Exceptional sequences in the derived category of a gentle algebra

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Abstract

In this paper, using the correspondence of gentle algebras and dissections of marked surfaces, we study full exceptional sequences in the perfect derived category $K^b(A)$ of a gentle algebra A. We show that full exceptional sequences in $K^b(A)$ exist if and only if the associated marked surface has no punctures and has at least two marked points on the boundary. Furthermore, by using induction on cuts of surfaces, we characterize when an exceptional sequence can be completed to a full exceptional sequence. If the genus of the associated surface is zero then we show that the action of the braid group together with the grading shift on full exceptional sequences in $K^b(A)$ is transitive. For the case of surfaces of higher genus, we reduce the problem of transitivity to the problem of the existence of certain sequences of pairs of exceptional objects. Finally, we interpret the duality of a full exceptional sequence induced by the longest element in the associated symmetric group using Koszul duality.

Keywords Derived category · Braid group action · Exceptional sequence · Intersection number

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Introduction

The notion of an exceptional object was introduced by Drezet and Le Potier in [17] to classify vector bundles on the projective plane $\mathbb{P}^2$. The theory fast developed when the group of algebraic geometers around Rudakov in Moscow systematically studied exceptional objects [26]. They introduced exceptional sequences in derived categories and connected these sequences through operations called mutations. An exceptional sequence is called a full exceptional sequence if its length equals n, the rank of the Grothendieck group of the category. General techniques of mutations for exceptional sequences in triangulated categories were developed by Bondal [8]. In particular, he shows that mutations give rise to an action of the braid group B_n on the set of full exceptional sequences.

While for some special derived categories the theory of exceptional sequences is well-studied, for example, the derived categories of hereditary algebras [15] and the derived categories of coherent sheaves over some special varieties such as weighted projective lines [23], for a general derived category, we know little. Even essential questions such as the the existence of full exceptional sequences is not known.

In this paper we study full exceptional sequences in the perfect derived category $\mathsf{K}^b(\mathsf{A})$ of a gentle algebra A. In particular, we consider the following questions in this context: When do full exceptional sequences exist? When can an exceptional sequence be completed to a full exceptional sequence? Is the braid group action together with the grading shift transitive on the set of full exceptional sequences? And finally, what are the relations between Koszul duality, Serre duality and dualities of full exceptional sequences induced by the longest element of a symmetric group associated to the sequence?

Gentle algebras are classical objects in the representation theory of associative algebras. They were introduced in the 1980s as a generalization of iterated tilted algebras of type A_n [2], and affine type $\widetilde{A}_n$ [3]. Remarkably, they connect to many other areas of mathematics. For example, they play an important role in the homological mirror symmetry of surfaces, see e.g. [7, 14, 19]. In particular, it has been shown that the perfect derived category of a homologically smooth gentle algebra is triangle

equivalent to the partially wrapped Fukaya category of a graded oriented smooth surface with marked points on the boundary [19, 22].

From the perspective of representation theory, it is shown in [24] that any gentle algebra A gives rise to a dissection Δ_A of a marked surface (S, M), where S is an oriented surface with boundary, M is a union of ○-marked points and ●-marked points which alternatively appear on the boundary ∂S as well as a finite set of marked points (called punctures) in the interior of S. Conversely, any gentle algebra arises in this way. The objects and morphisms in $K^b(A)$ can also been interpreted on the surface model (S, M, Δ_A). In particular, any indecomposable object $P^\bullet_{(\gamma, f)}$ corresponds to a graded curve (γ, f) on the surface, where γ is a closed curve or an arc with ○-points as endpoints and f is an integer valued function.

Although gentle algebras have been a constant object of study and much about their representation theory is known, except for some particular cases, such as, for example, those considered in [13] and [21], we know little in general about exceptional sequences in their derived categories. The aim of this paper is to use the surface model established in [24] to determine the existence of full exceptional sequences in $K^b(A)$ and to then study their properties. In [19, 22] it is shown that $K^b(A)$ is a particular realisation of a partially wrapped Fukaya category of a surface with stops and our results imply that full exceptional sequences exist in these partially wrapped Fukaya categories corresponding to $K^b(A)$ except for certain small exceptions.

More precisely, our first result is a characterisation of gentle algebras that admit full exceptional sequences in their perfect derived categories in terms of their surface models. For convenience, we assume the algebras to be connected, so that the associated surface is also connected. We denote by $\mathbb{T}_{(g,1,1)}$ the marked surface of genus $g \geq 1$ with exactly one boundary component and exactly one ○-marked point on it.

Theorem A (Theorem 3.7) *Let* A *be a gentle algebra with surface model* (S, M, Δ_A). *The following are equivalent:*

(1) There exists a full exceptional sequence in $K^b(A)$;
(2) There are at least two ○-points in M and there are no ●-marked points in $S \backslash \partial S$;
(3) (S, M) is not homeomorphic to $\mathbb{T}_{(g,1,1)}$ and there are no ●-marked points in $S \backslash \partial S$.

Theorem A translates into the following criterion to determine the existence of full exceptional sequences in $K^b(A)$ by directly looking at the quiver and the relations of the algebra A. For this we denote by Q_0 the set of vertices and by Q_1 the set of arrows of Q.

Theorem B (Corollary 3.10) *Let* A $= kQ/I$ *be a gentle algebra. Then there exists a full exceptional sequence in* $K^b(A)$ *if and only if the global dimension of A is finite and the pair* $(|Q_0|, |Q_1|)$ *is not equal to* $(2g, 4g - 1)$, *for any integer* $g \geq 1$.

The above theorem also confirms the existence of full exceptional sequences in the derived category of a derived-discrete algebra, see Corollary 3.9, which has been shown in [13] by considering a semi-orthogonal decomposition induced by an exceptional object.

By considering cuts of marked surfaces, we can use induction techniques to determine when an exceptional sequence can be completed to a full exceptional sequence. More precisely, we show the following.

Theorem C (Theorem 4.12) *Assume that there exists a full exceptional sequence in $K^b(A)$ for a gentle algebra A arising from a marked surface (S, M) and let $\mathbf{P}^\bullet = (P^\bullet_{(\gamma_1, f_1)}, \ldots, P^\bullet_{(\gamma_m, f_m)})$ be an exceptional sequence arising from a collection $\Delta = \{\gamma_1, \ldots, \gamma_m\}$ of arcs on (S, M). Then $\mathbf{P}^\bullet$ can be completed to a full exceptional sequence if and only if the cut surface of (S, M) along the arcs in Δ has no connected component of the form $\mathbb{T}_{(g,1,1)}$.*

For any triangulated category with a full exceptional sequence, it is conjectured by Bondal and Polishchuk in [11] that the action of $\mathbb{Z}^n \rtimes B_n$ on the set of full exceptional sequences is transitive, where the group $\mathbb{Z}^n$ acts on full exceptional sequences by shifting the grading of the objects. This has been shown to hold for derived categories of hereditary algebras [15], and also for derived categories of coherent sheaves over some special varieties such as del Pezzo surfaces, projective planes, and weighted projective lines, see more details in [23]. For the case of the derived category of gentle algebras, we can show the following transitivity result.

Theorem D (Theorem 5.1, Corollary 5.15) *Let A be a gentle algebra arising from a marked surface with genus zero. The action of $\mathbb{Z}^n \rtimes B_n$ on the set of full exceptional sequences in $K^b(A)$ is transitive. In particular, if A is a derived-discrete algebra, then the action of $\mathbb{Z}^n \rtimes B_n$ on the set of full exceptional sequences in $K^b(A)$ is transitive.*

In fact, we show that the braid group action is transitive in general for any gentle algebra with full exceptional sequences if a certain reachability condition, Condition **RCEA** (see Definition 5.9), holds for any two arcs corresponding to indecomposable exceptional objects (see Proposition 5.10). However, it seems difficult to establish in general for which surfaces (other than genus zero) the **RCEA** condition holds.

Finally, we determine the connection between Koszul duality and the right duality **R** and the left duality **L** induced by the longest element in the associated symmetric group (see Definition 6.1). For this, let $\mathbf{X} = (X_1, \ldots, X_n)$ be a full exceptional sequence in $K^b(A)$ associated with an ordered exceptional dissection $\Gamma = (\gamma_1, \ldots, \gamma_n)$. Let S_n be the symmetric group on n elements and let w_0 be the longest element in S_n. We define the right dual of $\mathbf{X}$ by setting $\mathbf{RX} = \omega_0 \mathbf{X}$ (see Definition 6.1 and Lemma 6.2). Let $\Gamma^* = (\gamma_n^*, \ldots, \gamma_1^*)$ be the dual of Γ, where γ_i^*, $1 \le i \le n$, is the unique $\bullet$-arc which intersects γ_i exactly once and intersects no other arcs in Γ. Denote by $A(\Gamma)$ and $(A(\Gamma)^!)^{\mathrm{op}} = A(\Gamma^*)$ the gentle algebra and its Koszul dual arising from Γ and Γ^* respectively (we recall the constructions in subsection 1.2). The next result shows that the right dual $\mathbf{RX}$ of $\mathbf{X}$ is induced by a twist to the next $\circ$-point of every arc γ^* in the dissection Γ^* corresponding to the Koszul dual $(A(\Gamma)^!)^{\mathrm{op}}$ of $A(\Gamma)$.

Theorem E (Theorem 6.5) *Let $\mathbf{R}\Gamma$ be the ordered exceptional dissection associated to the right dual $\mathbf{RX}$ of $\mathbf{X}$. Then $\mathbf{R}\Gamma = D(\Gamma^*)$, where $D(\Gamma^*)$ is the twist of Γ^*, which is obtained by rotating both endpoints of each arc in Γ^* anticlockwise to the next respective $\circ$-point.*

Using our geometric interpretation of the left (and right) duality of a full exceptional sequence, we recover a result which Bondal in [10] proves for a general triangulated category with a full exceptional sequence.

Theorem F (Theorem 6.6) *Let $\mathbb{S}$ be the Serre functor in $K^b(A)$. Then $\mathbb{S}(\mathbf{X}) = \mathbf{L}^2(\mathbf{X})$ for any full exceptional sequence $\mathbf{X}$ in $K^b(A)$.*

The paper is organized as follows. We recall some background material on marked surfaces, gentle algebras and their derived categories, as well as the theory of exceptional sequences in Sect. 1. In Sect. 2, we interpret exceptional sequences in the derived category of a gentle algebra in terms of ordered exceptional collections on the associated marked surface. We prove the existence of full exceptional sequences in Sect. 3. Section 4 is devoted to the study of cutting surfaces and the characterisation of when an exceptional sequence can be completed to a full exceptional sequence. In Sect. 5 we give a geometric realization of the braid group action on the set of full exceptional sequences, and study the transitivity of this action. Finally, in Sect. 6, we describe the relation between the left and right duality of exceptional sequences, Koszul duality and Serre duality.

1 Preliminaries

In this paper, an algebra will be assumed to be basic of finite dimension over a base field k. A quiver will be denoted by $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices and Q_1 is the set of arrows. Arrows in a quiver are composed from left to right as follows: for arrows a and b we write ab for the path from the source $s(a)$ of a to the target $t(b)$ of b. We adopt the convention that maps are also composed from left to right, that is if $f : X \to Y$ and $g : Y \to Z$ then $fg : X \to Z$. In general, we consider left modules. We denote by $\mathbb{Z}$ the set of integer numbers, and by $\mathbb{Z}^*$ the set of non-zero integer numbers. For a finite set M, we denote by $|M|$ its cardinality.

1.1 Marked surfaces

We recall some concepts about marked surfaces associated to gentle algebras, for which there are many references such as [19, 22, 25], in this paper we closely follow [24] and [1].

Definition 1.1 A triple (S, M, P) is called a *marked surface* if

(1) S is an oriented compact surface with non-empty boundary with connected components $\partial S = \sqcup_{i=1}^{b} \partial_i S$;
(2) $M = M^\circ \cup M^\bullet$ is a finite set of *marked points* on ∂S. The elements of M° and $M^\bullet$ will be respectively represented by symbols $\circ$ and $\bullet$. Each connected component $\partial_i S$ is required to contain at least one marked point of each colour, where in general the points $\circ$ and $\bullet$ are alternating on $\partial_i S$;
(3) $P = P^\bullet$ is a finite set of marked points in the interior of S. We refer to these points as *punctures*, and we will also represent them by the symbol $\bullet$.

Unless otherwise stated, we always assume that a marked surface is connected and that in the case of a disk with no punctures there are at least two $\circ$-marked points and two $\bullet$-marked points on the boundary.

Definition 1.2 Let (S, M, P) be a marked surface.

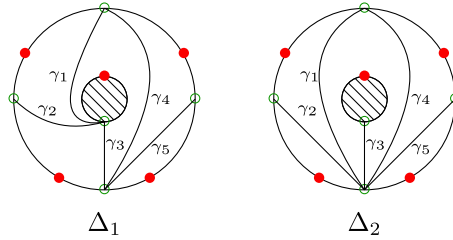- An $\circ$-*arc* is a non-contractible curve in $S \backslash P$ with endpoints in M°.

Fig. 1 Two admissible dissections on an annulus

- A *loop* is an ○-arc whose endpoints coincide.
- An •-*arc* is a non-contractible curve, with endpoints in $M^\bullet \cup P^\bullet$.
- An *infinite-arc* is a non-contractible curve with endpoints in $M^\circ \cup P^\bullet$, which has at least one endpoint in $P^\bullet$.
- An *admissible arc* is an ○-arc or an infinite-arc.
- A *closed curve* is a non-contractible curve in the interior of S whose endpoints coincide. We always assume a closed curve to be primitive, that is, it is not a non-trivial power of a closed curve in the fundamental group of S.
- A *simple closed curve* is a closed curve without self-intersections.

We have the following lemma about simple closed curves, which can be found, for example in [18, Section 1.3.1].

Lemma 1.3 *If α and β are any two non-separating simple closed curves in a surface S, then there is a homeomorphism $\phi : S \mapsto S$ with $\phi(\alpha) = \beta$.*

On the surface, all curves are considered up to homotopy, and all intersections of curves are required to be transversal and in minimal position. For simplicity, we denote a marked surface without punctures by (S, M).

Definition 1.4 A collection of ○-arcs $\{\gamma_1, \ldots, \gamma_r\}$ is *admissible* if the only possible intersections of these arcs are at the endpoints, and each subsurface enclosed by the arcs contains at least one •-point from $M^\bullet \cup P^\bullet$. A maximal admissible collection Δ of ○-arcs is called an *admissible ○ -dissection*. The notion of *admissible • -dissection* is defined in a similar way. For convenience, we will often simply write admissible dissection instead of admissible ○-dissection.

Note that an admissible dissection cuts the surface into polygons such that each polygon contains exactly one •-marked point. Furthermore, whenever we write (S, M, P, Δ), we implicitly assume that Δ is an admissible dissection on (S, M, P).

Example 1.5 The pictures in Fig. 1 are two admissible dissections on an annulus.

Denote by g the genus of S and by b the number of connected components of ∂S. It is shown in [1], see also [25], that an admissible collection of ○-arcs is an admissible dissection if and only if it is maximal, that is, it contains exactly $|M^\circ| - \chi$ arcs, where $\chi = 2 - 2g - b - |P|$ is the Euler characteristic of the marked surface. Furthermore, let Δ be an admissible dissection. Then there exists a unique admissible •-dissection Δ^* (up to homotopy) such that each arc of Δ^* intersects exactly one arc of Δ. We

call Δ^* the *dual* • *-dissection* of Δ. For a fixed admissible dissection Δ and its dual •-dissection Δ^*, unless otherwise stated, all curves on the surface are always assumed to be in minimal position with respect to both these dissections.

A *graded curve* (γ, f) on (S, M, P, Δ) is an admissible arc or a closed curve γ, together with a function

$$f : \gamma \cap \Delta^* \longrightarrow \mathbb{Z},$$

where $\gamma \cap \Delta^*$ is the totally ordered set of intersection points of γ with Δ^*, where the order is induced by the direction of γ. The function f is defined as follows: If p and q are in $\gamma \cap \Delta^*$ and q is the direct successor of p, then γ enters a polygon enclosed by •-arcs of Δ^* via p and leaves it via q. If the unique ∘-point in this polygon is to the left of γ, then $f(q) = f(p) + 1$; otherwise, $f(q) = f(p) - 1$. Note that not all closed curves are gradable.

For a grading f of γ, the map $f[n] : l \to f(l) - n$ with $n \in \mathbb{Z}$ is also a grading on γ, and all gradings on γ are of this form. We call [1] the *shift* of the grading f. The grading shift on curves corresponds to the shift functor in the derived category of the associated gentle algebra.

1.2 Derived categories of gentle algebras

We recall in this section the derived category of a gentle algebra, and its geometric realization given in [24] using a marked surface.

Definition 1.6 [3] We call an algebra A a *gentle algebra*, if A is given by kQ/I, where $Q = (Q_0, Q_1)$ is a finite quiver and I an admissible ideal of kQ satisfying the following conditions:

(1) Each vertex in Q_0 is the source of at most two arrows and the target of at most two arrows.
(2) For each arrow a in Q_1, there is at most one arrow b' such that $ab' \notin I$; at most one arrow c' such that $c'a \notin I$; at most one arrow b such that ab is a path in Q and $ab \in I$; at most one arrow c such that ca is a path in Q and $ca \in I$.
(3) I is generated by paths of length two.

For a finite dimensional algebra A, it is well known that the bounded derived category is triangle equivalent to the homotopy category $\mathsf{K}^{-,b}(A)$ of complexes of projective A-modules bounded on the right and bounded in homology. In the following, we will not distinguish these two categories. Denote by $\mathsf{K}^b(A)$ the full subcategory of $\mathsf{K}^{-,b}(A)$, which is the homotopy category of bounded complexes of projective A-modules.

The derived category of a gentle algebra is well-studied. In particular, the authors in [6] classified the indecomposable objects in the category in terms of *(homotopy) string objects* and *(homotopy) band objects*. The morphisms between these indecomposable objects are explicitly described in [5].

Starting with an admissible collection Δ on (S, M, P), we define an algebra $A(\Delta) = kQ(\Delta)/I(\Delta)$ as follows:

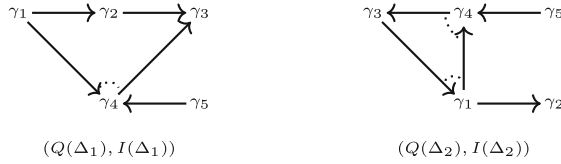$$(Q(\Delta_1), I(\Delta_1)) \qquad\qquad (Q(\Delta_2), I(\Delta_2))$$

Fig. 2 The quivers with relations associated to the two admissible dissections in Fig. 1, where we denote the relations by dotted lines

(1) The vertices of $Q(\Delta)$ are given by the arcs in Δ.

(2) There is an arrow $a : \alpha \to \beta$ for each endpoint q that α and β share, and β directly follows α anticlockwise at q.

(3) The ideal $I(\Delta)$ is generated by the following relations: whenever α and β intersect at a marked point as above, and the other end of β intersects γ at a marked point as above, then the composition ab of the corresponding arrows $a : \alpha \to \beta$ and $b : \beta \to \gamma$ is a relation.

Then $A(\Delta)$ is a gentle algebra. In particular, if Δ is an admissible dissection, then this establishes a bijection between the set of homeomorphism classes of marked surfaces (S, M, P, Δ) and the set of isomorphism classes of gentle algebras $A(\Delta)$, see [24], and also [9].

Example 1.7 The pictures in Fig. 2 are the quivers with relations associated to the two admissible dissections in Fig. 1.

Remark 1.8 Similarly, one can associate an algebra $A(\Delta^*)$ to an •-admissible collection Δ^*, which is the Koszul dual of $A(\Delta)$ by [24, Section 1.7].

Let A be a gentle algebra associated with a marked surface (S, M, P, Δ_A). It is shown in [24] that there is a correspondence between the indecomposable objects in $\mathsf{K}^{-,b}(\mathsf{A})$ and graded curves on the surface. More precisely, the graded admissible arcs (γ, f) on (S, M, P, Δ_A) are in bijection with the isomorphism classes of the indecomposable string objects $P^\bullet_{(\gamma,f)}$ in $\mathsf{K}^{-,b}(\mathsf{A})$; the graded closed curves (γ, f) on (S, M, P, Δ_A) together with an indecomposable $k[x, x^{-1}]$-module $\mathfrak{m}$ are in bijection with the isomorphism classes of the indecomposable band objects $P^\bullet_{(\gamma,f,\mathfrak{m})}$ in $\mathsf{K}^{-,b}(\mathsf{A})$. Furthermore, under the above bijections, the indecomposable objects in $\mathsf{K}^b(\mathsf{A})$ correspond to the graded ∘-arcs and the pairs of graded closed curves and indecomposable $k[x, x^{-1}]$-modules. For convenience of notation, we often will drop the $\mathfrak{m}$ from the notation, so that $P_{(\gamma,f)}$ denotes both a string or band object.

The morphisms in $\mathsf{K}^{-,b}(\mathsf{A})$ are described in terms of graded oriented intersections, see [24, Theorem 3.3]. Roughly speaking, each intersection of two curves on the boundary (i.e. at a ∘-marked point) gives rise to exactly one morphism between the corresponding indecomposable objects. Each intersection in the interior (not at a •-puncture) of S gives rise to two morphisms, while an intersection at a •-puncture gives rise to infinitely many morphisms. Since the geometric description of the morphisms is a crucial part of many of the proofs, we will now give a more detailed description. For
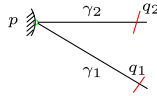
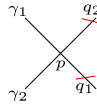**Fig. 3** Boundary intersection of γ_1 and γ_2 giving rise to exactly one morphism

Fig. 4 Interior intersection of γ_1 and γ_2 giving rise to two morphisms in opposite directions

that, let $P^\bullet_{(\gamma_1, f_1)}$ and $P^\bullet_{(\gamma_2, f_2)}$ be two indecomposable objects in $\mathsf{K}^{-,b}(\mathsf{A})$ corresponding to admissible graded arcs or graded closed curves (γ_1, f_1) and (γ_2, f_2).

Assume first that γ_1 and γ_2 intersect at a $\circ$-point p on the boundary as depicted in Fig. 3, where for $i \in \{1, 2\}$, q_i is the intersection in $\gamma_i \cap \Delta^*_A$ which is nearest to p. If $f_1(q_1) = f_2(q_2)$, then there is a morphism from $P^\bullet_{(\gamma_1, f_1)}$ to $P^\bullet_{(\gamma_2, f_2)}$. Note that there is no morphism from $P^\bullet_{(\gamma_2, f_2)}$ to $P^\bullet_{(\gamma_1, f_1)}[j]$ for any $j \in \mathbb{Z}$ arising from this intersection at p.

Assume now that γ_1 and γ_2 intersect at some point p in the interior of the surface, which is not a $\bullet$-puncture. Suppose that we have the following local configuration in S depicted in Fig. 4 below. If $f_1(q_1) = f_2(q_2)$, then there is one morphism from $P^\bullet_{(\gamma_1, f_1)}$ to $P^\bullet_{(\gamma_2, f_2)}$, and one morphism from $P^\bullet_{(\gamma_2, f_2)}$ to $P^\bullet_{(\gamma_1, f_1)}[1]$ (or to $P^\bullet_{(\gamma_1, f_1)}[-1]$ depending on the position of the corresponding $\circ$-marked point).

The Auslander-Reiten translation τ in $\mathsf{K}^b(\mathsf{A})$ is described in [24] in terms of arcs in the surface. We will rephrase the result by introducing a slightly different notion of twisting an arc as follows. The *direct twist* (resp. *inverse twist*) of an $\circ$-arc γ in (S, M, P) is an $\bullet$-arc $D\gamma$ (resp. $D^{-1}\gamma$) obtained from γ by rotating both endpoints anticlockwise (resp. clockwise) to the next $\bullet$-point. Similarly, we define the *direct twist $D\gamma^*$* and the *inverse twist $D^{-1}\gamma^*$* of an $\bullet$-arc γ^* in (S, M, P). Note that there is a canonical intersection between γ and $D^2\gamma$ arising from the rotation.

Lemma 1.9 ([24] Corollary 5.2) *Let $P^\bullet_{(\gamma, f)}$ be an indecomposable string object in $\mathsf{K}^b(\mathsf{A})$ corresponding to a graded $\circ$-arc (γ, f). Then $\tau^{-1} P^\bullet_{(\gamma, f)} = P^\bullet_{(D^2\gamma, f_{D^2\gamma})}$, where $f_{D^2\gamma}$ is a grading on $D^2\gamma$ which is uniquely determined by the fact that the canonical intersection between γ and $D^2\gamma$ gives rise to a map from $P^\bullet_{(D^2\gamma, f_{D^2\gamma})}$ to $P^\bullet_{(\gamma, f)}[1]$.*

1.3 Silting objects and exceptional sequences

We now recall some background on silting theory and the theory of exceptional sequences in a general triangulated category as well as for the particular case of the bounded derived categories of gentle algebras.

Let $\mathcal{T}$ be a triangulated category. We call a full subcategory $\mathcal{P}$ in $\mathcal{T}$ *pre-silting* if for all $i > 0$, $\mathrm{Hom}_\mathcal{T}(\mathcal{P}, \mathcal{P}[i]) = 0$. It is *silting* if in addition $\mathcal{T} = \mathsf{thick}\mathcal{P}$. An object P of $\mathcal{T}$ is said to be *pre-silting* if $\mathsf{add}P$ is a pre-silting subcategory and *silting* if $\mathsf{add}P$

is a silting subcategory. We always assume that pre-silting objects as well as silting objects are basic.

In the triangulated category $\mathcal{T}$, we call an object $X \in \mathcal{T}$ *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(X, X[\neq 0]) = 0$ and $\mathrm{End}_{\mathcal{T}}(X)$ is a division algebra. We call an (ordered) sequence $(X_1, \ldots, X_n)$ of exceptional objects in $\mathcal{T}$ an *exceptional sequence* if $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[\mathbb{Z}]) = 0$ for any $1 \leq j < i \leq n$, which is said to be *full* if in addition $\mathrm{thick}_{\mathcal{T}}(\bigoplus_{i=1}^n X_i) = \mathcal{T}$. Note that the grading does not have any bearing on whether a sequence is exceptional or not. In general, there is an action of $\mathbb{Z}^n$ on the set of full exceptional sequences $\exp\mathcal{T}$ in $\mathcal{T}$ given as follows

$$(\ell_1, \ldots, \ell_n)(X_1, \ldots, X_n) := (X_1[\ell_1], \ldots, X_n[\ell_n]).$$

The following proposition shows that there are close relations between exceptional sequences and silting objects.

Proposition 1.10 (Proposition 3.5 [4]) *Let $\mathcal{T}$ be a totally Hom-finite triangulated category, that is, for any $X, Y \in \mathcal{T}$ we have $\mathrm{Hom}_{\mathcal{T}}(X, Y[\ell]) = 0$ for any $|\ell| \gg 0$. Let $(X_1, \ldots, X_n)$ be an exceptional sequence in $\mathcal{T}$. Then*

(1) there exists $a \in \mathbb{Z}$ such that $X_1[\ell_1] \oplus \cdots \oplus X_n[\ell_n]$ is a pre-silting object for any integers $\ell_1 \ldots, \ell_n \in \mathbb{Z}$ satisfying $\ell_i + a \leq \ell_{i+1}$, for all $1 \leq i < n$.

(2) the exceptional sequence $(X_1, \ldots, X_n)$ is a full exceptional sequence if and only if $X_1[\ell_1] \oplus \cdots \oplus X_n[\ell_n]$ is a silting object.

The category $\mathsf{K}^b(\mathsf{A})$ always has silting objects. The category $\mathsf{K}^{-,b}(\mathsf{A})$ contains silting objects if and only if the global dimension of A is finite, that is, $\mathsf{K}^{-,b}(\mathsf{A})$ is triangle equivalent to $\mathsf{K}^b(\mathsf{A})$. For a gentle algebra A, the silting objects in $\mathsf{K}^b(\mathsf{A})$ are described in [1, Theorem 3.2]. Namely, any basic silting object X in $\mathsf{K}^b(\mathsf{A})$ is of the form $P^\bullet_{(\Delta, f)} = \bigoplus_{i=1}^n P^\bullet_{(\gamma_i, f_{\gamma_i})}$, where $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ is an admissible dissection of (S, M, P) and f is a set of gradings f_{γ_i} on γ_i, $1 \leq i \leq n$.

Note that, in general, for an arbitrary admissible dissection Δ, there currently is no characterisation of when there exists a set of gradings f over Δ such that (Δ, f) gives rise to a silting object.

2 Exceptional sequences in terms of ordered exceptional collections

In this section we describe exceptional sequences in terms of surface dissections. Throughout this section, let A be a gentle algebra associated with a marked surface (S, M, P, Δ_A). Let Δ_A^* be the dual admissible •-dissection of Δ_A.

We begin by describing indecomposable exceptional objects by the surface.

Lemma 2.1 *Let $P^\bullet_{(\gamma, f)}$ be an indecomposable object in $\mathsf{K}^b(\mathsf{A})$ arising from a graded curve (γ, f).*

(1) Then $P^\bullet_{(\gamma, f)}$ is exceptional if and only if γ is an ○-arc without self-intersections. In particular, γ is not a loop.

(2) If γ is an ○-arc without self-intersections, then for any grading f of γ, $P^\bullet_{(\gamma, f)}$ is an indecomposable exceptional object in $\mathsf{K}^b(\mathsf{A})$.
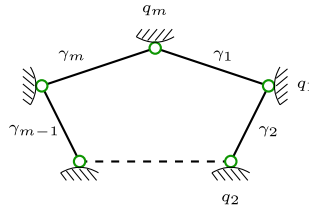
Fig. 5 A non-exceptional cycle

Proof (1) Assume $P^\bullet_{(\gamma,f)}$ is exceptional. Note that if γ is a closed curve, then $P^\bullet_{(\gamma,f)}$ is a band object which has self-extension and so cannot be exceptional. If γ is an infinite arc, that is at least one end of γ wraps infinitely many times around a puncture, then following [24, Remark 3.8] there exist infinitely many maps from $P^\bullet_{(\gamma,f)}$ to positive shifts of $P^\bullet_{(\gamma,f)}$. So γ can only be an ∘-arc. Moreover, if γ has an interior self-intersection, then we have two linearly independent maps arising from this intersection, and $\mathrm{Hom}(P^\bullet_{(\gamma,f)}, P^\bullet_{(\gamma,f)}[m]) \neq 0$ for some $m \in \mathbb{Z}^*$. Therefore $P^\bullet_{(\gamma,f)}$ is not exceptional. On the other hand, if γ is a loop without interior self-intersection, then we have two cases. If the two gradings near the endpoint are equal, then we have two linearly independent maps from $P^\bullet_{(\gamma,f)}$ to itself: the identity and the map corresponding to the intersection. Then the endomorphism algebra of $P^\bullet_{(\gamma,f)}$ is isomorphic to $k[x]/(x^2)$, which is not a division algebra. Otherwise, if the gradings are different, then there is some $m \in \mathbb{Z}^*$ such that there is a non-zero element in $\mathrm{Hom}(P^\bullet_{(\gamma,f)}, P^\bullet_{(\gamma,f)}[m])$ arising from the intersection. In both cases $P^\bullet_{(\gamma,f)}$ is not exceptional.

Conversely, it is clear that if γ is an ∘-arc without self-intersections, then $\mathrm{Hom}(P^\bullet_{(\gamma,f)}, P^\bullet_{(\gamma,f)}[\neq 0]) = 0$ and $\mathrm{End}(P^\bullet_{(\gamma,f)}) \cong k$ is a division ring, so that $P^\bullet_{(\gamma,f)}$ is exceptional.

(2) This is clear by the first statement and an observation that shifting the grading of objects does not change the fact whether an object is exceptional or not. □

Following the above lemma, we will call an ∘-arc an *exceptional arc* if it has no self-intersections.

Definition 2.2 Let Δ be an admissible ∘-collection on (S, M, P) consisting of exceptional arcs. We call Δ an *exceptional collection*, if it has no *non-exceptional cycle*, that is a subgraph consisting of arcs in Δ as depicted in Fig. 5, where γ_{i+1} (not necessarily directly) follows γ_i in the anticlockwise order at the endpoint q_i and where the index $1 \leq i \leq m$ is considered modulo m. We call an exceptional collection an *exceptional dissection* if it is an admissible dissection.

The quiver $Q(\Delta)$ associated to an admissible collection Δ gives an easy way of checking whether this collection is exceptional or not.

Lemma 2.3 *An admissible collection Δ is an exceptional collection if and only if there exists no oriented cycle in the associated quiver $Q(\Delta)$.*

Proof Clearly every non-exceptional cycle in Δ gives rise to an oriented cycle in $Q(\Delta)$. On the other hand suppose that we have an oriented cycle $a_1 a_2 \ldots a_m$ in $Q(\Delta)$. Then

the subgraph of Δ corresponding to all vertices $t(a_i)$ of $Q(\Delta)$ such that $a_i a_{i+1} \in I$, considering the indexing set modulo m, form a non-exceptional cycle in Δ. Thus Δ is not an exceptional collection. $\qquad\square$

Example 2.4 In Fig. 1, Δ_1 is an exceptional dissection, while Δ_2 is not. This can be seen from the quivers of the two algebras in Fig. 2: the quiver $Q(\Delta_1)$ does not have an oriented cycle whereas the quiver $Q(\Delta_2)$ has an oriented cycle.

Let $\Delta = \{\gamma_i, 1 \le i \le n\}$ be an exceptional collection. We define a partial order on Δ as follows. For two arcs $\gamma_i, \gamma_j \in \Delta$, set $\gamma_i \preceq \gamma_j$ if γ_i and γ_j share an endpoint q and if γ_i follows γ_j in the clockwise order at q or if $\gamma_i = \gamma_j$. We consider the transitive closure of this relation which we will still denote by $\preceq$. Then it is clear that $\preceq$ is reflexive. On the other hand, it follows from the definition of an exceptional collection that if $\gamma_i \preceq \gamma_j$ and $\gamma_i \ne \gamma_j$, then $\gamma_j \not\preceq \gamma_i$, since there are no 'oriented' subgraphs as depicted above. Thus this order relation is antisymmetric, that is $\gamma_i \preceq \gamma_j$ and $\gamma_j \preceq \gamma_i$ implies $\gamma_i = \gamma_j$. Therefore $\preceq$ is a well-defined partial order on Δ.

Definition 2.5 We call an ordered set of arcs $(\gamma_1, \ldots, \gamma_m)$ in (S, M, P) an *ordered exceptional collection* if it is an exceptional collection and the order of the arcs are compatible with the partial order $\preceq$ introduced above, that is, $\gamma_i \preceq \gamma_j$ implies $i \le j$. We call an ordered exceptional collection an *ordered exceptional dissection* if it is an admissible dissection. We will use Γ to denote an ordered exceptional dissection. We define $\Gamma^* = (\gamma_m^*, \ldots, \gamma_1^*)$ as the *dual ordered exceptional dissection* of Γ. Note that we reverse the indices of arcs in Γ^*.

While there is no geometric criterion to determine which admissible dissections give rise to silting objects, we can give such a criterion to determine the admissible dissections which correspond to full exceptional sequences as follows.

Proposition 2.6 *(1) If $(X_1, \ldots, X_n)$ is a full exceptional sequence in $K^b(A)$, then $X_i = P^\bullet_{(\gamma_i, f_i)}$ for some graded ∘-arcs (γ_i, f_i) such that $\Delta = (\gamma_1, \ldots, \gamma_n)$ is an ordered exceptional dissection of (S, M, P).*

(2) Let $\Delta = (\gamma_1, \ldots, \gamma_n)$ be an ordered exceptional dissection of (S, M, P). Then for any grading f_i over γ_i, $1 \le i \le n$, $(P^\bullet_{(\gamma_1, f_1)}, \ldots, P^\bullet_{(\gamma_n, f_n)})$ is a full exceptional sequence in $K^b(A)$.

Proof (1) Since each X_i, $1 \le i \le n$, is an indecomposable exceptional object, by Lemma 2.1, we may assume that $X_i = P^\bullet_{(\gamma_i, f_i)}$ for some graded arc (γ_i, f_i), where γ_i is an ∘-arc without self-intersections. On the other hand, since $K^b(A)$ is totally Hom-finite, there exists integers $\ell_1 \ldots, \ell_n \in \mathbb{Z}$ such that $X_1[\ell_1] \oplus \cdots \oplus X_n[\ell_n]$ is a silting object by Proposition 1.10. Thus $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ is an admissible dissection of (S, M, P). Suppose Δ is not an exceptional dissection. Then there is a cycle formed by arcs $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_m}$ in Δ. Without loss of generality, we may assume that this is

a cycle as follows.
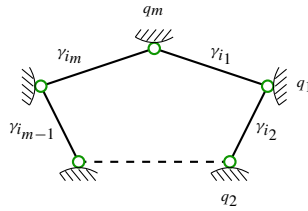
Then we have

$$\mathrm{Hom}(P^\bullet_{(\gamma_{i_j}, f_{i_j})}, P^\bullet_{(\gamma_{i_{j+1}}, f_{i_{j+1}})}[\mathbb{Z}]) \neq 0,$$

for $1 \leq j \leq m-1$, and

$$\mathrm{Hom}(P^\bullet_{(\gamma_{i_m}, f_{i_m})}, P^\bullet_{(\gamma_{i_1}, f_{i_1})}[\mathbb{Z}]) \neq 0.$$

This contradicts that $(X_1, \ldots, X_n)$ is an exceptional sequence. Therefore Δ is an exceptional collection, and thus an exceptional dissection.

(2) Let $\Delta = (\gamma_1, \ldots, \gamma_n)$ be an ordered exceptional dissection of (S, M, P). Since the order of Δ is compatible with the partial order $\preceq$ of the arcs on (S, M, P), $i < j$ implies that $\gamma_i \preceq \gamma_j$ or γ_i and γ_j are not comparable by $\preceq$. If $\gamma_i \preceq \gamma_j$, then $\mathrm{Hom}(P^\bullet_{(\gamma_j, f_j)}, P^\bullet_{(\gamma_i, f_i)}[\mathbb{Z}]) = 0$ for any grading f_i and f_j. On the other hand, if γ_i and γ_j are not comparable by $\preceq$, then in particular, there exists no inter-sections between γ_i and γ_j, and thus $\mathrm{Hom}(P^\bullet_{(\gamma_j, f_j)}, P^\bullet_{(\gamma_i, f_i)}[\mathbb{Z}]) = 0$. Therefore $\mathrm{Hom}(P^\bullet_{(\gamma_j, f_j)}, P^\bullet_{(\gamma_i, f_i)}[\mathbb{Z}]) = 0$ when $i < j$. Furthermore, since γ_i has no self-intersection, $P^\bullet_{(\gamma_i, f_i)}$ is an exceptional object for all $1 \leq i \leq n$ by Lemma 2.1. Thus $(P^\bullet_{(\gamma_1, f_1)}, P^\bullet_{(\gamma_2, f_2)}, \ldots, P^\bullet_{(\gamma_n, f_{i_n})})$ is an exceptional sequence, which is full since Δ is maximal. $\qquad\square$

We have the following immediate corollary.

Corollary 2.7 *(1) There exists a full exceptional sequence in $K^b(A)$ if and only if there exists an exceptional dissection on (S, M, P).*

(2) Let A_1 and A_2 be two gentle algebras with the same surface model, then there exists a full exceptional sequence in $K^b(A_1)$ if and only if there exists a full exceptional sequence in $K^b(A_2)$.

3 The existence of full exceptional sequences

In this section, we consider the existence of full exceptional sequences in $K^b(A)$ by considering the existence of exceptional dissections on the associated surface. We

Fig. 6 The quiver of an exceptional dissection on $\mathbb{T}_{(g,1,2)}$

Fig. 7 The quiver of an exceptional dissection on $\mathbb{T}_{(g,2,2)}$

begin by considering the case of surfaces containing one or two boundary components and one or two marked points.

(1) $\mathbb{T}_{(g,1,1)}$ is the marked surface of genus $g \geq 1$ with only one boundary component and exactly one ∘-marked point.

(2) $\mathbb{T}_{(g,1,2)}$ is the marked surface of genus $g \geq 1$ with only one boundary component and exactly two ∘-marked point.

(3) $\mathbb{T}_{(g,2,2)}$ is the marked surface of genus $g \geq 1$ with two boundary components each of which has one ∘-marked point.

Lemma 3.1 *There exists no exceptional dissection in* $\mathbb{T}_{(g,1,1)}$.

Proof Note that if there exists only one ∘-marked point on the surface, then any ∘-arc is a loop, and thus can not be exceptional. Therefore there exists no exceptional dissections in $\mathbb{T}_{(g,1,1)}$. □

Lemma 3.2 *There exist exceptional dissections in both* $\mathbb{T}_{(g,1,2)}$ *and* $\mathbb{T}_{(g,2,2)}$.

Proof On $\mathbb{T}_{(g,1,2)}$ and $\mathbb{T}_{(g,2,2)}$ there are admissible dissections such that the associated gentle algebras are of the form as in Fig. 6 and Fig. 7, respectively.

The existence of such admissible dissections follows from the computation of the ribbon graphs of these quivers with relations, see [24, 27], and from showing that the surfaces given by the ribbon graphs are exactly the surfaces $\mathbb{T}_{(g,1,2)}$ and $\mathbb{T}_{(g,2,2)}$, respectively. On the other hand, since the quivers have no oriented cycles, by Lemma 2.3, the associated admissible dissections are exceptional dissections on $\mathbb{T}_{(g,1,2)}$ and $\mathbb{T}_{(g,2,2)}$, respectively. □

In Proposition 4.7, we will show that any quiver with relations arising from an exceptional dissection on $\mathbb{T}_{(g,1,2)}$ and $\mathbb{T}_{(g,2,2)}$ is isomorphic to the quiver with relations depicted in Fig. 6 and Fig. 7 respectively. The following lemma describes a numerical property of the quiver arising from these three special surfaces.

Lemma 3.3 *Let* $A = kQ/I$ *be a gentle algebra arising from* (S, M, P, Δ_A). *Then*

(1) (S, M) *is of the form* $\mathbb{T}_{(g,1,1)}$ *if and only if* $(|Q_0|, |Q_1|) = (2n, 4n - 1)$, *for some* $n \geq 1$.

(2) (S, M) *is of the form* $\mathbb{T}_{(g,1,2)}$ *if and only if* $(|Q_0|, |Q_1|) = (2n + 1, 4n)$ *for some* $n \geq 1$.

(3) (S, M) *is of the form* $\mathbb{T}_{(g,2,2)}$ *if and only if* $(|Q_0|, |Q_1|) = (2n + 2, 4n + 2)$ *for some* $n \geq 1$.

Furthermore, if (1), (2) *or* (3) *holds then* $n = g$.

Proof For a marked surface (S, M), denote by $\chi = 2 - 2g - b$ the Euler characteristic of S, where g is the genus of S and b is the number of boundary components of S. Then we have the following equalities:

$$|Q_0| = |\Delta| = |M^\circ| - \chi \tag{3.1}$$

$$\chi = |Q_0| - |Q_1|, \tag{3.2}$$

see for example in [1, 22].

It follows directly from these equalities that if $(S, M) = \mathbb{T}_{(g,1,1)}$, then $(|Q_0|, |Q_1|) = (2g, 4g - 1)$; if $(S, M) = \mathbb{T}_{(g,1,2)}$, then $(|Q_0|, |Q_1|) = (2g + 1, 4g)$; if $(S, M) = \mathbb{T}_{(g,2,2)}$, then $(|Q_0|, |Q_1|) = (2g + 2, 4g + 2)$.

Now we show the converse.

(1) If $(|Q_0|, |Q_1|) = (2n, 4n - 1)$ for some $n \geq 1$, then by equality (3.2), the Euler characteristic of S equals $1 - 2n$. Thus by equality (3.1) there exists only one $\circ$-marked point on (S, M). Furthermore, there is only one boundary component on S, since there exists at least one $\circ$-marked point on each boundary component. Therefore (S, M) is of the form $\mathbb{T}_{(g,1,1)}$, and we have $g = n$.

(2) If $(|Q_0|, |Q_1|) = (2n + 1, 4n)$ for some $n \geq 1$, then by equality (3.2), the Euler characteristic of S equals $1 - 2n$. Thus by equality (3.1) there are exactly two $\circ$-marked points on (S, M). Hence S has only one or two boundary components. On the other hand, note that since $2 - 2g - b = \chi = 1 - 2n$, we have that b is odd. Therefore there is only one boundary component on S. So (S, M) is of the form $\mathbb{T}_{(g,1,1)}$, and $g = n$.

(3) The proof is similar to the proof of the second part. $\qquad\square$

Before giving equivalent conditions for the existence of full exceptional dissections on a marked surface, we show the following technical lemma.

Lemma 3.4 *Let (S, M, P) be a marked surface. Let (S', M', P) be a marked surface obtained from (S, M, P) by adding a $\circ$-point and a $\bullet$-point on some boundary component or by adding a new boundary component with exactly one $\circ$-point and one $\bullet$-point. If there exists an exceptional dissection on (S, M, P) then there exists an exceptional dissection on (S', M', P).*

Proof Let γ be a minimal element in an exceptional dissection Δ on (S, M, P) with respect to the partial order introduced in Sect. 2. Denote the new $\circ$-marked point by p and the new $\bullet$-marked point by q. The two generic possible ways to add p and q is as in Fig. 8.

We now construct an admissible collection of arcs Δ' on (S', M', P) by adding new arcs to Δ as shown in Fig. 8, where we add an arc α as in the case of the left of Fig. 8 and we add two arcs α_1 and α_2 as in the case of the right of Fig. 8. Then Δ' is an admissible dissection on (S', M', P), since it is maximal, that is it has precisely $|\Delta| + 1$ arcs in the first case and $|\Delta| + 2$ arcs in the second case.

Furthermore, note that the quiver $Q(\Delta')$ is obtained from $Q(\Delta)$ by adding an arrow from α to γ in the first case, and by adding two arrows from α_1 to α_2 and one arrow from α_2 to γ in the second case. On the other hand, since Δ is exceptional, there is no
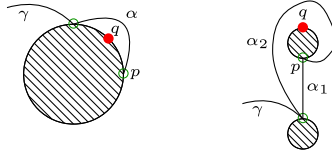
Fig. 8 Local configuration of the new surface (S', M', P) obtained from a surface (S, M, P) by adding marked points or a boundary with marked points. In the left case we add a ∘-marked point p and a ●-marked point q on a boundary component of S, while in the second case we add a new boundary component (upper one) with one ∘-marked point p and one ●-marked point q on (S, M, P)

oriented cycle in $Q(\Delta)$ by Lemma 2.3. Thus there is no oriented cycle in $Q(\Delta')$. So Δ' is exceptional again by Lemma 2.3.   □

Proposition 3.5 *Let (S, M, P) be a marked surface. Then the following are equivalent:*

(1) *There exists an exceptional dissection on (S, M, P);*
(2) *$P = \emptyset$ and $|M^\circ| \geq 2$;*
(3) *$P = \emptyset$ and (S, M, P) is not homeomorphic to $\mathbb{T}_{(g,1,1)}$.*

Proof Clearly (2) and (3) are equivalent. We now show that (1) implies (3). Suppose $P \neq \emptyset$ in (S, M, P). Let Δ be any admissible dissection of (S, M, P). Then there exists a polygon in Δ containing a ●-puncture and this polygon is a non-exceptional cycle in Δ. Thus Δ can not be an exceptional dissection. Furthermore, by Lemma 3.1, a surface of the form $\mathbb{T}_{(g,1,1)}$ has no exceptional dissection.

We now show that (2) implies (1). So assume that $P = \emptyset$ and $|M^\circ| \geq 2$. If the genus of S is greater than one then the surfaces $\mathbb{T}_{(g,1,2)}$ and $\mathbb{T}_{(g,2,2)}$ are the marked surfaces with minimal number of ∘-marked points containing at least two marked points. Lemma 3.2 shows that there exists at least one exceptional dissection on both $\mathbb{T}_{(g,1,2)}$ and $\mathbb{T}_{(g,2,2)}$. Then the result follows from Lemma 3.4 by induction on the number of marked points of the surface. If the genus of S is zero, we can also use an inductive argument based on iteratively using Lemma 3.4, and noticing that the disk with two ∘-marked points is the minimal surface with genus zero, which has an exceptional dissection.   □

The following result shows that if there is an exceptional dissection on (S, M) then there are infinitely many, unless S is a disk.

Proposition 3.6 *Let (S, M) be a marked surface with an exceptional dissection. Then the number of exceptional dissections is finite if and only if S is a disk.*

Proof Recall that the twist of an ∘-arc γ in (S, M) is an ●-arc $D\gamma$ obtained from γ by rotating anticlockwise both endpoints to the next respective ●-point. Then the result follows from the following two observations: any square of a twist of an exceptional dissection is again an exceptional dissection; the group generated by homeomorphisms of twisting each boundary component of (S, M) which maps a ∘-marked point to the (anticlockwise) next ∘-marked point is finite if and only S is a disk. Furthermore, if S is not a disk then the action of this group is faithful.   □

From Proposition 3.5 and Proposition 2.6 (2), we now obtain a complete description of the existence of full exceptional sequences in $K^b(A)$ for a gentle algebra A.

Theorem 3.7 *Let A be a gentle algebra with marked surface (S, M, P, Δ_A). The following are equivalent:*

(1) *There exists a full exceptional sequence in $K^b(A)$;*
(2) *$P = \emptyset$ and $|M^\circ| \geq 2$;*
(3) *$P = \emptyset$ and (S, M, P) is not homeomorhpic to $\mathbb{T}_{(g,1,1)}$.*

By the above Theorem and the fact that the global dimension of a gentle algebra is finite if and only if the associated marked surface has no punctures, we have the following.

Corollary 3.8 *Let A be a gentle algebra. If the global dimension of A is infinite then there exists no full exceptional sequence in $K^b(A)$.*

In particular, noting that the surface model of a non-hereditary discrete-derived algebra is an annulus, Theorem 3.7 gives a new proof of the following result on the existence of full exceptional sequences for the derived category of derived-discrete algebras which was first proved in [13, Proposition 6.6]. We refer the reader to [13, 28] for details on derived-discrete algebras.

Corollary 3.9 *Let A be a non-hereditary derived-discrete algebra. Then there exists a full exceptional sequence in $K^b(A)$.*

We now give a criterion to determine the existence of full exceptional sequences in $K^b(A)$ of a gentle algebra A by directly looking at the quiver and the relations of the algebra.

Corollary 3.10 *Let $A = kQ/I$ be a gentle algebra, then there exists a full exceptional sequence in $K^b(A)$ if and only if the following two conditions are satisfied*

(1) *the pair $(|Q_0|, |Q_1|)$ is not of the form $(2g, 4g - 1)$ for any $g \geq 1$;*
(2) *there exist no oriented cycles in Q such that the composition of any two neighboring arrows belongs to I.*

Proof Lemma 3.3 (1) says that a gentle algebra $A = kQ/I$ arising from an admissible dissection on a surface of the form $\mathbb{T}_{(g,1,1)}$ if and only if $(|Q_0|, |Q_1|) = (2g, 4g - 1)$. Furthermore, the global dimension of a gentle algebra is finite if and only if there exist no oriented cycles with relations at each vertex. This is further equivalent to the surfaces having not punctures, that is $P = \emptyset$. Then the result follows from Theorem 3.5. □

4 Cutting surfaces and completing exceptional sequences

We now define the notion of a cut surface which will play an important role in the inductive arguments later in the paper.
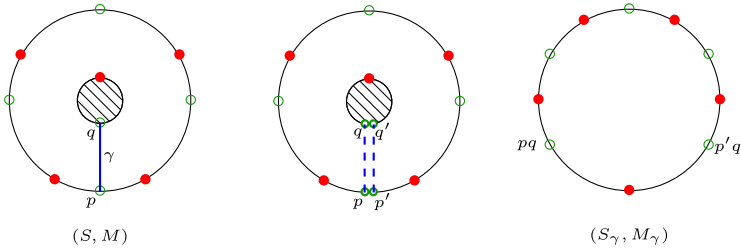
Fig. 9 An example of cut surface

Definition 4.1 Let γ be an exceptional ○-arc on a marked surface (S, M) which has two distinct endpoints p and q. We define (S_γ, M_γ) to be the marked surface obtained from (S, M) by cutting along γ and then contracting along the cut, and we set $M_\gamma = M\backslash\{p, q\}\sqcup\{pq, p'q'\}$, where pq and $p'q'$ are new marked points obtained from p and q after cutting and contracting along γ. We call (S_γ, M_γ) the *cut surface* of (S, M) along γ.

Note that whenever we consider a cut surface (S_γ, M_γ), we implicitly assume that γ is an exceptional ○-arc. When cutting the surface, we always ignore components corresponding to disks with only one ○-point and one ●-point. If the arc γ is a separating arc, then the cut surface (S_γ, M_γ) may not be connected anymore and (S_γ, M_γ) is in fact a union of marked surfaces.

Example 4.2 Fig. 9 gives an example of a cut surface. Namely, we cut an annulus along an arc γ connecting the two boundary components and then contract along the cut. From this we obtain the cut surface which is a disk.

Let α be an ○-arc in (S, M). It is easy to see that if α does not intersect γ, then α is still an ○-arc in S_γ and there are no other arcs in S which are identified with α in (S_γ, M_γ). While if α and γ intersect in the interior of S then α disappears after cutting, and it does not give rise to a curve in (S_γ, M_γ) anymore. When α and γ only intersect at endpoints, the situation becomes more complicated. In particular, in that case, two distinct arcs in S might be identified in S_γ, see for example in Fig. 10, α_1 and α_2 are identified in (S_γ, M_γ).

More precisely, for an ○-arc α in (S, M), we denote by $\overline{\alpha}$ the ○-arc in (S_γ, M_γ) induced by α. For example $\overline{\alpha_1} = \overline{\alpha_2}$ for α_1 and α_2 in Fig. 10. In particular, if α intersects γ in the interior, then we view $\overline{\alpha}$ as an empty element. For example, $\overline{\gamma}$ is empty. We refer to [16, Proposition 4.6] for a complete description of when two ○-arcs are identified in the cut surface.

The following two lemmas show that any admissible (resp. exceptional) collection on (S, M) induces an admissible (resp. exceptional) collection on (S_γ, M_γ).

Lemma 4.3 *Let Δ be an admissible collection on (S, M) and γ an exceptional arc in Δ. Denote respectively by Q and I the quiver and the set of relations associated to Δ. Then $\overline{\Delta} := \{\overline{\alpha}, \alpha \in \Delta\}$ is an admissible collection on (S_γ, M_γ), whose quiver $\overline{Q}$ and the set of relations $\overline{I}$ are obtained from Q and I as follows:*
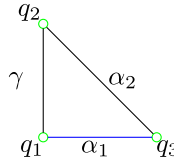
Fig. 10 An example of how arcs are identified in the cut surface, where on the cut surface (S_γ, M_γ), we have $\overline{\alpha_1} = \overline{\alpha_2}$

(1) *The vertex set $\overline{Q}_0$ is given by $Q_0\backslash\{\gamma\}$. The set $\overline{Q}_1$ of arrows is the union*

$$\{c : s \to t \in Q_1 \mid s \neq \gamma \text{ and } t \neq \gamma\} \sqcup \{[c_1 c_2] = j \xrightarrow{c_1} \gamma \xrightarrow{c_2} k \mid c_1 c_2 \in I\}.$$

(2) *$\overline{I}$ is generated by*

$$\{c_1 c_2 \mid c_1 c_2 \in I, c_1, c_2 \in Q_1 \cap \overline{Q}_1\} \sqcup \{c_0[c_1 c_2] \mid c_0 c_1 \in I\} \sqcup \{[c_1 c_2]c_3 \mid c_2 c_3 \in I\}.$$

Furthermore, if Δ is an admissible dissection, then $\overline{\Delta}$ is also an admissible dissection.

Proof The proof is straightforward. □

Lemma 4.4 *Assuming the notations in Lemma 4.3, we have that if Δ is an exceptional collection, then $\overline{\Delta}$ is an exceptional collection. Furthermore, if Δ is an exceptional dissection, then $\overline{\Delta}$ also is an exceptional dissection.*

Proof By Lemma 4.3, we know that $\overline{\Delta}$ is an admissible collection on (S_γ, M_γ). Now we show that $\overline{\Delta}$ is also exceptional. By Lemma 2.3, we only need to show that there is no oriented cycle in $\overline{Q}$. Assume for contradiction that ω is an oriented cycle in $\overline{Q}$. Then ω must contain a new arrow of the form $[c_1 c_2]$ described in Lemma 4.3, otherwise ω would already be an oriented cycle in Q, which contradicts to the fact that Δ is exceptional. But then after replacing all the new arrows $[c_1 c_2]$ in ω by the path $c_1 c_2$, we have an oriented cycle in Q, which is again a contradiction. Thus there is no oriented cycle in $\overline{Q}$, and thus $\overline{\Delta}$ is exceptional.

Now we prove the second statement. Let $\alpha \neq \beta$ be two arcs in $\Delta\backslash\{\gamma\}$, then $\overline{\alpha} \neq \overline{\beta}$ since otherwise α, β and γ form a triangle on the surface which has no •-point, which contradicts the fact that Δ is admissible. On the other hand, note that if $\overline{\alpha}$ is empty for an arc α in Δ, then $\alpha = \gamma$. Thus $\overline{\Delta}$ has exact $|\Delta| - 1$ arcs. Therefore by [1, Proposition 1.11], $\overline{\Delta}$ is a maximal admissible collection on (S_γ, M_γ), that is, $\overline{\Delta}$ is an exceptional dissection. □

Now we discuss how to lift an admissible (resp. exceptional) collection on (S_γ, M_γ) to an admissible (resp. exceptional) collection on (S, M). Let Δ_γ be a collection of arcs on (S_γ, M_γ). Each arc on (S_γ, M_γ) can be viewed as an arc induced by an arc (which may not be unique) on (S, M). Thus there always exists a collection of arcs $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ on (S, M) such that $\Delta_\gamma = \overline{\Delta}$. So we can always denote a collection of arcs on (S_γ, M_γ) by $\Delta_\gamma = \{\overline{\alpha}, \alpha \in \Delta\}$ for some collection of arcs Δ on (S, M), and we will always assume that Δ contains γ, noticing that $\overline{\gamma}$ is empty. We call Δ a *lift* of Δ_γ.
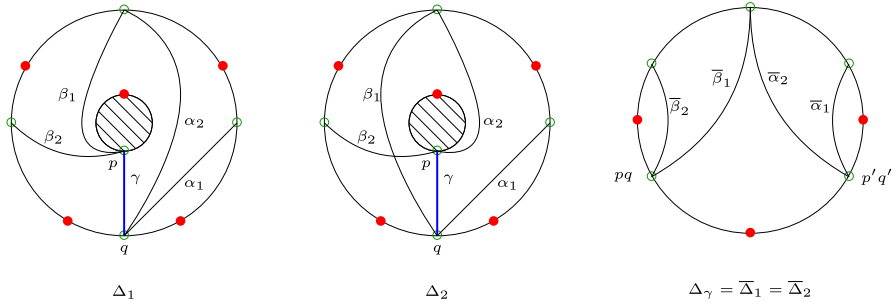
Fig. 11 Two lifts Δ_1 and Δ_2 of an admissible dissection Δ_γ on (S_γ, M_γ), where Δ_1 is an admissible dissection on (S, M) while Δ_2 is not
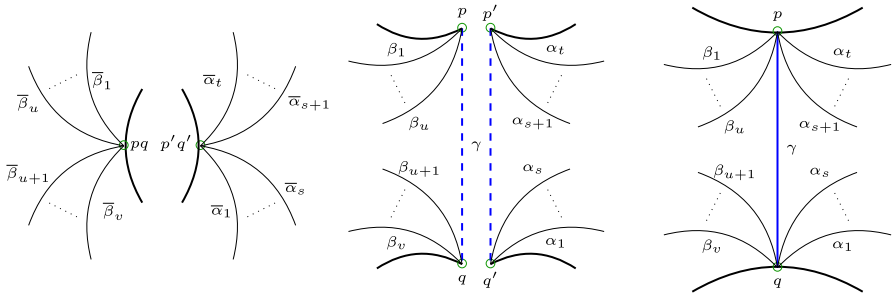

Fig. 12 Local configuration of lifting of arcs with endpoints pq or $p'q'$ in an admissible collection Δ_γ of the cut surface. For different i and j, $\overline{\beta}_i$ may coincide with $\overline{\beta}_j$. Similar for the arcs $\overline{\alpha}_i$ and $\overline{\alpha}_j$. There may also exist arcs $\overline{\beta}_i$ and $\overline{\alpha}_j$ which coincide

Let $\Delta_\gamma = \{\overline{\gamma}_i, 1 \le i \le n, \gamma_i \in \Delta\}$ be an admissible collection on (S_γ, M_γ) where Δ is a lift of Δ_γ. Note that the collection Δ is not necessarily admissible, see Fig. 11 for an example.

We will give conditions which guarantee that Δ is admissible. For this we denote by $(\overline{\beta}_1, \ldots, \overline{\beta}_v)$ and $(\overline{\alpha}_1, \ldots, \overline{\alpha}_t)$, respectively, the ordered subsets of Δ_γ of all arcs with endpoints at pq, respectively at $p'q'$. Here by our conventions the orders of $(\overline{\beta}_1, \ldots, \overline{\beta}_v)$ and $(\overline{\alpha}_1, \ldots, \overline{\alpha}_t)$ are compatible with the clockwise orders at the endpoints, that is, $\overline{\beta}_1 \preceq \cdots \preceq \overline{\beta}_v$ and $\overline{\alpha}_1 \preceq \cdots \preceq \overline{\alpha}_t$, see the left picture in Fig. 12.

Lemma 4.5 *Let Δ be a collection of arcs on (S, M) such that $\Delta_\gamma = \{\overline{\gamma}_i, 1 \le i \le n, \gamma_i \in \Delta\}$ is an admissible collection on (S_γ, M_γ). With the above notations, the following statements hold.*

(1) The collection Δ is an admissible collection if and only if there are integers $0 \le u \le v$ and $0 \le s \le t$ such that the lifts of pq and $p'q'$ respectively are such that, p is an endpoint of β_i for $1 \le i \le u$, p' is an endpoint of α_i for $1 \le i \le s$, q is an endpoint of β_j for $u + 1 \le j \le v$ and q' is an endpoint of α_j for $s + 1 \le j \le t$, see the two rightmost pictures in Fig. 12. Furthermore, if Δ_γ is an admissible dissection, then Δ is an admissible dissection if and only if the above conditions are satisfied.

(2) The collection Δ is an exceptional collection (resp. dissection) if and only if the following conditions are satisfied:

(2.1) Δ_γ *is an exceptional collection (resp. dissection);*

(2.2) Δ *is an admissible collection consisting of exceptional arcs;*

(2.3) for any two lifts β_i and α_j with common endpoint p (resp. q), either $\overline{\beta}_i$ and $\overline{\alpha}_j$ are not comparable or $\overline{\beta}_i \preceq \overline{\alpha}_j$ (resp. $\overline{\alpha}_j \preceq \overline{\beta}_i$).

We will denote by $\Delta_{(u,s)}$ the admissible dissection in (S, M) as in Lemma 4.5 (1) determined by a pair of integers (u, s).

Proof (1) If there exist integers $0 \leq u \leq v$ and $0 \leq s \leq t$ satisfying the conditions in the first statement, then the lifting of arcs with endpoints pq or $p'q'$ will not give rise to new interior intersections. Thus Δ is admissible since Δ_γ is. Conversely, assume that there exists no integer $0 \leq u \leq v$ or $0 \leq s \leq t$ satisfying the conditions in the first statement. Without loss of generality assume that such an integer u does not exist and that β_u has endpoint q and $\beta_{u'}$ has endpoint p with $u < u'$. Then β_u and $\beta_{u'}$ intersect in the interior and Δ is not admissible, see for example the right and middle pictures of Fig. 11.

(2) If Δ is an exceptional collection, then in particular it is an admissible collection and any arc in Δ is an exceptional arc with two distinct endpoints. By Lemma 4.4, Δ_γ is an exceptional collection. So conditions (2.1) and (2.2) hold. Furthermore, note that the condition $\overline{\alpha}_j \preceq \overline{\beta}_i$ implies that $\alpha_j \preceq \beta_i$ in Δ. On the other hand, the fact that β_i and α_j share a common endpoint p shows that $\beta_i \preceq \alpha_j$ in Δ. So (β_i, α_j) cannot be an exceptional pair, and Δ will not be exceptional. A similar discussion works for the case that β_i and α_j share a common endpoint q.

Conversely, assume that the three conditions in the second statement hold. We prove that Δ is exceptional. Suppose for contradiction that there is a non-exceptional cycle in Δ, or equivalently, that there is an oriented cycle $\mathbf{c}$ in the quiver Q associated to Δ. Since Q is the quiver of a gentle algebra, there are at most four arrows at the vertex in Q corresponding to γ, depicted as follows:
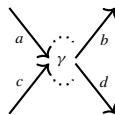
Note that since Δ_γ is exceptional, the associated quiver $\overline{Q}$ has no oriented cycles. On the other hand, $\overline{Q}$ is obtained from Q by deleting the arrows attached to γ and then adding new arrows $[ab]$ and $[cd]$ (if a, b and c, d, respectively, exist in Q). Thus the cycle $\mathbf{c}$ goes through γ and contains a deleted arrow in Q.

Assume $\mathbf{c}$ contains a. Then $\mathbf{c}$ also contains b or d, since it is a cycle. In the following we always assume that γ appears in $\mathbf{c}$ once, and each deleted arrow appears at most once in $\mathbf{c}$ and we call $\mathbf{c}$ a minimal cycle.

Case 1. If b belongs to $\mathbf{c}$, since $\mathbf{c}$ is minimal, a and b appear in $\mathbf{c}$ just once and no other deleted arrow appears in $\mathbf{c}$. After replacing ab in $\mathbf{c}$ by $[ab]$, we obtain a path in $\overline{Q}$, which is still an oriented cycle. This contradicts the assumption that Δ_γ is exceptional.

Case 2. If d belongs to $\mathbf{c}$, then we have two subcases. If $ad = \mathbf{c}$, then the source of a and the target of d coincide and we denote this vertex by α. Then the corresponding arc α on (S, M) is a loop in Δ, which contradicts the assumption that any arc in Δ is exceptional. Now assume $ad \neq \mathbf{c}$. We denote by β the source of a and by α the target

of d. Since the composition of a and d is nonzero, β and α share an endpoint p or q. Without loss of generality, we assume that they share the endpoint p. On the other hand, under the assumption that $\mathbf{c}$ is minimal, the path obtained from $\mathbf{c}$ by deleting the arrows a and d is a path in $\overline{Q}$, which is still connected and gives rise to the relation $\overline{\alpha} \preceq \overline{\beta}$. This contradicts condition (2.3) in the second statement.

The proof for the case that $\mathbf{c}$ contains c is similar. To sum up, if the three conditions in the second statement hold, then Δ is exceptional. □

The following corollary is a direct consequence of Lemma 4.5.

Corollary 4.6 *For any exceptional collection Δ_γ on (S_γ, M_γ), the admissible collection $\Delta_{(v,t)}$ is the unique exceptional collection on (S, M) such that $\Delta_\gamma = \overline{\Delta}_{(v,t)}$ and such that γ is a maximal element in $\Delta_{(v,t)}$. Furthermore, if Δ_γ is an exceptional dissection, then $\Delta_{(v,t)}$ is also an exceptional dissection.*

We apply iterated cuts of surfaces to prove the following.

Proposition 4.7 *The quiver with relations associated to any exceptional dissection on the surface $\mathbb{T}_{(g,1,2)}$ is of the form*

$$2g+1 \Rightarrow 2g \Rightarrow \cdots\cdots \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \tag{4.1}$$

and on the surface $\mathbb{T}_{(g,2,2)}$ it is of the form

$$2g+2 \Rightarrow 2g+1 \Rightarrow \cdots\cdots \Rightarrow 3 \Rightarrow 2 \Rightarrow 1. \tag{4.2}$$

Proof At first we show that any quiver Q associated to an exceptional dissection on the surface $\mathbb{T}_{(1,1,2)}$ is isomorphic to a quiver as in (4.1). By Lemma 3.3 there are three vertices and four arrows in Q and there must exist two vertices, for example vertices 3 and 2, such that there are two arrows between them. On the other hand, since there is no oriented cycle in Q, these two arrows have the same direction. So we can assume that there are two arrows from 3 to 2 in Q. Furthermore, since Q is the quiver of a gentle algebra, there are at most two arrows starting and at most to arrow ending at a given vertex, so the other two arrows must either start at 2 and end at 1, or start at 1 and end at 3. So Q is isomorphic to a quiver as (4.1). Furthermore, for such a quiver, up to isomorphism there is a unique choice of relations, depicted in (4.1), such that the resulting quotient algebra is gentle.

Let Δ be an exceptional dissection on the surface $(S, M) = \mathbb{T}_{(1,2,2)}$. Let γ be any arc in Δ, then the cut surface (S_γ, M_γ) is homeomorphic to the surface $\mathbb{T}_{(1,1,2)}$ with an exceptional dissection $\overline{\Delta}$, whose quiver $\overline{Q}$ is then necessarily as in (4.1). Furthermore, we may assume that γ is maximal in Δ with respect to the partial order of Δ. Then by Lemma 4.3, the quiver Q of Δ is obtained from $\overline{Q}$ by adding a vertex corresponding

to γ and two new arrows. Note that the latter follows from the fact that by Lemma 3.3, there are 6 arrows in Q. By the maximality of γ the two new arrows start at γ. Since Δ is an exceptional dissection, there are no oriented cycles in Q and thus in order for Q to give rise to a gentle algebra, it must be as in (4.2). Thus the statement holds for surfaces of genus one.

For the general case, the following two claims guarantee that we can use inductions on the genus of the surface, where the first claim is clear and we will prove the second one.

Claim 1. Let γ be any arc in an exceptional dissection Δ of the surface $(S, M) = \mathbb{T}_{(g,2,2)}$. Then the cut surface (S_γ, M_γ) is homeomorphic to $\mathbb{T}_{(g,1,2)}$.

Claim 2. Let γ be a maximal arc in an exceptional dissection Δ of the surface $(S, M) = \mathbb{T}_{(g,1,2)}$. Then the cut surface (S_γ, M_γ) is homeomorphic to $\mathbb{T}_{(g-1,2,2)}$.

We now show Claim 2. Since γ is a minimal arc in Δ, by Lemma 4.3, the quiver $\overline{Q}$ is obtained from Q by deleting the vertex corresponding to γ (which is a source) and the arrows incident with it. Note that since the surface (S, M) is connected, the quiver Q is connected. So the quiver $\overline{Q}$ is connected, and thus the cut surface (S_γ, M_γ) is also connected. Therefore γ is non-separating. Denote by $\tilde{S}$ the compact surface obtained from S by contracting the boundary to a point. Then γ becomes a simple closed curve on $\tilde{S}$, which is denoted by $\tilde{\gamma}$. By Lemma 1.3, there is a homeomorphism $\tilde{\phi} : \tilde{S} \mapsto \tilde{S}$ which maps $\tilde{\gamma}$ to a simple closed curve $\tilde{\phi}(\tilde{\gamma})$ which is a curve going around a handle of $\tilde{S}$ once. In fact, $\tilde{\phi}$ can be lifted to a homeomorphism ϕ of (S, M) by keeping the local configuration of the boundary, where $\phi(\gamma)$ is a curve going around a handle of (S, M) once. Thus the cut surface of (S, M) along $\phi(\gamma)$ is homeomorphic to $\mathbb{T}_{(g-1,2,2)}$. Furthermore, the cut surface of (S, M) along γ is also homeomorphic to $\mathbb{T}_{(g-1,2,2)}$, since ϕ is a homeomorphism of (S, M). $\qquad\square$

Corollary 4.8 *(1) Any arc in an exceptional dissection on the surfaces $\mathbb{T}_{(g,1,2)}$ and on $\mathbb{T}_{(g,2,2)}$ is a non-separating arc.*

(2) Let Δ_1 and Δ_2 be two exceptional dissections on $\mathbb{T}_{(g,1,2)}$ (on $\mathbb{T}_{(g,2,2)}$ resp.), then there is homeomorphism ϕ of $\mathbb{T}_{(g,1,2)}$ (of $\mathbb{T}_{(g,2,2)}$ resp.) such that $\phi(\Delta_1) = \Delta_2$.

Proof (1) Let γ be an arc in an exceptional dissection Δ on the surface $\mathbb{T}_{(g,1,2)}$ or on $\mathbb{T}_{(g,2,2)}$. By Proposition 4.7, the quiver with relations associated to Δ is as in (4.1) or (4.2) respectively. On the other hand, by the construction in Lemma 4.3 of the quiver $\overline{Q}$ associated to the exceptional dissection $\overline{\Delta}$, $\overline{Q}$ is still connected. Thus γ is non-separating.

(2) This follows from Proposition 4.7 and the bijective correspondence between gentle quivers and admissible dissections of the surfaces. $\qquad\square$

As another application of cutting marked surfaces, we will describe when an exceptional sequence in $\mathsf{K}^b(A)$ can be completed to a full exceptional sequence. We begin with the following straightforward observation.

Lemma 4.9 *An exceptional sequence $(P^\bullet_{(\gamma_1, f_1)}, \ldots, P^\bullet_{(\gamma_m, f_m)})$ in $\mathsf{K}^b(A)$ can be completed to a full exceptional sequence if and only if the associated exceptional collection $\{\gamma_1, \ldots, \gamma_m\}$ on (S, M) can be completed to an exceptional dissection.*

We now give an example showing that not every exceptional sequence can be completed to a full exceptional sequence.
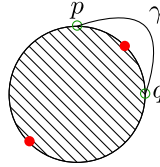
Fig. 13 The local configuration of an arc γ on $\mathbb{T}_{(1,1,2)}$

Example 4.10 Consider the marked surface $\mathbb{T}_{(g,1,2)}$ of genus $g \geq 1$ with one boundary component and two o-marked point p and q. Let γ be an arc connecting p and q, and such that γ is homotopic to a boundary segment, see Fig. 13. Then for any grading f_γ on γ, the associated object $P^\bullet_{(\gamma, f_\gamma)}$ is exceptional. However, it cannot be completed to a full exceptional sequence: Let α be an o-arc which has no self-intersections. So in particular, α must be have endpoints p and q. If α is not homotopic to γ, then γ and α form a non-exceptional cycle, and $\{\gamma, \alpha\}$ cannot be an exceptional collection. Thus by Lemma 4.9, $P^\bullet_{(\gamma, f_\gamma)}$ can not be completed to a full exceptional sequence.

Let $\{\gamma_1, \gamma_2\}$ be an exceptional collection, it is easy to see that

$$((S_{\gamma_1})_{\gamma_2}, (M_{\gamma_1})_{\gamma_2}) = ((S_{\gamma_2})_{\gamma_1}, (M_{\gamma_2})_{\gamma_1}).$$

This ensures that for any exceptional collection Δ, we may define the cut surface (S_Δ, M_Δ) by successively cutting the arcs in Δ in any order.

Proposition 4.11 *Let* $\Delta = \{\gamma_1, \ldots, \gamma_m\}$ *be an exceptional collection on* (S, M). *Then* Δ *can be completed to an exceptional dissection if and only if* (S_Δ, M_Δ) *has no connected component which is homeomorphic to* $\mathbb{T}_{(g,1,1)}$.

Proof We prove this by induction on m. We begin with the case $m = 1$. In this case $\Delta = \{\gamma_1\}$. Suppose that there is no connected component of (S_Δ, M_Δ) homeomorphic to $\mathbb{T}_{(g,1,1)}$. Then by Proposition 3.5, there exists an exceptional dissection Δ_{γ_1} on (S_Δ, M_Δ). Furthermore, by Corollary 4.6, there exists an exceptional dissection Δ of (S, M) containing γ_1, that is, Δ can be completed to an exceptional dissection on (S, M).

For the general case, let $\Delta = \{\gamma_1, \ldots, \gamma_m\}$ be an exceptional collection on (S, M) and suppose that (S_Δ, M_Δ) has no connected component homeomorphic to $\mathbb{T}_{(g,1,1)}$. Then $\{\gamma_2, \ldots, \gamma_m\}$ is an exceptional collection on $(S_{\gamma_1}, M_{\gamma_1})$ by Lemma 4.4. By induction, it can be completed to an exceptional dissection on $(S_{\gamma_1}, M_{\gamma_1})$. Then again by Corollary 4.6, there exists an exceptional dissection on (S, M) containing $\{\gamma_1, \ldots, \gamma_m\}$.

The converse directly follows from the fact that if there is a connected component of the form $\mathbb{T}_{(g,1,1)}$, then there are no exceptional arcs on $\mathbb{T}_{(g,1,1)}$ since this connected component has a single marked point on its unique boundary. $\square$

Now we have the following geometric characterisation of when an exceptional sequence can be completed to a full exceptional sequence.

Theorem 4.12 *Let A be a gentle algebra associated with a marked surface (S, M, Δ_A) with associated admissible dissection Δ_A. Assume there exist full exceptional sequences in $K^b(A)$ and let $(P^\bullet_{(\gamma_1, f_1)}, \ldots, P^\bullet_{(\gamma_m, f_m)})$ be an exceptional sequence arising from an exceptional collection $\Delta = \{\gamma_1, \ldots, \gamma_m\}$. Then it can be completed to a full exceptional sequence if and only if the cut surface (S_Δ, M_Δ) has no subsurface of the form $\mathbb{T}_{(g,1,1)}$.*

Proof This follows directly from Lemma 4.9 and Proposition 4.11.   □

5 Transitivity of the braid group action

In this section we will use the geometric model to show that the induced action of the Artin braid group is transitive on the set of full exceptional sequences in the perfect derived category of a gentle algebra corresponding to a genus zero surface. More precisely, denote by B_n the Artin braid group generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations $\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. Then we show the following.

Theorem 5.1 *Let A be a gentle algebra arising from a marked surface of genus zero, let n be the rank of $K_0(K^b(A))$ and B_n the Artin braid group on n strands. Then the action of $\mathbb{Z}^n \rtimes B_n$ on the set of full exceptional sequences in $K^b(A)$ is transitive.*

The proof is based on the following strategy: In the first half of this section we introduce an action of B_n on the set of ordered exceptional dissections of a marked surface (S, M). We then discuss the transitivity of this action and show that it holds for surfaces satisfying a certain condition (**RCEA** condition). In the second half of this section we prove the theorem by showing how this geometric interpretation translates to the braid group action on full exceptional sequences in the associated derived category $K^b(A)$.

5.1 Braid group action on exceptional surface dissections

We start with the following observation.

Lemma 5.2 *Let (γ_1, γ_2) be an ordered exceptional pair in (S, M). Then γ_1 and γ_2 either do not intersect or they intersect at one or two common endpoints, as shown in Figure 14.*

Definition 5.3 (1) Let (γ_1, γ_2) be an ordered exceptional pair in (S, M). We now define *the left (resp. right) mutation of γ_2 at γ_1* denoted by $L_{\gamma_1}\gamma_2$ (resp. $R_{\gamma_1}\gamma_2$) as follows:

- If γ_1 and γ_2 do not intersect, then we define $L_{\gamma_1}\gamma_2$ as γ_2 and define $R_{\gamma_2}\gamma_1$ as γ_1.
- If γ_1 and γ_2 share one endpoint q_1 then $L_{\gamma_1}\gamma_2$ is the smoothing of the crossing of γ_1 and γ_2 at q_1.
- If γ_1 and γ_2 share both endpoints, then $L_{\gamma_1}\gamma_2$ is obtained by first smoothing the crossing of γ_1 and γ_2 at one of the endpoints and then smoothing the crossing of the resulting arc with γ_1 at the other endpoint of γ_1, see Fig. 14.
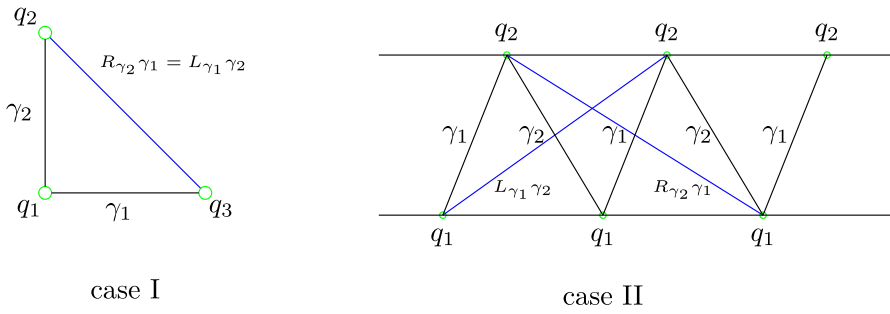
Fig. 14 Possible intersections of γ_1 and γ_2 in an ordered exceptional pair (γ_1, γ_2), where case II is depicted in the universal covering. This figure also illustrates the mutations of the ordered exceptional pair (γ_1, γ_2) (in blue)

Dually, we define $R_{\gamma_2\gamma_1}$ when γ_1 and γ_2 intersect in one or both endpoints, see Fig. 14.

(2) Let $\Gamma = (\gamma_1, \ldots, \gamma_m)$ be an ordered exceptional collection of arcs in (S, M), for any $1 \leq i \leq m-1$, we define

$$\sigma_i \Gamma = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, R_{\gamma_{i+1}}\gamma_i, \gamma_{i+2}, \ldots, \gamma_m),$$

$$\sigma_i^{-1} \Gamma = (\gamma_1, \ldots, \gamma_{i-1}, L_{\gamma_i}\gamma_{i+1}, \gamma_i, \gamma_{i+2}, \ldots, \gamma_m).$$

Theorem 5.4 *Let $\Gamma = (\gamma_1, \ldots, \gamma_m)$ be an ordered exceptional collection on (S, M). Then $\sigma_i \Gamma$ and $\sigma_i^{-1} \Gamma$ are ordered exceptional collections on (S, M) of length m, for all $1 \leq i \leq m-1$. Furthermore, we have*

$$\sigma_i^{-1}\sigma_i = \sigma_i\sigma_i^{-1} = id,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| > 1,$$

and this induces an action of the braid group B_m on the set of ordered exceptional collections on (S, M) of length m.

Proof The statement that $\sigma_i \Gamma$ and $\sigma_i^{-1} \Gamma$ are ordered exceptional collections on (S, M) of length m can be checked directly using the definitions. It is also straightforward to check the equalities $\sigma_i^{-1}\sigma_i = \sigma_i\sigma_i^{-1} = id$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| > 1$. Now we consider the braid relations. Without loss of generality, we may assume $m = 3$. Then

$$\sigma_1\sigma_2\sigma_1(\gamma_1, \gamma_2, \gamma_3) = (\gamma_3, R_{\gamma_3}\gamma_2, R_{\gamma_3}R_{\gamma_2}\gamma_1),$$
$$\sigma_2\sigma_1\sigma_2(\gamma_1, \gamma_2, \gamma_3) = (\gamma_3, R_{\gamma_3}\gamma_2, R_{R_{\gamma_3}\gamma_2}R_{\gamma_3}\gamma_1).$$

So we have to show that $R_{\gamma_3}R_{\gamma_2}\gamma_1 = R_{R_{\gamma_3}\gamma_2}R_{\gamma_3}\gamma_1$.

If there exists an arc γ_j, $1 \leq j \leq 3$ which shares no endpoints with the other two arcs, then it can be easily checked that $R_{\gamma_3}R_{\gamma_2}\gamma_1 = R_{R_{\gamma_3}\gamma_2}R_{\gamma_3}\gamma_1 = \gamma_1$ if $j = 1$, $R_{\gamma_3}R_{\gamma_2}\gamma_1 = R_{R_{\gamma_3}\gamma_2}R_{\gamma_3}\gamma_1 = R_{\gamma_3}\gamma_1$ if $j = 2$, $R_{\gamma_3}R_{\gamma_2}\gamma_1 = R_{R_{\gamma_3}\gamma_2}R_{\gamma_3}\gamma_1 = R_{\gamma_2}\gamma_1$ if $j = 3$.

If any two arcs intersect at one or both of the endpoints then there are several cases to consider depending on the number of the intersections as well as the positions of the different arcs. The result then follows from a straightforward verification of the definitions. We prove one of the cases in the following figure, the other cases being similar.
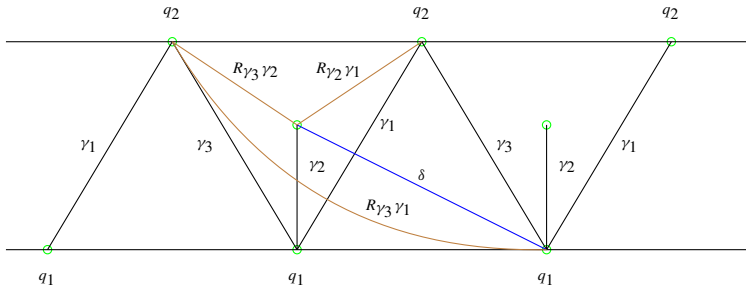
In this situation it directly follows from the figure that $R_{\gamma_3} R_{\gamma_2} \gamma_1 = R_{R_{\gamma_3}\gamma_2} R_{\gamma_3}\gamma_1$, which corresponds to the blue arc δ for the case when γ_2 shares one endpoint both with γ_1 and γ_3, and γ_1, γ_3 share two endpoints. □

In the following we consider the transitivity of the braid group action on the set of ordered exceptional dissections on (S, M). For this we first show some technical lemmas.

We observe that different choices of order for a given exceptional collection on (S, M) gives rise to distinct ordered exceptional collections. However, we show that they are all related through the braid group action.

Lemma 5.5 *Let $\Gamma = (\gamma_1, \ldots, \gamma_m)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_m)$ be two ordered exceptional collections on (S, M) arising from the same exceptional collection Δ. Then there is an element σ in B_m such that $\sigma\Gamma = \Gamma'$.*

Proof Suppose that $\gamma_i = \gamma'_m$ for some $1 \leq i \leq m$. Then note that γ_i is a maximal element with respect to the partial order on Δ. Thus there are no intersections between γ_i and γ_j, for $i + 1 \leq j \leq m$ and $R_{\gamma_j}\gamma_i = \gamma_i$. So $\sigma_{m-1} \cdots \sigma_i \Gamma = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_m, \gamma_i = \gamma'_m)$ which is obtained from Γ by moving γ'_m to the last position and shifting the arcs $\gamma_{i+1}, \ldots, \gamma_m$ one position to the left. Then we consider the position of γ'_{m-1} in $\sigma_{m-1} \cdots \sigma_i \Gamma$, and similarly mutate it to the position before γ'_m. Continuing in this way, we can mutate Γ to Γ', that is, we have inductively constructed an element σ such that $\sigma\Gamma = \Gamma'$. □

For the rest of this section we fix the following conventions. Let (S, M) be a marked surface with exceptional dissections. Let γ be an exceptional ∘-arc such that there exist exceptional dissections on the cut surface (S_γ, M_γ). We extend the definition of the braid group action to a non-connected marked surface. Note that if the braid group B_m acts on the set of ordered exceptional dissections of a marked surface (S, M), then the braid group B_{m-1} acts on the cut surface (S_γ, M_γ), independently from whether (S_γ, M_γ) is connected or not.

Let Δ_γ be an admissible dissection on (S_γ, M_γ). Recall from Lemma 4.5 that $\Delta_{(u,s)}$ is the unique lift of Δ_γ to (S, M) which is an admissible dissection determined by

the pair (u, s) such that $0 \leq u \leq v, 0 \leq s \leq t$, where v and t are as in the paragraph preceding Lemma 4.5. For two pairs (u, s) and (u', s'), we say $(u, s) < (u', s')$ if $u \leq u', s \leq s'$ and at least one of the inequalities is strict. The following lemma allows us to later use induction.

Lemma 5.6 *Let $\Delta_{(u,s)}$ be a lift of Δ_γ which is exceptional. If $(u, s) < (v, t)$, then there exists an ordered exceptional dissection $\Gamma_{(u,s)}$ arising from $\Delta_{(u,s)}$ and an element σ in B_n such that $\sigma \Gamma_{(u,s)}$ is an ordered exceptional dissection with underlying set $\Delta_{(u',s')}$ for some (u', s'), which is a lift of Δ_γ with $(u, s) < (u', s')$.*

Proof We first note the following useful fact: Using the notations in Fig. 12, α_{s+1} and β_{u+1} are the only two minimal elements in $\Delta_{(u,s)}$ which are greater than γ, that is, if $\gamma \preceq \alpha$ for some $\alpha \in \Delta_{(u,s)}$, then at least one of $\alpha_{s+1} \preceq \alpha$ or $\beta_{u+1} \preceq \alpha$ holds.

Case 1. $\alpha_{s+1} = \beta_{u+1}$. Let $\Gamma_{(u,s)}$ be an ordered exceptional dissection arising from $\Delta_{(u,s)}$ such that α_{s+1} directly follows γ. This is possible, since for any $\alpha \in \Delta_{(u,s)}$ with $\gamma \preceq \alpha$, we have $\alpha_{s+1} = \beta_{u+1} \preceq \alpha$ by the above fact. Assume γ is at the i-th position in $\Gamma_{(u,s)}$, then $\sigma_i^{-1}\Gamma_{(u,s)}$ is an ordered exceptional dissection obtained from $\Gamma_{(u,s)}$ by exchanging the positions of γ and α_{s+1}, and then replacing α_{s+1} by $L_\gamma \alpha_{s+1}$. Then $\Gamma_{(u',s')} := \sigma_i^{-1}\Gamma_{(u,s)}$ is what we want, noticing that on the one hand $(u', s') = (u+1, s+1)$, on the other hand $\overline{L_\gamma \alpha_{s+1}} = \overline{\alpha_{s+1}}$ and thus the underlying set of $\Gamma_{(u',s')}$ is a lift of Δ_γ.

Case 2. $\alpha_{s+1} \neq \beta_{u+1}$. Then at most one of $\alpha_{s+1} \preceq \beta_{u+1}$ and $\beta_{u+1} \preceq \alpha_{s+1}$ is satisfied. Otherwise, we have $\overline{\beta_{u+1}} \preceq \overline{\alpha_{s+1}} \preceq \overline{\beta_{u+1}}$, which contradicts the fact that Δ_γ is exceptional. Without loss of generality, assume that $\beta_{u+1} \preceq \alpha_{s+1}$ does not hold. Then for $\alpha \in \Delta$ such that $\gamma \preceq \alpha$, we have $\alpha_{s+1} \preceq \alpha$. Otherwise, if $\alpha \preceq \alpha_{s+1}$, then by the above fact, we have $\beta_{u+1} \preceq \alpha$, thus $\beta_{u+1} \preceq \alpha_{s+1}$. A contradiction. Then similar to the first case, we define $\Gamma_{(u,s)}$ as an ordered exceptional dissection arising from $\Delta_{(u,s)}$ such that α_{s+1} directly follows γ. And then define $\Gamma_{(u',s')}$ as $\sigma_i^{-1}\Gamma_{(u,s)}$, where $(u', s') = (u, s+1)$. □

The next corollary follows from the two previous lemmas.

Corollary 5.7 *Let Γ_1 and Γ_2 be two ordered exceptional dissections on (S, M) containing γ. Let $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ be the ordered exceptional dissections on (S_γ, M_γ) induced by Γ_1 and Γ_2, respectively. Suppose $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ have the same underlying set Δ_γ. Then there is an element σ in B_n such that $\sigma \Gamma_1 = \Gamma_2$.*

Proof Denote by $\Delta_{(v,t)}$ the unique exceptional dissection on (S, M) such that $\Delta_\gamma = \overline{\Delta}_{(v,t)}$ and γ is a maximal element in $\Delta_{(v,t)}$. Let $\Gamma_{(v,t)}$ be any ordered exceptional dissection arising from $\Delta_{(v,t)}$. Denote by $\Delta_{(u_1,s_1)}$ and $\Delta_{(u_2,s_2)}$ the underlying sets of Γ_1 and Γ_2 respectively. Then by alternatively using Lemma 5.6 and Lemma 5.5, for $1 \leq i \leq 2$, we have an element σ_i' in B_n and an ordered exceptional dissection Γ_i' on (S, M) with underlying set $\Delta_{(u_i,s_i)}$, such that $\sigma_i'\Gamma_i' = \Gamma_{(v,t)}$. On the other hand, by Lemma 5.5, for $1 \leq i \leq 2$, we have an element σ_i in B_n such that $\sigma_i \Gamma_i = \Gamma_i'$. Then by setting $\sigma = \sigma_2^{-1}\sigma_2'^{-1}\sigma_1'\sigma_1$, we have $\sigma \Gamma_1 = \Gamma_2$. □

The next lemma sets up the induction step for the proof of Theorem 5.1.

Lemma 5.8 *Let Γ and Γ' be two ordered exceptional dissections on (S, M) containing γ. If B_{n-1} is transitive on the set of ordered exceptional dissections on (S_γ, M_γ), then there is an element σ in B_n such that $\sigma \Gamma = \Gamma'$.*

Proof At first, we assume that γ is a maximal element in both Γ and Γ'. Then due to Lemma 5.5, without loss of generality, we assume that γ is the last element in Γ and Γ'. Note that by Lemma 4.4 the induced ordered sets $\overline{\Gamma}$ and $\overline{\Gamma}'$ are both ordered exceptional dissections on (S_γ, M_γ). So by assumption, there is an element $\sigma = \delta_m \cdots \delta_1$ in B_{n-1} such that $\sigma \overline{\Gamma} = \overline{\Gamma}'$, where each δ_j is some generator σ_i or σ_i^{-1} of B_{n-1} for some $1 \leq i \leq n-2$. By Corollary 4.6, there is an unique ordered exceptional dissection Γ_1 on (S, M) such that $\overline{\Gamma}_1 = \delta_1 \overline{\Gamma}$ and such that γ is its last element. By viewing δ_1 as an element in B_n by the natural embedding of braid groups $B_{n-1} \subseteq B_n$, the set $\delta_1 \Gamma$ is an ordered exceptional dissection on (S, M). Note that Γ_1 and $\delta_1 \Gamma$ are both lifts of $\delta_1 \overline{\Gamma}$ containing γ as their last element. Thus we have $\Gamma_1 = \delta_1 \Gamma$. Similarly, we construct ordered exceptional dissections Γ_j, $2 \leq j \leq m$, on (S, M) such that $\overline{\Gamma}_j = \delta_j \cdots \delta_1 \overline{\Gamma}$ and $\Gamma_j = \delta_j \cdots \delta_1 \Gamma$. Then we have

$$\Gamma' = \delta_m \Gamma_{m-1} = \delta_m \delta_{m-1} \Gamma_{m-2} = \cdots = \delta_m \cdots \delta_1 \Gamma = \sigma \Gamma,$$

where δ_j, $1 \leq j \leq m$ and σ are viewed as elements in B_n.

Finally, for the general case, that is, the case when γ is not necessary a maximal element in Γ or Γ', the proof follows from the above and Corollary 5.7. $\square$

Next we formulate a condition on a marked surface such that for surfaces satisfying this condition, the action of the braid group is transitive on the set of ordered exceptional dissections. We say that a marked surfaces (S, M) satisfies the 'reachable condition for exceptional arcs' if the following holds.

Definition 5.9 (Condition RCEA) We say that a marked surface (S, M) satisfies the RCEA condition if for any two exceptional arcs γ and γ' on (S, M), there is a sequence of exceptional arcs $\gamma_0, \ldots, \gamma_{m+1}$ on (S, M), such that each pair $\{\gamma_i, \gamma_{i+1}\}$ is an exceptional collection, where $0 \leq i \leq m$, $\gamma_0 = \gamma$, and $\gamma_{m+1} = \gamma'$.

Theorem 5.10 *Suppose that the RCEA condition holds for all marked surfaces. Then for any marked surface with exceptional dissections, the braid group action on the set of ordered exceptional dissections is transitive.*

Proof Let (S, M) be a connected marked surface. We prove the result by using induction on the number $n = |M^\circ| + 2g + b - 2$ of elements in an exceptional dissection on (S, M). The base case of the induction corresponds to marked surfaces which are unions of non-punctured-disks each with two $\circ$-marked points, where the statement is clearly true. We now show the induction step. Assume the result holds for surfaces with less than $n - 1$ elements in an exceptional dissection.

Let Γ and Γ' be any two ordered exceptional dissections on (S, M). We will show that there is an element σ in B_n such that $\sigma \Gamma = \Gamma'$. Let γ and γ' be any two arcs in Γ and Γ' respectively. Then by the RCEA condition, there is a sequence of exceptional arcs $\gamma_0, \ldots, \gamma_{m+1}$ on (S, M), such that each pair $\{\gamma_i, \gamma_{i+1}\}$ is an exceptional collection, where $0 \leq i \leq m$ and $\gamma_0 = \gamma$, $\gamma_{m+1} = \gamma'$. We have two cases.
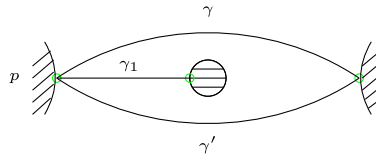
Fig. 15 The exceptional arcs γ and γ' share two endpoints and (γ, γ') is a non-exceptional cycle, where the left and right boundary components may coincide

Case 1. (S, M) is not a surface of the form $\mathbb{T}_{(g,1,2)}$. Note that in this case, for each $1 \le i \le m$, the cut surface $(S_{\gamma_i}, M_{\gamma_i})$ does not have a connected component of the form $\mathbb{T}_{(g,1,1)}$. Then by Proposition 4.11, there exists an exceptional dissection Γ_i on (S, M) which contains γ_i and γ_{i-1} for each $1 \le i \le m$. Furthermore, by Lemma 5.8, there is an element $\sigma_i, 0 \le i \le m$ in B_n such that $\sigma_i \Gamma_i = \Gamma_{i+1}$, where we denote Γ by Γ_0 and denote Γ' by Γ_{m+1} for simplicity. Then we have $\sigma \Gamma = \Gamma'$, where $\sigma = \sigma_m \cdots \sigma_0 \in B_n$.

Case 2. (S, M) is a surface of the form $\mathbb{T}_{(g,1,2)}$. Note that in this case, since each $\gamma_i, 1 \le i \le m$ is contained in an exceptional pair, it can not be an arc homotopic to a boundary segment which contains a •-marked point, see in Fig. 13. Thus there exists an exceptional dissection Γ_i which contains γ_i for each $1 \le i \le m$. Then similar to the proof of the first case, we have an element σ in B_n such that $\sigma \Gamma = \Gamma'$.

□

In general, it seems a difficult problem to determine, for which surfaces the **RCEA** condition holds. In the following, we show that the **RCEA** condition does hold for surfaces of genus zero.

Theorem 5.11 *The* **RCEA** *condition is satisfied for any marked surface of zero genus.*

Proof Let γ and γ' be two exceptional arcs on (S, M). Assume that (γ, γ') is not an exceptional collection, otherwise there is nothing to prove. Then γ and γ' share two endpoints or they have at least one intersection in the interior of S. Suppose first that γ and γ' share two endpoints and that they do not intersect in the interior. Then they surround a bigon which contains at least one boundary component, see in Fig. 15, noticing that the genus of the surface is zero. Then there exists an exceptional arc γ_1 which connects an endpoint of γ and a ○-marked point on a boundary component in the bigon, as depicted in Fig. 15. Then (γ, γ_1) and (γ_1, γ') are both exceptional collections.

In the following, we assume that γ and γ' intersect in the interior of the surface. Note that since the genus of the surface is zero there are three possible cases as shown in Figs. 16, 17, and 18.

Case 1. All the endpoints of γ and γ' are on the same boundary component. Then the local configuration of the intersection nearest to an endpoint p of γ' is depicted in Fig. 16. We choose γ_1 as showed in the figure, then (γ, γ_1) and (γ_1, γ') are both exceptional pairs.

Case 2. Both endpoints of γ (resp. γ') are on the same boundary component B (resp. B'), where B and B' are different. By assumption γ and γ' have an interior intersection
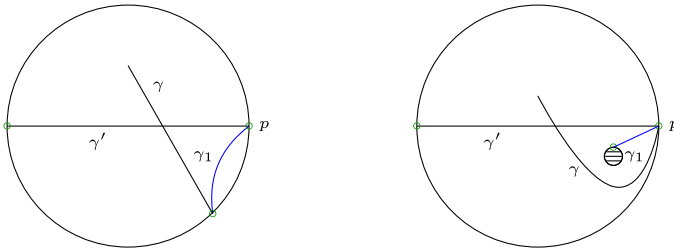
Fig. 16 Local configurations of the intersection nearest to an endpoint p of γ' on a surface of genus zero, where the endpoints of the two exceptional arcs γ and γ' are on the same boundary component
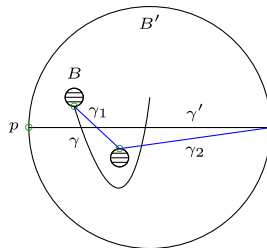
Fig. 17 Local configuration of the intersection of two exceptional arcs γ and γ' nearest to an endpoint p of γ on a surface of genus zero, where the endpoints of γ and γ' are on different boundary components B and B' respectively

which implies that there are at least three boundary components on (S, M). Then up to homotopy, the local configuration of the intersection nearest to one of the endpoints of γ, which will denote by p, is as in Fig. 17. We choose γ_1 and γ_2 as shown in Fig. 17. Then (γ, γ_1), (γ_1, γ_2) and (γ_2, γ') are exceptional pairs.

Case 3. There is at least one arc, for example γ, with endpoints on two different boundary components. Denote one of these by B. Assume that locally γ and γ' form a bigon containing at least one boundary component as shown in the left picture of Fig. 18. We construct a new arc γ_1 obtained from γ by smoothing the intersections of γ and γ' at this bigon, as shown in the left picture of Fig. 18. Then there are no interior intersections of γ and γ_1, and $|\gamma_1 \cap \gamma'| = |\gamma \cap \gamma'| - 2$. But γ_1 and γ do not form an exceptional pair, since γ and γ_1 intersect each other at both endpoints in such a way that they form a non-exceptional cycle.

Now we rotate γ_1 anti-clockwise one time around the boundary component B. Then γ_1 and γ form an exceptional pair, noticing that the endpoints of γ are on two different boundary components and that in this case this process does not give rise to interior intersections of these arcs. On the other hand, such a rotation may produce new intersections between γ_1 and γ'. However, we may assume that at most one endpoint of γ' lies on B, otherwise we choose the boundary component of the other endpoint of γ. Under such assumption, there may appear at most one new intersection between γ_1 and γ'. So that we have $|\gamma_1 \cap \gamma'| \leq |\gamma \cap \gamma'| - 1$, and (γ, γ_1) is an exceptional pair.

Then we replace γ by γ_1, and continue this process until there are no bigons surrounded by γ' and the successive replacements of γ. By abuse of notation, we will
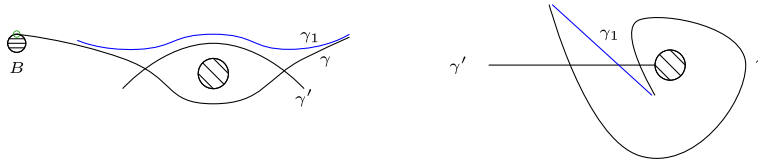
Fig. 18 Local configurations of intersections between two exceptional arcs on a surface of genus zero, where in the right picture the arc γ may also surround the boundary component in another direction

denote this new arc replacing γ again by γ. Then there are two cases. Either γ and γ' share at most one endpoint (with possible interior intersections) and an argument as in case 1 finishes the proof or γ and γ' share both endpoints. But then, since the genus of the surface is zero, the only possible intersections between γ and γ' come from rotations of one of the arcs, for example γ, around a boundary component which is incident to the other arc γ', see the right picture in Fig. 18. Then define γ_1 as the new arc which is obtained from γ by a local surgery, where locally we replace the shown segment of γ by the blue segment as in the figure. Note that we have $|\gamma_1 \cap \gamma'| = |\gamma \cap \gamma'| - 1$, and (γ, γ_1) is an exceptional pair. Continue this process until there are no intersections between the new arc and γ'. Then we have a sequence of exceptional arcs as desired.   □

Combining Theorems 5.10 and 5.11, we obtain the following.

Corollary 5.12 *The action of the braid group on the set of ordered exceptional dissections on a surface of genus zero is transitive.*

5.2 Braid group action on full exceptional sequences in $\mathsf{K}^b(\mathsf{A})$

In the following, by using the geometric model established above, we will show how the braid group B_n acts on the full exceptional sequences in the perfect derived category $\mathsf{K}^b(\mathsf{A})$ of a gentle algebra A. This allows us to show the transitivity of the braid group action when the associated surface is of genus zero.

We start by recalling the braid group action on a triangulated category $\mathcal{T}$. Let (X, Y) be an exceptional pair in $\mathcal{T}$. Define objects $R_Y X$ and $L_X Y$ in $\mathcal{T}$ through the following triangles

$$X \longrightarrow \coprod_{\ell \in \mathbb{Z}} D\mathrm{Hom}_{\mathcal{T}}(X, Y[\ell]) \otimes_k Y[\ell] \longrightarrow R_Y X[1] \longrightarrow X[1] \qquad (5.1)$$

$$Y[-1] \longrightarrow L_X Y[-1] \longrightarrow \coprod_{\ell \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X[\ell], Y) \otimes_k X[\ell] \longrightarrow Y. \qquad (5.2)$$

Then $(Y, R_Y X)$ and $(L_X Y, X)$ are again exceptional pairs in $\mathcal{T}$. We call $R_Y X$ the *right mutation* of X at Y, and and $L_X Y$ the *left mutation* of Y at X, where the two distinguished triangles (5.1) and (5.2) are called *exchange triangles*.

We denote by $\exp \mathcal{T}$ the set of isomorphism classes of full exceptional sequences in $\mathcal{T}$, where the length of each full exceptional sequence is n which is the rank of the Grothendieck group $K_0(\mathcal{T})$.
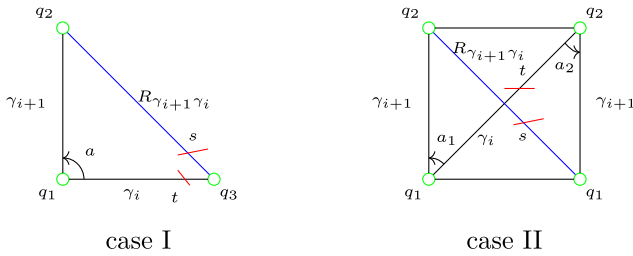
Fig. 19 Possible relative positions of γ_i and γ_{i+1} with regards to the right mutation
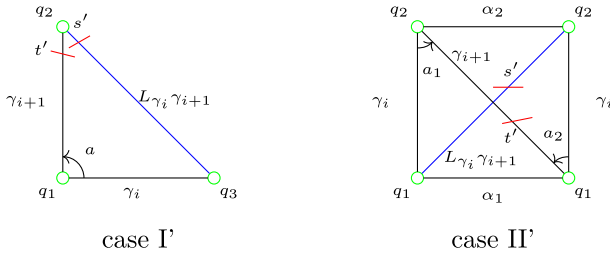

Fig. 20 Possible relative positions of γ_i and γ_{i+1} with regards to the left mutation

Then by [20], B_n acts on $\exp \mathcal{T}$ as follows: For a full exceptional sequence $\mathbf{X} :=$ $(X_1, \ldots, X_n)$ and $1 \le i < n$, set

$$\sigma_i \mathbf{X} := (X_1, \ldots, X_{i-1}, X_{i+1}, R_{X_{i+1}} X_i, X_{i+2}, \ldots, X_n)$$
$$\sigma_i^{-1} \mathbf{X} := (X_1, \ldots, X_{i-1}, L_{X_i} X_{i+1}, X_i, X_{i+2}, \ldots, X_n).$$

Let $(X_1, \ldots, X_n)$ be a full exceptional sequence in $\mathsf{K}^b(A)$ where, for all $1 \le i \le n$, $X_i = P^\bullet_{(\gamma_i, f_{\gamma_i})}$ for some graded ∘-arc (γ_i, f_{γ_i}) in the surface model (S, M, Δ_A) of A. Then by Proposition 2.6 the set $(\gamma_1, \ldots, \gamma_n)$ is an ordered exceptional dissection. In particular, for any $1 \le i \le n-1$, (γ_i, γ_{i+1}) is an exceptional pair. So the possible relative positions of γ_i and γ_{i+1} are shown in Fig. 14, where we view γ_1 as γ_i and view γ_2 as γ_{i+1}.

Note that for the right mutation, we need to consider the maps from X_i to $X_{i+1}[m]$ for any integer m, while for the left mutation, we need to consider all the maps from $X_i[m]$ to X_{i+1}, see the exchange triangles (5.1) and (5.2) respectively. So in both cases we need to associate gradings to the arcs. For the sake of clarity, we redraw and relabel the picture in Fig. 14 as four pictures in Figs. 19 and 20, where s and t (s' and t' respectively) denote the intersections of the ∘-arcs $R_{\gamma_{i+1}\gamma_i}$ and γ_i ($L_{\gamma_i}\gamma_{i+1}$ and γ_{i+1} respectively) with the initial dual •-dissection Δ_A^* closest to the intersection of $R_{\gamma_{i+1}\gamma_i}$ and γ_i ($L_{\gamma_i}\gamma_{i+1}$ and γ_{i+1} respectively).

Theorem 5.13 *Let A be a gentle algebra with associated surface model (S, M, Δ_A). Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a full exceptional sequence in $\mathsf{K}^b(A)$, and assume that $X_i = P^\bullet_{(\gamma_i, f_{\gamma_i})}$ for some graded ∘-arc (γ_i, f_{γ_i}), for all $1 \le i \le n$.*

(1) If γ_i and γ_{i+1} do not intersect then $R_{X_{i+1}} X_i = X_i$ and $L_{X_i} X_{i+1} = X_{i+1}$. So

$$\sigma_i \mathbf{X} = \sigma_i^{-1} \mathbf{X} = (X_1, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, \ldots, X_n),$$

that is, the actions of σ_i and σ_i^{-1} change the order of X_i and X_{i+1}.

(2) Suppose that γ_i and γ_{i+1} intersect. Then

- *$R_{X_{i+1}} X_i = P^{\bullet}_{(R_{\gamma_{i+1}} \gamma_i, f_{R_{\gamma_{i+1}} \gamma_i})}$ with $f_{R_{\gamma_{i+1}} \gamma_i}(s) = f_{\gamma_i}(t)$, where the arc $R_{\gamma_{i+1}} \gamma_i$ and the intersections s and t are depicted in Fig. 19.*
- *$L_{X_i} X_{i+1} = P^{\bullet}_{(L_{\gamma_i} \gamma_{i+1}, f_{L_{\gamma_i} \gamma_{i+1}})}$ with $f_{L_{\gamma_i} \gamma_{i+1}}(s') = f_{\gamma_{i+1}}(t')$, where the arc $L_{\gamma_i} \gamma_{i+1}$ and the intersections s' and t' are depicted in Fig. 20.*

In this case,

$$\sigma_i \mathbf{X} = (X_1, \ldots, X_{i-1}, X_{i+1}, P^{\bullet}_{(R_{\gamma_{i+1}} \gamma_i, f_{R_{\gamma_{i+1}} \gamma_i})}, X_{i+2}, \ldots, X_n),$$

$$\sigma_i^{-1} \mathbf{X} = (X_1, \ldots, X_{i-1}, P^{\bullet}_{(L_{\gamma_i} \gamma_{i+1}, f_{L_{\gamma_i} \gamma_{i+1}})}, X_i, X_{i+2}, \ldots, X_n).$$

Proof (1) Since γ_i and γ_{i+1} do not intersect, there exists no map from X_i to $X_{i+1}[\mathbb{Z}]$. Therefore the distinguished triangles

$$X_i \longrightarrow 0 \longrightarrow X_i[1] \longrightarrow X_i[1],$$

$$X_{i+1}[-1] \longrightarrow X_{i+1}[-1] \longrightarrow 0 \longrightarrow X_{i+1}$$

are the respective exchange triangles of the right and left mutations in (5.1) and (5.2). So $R_{X_{i+1}} X_i = X_i$, $L_{X_i} X_{i+1} = X_{i+1}$, and

$$\sigma_i \mathbf{X} = \sigma_i^{-1} \mathbf{X} = (X_1, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, \ldots, X_n).$$

(2) We only prove the case for the right braid group action in detail, the proof for left braid group action is similar. For this we consider the two cases in Fig. 19.

For case I, note that there exists a unique $m \in \mathbb{Z}$ such that the intersection q_1 induces a map

$$a : P^{\bullet}_{(\gamma_i, f_{\gamma_i})} \longrightarrow P^{\bullet}_{(\gamma_{i+1}, f_{\gamma_{i+1}})}[m].$$

Moreover, note that up to isomorphism this is the unique map from $P^{\bullet}_{(\gamma_i, f_{\gamma_i})}$ to $P^{\bullet}_{(\gamma_{i+1}, f_{\gamma_{i+1}})}[\mathbb{Z}]$, since there are no other intersections of γ_i and γ_{i+1}. Thus for some proper grading g of $R_{\gamma_{i+1}} \gamma_i$, the distinguished triangle

$$X_i \xrightarrow{a} X_{i+1}[m] \longrightarrow P^{\bullet}_{(R_{\gamma_{i+1}} \gamma_i, g)} \longrightarrow X_i[1]$$

is exactly the triangle in (5.1) defining the right mutation. So $R_{X_{i+1}} X_i[1] = P^{\bullet}_{(R_{\gamma_{i+1}} \gamma_i, g)}$. Note that by [16, Lemma 2.3], we have $g(s) = f_{\gamma_i}(t) - 1$. Set $f_{R_{\gamma_{i+1}} \gamma_i} = g[-1]$, then $R_{X_{i+1}} X_i = P^{\bullet}_{(R_{\gamma_{i+1}} \gamma_i, f_{R_{\gamma_{i+1}} \gamma_i})}$ with $f_{R_{\gamma_{i+1}} \gamma_i}(s) = f_{\gamma_i}(t)$.

For case II, there are unique $m, n \in \mathbb{Z}$ such that we have homomorphisms

$$a_1 : P^\bullet_{(\gamma_i, f_{\gamma_i})} \longrightarrow P^\bullet_{(\gamma_{i+1}, f_{\gamma_{i+1}})}[m],$$
$$a_2 : P^\bullet_{(\gamma_i, f_{\gamma_i})} \longrightarrow P^\bullet_{(\gamma_{i+1}, f_{\gamma_{i+1}})}[n]$$

arising from the intersections at q_1 and q_2 respectively. Then by [16, Proposition 2.5], there is a grading g over $R_{\gamma_{i+1}}\gamma_i$ such that

$$X_i \xrightarrow{(a_1, a_2)} X_{i+1}[m] \oplus X_{i+1}[n] \longrightarrow P^\bullet_{(R_{\gamma_{i+1}}\gamma_i, g)} \longrightarrow X_i[1]$$

is a distinguished triangle. On the other hand, note that a_1 and a_2 form a basis of the homomorphism space from $P^\bullet_{(\gamma_i, f_{\gamma_i})}$ to $P^\bullet_{(\gamma_{i+1}, f_{\gamma_{i+1}})}[\mathbb{Z}]$ since there are no other intersections of γ_i and γ_{i+1}. So the above triangle is again exactly the exchange triangle in (5.1) defining the right mutation. Therefore $R_{X_{i+1}} X_i[1] = P^\bullet_{(R_{\gamma_{i+1}}\gamma_i, g)}$ with $g(s) = f_{\gamma_i}(t) - 1$. Then $R_{X_{i+1}} X_i = P^\bullet_{(R_{\gamma_{i+1}}\gamma_i, f_{R_{\gamma_{i+1}}\gamma_i})}$ if we set $f_{R_{\gamma_{i+1}}\gamma_i} = g[-1]$, where $f_{R_{\gamma_{i+1}}\gamma_i}(s) = f_{\gamma_i}(t)$.

To sum up, for both cases, we have $R_{X_{i+1}} X_i = P^\bullet_{(R_{\gamma_{i+1}}\gamma_i, f_{R_{\gamma_{i+1}}\gamma_i})}$ with $f_{R_{\gamma_{i+1}}\gamma_i} = g[-1]$, and thus

$$\sigma_i \mathbf{X} = (X_1, \ldots, X_{i-1}, X_{i+1}, P^\bullet_{(R_{\gamma_{i+1}}\gamma_i, f_{R_{\gamma_{i+1}}\gamma_i})}, X_{i+2}, \ldots, X_n).$$

$\square$

The following proposition can be directly derived from above theorem and Proposition 5.10.

Proposition 5.14 *If the* **RCEA** *condition is satisfied for any marked surface, then for any gentle algebra* A *such that there are full exceptional sequences in* $K^b(A)$, *the action of* $\mathbb{Z}^n \ltimes B_n$ *on the set of full exceptional sequences in* $K^b(A)$ *is transitive.*

Then Theorem 5.1 follows from above proposition and Theorem 5.11. We have the following straightforward corollary from Theorem 5.1, noticing that for those derived-discrete algebra A which are hereditary, [15] has proven the transitivity of the action of $\mathbb{Z}^n \ltimes B_n$ on the set of ordered exceptional dissections in $K^b(A)$ and that the remaining derived-discrete algebras are gentle algebras associated to surfaces of genus zero.

Corollary 5.15 *Let* A *be a derived-discrete algebra. The action of* $\mathbb{Z}^n \ltimes B_n$ *on the set of ordered exceptional dissections in* $K^b(A)$ *is transitive.*

6 Three dualities: exceptional, Koszul and Serre dualities

In this section we study the relation between the duality of a full exceptional sequence introduced in [26, Section 8] and Koszul duality for gentle algebras. We further show that the duality of exceptional sequences can also be explained using the Auslander-Reiten translation in $K^b(A)$, thereby giving a new proof of this result by Bondal [10] in the case of the perfect derived categories of gentle algebras.

For the rest of this section, we fix the following notation. Denote by A a gentle algebra associated to a marked surface (S, M, Δ_A) such that there exists a full exceptional sequence in $\mathsf{K}^b(A)$ and let $\mathbf{X} = (X_1, \ldots, X_n)$ be such a full exceptional sequence associated to an ordered exceptional dissection $\Gamma = (\gamma_1, \ldots, \gamma_n)$ on (S, M). We define the ordered dual exceptional dissection of Γ to be the dual dissection Γ^* with the following order $\Gamma^* = (\gamma_n^*, \ldots, \gamma_1^*)$. Denote by $\mathsf{A}(\Gamma)$ and $\mathsf{A}(\Gamma^*)$ the gentle algebras arising from Γ and Γ^* respectively. Then $\mathsf{A}(\Gamma^*)$ is the Koszul dual of $\mathsf{A}(\Gamma)$, see Remark 1.8. Furthermore, Γ^* is exceptional, this follows directly from the fact that the quiver of $\mathsf{A}(\Gamma^*)$ is the opposite quiver of $\mathsf{A}(\Gamma)$.

For any $i \leq j \leq n$, set

$$R^{j-i} X_i = R_{X_j} \cdots R_{X_{i+1}} X_i,$$

and for any $1 \leq k \leq i$, set

$$L^{i-k} X_i = L_{X_k} \cdots L_{X_{i-1}} X_i.$$

In particular, we have $R^{j-i} X_i = X_i$ and $L^{i-k} X_i = X_i$ if $j = i$ and $k = i$ respectively. We will denote by $R^{j-i} \gamma_i$ the arc associated to $R^{j-i} X_i$, and denote by $L^{i-k} \gamma_i$ the arc associated to $L^{i-k} X_i$.

Definition 6.1 [26] Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a full exceptional sequence in $\mathsf{K}^b(A)$. We call $\mathbf{RX} = (X_n, R^1 X_{n-1} \ldots, R^{n-1} X_1)$ the *right dual* of $\mathbf{X}$ and $\mathbf{LX} = (L^{n-1} X_n, L^{n-2} X_{n-1} \ldots, X_1)$ *left dual* of $\mathbf{X}$.

The following lemma directly follows from the definitions.

Lemma 6.2 *Let* $\omega_0 = \sigma_1(\sigma_2\sigma_1) \cdots (\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1)$ *be the element in* B_n *corresponding to the longest element in the associated symmetric group* S_n. *Then* $\mathbf{RX} = \omega_0 \mathbf{X}$ *and* $\mathbf{LX} = \omega_0^{-1} \mathbf{X}$. *So we have* $\mathbf{R} = \mathbf{L}^{-1}$ *on the set of full exceptional sequences in* $\mathsf{K}^b(A)$.

We now give an interpretation of the left and right dual of a full exceptional sequence in terms the geometric surface model of A. For this, note that in an exceptional dissection we cannot have polygons such that if the polygon is unfolded one of its edges appears twice since such an arc must be contained in a non-exceptional cycle. An example of such a non-exceptional cycle is given in Fig. 15 where the arc γ_1 would appear twice in the unfolded polygon. Therefore, in particular, for any $1 \leq i \leq n$, γ_i belongs to two polygons of Γ, as in Fig. 21, where k and m are allowed to be zero, which means that the •-marked points q and q' in the polygons lie in the same boundary component, otherwise we have $1 \leq j \leq k \leq n$, and $1 \leq l \leq m \leq n$. Denote the sets of arcs in Fig. 21 by $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ and $\Sigma' = \{\alpha'_1, \ldots, \alpha'_m\}$.

Let s be the intersection $\gamma_i \cap \gamma_i^*$, noting that following our conventions, this is the unique intersection of γ_i and γ_i^*. Recall that $D\gamma_i^*$ and $D^{-1}\gamma_i^*$ are the direct and inverse twists of γ_i^* to the next ∘-point, respectively. We also have the following intersections, which are in fact the unique intersections between the two arcs in consideration,

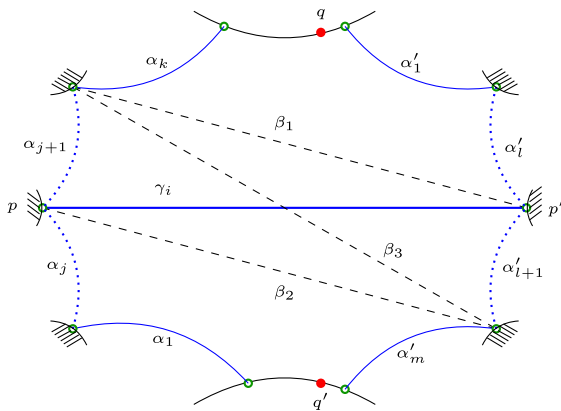$$u = \gamma_i \cap D\gamma_i^*, \ u' = \gamma_i \cap D^{-1}\gamma_i^*, \ u'' = D\gamma_i^* \cap D^{-1}\gamma_i^*,$$

Fig. 21 An ○-arc γ_i of an exceptional dissection belongs to two polygons, where two marked points and two ○-arcs may coincide
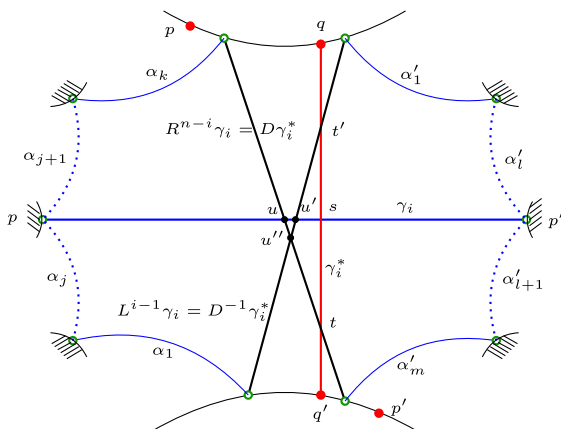

Fig. 22 $R^{n-i}\gamma_i = D\gamma_i^*$ and $L^{i-1}\gamma_i = D^{-1}\gamma_i^*$ for an ○-arc γ_i in an exceptional dissection

where u'' always is in the interior of S, while u and u' may be on the boundary, see Fig. 22.

The following proposition gives an explicit description of the object $R^{n-i}X_i$ in terms of the twist of the dual γ_i^* of the arc γ_i corresponding to X_i.

Proposition 6.3 *With the notation above, we have $R^{n-i}X_i = P^\bullet_{(D\gamma_i^*,f)}$, where f is the grading of $D\gamma_i^*$ such that there is a map from $P^\bullet_{(D\gamma_i^*,f)}$ to X_i arising from the intersection u of $D\gamma_i^*$ and γ_i.*

Proof At first we show that $R^{n-i}\gamma_i = D\gamma_i^*$. Following our conventions, see Fig. 21, we have

$$\alpha_1 \preceq \cdots \preceq \alpha_j \preceq \gamma_i \preceq \alpha_{j+1} \preceq \cdots \preceq \alpha_k,$$
$$\alpha_1' \preceq \cdots \preceq \alpha_l' \preceq \gamma_i \preceq \alpha_{l+1}' \preceq \cdots \preceq \alpha_m'.$$

Then either $\gamma_{i+1} = \alpha_{j+1}$ or α'_{l+1} if $\gamma_{i+1} \in \Sigma \sqcup \Sigma'$, or $\gamma_{i+1} \in \Gamma \backslash (\Sigma \sqcup \Sigma')$. In either case, $\gamma_{i+1} \in \{\alpha_{j+1}, \alpha'_{l+1}\} \sqcup (\Gamma \backslash (\Sigma \sqcup \Sigma'))$.

Case 1. If $\gamma_{i+1} = \alpha_{j+1} \neq \alpha'_{l+1}$, then by Theorem 5.13 (2), $R\gamma_i = \beta_1$ as shown in Fig. 21 (compare also to the left picture in Fig. 19).

Case 2. If $\gamma_{i+1} = \alpha'_{l+1} \neq \alpha_{j+1}$, then similar to the first case, $R\gamma_i = \beta_2$ as shown in Fig. 21.

Case 3. If $\gamma_{i+1} = \alpha_{j+1} = \alpha'_{l+1}$, then by Theorem 5.13 (2), $R\gamma_i = \beta_3$ as in Fig. 21 (compare to the right picture in Fig. 19).

Case 4. If $\gamma_{i+1} \in \Gamma \setminus (\Sigma \sqcup \Sigma')$ then we show that $\mathrm{Hom}(X_i, X_{i+1}) = 0$ and by Theorem 5.13 (1), $R\gamma_i = \gamma_i$. Suppose for contradiction that $\mathrm{Hom}(X_i, X_{i+1}) \neq 0$. Then $\gamma_i \cap \gamma_{i+1} \in \{p, p'\}$. If $p \in \gamma_i \cap \gamma_{i+1}$, then γ_{i+1} follows γ_i anticlockwise at p. On the other hand, α_{j+1} directly follows γ_i at p in the anticlockwise direction, thus $\alpha_{j+1} \preceq \gamma_{i+1}$. This is a contradiction since $\gamma_{i+1} \preceq \alpha_{j+1}$ (γ_{i+1} is a minimal element which is larger than γ_i) and $\gamma_{i+1} \neq \alpha_{j+1}$. The argument for the case $p' \in \gamma_i \cap \gamma_{i+1}$ is similar.

To sum up, for all the cases, $R\gamma_i = \gamma_i$ or $R\gamma_i$ is obtained by rotating γ_i clockwise at one endpoint or at both endpoints. Now let

$$\sigma_i \mathbf{X} = \{X_1, \ldots, X_{i-1}, X_{i+1}, RX_i, X_{i+2}, \ldots, X_n\}$$

be the new full exceptional sequence. Using a similar discussion for $R\gamma_i$ as for γ_i above, we know that $R^2\gamma_i = R\gamma_i$ or $R^2\gamma_i$ is obtained by rotating $R\gamma_i$ clockwise at one endpoint or at both endpoints. Continue this process until we obtain a full exceptional dissection

$$\sigma_{i+r-1} \cdots \sigma_i(\mathbf{X}) = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{i+r}, R^r X_i, X_{i+r+1}, \ldots, X_n)$$

such that we can not continue the process any more, that is, q is the $\bullet$-point which directly follows (anticlockwise) an endpoint of $R^r\gamma_i$ and q' is the $\bullet$-point which directly follows (anticlockwise) the other endpoint of $R^r\gamma_i$. In this case, there is no intersection between γ_v and $R^r\gamma_i$, for any $i+r+1 \leq v \leq n$. Otherwise, they intersect at the endpoints of $R^r\gamma_i$ which implies $\gamma_v \preceq R^r\gamma_i$, see Fig. 22. But this contradicts the fact that $\sigma_{i+r-1} \cdots \sigma_i(\mathbf{X})$ is an exceptional sequence. Therefore $\mathrm{Hom}(R^r X_i, X_v) = 0$ for any $i+r+1 \leq v \leq n$ and by inductively using Theorem 5.13 (1), we obtain $R^r X_i = R^{n-i} X_i$ and thus $R^r\gamma_i = R^{n-i}\gamma_i$. On the other hand, since $R^r\gamma_i = D\gamma_i^*$, so $R^{n-i}\gamma_i = D\gamma_i^*$.

Now we consider the grading f. By the definition of $\sigma_j, i \leq j \leq i+r-1$, we have the following exchange triangle

$$R^{j-i} X_i \longrightarrow \coprod_{\ell \in \mathbb{Z}} D\mathrm{Hom}(R^{j-i} X_i, R^{j-i+1} X_i[\ell]) \otimes_k R^{j-i+1} X_i[\ell] \longrightarrow$$

$$R^{j-i+1} X_i[1] \xrightarrow{a_j} R^{j-i} X_i[1],$$

where the last map a_j is nonzero. Furthermore, the composition $a_i \cdots a_n$ is again a nonzero map from $R^{n-i} X_i$ to X_i, which corresponds to the intersection u of $R^{n-i}\gamma_i =$

$D\gamma^*$ and γ_i. This implies that f is the grading such that there is a non-zero map from $R^{n-i}X_i = P^\bullet_{(R^{n-i}\gamma_i, f)}$ to X_i.   □

The following proposition gives an explicit description of the object $L^{i-1}X_i$. The proof is similar to the proof of Proposition 6.3.

Proposition 6.4 *With the notation above we have* $L^{i-1}X_i = P^\bullet_{(L^{i-1}\gamma_i, g)}$, *where* $L^{i-1}\gamma_i = D^{-1}(\gamma_i^*)$ *and g is the grading such that there is a map from* X_i *to* $L^{i-1}X_i$ *arising from the intersection* u'.

The following theorem shows that the right (resp. left) duality of an exceptional sequence corresponds to Koszul duality followed by twisting (resp. inversely twisting) the arcs in the corresponding dual exceptional dissection.

Theorem 6.5 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a full exceptional sequence in* $K^b(A)$ *associated to an ordered exceptional dissection* $\Gamma = (\gamma_1, \ldots, \gamma_n)$ *of* (S, M), *that is* $X_i = P^\bullet_{(\gamma_i, f_{\gamma_i})}$ *for some grading* f_{γ_i}. *Then the right dual* $\mathbf{RX}$ *is induced by the equality*

$$\mathbf{R}\Gamma = D(\Gamma^*)$$

where $D(\Gamma^*)$ *is the twist of the ordered dual exceptional dissection* Γ^*.
The left dual $\mathbf{LX}$ *is induced by the equality*

$$\mathbf{L}\Gamma = D^{-1}(\Gamma^*)$$

where $D^{-1}(\Gamma^*)$ *is the inverse twist of the ordered dual exceptional dissection* Γ^*.

Proof The proof directly follows from Lemma 6.2 and Proposition 6.3.   □

Note that in Theorem 6.5 when we say that $\mathbf{RX}$ is induced by the equality $\mathbf{R}\Gamma = D(\Gamma^*)$ we mean that the grading of $\mathbf{RX}$ is induced by the gradings as described in Proposition 6.3 and similarly for $\mathbf{LX}$ the grading is the one induced by the gradings in Proposition 6.4.

It is shown in [12] that if a triangulated category has full exceptional sequences, then it has a Serre functor, and in [10, Proposition 4.2] that the duality of the exceptional sequences can be obtained using the Serre functor. In the following we give a new proof of this result for $K^b(A)$.

Theorem 6.6 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a full exceptional sequence in* $K^b(A)$ *and let* $\mathbb{S}$ *be the Serre functor in* $K^b(A)$. *Then* $\tau(\mathbf{X}) = \mathbf{L}^2[-1](\mathbf{X})$ *and thus* $\mathbb{S}(\mathbf{X}) = \mathbf{L}^2(\mathbf{X})$.

Proof We only need to show $\tau(\mathbf{X}) = \mathbf{L}^2[-1](\mathbf{X})$, since $\tau = \mathbb{S}[-1]$. Let $X_i = P^\bullet_{(\gamma_i, f_i)}$ so that $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is the ordered exceptional dissection corresponding to $\mathbf{X}$ and set $\mathbf{L}^2(\mathbf{X}) = (Y_1, Y_2, \ldots, Y_n)$. Then by Theorem 6.5, $\mathbf{L}\Gamma = D^{-1}(\Gamma^*)$. Furthermore, it is easy to see that the twist operator D (as well as D^{-1}) and the dual operator $*$ commute, that is, $D^{-1}(\Gamma^*) = (D^{-1}(\Gamma))^*$. So we have $\mathbf{L}^2(\Gamma) = D^{-2}(\Gamma)$. On the other hand, we know from Lemma 1.9 that the arc of τX_i is exactly $D^{-2}\gamma_i$. Thus τX_i and Y_i correspond to the same arc $D^{-2}\gamma_i$.

Now we consider the grading. On the one hand, there is a non-zero map F from $X_i[-1]$ to τX_i arising from the Auslander-Reiten triangle for X_i. On the other hand, by Proposition 6.4, there is a non-zero map G from X_i to Y_i. Note that both maps F and G arise from the unique intersection between $D^{-2}\gamma_i$ and γ_i. Thus we have $\tau X_i = Y_i[-1]$ and $\tau(\mathbf{X}) = \mathbf{L}^2[-1](\mathbf{X})$.   □

The following corollary directly follows from Theorem 6.6.

Corollary 6.7 *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a full exceptional sequence in $K^b(A)$. Then $L^{n-1}X_n = \mathbb{S}X_n = \tau X_n[1]$ and $R^{n-1}X_1 = \mathbb{S}^{-1}X_1 = \tau^{-1}X_1[-1]$.*

References

1. Amiot, C., Plamondon, P.-G., Schroll, S.: A complete derived invariant for gentle algebras via winding numbers and Arf invariants. Sel. Math. New Ser. **29**, 30 (2023)
2. Assem, I., Happel, D.: Generalized tilted algebras of type A_n. Commun. Algebra **9**(20), 2101–2125 (1981)
3. Assem, I., Skowroński, A.: Iterated tilted algebras of type affine A_n. Math. Z. **195**(2), 269–290 (1987)
4. Aihara, T.T., Iyama, O.: Silting mutation in triangulated categories. J. Lond. Math. Soc. (2) **85**(3), 633–668 (2012)
5. Arnesen, K.-K., Laking, R., Pauksztello, D.: Morphisms between indecomposable complexes in the bounded derived category of a gentle algebra. J. Algebra **467**, 1–46 (2016)
6. Bekkert, V., Merklen, H.A.: Indecomposables in derived categories of gentle algebras. Algebr. Represent. Theory **6**(3), 285–302 (2003)
7. Bocklandt, R.: Noncommutative mirror symmetry for punctured surfaces. With an appendix by Mohammed Abouzaid. Trans. Am. Math. Soc. **368**(1), 429–469 (2016)
8. Bondal, A.: Representations of associative algebras and coherent sheaves. Math. USSR Izv. **34**, 23–42 (1990)
9. Baur, K., Coelho Simões, R.: A geometric model for the module category of a gentle algebra. Int. Math. Res. Not. **15**, 11357–11392 (2021)
10. Bondal, A.: Helices, representations of quivers and Koszul algebras. Helices and vector bundles. London Math. Soc. Lecture Note Ser., vol. 148. Cambridge Univ. Press, Cambridge, pp. 75–95 (1990)
11. Bondal, A., Polishchuk, A.: Homological properties of associated algebras: The method of helices. Russ. Acad. Sci. Izv. Math. **42**, 219–260 (1994)
12. Bondal, A., Kapranov, M.: Representable functors, Serre functors, and reconstructions. Math. USSR-Izv. **35**(3), 519–541 (1990)
13. Broomhead, N., Pauksztello, D., Ploog, D.: Derived-discrete categories I: homomorphisms, autoequivalences and t-structures. Math. Z. **285**(1–2), 39–89 (2017)
14. Burban, I., Drozd, Y.: Non-commutative nodal curves and derived tame algebras. Preprint arXiv:1805.05174 (2018)

15. Crawley-Boevey, W.: Exceptional sequences of representations of quivers. Representations of algebras (Ottawa, ON, 1992), 117-124, CMS Conf. Proc., 14, Amer. Math. Soc., Providence, RI (1993)
16. Chang, W., Schroll, S.: A geometric realization of silting theory for gentle algebras. Math. Z. **303**(3), 67 (2023)
17. Drezet, J.M., Le Potier, J.: Fibrés stables et fibrés exceptionnels sur $\mathbb{P}^2$. Ann. Sci. Ec. Norm. Sup. (4) **18**, 193–243 (1985)
18. Farb, B., Margalit, D.: A Primer on Mapping Class Groups (PMS-49). Princeton University Press, Princeton (2012)
19. Haiden, F., Katzarkov, L., Kontsevich, M.: Flat surfaces and stability structures. Publ. Math. Inst. Hautes Études Sci. **126**, 247–318 (2017)
20. Gorodentsev, A.L., Rudakov, A.N.: Exceptional vector bundles on projective spaces. Duke Math. J. **54**(1), 115–130 (1987)
21. Lekili, Y., Polishchuk, A.: Auslander orders over nodal stacky curves and partially wrapped Fukaya categories. J. Topol. **11**(3), 615–644 (2018)
22. Lekili, Y., Polishchuk, A.: Derived equivalences of gentle algebras via Fukaya categories. Math. Ann. **376**(1–2), 187–225 (2020)
23. Meltzer, H.: Exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines. Mem. Am. Math. Soc. **171**(808), viii+139 pp (2004)
24. Opper, S., Plamondon, P.-G., Schroll, S.: A geometric model for the derived category of gentle algebras. arXiv:1801.09659 (2018)
25. Palu, Y., Pilaud, V., Plamondon, P.-G.: Non-kissing and non-crossing complexes for locally gentle algebras. J. Comb. Algebra **3**(4), 401–438 (2019)
26. Rudakov, Helices and vector bundles. Seminaire Rudakov. Translated from the Russian by A. D. King, P. Kobak and A. Maciocia. London Mathematical Society Lecture Note Series, 148. Cambridge University Press, Cambridge, 1990. iv+143 pp
27. Schroll, S.: Trivial extensions of gentle algebras and Brauer graph algebras. J. Algebra **444**, 183–200 (2015)
28. Vossieck, D.: The algebras with derived-discrete category. J. Algebra **243**(1), 168–176 (2001)