'$, then $(\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^{K'} = (\underline{X}, \underline{A})^{K \star K'}$.*

Proof. Since $\mathcal{C}$ is cartesian closed, cartesian products preserve colimits. Thus there is there is a chain of equivalences

$$(\underline{X}, \underline{A})^K \times (\underline{X}, \underline{A})^{K'} \simeq \operatorname*{colim}_{\sigma' \in K'} \left((\underline{X}, \underline{A})^K \times D(\sigma')\right) \simeq \operatorname*{colim}_{\sigma' \in K'} \left(\operatorname*{colim}_{\sigma \in K} \left(D(\sigma) \times D(\sigma')\right) \right).$$

The iterated colimits can be rewritten as one colimit iterating over $\sigma \in K$ and $\sigma' \in K'$. This yields the equivalences

$$\operatorname*{colim}_{\sigma' \in K'} \left(\operatorname*{colim}_{\sigma \in K} \left(D(\sigma) \times D(\sigma')\right) \right) \simeq \operatorname*{colim}_{\sigma \in K, \sigma' \in K'} \left(D(\sigma) \times D(\sigma')\right) \simeq (\underline{X}, \underline{A})^{K * K'}. \qquad \square$$

We will also need a space called the polyhedral smash product. It was first defined for topological spaces in [6].

Definition 3.10. Let $(\underline{X}, \underline{A})$ be a family of pairs and let K be a simplicial complex on the vertex set $[m]$. Let $\mathcal{K}$ be the face poset category of K ordered by inclusions. We define the *polyhedral smash product* $\widehat{(\underline{X}, \underline{A})}^K$ as

$$\widehat{(\underline{X}, \underline{A})}^K := \operatorname*{colim}_{\sigma \in K} \widehat{D}(\sigma),$$

with $\widehat{D}(\sigma)$ defined as follows:

$$\widehat{D}(\sigma) = \bigwedge_{i=1}^{m} Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

For any pair of simplices $\sigma \subset \tau \in K$ the map from $\widehat{D}(\sigma)$ to $\widehat{D}(\tau)$ is induced by the maps ι_i and the identity.

Definition 3.11. Let $(\underline{X}, \underline{A})$ be a family of pairs, K a simplicial complex, and $I \subset [m]$. Define

$$(\underline{X}_I, \underline{A}_I) = ((X_{i_j}, A_{i_j}))_{j=1}^{|I|}$$

as the subfamily of $(\underline{X}, \underline{A})$ determined by I. We define $(\underline{X}, \underline{A})^{K_I} := (\underline{X}_I, \underline{A}_I)^{K_I}$ and similarly for the polyhedral smash product $\widehat{(\underline{X}, \underline{A})}^{K_I}$.

We will now see how pushouts of simplicial complexes induce pushouts of polyhedral products. Let K be a simplicial complex, and suppose there exists subcomplexes K_1, K_2, and L such that $K = K_1 \cup_L K_2$. To be able to relate the various polyhedral products, it is important that K, K_1, K_2, and L are all on the same vertex set. If K is a simplicial complex on the vertex set $[m]$ we write $\overline{K_1}, \overline{K_2}, \overline{L}$ for the simplicial complexes K_1, K_2, and L regarded as simplicial complexes on the vertex set $[m]$.

Proposition 3.12. *Let K be a simplical complex on the vertex set $[m]$ with subcomplexes K_1, K_2, and L such that $K = K_1 \cup_L K_2$. Let $(\underline{X}, \underline{A})$ be a family of m pairs. Then there is a pushout of polyhedral products*

$$\begin{array}{ccc} (\underline{X}, \underline{A})^{\overline{L}} & \longrightarrow & (\underline{X}, \underline{A})^{\overline{K_1}} \\ \downarrow & \ulcorner & \downarrow \\ (\underline{X}, \underline{A})^{\overline{K_2}} & \longrightarrow & (\underline{X}, \underline{A})^{K}. \end{array}$$

Proof. Denote the face categories of K_1, K_2, and L by $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{L}$. Let D_1, D_2 and D_L be the diagram D restricted to $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{L}$ respectively. Hence, the colimits of D_1, D_2, and D_L are $(\underline{X}, \underline{A})^{\overline{K_1}}, (\underline{X}, \underline{A})^{\overline{K_2}}$, and $(\underline{X}, \underline{A})^{\overline{L}}$ respectively. Let D' be the following diagram of diagrams

$$D_1 \leftarrow D_L \rightarrow D_2.$$

This diagram is a left Kan extension the diagram D, and hence has the same colimit as D, which is $(\underline{X}, \underline{A})^K$. By [30, Proposition 4.4.2.2], we may compute the colimits termwise in the diagram, which yields the desired pushout square. □

Let K, L be simplicial complexes. We denote the disjoint union by $K \sqcup L$.

Corollary 3.13. *Let K_1, K_2 be simplicial complexes on the vertex sets $[m]$ and $[n]$. Let $(\underline{X}, \underline{*})$ be a family of $m + n$ pairs.*

$$(\underline{X}, \underline{*})^{K_1 \sqcup K_2} \simeq (\underline{X}_{\{1,\ldots,m\}}, \underline{*})^{K_1} \vee (\underline{X}_{\{m+1,\ldots,m+n\}}, \underline{*})^{K_2}.$$

Remark 3.14. A version of Proposition 3.12 in the ∞-category of spaces was used by Grbić and Theriault in [20] to determine that moment-angle complexes have the homotopy type of a wedge of spheres when K is a shifted simplicial complex.

4 Stable splitting of polyhedral products

In this section we will prove some stable splitting results for polyhedral products. The results are ∞-categorical generalizations of work by Bahri, Bendersky, Cohen, and Gitler in [6]. For the rest of this section, unless stated otherwise, fix $\mathcal{C}$ to be a cartesian closed ∞-category.

Definition 4.1. For a poset category $\mathcal{D}$, let $|\mathcal{D}|$ be the realization of $\mathcal{D}$ in $\mathcal{C}$. That is, the colimit over the $\mathcal{D}$-shaped diagram, with constant value $* \in \mathcal{C}$.

Definition 4.2. Let $\mathcal{D}$ be a poset category. For an object $a \in \mathcal{D}$ let $\mathcal{D}_{\leq a}$ be the undercategory of a. Let $\mathcal{D}_{<a}$ be the category of objects in $\mathcal{D}$) that are strictly smaller than a.

Definition 4.3. When K is a simplicial complex and $\mathcal{K}$ is its face category, we will write $|K| := |\mathcal{K}_{<\emptyset}|$.

Remark 4.4. When $\mathcal{C}$ is the ∞-category of spaces, the realization $|K|$ of Definition 4.3 and the geometric realization of K as a topological space from Definition 2.3 agrees.

The main result of this section is the following theorem.

Theorem 4.5. *Let K be a simplicial complex with m vertices and let $(\underline{X}, \underline{A})$ have the property that each map $\iota_i \colon A_i \to X_i$ is null. Then there is an equivalence*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \left(\bigvee_{I \subset [m]} \left(\bigvee_{\sigma \in K_I} |(\mathcal{K}_I)_{<\sigma}| \star \widehat{D}(\sigma) \right) \right).$$

We will postpone the proof of the theorem until the end of the section. When each X_i is contractible Theorem 4.5 simplifies to the following.

Corollary 4.6. *Let K be a simplicial complex with m vertices and let $(\underline{X}, \underline{A})$ be a family of pairs where each X_i is contractible. Then there is an equivalence*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \bigvee_{I \notin K} |K_I| \star \widehat{A}^I.$$

Remark 4.7. Proposition 4.10 and Theorem 4.5 are ∞-categorical versions of Theorem 2.10 and 2.13 in [6].

To prove Theorem 4.5 and Corollary 4.6 a collection of results will be needed. We will follow the proof strategy of [6]. The following result is known as the Ganea splitting.

Lemma 4.8 ([16, Corollary 2.24.2])**.** *Let $\mathcal{C}$ be an ∞-category with finite limits and pushouts. Then, for every pair of pointed objects $X, Y \in \mathcal{C}$, there is a natural equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$.*

The stable splitting of a product gives a nice description of the stable splitting of larger products. Consecutive applications of the Ganea splitting yields the following result.

Corollary 4.9. *Let Y_i be pointed objects in $\mathcal{C}$. There is an equivalence*

$$\Sigma(Y_1 \times \ldots \times Y_m) \simeq \Sigma \bigvee_{I \subset [m]} \widehat{Y}^I.$$

Recall that $\widehat{(\underline{X}, \underline{A})}^K$ denotes the polyhedral smash product from Definition 3.10.

Proposition 4.10. *Given a simplicial complex K with m vertices and a family of pairs $(\underline{X}, \underline{A})$, we have the following natural equivalence*

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \left(\bigvee_{I \subset [m]} \widehat{(\underline{X}, \underline{A})}^{K_I} \right).$$

Proof. For $I \in [m]$ and $\sigma \in K$, define

$$\widehat{D}_I(\sigma) := \bigwedge_{i \in I} Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Since suspending commutes with colimits, there is an equivalence

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \operatorname*{colim}_{\sigma \in K} D(\sigma) \simeq \operatorname*{colim}_{\sigma \in K} \Sigma D(\sigma).$$

By Corollary 4.9 we can describe $\Sigma D(\sigma)$ for $\sigma \in K$. For each $\sigma \in K$, there is an equivalence

$$\Sigma D(\sigma) = \Sigma \bigvee_{I \subset [m]} \widehat{D}_I(\sigma).$$

Since the equivalence of Corollary 4.9 was natural and colimits commute, there is an equivalence

$$\Sigma(\underline{X}, \underline{A})^K \simeq \operatorname*{colim}_{\sigma \in K} \left(\Sigma \bigvee_{I \subset [m]} \widehat{D}_I(\sigma) \right) \simeq \Sigma \left(\bigvee_{I \subset [m]} \operatorname*{colim}_{\sigma \in K} \widehat{D}_I(\sigma) \right).$$

Now fix some $I \subset [m]$ and consider the colimit $\operatorname*{colim}_{\sigma \in K} \widehat{D}_I(\sigma)$. For each $i \notin I$, the maps induced by $\tau \subset \sigma$ where τ is obtained from σ by removing vertex i are identity maps. Let $K \ \setminus \{i\}$ denote the full subcomplex $K_{\{1,\ldots,i-1,i+1,\ldots,m\}}$. There is an equivalence

$$\operatorname*{colim}_{\sigma \in K} \widehat{D}_I(\sigma) \simeq \operatorname*{colim}_{\sigma \in K \setminus \{i\}} \widehat{D}(\sigma).$$

Iterating this process for each $i \notin I$ yields,

$$\operatorname*{colim}_{\sigma \in K} \widehat{D}_I(\sigma) \simeq \operatorname*{colim}_{\sigma \in K\setminus\{i\}} \widehat{D}(\sigma) \simeq \operatorname*{colim}_{\sigma \in K_I} \widehat{D}(\sigma) \simeq \widehat{(\underline{X}, \underline{A})}^{K_I}. \qquad \square$$

The previous proposition reduces the question about the stable homotopy type of a polyhedral product to understanding the homotopy type of $\widehat{(\underline{X}, \underline{A})}^{K}$.

Proposition 4.11. *Let K be a simplicial complex and consider a family of pairs $(\underline{X}, \underline{A})$ where the map $\iota_i \colon A_i \to X_i$ is null for all i. Then there is an equivalence*

$$\widehat{(\underline{X}, \underline{A})}^{K} \simeq \bigvee_{\sigma \in K} |\mathcal{K}_{<\sigma}| \star \widehat{D}(\sigma).$$

We will postpone the proof of Proposition 4.11 until Section 4.1. We can now prove Theorem 4.5 and Corollary 4.6.

Proof of Theorem 4.5. The assertion follows by applying the result of Proposition 4.11 to the right-hand side of

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \left(\bigvee_{I \subset [m]} \widehat{(\underline{X}, \underline{A})}^{K_I} \right).$$

from Proposition 4.10. $\square$

Proof of Corollary 4.6. Fix $I \subset [m]$. Since X_i is contractible, it follows that $\widehat{D}(\sigma)$ is contractible for all $\sigma \in K$ where $\sigma \neq \emptyset$. Consequently, there is an equivalence

$$\Sigma(\underline{X}, \underline{A})^K \simeq \Sigma \left(\bigvee_{I \subset [m]} \left(|(\mathcal{K}_I)_{<\emptyset}| \star \widehat{D}(\emptyset) \right) \right) \simeq \Sigma \bigvee_{I \in K} |K_I| \star \widehat{A}^I.$$

The space $|K_I|$ is contractible whenever $I \in K$, so we only need to consider $I \notin K$. $\square$

The rest of this section is dedicated to proving Proposition 4.11.

4.1 The proof of Proposition 4.11

We need to introduce some notation before we can get to the technical lemmas.

Definition 4.12. Let $\mathcal{D}$ be a poset category. A diagram $\mathfrak{X} \colon \mathcal{D} \to \mathcal{C}$ is called a diagram with *constant maps* if for all objects $a, b \in \mathcal{D}$ and any nonidentity morphism $f \colon a \to b, f \neq \mathrm{id}_a$ the map $\mathfrak{X}(f) \colon \mathfrak{X}(a) \to \mathfrak{X}(b)$ is a constant map. In other words, the morphism f can be factored as a composition of maps $\mathfrak{X}(a) \to * \to \mathfrak{X}(b)$.

Definition 4.13. Let $\mathcal{D}$ be a poset category. For an object $c \in \mathcal{C}$ let $\mathcal{D}_c$ be the diagram with the shape of $\mathcal{D}$, but every object is mapped to the object c. Note that there is an equivalence $\operatorname{colim} \mathcal{D}_* \simeq |\mathcal{D}|$ by Definition 4.1. Because $\mathcal{C}$ is cartesian closed, there is an equivalence $\operatorname{colim} \mathcal{D}_c \simeq c \times |\mathcal{D}|$.

The following is an ∞-categorical version of the *initial diagram lemma* found in [39, Lemma 3.4].

Lemma 4.14. *Let $\mathfrak{X}$ be an initial diagram with constant maps over a poset category $\mathcal{D}$ with initial object a. If $\mathfrak{X}(b) = *$ for each $b \neq a$ then there is the following equivalence*

$$\operatorname*{colim}_{\mathcal{D}} \mathfrak{X} \simeq \mathfrak{X}(a) \star |\mathcal{D}_{<a}|.$$

Proof. The diagram category $\mathcal{D}$ is the category $\mathcal{D}_{<a}$, but with an initial object. Let $\{a\}$ denote the single object category. We have an equivalence of categorties $\mathcal{D} \simeq \{a\} \star \mathcal{D}_{<a}$, where $\star$ denotes the join of categories as in [30, §1.2.8]. To model the join, we have have following pushout of categories

$$\{a\} \longleftarrow \mathcal{D}_{<a} \times \{a\} \longrightarrow \mathcal{D}_{<a} \times \Delta^1.$$

The diagram $\mathfrak{X}$ is induced by the maps

$$\begin{array}{ccc}
\mathcal{D}_{<a} \times \{a\} & \longrightarrow & \mathcal{D}_{<a} \times \Delta^1 \\
\big\downarrow & \searrow^{\mathrm{const}_{\mathfrak{X}(a)}} & \big\downarrow{\scriptstyle \mathrm{const}_*} \\
\{a\} & \xrightarrow{\quad \mathfrak{X}(a) \quad} & \mathcal{C}.
\end{array}$$

The colimit of $\mathfrak{X}$ is equivalent to the colimit of the diagram of diagrams

$$\mathfrak{X}(a) \longleftarrow (\mathcal{D}_{<a})_{\mathfrak{X}(a)} \longrightarrow ((\mathcal{D}_{<a}) \times \Delta^1)_*.$$

We have written it as a pushout of diagrams to ease notation. By [30, Proposition 4.4.2.2] we may first compute the colimits termwise in the pushout diagram. Thus, we are left we with a diagram

$$\mathfrak{X}(a) \longleftarrow \mathfrak{X}(a) \times |\mathcal{D}_{<a}| \longrightarrow |\mathcal{D}_{<a}|,$$

which by definition has colimit equal to the join $\mathfrak{X}(a) \star |\mathcal{D}_{<a}|$. ☐

Lemma 4.15. *Suppose that $\mathfrak{X}$ is a diagram where each morphism is null over an indexing poset category $\mathcal{D}$ with initial object a. The colimit of the diagram $\mathfrak{X}$ has the wedge decomposition*

$$\operatorname{colim} \mathfrak{X} \simeq \bigvee_{a \in \mathrm{Obj}(\mathcal{D})} \left(|\mathcal{D}_{<a}| \star \mathfrak{X}(a) \right).$$

Proof. We will start by defining a couple of necessary diagrams. For each object $a \in \mathcal{D}$, let $X[a] \colon \mathcal{D} \to \mathcal{C}$ be the diagram such that $a \mapsto \mathfrak{X}(a)$ and $b \mapsto *$ for each $b \in \mathcal{D}$ where $b \not\simeq a$. Similarly, for each object $a \in \mathcal{D}$, let $X[a]' \colon \mathcal{D}_{\leq a} \to \mathcal{C}$ be the diagram such that $a \mapsto \mathfrak{X}(a)$ and $b \mapsto *$ for each $b \in \mathcal{D}_{\leq a}$ where $b \not\simeq a$. Since each morphism in the diagram $\mathfrak{X}$ is null, it can be decomposed as a wedge of diagrams $X[a]$ for each object $a \in \mathcal{D}$. Thus, we have an equivalence

$$\operatorname{colim} \mathfrak{X} \simeq \bigvee_{a \in \mathrm{Obj}(\mathcal{D})} \operatorname{colim} X[a].$$

For each object $a \in \mathcal{D}$ the diagram $X[a]$ is a left Kan extension the functors

$$X[a]' \colon \mathcal{D}_{\leq a} \to \mathcal{C} \quad \text{and} \quad i \colon \mathcal{D}_{\leq a} \to \mathcal{D}.$$

Since left Kan extensions preserve colimits, there is an equivalence $\operatorname{colim} X[a] \simeq \operatorname{colim} X[a]'$. The diagram $X[a]'$ satisfies the conditions of Lemma 4.14. Hence there is an equivalence

$$\operatorname{colim} X[a]' \simeq |\mathcal{D}_{<a}| \star \mathfrak{X}(a). \qquad \square$$

We can now prove Proposition 4.11.

Proof of Proposition 4.11. Define the diagram $\widehat{E} \colon \mathcal{K} \to \mathcal{C}$ to be given by $\widehat{E}(\sigma) = \widehat{D}(\sigma)$ for all $\sigma \in K$, and for all $\sigma \subsetneq \tau$ the maps $\widehat{e}_{\sigma,\tau} \colon \widehat{E}(\sigma) \to \widehat{E}(\tau)$ to be the constant map to the basepoint. Since the maps $f_i \colon A_i \to X_i$ are null by assumption, the maps between $\widehat{D}(\sigma)$ and $\widehat{D}(\tau)$ for $\sigma \subsetneq \tau$ will also be null-homotopic. We get the following equivalences

$$\widehat{(\underline{X}, \underline{A})}^K \simeq \operatorname{colim} \widehat{D} \simeq \operatorname{colim} \widehat{E} \simeq \bigvee_{\sigma \in K} |\mathcal{K}_{<\sigma}| \star \widehat{D}(\sigma)$$

since $\widehat{E}$ satisfies the conditions of Lemma 4.15. $\square$

5 Example categories

In this section we discuss polyhedral products in several categories. In particular in Section 5.3, we will introduce the category of motivic spaces and motivic moment-angle complexes, which will be our main focus for the rest of the paper.

5.1 The category of spaces

Let $\mathcal{S}$ be the ∞-category of spaces. In Section 2.2 we defined polyhedral products in topological spaces as a union of topological spaces. As long as the map $A_i \to X_i$ is a cofibration, for each pair of spaces $(X_i, A_i) \in (\underline{X}, \underline{A})$, then by [39, Lemma 3.1] there is a homotopy equivalence

$$(\underline{X}, \underline{A})^K := \operatorname*{colim}_{\sigma \in K} D(\sigma) \simeq \bigcup_{\sigma \in K} D(\sigma).$$

It is important to emphasise that the colimit in $\mathcal{S}$ is a higher categorical colimit, and thus corresponds to a homotopy colimit in classical homotopy theory. Thus, the results of Section 2.2 all follow from Section 3 and 4 by letting $\mathcal{C} = \mathcal{S}$.

5.2 The category of G-equivariant spaces

For a discrete group G, one can define the category $\mathcal{S}^G$ of G-equivariant topological spaces. For a G-space X, denote the fixed points by X^G. A map $f \colon X \to Y$ is a weak equivalence of G-spaces if for every subgroup $H \leq G$ the restricted map $f^H \colon X^H \to Y^H$

is a weak equivalence of spaces. Due to Elmendorf's theorem [18], the category of G-spaces can be seen as a presheaf category, and is in particular an ∞-topos. This makes it possible to define G-equivariant polyhedral products coming from families of pairs $(\underline{X}, \underline{A})$ of G-spaces. The stable splitting results from Section 4 also hold.

The moment-angle complex $Z_K = (D^2, S^1)^K$ can be endowed with both a C_2-action (reflection) and a S^1-action (rotation). Similarly, the pair $\mathbb{R}Z_K = (D^1, S^0)^K$ can be endowed with a C_2-action (reflection). There is a relation between Z_K and $\mathbb{R}Z_K$ through the fixed points of Z_K under the reflection.

$$Z_K^{C_2} = ((D^2, S^1)^K)^{C_2} \simeq ((D^2)^{C_2}, (S^1)^{C_2})^K \simeq (D^1, S^0)^K = \mathbb{R}Z_K.$$

When K is a simplicial complex with m vertices, there is an action of the torus $T = (S^1)^{\times m}$ on $Z_K^{\mathbb{A}^1}$ induced by the S^1-action on each pair (D^2, S^1). For a freely acting subtorus L of T, one can define the *partial quotient* as the quotient Z_K/L. The partial quotients are topological versions of smooth not necessarily projective toric varieties.

Actions of a group G on the simplicial complex K can also product interesting equivariant examples. This approach does not require the pairs of spaces $(\underline{X}, \underline{A})$ to be G-spaces. However, this case does not allow for use of the results developed in Section 4. Some of the results do still hold, but with modifications. Fu and Grbić showed [19, Theorem 3.3] that if K is a simplicial complex with a G-action, then there is a homotopy G-equivalence

$$\Sigma^2(X, A)^K \simeq \Sigma^2 \bigvee_{I \in K} \widehat{(X, A)}^K.$$

This is a similar result as Proposition 4.11, but with an extra suspension.

5.3 The category of motivic spaces

The category of motivic spaces over a base field k was introduced by Morel and Voevodsky in [35]. Roughly speaking, the category of motivic spaces, also known as the $\mathbb{A}^1$-homotopy theory of k, is a homotopy theory for smooth schemes. Methods and concepts from algebraic topology had been in use in algebraic geometry for a long time before motivic homotopy theory, but the category of motivic spaces allowed for a framework where techniques from algebraic topology could systematically be lifted to algebraic geometry.

Let k be a perfect field of characteristic different from 2. Let Sm_k be the category of smooth schemes of finite type over k. We denote by $\mathrm{PreSh}(\mathrm{Sm}_k)$, the category of simplicial presheaves on Sm_k. We denote by $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, the category of simplicial Nisnevich sheaves on Sm_k. In motivic homotopy theory the affine line $\mathbb{A}^1$ takes on the role of the interval. We say that a presheaf $\mathcal{F} \in \mathrm{PreSh}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-invariant if there is an equivalence $\mathcal{F}(X) \simeq \mathcal{F}(X \times \mathbb{A}^1)$. The category of motivic spaces $\mathcal{H}(k)$ is the full subcategory of $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ spanned by $\mathbb{A}^1$-invariant Nisnevich sheaves $\mathcal{F}$. In particular, there is a localization functor $\mathrm{L}_{\mathrm{Mot}} \colon \mathrm{PrSh}(\mathrm{Sm}_k) \to \mathcal{H}(k)$, which is a left adjoint of the inclusion $\mathcal{H}(k) \subset \mathrm{PrSh}(\mathrm{Sm}_k)$. Small colimits are universal in the $\mathcal{H}(k)$ [24, Proposition

3.15], which shows that it is cartesian closed. A more detailed explanation of the ∞-categorical construction of the category of motivic spaces can be found in [23, Appendix C] or [24, §3]. A model categorical survey and introduction of unstable motivic homotopy theory can be found in [2].

We will now survey some results about the motivic homotopy theory which can be found in Chapter 3 of [35]. The category of motivic spaces contains both geometric objects (schemes) and topological objects (simplicial sets). A scheme $X \in \mathrm{Sm}_k$ can be seen as an element of $\mathcal{H}(k)$ in the following way. For any $Y \in \mathrm{Sm}_k$, the scheme X can be viewed as a simplicial presheaf by letting $X(Y) := \mathrm{Hom}_k(Y, X)$, the scheme morphisms of Y to X. By motivic localization the presheaf represented by X can be considered an object of $\mathcal{H}(k)$. Let S be a simplicial set. One can consider the constant simplicial presheaf const_S. When it is clear that we are working with motivic spaces, we will abuse notation and write X for $\mathrm{L_{Mot}Hom}_k(-, X) \in \mathcal{H}(k)$ and S for the motivic localization $\mathrm{L_{Mot}const}_S$.

We will now look at a some motivic spaces. Recall that the affine line $\mathbb{A}^1$ plays the role of the interval in $\mathcal{H}(k)$ and is contractible. That is, there is an equivalence $\mathbb{A}^1 \simeq \mathrm{Spec}(k)$ in $\mathcal{H}(k)$. A prominent feature of motivic homotopy theory is that the spheres are bigraded. There is the simplicial circle, which is given by the constant simplicial presheaf to a simplicial set model of S^1. There is also a geometric circle, the punctured affine line $\mathbb{G}_\mathrm{m}$, which is given by the scheme $\mathbb{G}_\mathrm{m} := \mathbb{A}^1 \setminus 0$. The motivic spheres are created by smashing copies of the geometric and simplicial circles. Let $S^{1,0}$ be the simplicial circle, and $S^{1,1} := \mathbb{G}_\mathrm{m}$. Thus for $a \geq b \geq 0$, we have $S^{a,b} = (S^1)^{\wedge(a-b)} \wedge \mathbb{G}_\mathrm{m}^{\wedge b}$. Some of the higher dimensional spheres can be represented by schemes as well. A standard way of constructing the projective line $\mathbb{P}^1$ in algebraic geometry is by gluing two affine lines along a common $\mathbb{G}_\mathrm{m}$. Thus $\mathbb{P}^1$ is the colimit of the following diagram

$$\mathbb{A}^1 \longleftarrow \mathbb{G}_\mathrm{m} \longrightarrow \mathbb{A}^1.$$

Since $\mathbb{A}^1$ is contractible in $\mathcal{H}(k)$, $\mathbb{P}^1$ is equivalent to the colimit of the diagram

$$\mathrm{Spec}(k) \longleftarrow \mathbb{G}_\mathrm{m} \longrightarrow \mathrm{Spec}(k),$$

which is $\Sigma\mathbb{G}_\mathrm{m} \simeq S^{1,0} \wedge \mathbb{G}_\mathrm{m}$. In other words, $\mathbb{P}^1$ is equivalent to $S^{2,1}$. Higher dimensional punctured affine spaces are also models for motivic spheres. Let $n > 0$, then $\mathbb{A}^n \setminus 0 \simeq S^{2n-1,n}$.

There are also ways of relating motivic homotopy theory to classical homotopy theory. When k is a subfield of $\mathbb{C}$ (resp. $\mathbb{R}$) and $X \in \mathrm{Sm}_k$, we will write $X(\mathbb{C})$ (resp. $X(\mathbb{R})$) for its complex (resp. real) points as a topological space. This yields a realization functor from $\mathcal{H}(\mathbb{C}) \to \mathcal{S}$. Furthermore, there is a second realization functor when k is a subfield of $\mathbb{R}$. Whenever X is a smooth scheme over $\mathbb{R}$, then $X(\mathbb{C})$ is a C_2-space with the action of complex conjugation and we have the relation $X(\mathbb{C})^{C_2} \simeq X(\mathbb{R})$ as topological spaces. Thus we can describe the two realization functors as

$$\mathrm{Re}_\mathbb{C} \colon \mathcal{H}(\mathbb{C}) \to \mathcal{S} \quad \text{and} \quad \mathrm{Re}_\mathbb{R} \colon \mathcal{H}(\mathbb{R}) \to \mathcal{S}^{C_2}.$$

Let S be a simplicial set, then both its complex and real realization of its associated object in $\mathcal{H}(k)$ is the realization S as a topological space, with trivial C_2-action under real realization. A reader not familiar with algebraic geometry might not understand why $\mathbb{G}_{\mathrm{m}}$ could be an algebraic sphere at first glance. Consider the real number line, when we remove the origin we get a topological space which has the homotopy type of S^0. Now consider the complex numbers, the space $\mathbb{C} \setminus 0$ has the homotopy type of S^1. So $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) \simeq S^1$ and $\mathbb{G}_{\mathrm{m}}(\mathbb{R}) \simeq S^0$. For an arbitrary motivic sphere $S^{a,b}$, with $a \geq b \geq 0$, we have $\mathrm{Re}_{\mathbb{C}}(S^{a,b}) \simeq S^a$ and $(\mathrm{Re}_{\mathbb{R}}(S^{a,b}))^{C_2} \simeq S^{a-b}$.

As described in Section 2.3, the moment-angle complex $(D^2, S^1)^K$ and the real moment-angle complex $(D^1, S^0)^K$ have both been extensively studied in the category of spaces. We will now introduce the *motivic moment-angle complex*.

Definition 5.1. Let K be a simplicial complex, we define the *motivic moment-angle complex* $Z_K^{\mathbb{A}^1}$ to be the polyhedral product

$$Z_K^{\mathbb{A}^1} := (\mathbb{A}^1, \mathbb{G}_{\mathrm{m}})^K$$

in the ∞-category $\mathcal{H}(k)$.

Remark 5.2. When k is a subfield of $\mathbb{C}$, using complex realization yields the equivalence $Z_K^{\mathbb{A}^1}(\mathbb{C}) \simeq (\mathbb{C}, \mathbb{C}^\times)^K$ which deformation retracts onto Z_K [11, Theorem 4.7.5]. Furthermore, if k is a subfield of $\mathbb{R}$, there is a deformation retraction $Z_K^{\mathbb{A}^1}(\mathbb{R}) \simeq \mathbb{R}Z_K$.

As noted earlier, the $\mathcal{H}(k)$ is cartesian closed and has all small colimits. This makes it possible to apply Theorem 4.5. Since $\mathbb{A}^1$ is contractible, the following result an application of Corollary 4.6.

Theorem 5.3. *Let K be a simplicial complex. Then there is an equivalence in $\mathcal{H}(k)$*

$$\Sigma Z_K^{\mathbb{A}^1} \simeq \Sigma \left(\bigvee_{I \notin K} |K_I| \star \mathbb{G}_{\mathrm{m}}^{\wedge |I|} \right) \simeq \bigvee_{I \notin K} |K_I| \wedge S^{|I|+2,|I|}.$$

Remark 5.4. All of the results from Section 4 can be proven for $\mathcal{C} = \mathcal{H}(k)$ using Morel and Voevodsky's model structure and proving it locally on the value of simplicial presheaves using the results for topological spaces.

6 Affine models for motivic moment-angle complexes and toric varieties

In this section we will provide various models of $Z_K^{\mathbb{A}^1}$ in Sm_k. We will also use affine models of $Z_K^{\mathbb{A}^1}$ to give affine models of smooth projective toric varieties. We begin with identifying $Z_K^{\mathbb{A}^1}$ with a smooth scheme.

Proposition 6.1. *Let K be a simplicial complex, and let $\sigma_1, \dots, \sigma_n$ be the maximal simplices of K. There is an equivalence*

$$Z_K^{\mathbb{A}^1} \simeq \mathbb{A}^m \setminus L,$$

where L is the variety cut out by the monomial ideal

$$(\prod_{i \notin \sigma_1} x_i, \dots, \prod_{i \notin \sigma_n} x_i) \subset k[x_1, \dots, x_m].$$

Proof. The colimit presentation of $Z_K^{\mathbb{A}^1}$ gives a Zariski cover of an algebraic variety. For each simplex $\sigma \in K$, we have an equivalence $D(\sigma) = \mathbb{A}^m \setminus (\prod_{i \notin \sigma} x_i)$. The algebraic variety $Z_K^{\mathbb{A}^1}$ can be described as the union

$$\bigcup_{\sigma \in K} D(\sigma) = \bigcup_{\sigma \in K} \mathbb{A}^m \setminus (\prod_{i \notin \sigma} x_i) = \mathbb{A}^m \setminus L.$$

We may enumerate the maximal simplices of K as $\sigma_1, \dots, \sigma_n$. The variety L is cut out by the ideal

$$\bigcap_{1 \leq j \leq n} (\prod_{i \notin \sigma_j} x_i) = (\prod_{i \notin \sigma_1} x_i, \dots, \prod_{i \notin \sigma_n} x_i).$$ □

Remark 6.2. The ideal cutting out L is the Stanley–Reisner ideal $I_{K^\vee}$ of Alexander dual $K^\vee$.

Remark 6.3. From now on, whenever we speak about $Z_K^{\mathbb{A}^1}$ as a scheme, we will always mean $\mathbb{A}^m \setminus L$.

The scheme $\mathbb{A}^m \setminus L$ can be identified with a toric variety generated by the following fan. Let e_i denote the ith coordinate unit vector of $\mathbb{R}^m$. For each $\sigma \in K$, let $C_\sigma := \mathrm{Cone}(e_{i_1}, \dots, e_{i_n})$ with $i_j \in \sigma$. The fan for $Z_K^{\mathbb{A}^1}$ is the collection of the cones C_σ for each $\sigma \in K$. Motivic moment-angle complexes have been studied before as toric varieties. In [40] Wendt computes the $\mathbb{A}^1$-fundamental group of smooth toric varieties, this includes motivic-moment-angle complexes.

Our goal is to identify $Z_K^{\mathbb{A}^1}$ with an affine scheme. To do this we will need the following family of affine schemes.

Definition 6.4. Let $L \subset \mathbb{A}^m$ be a closed subvariety cut out by the ideal $I = (f_1, \dots, f_n)$. We define

$$Q_I := \mathrm{Spec}\left(\frac{k[x_1, \dots x_m, y_1, \dots y_n]}{(f_1 y_1 + \ldots + f_n y_n - 1)}\right)$$

and a morphism

$$\pi \colon Q_I \to \mathbb{A}^m \setminus L$$

given by projection onto $(x_1, \dots, x_m) \in \mathbb{A}^m$.

Lemma 6.5. *Let $L \subset \mathbb{A}^m$ be a closed subvariety cut out by the ideal $I = (f_1, \dots, f_n)$. The map $\pi \colon Q_I \to \mathbb{A}^m \setminus L$ is an $\mathbb{A}^1$-equivalence.*

Proof. The variety $\mathbb{A}^m \setminus L$ is covered by the opens U_i, where $f_i \neq 0$. Thus, locally for each U_i,

$$U_i \cong \operatorname{Spec}(k[x_1, \ldots, x_m][f_i^{-1}]).$$

Computing the preimage of U_i yields

$$\pi^{-1}(U_i) \cong \operatorname{Spec}\left(\frac{k[x_1, \ldots x_m, y_1, \ldots y_n][f_i^{-1}]}{(f_1 y_1 + \ldots + f_n y_n - 1)}\right).$$

We see that π trivializes, and we get $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^{n-1}$. The opens $\pi^{-1}(U_i)$ cover Q_I as well since the ideal I is a unit ideal in the coordinate ring of Q_I. The fibers of π are trivial and the morphism is smooth, which implies that Q_I is a Zariski locally trivial bundle over $\mathbb{A}^m \setminus L$ and π is an $\mathbb{A}^1$-equivalence. ☐

Proposition 6.1 shows that $Z_K^{\mathbb{A}^1}$ is homotopy equivalent to $\mathbb{A}^m \setminus L$ where L is some intersection of coordinate hyperplanes. As noted in Remark 6.2, the variety L is cut out by a monomial ideal $I_{K^\vee} = (f_1, \ldots, f_n) \subset k[x_1, \ldots, x_m]$.

Corollary 6.6. *Let K be a simplicial complex. Then there is an $\mathbb{A}^1$-equivalence*

$$Z_K^{\mathbb{A}^1} \simeq Q_{I_{K^\vee}}.$$

We will also give a different affine model of $Z_K^{\mathbb{A}^1}$ that uses the Stanley–Reisner ideal $I_K = (f_1, \ldots, f_n)$. For simplicity, we will write $i \in f_j$ when x_i is a factor of f_j.

Proposition 6.7. *Let K be a simplicial complex on the vertex set $[m]$ and $I_K = (f_1, \ldots, f_n)$ its Stanley–Reisner ideal. There is an equivalence*

$$Z_K^{\mathbb{A}^1} \simeq \operatorname{Spec}\left(\frac{k[x_1, \ldots, x_m, y_{ij}]}{(\sum_{i\in f_1} x_i y_{i1} - 1, \ldots, \sum_{i \in f_n} x_i y_{il} - 1)}\right).$$

Proof. One can cover $\mathbb{A}^m \setminus L$ by the affine opens corresponding to the maximal simplices of K, i.e. $D(\sigma)$ (as in Definition 3.7) where σ is maximal in K. Using the same strategy as the proof as Lemma 6.5 with this open cover yields the result. ☐

Remark 6.8. For any $n > 0$, the two affine representatives for $Z_K^{\mathbb{A}^1}$ from Corollary 6.6 and Proposition 6.7 are equal when $K = \partial\Delta^n$.

In some special cases, it is possible to give an affine model of $\Sigma Z_K^{\mathbb{A}^1}$. In the special case where $K = \partial\Delta^{n-1}$ we have $Z_{\partial\Delta^{n-1}}^{\mathbb{A}^1} \simeq S^{2n-1,n}$, and there is an equivalence due to Asok, Doran, and Fasel [3, Theorem 2.2.5]

$$\Sigma Z_{\partial\Delta^{n-1}}^{\mathbb{A}^1} \simeq S^{2n,n} \simeq \operatorname{Spec}\left(\frac{k[x_1, \ldots, x_n, y_1, \ldots, y_n, z]}{(x_1 y_1 + \ldots x_n y_n - z(1-z))}\right).$$

However, we are able to say more about $\Sigma_{\mathbb{P}^1} Z_K^{\mathbb{A}^1}$. By [4, Corollary 4.16], the motivic space $\Sigma_{\mathbb{P}^1} Z_K^{\mathbb{A}^1}$ admits an affine model because $Z_K^{\mathbb{A}^1}$ has the homotopy type of an affine scheme. The following remark shows how the $\mathbb{P}^1$-suspension of $Z_K^{\mathbb{A}^1}$ is a motivic moment-angle complex.

Remark 6.9. Let K and Δ^{m-1} be a simplicial complex on the vertex set $[m]$. We define $M = K \star \{m+1\}$ on the vertex set $[m+1]$ Consider the pushout of simplicial complexes

$$\begin{array}{ccc} \overline{K} & \longrightarrow & \overline{M} \\ \downarrow & \ulcorner & \downarrow \\ \overline{\Delta^{m-1}} & \longrightarrow & K'. \end{array}$$

By Proposition 3.12, this induces a pushout of motivic moment-angle complexes

$$\begin{array}{ccc} Z^{\mathbb{A}^1}_{\overline{K}} & \longrightarrow & Z^{\mathbb{A}^1}_{\overline{M}} \\ \downarrow & \ulcorner & \downarrow \\ Z^{\mathbb{A}^1}_{\overline{\Delta^{m-1}}} & \longrightarrow & Z^{\mathbb{A}^1}_{K'}. \end{array}$$

We can express the motivic moment-angle complexes in the diagram in terms of $Z^{\mathbb{A}^1}_K$, and this gives

$$\begin{array}{ccc} Z^{\mathbb{A}^1}_K \times \mathbb{G}_{\mathrm{m}} & \longrightarrow & Z^{\mathbb{A}^1}_K \times \mathbb{A}^1 \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{A}^m \times \mathbb{G}_{\mathrm{m}} & \longrightarrow & Z^{\mathbb{A}^1}_{K'}. \end{array}$$

Since $\mathbb{A}^1$ is contractible, we get an equivalence

$$Z^{\mathbb{A}^1}_{K'} \simeq Z^{\mathbb{A}^1}_K \star \mathbb{G}_{\mathrm{m}} \simeq \Sigma_{\mathbb{P}^1} Z^{\mathbb{A}^1}_K.$$

Thus $\Sigma_{\mathbb{P}^1} Z^{\mathbb{A}^1}_K$ is a motivic moment-angle complex and has the homotopy type of a smooth affine scheme.

Recall that any smooth projective toric variety X can be realized as a quotient of a motivic-moment-angle complex under the action of a torus [13]. The torus action induces an action on Q_K. By computing the ring of invariants of the coordinate ring of Q_K under the torus action, we can find an explicit smooth affine description of X. Concretely, if $X = Z^{\mathbb{A}^1}_K/T$ is a smooth projective toric variety for some simplicial complex K and torus T, then Q_K/T is an affine bundle over X with trivial fibers.

Example 6.10. Let K be two disjoint points, then $Z^{\mathbb{A}^1}_K \simeq \mathbb{A}^2 \setminus 0$. We make $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{A}^2 \setminus 0$ by scalar multiplication. That is, for $\lambda \in \mathbb{G}_{\mathrm{m}}$, we define $\lambda \cdot (x_1, x_2) = (\lambda x_1, \lambda x_2)$. By Corollary 6.6, there is an equivalence

$$Z^{\mathbb{A}^1}_K \simeq \mathbb{A}^2 \setminus 0 \simeq \operatorname{Spec}\left(\frac{k[x_1, x_2, f_1, f_2]}{(x_1 f_1 + x_2 f_2 - 1)}\right) = \mathrm{SL}_2.$$

The action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^2 \setminus 0$ extends to an action on SL_2 as follows

$$\lambda \cdot (x_1, x_2, f_1, f_2) = (\lambda x_1, \lambda x_2, \frac{f_1}{\lambda}, \frac{f_2}{\lambda}).$$

Computing the ring of invariants of yields the generators x_1f_1, x_2f_2, x_1f_2, and x_2f_1. There are also relations $x_1f_1 + x_2f_2 = 1$ and $x_1f_1 \cdot x_2f_2 = x_1f_2 \cdot x_2f_1$. Thus the ring of invariants is isomorphic to the ring

$$R = \frac{k[a,b,c,d]}{(a+d-1, ad-bc)}.$$

Thus $\mathbb{P}^1 \simeq \mathrm{Spec}(R)$. One can view R as a (2×2)-matrix with trace 1 and determinant 0. This is equivalent to a rank 1 idempotent matrix. Similar techniques can be applied to identify $\mathbb{P}^{n-1}$ with an idempotent $(n \times n)$ matrices of rank 1.

Let K be a square, then $Z_K^{\mathbb{A}^1} \simeq \mathbb{A}^2 \setminus 0 \times \mathbb{A}^2 \setminus 0$. We can consider a 2-dimensional torus acting on $Z_K^{\mathbb{A}^1}$ in the following way. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{G}_{\mathrm{m}}^{\times 2}$, we let $\mathbb{G}_{\mathrm{m}}^{\times 2}$ act on $Z_K^{\mathbb{A}^1}$ in the following way

$$\lambda \cdot (x_1, x_2, x_3, x_4) = (\lambda_1 x_1, \lambda_1 x_2, \lambda_2 x_3, \lambda_2 x_4).$$

The GIT quotient of the action is $\mathbb{P}^1 \times \mathbb{P}^1$. The following two examples show how the two different affine models for $Z_K^{\mathbb{A}^1}$ yield different affine models for a toric variety.

Example 6.11. By Corollary 6.6, there is an equivalence $Z_K^{\mathbb{A}^1} \simeq Q_{I_{K^\vee}}$. In the case where K is a square, this yields

$$Z_K^{\mathbb{A}^1} \simeq \mathrm{Spec}\left(\frac{k[x_1,x_2,x_3,x_4,f_{13},f_{14},f_{23},f_{24}]}{(x_1x_3f_{13} + x_1x_4f_{14} + x_2x_3f_{23} + x_2x_4f_{24} - 1)}\right).$$

The action of $\mathbb{G}_{\mathrm{m}}^{\times 2}$ on $Z_K^{\mathbb{A}^1}$ by $\mathbb{G}_{\mathrm{m}}^{\times 2}$ as previously described extends to an action on $Q_{I_{K^\vee}}$ as follows

$$\lambda \cdot (x_1,x_2,x_3,x_4,f_{13},f_{14},f_{23},f_{24}) = (\lambda_1 x_1, \lambda_1 x_2, \lambda_2 x_3, \lambda_2 x_4, \frac{f_{13}}{\lambda_1\lambda_2}, \frac{f_{14}}{\lambda_1\lambda_2}, \frac{f_{23}}{\lambda_1\lambda_2}, \frac{f_{24}}{\lambda_1\lambda_2}).$$

The 16 generators of the ring of invariants under the action of the torus are $x_ix_jf_{pq}$ for $i = 1,2$, $j = 3,4$, $p = 1,2$, and $q = 3,4$. The ring of invariants has the relation

$$x_1x_3f_{13} + x_1x_4f_{14} + x_2x_3f_{23} + x_2x_4f_{24} = 1$$

as well as

$$x_ix_jf_{pq} \cdot x_{i'}x_{j'}f_{p'q'} = x_ix_jf_{p'q'} \cdot x_{i'}x_{j'}f_{pq}.$$

The variety $Q_{I_{K^\vee}}/T$ can be seen as the pullback of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ and the morphism $\pi\colon \tilde{\mathbb{P}}^3 \to \mathbb{P}^1$, where $\tilde{\mathbb{P}}^3$ is the affine replacement for $\mathbb{P}^3$.

Example 6.12. When K is a square, there is an equivalence by Proposition 6.7

$$Z_K^{\mathbb{A}^1} \simeq \mathrm{Spec}\left(\frac{k[x_1,x_2,x_3,x_4,f_1,f_2,f_3,f_4]}{(x_1f_1 + x_2f_2 - 1, x_3f_3 + x_4f_4 - 1)}\right).$$

In this case, the action of $\mathbb{G}_{\mathrm{m}}^{\times 2}$ on $Z_K^{\mathbb{A}^1}$ by $\mathbb{G}_{\mathrm{m}}^{\times 2}$ extends as follows

$$\lambda \cdot (x_1, x_2, x_3, x_4, f_{13}, f_{14}, f_{23}, f_{24}) = (\lambda_1 x_1, \lambda_1 x_2, \lambda_2 x_3, \lambda_2 x_4, \frac{f_1}{\lambda_1}, \frac{f_2}{\lambda_1}, \frac{f_3}{\lambda_2}, \frac{f_4}{\lambda_2}).$$

The eight generators of the ring of invariants are

$$x_i f_j, x_p f_q \quad \text{for} \quad i = 1, 2,\ j = 1, 2,\ p = 3, 4,\ q = 3, 4.$$

We have relations $x_1 f_1 + x_2 f_2 = 1$ and $x_3 f_3 + x_4 f_4 = 1$, as well as $x_i f_i \cdot x_j f_j = x_i f_j \cdot x_j f_i$ for $(i, j) = (1, 2)$ and $(i, j) = (3, 4)$. The ring of invariants is isomorphic to the tensor product of two copies of the coordinate ring of the affine replacement of $\mathbb{P}^1$ from Example 6.10.

7 Invariants of motivic polyhedral products

In this section we will consider various invariants for objects in the motivic homotopy category and apply them to motivic moment-angle complexes.

7.1 Cellular $\mathbb{A}^1$-homology

In [34] Morel and Sawant define cellular $\mathbb{A}^1$-homology for cellular varieties. Cellular varieties are smooth schemes that admits a nice stratification. The cellular $\mathbb{A}^1$-homology takes values in the derived category of strictly $\mathbb{A}^1$-invariant Nisnevich sheaves of abelian groups on Sm_k, which we will denote by $D(Ab_{\mathbb{A}^1}(k))$. Morel and Sawant define a scheme X to be *cohomologically trivial* if $\mathbf{H}_n^{\mathrm{Nis}}(X, \mathbf{M}) = 0$, for every $n \geq 1$ and $\mathbf{M} \in Ab_{\mathbb{A}^1}(k)$ [34, Definition 2.9]. Examples of cohomologically trivial schemes are $\mathbb{A}^1, \mathbb{G}_{\mathrm{m}}$, and products of cohomologically trivial schemes [34, Remark 2.10].

Definition 7.1 ([34, Definition 2.11])**.** Let $X \in \mathrm{Sm}_k$ be a smooth k-scheme. A *cellular structure* on X consists of an increasing filtration

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \ldots \subsetneq \Omega_n = X$$

by open subschemes of X such that for each $0 \leq i \leq n$, the reduced induced subscheme $X_i := \Omega_i \setminus \Omega_{i-1}$ of Ω_i is smooth, affine, everywhere of codimension i, and cohomologically trivial. We say that X is a *cellular scheme* if X admits a cellular structure.

This definition is meant to imitate the CW-structure of a topological space. See [34, Remark 2.12.(2)] for further details.

Proposition 7.2. *Let K be a simplicial complex, and let K_i denote the i-skeleton of K. Let s be the smallest integer such that $K_s = K$. Then*

$$\emptyset \subset \mathbb{G}_{\mathrm{m}}^{\times m} \subsetneq Z_{K_0}^{\mathbb{A}^1} \subsetneq Z_{K_1}^{\mathbb{A}^1} \subsetneq \ldots \subsetneq Z_{K_s}^{\mathbb{A}^1} = Z_K^{\mathbb{A}^1}$$

is a cellular structure on $Z_K^{\mathbb{A}^1}$.

Proof. By Proposition 6.1, for each $0 \le i \le s$ the variety $Z_{K_i}^{\mathbb{A}^1}$ is the open complement of the variety $V(I_{K_i^\vee})$ cut out by a monomial ideal $I_{K_i^\vee}$ in $\mathbb{A}^m$. Thus,

$$X_{i+1} = Z_{K_i}^{\mathbb{A}^1} \setminus Z_{K_{i-1}}^{\mathbb{A}^1} = V(I_{K_i^\vee}) \setminus V(I_{K_{i-1}^\vee}) = \bigsqcup_{\sigma \in K, |\sigma| = i} \mathbb{G}_{\mathrm{m}}^{\times(m-i)}.$$

Since $\mathbb{G}_{\mathrm{m}}$ is cohomologically trivial, $Z_K^{\mathbb{A}^1}$ admits a cellular structure. □

Using the cellular structure of cellular varieties, Morel and Sawant define an $\mathbb{A}^1$-chain complex. From the cellular structure on $Z_K^{\mathbb{A}^1}$, one can create a cellular $\mathbb{A}^1$-chain complex in the fashion of Morel and Sawant. However, if we want to exploit the homotopical properties of polyhedral products, we are going to need a different cellular $\mathbb{A}^1$-chain complex. This is no problem since [34, Corollary 2.42] shows that any two cellular $\mathbb{A}^1$-chain complexes of a $Z_K^{\mathbb{A}^1}$ will be homotopy equivalent in $D(Ab_{\mathbb{A}^1}(k))$. We will now show that the functor $C_*^{\mathrm{cell}}(-)$ sends motivic moment-angle complexes to a polyhedral product in $D(Ab_{\mathbb{A}^1}(k))$.

Proposition 7.3. *The functor $C_*^{\mathrm{cell}}(-)$ preserves colimits of cohomologically trivial objects.*

Proof. The functor $C_*^{\mathrm{cell}}(-)$ is a pro-left adjoint to the category pro-$D(Ab_{\mathbb{A}^1}(k))$. When the objects are cohomologically trivial, their image in pro-$D(Ab_{\mathbb{A}^1}(k))$ are constant, and can be represented as elements of $D(Ab_{\mathbb{A}^1}(k))$. See [34, Corollary 2.38 and Remark 2.39] for further details. □

By [34, Lemma 2.31], for X, Y smooth cellular schemes, there is an isomorphism of chain complexes

$$C_*^{\mathrm{cell}}(X \times Y) \cong C_*^{\mathrm{cell}}(X) \otimes C_*^{\mathrm{cell}}(Y).$$

We say that a chain complex C_* of strictly $\mathbb{A}^1$-invariant sheaves is *pointed*, if C_0 admits $\mathbb{Z}$ as a direct summand. We denote the reduced chain complex of C by $\widetilde{C}$ and there are isomorphisms $C_0 \cong \mathbb{Z} \oplus \widetilde{C}_0$ and $C_i \cong \widetilde{C}_i$ for $i > 0$. In the case where C_* and D_* are two pointed chain complexes concentrated in degree 0, we get the following splitting

$$C_* \otimes D_* \simeq \mathbb{Z} \oplus \widetilde{C}_0 \oplus \widetilde{D}_0 \oplus (\widetilde{C}_0 \otimes \widetilde{D}_0).$$

When X is a pointed space that admits a cellular structure then $C_*^{\mathrm{cell}}(X)$ is a pointed chain complex. Because motivic moment-angle complexes are built out of products of $\mathbb{A}^1$'s and $\mathbb{G}_{\mathrm{m}}$'s, it is important to understand the cellular structure of those pieces. Denote the n-th unramified Milnor–Witt K-theory sheaf by $\mathbf{K}_n^{\mathrm{MW}}$ (see [33, §3.2] for a definition). Going forward, we will need the following property of the Milnor–Witt K-theory sheaves. For $i, j \ge 0$ there is an isomorphism $\mathbf{K}_i^{\mathrm{MW}} \otimes \mathbf{K}_j^{\mathrm{MW}} \cong \mathbf{K}_{i+j}^{\mathrm{MW}}$. The following result is due to Morel and Sawant.

Proposition 7.4. *The cellular $\mathbb{A}^1$-chain complex for $\mathbb{A}^1$ and $\mathbb{G}_{\mathrm{m}}$ are given by*

$$C_i^{\mathrm{cell}}(\mathbb{A}^1) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i > 0, \end{cases} \quad \text{and} \quad C_i^{\mathrm{cell}}(\mathbb{G}_{\mathrm{m}}) = \begin{cases} \mathbb{Z} \oplus \mathbf{K}_1^{\mathrm{MW}} & i = 0, \\ 0 & i > 0. \end{cases}$$

When $S \in Ab_{\mathbb{A}^1}(k)$, we will abuse notation and write S for the chain complex concentrated in degree 0 with value S. Since $C^{\text{cell}}(-)$ preserves products and colimits, it sends the motivic moment-angle complex to a polyhedral product in $D_{\mathbb{A}^1}(Ab(k))$. We can now define a second $\mathbb{A}^1$-chain complex for $Z_K^{\mathbb{A}^1}$ as a polyhedral product

$$C_*^{\text{cell}}(Z_K^{\mathbb{A}^1}) \simeq (C_*^{\text{cell}}(\mathbb{A}^1), C_*^{\text{cell}}(\mathbb{G}_{\text{m}}))^K = (\mathbb{Z}, \mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}})^K.$$

Recall that for a simplical complex K, we denote the geometric realization of K by $|K|$ as in Definition 4.3. In $D_{\mathbb{A}^1}(Ab(k))$, the geometric realization $|K|$ is represented by any singular chain complex that computes the singular homology of K as a topological space.

Proposition 7.5. *There the following is an equivalence of of chain complexes in $D(Ab_{\mathbb{A}^1}(k))$*

$$(C_*^{\text{cell}}(\mathbb{A}^1), C_*^{\text{cell}}(\mathbb{G}_{\text{m}}))^K \simeq \bigvee_{I \notin K} \Sigma |K_I| \wedge (\mathbb{Z} \oplus \mathbf{K}_{|I|}^{\text{MW}}).$$

Proof. We saw earlier how tensor products of pointed complexes concentrated in degree zero splits into a wedge of complexes. Since $(C_*^{\text{cell}}(\mathbb{A}^1), C_*^{\text{cell}}(\mathbb{G}_{\text{m}}))^K$ is a colimit of tensor products of complexes concentrated in degree 0, we get the following chain of equivalences

$$(C_*^{\text{cell}}(\mathbb{A}^1), C_*^{\text{cell}}(\mathbb{G}_{\text{m}}))^K \simeq \operatorname*{colim}_{\sigma \in K} D(\sigma) \simeq \operatorname*{colim}_{\sigma \in K} (\mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}})^{\otimes (m - |\sigma|)}.$$

We can now split the diagram into a wedge of diagrams. For each $I \subseteq [m]$ and $\sigma \in K$ define

$$E_I(\sigma) = \begin{cases} \mathbb{Z} \oplus \mathbf{K}_{|I|}^{\text{MW}} & \sigma \not\subseteq I, \\ \mathbb{Z} & \sigma \subseteq I. \end{cases}$$

There is an equivalence

$$\operatorname*{colim}_{\sigma \in K} (\mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}})^{\otimes (m - |\sigma|)} \simeq \bigvee_{I \subseteq [m]} \operatorname*{colim}_{\sigma \in K} E_I(\sigma).$$

We are now in a similar situation as in the proof of Proposition 4.11. Fix $I \subset [m]$ and look at the colimit

$$\operatorname*{colim}_{\sigma \in K} E_I(\sigma).$$

For each $i \notin I$, the maps induced by $\tau \subset \sigma$ obtained by removing vertex i are identity maps. We get an equivalence of colimits

$$\operatorname*{colim}_{\sigma \in K} E_I(\sigma) \simeq \operatorname*{colim}_{\sigma \in K_I} E'_I(\sigma),$$

where $E'_I(\sigma) = \mathbb{Z} \oplus \mathbf{K}_{|I|}^{\text{MW}}$ if $\sigma = \emptyset$ and $E'_I(\sigma) = \mathbb{Z}$ otherwise. This is a diagram that satisfies Lemma 4.14, and we get an equivalence

$$\operatorname*{colim}_{\sigma \in K} E_I(\sigma) \simeq \Sigma |K_I| \wedge (\mathbb{Z} \oplus \mathbf{K}_{|I|}^{\text{MW}}).$$

With the colimit of each wedge summand computed, we get

$$(C_*^{\mathrm{cell}}(\mathbb{A}^1), C_*^{\mathrm{cell}}(\mathbb{G}_\mathrm{m}))^K \simeq \bigvee_{I\subseteq[m]} \operatorname*{colim}_{\sigma\in K} E_I(\sigma) \simeq \bigvee_{I\subseteq[m]} \Sigma|K_I| \wedge (\mathbb{Z} \oplus \mathbf{K}^{\mathrm{MW}}_{|I|}).$$

Since $|K_I|$ is contractible if $I \in K$, we only need to consider $K \notin I$. This proves the claim. □

This splitting result can be seen as an unstable version of Theorem 4.5. Computing the homology of the chain complex from Proposition 7.5 yields the following.

Corollary 7.6. *Let K be a simplicial complex. There is an isomorphism*

$$\widetilde{\mathbf{H}}_*^{\mathrm{cell}}(Z_K^{\mathbb{A}^1}) \cong \bigoplus_{I\notin K} \widetilde{\mathbf{H}}_*(\Sigma|K_I| \wedge (\mathbb{Z} \oplus \mathbf{K}^{\mathrm{MW}}_{|I|}).$$

We wish to describe the cellular $\mathbb{A}^1$-homology of $Z_K^{\mathbb{A}^1}$ in terms of K, so the next natural step is to understand what smashing with $(\mathbb{Z}\oplus\mathbf{K}^{\mathrm{MW}}_{|I|})$ and suspending $|K_I|$ does to the homology.

Definition 7.7. For a chain complex C with differential d, the cone of C is defined as the chain complex $\mathrm{Cone}(C)_n := C_n \oplus C_{n-1} \oplus C_n$, with differential

$$\delta_n(a, b, c) = (da + b, -db, dc - b).$$

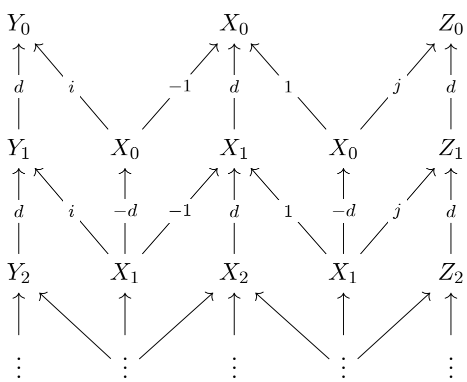
For a diagram $(Y \longleftarrow X \longrightarrow Z)$ of chain complexes X, Y, and Z, with chain maps $i\colon X \to Y$ and $j\colon X \to Z$, we can create the following chain complex modeling the homotopy pushout in a category of chain complexes

$$C_n = Y_n \oplus X_{n-1} \oplus X_n \oplus X_{n-1} \oplus Z_n$$

with differential

$$\partial(y, x_{n-1}, x_n, x'_{n-1}, z_n) = \big(dy + i(x_{n-1}), -dx_{n-1}, dx_n - x_{n-1} + x'_{n-1}, -dx'_{n-1}, dz + j(x'_{n-1})\big)\,.$$

A visualization of the complex can be seen below. The complex C is the homotopy pushout $(Y \longleftarrow X \longrightarrow Z)$.



In the category of pointed chain complexes of strictly $\mathbb{A}^1$-invariant sheaves, the constant sheaf $\mathbb{Z}$ is the terminal object. We can now compute the wedge of two pointed complexes.

Proposition 7.8. *Let $C, D \in \mathcal{D}(Ab_{\mathbb{A}^1}(k))$ be pointed chain complexes with $C_0 = \mathbb{Z} \oplus \widetilde{C}_0$ and $D_0 = \mathbb{Z} \oplus \widetilde{D}_0$. Then the wedge $C \vee D$ is the chain complex*

$$(C \vee D)_n = \begin{cases} \mathbb{Z} \oplus \widetilde{C}_0 \oplus \widetilde{D}_0 & n = 0, \\ C_n \oplus D_n & n > 0. \end{cases}$$

Proof. Compute the homotopy pushout as in Definition 3.2 using a cone on the chain complex of the point. ☐

We can now compute the smash product of $C \wedge D \in \mathcal{D}(Ab_{\mathbb{A}^1}(k))$ as the homotopy pushout of the square

$$* \leftarrow C \vee D \rightarrow C \otimes D.$$

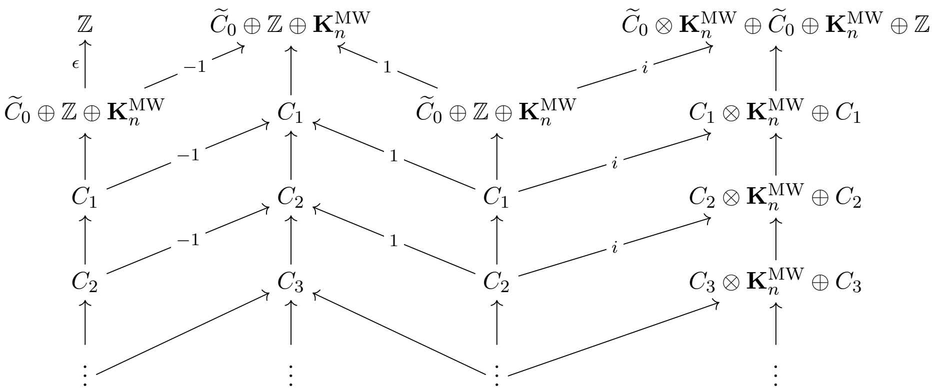
Lemma 7.9. *Let C be a pointed chain complex of strictly $\mathbb{A}^1$-invariant sheaves and let $n > 0$, then there is an isomorphism*

$$\mathbf{H}_i(C \wedge (\mathbb{Z} \oplus \mathbf{K}_n^{\mathrm{MW}})) \cong \begin{cases} \mathbb{Z} \oplus (\widetilde{\mathbf{H}}_0(C) \otimes \mathbf{K}_n^{\mathrm{MW}}) & i = 0, \\ \mathbf{H}_i(C) \otimes \mathbf{K}_n^{\mathrm{MW}} & i > 0. \end{cases}$$

Proof. To prove the quasi-isomorphism, we will compute the homology. We have the equivalence

$$C \wedge (\mathbb{Z} \oplus \mathbf{K}_n^{\mathrm{MW}}) \simeq \mathrm{colim}(* \longleftarrow C \vee (\mathbb{Z} \oplus \mathbf{K}_n^{\mathrm{MW}}) \longrightarrow C \otimes (\mathbb{Z} \oplus \mathbf{K}_n^{\mathrm{MW}}).$$

The chain complex for $C \wedge (\mathbb{Z} \oplus \mathbf{K}_n^{\mathrm{MW}})$ can be modeled as the following homotopy pushout:



All vertical arrows except ϵ are inherited from the differential d on C. The map ϵ is projection onto $\mathbb{Z}$. The map i is the inclusion. We may restrict ∂_1 to the factors $\mathbb{Z}, \mathbf{K}_n^{\mathrm{MW}}$, and $\widetilde{C}_0$.

$$\partial_1^{\mathbb{Z}} = \begin{pmatrix} \mathrm{Id}_{\mathbb{Z}} & -\mathrm{Id}_{\mathbb{Z}} & 0 \\ 0 & \mathrm{Id}_{\mathbb{Z}} & \mathrm{Id}_{\mathbb{Z}} \end{pmatrix} \qquad \partial_1^{\mathrm{MW}} = \begin{pmatrix} -\mathrm{Id}_{\mathbf{K}_n^{\mathrm{MW}}} & 0 \\ \mathrm{Id}_{\mathbf{K}_n^{\mathrm{MW}}} & -\mathrm{Id}_{\mathbf{K}_n^{\mathrm{MW}}} \end{pmatrix} \qquad \partial_1^{\widetilde{C}_0} = \begin{pmatrix} -\mathrm{Id}_{\widetilde{C}_0} & 0 \\ \mathrm{Id}_{\widetilde{C}_0} & -\mathrm{Id}_{\widetilde{C}_0} \end{pmatrix}$$

It is straightforward to check that all of the three restricted differentials above are surjective. The map $d_1^{\mathrm{MW}} \colon C_1 \otimes \mathbf{K}_n^{\mathrm{MW}} \to \widetilde{C}_0 \otimes \mathbf{K}_n^{\mathrm{MW}}$ has image $\mathrm{Im}(d_1) \otimes \mathbf{K}_n^{\mathrm{MW}}$. Thus

$$\mathbf{H}_0(C \wedge (\mathbb{Z} \oplus \mathbf{K}_n^{\mathrm{MW}})) \cong \mathbb{Z} \oplus (\widetilde{\mathbf{H}}_0(C) \otimes \mathbf{K}_n^{\mathrm{MW}}).$$

To compute $\ker \partial_1$, we first note that $\ker \partial_1^{\mathbb{Z}}$, $\ker \partial_1^{\mathrm{MW}}$, and $\ker \partial_1^{\widetilde{C}_0}$ are all trivial. We get $\ker d_1^{\mathrm{MW}} = \ker(d_1) \otimes \mathbf{K}_n^{\mathrm{MW}}$. The map onto $\widetilde{C}_0 \oplus \widetilde{C}_0$ is given by the matrix

$$\partial_1^C = \begin{pmatrix} -\mathrm{Id}_{\widetilde{C}_0} & 0 \\ \mathrm{Id}_{\widetilde{C}_0} & \mathrm{Id}_{\widetilde{C}_0} \\ d_1 & 0 \\ 0 & d_1 \end{pmatrix} : \widetilde{C}_0 \oplus \widetilde{C}_0 \oplus C_1 \oplus C_1 \to \widetilde{C}_0 \oplus \widetilde{C}_0.$$

Since $\mathrm{Im}\, d_1 \subset \widetilde{C}_0$, the kernel is $\ker \partial_1^C = C_1 \oplus C_1$. Thus $\ker \partial_1 = \ker \partial_1^C \oplus \ker d_1^{\mathrm{MW}}$. This result also extends to $\ker \partial_i = C_i \oplus C_i \oplus (\ker d_i \otimes \mathbf{K}_n^{\mathrm{MW}})$. Computing image of ∂_{i+1} yields $\mathrm{Im}\, \partial_i = C_i \oplus C_i \oplus (\mathrm{Im}\, d_{i+1} \otimes \mathbf{K}_n^{\mathrm{MW}})$. The homology of the complex is $\mathbf{H}_i(C) \otimes \mathbf{K}_n^{\mathrm{MW}}$ for $i \geq 1$. ☐

A similar proof strategy yields the following result.

Lemma 7.10. *Let C be a pointed chain complex of strictly $\mathbb{A}^1$-invariant sheaves, then there is an isomorphism*

$$\mathbf{H}_i(\Sigma C) \cong \begin{cases} \mathbb{Z} & i = 0, \\ \widetilde{\mathbf{H}}_{i-1}(C) & i > 0. \end{cases}$$

Applying Lemmas 7.9 and 7.10 to Corollary 7.6 yields the following theorem.

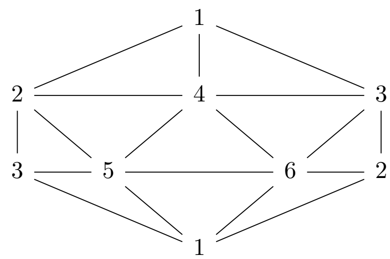
Theorem 7.11. *Let K be a simplicial complex. Then $\mathbf{H}_0^{\mathrm{cell}}(Z_K^{\mathbb{A}^1}) = \mathbb{Z}$ and for $i > 0$*

$$\mathbf{H}_i^{\mathrm{cell}}(Z_K^{\mathbb{A}^1}) \cong \bigoplus_{I \notin K} \widetilde{\mathbf{H}}_{i-1}(|K_I|) \otimes \mathbf{K}_{|I|}^{\mathrm{MW}}.$$

Example 7.12. Let $K = \partial \Delta^{m-1}$, then

$$\mathbf{H}_i^{\mathrm{cell}}(Z_K^{\mathbb{A}^1}) \cong \begin{cases} \mathbb{Z} & i = 0, \\ \mathbf{K}_m^{\mathrm{MW}} & i = m-1, \\ 0 & i \neq 0, m-1. \end{cases}$$

Example 7.13. Let K be the following triangulation of $\mathbb{RP}^2$. Vertices with the same labels are identified, and all triangles are filled in.



By Theorem 7.11, we get the following decomposition of the cellular $\mathbb{A}^1$-homology of $Z_K^{\mathbb{A}^1}$.

$$\mathbf{H}_i^{\text{cell}}(Z_K^{\mathbb{A}^1}) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 1, \\ (\mathbf{K}_3^{\text{MW}})^{\oplus 10} \oplus (\mathbf{K}_4^{\text{MW}})^{\oplus 15} \oplus (\mathbf{K}_5^{\text{MW}})^{\oplus 6} \oplus (\mathbb{Z}_2 \otimes \mathbf{K}_6^{\text{MW}}) & i = 2, \\ 0 & i \geq 3. \end{cases}$$

This is an example of a space with integral torsion in its cellular $\mathbb{A}^1$-homology.

Remark 7.14. Theorem 7.11 could be computed using Corollary 4.6 as follows. By Corollary 4.6 there is an equivalence

$$\Sigma(C_*^{\text{cell}}(\mathbb{A}^1), C_*^{\text{cell}}(\mathbb{G}_{\text{m}}))^K \simeq \bigvee_{I \notin K} \Sigma^2 |K_I| \wedge C_*^{\text{cell}}(\mathbb{G}_{\text{m}})^{\wedge |I|}.$$

Since Lemma 7.10 tells us how the homology changes after suspending and Lemma 7.9 in the case of $n = 1$ tells us what smashing with $C_*^{\text{cell}}(\mathbb{G}_{\text{m}})$ does to the homology of a chain complex. Computing the homology and accounting for the extra suspension recovers Theorem 7.11.

7.2 Cohomology

Motivic cohomology is a bigraded cohomology theory for $\mathcal{H}(k)$. For a motivic space X and a commutative ring $\mathbf{k}$, write $H_{\text{Mot}}^{i,j}(X; \mathbf{k})$ for the motivic cohomology of X with coefficients in $\mathbf{k}$. We denote the cohomology of the point by $A = H_{\text{Mot}}^{*,*}(\text{Spec}(k); \mathbf{k})$, thus for $p, q \in \mathbb{Z}$ we have $A^{p,q} = H_{\text{Mot}}^{p,q}(\text{Spec}(k); \mathbf{k})$. For a bigraded A-module M, and two integers $i, j \in \mathbb{Z}$ we define $M[i, j]$ to be M, but with the grading shifted to by (i, j), that is $M[i, j]^{p,q} = M^{p-i,q-j}$. With this, the cohomology of motivic spheres can be described as an A-module

$$H_{\text{Mot}}^{*,*}(S^{p,q}; \mathbf{k}) \cong A \oplus A[p, q].$$

We may use the stable splitting of $Z_K^{\mathbb{A}^1}$ to describe the cohomology groups of $Z_K^{\mathbb{A}^1}$ for certain simplicial complexes K. We denote the reduced motivic cohomology of a motivic space X by $\widetilde{H}_{\text{Mot}}^{i,j}(X; \mathbf{k})$.

Theorem 7.15. *Let K be a simplicial complex. The reduced motivic cohomology groups of $Z_K^{\mathbb{A}^1}$ are given by the isomorphism*

$$\widetilde{H}^{p,q}_{\mathrm{Mot}}(Z_K^{\mathbb{A}^1};\mathbf{k}) \cong \bigoplus_{I\notin K, |I|=j} \widetilde{H}^{p-j-1,q-j}_{\mathrm{Mot}}(|K_I|;\mathbf{k}).$$

Proof. For a motivic space X, there is the relation $\widetilde{H}^{p,q}_{\mathrm{Mot}}(X;\mathbf{k}) = \widetilde{H}^{p+i,q+j}_{\mathrm{Mot}}(X \wedge S^{i,j};\mathbf{k})$. Combining this with the stable splitting from Theorem 5.3 yields the desired result. □

In the case where $\Sigma|K_I|$ (as a topological space) is a wedge of spheres, we can express the cohomology of $Z_K^{\mathbb{A}^1}$ just in terms of shifted copies of A.

Proposition 7.16. *Let K be a simplicial complex on the vertex set $[m]$ such that $\Sigma|K_I|$ is a wedge of spheres for all $I \notin K$. Then there is an isomorphism of A-modules*

$$\widetilde{H}^{*,*}_{\mathrm{Mot}}(Z_K^{\mathbb{A}^1};\mathbf{k}) \cong \bigoplus_{\substack{I\notin K, |I|=j,\\ 0\le i\le m-2}} A[i+j+1,j]^{\oplus \mathrm{rank} \widetilde{H}^{i-j-1}(K_I;\mathbb{Z})}.$$

Proof. Fix $I \in K$. If $\Sigma|K_I|$ splits into a wedge of spheres, then its homotopy type is solely determined by the rank of its singular homology groups. This allows us to express the motivic cohomology of $|K_I|$ in terms of A.

$$\widetilde{H}^{*,*}_{\mathrm{Mot}}(|K_I|;\mathbf{k}) = \bigoplus_{i=0}^{m-2} A[i,0]^{\oplus \mathrm{rank} \widetilde{H}^{i}(K_I;\mathbb{Z})}$$

By summing over each $I \notin K$ and shifting according to suspensions by $\mathbb{G}_{\mathrm{m}}$, we recover the result. The reason the sum is not infinite is because K is a simplicial complex is finite dimensional. □

Remark 7.17. Simplicial complexes such as flag complexes and triangulations of spheres satisfy the conditions of Proposition 7.16.

Classically, figuring out the ring structure of the cohomology of the moment-angle complex can be done with the Eilenberg–Moore spectral sequence. Unfortunately, we run into some problems when using the same approach motivically as there is no suitable Eilenberg–Moore spectral sequence available for us to use.

Remark 7.18. There is a version of the Eilenberg-Moore spectral sequence due to Krishna [25], but it only computes cohomology groups. In particular, Krishna provides for each integer j, a spectral sequences converging to $H^{*,j}_{\mathrm{Mot}}$.

When the base field $k = \mathbb{C}$ we are able to say some things due to complex realization. Work by Levine [27] shows that complex realization is a symmetric monoidal functor from the stable motivic homotopy category to the stable homotopy category (of topological spaces). It follows that complex realization induces for any commutative ring $\mathbf{k}$ a $\mathbf{k}$-algebra homomorphism

$$\phi\colon \bigoplus_{i,j} H^{i,j}_{\mathrm{Mot}}(X;\mathbf{k}) \to \bigoplus_{i} H^{i}(X(\mathbb{C});\mathbf{k}).$$

Since ϕ is a $\mathbf{k}$-algebra morphism lets us pull back cup products from $H^i(X(\mathbb{C});\mathbf{k})$ to $H^{i,j}_{\mathrm{Mot}}(X;\mathbf{k})$. That is, if there exists $\alpha,\beta \in H^{*,*}_{\mathrm{Mot}}(Z_K^{\mathbb{A}^1};\mathbf{k})$ such that $\phi(\alpha) \smile \phi(\beta) \neq 0$, then there exists $\gamma \in H^{*,*}_{\mathrm{Mot}}(Z_K^{\mathbb{A}^1};\mathbf{k})$ such that $\alpha \smile \beta = \gamma$.

7.3 Betti numbers

A much coarser invariant than (co)homology are Betti numbers. By Theorem 2.19 the cohomology of the moment-angle complex $Z_K = (D^2, S^1)^K$ is isomorphic to a bigraded Tor-algebra

$$H^{2j-i}(Z_K) \cong \mathrm{Tor}^{i,j}_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z},\mathbb{Z}[K]).$$

The *bigraded Betti numbers* $b^{i,j}$ of Z_K are defined as follows

$$b^{i,j}(Z_K) := \mathrm{rank}\ \mathrm{Tor}^{i,j}_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z},\mathbb{Z}[K]).$$

When K is a simplicial complex where $\Sigma|K_I|$ is a wedge of spheres for all I, the description of the cohomology of $Z_K^{\mathbb{A}^1}$ from Proposition 7.16 allows us to define Betti numbers.

Definition 7.19. The (i,j)th $\mathbb{A}^1$-Betti number of $Z_K^{\mathbb{A}^1}$ is defined as follows

$$b^{i,j}_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) := \begin{cases} \sum_{I\notin K, |I|=j} \mathrm{rank}\ \widetilde{H}^{i-j-1}(K_I;\mathbb{Z}) & (i,j)\neq(0,0), \\ 1 & (i,j)=(0,0). \end{cases}$$

The following example highlights the choice of definition.

Example 7.20. Let $K = \partial\Delta^n$, then $Z_K^{\mathbb{A}^1} \simeq S^{2n,n-1}$ and

$$b^{i,j}_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \begin{cases} 1 & (i,j)=(0,0) \text{ or } (i,j)=(2n,n-1), \\ 0 & \text{otherwise.} \end{cases}$$

We now have two different bigraded Betti numbers related to (motivic) moment-angle complexes. The next step is to compare the two notions.

Theorem 7.21. *Let K be a simplicial complex. Then there is an equality*

$$b^{i,j}_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = b^{-j,2i}(Z_K).$$

Proof. In the classical case, we have the following well known isomorphism of groups due to Hochster [22, Theorem 5.1],

$$\mathrm{Tor}^{-j,2i}_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z},\mathbb{Z}[K]) \cong \bigoplus_{I\subset[m]:|I|=i} \widetilde{H}^{i-j-1}(K_I).$$

Thus the bigraded Betti numbers of Z_K can be expressed as

$$b^{-j,2i}(Z_K) = \sum_{I\subset[m]:|I|=i} \mathrm{rank}\ \widetilde{H}^{i-j-1}(K_I;\mathbb{Z}) = b^{i,j}_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}).$$

□

7.4 Euler characteristics

For any symmetric monoidal category $\mathcal{C}$, there is a notion of categorical Euler characteristic of a dualizable object [17]. Let $1_{\mathcal{C}}$ be the unit of $\mathcal{C}$. If X is a dualizable object in $\mathcal{C}$, then there exists a dual object $X^\vee$, an evaluation map $\epsilon\colon X \otimes X^\vee \to 1_{\mathcal{C}}$, and a coevaluation map $\eta\colon 1_{\mathcal{C}} \to X \otimes X^\vee$. The categorical Euler characteristic of a dualizable object X is the composition

$$1_{\mathcal{C}} \xrightarrow{\eta} X \otimes X^\vee \xrightarrow{\mathrm{id}_X \otimes \mathrm{id}_{X^\vee}} X \otimes X^\vee \xrightarrow{\epsilon} 1_{\mathcal{C}}.$$

We denote the categorical Euler characteristic by is written as $\chi_{\mathcal{C}}(X)$. We see that the categorical Euler characteristic takes values in $\mathrm{End}(1_{\mathcal{C}})$ endomorphism of the unit object of $\mathcal{C}$.

We denote the stable motivic homotopy category over a field k by $\mathrm{SH}(k)$. In the case of the stable motivic homotopy category $\mathrm{End}(1_{\mathrm{SH}(k)}) = \mathrm{GW}(k)$ [32, Theorem 6.4.1], where $\mathrm{GW}(k)$ denotes the Grothendieck–Witt ring of quadratic forms over k. The elements of $\mathrm{GW}(k)$ are formal differences of k-valued, non-degenerate, quadratic forms on finite dimensional k-vector spaces. For a unit $u \in k^\times$, we let $\langle u\rangle \in \mathrm{GW}(k)$ denote the rank one quadratic form $x \mapsto ux^2$. In addition, $\mathrm{GW}(k)$ is generated by the rank one forms as a group. For units $u, v \in k^\times$, we have $\langle u\rangle \cdot \langle v\rangle = \langle uv\rangle$. For any $u \in k^\times$, there is an equivalence $\langle u^2\rangle = \langle 1\rangle$. Thus $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ and $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$. Examples of computations of categorical Euler characteristics in the stable homotopy category can be found in [27, 29, 23].

In motivic homotopy theory, there is an $\mathbb{A}^1$-Euler characteristic closely related to $\chi_{\mathrm{SH}(k)}$. We say that a motivic space $X \in \mathcal{H}(k)$ is *dualizable* if the $\mathbb{P}^1$-suspension spectrum $\Sigma^\infty_{\mathbb{P}^1} X_+$ is dualizable in $\mathrm{SH}(k)$. Let X be a dualizable object in $\mathcal{H}(k)$, then the $\mathbb{A}^1$-Euler characteristic $\chi_{\mathbb{A}^1}(X)$ is defined as follows

$$\chi_{\mathbb{A}^1}(X) := \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X_+).$$

For a finite CW-space X, let $\chi(X)$ denote its Euler characteristic. Real and complex realization allows us to relate the $\mathbb{A}^1$-Euler characteristic to its classical counterpart. When k is a subfield of $\mathbb{R}$, we have the following relation between the $\mathbb{A}^1$-Euler characteristic and the Euler characteristic of the real and complex points of X [28, Remark 1.3].

$$\chi(X(\mathbb{C})) = \mathrm{rank}\ \chi_{\mathbb{A}^1}(X) \quad \text{and} \quad \chi(X(\mathbb{R})) = \mathrm{sign}\ \chi_{\mathbb{A}^1}(X).$$

If k is just a subfield of $\mathbb{C}$, we only have the first relation for the complex points.

Example 7.22. Recall that $\mathbb{G}_m(\mathbb{C}) \simeq S^1$ and $\mathbb{G}_m(\mathbb{R}) \simeq S^0$. Let k be a field of characteristic different from 2. Then $\chi_{\mathbb{A}^1}(\mathbb{G}_\mathrm{m}) = \langle 1\rangle - \langle -1\rangle$. When $k = \mathbb{C}$, all units are squares. Thus $\langle -1\rangle = \langle 1\rangle$, and $\chi_{\mathbb{A}^1}(\mathbb{G}_\mathrm{m}) = 0 = \chi(S^1)$. When $k = \mathbb{R}$, the signature of $\langle 1\rangle - \langle -1\rangle$ is 2 which coincides with $\chi(S^0)$.

The Euler characteristic of a moment-angle complex is not an interesting invariant, because by Example 2.18 $\chi(Z_K) = 0$ for any K. However, the real moment-angle complex can have nonzero Euler characteristic. The realization result above thus suggests that the motivic moment-angle complex does not have trivial Euler characteristic when k is a subfield of the real numbers.

The first thing we need is to show that $Z_K^{\mathbb{A}^1}$ admits an $\mathbb{A}^1$-Euler characteristic.

Lemma 7.23. *Let K be a simplicial complex. Then $Z_K^{\mathbb{A}^1}$ is dualizable in $\mathcal{H}(k)$.*

Proof. Using Theorem 5.3, the $\mathbb{P}^1$-suspension spectrum of $Z_K^{\mathbb{A}^1}$ can be written as

$$\Sigma^\infty_{\mathbb{P}^1} {Z_K^{\mathbb{A}^1}}_+ = \Sigma^\infty_{\mathbb{P}^1}(S^{0,0}) \vee \Sigma^{-1,0} \bigvee_{I \notin K} \Sigma^{|I|+2,|I|} \Sigma^\infty_{\mathbb{P}^1} |K_I|.$$

Let SH be the stable homotopy category of topological spaces. There is a symmetric monoidal functor SH $\to$ SH(k), which in particular preserves dualizable objects. Since the spaces $\Sigma^\infty_{\mathbb{P}^1}|K_I|$ are images of finite CW-spectra under this map, they are also dualizable. Since spheres and wedges of dualizable spaces are dualizable, $Z_K^{\mathbb{A}^1}$ is dualizable. $\square$

Earlier, we saw that Davis had computed the Euler characteristic of polyhedral products (Theorem 2.17). We can recover the result for motivic moment-angle complexes.

Theorem 7.24. *The $\mathbb{A}^1$-Euler characteristic of the motivic moment-angle complex is*

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \sum_{\sigma \in K} \langle -1 \rangle^{|\sigma|} (\langle 1 \rangle - \langle -1 \rangle)^{m-|\sigma|} = \sum_{\sigma \in K} (-1)^{|\sigma|} 2^{m-|\sigma|-1} (\langle 1 \rangle - \langle -1 \rangle).$$

Proof. By using [28, Lemma 1.4(3)], we can express the $\mathbb{A}^1$-Euler characteristic of a smooth scheme in terms of an open subscheme and its complement. By applying this to the cellular structure of $Z_K^{\mathbb{A}^1}$ from Proposition 7.2, we recover the theorem. $\square$

We will also give a computation of $\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1})$ that uses the stable splitting, and hence describes the $\mathbb{A}^1$-Euler characteristic in terms of the Euler characteristic of full subcomplexes of K. For a simplicial complex K, we write $\chi(K)$ for it classical Euler characteristic.

Theorem 7.25. *Let K be a simplicial complex. The $\mathbb{A}^1$-Euler characteristic of the motivic moment-angle complex is*

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \langle 1 \rangle - \sum_{I \notin K} (-1)^{|I|} (\chi(K_I) - 1) \cdot \langle -1 \rangle^{|I|}.$$

We postpone the proof of the theorem until the end of the section.

Example 7.26. Let K be a square, as in Example 2.7. Then K has four vertices and four edges. Theorem 7.24, we compute

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = (2^3 - 4 \cdot 2^2 + 4 \cdot 2) \cdot (\langle 1 \rangle - \langle -1 \rangle) = 0.$$

Using Theorem 7.25, we have three full subcomplexes of K corresponding to the cases where $I = \{1,2\}, \{3,4\}$ or $\{1,2,3,4\}$.

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \langle 1 \rangle - 2(\chi(S^0) - 1) \cdot \langle 1 \rangle - (\chi(S^1) - 1) \cdot \langle 1 \rangle = (1 - 2 + 1)\langle 1 \rangle = 0.$$

The $\mathbb{A}^1$-Euler characteristic also exhibits some nice properties like the classical Euler characteristic.

Lemma 7.27. $\chi_{\mathbb{A}^1}(X \vee Y) = \chi_{\mathbb{A}^1}(X) + \chi_{\mathbb{A}^1}(Y) - \langle 1 \rangle$.

Proof. The wedge $X \vee Y$ may be written as the homotopy pushout

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & X \vee Y. \end{array}$$

In [31] May proved that the following holds for the categorical Euler characteristic.

$$\chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X \vee Y) = \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X) + \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} Y) - \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} \mathrm{Spec}(k))$$

The diagram above is still a homotopy pushout after adding a disjoint basepoint, thus we get

$$\chi_{\mathbb{A}^1}(X \vee Y) = \chi_{\mathbb{A}^1}(X) + \chi_{\mathbb{A}^1}(Y) - \langle 1 \rangle. \qquad \square$$

In classical topology we have the following relation between the Euler characteristic of a space X and its suspension ΣX

$$\chi(\Sigma X) = 2 - \chi(X).$$

Similar to the classical relation, we have the following relation of $\mathbb{A}^1$-Euler characteristics.

Lemma 7.28. $\chi_{\mathbb{A}^1}(X \wedge S^{p,q}) = \langle 1 \rangle + (-1)^p \langle -1 \rangle^q (\chi_{\mathbb{A}^1}(X) - \langle 1 \rangle)$.

Proof. We have

$$\chi_{\mathbb{A}^1}(X \wedge S^{p,q}) = \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}(X \wedge S^{p,q})) + \langle 1 \rangle.$$

For the categorical Euler characteristic, May [31] proved that

$$\chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X \wedge S^{p,q}) = \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X) \cdot \chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} S^{p,q}).$$

We may rewrite $\chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X) = \chi_{\mathbb{A}^1}(X) - \langle 1 \rangle$ and use the $\chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} S^{p,q}) = (-1)^p \langle -1 \rangle^q$ by [28, Lemma 1.2] to get the claimed result. $\square$

We can now prove Theorem 7.25.

Proof of Theorem 7.25. By Lemma 7.28, the $\mathbb{A}^1$-Euler characteristic of $Z_K^{\mathbb{A}^1}$ can be expressed in the following way

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = 2\langle 1 \rangle - \chi_{\mathbb{A}^1}(\Sigma Z_K^{\mathbb{A}^1}).$$

Since $\Sigma Z_K^{\mathbb{A}^1}$ splits into a wedge sum by Theorem 5.3, we get

$$\begin{aligned}\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) &= 2\langle 1\rangle - \chi_{\mathbb{A}^1}(\bigvee_{I\notin K} \Sigma^2|K_I| \wedge \mathbb{G}_{\mathrm{m}}^{\wedge|I|}) \\ &= 2\langle 1\rangle - \chi_{\mathbb{A}^1}(\bigvee_{I\notin K} |K_I| \wedge S^{|I|+2,|I|}).\end{aligned}$$

We continue by applying Lemma 7.27 to the wedge sum resulting in

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \langle 1\rangle - \sum_{I\notin K} \left(\chi_{\mathbb{A}^1}(|K_I| \wedge S^{|I|+2,|I|}) - \langle 1\rangle\right).$$

We then apply Lemma 7.28 to get

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \langle 1\rangle - \sum_{I\notin K}(-1)^{|I|}\langle -1\rangle^{|I|}(\chi_{\mathbb{A}^1}(|K_I|) - \langle 1\rangle).$$

For any simplicial complex K, we have $\chi_{\mathbb{A}^1}(|K|) = \chi(K)\cdot\langle 1\rangle$. This allows us to rewrite the result above to

$$\chi_{\mathbb{A}^1}(Z_K^{\mathbb{A}^1}) = \langle 1\rangle - \sum_{I\notin K}(-1)^{|I|}(\chi(K_I) - 1)\cdot\langle -1\rangle^{|I|}. \qquad \square$$

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Paper III

POLYHEDRAL COPRODUCTS

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Abstract. Dualising the construction of a polyhedral product, we introduce the notion of a *polyhedral coproduct* as a certain homotopy limit over the face poset of a simplicial complex. We begin a study of the basic properties of polyhedral coproducts, surveying the Eckmann–Hilton duals of various familiar examples and properties of polyhedral products. In particular, we show that polyhedral coproducts give a functorial interpolation between the wedge and cartesian product of spaces which differs from the one given by polyhedral products, and we establish a general loop space decomposition for these spaces which is dual to the suspension splitting of a polyhedral product due to Bahri, Bendersky, Cohen and Gitler.

Contents

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1. Introduction

Polyhedral products are natural subspaces of cartesian products defined as certain colimits over the face poset of a finite simplicial complex K. This construction generalises and unifies into a common combinatorial framework many familiar methods of constructing new topological spaces from given ones—for example, products, wedge sums, joins, half-smash products and the fat wedge construction are all special cases. Since their introduction by Bahri, Bendersky, Cohen and Gitler [BBCG1], the topology of polyhedral products has become a growing topic of investigation within homotopy theory and has made fruitful contact with many other areas of mathematics. Notable examples include toric topology, following Buchstaber–Panov's [BP] formulation of moment-angle complexes as polyhedral products; commutative algebra, where polyhedral products give geometric realisations of Stanley–Reisner rings and their Tor algebras; and geometric group theory, where polyhedral products model the classifying spaces of right-angled Artin and Coxeter groups. Other examples include robotics [HCK, KT] and topological data analysis [BLPSS]. For more on the history and far-reaching applications of polyhedral products, we recommend the excellent survey [BBC] and references therein.

Motivated by the ubiquity and utility of polyhedral products, the purpose of this paper is to propose a definition for the dual notion of a *polyhedral coproduct* and begin a study of its basic properties. Before describing the main results, we first review the construction of polyhedral products more precisely.

Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be an m-tuple of pointed CW-pairs. The *polyhedral product* associated to $(\underline{X}, \underline{A})$ is the functor

$$(\underline{X}, \underline{A})^{(-)} \colon \mathbf{SCpx}_m \to \mathbf{Top}_*$$

which associates to each simplicial complex K on the vertex set $[m] = \{1, \ldots, m\}$ the (homotopy) colimit

$$(\underline{X}, \underline{A})^K = \operatorname*{hocolim}_{\sigma \in K} \prod_{i=1}^m Y_i(\sigma),$$

where $Y_i \colon \mathrm{cat}(K) \to \mathbf{Top}_*$ is the diagram defined for each $i \in [m]$ by

$$Y_i(\sigma) = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Here $\mathrm{cat}(K)$ denotes the face poset of K, regarded as a small category with objects given by faces $\sigma \in K$ and morphisms given by face inclusions $\tau \subset \sigma$. We denote the initial object of $\mathrm{cat}(K)$ by $\varnothing$, which corresponds to the empty face of K.

As has been pointed out in [KL, NR, WZZ], for example, the homotopy colimit above agrees up to homotopy with the usual colimit $\bigcup_{\sigma \in K} \prod_{i=1}^m Y_i(\sigma)$ since each (X_i, A_i) is an NDR-pair. In particular, the polyhedral product $(\underline{X}, \underline{A})^K$ is a cellular subcomplex of $\prod_{i=1}^m X_i$ for all K. In the case that $A_i = *$ for all $i \in [m]$, this subcomplex $(\underline{X}, \underline{*})^K$ naturally interpolates between the wedge $\bigvee_{i=1}^m X_i$ (when K consists of m disjoint vertices) and the product $\prod_{i=1}^m X_i$ (when $K = \Delta^{m-1}$ is the simplex on m vertices).

Dualising the definition of a polyhedral product as a homotopy colimit of products, we define a polyhedral coproduct as a homotopy limit of coproducts, as follows.

Definition 1.1. Let $\underline{f} = (f_1, \dots, f_m)$ be an m-tuple of maps $f_i \colon X_i \to A_i$ of pointed spaces. We define the *polyhedral coproduct* associated to $\underline{f}$ as the functor

$$\underline{f}_{\mathrm{co}}^{(-)} \colon \mathbf{SCpx}_m \to \mathbf{Top}_*$$

which associates to each simplicial complex K on $[m]$ the homotopy limit

$$\underline{f}_{\mathrm{co}}^K = \operatorname*{holim}_{\sigma \in K} \bigvee_{i=1}^m Y_i(\sigma),$$

where $Y_i \colon \mathrm{cat}(K)^{\mathrm{op}} \to \mathbf{Top}_*$ is the diagram defined for each $i \in [m]$ by

$$Y_i(\sigma) = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Note that for a face inclusion $\tau \subset \sigma \in K$, there are maps $Y_i(\sigma) \to Y_i(\tau)$ defined for each $i \in [m]$ by f_i, if $i \in \sigma \backslash \tau$ and by the identity map otherwise, and hence there is an induced map

$$\bigvee_{i=1}^m Y_i(\sigma) \to \bigvee_{i=1}^m Y_i(\tau).$$

For a family $(\underline{X}, \underline{A})$ of pairs of spaces, if the maps $f_i \colon X_i \to A_i$ are clear from context, we will sometimes denote $\underline{f}_{\mathrm{co}}^K$ by $(\underline{X}, \underline{A})_{\mathrm{co}}^K$. One example is the case that $A_i = *$ is a point, and f_i is the constant map for all $i \in [m]$. In this case, as we show in Section 2, the polyhedral coproduct $(\underline{X}, \underline{*})_{\mathrm{co}}^K$ naturally interpolates between $\bigvee_{i=1}^m X_i$ (when $K = \Delta^{m-1}$) and $\prod_{i=1}^m X_i$ (when K is m disjoint vertices).

Although we restrict our attention to constructions in $\mathbf{Top}_*$ in this paper, note that polyhedral (co)products could be defined more generally in any model category $\mathcal{C}$, for example, by replacing the category of pointed spaces with $\mathcal{C}$ in the definitions above. Since any (closed) model category has an initial object and a terminal object, the polyhedral products and coproducts of the form $(\underline{X}, \underline{*})^K$ and $(\underline{X}, \underline{*})_{\mathrm{co}}^K$ can be defined in this setting to yield functorial interpolations between the categorical product and coproduct in $\mathcal{C}$.

For polyhedral products, the relationship between the combinatorics of K and the homotopy type of the space $(\underline{X}, \underline{*})^K$ interpolating between the m-fold wedge and m-fold product is made clear after suspending. By [BBCG2, Theorem 2.15], there is a natural homotopy equivalence

$$\Sigma(\underline{X}, \underline{*})^K \simeq \bigvee_{\sigma \in K} \Sigma X^{\wedge \sigma}, \tag{1}$$

where $X^{\wedge\sigma} = X_{i_1} \wedge \cdots \wedge X_{i_k}$ for each face $\sigma = \{i_1, \ldots, i_k\} \in K$. Notice that this generalises the well-known splitting of $\Sigma\big(\prod_{i=1}^m X_i\big)$ when $K = \Delta^{m-1}$, in which case the wedge above is indexed over all subsets of the vertex set $[m]$. For polyhedral coproducts, we dualise the suspension splitting (1) by establishing a loop space decomposition for $(\underline{X}, \underline{*})^K_{\mathrm{co}}$ involving a product indexed by the faces of the simplicial complex K (see Theorem 4.3). This similarly generalises a product decomposition due to Porter for $\Omega\big(\bigvee_{i=1}^m X_i\big)$ when $K = \Delta^{m-1}$.

The equivalence (1) is a special case of the more general Bahri–Bendersky–Cohen–Gitler splitting (henceforth, BBCG splitting) which identifies the homotopy type of any polyhedral product $(\underline{X}, \underline{A})^K$ as a certain wedge after suspending once. In [BBCG2], the authors describe the BBCG splitting as a generalisation of a lemma regarding homotopy colimits of certain diagrams due to Welker, Ziegler, and Živaljević [WZZ]. We first dualise the Welker–Ziegler–Živaljević lemma (see Lemma 3.7), and then use this to dualise the BBCG splitting. This gives a general loop space decomposition for an arbitrary polyhedral coproduct $\underline{f}^K_{\mathrm{co}}$ (see Theorem 4.2). We investigate special cases analogous to important examples of the BBCG splitting, and speculate on potential deloopings in Section 4.

Definition 1.1 is alternate to Theriault's definition of a *dual polyhedral product*, which was introduced in [T] and used to identify the Lusternik–Schnirelmann cocategory of a simply connected space X with the homotopy nilpotency of its loop space ΩX. Although the two notions coincide in some special cases (see Remark 2.2), the diagrams defining polyhedral coproducts and dual polyhedral products are very different in general, and our definition is more suitable for dualising the BBCG splitting of $\Sigma(\underline{X}, \underline{A})^K$ (see Section 4).

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2. Basic properties

2.1. **Basic examples.** We begin by computing some basic examples of polyhedral coproducts, in each case illustrating the Eckmann–Hilton duality between these constructions and their corresponding polyhedral products.

Example 2.1 (The $\underline{A} = \underline{*}$ case)**.**

(1) Let K be m disjoint vertices. In this case, the polyhedral product associated to the m-tuple of pairs $(\underline{X}, \underline{*}) = \{(X_i, *)\}_{i=1}^m$ is the wedge

$$(\underline{X}, \underline{*})^K \simeq X_1 \vee \cdots \vee X_m.$$

Dually, if $f_i \colon X_i \to *$ is the constant map for each $i = 1, \dots, m$, then by definition the corresponding polyhedral coproduct is given by

$$(\underline{X}, \underline{*})^K_{\text{co}} \simeq X_1 \times \cdots \times X_m.$$

(2) On the other extreme, let $K = \Delta^{m-1}$. The polyhedral product associated to $(\underline{X}, \underline{*})$ in this case is

$$(\underline{X}, \underline{*})^K \simeq X_1 \times \cdots \times X_m.$$

Since the diagram defining $(\underline{X}, \underline{*})^K_{\text{co}}$ has an initial object corresponding to the maximal face of the simplex Δ^{m-1},

$$(\underline{X}, \underline{*})^K_{\text{co}} \simeq X_1 \vee \cdots \vee X_m.$$

(3) Let $K = \partial\Delta^{m-1}$. The polyhedral product $(\underline{X}, \underline{*})^K$ in this case is precisely the *fat wedge* of the spaces $X_1, \dots, X_m$, which is defined as

$$FW(X_1, \dots, X_m) = \{(x_1, \dots, x_m) \mid x_i = * \text{ for at least one } i\}.$$

Dual to the fat wedge is the *thin product* of $X_1, \dots, X_m$, as defined by Hovey in [Ho]. This construction is realised by the polyhedral coproduct $(\underline{X}, \underline{*})^K_{\text{co}}$.

Remark 2.2. The dual polyhedral product, denoted $(\underline{X}, \underline{A})^K_D$, defined by Theriault [T] also models some of the spaces in Example 2.1. In particular, when K is m disjoint points, $(\underline{X}, \underline{*})^K_D$ is equal to the thin product of $X_1, \dots, X_m$. When $K = \partial\Delta^{m-1}$, $(\underline{X}, \underline{*})^K_D \simeq X_1 \times \cdots \times X_m$. Outside of these cases, it is not clear whether there is any correspondence between the dual polyhedral product, and the polyhedral coproduct. Theriault also used the dual polyhedral product to give a loop space decomposition of the thin product. An alternate loop space decomposition of the thin product can be recovered in the context of polyhedral coproducts by Theorem 4.3.

Just like the polyhedral product $(\underline{X}, \underline{*})^K$, the polyhedral coproduct $(\underline{X}, \underline{*})^K_{\text{co}}$ interpolates between the categorical product $X_1 \times \cdots \times X_m$ and coproduct $X_1 \vee \cdots \vee X_m$ as K interpolates between a discrete set of vertices and a full simplex. Next, we compute two further examples of $\underline{f}^K_{\text{co}}$ where the m-tuple $\underline{f}$ involves maps other than the constant map $X_i \to *$. An important class of polyhedral products (which includes generalised moment-angle complexes $(D^n, S^{n-1})^K$) is given by those associated to CW-pairs $(\underline{CX}, \underline{X}) = \{(CX_i, X_i)\}_{i=1}^m$ consisting of cones and their bases. The first example below dualises this case by replacing the cofibrations $X_i \hookrightarrow CX_i$ with path space fibrations $PX_i \to X_i$.

Example 2.3 (Dual of the join)**.** Let $K = \partial\Delta^1$ be two disjoint vertices so that the only faces of K are $\varnothing$, $\{1\}$ and $\{2\}$, and its face poset is given by $\{1\} \leftarrow \varnothing \rightarrow \{2\}$. In this case the polyhedral product $(\underline{CX}, \underline{X})^K$ recovers the join of X_1 and X_2 as a pushout:

$$(\underline{CX}, \underline{X})^K = CX_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times CX_2 \simeq X_1 \star X_2.$$

For $i \in \{1, 2\}$, let $f_i \colon PX_i \to X_i$ be the path space fibration over X_i. The polyhedral coproduct $\underline{f}^K_{\mathrm{co}} = (\underline{PX}, \underline{X})^K_{\mathrm{co}}$ is then the homotopy limit of the middle column of the commutative diagram

$$\begin{array}{ccccc}
* & \longrightarrow & X_1 & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & X_1 \vee X_2 & \longrightarrow & X_1 \times X_2 \\
\uparrow & & \uparrow & & \uparrow \\
* & \longrightarrow & X_2 & \longrightarrow & X_2,
\end{array}$$

where the vertical maps are inclusions and the rows are homotopy fibrations. The homotopy limit of the right column is contractible, so by taking homotopy limits of the columns we obtain a homotopy equivalence $(\underline{PX}, \underline{X})^K_{\mathrm{co}} \simeq \Omega F$. By [G], there is a homotopy equivalence $F \simeq \Sigma(\Omega X_1 \wedge \Omega X_2)$, and so there is a homotopy equivalence $(\underline{PX}, \underline{X})^K_{\mathrm{co}} \simeq \Omega\Sigma(\Omega X_1 \wedge \Omega X_2)$. The space $(\underline{PX}, \underline{X})^K_{\mathrm{co}}$ is known as the *cojoin* of X_1 and X_2.

Example 2.4 (Dual of the half-smash)**.** Let $K = \partial\Delta^1$ be two disjoint vertices and consider the CW-pairs $(\underline{X}, \underline{A}) = \{(CX, X), (Y, *)\}$. As in the previous example, the polyhedral product is a pushout $(\underline{X}, \underline{A})^K = CX \times * \cup_{X \times *} X \times Y$. Since $CX \times *$ is contractible, this is simply the cofibre of the inclusion $X \times * \hookrightarrow X \times Y$, which by definition is the *half-smash product*

$$(\underline{X}, \underline{A})^K \simeq X \ltimes Y.$$

To dualise this example, let $\underline{f} = (f_1, f_2)$ where $f_1 \colon PX \to X$ is the path space fibration and $f_2 \colon Y \to *$ is the constant map. Then by definition, the polyhedral coproduct is given by

$$\begin{aligned}
\underline{f}^K_{\mathrm{co}} &= \operatorname{holim}(PX \vee * \to X \vee * \leftarrow X \vee Y) \\
&\simeq \operatorname{hofib}(X \vee Y \xrightarrow{\pi_X} X),
\end{aligned}$$

the expected Eckmann–Hilton dual of the cofibre $(\underline{X}, \underline{A})^K = \operatorname{hocofib}(X \xrightarrow{i_X} X \times Y)$ above. The homotopy fibre of the projection onto a wedge summand can be identified using Mather's Cube Lemma [Ma], and we therefore obtain that the dual of the half-smash is given by

$$\underline{f}^K_{\mathrm{co}} \simeq \Omega X \ltimes Y.$$

Moreover, by Mather's Cube Lemma or [G, Theorem 1.1], there is a homotopy fibration

$$\Sigma\Omega X \wedge \Omega Y \to \Omega X \ltimes Y \to Y,$$

where the right map is the projection map. The projection has a right homotopy inverse, implying there is a homotopy equivalence

$$\Omega(\Omega X \ltimes Y) \simeq \Omega Y \times \Omega(\Sigma\Omega X \wedge \Omega Y).$$

This result can be recovered in the context of polyhedral coproducts by Theorem 4.2.

2.2. **Functorial properties.** The polyhedral product is a bifunctor (see [BBCG2, Remark 2.3]). Namely, it defines a functor from the category of (m-tuples of) CW-pairs to the category of CW-complexes, and it also defines a functor from the category of simplicial complexes to the category of CW-complexes. In this section, we prove that the polyhedral coproduct enjoys similar functorial properties. First, we show naturality with respect to maps of spaces.

Theorem 2.5. *Let K be a simplicial complex on $[m]$. For $1 \leqslant i \leqslant m$, let $f_i\colon X_i \to A_i$ and $f'_i\colon X'_i \to A'_i$ be maps. If there are maps $g_i\colon X_i \to X'_i$ and $h_i\colon A_i \to A'_i$ such that the diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{g_i} & X'_i \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f'_i} \\ A_i & \xrightarrow{h_i} & A'_i \end{array} \tag{2}$$

homotopy commutes, then there is an induced map $\underline{f}^K_{\mathrm{co}} \to \underline{f'}^K_{\mathrm{co}}$.

Proof. Let D_K and D'_K be the diagrams defining $\underline{f}^K_{\mathrm{co}}$ and $\underline{f'}^K_{\mathrm{co}}$ respectively. For a face $\sigma \in K$, define a map $F_\sigma : D_K(\sigma) \to D'_K(\sigma)$, defined by

$$F_\sigma : D_K(\sigma) = \bigvee_{i=1}^m Y_i \xrightarrow{\bigvee_{i=1}^m \phi_i} \bigvee_{i=1}^m Y'_i = D'_K(\sigma),$$

where $\phi_i = g_i$ if $i \in \sigma$, and $\phi_i = h_i$ if $i \notin \sigma$. By (2), F_σ induces a natural transformation $D_K \to D'_K$, which in turn induces a map $\underline{f}^K_{\mathrm{co}} \to \underline{f'}^K_{\mathrm{co}}$. □

The definition of $\underline{f}^K_{\mathrm{co}}$ is also natural with respect to simplicial inclusions.

Theorem 2.6. *Let K be a simplicial complex on $[m]$, and let L be a subcomplex of K on $[n]$ with $n \leqslant m$. Then the simplicial inclusion $L \to K$ induces a map $\underline{f}^K_{\mathrm{co}} \to \underline{f}^L_{\mathrm{co}}$.*

Proof. Let D_K and D_L be the diagrams defining $\underline{f}^K_{\mathrm{co}}$ and $\underline{f}^L_{\mathrm{co}}$ respectively. Let D^K_L be the diagram indexed by $\mathrm{cat}(L)$ which is defined by $D^K_L(\sigma) = \bigvee\limits_{i=1}^m Y_i$, where $Y_i = X_i$ if $i \in \sigma$, and $Y_i = A_i$ if $i \notin \sigma$. By definition of $\underline{f}^K_{\mathrm{co}}$ as a homotopy limit, there are canonical maps $\underline{f}^K_{\mathrm{co}} \to D^K_L(\sigma)$ for all $\sigma \in L$, and so the inclusion $\mathrm{cat}(L) \to \mathrm{cat}(K)$ induces a map $\underline{f}^K_{\mathrm{co}} \to \operatorname{holim} D^K_L$.

Now define a natural transformation of diagrams $D^K_L \to D_L$ by the pinch map

$$D^K_L(\sigma) = \bigvee_{i=1}^m Y_i \to \bigvee_{i=1}^n Y_i = D_L(\sigma).$$

This induces a map $\operatorname{holim} D_L^K \to \underline{f}_{\text{co}}^L$. Therefore, the simplicial inclusion induces the composite

$$\underline{f}_{\text{co}}^K \to \operatorname{holim} D_L^K \to \underline{f}_{\text{co}}^L. \qquad \square$$

Remark 2.7. The map $\underline{f}_{\text{co}}^K \to \underline{f}_{\text{co}}^L$ can be represented as the homotopy limit of a map of diagrams $D_K \to D_L$. For each $\sigma \in L$, we have a pinch map $D_K(\sigma) \to D_L(\sigma)$. By computing $\operatorname{holim} D_K$, one can see that the maps $\underline{f}_{\text{co}}^K \to D_L(\sigma)$ for $\sigma \in L$ are the maps described in the proof of Theorem 2.6.

2.3. **Retractions.** Let K be a simplicial complex and L a full subcomplex of K. For polyhedral products, by [DS, Lemma 2.2.3], there is a map $(\underline{X}, \underline{A})^K \to (\underline{X}, \underline{A})^L$ which is a left inverse for the map $(\underline{X}, \underline{A})^L \to (\underline{X}, \underline{A})^K$. In the case of polyhedral coproducts, there is an analogous statement.

Theorem 2.8. *Let K be a simplicial complex on $[m]$ and L be a full subcomplex of K on $[n]$, with $n < m$. Then there is a right homotopy inverse for the map $\underline{f}_{\text{co}}^K \to \underline{f}_{\text{co}}^L$ induced by the simplicial inclusion $L \to K$.*

Proof. Let D_K and D_L be the diagrams defining $\underline{f}_{\text{co}}^K$ and $\underline{f}_{\text{co}}^L$ respectively. Recall from the proof of Theorem 2.6 the diagram D_L^K indexed by $\operatorname{cat}(L)$, which is defined by $D_L^K(\sigma) = \bigvee_{i=1}^m Y_i$, where $Y_i = X_i$ if $i \in \sigma$, and $Y_i = A_i$ if $i \notin \sigma$. Define a natural transformation $D_L \to D_L^K$ by the inclusion

$$D_L(\sigma) = \bigvee_{i=1}^n Y_i \hookrightarrow \bigvee_{i=1}^m Y_i = D_L^K(\sigma)$$

This induces a map $\underline{f}_{\text{co}}^L \xrightarrow{f} \operatorname{holim} D_L^K$. Define a functor $F : \operatorname{cat}(K) \to \operatorname{cat}(L)$ by sending $\sigma \in K$ to the face $\tau \in L$, where τ is obtained from σ by removing any instances of the vertices $\{n+1, \ldots, m\}$. Since L is a full subcomplex, F is well defined. The functor F induces a map $\operatorname{holim} D_L^K \xrightarrow{g} \underline{f}_{\text{co}}^K$. Therefore, we obtain a composite

$$\underline{f}_{\text{co}}^L \xrightarrow{f} \operatorname{holim} D_L^K \xrightarrow{g} \underline{f}_{\text{co}}^K.$$

Now consider the composite

$$\phi : \underline{f}^L \xrightarrow{f} \operatorname{holim} D_L^K \xrightarrow{g} \underline{f}_{\text{co}}^K \xrightarrow{h} \operatorname{holim} D_L^K \xrightarrow{k} \underline{f}_{\text{co}}^L,$$

where the composite $\underline{f}_{\text{co}}^K \xrightarrow{h} \operatorname{holim} D_L^K \xrightarrow{k} \underline{f}_{\text{co}}^L$ is defined as in Theorem 2.6. By definition of the functor F, the composite $\operatorname{cat}(L) \hookrightarrow \operatorname{cat}(K) \xrightarrow{F} \operatorname{cat}(L)$ is the identity, and so the composite $h \circ g$ is the identity. For a face σ, the natural transformation inducing the composite $k \circ f$ is the identity on $D_L^K(\sigma)$, and so $k \circ f$ is the identity. Hence, ϕ is the identity map, and so the composite $g \circ f$ is a right homotopy inverse for the map induced by $L \to K$. $\square$

2.4. **Homotopy cofibrations.** For polyhedral products, it was shown in [DS, Lemma 2.3.1] that there exists a homotopy fibration

$$(\underline{C\Omega X}, \underline{\Omega X})^K \to (\underline{X}, \underline{*})^K \to \prod_{i=1}^{m} X_i,$$

which splits after looping. More generally, it was shown in [HST, Theorem 2.1] that there is a homotopy fibration

$$(\underline{CY}, \underline{Y})^K \to (\underline{X}, \underline{A})^K \to \prod_{i=1}^{m} X_i,$$

where Y_i is the homotopy fibre of the inclusion $A_i \to X_i$. Moreover, this homotopy fibration also splits after looping, giving a homotopy equivalence

$$\Omega(\underline{X}, \underline{A})^K \simeq \prod_{i=1}^{m} \Omega X_i \times \Omega(\underline{CY}, \underline{Y})^K.$$

This implies that to understand the loop spaces of polyhedral products, and therefore their homotopy groups, it suffices to study polyhedral products of the form $\Omega(\underline{CY}, \underline{Y})^K$. Loop space decompositions of certain polyhedral products of this form have been studied in [PT, S]. For polyhedral coproducts, one might hope there is a homotopy cofibration

$$\bigvee_{i=1}^{m} X_i \to (\underline{X}, \underline{*})^K_{\text{co}} \to (\underline{P\Sigma X}, \underline{\Sigma X})^K,$$

or more generally,

$$\bigvee_{i=1}^{m} X_i \to (\underline{X}, \underline{A})^K_{\text{co}} \to (\underline{PY}, \underline{Y})^K,$$

where Y_i is the homotopy cofibre of $f_i : X_i \to A_i$. This would allow us to understand the suspension of polyhedral coproducts, and therefore their homology. However, we show that in general, these homotopy cofibrations do not exist. This is reminiscent of how Ganea's theorem [G, Theorem 1.1] does not dualise canonically; see [G, Remark 3.5].

Let $(\underline{X}, \underline{X})^K_{\text{co}}$ be the polyhedral coproduct defined via the identity map on X_i, and denote by D_{id} the corresponding diagram. Observe that $(\underline{X}, \underline{X})^K_{\text{co}} = \bigvee_{i=1}^{m} X_i$. By Theorem 2.5, there is a map $\bigvee_{i=1}^{m} X_i \to \underline{f}^K_{\text{co}}$ defined by the commutative diagram

$$\begin{array}{ccc} X_i & = & X_i \\ \| & & \downarrow f_i \\ X_i & \xrightarrow{f_i} & A_i. \end{array}$$

Consider the case where K is two disjoint points. Then by part (1) of Example 2.1, the map $X_1 \vee X_2 \to \underline{f}^K_{\text{co}}$ is the inclusion $X_1 \vee X_2 \to X_1 \times X_2$, which has cofibre $X_1 \wedge X_2$. Now consider the polyhedral coproduct $(\underline{P\Sigma X}, \underline{\Sigma X})^K$. By definition, this is the homotopy limit of the diagram

$$\begin{array}{ccc} & & P\Sigma X_1 \vee \Sigma X_2 \\ & & \downarrow \\ \Sigma X_1 \vee P\Sigma X_2 & \longrightarrow & \Sigma X_1 \vee \Sigma X_2. \end{array}$$

Since $P\Sigma X_1$ and $P\Sigma X_2$ are contractible, this can be written, up to homotopy, as the homotopy pullback

$$\begin{array}{ccc} & & \Sigma X_2 \\ & & \big\downarrow \\ \Sigma X_1 & \hookrightarrow & \Sigma X_1 \vee \Sigma X_2. \end{array}$$

By Example 2.3, the homotopy type of this pullback is $\Omega\Sigma(\Omega\Sigma X_1 \wedge \Omega\Sigma X_2)$. Hence, there is not a homotopy cofibration dual to the homotopy fibration for polyhedral products. This gives rise to the following problem.

Problem 2.9. For certain classes of polyhedral coproduct, determine a decomposition for its suspension.

3. Preliminary Results

3.1. **Preliminary decompositions.** To decompose the loop space of a polyhedral coproduct, we will use a result known as the Porter decomposition. Let K be m disjoint points. By [DS, Lemma 2.3.1], there is a homotopy fibration

$$(\underline{C\Omega X}, \underline{\Omega X})^K \to \bigvee_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i.$$

A result of Porter [P, Theorem 1] identifies the homotopy type of $(\underline{C\Omega X}, \underline{\Omega X})^K$ in the case that each X_i is simply connected. For a space X and $k \geqslant 1$, let $X^{\vee k}$ be the k-fold wedge of X.

Theorem 3.1. *Let $X_1, \dots, X_m$ be pointed, simply connected CW-complexes, and let K be m disjoint points. There is a homotopy equivalence*

$$(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \bigvee_{k=2}^{m} \bigvee_{1 \leqslant i_1 < \cdots < i_k \leqslant m} (\Sigma\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)}.$$

Moreover, this homotopy equivalence is natural for maps $X_i \to Y_i$. □

There is a special case of the naturality in Theorem 3.1 which will be important. Let $n < m$ and let $Y_i = X_i$ for $1 \leqslant i \leqslant n$, and let $Y_i = CX_i$ for $n+1 \leqslant i \leqslant m$. In this case, we obtain the following.

Proposition 3.2. *Let $n < m$, and let $X_1, \dots, X_m$ be pointed, simply connected CW-complexes. There is a homotopy commutative diagram*

$$\begin{array}{ccccc} \bigvee_{k=2}^{m} \bigvee_{1 \leqslant i_1 < \cdots < i_k \leqslant m} \Sigma(\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)} & \longrightarrow & \bigvee_{i=1}^{m} X_i & \hookrightarrow & \prod_{i=1}^{m} X_i \\ \big\downarrow p' & & \big\downarrow p & & \big\downarrow \pi \\ \bigvee_{k=2}^{n} \bigvee_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \Sigma(\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)} & \longrightarrow & \bigvee_{i=1}^{n} X_i & \hookrightarrow & \prod_{i=1}^{n} X_i, \end{array}$$

where p and p' are pinch maps and π is the projection. □

Recall that K is m disjoint points and there is a homotopy fibration

$$(\underline{C\Omega X}, \underline{\Omega X})^K \to \bigvee_{i=1}^m X_i \xrightarrow{i} \prod_{i=1}^m X_i,$$

where i is the inclusion. After looping, there is a natural right homotopy inverse s for i, given by multiplying the inclusions $X_i \to \prod_{i=1}^m X_i$. The naturality of s and the homotopy fibration in Theorem 3.1 imply the following.

Theorem 3.3. *Let $X_1, \ldots, X_m$ be pointed, simply connected spaces. There is a homotopy equivalence*

$$\Omega\left(\bigvee_{i=1}^m X_i\right) \simeq \prod_{i=1}^m \Omega X_i \times \Omega\left(\bigvee_{k=2}^m \bigvee_{1\leqslant i_1<\cdots<i_k\leqslant m} (\Sigma\Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)}\right).$$

Moreover, this homotopy equivalence is natural for maps $X_i \to Y_i$. □

Let K be a simplicial complex on $[m]$, and let $X_1, \ldots, X_m$ be spaces. For a face $\sigma \in K$, denote by $X^{\wedge\sigma} = X_{i_1} \wedge \cdots \wedge X_{i_k}$, where $\sigma = \{i_1, \ldots, i_k\}$.

Remark 3.4. Observe that in Theorem 3.3, the wedge summand in the right hand product term can be indexed as

$$\bigvee_{\sigma\in\Delta^{m-1}, |\sigma|\geqslant 2} (\Sigma(\Omega X)^{\wedge\sigma})^{\vee(|\sigma|-1)}.$$

Now we recall the Hilton–Milnor theorem. Let L be the free (ungraded) Lie algebra over $\mathbb{Z}$ on the elements $x_1, \ldots, x_m$, and let B be a Hall basis of L. For a bracket $b \in B$, let $k_i(b)$ be the number of instances of x_i in b. For a space X and $k \geqslant 0$, denote by $X^{\wedge k}$ to be the k-fold smash of X. The following is from [Hi, Mi]. We will define the 0-fold smash of X to be omission of the corresponding term, rather than a trivial space.

Theorem 3.5. *Let $X_1, \ldots, X_m$ be connected topological spaces. Then there is a homotopy equivalence*

$$\Omega\left(\bigvee_{i=1}^m \Sigma X_i\right) \simeq \prod_{b\in B} \Omega\Sigma(X_1^{\wedge k_1(b)} \wedge \cdots \wedge X_m^{\wedge k_m(b)}).$$

Moreover, this homotopy equivalence is natural for maps $X_i \to Y_i$. □

As in the case of the Porter decomposition, there is a special case which will be important. Let $n < m$ and let $Y_i = X_i$ for $1 \leqslant i \leqslant n$, and let $Y_i = CX_i$ for $n+1 \leqslant i \leqslant m$. By contracting out the CX_i terms, we obtain the following.

Corollary 3.6. *Let $n < m$, let B_n be a Hall basis on the free Lie algebra generated by $x_1, \ldots, x_n$, and let B_m be a Hall basis on the free Lie algebra generated by $x_1, \ldots, x_m$. Then the diagram*

$$\begin{array}{ccc} \Omega\Sigma\left(\bigvee_{i=1}^{m} X_i\right) & \xrightarrow{\simeq} & \prod_{b\in B_m} \Omega\Sigma(X_1^{\wedge k_1(b)} \wedge \cdots \wedge X_m^{\wedge k_m(b)}) \\ \downarrow{\Omega p} & & \downarrow{\pi} \\ \Omega\Sigma\left(\bigvee_{i=1}^{n} X_i\right) & \xrightarrow[\simeq]{} & \prod_{b\in B_n} \Omega\Sigma(X_1^{\wedge k_1(b)} \wedge \cdots \wedge X_n^{\wedge k_n(b)}). \end{array}$$

homotopy commutes. □

3.2. **Preliminary homotopy limit decompositions.** In this section, we prove some decompositions of certain homotopy limits indexed by the opposite of the face category of a simplicial complex. The first lemma is the dual statement of the "Wedge Lemma" from [WZZ, Proposition 3.5].

Lemma 3.7. *Let K be a simplicial complex. Let X be a space and let $\mathcal{D}$ be a diagram with the shape of $\mathrm{cat}(K)^{\mathrm{op}}$ with $\mathcal{D}(\varnothing) = X$ and $\mathcal{D}(\sigma) = *$ for all $\sigma \neq \varnothing$. Then*

$$\operatorname*{holim}_{\sigma\in K} \mathcal{D} \simeq \mathrm{Map}_*(\Sigma|K|, X).$$

Proof. Let $\mathcal{X}$ be the diagram with the shape of $\mathrm{cat}(K)^{\mathrm{op}}$ with $\mathcal{X}(\varnothing) = *$ and $\mathcal{X}(\sigma) = X$ for all $\sigma \neq \varnothing$. Let $\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing}$ denote the over category (slice category) over $\varnothing$. For a topological space A and an indexing category $\mathcal{C}$, let $\mathcal{C}_A$ be the $\mathcal{C}$-shaped diagram with $\mathcal{C}(c) = A$ for all $c \in \mathcal{C}$. The diagram $\mathcal{D}$ can be written as the homotopy pullback of the diagram

$$\mathrm{cat}(K)^{\mathrm{op}}_* \longrightarrow \mathcal{X} \longleftarrow \mathrm{cat}(K)^{\mathrm{op}}_X, \tag{3}$$

where the right hand map is the constant map to the basepoint for $\sigma = \varnothing$, and the identity on X for $\sigma \neq \varnothing$, and for each $\sigma \in K$, the lefthand map is the inclusion of the basepoint. By [WZZ, Proposition 4.1], there is a homotopy equivalence $|K| \simeq \mathrm{hocolim}(\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing})_*$, and so there are homotopy equivalences $\mathrm{Map}(|K|, X) \simeq \mathrm{Map}(\mathrm{hocolim}(\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing})_*, X) \simeq \mathrm{holim}\left((\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing})_X\right)$. The diagram $\mathcal{X}$ is equivalent to the diagram $(\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing})_X \rightarrow *$. There is a homotopy equivalence $* \simeq \mathrm{Map}(|K|, *)$, thus the diagram $\mathcal{X}$ can be written as the homotopy limit of the diagram $(\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing})_X \rightarrow (\mathrm{cat}(K)^{\mathrm{op}}_{>\varnothing})_*$. This is in fact an iterated homotopy limit, and we obtain homotopy equivalences

$$\mathrm{holim}\,\mathcal{X} \simeq \mathrm{holim}\left(\mathrm{Map}(|K|, X) \rightarrow \mathrm{Map}(|K|, *)\right) \simeq \mathrm{Map}(|K|, \mathrm{holim}(X \rightarrow *)) \simeq \mathrm{Map}(|K|, X).$$

Recall that the diagram $\mathcal{D}$ was equivalent to the diagram (3). Using that $\mathrm{cat}(K)^{\mathrm{op}}_*$ is contractible and the previous observations about $\mathcal{X}$ yields the homotopy equvalence

$$\mathrm{holim}\,\mathcal{D} \simeq \mathrm{holim}\left(* \longrightarrow \mathrm{Map}(|K|, X) \longleftarrow X\right). \tag{4}$$

Consider the composition of squares

$$\begin{array}{ccccc} \mathrm{Map}_*(\Sigma|K|, X) & \longrightarrow & \mathrm{Map}(\Sigma|K|, X) & \longrightarrow & \mathrm{Map}(*, X) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}(*, X) & \longrightarrow & \mathrm{Map}(|K|, X). \end{array}$$

We wish to show that the outer square is a pullback. The right square is a pullback because $\mathrm{Map}(\mathrm{hocolim}_i A_i, Y) \simeq \mathrm{holim}_i \mathrm{Map}(A_i, Y)$ in the category of spaces. The left square is a pullback, and is the definition of $\mathrm{Map}_*(\Sigma|K|, X)$. By the pasting law for pullbacks, this implies that the outer square is a pullback. Note that the outer pullback square coincides with (4), and so we obtain a homotopy equivalence

$$\mathrm{holim}\, \mathcal{D} \simeq \mathrm{Map}_*(\Sigma|K|, X). \qquad \square$$

Lemma 3.8. *Let K be a simplicial complex on $[m]$. Let $I \subseteq [m]$, and let $\mathcal{D}$ be a diagram with the shape of $\mathrm{cat}(K)^{\mathrm{op}}$. Suppose that all maps induced by $\sigma \subset \tau$, where σ is obtained from τ by removing a single vertex not contained in I, are identity maps. Then the homotopy limit of $\mathcal{D}$ is equivalent to a diagram $\mathcal{D}'$ with the shape of $\mathrm{cat}(K_I)$, where $\mathcal{D}'(\sigma_I) = \mathcal{D}(\sigma)$.*

Proof. For any $i \notin I$, consider all pairs of simplices $\tau \subset \sigma$ where σ is obtained from τ by removing vertex i. We may contract all those arrows in the diagram simultaneously without changing the homotopy limit of $\mathcal{D}$. We do this for all $i \notin I$. Thus we are left with a diagram $\mathcal{D}'$ with shape of $\mathrm{cat}(K_I)$ with $D'(\varnothing) = X$ and for $\sigma \in K$, we have $\mathcal{D}'(\sigma_I) = \mathcal{D}(\sigma)$. $\square$

4. Loop spaces of polyhedral coproducts

4.1. **A general loop space decomposition.** In [BBCG2, Definition 2.2], for a simplicial complex K, a construction known as the *polyhedral smash product* is defined and denoted by $\widehat{(\underline{X}, \underline{A})}^K$. By [BBCG2, Theorem 2.10], there is a homotopy equivalence

$$\Sigma(\underline{X}, \underline{A})^K \simeq \bigvee_{I \subseteq [m]} \Sigma\widehat{(\underline{X}, \underline{A})}^{K_I}.$$

In this subsection, we show a dual statement for polyhedral coproducts.

Definition 4.1. The *polyhedral smash coproduct* is defined as the homotopy limit

$$\hat{\underline{f}}^K_{\mathrm{co}} = \underset{\sigma \in K}{\mathrm{holim}}\, \Sigma\hat{D}(\sigma), \quad \text{where } \hat{D}(\sigma) = \bigwedge_{i=1}^m \Omega Y_i \quad \text{and} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

For a set of positive integers $N = \{k_1(N), \dots, k_m(N)\}$, we define *the weighted polyhedral smash coproduct* as

$$\hat{\underline{f}}^K_{N,\mathrm{co}} = \underset{\sigma \in K}{\mathrm{holim}}\, \Sigma\hat{D}^N(\sigma), \quad \text{where } \hat{D}^N(\sigma) = \bigwedge_{i=1}^m (\Omega Y_i)^{\wedge k_i(N)} \quad \text{and} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Before stating the result, we set up some notation which will be used throughout the rest of Section 4. Let K be a simplicial complex on $[m]$, and let $\Delta(i_1, \ldots, i_k)$ denote a simplex on the vertices $i_1, \ldots, i_k$. For a face $\sigma = \{i_1, \ldots, i_k\} \in K$, let J_σ be the set

$$\{a \cdot I \mid I \subseteq \Delta(i_1, \ldots, i_k), |I| \geqslant 2, 1 \leqslant a \leqslant |I| - 1\}.$$

Denote by B_σ a Hall basis of the free ungraded Lie algebra on the set J_σ. For a bracket $b \in B_\sigma$ and $I \subseteq \sigma$, let $b(I)$ be the sum of the number of instances of $a \cdot I$ in b for each $1 \leqslant a \leqslant |I| - 1$. For $1 \leqslant i \leqslant m$ and a bracket $b \in B_{\Delta^{m-1}}$, we define

$$l_i(b) := \sum_{I \subseteq [m], i \in I} b(I),$$

which counts the number of instances of each vertex i in the faces in b. Let $L_b = (l_1(b), \ldots, l_m(b))$. For any $I \subseteq [m]$ and $b \in B_{\Delta^{m-1}}$, define

$$I_b := I \cap \{j \mid 1 \leqslant j \leqslant m, l_j(b) \neq 0\}.$$

This set contains the vertices which appear in the faces in b. To ensure that ΩX_i is connected in order to apply Theorem 3.5, we need the hypothesis that each X_i is simply connected.

Theorem 4.2. *Let $f_i \colon X_i \to A_i$ be a map of pointed, simply connected CW-complexes for all $1 \leqslant i \leqslant m$. There is a homotopy equivalence*

$$\Omega \underline{f}^K_{\mathrm{co}} \simeq \prod_{i=1}^m \Omega X_i \times \prod_{b \in B_{\Delta^{m-1}}} \Omega \underline{\hat{f}}^{K_{I_b}}_{L_b, \mathrm{co}}.$$

Proof. Since taking loops commutes with homotopy limits, we first consider $\Omega D(\sigma)$ for each $\sigma \in K$. By Theorem 3.3 and Remark 3.4, there is a homotopy equivalence

$$\Omega D(\sigma) \simeq \prod_{i=1}^m \Omega Y_i \times \Omega\Sigma \left(\bigvee_{\sigma \in \Delta^{m-1}, |\sigma| \geqslant 2} ((\Omega Y)^{\wedge \sigma})^{\vee (|\sigma| - 1)} \right).$$

We can apply the Hilton–Milnor theorem (Theorem 3.5) to the right hand product term to obtain the natural homotopy equivalence

$$\Omega \left(\bigvee_{\sigma \in \Delta^{m-1}, |\sigma| \geqslant 2} \Sigma((\Omega Y)^{\wedge \sigma})^{\vee (|\sigma| - 1)} \right) \simeq \prod_{b \in B_{\Delta^{m-1}}} \Omega\Sigma \left(\bigwedge_{\sigma \in \Delta^{m-1}, |\sigma| \geqslant 2} ((\Omega Y)^{\wedge \sigma})^{\wedge b(\sigma)} \right).$$

Note that for any $b \in B_{\Delta^{m-1}}$, by definition

$$\Sigma \left(\bigwedge_{\sigma \in \Delta^{m-1}, |\sigma| \geqslant 2} ((\Omega Y)^{\wedge \sigma})^{\wedge b(\sigma)} \right) = \Sigma \hat{D}^{L_b}(\sigma).$$

The diagram defining $\Omega \underline{f}^K_{\mathrm{co}}$ may now be described as the homotopy limit

$$\Omega \underline{f}^K_{\mathrm{co}} \simeq \operatorname*{holim}_{\sigma \in K} \Omega D(\sigma) \simeq \operatorname*{holim}_{\sigma \in K} \left(\prod_{i=1}^m \Omega Y_i \times \prod_{b \in B_{\Delta^{m-1}}} \Omega\Sigma \hat{D}^{L_b}(\sigma) \right).$$

Due to the naturality in Theorem 3.5, we can consider the homotopy limit termwise and there is a homotopy equivalence

$$\Omega \underline{f}_{\mathrm{co}}^{K} \simeq \prod_{i=1}^{m} \left(\operatorname*{holim}_{\sigma\in K} \Omega Y_i \right) \times \prod_{b\in B_{\Delta^{m-1}}} \left(\operatorname*{holim}_{\sigma\in K} \Omega\Sigma \hat{D}^{L_b}(\sigma) \right).$$

Fix $i \in [m]$ and consider the diagram for the term ΩY_i. The maps induced by $\sigma \subset \tau$, $\sigma \neq \varnothing$ are the identity. Contracting these arrows, we are left with the diagram $\Omega X_i \to \Omega A_i$, whose homotopy limit is ΩX_i. For any $b \in B_{\Delta^{m-1}}$, the maps induced by $\sigma \subset \tau$ where σ is obtained from τ by removing a vertex not in I_b in the diagram

$$\operatorname*{holim}_{\sigma\in K} \Sigma \hat{D}^{L_b}(\sigma)$$

are identity maps. Therefore, Lemma 3.8 implies that

$$\operatorname*{holim}_{\sigma\in K} \Sigma \hat{D}^{L_b}(\sigma) \simeq \operatorname*{holim}_{\tau\in K_{I_b}} \Sigma \hat{D}^{L_b}(\tau) \simeq \hat{\underline{f}}^{K_{I_b}}_{L_b,\mathrm{co}}. \qquad \square$$

4.2. **Loop space decompositions of $(\underline{X}, \underline{*})^K_{\mathrm{co}}$.** For polyhedral products of the form $(\underline{X}, \underline{*})^K$, by [BBCG2, Theorem 2.15], there is a homotopy equivalence

$$\Sigma(\underline{X}, \underline{*})^K \simeq \bigvee_{\sigma\in K} \Sigma X^{\wedge\sigma}.$$

In this subsection, we prove a dual statement for polyhedral coproducts of the form $(\underline{X}, \underline{*})^K_{\mathrm{co}}$. Let $\mathcal{F}$ and $\mathcal{M}$ be the set of faces and maximal faces of K on 2 or more vertices respectively. The following result could be shown using Theorem 4.2 by showing that certain polyhedral smash coproducts are contractible in this case. However, this would then involve a technical argument involving choices of vector space bases for free Lie algebras. To avoid these technicalities, and make clearer the connection to Hall bases, we provide a proof using Corollary 3.6.

Theorem 4.3. *Let $X_1, \ldots, X_m$ be pointed, simply connected CW-complexes. There is a homotopy equivalence*

$$\Omega(\underline{X}, \underline{*})^K_{\mathrm{co}} \simeq \prod_{i=1}^{m} \Omega X_i \times \prod_{b\in \bigcup_{\sigma\in\mathcal{M}} B_\sigma} \Omega\Sigma \left(\bigwedge_{\tau\in\mathcal{F}} ((\Omega X)^{\wedge\tau})^{\wedge b(\tau)} \right).$$

Proof. By definition of the polyhedral coproduct, $(\underline{X}, \underline{*})^K_{\mathrm{co}} = \operatorname{holim}_{\sigma\in K} D(\sigma)$, where, if $\sigma = \{i_1, \ldots, i_k\}$, $D(\sigma) = \bigvee_{j=1}^{k} X_{i_j}$, and for each $\tau \subset \sigma$, the map $D(\sigma) \to D(\tau)$ is the pinch map. Since looping commutes with homotopy limits, we obtain a homotopy equivalence $\Omega \operatorname{holim}_{\sigma\in K} D(\sigma) \simeq \operatorname{holim}_{\sigma\in K} \Omega D(\sigma)$. By Theorem 3.3 and Remark 3.4, there is a natural homotopy equivalence

$$\Omega \left(\bigvee_{j=1}^{k} X_{i_j} \right) \simeq \prod_{j=1}^{k} \Omega X_{i_k} \times \Omega \left(\bigvee_{\tau\in\Delta(i_1,\ldots,i_k), |\tau|\geqslant 2} (\Sigma(\Omega X)^{\wedge\tau})^{\vee|\tau|-1} \right). \tag{5}$$

Under this equivalence, it follows from Proposition 3.2 that the maps $\Omega D(\sigma) \to \Omega D(\tau)$ are given by $\pi \times \Omega p$ up to homotopy, where π is the projection, and p is the pinch map.

Applying the Hilton–Milnor theorem to the right hand product in (5), we obtain a natural homotopy equivalence

$$\Omega\left(\bigvee_{\tau\in\Delta(i_1,\ldots,i_k),|\tau|\geqslant 2}(\Sigma(\Omega X)^{\wedge\tau})^{\vee|\tau|-1}\right)\simeq\prod_{b\in B_\sigma}\Omega\Sigma\left(\bigwedge_{\tau\in\Delta(i_1,\ldots,i_k),|\tau|\geqslant 2}((\Omega X)^{\tau})^{\wedge b(\tau)}\right).$$

By Theorem 3.6, the map Ωp becomes the projection onto the respective terms. Therefore, we obtain a diagram where each object is a product of spaces, and each of the maps is a projection. Hence $\Omega(\underline{X},\underline{*})^K_{\mathrm{co}}$ is the product of each of the distinct factors that appear in the diagram. For $\tau\subseteq\sigma$, the product terms appearing in the decomposition for $\Omega D(\sigma)$ strictly contains the product terms in the decomposition for $\Omega D(\tau)$. Therefore, enumerating the distinct factors that appear for the maximal faces, we obtain a homotopy equivalence

$$\Omega(\underline{X},\underline{*})^K_{\mathrm{co}}\simeq\prod_{i=1}^m\Omega X_i\times\prod_{b\in\bigcup_{\sigma\in\mathcal{M}}B_\sigma}\Omega\Sigma\left(\bigwedge_{\tau\in\mathcal{F}}((\Omega X)^{\wedge\tau})^{\wedge b(\tau)}\right).$$

□

Example 4.4. Let K be a 1-dimensional simplicial complex on $[m]$. In this case, the set $\mathcal{M}$ consists of all the 1-simplices in K. For each $\sigma=\{i,j\}\in\mathcal{M}$, $B_\sigma=\{\sigma\}$. Therefore, Theorem 4.3 implies there is a homotopy equivalence

$$\Omega(\underline{X},\underline{*})^K_{co}\simeq\prod_{i=1}^m\Omega X_i\times\prod_{\sigma\in\mathcal{M}}\Omega\Sigma(\Omega X_i\wedge\Omega X_j).$$

4.3. **Loop space decompositions when the domain is contractible.** For a simplicial complex K, let $|K|$ be the geometric realisation of K as a topological space. For polyhedral products of the form $(\underline{CX},\underline{X})^K$, by [BBCG2, Theorem 2.21], there is a homotopy equivalence

$$\Sigma(\underline{CX},\underline{X})^K\simeq\bigvee_{I\notin K}\Sigma(|K_I|\wedge X^{\wedge I}). \tag{6}$$

In this subsection, we prove a dual statement for polyhedral coproducts of the form $\underline{f}^K_{\mathrm{co}}$ where the domain of each f_i is contractible.

Theorem 4.5. *Let K be a simplicial complex on $[m]$ and $f_i\colon X_i\to A_i$ where X_i is contractible and A_i is a pointed, simply connected CW-complex for $1\leqslant i\leqslant m$. Then there is a homotopy equivalence*

$$\Omega\underline{f}^K_{\mathrm{co}}\simeq\prod_{b\in B_{\Delta^{m-1}},I_b\notin K}\Omega\mathrm{Map}_*(\Sigma|K_{I_b}|,\Sigma\Omega A_1^{\wedge l_1(b)}\wedge\cdots\wedge\Omega A_m^{\wedge l_m(b)}).$$

To prove Theorem 4.5, we will use the following consequence of Theorem 4.2.

Lemma 4.6. *Assume that X_i is contractible and A_i is a pointed, simply connected CW-complex for all i and $N\in\mathbb{N}^m$. There is a homotopy equivalence*

$$\hat{\underline{f}}^{K_{I_b}}_{N,\mathrm{co}}\simeq\mathrm{Map}_*(\Sigma|K|,\Sigma\Omega A_1^{\wedge k_1(N)}\wedge\cdots\wedge\Omega A_m^{\wedge k_m(N)}).$$

Proof. Since all the X_i are contractible, $\hat{D}^N(\sigma) \simeq *$ for all $\sigma \neq \varnothing$. Thus, the diagram defining $\hat{\underline{f}}^{K_{I_b}}_{N,\mathrm{co}}$ satisfies the conditions of Lemma 3.7. □

With the lemma above, it is straightforward to prove Theorem 4.5

Proof of Theorem 4.5. By Lemma 4.6, if $I_b \in K$, then $\hat{\underline{f}}^{K_{I_b}}_{L_b,\mathrm{co}}$ is contractible. One can then apply Lemma 4.6 to the decomposition in Theorem 4.2 to prove the statement. □

Example 4.7. Let $K = \partial\Delta^{m-1}$. In this case, the only missing face of K is $\{1, \cdots, m\}$. By Theorem 4.5, there is a homotopy equivalence

$$\Omega\underline{f}^K_{\mathrm{co}} \simeq \prod_{b \in B_{\Delta^{m-1}}, I_b = \{1,\cdots,m\}} \Omega\mathrm{Map}_*(\Sigma|K_{I_b}|, \Sigma\Omega A_1^{\wedge l_1(b)} \wedge \cdots \wedge \Omega A_m^{\wedge l_m(b)}),$$

where the indexing set of the product consists of brackets b such that for each $i \in [m]$, there is a face $\sigma \in K$ in b which contains i.

In the case of polyhedral products, it is known that the decomposition in (6) desuspends in certain cases. For example, when K is a shifted complex [GT1, IK1], a flag complex with chordal 1-skeleton [PT, Theorem 6.4], or more generally, a totally fillable simplicial complex [IK2, Corollary 7.3]. Specialising, polyhedral products of the form $(D^2, S^1)^K$ are known as *moment-angle complexes*, which are denoted $\mathcal{Z}_K$. In the aforementioned cases, $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres.

Consider the case where K is a simplicial complex on $[m]$, and is either a shifted complex, or a flag complex with chordal 1-skeleton. The dual of the polyhedral product $(\underline{CX}, \underline{X})^K$ is the polyhedral coproduct $(\underline{PX}, \underline{X})^K_{\mathrm{co}}$. In the first case, $|K_I|$ is homotopy equivalent to a wedge of spheres for all $I \subseteq [m]$, and in the second case, $|K_I|$ is homotopy equivalent to a set of disjoint points for all $I \subseteq [m]$. Therefore, in the case where each X_i is a simply connected sphere, Theorem 4.5 implies that $\Omega(\underline{PX}, \underline{X})^K_{\mathrm{co}}$ is homotopy equivalent to a product of iterated loop spaces of spheres. Dual to the polyhedral product case, we give the following conjecture.

Conjecture 4.8. Let K be a shifted complex or a flag complex with chordal 1-skeleton. Then the decomposition in Theorem 4.5 deloops.

5. Polyhedral coproducts under operations on simplicial complexes

5.1. **Joins of simplicial complexes.** For any polyhedral product, if $K = K_1 \star K_2$ is the join of K_1 and K_2, then $(\underline{X}, \underline{A})^K \cong (\underline{X}, \underline{A})^{K_1} \times (\underline{X}, \underline{A})^{K_2}$. Therefore, we may expect a homotopy equivalence $(\underline{X}, \underline{A})^K_{\mathrm{co}} \simeq (\underline{X}, \underline{A})^{K_1}_{\mathrm{co}} \vee (\underline{X}, \underline{A})^{K_2}_{\mathrm{co}}$. However, this does not hold in general for polyhedral coproducts.

For $1 \leqslant i \leqslant 4$, let $X_i = \mathbb{C}P^\infty$, and let $K = \{1,2\} \star \{3,4\}$ be the boundary of a square. Since $(\mathbb{C}P^\infty, *)^{\{1,2\}}_{\mathrm{co}}$ and $(\mathbb{C}P^\infty, *)^{\{3,4\}}_{\mathrm{co}}$ are homotopy equivalent to $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ by Example 2.1, suppose that $(\mathbb{C}P^\infty,*)^K_{\text{co}} \simeq (\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \vee (\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Since $\mathbb{C}P^\infty$ is simply connected, and $\Omega\mathbb{C}P^\infty \simeq S^1$, by Theorem 4.3, there is a homotopy equivalence

$$\Omega(\mathbb{C}P^\infty,*)^K_{\text{co}} \simeq \prod_{i=1}^{4}(S^1 \times \Omega S^3).$$

Now by Theorem 3.3 applied to $(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \vee (\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, there is a homotopy equivalence

$$\Omega((\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \vee (\mathbb{C}P^\infty \times \mathbb{C}P^\infty)) \simeq \prod_{i=1}^{4} S^1 \times \Omega\Sigma\left((S^1 \times S^1) \wedge (S^1 \times S^1)\right).$$

For spaces X and Y, there is a well-known homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. By shifting the suspension coordinate, we obtain homotopy equivalences

$$\prod_{i=1}^{4} S^1 \times \Omega\Sigma\left((S^1 \vee S^1 \vee S^2) \wedge (S^1 \vee S^1 \vee S^2)\right) \simeq \prod_{i=1}^{4} S^1 \times \Omega\Sigma\left(\bigvee_{i=1}^{4} S^2 \vee \bigvee_{i=1}^{4} S^3 \vee S^4\right).$$

By Theorem 3.5, $\Omega\Sigma\left(\bigvee_{i=1}^{4} S^2 \vee \bigvee_{i=1}^{4} S^3 \vee S^4\right)$ decomposes as an infinite, finite type product of spheres and loops on spheres. However, since $\Omega(\mathbb{C}P^\infty,*)^K_{\text{co}}$ is homotopy equivalent to a finite product of spheres and loops on spheres,

$$\Omega(\mathbb{C}P^\infty,*)^K_{\text{co}} \not\simeq \Omega((\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \vee (\mathbb{C}P^\infty \times \mathbb{C}P^\infty)),$$

which implies that

$$(\mathbb{C}P^\infty,*)^K_{\text{co}} \not\simeq (\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \vee (\mathbb{C}P^\infty \times \mathbb{C}P^\infty).$$

However, it is possible to say something about certain joins.

Proposition 5.1. *Let K be a simplicial complex on the vertex set $[m]$ and let $\underline{f}^K_{\text{co}}$ be any polyhedral coproduct. Let $K' = K \star \{m+1\}$ where $f_{m+1}\colon * \to Y$ for some space Y. Then $\underline{f}^{K'}_{\text{co}} \simeq \underline{f}^K_{\text{co}}$.*

Proof. Let D denote the diagram defining $\underline{f}^K_{\text{co}}$ and let $D \vee Y$ (resp. $D \times Y$) be the diagram where for each $\sigma \in K$, $(D \vee Y)(\sigma) = D(\sigma) \vee Y$ (resp. $D(\sigma) \times Y$). Let D_* (resp. D_Y) be the diagram with the shape of D and $D_*(\sigma) = *$ (resp. $D_Y(\sigma) = Y$) for all $\sigma \in K$. Let F be the D-shaped diagram with $F(\sigma) = \text{hofib}(D(\sigma) \vee Y \to D(\sigma) \times Y)$ for each $\sigma \in K$. For each $\sigma \in K$, there exists a simplex $\sigma' \in K'$ that is the join of σ and the vertex $\{m+1\}$. Thus the diagram for $\underline{f}^{K'}_{\text{co}}$ can be written as the iterated homotopy limit $\text{holim}(D \vee Y \leftarrow D)$, where the maps $D(\sigma) \to (D \vee Y)(\sigma)$ are inclusions for all $\sigma \in K$. We have the following homotopy fibration

$$\text{holim}\,(F \leftarrow D_*) \to \text{holim}\,(D \vee Y \leftarrow D) \to \text{holim}\,(D \times Y \leftarrow D).$$

The space $\text{holim}\,(F \leftarrow D_*)$ is contractible since

$$\text{holim}\,(F \leftarrow D_*) \simeq \text{holim}\,(\text{holim}(F) \leftarrow *) \simeq *.$$

Since the fibre is contractible, there is a homotopy equivalence $\underline{f}_{\mathrm{co}}^{K'} \simeq \operatorname{holim}(D \times Y \leftarrow D)$. The right-hand side decomposes as a product of diagrams, and so we obtain,

$$\begin{aligned}\operatorname{holim}(D \times Y \leftarrow D) &\simeq \operatorname{holim}(D \leftarrow D) \times \operatorname{holim}(D_Y \leftarrow D_*)\\ &\simeq \operatorname{holim}(D) \times \operatorname{holim}(Y \leftarrow *)\\ &\simeq \underline{f}_{\mathrm{co}}^{K}.\end{aligned}$$

□

5.2. **Pullbacks of polyhedral coproducts.** Let K_1 be a simplicial complex on $\{1, \ldots, n\}$ and K_2 be a simplicial complex on $\{l, \ldots, m\}$ with $n < m$ and $l \leqslant m$, and let L be a subcomplex (possibly empty) of K_1 and K_2 on $\{l, \ldots, n\}$. Define $K = K_1 \cup_L K_2$, and for M one of K_1, K_2 or L, let $\overline{M}$ be the simplicial complex considered on the vertex set $\{1, \ldots, m\}$. For polyhedral products, by [GT1, Proposition 3.1], there is a pushout

$$\begin{array}{ccc}(\underline{X}, \underline{A})^{\overline{L}} & \longrightarrow & (\underline{X}, \underline{A})^{\overline{K_1}}\\ \downarrow & & \downarrow\\ (\underline{X}, \underline{A})^{\overline{K_2}} & \longrightarrow & (\underline{X}, \underline{A})^{K}.\end{array}$$

For polyhedral coproducts, we can prove a dual statement.

Proposition 5.2. *Let K_1 be a simplicial complex on $\{1, \ldots, n\}$ and K_2 be a simplicial complex on $\{l, \ldots, m\}$ with $n < m$ and $l \leqslant m$, and let L be a subcomplex (possibly empty) of K_1 and K_2 on $\{l, \ldots, n\}$. Define $K = K_1 \cup_L K_2$. Then there is a homotopy pullback of polyhedral coproducts*

$$\begin{array}{ccc}\underline{f}_{\mathrm{co}}^{K} & \longrightarrow & \underline{f}_{\mathrm{co}}^{\overline{K_2}}\\ \downarrow & & \downarrow\\ \underline{f}_{\mathrm{co}}^{\overline{K_1}} & \longrightarrow & \underline{f}_{\mathrm{co}}^{\overline{L}}\end{array}$$

where the maps $\underline{f}_{\mathrm{co}}^{\overline{K_1}} \to \underline{f}_{\mathrm{co}}^{\overline{L}}$, and $\underline{f}_{\mathrm{co}}^{\overline{K_2}} \to \underline{f}_{\mathrm{co}}^{\overline{L}}$ are induced by the simplicial inclusions.

Proof. By Remark 2.7, one may write the elements of the pullback

$$\begin{array}{ccc} & & \underline{f}_{\mathrm{co}}^{\overline{K_2}}\\ & & \downarrow\\ \underline{f}_{\mathrm{co}}^{\overline{K_1}} & \longrightarrow & \underline{f}_{\mathrm{co}}^{\overline{L}}\end{array}$$

as diagrams and we are left with a diagram $\mathcal{D}$ that almost has the shape of $\mathrm{cat}(M)$, but with each $\sigma \in L$ showing up thrice. For each $\sigma \in L$, let σ_{K_1} (resp. σ_{K_2}) denote the copy in $\mathcal{D}$ in $\underline{f}_{\mathrm{co}}^{\overline{K_1}}$ (resp. $\underline{f}_{\mathrm{co}}^{\overline{K_2}}$). For each σ, the maps $\mathcal{D}(\sigma_{K_1}) \to \mathcal{D}(\sigma)$ and $\mathcal{D}(\sigma_{K_2}) \to \mathcal{D}(\sigma)$ are the identity map. Therefore, for all $\sigma \in L$, we may contract these edges in the diagram without changing the homotopy limit. The resulting diagram $\mathcal{D}'$ has the shape of $\mathrm{cat}(K)$ and is the diagram with homotopy limit $\underline{f}_{\mathrm{co}}^{K}$ by definition. □

Let K_1 and K_2 be simplicial complexes and let $K = K_1 \sqcup K_2$. By definition of the polyhedral product, $(\underline{X}, \underline{*})^K = (\underline{X}, \underline{*})^{K_1} \vee (\underline{X}, \underline{*})^{K_2}$. In the case of a polyhedral coproduct $(\underline{X}, \underline{*})^K_{\text{co}}$, using Proposition 5.2, we show that the dual holds in this case.

Theorem 5.3. *Let K_1 and K_2 be simplicial complexes, and let $K = K_1 \sqcup K_2$. There is a homotopy equivalence*

$$(\underline{X}, \underline{*})^K_{\text{co}} \simeq (\underline{X}, \underline{*})^{K_1}_{\text{co}} \times (\underline{X}, \underline{*})^{K_2}_{\text{co}}.$$

Proof. By definition, since each $A_i = *$, $(\underline{X}, \underline{*})^{\varnothing} = *$, and $(\underline{X}, \underline{*})^{\overline{K_i}} = (\underline{X}, \underline{*})^{K_i}$ for $i \in \{1, 2\}$. Therefore, Proposition 5.2 implies there is a homotopy pullback

$$\begin{array}{ccc} (\underline{X}, \underline{*})^K_{\text{co}} & \longrightarrow & (\underline{X}, \underline{*})^{K_2}_{\text{co}} \\ \downarrow & & \downarrow \\ (\underline{X}, \underline{*})^{K_1}_{\text{co}} & \longrightarrow & *. \end{array}$$

Hence, there is a homotopy equivalence

$$(\underline{X}, \underline{*})^K_{\text{co}} \simeq (\underline{X}, \underline{*})^{K_1}_{\text{co}} \times (\underline{X}, \underline{*})^{K_2}_{\text{co}}.$$ □

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