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Decay and symmetry of solitary waves

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ABSTRACT

In this paper we consider the decay rate of solitary-wave solutions to some classes of non-linear and non-local dispersive equations, including for example the Whitham equation and a Whitham–Boussinesq system. The dispersive term is represented by a Fourier multiplier operator that has a real analytic symbol that either decays/grows, and we show that all supercritical/subcritical solitary-wave solutions decay exponentially, and moreover provide the exact decay rate, which in general will depend on the speed of the wave. We also prove that solitary waves have only one crest and are symmetric for some class of equations.

1. Introduction

This paper is devoted to the study of decaying solutions $u\colon \mathbb{R} \to \mathbb{R}$ to equations of the form

$$cu - L(u) - G_c(u) = 0, \tag{1.1}$$

where $c > 0$ is a parameter, L is a Fourier multiplier operator with symbol $m\colon \mathbb{R} \to \mathbb{R}$, meaning that

$$\widehat{L\varphi}(\xi) = m(\xi)\widehat{\varphi}(\xi),$$

and G_c is some non-linear function that may depend on the parameter c (see examples and assumptions below). In general we will suppress the potential dependency of G_c on c and simply write G. By decaying solutions, we mean that

$$\lim_{|x|\to\infty} u(x) = 0.$$

The goal of this paper is to determine the rate of decay of the solutions, under some assumptions on m and G.

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Equations of the form (1.1) are of interest as solitary-wave solutions to a wide variety of model equations for water waves can be represented as decaying solutions to (1.1). Consider for instance the non-linear dispersive equation

$$u_t + L(u)_x + G(u)_x = 0. \tag{1.2}$$

This is a typical form of model equations for the water-wave problem, and many of the most prominent models can be cast in this form. For instance, if $G(u) = u^2$ and $m(\xi) = 1 - \xi^2$ we get the Korteweg-de Vries equation, and if $m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}$ we get the Whitham equation [14], to mention a few. Assuming that u is a solitary-wave solution to (1.2) moving to the right with speed c, that is, $u(x,t) = u(x-ct)$ and $\lim_{|x-ct|\to\infty} u(x-ct) = 0$, we can integrate (1.2) to get (1.1).

Another example is solitary-wave solutions to the Whitham-Boussinesq type system:

$$\begin{aligned} \eta_t &= -L(u)_x - (\eta u)_x \\ u_t &= -\eta_x - u u_x, \end{aligned} \tag{1.3}$$

where η is the surface elevation and u is the velocity at the surface in the rightwards direction. A solitary-wave solution to (1.3) with speed $c > 0$ is a solution of the form $\eta(x,t) = \eta(x-ct)$, $u(x,t) = u(x-ct)$ such that $u(\zeta), \eta(\zeta) \to 0$ as $|\zeta| \to \infty$. Under this ansatz, one finds that (see [13]) $\eta = u(c - \frac{u}{2})$ and

$$L(u) - u(u-c)\left(\frac{u}{2} - c\right) = 0. \tag{1.4}$$

This can be written in the form (1.1) with $G(u) = \frac{u^2}{2}(3c - u)$ and c replaced by c^2 in the first term.

We will not concern ourselves with existence theory in this paper, but simply establish the decay properties of solutions, should they exist. For results on existence of solitary-wave solutions to equations of the form (1.2), see for instance [7] for weak dispersion (i.e. when L is a smoothing operator), and [2] for when L is a differentiating operator.

If $m(\xi) \neq c$ for all $\xi \in \mathbb{R}$, we can formally write (1.1) as

$$u = K_c * G(u), \ K_c = \mathscr{F}^{-1}\left(\frac{1}{c-m}\right). \tag{1.5}$$

However, if m decays (that is, L is a smoothing operator), then $(c-m)^{-1}$ tends to $c^{-1} > 0$ at infinity, and K_c exists only in a distributional sense. To remedy this, we can apply the operator L to both sides of (1.5), and from (1.1) we get

$$u\left(c - \frac{G(u)}{u}\right) = H_c * G(u), \tag{1.6}$$

where

$$H_c = \mathscr{F}^{-1}\left(\frac{m}{c-m}\right). \tag{1.7}$$

We will work under the assumption that L is a smoothing operator (see assumption (A1) below), but the results can be applied to differentiating operators as well, if the inverse (which will be a smoothing operator) satisfies our assumptions; see section 4.3. In fact, this case is even simpler.

The idea of formulating the equation as a convolution equation in order to study decay is taken from the classical paper [4], where the authors study the decay of solutions to equations of the form

$$u = K * G(u),$$

under a mild assumption on G (see assumption (A3) below) and for $\widehat{K} \in H^s(\mathbb{R})$ for some $s \geq 0$. Philosophically the idea is natural: the decay rate of K should decide the decay rate of u, and if one can prove that, the problem is reduced to investigating the kernel K. In [4] they show, under some integrability assumptions on K, that a solution u that tends to 0 at infinity decays at least as fast as K, and we will show that it will not decay faster. From Fourier analysis it is known that a requirement for K to be exponentially decaying is that $\mathscr{F}(K)$ is analytical in a strip in the complex plane. Hence one would expect that if the symbol m of L is not smooth, solitary waves will decay only algebraically. This has been observed for instance for the Benjamin-Ono equation [3], for which there is only one solitary wave and that one decays algebraically [1], and also for generalized KP equations [6], both of which have Fourier symbols of finite smoothness. A more general result about the relation between finite smoothness and algebraic decay can be found in [5]. We will assume smoothness of the symbol m in this paper. To be precise, we will study (1.6) under the following assumptions:

Assumptions.

(A1) There is an $m_0 < 0$ such that

$$|m^{(n)}(\xi)| \leq C_n(1 + |\xi|)^{m_0 - n}, \ n \in \mathbb{N}_0.$$

(A1*) The function m is real analytic and the constants $C_n \geq 0$ in (A1) can be chosen such that $\lim_{n\to\infty} \frac{C_{n+1}/(n+1)!}{C_n/n!} = k$ for some $k \geq 0$.

(A2) The function m is even and the parameter c satisfies

$$\max_{\xi\in\mathbb{R}} m(\xi) < c.$$

(A3) $G\colon \mathbb{R} \to \mathbb{R}$ is bounded on compact sets, and for all small values of u, we have that $|G(u)| \lesssim |u|^r$ for some $r > 1$.

Remark 1.1. The assumptions (A1) and (A2) imply that H_c decays algebraically of arbitrary order (cf. Section 3.1), while assumption (A1*) is needed for exponential decay. Indeed, if m is real analytic then it admits a local extension to a complex analytic function around every point in $\mathbb{R}$, and the condition on the constants C_n in (A1*) implies that there is a uniform lower bound on the radius where the local extension is valid. This gives that m can be extended to a strip in the complex plane and Paley-Wiener theory can then be used to show exponential decay - see Section 3.2.

Under these assumptions we have the following result on decay:

Theorem 1.2. *Let (A1), (A2) and (A3) be satisfied and suppose that* $u \in L^\infty(\mathbb{R})$ *with* $\lim_{|x|\to\infty} u(x) = 0$ *is a non-trivial solution to* (1.6)*. Then the following holds:*

(i) $|\cdot|^l u(\cdot) \in L^\infty(\mathbb{R})$ *for any* $l \geq 0$.

(ii) *If* m *satisfies (A1*) in addition, then* u *has the exact same rate of asymptotic decay as* H_c*. In particular, there is a number* $\delta_c > 0$ *depending on* m *and* c*, such that for all* $\delta \in (0, \delta_c)$*,*

$$\mathrm{e}^{\delta|\cdot|}u(\cdot) \in L^1 \cap L^\infty(\mathbb{R}).$$

Moreover,

$$\mathrm{e}^{\delta_c|\cdot|}|u(\cdot)|$$

does not decay to 0*, but it can be bounded, depending on* m *and* c *(see Lemma 3.4 for more details).*

It is also worth noting that while our inspiration comes from equations and systems for which solitary-wave solutions are solutions to equations of the form (1.1) and we therefore work with (1.6), the results apply to more general equations. Indeed, it is straightforward to extend the arguments to equations which can be cast in the form

$$uF(u) = H_c * G(u),$$

as long as $F\colon \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{x\to 0} F(x) \neq 0$ and is such that Lemma 4.2 holds.

Under an assumption on H_c that is independent of (A1) and (A2), and some assumptions on the behaviour of G on the range of the solution, we have that decaying solutions to (1.6) are symmetric:

Theorem 1.3. *Assume that* $H_c \in L^1(\mathbb{R})$ *is non-negative, symmetric and monotonically decreasing on* $(0, \infty)$*, and that* G *satisfies (A3). Let* $u \in BC(\mathbb{R})$ *with* $\lim_{|x|\to\infty} u(x) = 0$ *be a solution to* (1.6) *and assume that* G *is non-negative and increasing on the range of* u*, and for all* $0 \le y < x < \max(u)$*,*

$$|G(x) - G(y)| < C(x)|x - y|,$$

where $C(x) < \tilde{c}$ *for some* $0 < \tilde{c} < c$ *on the range of* u*, and* $\lim_{x\to 0^+} C(x) = 0$*. Then* u *is symmetric about some point* $\lambda_0 \in \mathbb{R}$ *and has exactly one crest, located at* λ_0*.*

Remark 1.4. Some remarks on the assumptions:

- Note that we are requiring G to be non-negative, increasing and Lipschitz continuous with Lipschitz constant $\tilde{c} < c$ only on the range of u, so these are implicitly assumptions on the solution u itself.
- If $\frac{|G(x)-G(y)|}{|x-y|} \le \left|\frac{G(x)}{x} + \frac{G(y)}{y}\right|$, then it is not necessary to assume that $|G(x) - G(y)| \le \tilde{c}|x - y|$, as it follows from Lemma 5.1. This is the case if, for example, $G(u) = |u|^r$ for $1 < r \le 2$.
- If $f(\xi) = g(\xi^2)$ where $\lim_{x\to 0^+} g(x) < \infty$ and $\lim_{x\to\infty} g(x) = 0$ and g is completely monotone, then $\mathscr{F}^{-1}(f)$ is smooth outside the origin and monotone (Proposition 2.18 in [8]). As one can verify, if $m(\sqrt{\cdot})$ is completely monotone on $(0, \infty)$, then so is $\frac{m(\sqrt{\cdot})}{c-m(\sqrt{\cdot})}$. It follows that $m(0) > 0$ and $m(\sqrt{\cdot})$ completely monotone on $(0, \infty)$ is sufficient for H_c to be symmetric and monotone on $(0, \infty)$.

The paper is organized as follows. Section 3 is devoted to establishing integrability properties and the decay rate of H_c under assumptions (A1), (A2) (and (A1*)). An exact description of the asymptotic behaviour, depending on c and m, is given. In section 4 we prove Theorem 1.2. Part (i) is more or less a straightforward adaption of the proof of algebraic decay of solitary waves for the Whitham equation in [9] (see also [4]) and we do only part of the proof to show that the arguments of the aforementioned paper can indeed be applied. The proof of part (ii) is also an adaption of the arguments in [4] and [9], but we are able to give the exact rate of exponential decay. Moreover, in subsection 4.3, the simpler case when L is a differentiating operator, rather than smoothing as implied by assumption (A1), is discussed. In section 5 symmetry is discussed and Theorem 1.3 is proved. Finally, in section 6, the general results from the preceding sections are applied to some specific examples, in particular to the Whitham equation, and the bi-directional Whitham equation, giving the exact rate of exponential decay of solitary-wave solutions to these equations. This is an improvement on the results of [9], where exponential decay of solitary waves of the Whitham equation is proved, but the exact decay rate is not established.

2. Notation

As indicated by the very definition of L and H_c, we will make much use of the Fourier transform, for which we will use the normalization

$$\mathscr{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x) \mathrm{e}^{-ix\xi} \, \mathrm{d}x.$$

The inverse Fourier transform of φ will be denoted by $\mathscr{F}^{-1}$ or $\check{\varphi}$ and is defined as

$$\check{\varphi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\xi) \mathrm{e}^{ix\xi} \, \mathrm{d}\xi.$$

With this normalization, the Fourier transform is a unitary operator on $L^2(\mathbb{R})$.

For $s \geq 0$, the Sobolev space $H^s(\mathbb{R})$ is the space of all $L^2(\mathbb{R})$ functions f which satisfy

$$\|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} < \infty.$$

The definition can be extended to $s < 0$ by considering tempered distributions, but that is not relevant here.

3. The kernel H_c

In this section we establish some essential properties of H_c, in particular its decay rate. We start by establishing integrability and algebraic decay; as one could expect assumption (A1*) is not necessary for these properties, only (A1) and (A2).

3.1. Algebraic decay

Lemma 3.1. *Assume (A1) and (A2) are satisfied. Then, for all $l \geq 1$, we have that*

$$(\cdot)^l H_c(\cdot) \in L^p(\mathbb{R}), \quad \text{for all} \quad 2 \leq p \leq \infty. \tag{3.1}$$

Proof. Assumptions (A1) and (A2) imply that

$$\widehat{H_c}^{(j)} \in L^p(\mathbb{R}), \quad \text{for all} \quad 1 \leq p \leq \infty, \; j \in \mathbb{N}_+.$$

As $\widehat{H_c}^{(j)} = \widehat{(-i\cdot)^j H_c}$ and $\mathscr{F}\colon L^p(\mathbb{R}) \to L^q(\mathbb{R})$ for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, the result follows. □

Lemma 3.2. *Assume (A1) and (A2) are satisfied. Then, for $|x| \ll 1$, we have that*

$$|H_c(x)| \simeq \begin{cases} |x|^{-1-m_0}, & -1 < m_0 < 0, \\ |\ln(|x|)|, & m_0 = -1, \\ 1, & m_0 < -1. \end{cases}$$

That is $H_c \in L^\infty(\mathbb{R})$ when $m_0 < -1$.

Proof. If $m_0 < -1$, then $m \in L^1(\mathbb{R})$ and by (A2) so is $\frac{m}{c-m}$ and the result is clear. Assume therefore that $-1 < m_0 < 0$. Let

$$g = \left(\frac{m}{c-m}\right)' = \frac{cm'}{(c-m)^2}.$$

As m is even, we have that g is odd and for $x > 0$ (it is sufficient to consider $x > 0$ as m, and therefore H_c, is even),

$$\begin{aligned} xH_c(x) =& \mathrm{i}\,\mathscr{F}^{-1}(g)(x) \\ =& -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi)\sin(x\xi)\,\mathrm{d}\xi \\ =& -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(\frac{s}{x}\right) \frac{\sin(s)}{x}\,\mathrm{d}s \\ =& -\frac{2c}{\sqrt{2\pi}} \int_0^\infty \frac{\sin(s)}{x} \frac{1}{(c-m(s/x))^2} m'\left(\frac{s}{x}\right)\,\mathrm{d}s. \end{aligned}$$

By assumption (A1), we have that $|m'\left(\frac{s}{x}\right)| \lesssim \left(\frac{s}{x}\right)^{m_0-1}$. Moreover, $(c-m)^{-2}$ is bounded by assumption (A2). Hence

$$x|H_c(x)| \lesssim x^{-m_0} \int_0^\infty \frac{|\sin(s)|}{s^{1-m_0}}\,\mathrm{d}s = Cx^{-m_0}.$$

Dividing by x on both sides gives the desired result. Now if $m_0 = -1$, we use the estimate $|m'\left(\frac{s}{x}\right)| \lesssim \left(\frac{s}{x}\right)^{-2}$ for $s \geq x$ and the estimate $|m'\left(\frac{s}{x}\right)| \lesssim \left(\frac{s}{x}\right)^{-1}$ for $0 < s < x$. Hence

$$\begin{aligned} x|H_c(x)| \lesssim& \int_0^x \frac{|\sin(s)|}{s}\,\mathrm{d}s + x\int_x^\infty \frac{|\sin(s)|}{s^2}\,\mathrm{d}s \\ \simeq& x + x|\ln(x)|, \end{aligned}$$

and the conclusion follows. $\square$

From Lemmas 3.1 and 3.2, we have the following Corollary:

Corollary 3.3. *Assume (A1) and (A2) are satisfied. Then* $x \mapsto |x|^\alpha H_c(x) \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, *if* $\alpha > \max\{1+m_0, 0\} - \frac{1}{p}$.

Proof. By (3.1), $|\cdot|^\alpha H_c(\cdot) \in L^p(\mathbb{R} \setminus (-1,1))$ for all $1 \leq p \leq \infty$ and all $\alpha \in \mathbb{R}$. It remains to consider the behaviour around the origin. If $-1 < m_0 < 0$, then by Lemma 3.2, we have that $|x|^\alpha |H_c(x)| \simeq |x|^{-1-m_0+\alpha}$ which is in $L^p_{loc}(\mathbb{R})$ exactly when $\alpha > 1 + m_0 - \frac{1}{p}$. For $m_0 \leq -1$, $|x|^\alpha |H_c(x)| \lesssim |x|^\alpha |\ln(|x|)|$ which is in $L^p_{loc}(\mathbb{R})$ if and only if $\alpha > -\frac{1}{p}$. $\square$

The results above state that H_c decays algebraically with arbitrary order, is a bounded function away from the origin, with the behaviour at the origin being given by Lemma 3.2. As H_c decays faster than any polynomial, the natural question to ask is whether it decays exponentially. This is indeed the case, if assumption (A1*) is satisfied in addition to (A1) and (A2).

3.2. Exponential decay

Now we turn the exponential decay of H_c, under the additional assumption (A1*).

Lemma 3.4. *Let (A1), (A2) and (A1*) be satisfied. Then m can be considered as a function in the complex plane. Let $\delta_c > 0$ be the smallest number for which there exists $z_0 \in \mathbb{C}$ with $\operatorname{Im} z_0 = \delta_c$ such that $m(z_0) = c$ or m has an essential singularity at z_0. The number δ_c is well-defined and the kernel H_c can be expressed as*

$$H_c(x) = \mathrm{e}^{-\delta_c |x|} \left(v + P(|x|) \right), \; x \in \mathbb{R},$$

where $v \in L^p(\{x \in \mathbb{R} : |x| \geq 1\})$, $1 \leq p \leq \infty$ satisfies Lemma 3.2 and P is bounded near the origin and grows slower than any exponential. If $m(z_0) = c$ for all the $z_0 \in \mathbb{C}$ defined as above, that is, there are no essential singularities among them, then $P = P_n$ is a polynomial of order n, where n is the highest order of the zeros of m' at the z_0's. In particular, if $m'(z_0) \neq 0$, then P_n is a non-zero constant.

Proof. As m is an analytic function on $\mathbb{R}$, we have at each point $x_0 \in \mathbb{R}$ a local extension to the complex plane given by

$$m(z) = \sum_{n=0}^{\infty} \frac{m^{(n)}(x_0)}{n!} (z - x_0)^n,$$

which is valid for all z within a ball around x_0 with non-zero radius depending on x_0. Let

$$\sigma := \inf_{x_0 \in \mathbb{R}} \sup\{r : \sum_{n=0}^{\infty} \frac{|m^{(n)}(x_0)|}{n!} r^n < \infty\}.$$

That is, σ is the infimum of the convergence radius over all points in $\mathbb{R}$. By (A1*), for any $x_0 \in \mathbb{R}$ the convergence radius of the series above is greater than or equal to the convergence radius of

$$\sum_{n=0}^{\infty} \frac{C_n (1 + |x_0|)^{m_0 - n}}{n!} z^n,$$

which converges for all $|z| < \frac{1}{k(1+|x_0|)^{m_0}}$ where k is as in (A1*). Hence there is a lower bound on the convergence radius that is independent of the point x_0, and $\sigma \geq \frac{1}{k} > 0$. Moreover, the convergence radius goes to infinity as $|x_0| \to \infty$. In general, given a closed form expression for m, it can be considered as a function in the complex plane, barring singularities.

We want to use Paley-Wiener theory to show exponential decay. While m is not guaranteed to be in L^2, assumption (A1) assures that m' is, and we have

$$\mathscr{F}(i \cdot H_c(\cdot)) = \left(\frac{m}{c - m} \right)' = \frac{c m'}{(c - m)^2} =: g.$$

Clearly g is meromorphic in the strip $|\operatorname{Im} z| < \sigma$ with the only potential poles where $m(z) = c$. If there are any such points then, as m decays along all lines in the strip, $\delta_c := \inf\{|\operatorname{Im} z| : z \in \mathbb{C}, \; m(z) = c\} \in (0, \sigma]$ is well-defined and achieved at finitely many points, proving the existence of δ_c as claimed. Assume this is not the case. There are three possibilities for the singularities of m along the edge of the strip. If m has a pole, then $m/(c - m)$ is bounded and g has a removable singularity and can be extended further. If m has a branch cut, this has no impact for our purposes. To see this, let $h(z)$ be real valued on the real line and even there. By symmetry, we can without loss of generality assume that h has a branch cut on the imaginary axis. As h is even on the real line and holomorphic away from the branch cut, we have

$$h(-\bar{z}) = h(\bar{z}) = \overline{h(z)}.$$

Hence h has equal real part but opposite imaginary part on the two branches. Moreover, as h is even $\operatorname{Re} h'(z) = 0$ along the imaginary axis and

$$h'(-\bar{z}) = -h'(\bar{z}) = -\overline{h'(z)}.$$

This implies that h' has the same value on each part of the branch, hence it is not a necessary branch cut for h'. In general, if there are two or more branch cuts that are mirrored over the imaginary axis, the integrals of g around them will cancel each other out, and we can without loss of generality assume that there are no branch cuts. The final possibility is that m has an essential singularity, in which case so does g. At some point in the process of extending g, we must either reach an essential singularity or a pole where $m(z) = c$.

Let $y > 0$ be fixed and $|x| > |y|k$. By (A1*),

$$\begin{aligned} |m'(x+iy)| &\le \sum_{n=0}^{\infty} \left| \frac{m^{(n+1)}(x)}{n!}(iy)^n \right| \\ &\le \sum_{n=0}^{\infty} \frac{C_{n+1}}{n!} |y|^n (1+|x|)^{m_0-n-1} \\ &= (1+|x|)^{m_0-1} \sum_{n=0}^{\infty} \frac{C_{n+1}}{n!} \left(\frac{|y|}{1+|x|} \right)^n \\ &\le K(1+|x|)^{m_0-1}, \end{aligned} \tag{3.2}$$

for some $K > 0$ (that will in fact decrease as $|x|$ increases). As $m_0 < 0$, this is in $L^p(\mathbb{R})$ for all $1 \le p \le \infty$. This also implies that for $x > 0$,

$$\lim_{|\xi| \to \infty} \sup_{0 \le \eta \le \delta_c} |g(\xi + i\eta) \mathrm{e}^{ixz}| = 0, \tag{3.3}$$

as e^{ixz} is bounded for $x > 0$ and $\operatorname{Im} z \ge 0$, where $z = \xi + i\eta$. If $x < 0$ and $\operatorname{Im} z \le 0$, then e^{ixz} is also bounded and

$$\lim_{|\xi| \to \infty} \sup_{-\delta_c \le \eta \le 0} |g(\xi + i\eta) \mathrm{e}^{ixz}| = 0.$$

We consider first the case $x > 0$; the case when $x < 0$ is similar. For any $\delta \in (0, \delta_c)$, $g(\cdot + \imath\delta) \in L^p(\mathbb{R})$ for all $1 \le p \le \infty$ by (3.2). We can apply Cauchy's theorem to the rectangle with corners $\pm R$ and $\pm R + \mathrm{i}\delta$. By (3.3), the integral over the vertical lines vanish as $R \to \infty$. As $g(z)\mathrm{e}^{ixz}$ has no singularities in this domain, Cauchy's theorem then gives (letting $R \to \infty$)

$$ixH_c(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) \mathrm{e}^{ix\xi} \,\mathrm{d}\xi = -\mathrm{e}^{-\delta x} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi + i\delta) \mathrm{e}^{ix\xi} \,\mathrm{d}\xi,$$

where the integral on the right-hand side is a function of x that is in $L^p(\mathbb{R})$ for all $1 \le p \le \infty$. For $x < 0$ the arguments are similar, and we find that for any $\delta \in (0, \delta_c)$, $\mathrm{e}^{\delta|\cdot|} H_c(\cdot) \in L^p(\mathbb{R} \setminus [-1, 1])$ for all $1 \le p \le \infty$. This implies that the growth of $\mathrm{e}^{\delta_c|\cdot|} H_c(\cdot)$ must be less than exponential.

If along the lines $\{z = x + \pm i\delta_c : x \in \mathbb{R}\}$ there are no essential singularities, only points where $m(z) = c$, then we can calculate the expression for H_c more explicitly. Note that if $m(z_0) = c \in \mathbb{R}$, the symmetry of m on the real line and the analyticity of m around z_0 implies that

$$m(-z_0) = m(z_0) = c = \overline{m(z_0)} = m(\overline{z_0}) = m(-\overline{z_0}).$$

That is, the poles of g are symmetric with respect to the real axis and the imaginary axis. As g is the derivative of a function, we immediately get that the residue of g at the poles $z_0 = \xi \pm i\delta_c$ such that $m(z_0) = c$ is zero: for any $0 < r < s$ such that $g(z)$ is analytic for $0 < |z - z_0| < s$,

$$\frac{1}{2\pi i} \int\limits_{|z-z_0|=r} g(z)\,\mathrm{d}z = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{c - m(z)}\,\mathrm{d}z = 0,$$

as the integral is taken over a closed circle. Let n be the order of the zero of m' at z_0. Then, locally around z_0,

$$g(z) = c\frac{(n+1)!(n+1)}{m^{(n+1)}(z_0)}(z - z_0)^{-n-2} + O((z - z_0)^{-n-1}).$$

That is, g and therefore also $g(z)\mathrm{e}^{ixz}$, $x \in \mathbb{R} \setminus \{0\}$, has a pole of order $n + 2$ at z_0. It follows that

$$\begin{aligned}
\text{Residue}\,[g(z)\mathrm{e}^{ixz}, z_0] =& \frac{1}{(n+1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} [g(z)\mathrm{e}^{ixz}(z - z_0)^{n+2}] \\
=& \sum_{k=0}^{n+1} \frac{1}{k!(n+1-k)!} \left[\lim_{z \to z_0} \frac{\mathrm{d}^{n+1-k}}{\mathrm{d}z^{n+1-k}} g(z)(z - z_0)^{n+2} \right] i^k x^k \mathrm{e}^{ixz_0} \\
=:& \mathrm{e}^{ixz_0} Q_{n+1}(x),
\end{aligned} \tag{3.4}$$

where $Q_{n+1}(x)$ is a polynomial of order $n + 1$. As the residue of g is zero, $Q_{n+1}(0) = 0$, and we can write Q_{n+1} as

$$Q_{n+1}(x) = \sum_{k=1}^{n+1} a_k x^k, \quad a_{n+1} = i^{n+1} c \frac{n+1}{m^{(n+1)}(z_0)} \neq 0. \tag{3.5}$$

In particular, Residue $[g(z)\mathrm{e}^{ixz}, z_0] \neq 0$ for $x \neq 0$. Furthermore, as $g(-\overline{z}) = -\overline{g(z)}$, we see that the residue of $g(z)\mathrm{e}^{ixz}$ at z_0 is purely real if $z_0 = \pm i\delta_c$. If $z_0 = \xi_0 \pm i\delta_c$, $\xi_0 \neq 0$, then imaginary parts of the residues at z_0 and $-\overline{z_0}$ cancel each other, while the real parts add up. For simplicity we therefore assume that we only have poles at $\pm i\delta_c$. From (3.4) we then get

$$\text{Residue}\,[g(z)\mathrm{e}^{ixz}, \pm i\delta_c] = \mathrm{e}^{\mp x\delta_c} Q_{n+1}(x) \tag{3.6}$$

for any $x \in \mathbb{R}$.

The function $g(\cdot + i\delta_c)$ is not in $L^2(\mathbb{R})$, but for every $\varepsilon > 0$, we have $g(\cdot + i\delta_c) \in L^2(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$ by (3.2). To apply Cauchy's theorem, we consider the indented rectangle defined by the line segments $[-R, R]$, $[\pm R, \pm R + i\delta_c]$, $[-R + i\delta_c, -\varepsilon + i\delta_c]$, $[\varepsilon + i\delta_c, R + i\delta_c]$, and the half-circle

$$\Gamma_\varepsilon = \{z = i\delta_c + \varepsilon\mathrm{e}^{i\theta} : \pi \leq \theta \leq 2\pi\}.$$

By (3.3), the integral over the vertical lines vanish as $R \to \infty$. As $g(z)\mathrm{e}^{ixz}$ has no singularities in this domain, Cauchy's theorem then gives (letting $R \to \infty$)

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \mathrm{e}^{-\delta_c x}\frac{1}{\sqrt{2\pi}}\int_{|\xi|\geq\varepsilon} g(\xi+i\delta_c)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi + \frac{1}{\sqrt{2\pi}}\int_{\Gamma_\varepsilon} g(z)\mathrm{e}^{ixz}\,\mathrm{d}z. \tag{3.7}$$

Recall that $(2\pi)^{-1/2}\int_{\mathbb{R}} g(\xi)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = -ixH_c(x)$. Multiplying by $\mathrm{e}^{\delta_c x}$ on both sides, we get

$$-\mathrm{e}^{x\delta_c}ixH_c(x) = \frac{1}{\sqrt{2\pi}}\int_{|\xi|\geq\varepsilon} g(\xi+i\delta_c)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi + \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\delta_c x}\int_{\Gamma_\varepsilon} g(z)\mathrm{e}^{ixz}\,\mathrm{d}z.$$

The first term on the right hand side is in $L^2(\mathbb{R})$. For the second term, the (fractional) residue theorem gives that for $\varepsilon > 0$ small enough,

$$\begin{aligned}\mathrm{e}^{\delta_c x}\frac{1}{\sqrt{2\pi}}\int_{\Gamma_\varepsilon} g(z)\mathrm{e}^{ixz}\,\mathrm{d}z =& \mathrm{e}^{\delta_c x}\frac{1}{\sqrt{2\pi}}\left(i\pi\mathrm{Residue}\left[g(z)\mathrm{e}^{ixz}, i\delta_c\right] + O(\varepsilon)\right)\\ =& \sqrt{\frac{\pi}{2}}iQ_{n+1}(x) + O(\varepsilon).\end{aligned}$$

This is clearly not in $L^p(\mathbb{R})$ for any $p \in [1,\infty]$. These calculations were for $x > 0$; if $x < 0$ we consider the conjugate of the indented rectangle and we obtain the equivalent result. To get the expression for H_c, consider the function $\xi^{n+2}g(\xi+i\delta_c)$. Taylor expanding around $\xi = 0$, we get by (3.4) and (3.5) that

$$\xi^{n+2}g(\xi+i\delta_c) = \sum_{k=0}^{n}\frac{(n+1-k)!}{(n+1)!}i^{k-n-1}a_{n+1-k}\xi^k + O(\xi^{n+2}),$$

and hence

$$g(\xi+i\delta_c) = \sum_{k=0}^{n}(n+1-k)!i^{k-n-1}a_{n+1-k}\xi^{k-n-2} + O(1)$$

for small ξ. As $g(\cdot + i\delta_c) \in L^p(\mathbb{R}\setminus(-1,1))$ for all $1 \leq p \leq \infty$, we have that

$$g(\xi+i\delta_c) = \sum_{k=0}^{n}(n+1-k)!i^{k-n-1}a_{n+1-k}\xi^{k-n-2} + w(\xi),$$

where $w \in L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$. Hence the "ill-behaved" part of $\int_{|\xi|\geq\varepsilon} g(\xi+i\delta_c)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi$ can be explicitly calculated as $\varepsilon \to 0^+$, as the limit is symmetric (otherwise it is not defined). The calculation is straightforward calculus and the result can be found in any table of Fourier transforms, and we find that (again we are assuming $x > 0$)

$$\lim_{\varepsilon\to 0^+}\frac{1}{\sqrt{2\pi}}\int_{|\xi|\geq\varepsilon} g(\xi+i\delta_c)\mathrm{e}^{ix\xi}\,\mathrm{d}\xi = \sqrt{\frac{\pi}{2}}iQ_{n+1}(x) + \check{w}(x),$$

where $\check{w} \in L^p(\mathbb{R})$ for all $2 \leq p \leq \infty$. Hence, taking the limit $\varepsilon \to 0^+$ in (3.7), we get

$$-ixH_c(x) = \mathrm{e}^{-\delta_c|x|}\left(\check{w}(x) - i\sqrt{2\pi}Q_{n+1}(x)\right). \tag{3.8}$$

Dividing by $-ix$ for $x \neq 0$ gives the expression for H_c, with $v = -\frac{\check{w}}{ix}$ and $P_n(x) = \sqrt{2\pi}Q_{n+1}(x)x^{-1}$ (recall that $Q_{n+1}(0) = 0$, so that $Q_{n+1}(x)x^{-1}$ is indeed a polynomial). □

Corollary 3.5. *Let (A1), (A2) and (A1*) be satisfied, and let δ_c be as in Lemma 3.4. Then, for all $0 < \delta < \delta_c$, we have that $\mathrm{e}^{\delta|\cdot|}H_c(\cdot) \in L^p(\mathbb{R})$ for all $1 \le p < \frac{1}{1+m_0}$ if $-1 < m_0 < 0$, all $1 \le p < \infty$ if $m_0 = -1$, and all $1 \le p \le \infty$ if $m_0 < -1$.*

Proof. Let δ as in the assumptions. By Lemma 3.4, we have that

$$\mathrm{e}^{\delta|x|}H_c(x) = \mathrm{e}^{-(\delta_c-\delta)|x|}\left(v(x) + P(|x|)\right).$$

As $v \in L^p(\{x \in \mathbb{R} : |x| \ge 1\})$ for all $1 \le p \le \infty$ and $\delta_c - \delta > 0$, we get that $\mathrm{e}^{\delta|\cdot|}H_c(\cdot) \in L^p(\{x \in \mathbb{R} : |x| \ge 1\})$, hence we need only check the behaviour at 0. If $m_0 < -1$, then by Lemma 3.2, we have that $v \in L^\infty(\mathbb{R})$, and the conclusion follows. For $-1 \le m_0 < 0$, the result follows from Corollary 3.3 with $\alpha = 0$. □

We have established the precise decay rate of H_c, which is sufficient to establish the precise decay rate of (decaying) solutions to (1.6) (see Section 4 below).

4. Decay of solitary waves

With the properties of H_c established in Section 3, we can now establish the decay properties of solutions to (1.6), under assumption (A3) on G. We start with algebraic decay.

4.1. Algebraic decay of solitary waves

Theorem 4.1. *Let (A1), (A2) and (A3) be satisfied and suppose that $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} u(x) = 0$ is a solution to (1.6). Then*

$$(\cdot)^l u(\cdot) \in L^q(\mathbb{R})$$

for all $l \ge 0$ and all $q \in \left(\max\{2, -m_0^{-1}\}, \infty\right)$.

Proof. Choose $p \in (1,2)$ if $m_0 \le -\frac{1}{2}$ or $p \in \left(1, \frac{1}{1+m_0}\right)$ if $-\frac{1}{2} < m_0 < 0$ and let $\alpha = \alpha(p)$ be a constant satisfying

$$\alpha > \max\{1 + m_0, 0\} - \frac{1}{p}.$$

In particular, α satisfies the condition in Corollary 3.3, so that $(1 + |\cdot|)^\alpha H_c(\cdot) \in L^p(\mathbb{R})$. Let As $|G(u)| \lesssim |u|^r$ for some $r > 1$ and $\lim_{|x|\to\infty} u(x) = 0$, we get that for any $\delta > 0$, there exists an $R_\delta \ge 0$ such that

$$|G(u(x))| \le \delta |u(x)| \text{ for all } |x| \ge R_\delta.$$

Picking $0 < \delta < c$, we get that

$$c - \frac{G(u(x))}{u(x)} \ge c - \delta > 0 \text{ for all } |x| \ge R_\delta$$

As u is a solution to (1.6), we get that

$$\left| u(x)\left(c - \frac{G(u(x))}{u(x)}\right)\right| = \left| \int_{\mathbb{R}} H_c(x-y)(1 + |x-y|)^\alpha \frac{G(u(y))}{(1 + |x-y|)^\alpha}\,\mathrm{d}y \right|$$

$$\leq \int_{\mathbb{R}} |H_c(x-y)|(1+|x-y|)^\alpha \frac{|G(u)|}{(1+|x-y|)^\alpha}\,\mathrm{d}y.$$

Letting q be the conjugate of p, we get by Hölder's inequality that

$$|u(x)| \leq C\left(\int_{\mathbb{R}} \frac{|G(u)|^q}{(1+|x-y|)^{\alpha q}}\,\mathrm{d}y\right)^{1/q} \quad \text{for all } |x| \geq R_\delta, \tag{4.1}$$

where $C = C_{\alpha,p,\delta} = (c-\delta)^{-1}\|(1+|\cdot|)^\alpha H_c(\cdot)\|_{L^p(\mathbb{R})} < \infty$. That $(\cdot)^l u(\cdot) \in L^q(\mathbb{R})$ can now be proven in precisely the same way as in the proof of Theorem 3.9 in [9]. The only difference is that there $G(u) = u^2$, but the only properties needed are that $u \in L^\infty(\mathbb{R}) \Rightarrow G(u) \in L^\infty(\mathbb{R})$ and that for any $\delta > 0$, there is an R_δ such that $|G(u(x))| \leq \delta |u(x)|$ for all $|x| \geq R_\delta$. Both of these properties are guaranteed by (A3). As q is the conjugate of p and p can be picked arbitrarily within the specified interval, this gives the stated range for q. □

With this result it is simple to prove part (i) of Theorem 1.2:

Proof of Theorem 1.2 (i). As shown in the proof of Theorem 4.1, for every $0 < \delta < c$ there is a $R_\delta \geq 0$ such that

$$c - \frac{G(u(x))}{u(x)} \geq c - \delta > 0 \text{ for all } |x| \geq R_\delta.$$

Pick one such δ and R_δ. Since $u \in L^\infty(\mathbb{R})$, we have that $|\cdot|^l u(\cdot)$ is bounded on bounded sets, so it remains only to consider $|x| \geq R_\delta$. From (1.6) and repeated use of Hölder's inequality, we get

$$\begin{aligned}
|x|^l|u(x)| \leq& (c-\delta)^{-1}\int_{\mathbb{R}} |x-y|^l |H_c(x-y)||G(u(y))|\,\mathrm{d}y + \int_{\mathbb{R}} |H_c(x-y)||y|^l|G(u(y))|\,\mathrm{d}y \\
\lesssim& \||\cdot|^l H_c(\cdot)\|_{L^1(\mathbb{R})}\|G(u)\|_{L^\infty(\mathbb{R})} + \|u^{r-1}\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} |H_c(x-y)||y|^l|u(y)|\,\mathrm{d}y \\
\lesssim& \||\cdot|^l H_c(\cdot)\|_{L^1(\mathbb{R})}\|u\|^r_{L^\infty(\mathbb{R})} + \|u\|^{r-1}_{L^\infty(\mathbb{R})}\|H_c\|_{L^p(\mathbb{R})}\||\cdot|^l u(\cdot)\|_{L^q(\mathbb{R})},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By Corollary 3.3 the first term in the last line is bounded and we can find $p \in (1,2)$ such that $H_c \in L^p(\mathbb{R})$ and $q \in \left(\max\{2, -m_0^{-1}\}, \infty\right)$ and by Theorem 4.1 the last term is also bounded. The constant implied in the notation $\lesssim$ can be taken independently of x, and the conclusion follows. □

4.2. Exponential decay of solitary waves

In this section we will add assumption (A1*).

Lemma 4.2. *Let (A1), (A2), (A3) and (A1*) be satisfied. Suppose that* $u \in L^\infty(\mathbb{R})$ *with* $\lim_{|x|\to\infty} u(x) = 0$ *is a solution to* (1.6)*. Then there exists a constant* $C > 0$ *such that*

$$|u(x)| \leq C\int_{\mathbb{R}} |H_c(x-y)||G(u(y))|\,\mathrm{d}y$$

for almost every $x \in \mathbb{R}$*.*

Proof. From (1.6) we get that

$$|u(x)|\left|c-\frac{G(u(x))}{u(x)}\right| = \left|\int_{\mathbb{R}} H_c(x-y)G(u(y))\,\mathrm{d}y\right| \leq \int_{\mathbb{R}} |H_c(x-y)||G(u(y))|\,\mathrm{d}y.$$

If $\left|c-\frac{G(u(x))}{u(x)}\right| \geq \gamma > 0$, then

$$|u(x)| \leq \gamma^{-1}\int_{\mathbb{R}} |H_c(x-y)||G(u(y))|\,\mathrm{d}y.$$

As shown in the proof of Theorem 4.1, for any $\gamma \in (0,c)$, there exists an $R_\gamma > 0$ such that $\left|c-\frac{G(u(x))}{u(x)}\right| \geq \gamma > 0$ for all $|x| \geq R_\gamma$. It follows that the set

$$E_\gamma = \{x \in \mathbb{R} : \left|c-\frac{G(u(x))}{u(x)}\right| < \gamma\},$$

is contained in bounded interval, and moreover that $\inf_{x\in E_\gamma}|u(x)| \geq C > 0$ for some $C > 0$. We have that

$$|cu(x)-G(u(x))| < \gamma|u(x)| \leq \gamma\|u\|_{L^\infty(\mathbb{R})},\ x \in E_\gamma.$$

By (1.1), $cu - G(u) = L(u) \in C(\mathbb{R})$ (since L is smoothing), and it follows that $G(u(x))$ is non-zero in some interval around x for all $x \in E_\gamma$. As H_c is non-zero around the origin and E_γ is a subset of a compact set, it follows that

$$I_\gamma := \inf\{\int_{\mathbb{R}} |H_c(x-y)||G(u(y))|\,\mathrm{d}y : x \in E_\gamma\} > 0.$$

Hence, for any $\gamma \in (0,c)$, we have that $\max\{\gamma^{-1}, I_\gamma^{-1}\|u\|_{L^\infty(\mathbb{R})}\} < \infty$ and

$$|u(x)| \leq \max\{\gamma^{-1}, I_\gamma^{-1}\|u\|_{L^\infty(\mathbb{R})}\}\int_{\mathbb{R}} |H_c(x-y)||G(u(y))|\,\mathrm{d}y,$$

which guarantees the existence of a C such as in the statement. □

Now we will prove our main result, part (ii) of Theorem 1.2.

Proof of Theorem 1.2 (ii). First we want to show that

$$\mathrm{e}^{\delta|\cdot|}u(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ for any } \delta \in [0,\delta_c).$$

The proof of this follows largely the arguments of Corollary 3.1.4 in [4], with some adaptations (see also Theorem 3.12 in [9]), but we include the details for completeness. If $-\frac{1}{2} \leq m_0 < 0$, let $1 < p < \frac{1}{1+m_0}$; otherwise let $1 < p < 2$. Let q be the Hölder conjugate of p and let $\delta \in (0,\delta_c)$. Let M_1 be the smallest constant such that

$$|u(x)| \leq M_1\int_{\mathbb{R}} |H_c(x-y)||G(u(y))|\,\mathrm{d}y \text{ for all } x \in \mathbb{R}, \tag{4.2}$$

and set

$$M_2 = \|(\cdot)G(u)u^{-1}\|_{L^\infty(\mathbb{R})}$$
$$M_3 = \|\mathrm{e}^{\delta|\cdot|}H_c(\cdot)\|_{L^p(\mathbb{R})}.$$

The boundedness of M_1 and M_2 follows from Lemma 4.2 and Theorem 1.2 (i), respectively, and M_3 is bounded by Corollary 3.5. Let

$$D := \max\{1, \frac{\delta}{2}\|u\|_{L^1(\mathbb{R})}, M_1M_2M_3\delta^{1/p}\left(\frac{2}{q}\right)^{1/q}\}.$$

We claim that

$$\|(\cdot)^l u(\cdot)\|_{L^1(\mathbb{R})} \leq \frac{(l+2)!D^{l+1}}{\delta^{l+1}}, \text{ for all } l \in \mathbb{N}. \tag{4.3}$$

Clearly it is true for $l = 0$. Assume it is true for $l = 1, 2, ...n$. Recall the following identity that can be proved by induction:

$$x^n(f * g)(x) = \sum_{j=0}^{n} \binom{n}{j} \left((\cdot)^{n-j} f * (\cdot)^j g\right)(x).$$

Using this identity, Young's inequality and (4.2), we find that

$$\begin{aligned}
\|(\cdot)^{n+1}u(\cdot)\|_{L^1(\mathbb{R})} \leq & M_1\|(\cdot)^{n+1}(H_c * G(u))(\cdot)\|_{L^1(\mathbb{R})} \\
\leq & M_1 \sum_{j=0}^{n+1} \binom{n+1}{j} \|(\cdot)^{n+1-j}H_c(\cdot)\|_{L^1(\mathbb{R})}\|(\cdot)^j G(u(\cdot))\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Considering the term involving H_c first, we get by Hölder's inequality:

$$\begin{aligned}
\int_{\mathbb{R}} |x^{n+1-j}H_c(x)|\,\mathrm{d}x &\leq \int_{\mathbb{R}} |x^{n+1-j}\mathrm{e}^{-\delta|x|}||\mathrm{e}^{\delta|x|}H_c(x)|\,\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}} |\mathrm{e}^{\delta|x|}H_c(x)|^p\,\mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}} |x|^{q(n+1-j)}\mathrm{e}^{-q\delta|x|}\,\mathrm{d}x\right)^{1/q} \\
&= M_3 2^{1/q} \left(\int_0^\infty x^{q(n+1-j)}\mathrm{e}^{-q\delta|x|}\,\mathrm{d}x\right)^{1/q} \\
&= M_3 2^{1/q} \left(\frac{(q(n+1-j))!}{(q\delta)^{q(n+1-j)+1}}\right)^{1/q} \\
&\leq M_3 \left(\frac{2}{q}\right)^{1/q} \frac{(n+1-j)!}{\delta^{n+1-j+1/q}}.
\end{aligned}$$

And for the term involving G, we have that for $1 \leq j \leq n+1$,

$$\begin{aligned}
\|(\cdot)^j G(u(\cdot))\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} |x|^j \frac{G(u(x))}{u(x)} u(x)\,\mathrm{d}x \\
&\leq M_2\|(\cdot)^{j-1}u(\cdot)\|_{L^1(\mathbb{R})}
\end{aligned}$$

$$\le M_2 \frac{(j+1)! D^j}{\delta^j}.$$

Thus we get that

$$\begin{aligned}
\|(\cdot)^{n+1} u(\cdot)\|_{L^1(\mathbb{R})} &\le M_1 M_2 M_3 \left(\frac{2}{q}\right)^{1/q} \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{(n+1-j)!(j+1)! D^j}{\delta^{n+1+1/q}} \\
&= M_1 M_2 M_3 \delta^{1/p} \left(\frac{2}{q}\right)^{1/q} \sum_{j=0}^{n+1} \frac{(n+1)!(j+1) D^j}{\delta^{n+1+1/q+1/p}} \\
&\le \sum_{j=0}^{n+1} \frac{(n+1)!(j+1) D^{j+1}}{\delta^{n+2}} \\
&= \frac{(n+3)! D^{n+2}}{\delta^{n+2}},
\end{aligned}$$

which proves the claim. Applying (4.3),

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{e}^{\nu|x|} |u(x)| \,\mathrm{d}x &\le \sum_{l=0}^{\infty} \frac{\nu^l}{l!} \int_{\mathbb{R}} |x|^l |u(x)| \,\mathrm{d}x \\
&\le \sum_{l=0}^{\infty} \frac{\nu^l}{l!} \frac{(l+2)! D^{l+1}}{\delta^{l+1}} \\
&\le \sum_{l=0}^{\infty} \frac{\nu^l (l+2)(l+1) D^{l+1}}{\delta^{l+1}}.
\end{aligned}$$

Hence the integral converges if $0 < \nu < \frac{\delta}{D}$, and it follows that $\mathrm{e}^{\nu|\cdot|} u(\cdot) \in L^1(\mathbb{R})$ for some $0 < \nu < \delta$. Next we show that $\mathrm{e}^{\nu|\cdot|} u(\cdot) \in L^\infty(\mathbb{R})$. We have that

$$|u(x)| \mathrm{e}^{\nu|x|} \le M_1 \int_{\mathbb{R}} |H_c(x-y)| \mathrm{e}^{\nu|x-y|} |G(u)| \mathrm{e}^{\nu|y|} \,\mathrm{d}y = M_1 \left(H_c(\cdot) \mathrm{e}^{\nu|\cdot|} * |G(u(\cdot))| \mathrm{e}^{\nu|\cdot|} \right)(x).$$

As G is bounded on compact sets and $u \in L^\infty(\mathbb{R})$, (A3) implies that $\left\| \frac{G(u)}{|u|^r} \right\|_{L^\infty(\mathbb{R})} < \infty$. It follows that

$$|G(u)| \lesssim |u|^r \tag{4.4}$$

uniformly over $\mathbb{R}$, where $r > 1$. Pick a $p > 1$ such that $H_c(\cdot) \mathrm{e}^{\nu|\cdot|} \in L^p(\mathbb{R})$; Corollary 3.5 guarantees that this is possible. Using (4.4), Young's inequality and Hölder's inequality, we get that for $1 \le q \le \frac{p}{p-1}$,

$$\begin{aligned}
\|\mathrm{e}^{\nu|\cdot|} u(\cdot)\|_{L^{\frac{qp}{q+p-qp}}(\mathbb{R})} &\le M_1 \|H_c(\cdot) \mathrm{e}^{\nu|\cdot|}\|_{L^p(\mathbb{R})} \|G(u(\cdot)) \mathrm{e}^{\nu|\cdot|}\|_{L^q(\mathbb{R})} \\
&\lesssim M_1 \|u\|_{L^\infty(\mathbb{R})}^{r-1} \|H_c(\cdot) \mathrm{e}^{\nu|\cdot|}\|_{L^p(\mathbb{R})} \|u(\cdot) \mathrm{e}^{\nu|\cdot|}\|_{L^q(\mathbb{R})} \\
&\lesssim \|u(\cdot) \mathrm{e}^{\nu|\cdot|}\|_{L^q(\mathbb{R})},
\end{aligned}$$

uniformly in q, as $u \in L^\infty(\mathbb{R})$. We have already established that the right-hand side is bounded for $q = 1$. Setting $q_{i+1} = \min\left\{ \frac{q_i p}{q_i + p - q_i p}, \frac{p}{p-1} \right\}$ and starting from $q_1 = 1$, we get that $q_i = \frac{p}{p-1}$ eventually, in which case $\frac{q_i p}{q_i + p - q_i p} = \infty$, and it follows that $\mathrm{e}^{\nu|\cdot|} u(\cdot) \in L^\infty(\mathbb{R})$.

Let $\eta = \sup\{\nu : \mathrm{e}^{\nu|\cdot|}u(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\}$. Assume $\eta < \delta$, and choose ν such that

$$\frac{\eta}{r} < \nu < \min\{\eta, \frac{\delta}{r}\}.$$

Recalling, (4.4) we have that

$$\begin{aligned}|u(x)|\mathrm{e}^{r\nu|x|} \leq& M_1 \int_{\mathbb{R}} |H_c(x-y)|\mathrm{e}^{r\nu|x-y|}|G(u)|\mathrm{e}^{r\nu|y|}\,\mathrm{d}y \\ \lesssim& M_1 \int_{\mathbb{R}} |H_c(x-y)|\mathrm{e}^{r\nu|x-y|}\left(u(y)\mathrm{e}^{\nu|y|}\right)^r\,\mathrm{d}y \\ =& M_1\left(H_c(\cdot)\mathrm{e}^{r\nu|\cdot|} * \left(u(\cdot)\mathrm{e}^{\nu|\cdot|}\right)^r\right)(x).\end{aligned}$$

By Young's inequality, we get

$$\|\mathrm{e}^{r\nu|\cdot|}u(\cdot)\|_{L^1(\mathbb{R})} \leq M_1\|H_c(\cdot)\mathrm{e}^{r\nu|\cdot|}\|_{L^1(\mathbb{R})}\|\left(u(\cdot)\mathrm{e}^{\nu|\cdot|}\right)^r\|_{L^1(\mathbb{R})} < \infty,$$

and

$$\|\mathrm{e}^{r\nu|\cdot|}u(\cdot)\|_{L^\infty(\mathbb{R})} \leq M_1\|H_c(\cdot)\mathrm{e}^{r\nu|\cdot|}\|_{L^1(\mathbb{R})}\|\left(u(\cdot)\mathrm{e}^{\nu|\cdot|}\right)^r\|_{L^\infty(\mathbb{R})} < \infty.$$

But as $r > 1$ we have that $r\nu > \eta$, and this contradicts the definition of η. Hence the assumption that $\eta < \delta$ must be false, and it must be the case that $\eta \geq \delta$. As $\delta \in (0, \delta_c)$ was arbitrary, this shows that

$$\mathrm{e}^{\delta|\cdot|}u(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ for any } \delta \in [0, \delta_c).$$

Assume now that $n = 0$ in Lemma 3.4, so that $H_c(x) = \mathrm{e}^{-\delta_c|x|}(v + C)$ for some constant $C \neq 0$, and let $f = v + C$; then $f \in L^1_{loc}(\mathbb{R})$ and f is bounded for $|x| > 1$. We have that

$$\mathrm{e}^{\delta_c|x|}|u(x)| \lesssim \int_{\mathbb{R}} |H_c(x-y)|\mathrm{e}^{\delta_c|x-y|}G(u(y))\mathrm{e}^{\delta_c|y|}\,\mathrm{d}y \simeq \int_{\mathbb{R}} |f(x-y)|\left(u(y)\mathrm{e}^{\frac{\delta_c}{r}|y|}\right)^r\,\mathrm{d}y.$$

Splitting the integral into the integral over $|x-y| < 1$, and $|x-y| \geq 1$ and applying Hölder's inequality, we get

$$\begin{aligned}\mathrm{e}^{\delta_c|x|}|u(x)| \lesssim& \|f\|_{L^1((-1,1))}\|u(\cdot)\mathrm{e}^{\frac{\delta_c}{r}|\cdot|}\|^r_{L^\infty(\mathbb{R})} \\ &+ \|f\|_{L^\infty(\mathbb{R}\setminus(-1,1))}\|u(\cdot)\mathrm{e}^{\frac{\delta_c}{r}|\cdot|}\|_{L^\infty(\mathbb{R})}\|u(\cdot)\mathrm{e}^{\frac{\delta_c}{r}|\cdot|}\|_{L^1(\mathbb{R})}.\end{aligned}$$

The right hand side is finite and independent of x, hence we conclude that $\mathrm{e}^{\delta_c|\cdot|}u(\cdot) \in L^\infty(\mathbb{R})$. Now we want to show that this is optimal. Let $\varepsilon > 0$. By the decay of u and assumption (A3), we have that

$$|H_c(x-y)G(u(y))| \lesssim \mathrm{e}^{-\delta_c|x-y|}|f(x-y)|\mathrm{e}^{-r\delta_c|y|} \leq \mathrm{e}^{-\delta_c|x|}|f(x-y)|\mathrm{e}^{-(r-1)\delta_c|y|},$$

for all $|y|$ sufficiently large. As $r > 1$ and $f \in L^1_{loc}(\mathbb{R})$, we can find R_ε such that

$$\left|\int_{|y|>R_\varepsilon} H_c(x-y)G(u(y))\,\mathrm{d}y\right| < \varepsilon\mathrm{e}^{-\delta_c|x|}.$$

Now let $|x| > R_\varepsilon$ be such that $f(x-y) = C + O(\varepsilon)$ for all $|y| \leq R_\varepsilon$. This is possible as $\lim_{|x|\to\infty} f(x) = C \neq 0$. If $x > R_\varepsilon$, we get that

$$
\begin{aligned}
\mathrm{e}^{\delta_c|x|}u(x) \simeq& \mathrm{e}^{\delta_c|x|} \int_{\mathbb{R}} \mathrm{e}^{-\delta_c|x-y|} f(x-y) G(u(y))\,\mathrm{d}y \\
=& \mathrm{e}^{\delta_c|x|} \int_{|y|\leq R_\varepsilon} \mathrm{e}^{-\delta_c|x-y|} f(x-y) G(u(y))\,\mathrm{d}y \\
&+ \mathrm{e}^{\delta_c|x|} \int_{|y|> R_\varepsilon} \mathrm{e}^{-\delta_c|x-y|} f(x-y) G(u(y))\,\mathrm{d}y \\
=& C \int_{|y|\leq R_\varepsilon} \mathrm{e}^{\delta_c y} G(u(y))\,\mathrm{d}y + O(\varepsilon),
\end{aligned}
$$

and if $x < -R_\varepsilon$, we get

$$
\mathrm{e}^{\delta_c|x|}u(x) \simeq C \int_{|y|\leq R_\varepsilon} \mathrm{e}^{-\delta_c y} G(u(y))\,\mathrm{d}y + O(\varepsilon)
$$

As $G(u)$ is non-zero on a set of non-zero measure,

$$
\int_{|y|\leq R_\varepsilon} \mathrm{e}^{-\delta_c y} G(u(y))\,\mathrm{d}y \quad \text{and} \quad \int_{|y|\leq R_\varepsilon} \mathrm{e}^{\delta_c y} G(u(y))\,\mathrm{d}y
$$

cannot both converge to 0 as $\varepsilon \to 0^+$. This shows that $\mathrm{e}^{\delta_c|x|}u(x)$ does not decay to 0 as $|x| \to \infty$, and it also implies that $\mathrm{e}^{\delta_c|\cdot|}u(\cdot) \in L^p(\mathbb{R})$ only for $p = \infty$. This was for $n = 0$; by the same arguments we see that $\mathrm{e}^{\delta_c|\cdot|}|u(\cdot)|$ has the same growth as P in general. $\square$

4.3. When L is a differentiating operator

Assumption (A1) implies that L is a smoothing operator, and the dispersion in (1.1) is very weak. However, our results can easily be extended to the case with stronger dispersion as well, by making a few observations. As shown in the introduction, (1.1) can formally be written as

$$
u = \mathscr{F}^{-1}\left(\frac{1}{c-m}\right) * G(u).
$$

If $m > 0$ and $m(\xi) \to \infty$ as $\xi \to \infty$, then $\tilde{m} = \frac{1}{m}$ is bounded and $\lim_{\xi\to\pm\infty} \tilde{m}(\xi) = 0$. Moreover,

$$
\frac{1}{c-m} = \frac{1}{m}\frac{1}{c/m - 1} = -\frac{1}{c}\frac{\tilde{m}}{\frac{1}{c} - \tilde{m}}.
$$

Hence, letting H_c be defined by (1.7) as in Sections 3 and 4, with $\tilde{m}$ in place of m, that is, $H_c = \mathscr{F}^{-1}\left(\frac{\tilde{m}}{c-\tilde{m}}\right)$, we get that (1.1) can be written as

$$
cu = -H_{1/c} * G(u). \tag{4.5}
$$

Note that c is just a constant and the minus sign makes no difference to our results as we have no assumptions on the sign of G. Hence this equation is even simpler than (1.6), as we do not have the term $c - \frac{G(u)}{u}$ on the left-hand side, and all our results are therefore valid if assumptions (A1), (A2) and (A3) (and (A1*)) are satisfied for $\tilde{m}$, $\frac{1}{c}$ and G. We summarize the results in the following theorem:

Theorem 4.3. *Let (A3) be satisfied, $m\colon \mathbb{R} \to \mathbb{R}$ be even and strictly positive, and such that m^{-1} satisfies (A1), and let $0 < c < \min_{\xi\in\mathbb{R}} m(\xi)$. Suppose that $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} u(x) = 0$ is a non-trivial solution to (1.1). Then*

$$|\cdot|^l u(\cdot) \in L^\infty(\mathbb{R}),$$

for any $l \geq 0$. If m^{-1} satisfies (A1) in addition, then there is a number $\delta_{1/c}$ and an integer $n \geq 0$, depending on $1/m$ and $1/c$ (see Lemma 3.4) such that*

$$\mathrm{e}^{\delta_{1/c}|\cdot|} u(\cdot)$$

has algebraic growth of order n. That is, for all $\delta \in (0, \delta_{1/c})$,

$$\mathrm{e}^{\delta|\cdot|} u(\cdot) \in L^1 \cap L^\infty(\mathbb{R}).$$

Moreover, $\mathrm{e}^{\delta_{1/c}|\cdot|} u(\cdot) \notin L^p(\mathbb{R})$ for any $p \in [1, \infty)$ and any $n \geq 0$, and $\mathrm{e}^{\delta_{1/c}|\cdot|} u(\cdot) \in L^\infty(\mathbb{R})$ if and only if $n = 0$.

5. Symmetry of solitary waves

Now we will prove Theorem 1.3. The method is based on the method of moving planes. Our proof is an adaption of the proof of Theorem 4.4 in [9] (symmetry of solitary-wave solutions to the Whitham equation), which in turn was inspired by [12] (see also [11] and [10]).

Essential to the proof of symmetry is the following "touching" lemma:

Lemma 5.1. *Let H_c be as in Theorem 1.3, and let $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} u(x) = 0$ be a solution to (1.6) and assume that G is non-negative and increasing on the range of u. Denote by $u_\lambda(\cdot) := u(2\lambda - \cdot)$ the reflection of u about $\lambda \in \mathbb{R}$. If $u \geq u_\lambda$ on $[\lambda, \infty)$, then either*

- *$u = u_\lambda$, or*
- *$u > u_\lambda$ and $\frac{G(u)}{u} + \frac{G(u_\lambda)}{u_\lambda} < c$ for all $x > \lambda$.*

That is, if $u \geq u_\lambda$ on (λ, ∞), then either they are equal or they do not touch.

This lemma is essentially corollary 4.2 in [9] for a general class of equations and can be proved in a similar manner. For completeness we include the proof; some of the arguments will also be useful later.

Proof. Let $f \geq 0$ on $[\lambda, \infty)$ be odd about λ, that is, $f(x) = -f(2\lambda - x)$, and let $x \geq \lambda$. A simple change of variables and that f is odd with respect to λ gives that

$$\begin{aligned} H_c * f(x) &= \int_\lambda^\infty H_c(x-y) f(y)\,\mathrm{d}y + \int_{-\infty}^\lambda H_c(x-y) f(y)\,\mathrm{d}y \\ &= \int_\lambda^\infty H_c(x-y) f(y)\,\mathrm{d}y + \int_\lambda^\infty H_c(x+y-2\lambda) f(2\lambda - y)\,\mathrm{d}y \end{aligned}$$

$$= \int_{\lambda}^{\infty} (H_c(x-y) - H_c(x+y-2\lambda))\, f(y)\, \mathrm{d}y.$$

As H_c is symmetric and monotonically decreasing on $(0,\infty)$, and $f \geq 0$ on $[\lambda,\infty)$, we conclude that

$$H_c * f(x) \geq 0 \text{ for all } x \geq \lambda,$$

with equality if and only if $f = 0$ on (λ,∞). By the definition of u_λ, $G(u) - G(u_\lambda)$ is odd about λ, and as $u(x) \geq u_l(x)$ for $x \geq \lambda$, it follows from the assumption that G is increasing on the range of u that $G(u) - G(u_\lambda) \geq 0$ for $x \geq \lambda$. Hence G satisfies the same properties as f, and by the symmetry of H_c we have that u_λ is also a solution to (1.6). We therefore conclude that

$$(u-u_\lambda)\left(c - \frac{G(u)}{u} - \frac{G(u_\lambda)}{u_\lambda}\right) = H_c * (G(u) - G(u_\lambda)) > 0$$

for all $x > \lambda$ unless $u = u_\lambda$. □

With this result we can prove Theorem 1.3:

Proof of Theorem 1.3. Following [12], we define

$$\Sigma_\lambda := \{x \in \mathbb{R} : x > \lambda\}$$

and

$$\Sigma_\lambda^- := \{x \in \Sigma_\lambda : u(x) < u_\lambda(x)\}.$$

The first step is to show that there is a λ far enough to the left such that the open set Σ_λ^- is empty. A straightforward calculation similar to the one in Lemma 5.1 gives that

$$\begin{aligned} &c(u(x) - u_\lambda(x)) \\ &\quad = \int_{\Sigma_\lambda} (H_c(x-y) - H_c(x+y-2\lambda))\,(G(u(y)) - G(u_\lambda(y)))\,\mathrm{d}y + G(u(x)) - G(u_\lambda(x)). \end{aligned}$$

Let $x \in \Sigma_\lambda^-$ and let $r > 1$ be as in assumption (A3). Then

$$\begin{aligned} 0 <& c(u_\lambda(x) - u(x)) \\ &\leq \int_{\Sigma_\lambda^-} (H_c(x-y) - H_c(2\lambda - x - y))\,(G(u_\lambda(y)) - G(u(y)))\,\mathrm{d}y + G(u_\lambda(x)) - G(u(x)) \\ &\leq \int_{\Sigma_\lambda^-} H_c(x-y)\,(G(u_\lambda(y)) - G(u(y)))\,\mathrm{d}y + G(u_\lambda(x)) - G(u(x)). \end{aligned}$$

By Hölder's inequality we get that

$$\|u_\lambda - u\|_{L^\infty(\Sigma_\lambda^-)} \leq \frac{1}{c}\left(\|H_c\|_{L^1(\mathbb{R})} + 1\right)\|G(u_\lambda) - G(u)\|_{L^\infty(\Sigma_\lambda^-)}, \tag{5.1}$$

and by assumption, $|G(x) - G(y)| \leq C(x)|x-y|$ for $0 \leq y < x$, and hence

$$\|G(u_\lambda) - G(u)\|_{L^\infty(\Sigma_\lambda^-)} \le \|C(u_\lambda)\|_{L^\infty(\Sigma_\lambda^-)}\|u_\lambda - u\|_{L^\infty(\Sigma_\lambda^-)},$$

as $0 \le u < u_\lambda$ on Σ_λ^-. As long as $\|u_\lambda - u\|_{L^\infty(\Sigma_\lambda^-)} \ne 0$, we can divide by this term on both sides in (5.1), and we get that

$$1 \le \frac{1}{c}\left(\|H_c\|_{L^1(\mathbb{R})} + 1\right)\|C(u_\lambda)\|_{L^\infty(\Sigma_\lambda^-)} \tag{5.2}$$

Note that as $u_\lambda(x) = u(2\lambda - x)$ and u is decaying, $\lim_{\lambda\to-\infty}\|u_\lambda\|_{L^\infty(\Sigma_\lambda^-)} = 0$. As $C(x) \to 0$ as $x \to 0^+$, we get that the right-hand side in (5.2) goes to 0 as λ goes to infinity, which is a clear contradiction. Hence it must be the case that there exists an $N \in \mathbb{R}$, such that $\|u_\lambda - u\|_{L^\infty(\Sigma_\lambda^-)} = 0$ for all $\lambda \le -N$. It follows that Σ_λ^- is empty for all $\lambda \le -N$, and that u cannot have any crests to the left of $-N$.

The next step now is to move the plane $x = \lambda$ to the right from $\lambda = -N$ until the final point for which Σ_λ^- is empty. This process will stop at a crest of before. Assume the process stops at a point λ_0, where $u(x) \ge u_\lambda(x)$, but $u(x) \ne u_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$. That is, u is not symmetric about λ_0. By Lemma 5.1, we get that $u(x) > u_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$. As u is continuous, we have that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|\overline{\Sigma_\lambda^-}| < \varepsilon$ for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$. Let $\lambda > \lambda_0$ with $|\lambda - \lambda_0|$ sufficiently small such that Σ_λ^- is bounded (by assumption, Σ_λ^- is non-empty, otherwise the process of moving the plane would not have stopped at λ_0). Let $x \in \Sigma_\lambda^-$. By similar calculations as those preceding (5.1), we get that

$$\begin{aligned} 0 <& c(u_\lambda(x) - u(x)) \\ &\le \int_{\Sigma_\lambda^-} H_c(x-y)(u_\lambda(y) - u(y))\left(\frac{G(u_\lambda(y)) - G(u(y))}{u_\lambda(y) - u(y)}\right)\mathrm{d}y + G(u_\lambda(x)) - G(u(x)). \end{aligned}$$

Let $p \in (1, \infty)$. By Young's and Hölder's inequalities, we get that

$$\begin{aligned} c\|u_\lambda - u\|_{L^p(\Sigma_\lambda^-)} \le& \|H_c\|_{L^s(\mathbb{R})}\|(u_\lambda - u)\left(\frac{G(u_\lambda) - G(u)}{u_\lambda - u}\right)\|_{L^q(\Sigma_\lambda^-)} \\ &+ \|G(u_\lambda) - G(u)\|_{L^p(\Sigma_\lambda^-)} \\ \le& \|H_c\|_{L^s(\mathbb{R})}\|\frac{G(u_\lambda) - G(u)}{u_\lambda - u}\|_{L^{qp/(p-q)}(\Sigma_\lambda^-)}\|u_\lambda - u\|_{L^p(\Sigma_\lambda^-)} \\ &+ \|\frac{G(u_\lambda) - G(u)}{u_\lambda - u}\|_{L^\infty(\Sigma_\lambda^-)}\|u_\lambda - u\|_{L^p(\Sigma_\lambda^-)}, \end{aligned}$$

where $s, q \in [1, \infty)$ are chosen such that $1 + \frac{1}{p} = \frac{1}{s} + \frac{1}{q}$. Note that this choice can be made such that $q > p$ and hence $1 < \frac{qp}{p-q} < \infty$. Since Σ_λ^- is assumed to be non-empty, the continuity of u implies that $\|u_\lambda - u\|_{L^p(\Sigma_\lambda^-)} > 0$, so we can divide out this term and we get that

$$c \le \|H_c\|_{L^s(\mathbb{R})}\|\frac{G(u_\lambda) - G(u)}{u_\lambda - u}\|_{L^{qp/(p-q)}(\Sigma_\lambda^-)} + \|\frac{G(u_\lambda) - G(u)}{u_\lambda - u}\|_{L^\infty(\Sigma_\lambda^-)}. \tag{5.3}$$

As $|\overline{\Sigma_\lambda^-}| \to 0$ as $\lambda \to \lambda_0^+$, the first term on the right-hand side can be made arbitrarily small by taking $\lambda > \lambda_0$ close enough to λ_0. By assumption we have that $G(u_\lambda) - G(u) \le \tilde{c}(u_\lambda - u)$, so that

$$\|\frac{G(u_\lambda) - G(u)}{u_\lambda - u}\|_{L^\infty(\Sigma_\lambda^-)} \le \tilde{c} < c.$$

We have thus showed that there is a $\delta > 0$ such that the right-hand side of (5.3) is less than c for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$, which is clearly a contradiction. Hence it must be the case that $\|u_\lambda - u\|_{L^p(\Sigma_\lambda^-)} = 0$, which implies that Σ_λ^- is empty - a contradiction. It follows that the assumption that u is not symmetric about λ_0 is false and this completes the proof. □

6. Examples

In this section we apply our theory from the preceding sections to some equations of interest, for which the (precise) decay properties have not previously been established.

A Whitham–Boussinesq system

Let us return to the Whitham–Boussinesq system mentioned in the introduction (cf. (1.3)). Solitary-wave solutions to this system satisfy (see (1.4))

$$u\left(c^2 - \frac{G(u)}{u}\right) = H_c * G(u),$$

where $G(u) = \frac{u^2}{2}(3c - u)$ and

$$H_c = \mathscr{F}^{-1}\left(\frac{m}{c^2 - m}\right). \tag{6.1}$$

This is exactly of the form (1.6) only with c replaced by c^2. Clearly, G satisfies (A3) with $r = 2$ and hence, if (A2) is satisfied with c^2 in place of c, all the results of the previous sections are valid. A specific equation of particular interest within this class is when m is the bi-directional Whitham-Kernel:

$$m(\xi) = \frac{\tanh(\xi)}{\xi}. \tag{6.2}$$

For this m, the theory in the previous sections gives the following result:

Theorem 6.1. *Let $c > 1$, $m(\xi) = \frac{\tanh(\xi)}{\xi}$, and $\delta_c \in (0, \frac{\pi}{2})$ satisfy $\frac{\tan(\delta_c)}{\delta_c} = c^2$. Then, for H_c defined as in (6.1),*

$$H_c(x) = \mathrm{e}^{-\delta_c|x|}\left(v(x) + \sqrt{2\pi}\frac{\tan(\delta_c)\delta_c}{\delta_c\sec^2(\delta_c) - \tan(\delta_c)}\right),$$

for some even function v that satisfies $v \in L^p(\{x \in \mathbb{R} : |x| \geq 1\})$ for all $1 \leq p \leq \infty$, and $v(x) \simeq |\ln(|x|)|$ for $|x| \ll 1$.

Moreover, if $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} u(x) = 0$ is a non-trivial solution to (1.4), then

$$u(\cdot)\mathrm{e}^{\delta_c|\cdot|} \in L^p(\mathbb{R}) \text{ if and only if } p = \infty.$$

That is, $u(x)$ decays exactly like $\mathrm{e}^{-\delta_c|x|}$. Furthermore, if $u\colon \mathbb{R} \to [0, c - \frac{1}{3}]$, then u is symmetric.

Proof. It is straightforward to see that m satisfies (A1) and (A2) with $m_0 = -1$, so all results in Section 3 hold in the present case; in particular $H_c(x) \simeq |\ln(|x|)|$ for x near 0 (cf. Lemma 3.2). Moreover, $m(z)$ is analytic for all $z \in \mathbb{C}$ except $z \in i\frac{\pi}{2}\mathbb{Z} \setminus \{0\}$ and as m is even and monotonically decreasing on $(0, \infty)$, it is real-valued on, and only on, the real and the imaginary axis. Along the imaginary axis,

$$m(iy) = \frac{\tan(y)}{y},$$

which is even in y and a bijection from $[0, \frac{\pi}{2})$ to $[1, \infty)$. Hence, for all $c > 1$, the equation $\frac{\tan(y)}{y} = c^2$ has one solution in $(0, \frac{\pi}{2})$, which we denote by δ_c. By Lemma 3.4 we get that

$$H_c(x) = \mathrm{e}^{-\delta_c|x|}\left(v(x) + C\right),$$

with v as in the statement and some C. As the singularities are at $\pm i\delta_c$, we can use (3.8) to calculate C explicitly in terms of c (recall that $\frac{\tan(\delta_c)}{\delta_c} = c^2$) and we get the expression in the statement.

With the expression for H_c, the decay of u follows directly from Theorem 1.2. It remains only to show symmetry. It is straightforward to check that the function $G(x) = \frac{x^2}{2}(3c - x)$ is increasing on $[0, 2c]$, $G'(x) < c^2$ on $[0, c - \frac{1}{\sqrt{3}})$ and $\lim_{x\to 0} G'(x) = 0$, so that G satisfies the assumptions in 1.3. Similarly, it is straightforward to check that H_c also satisfies the assumptions in Theorem 1.3 (cf. Remark 1.4). The symmetry then follows from Theorem 1.3. □

The Whitham equation

Let us now turn to the Whitham equation

$$u_t + 2uu_x + Lu_x = 0, \tag{6.3}$$

where, $m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}$. In this case solitary wave solutions will satisfy the equation

$$u(c - u) = H_c * u^2. \tag{6.4}$$

Clearly m satisfies (A1) and (A2) with $m_0 = -1/2$.

In [9] they prove that for $c > 1$

$$\mathrm{e}^{\delta|\cdot|}(\cdot)H_c(\cdot) \in L^2(\mathbb{R}), \text{ for any } \delta \in (0, \delta_c),$$

where $\delta_c \in (0, \frac{\pi}{2})$ satisfies $\sqrt{\frac{\tan(\delta_c)}{\delta_c}} = c$, without showing whether or not this is optimal. Moreover, they prove that solitary waves satisfy

$$\mathrm{e}^{\eta|\cdot|}u(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \text{ for some } \eta \geq \delta.$$

With our results from Sections 3 and 4, we can improve upon these results by giving the precise rate of decay both for the kernel H_c and for a solitary-wave solution u:

Theorem 6.2. *Let $c > 1$, $m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}$, and $\delta_c \in (0, \frac{\pi}{2})$ satisfy $\sqrt{\frac{\tan(\delta_c)}{\delta_c}} = c$. Then*

$$H_c(x) = \mathrm{e}^{-\delta_c|x|}\left(v(x) + \sqrt{2\pi}\frac{2\tan(\delta_c)\delta_c}{\delta_c\sec^2(\delta_c) - \tan(\delta_c)}\right),$$

for some even function $v \in L^p(\{x \in \mathbb{R} : |x| \geq 1\})$ for all $1 \leq p \leq \infty$ that satisfies $v(x) \simeq |x|^{-1/2}$ for $|x| < 1$.

Moreover, if $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} u(x) = 0$ is a non-trivial solution to (6.4), then

$$\mathrm{e}^{\delta_c|\cdot|}u(\cdot) \in L^p(\mathbb{R}) \textit{ if and only if } p = \infty.$$

Proof. As noted above, m satisfies (A1) and (A2) with $m_0 = -\frac{1}{2}$, so all results in Section 3 hold in the present case; in particular $H_c(x) \simeq |x|^{-1/2}$ for x near 0 (cf. Lemma 3.2). Moreover, $m(z)^2$ is analytic for all $z \in \mathbb{C}$ except $z \in i\frac{\pi}{2}\mathbb{Z} \setminus \{0\}$. Hence $m(z)$ is analytic in the strip $|\mathrm{Im}\, z| < \frac{\pi}{2}$. Moreover, as m is even and, clearly, monotonically decreasing on $(0, \infty)$, it is real-valued on, and only on, the real and the imaginary axis. Along the imaginary axis,

$$m(iy) = \sqrt{\frac{\tan(y)}{y}},$$

which is even in y and a bijection from $[0, \frac{\pi}{2})$ to $[1, \infty)$. Hence, for all $c > 1$, the equation $\sqrt{\frac{\tan(y)}{y}} = c$ has one solution in $(0, \frac{\pi}{2})$, which we denote by δ_c, and g has two singularities within the strip $|\operatorname{Im} z| < \frac{\pi}{2}$, namely at $\pm i\delta_c$. It follows from Lemma 3.4

$$H_c(x) = \mathrm{e}^{-\delta_c|x|}\left(v(x) + C\right),$$

for v as in the statement and some C. However, as our singularities are at $\pm i\delta_c$, we can use (3.8) to calculate

$$C = -i\sqrt{2\pi}\frac{c}{m'(i\delta_c)} = \sqrt{2\pi}\frac{2\tan(\delta_c)\delta_c}{\delta_c\sec^2(\delta_c) - \tan(\delta_c)},$$

where we used that $c = \sqrt{\frac{\tan(\delta_c)}{\delta_c}}$. This proves the first part.

For the second part, note that $G(u) = u^2$ satisfies (A3) with $r = 2$. Having proved the first part, the second part now follows by Theorem 1.2. □

The capillary Whitham equation

The examples above were with very weak dispersion, but as shown in Section 4.3 the theory can also be applied to equations with stronger dispersion. We take the Capillary Whitham equation as an example. That is, we consider (6.3), now with

$$m(\xi) = \sqrt{\frac{(1+\beta\xi^2)\tanh(\xi)}{\xi}}, \tag{6.5}$$

where $\beta > 0$, called the Bond number, is the strength of the surface tension. In this case we have that all sub-critical solitary wave solutions are exponentially decaying:

Theorem 6.3. *Let $\beta > 0$ and m be defined by (6.5), and let $0 < c < \min_{\xi\in\mathbb{R}} m(\xi)$. Denoting by $\delta_{1/c} > 0$ the smallest positive number for which there exists a $z_0 \in \mathbb{C}$ with $\operatorname{Im} z_0 = \delta_{1/c}$ such that $m(z_0) = c$, we have that if $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} u(x) = 0$ is a non-trivial solution to (6.4), then*

$$\mathrm{e}^{\delta_{1/c}|\cdot|}u(\cdot) \in L^p(\mathbb{R}) \text{ if and only if } p = \infty.$$

Noting that (A3) is clearly satisfied and for all $\beta > 0$ and $0 < c < \min_{\xi\in\mathbb{R}} m(\xi)$,

$$\tilde{m}(\xi) := \frac{1}{m(\xi)} = \sqrt{\frac{\xi}{(1+\beta\xi^2)\tanh(\xi)}}$$

satisfies (A1), (A1*) and (A2) (with $\frac{1}{c}$ in place of c), the result follows directly from Theorem 4.3. However, it is still of interest to investigate some of the dynamics. Let $z_0 \in C$ with $\operatorname{Im} z_0 = \delta_{1/c}$ be such that $\tilde{m} = \frac{1}{c}$. We observe that $\tilde{m}$ is analytic in

$$\mathbb{C} \setminus \{iy : y \in \mathbb{R}\setminus\{0\},\ \operatorname{sign}(y)(1-\beta y^2)\tan(y) \le 0\},$$

and the intervals cut out from the imaginary axis are branch cuts. In particular it is analytic in the strip $|\operatorname{Im} z| < \min(\sqrt{\beta^{-1}}, \frac{\pi}{2})$. If $\beta > \frac{4}{\pi^2}$, then

$$\tilde{m}(iy) = \sqrt{\frac{y}{(1-\beta y^2)\tan(y)}} : [0, \beta^{-1/2}) \to [1, \infty),$$

is a bijection; in particular, z_0 lies on the imaginary axis within the strip where $\tilde{m}$ is analytic. If $\beta < \frac{4}{\pi}$, however, then the point z_0 does not lie within the strip, and not necessarily even on the imaginary axis (if z_0 is not purely imaginary, then $\tilde{m}(-\overline{z_0}) = \frac{1}{c}$ as well).

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