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Bounded gaps between product of two primes in imaginary quadratic number fields

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Abstract

We study the gaps between products of two primes in imaginary quadratic number fields using a combination of the methods of Goldston–Graham–Pintz–Yildirim (Proc Lond Math Soc 98:741–774, 2009), and Maynard (Ann Math 181:383–413, 2015). An important consequence of our main theorem is existence of infinitely many pairs α_1, α_2 which are product of two primes in the imaginary quadratic field K such that $|\sigma(\alpha_1 - \alpha_2)| \leq 2$ for all embeddings σ of K if the class number of K is one and $|\sigma(\alpha_1 - \alpha_2)| \leq 8$ for all embeddings σ of K if the class number of K is two.

Keywords: Quadratic number fields, Product of primes, Distribution of primes

Mathematics Subject Classification: 11N05, 11N36

1 Introduction

One of the classical problems in prime number theory is to study the gaps between prime numbers. The famous twin prime conjecture asserts that there are infinitely many pairs (p_1, p_2) of prime numbers such that $|p_1 - p_2| = 2$. Although this conjecture remains out of reach, study of this conjecture leads to several interesting results. The first and very important breakthrough in this direction is the result of Goldston et al. [3] who showed that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0\,,$$

where p_n denotes the nth prime. Zhang [16] has subsequently improved this result by showing that

$$\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 7 \times 10^7.$$

Very shortly afterwards, a further breakthrough was obtained by Maynard [9], who developed a multidimensional version of the Selberg sieve to obtain 600 instead of 7×10^7. DHJ polymath [13] group extends the methods of Maynard by generalizing the Selberg sieve further reduce it to 246 unconditionally, and 6 under the assumption of the generalized Elliott–Halberstam conjecture.

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Let $\{q_n\}_{n\geq 1}$ be the sequence of positive integers which are products of exactly two primes written in the increasing order. The members of this sequence are called E_2 numbers. Heuristically problems involving E_2- numbers are as difficult as problems involving prime numbers as sieve methods do not seem to distinguish between numbers with even numbers of prime factors and odd number of prime factors (parity principle of Selberg [14]). Hence it is interesting to study the gaps between E_2- numbers. This study was initiated by Goldston et al. [4] who showed that

$$\liminf_{n}(q_{n+1} - q_n) \leq 26\,. \tag{1.1}$$

Later developing the methods of [4], they are able to improve the constant on the right hand of (1.1) to 6 in [5].

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. We say that an element $\alpha \in \mathcal{O}_K$ is prime if the principle ideal $\alpha\mathcal{O}_K$ is a prime ideal. Castillo et al. [1] have initiated the study of gaps between primes in number fields. By extending the methods of Maynard-Tao they showed that for a totally real field K there are infinitely many primes α_1 and α_2 in $\mathcal{O}_K$ such that $|\sigma(\alpha_1 - \alpha_2)| \leq 600$ for every embedding σ of K. The case when K is imaginary is first considered by Vatwani [15]. In particular it is shown in [15] that there are infinitely many prime pairs $(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{Z}[i] \times \mathbb{Z}[i]$ such that $N(\mathbf{p}_1 - \mathbf{p}_2) < 246^2$, where $N(\cdot)$ denotes the norm on $\mathbb{Q}(i)$. The method of the proof can be generalized to cover any imaginary quadratic number field with class number 1.

In the spirit of [1,4,5,8] it is natural to consider gaps between products of two primes in number fields. Before stating our main result of this article, we will fix some notations. Let $\mathcal{P}$ be the set of prime numbers in $\mathcal{O}_K$. Let G_2^K be the set of all $\alpha \in \mathcal{O}_K$ which can be written as a product of two elements from $\mathcal{P}$. We say that a tuple $(\mathfrak{h}_1, \ldots, \mathfrak{h}_k) \in \mathcal{O}_K^k$ is admissible if it does not cover all the residue classes modulo $\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. Now we are in a position to state the main result of this paper.

Theorem 1 *Let K be an imaginary quadratic number field and let $r \geq 2$ be an integer. Then there exists a positive integer $\tilde{k} := \tilde{k}(r, K)$ such that for any admissible k-tuple $(\mathfrak{h}_1, \ldots, \mathfrak{h}_k) \in \mathcal{O}_K^k$ with $k \geq \tilde{k}$, there are infinitely many $\alpha \in \mathcal{O}_K$ such that at least r of $\alpha + \mathfrak{h}_1, \ldots, \alpha + \mathfrak{h}_k$ are G_2^K-numbers.*

It is clear from Theorem 1.1 that $\liminf |\sigma(\alpha - \beta)| \leq M(K)$ where $M(K)$ is a constant depends only on K and the lim inf is taken when α, β runs over all G_2^K numbers. It will be clear at the end of the proof that the constant depends only on the class number. In the following corollaries we will precisely give the value of $M(K)$ when the class number is 1 or 2.

Corollary 1 *Let $K_d := \mathbb{Q}\left(\sqrt{d}\right)$ be an imaginary quadratic field with class number one (there are exactly nine such fields corresponding to $d = -1, -2, -3, -7, -11, -19, -43, -67$ and -163). There exist infinitely many $G_2^{K_d}$-numbers α_1, α_2 such that $|\sigma(\alpha_1 - \alpha_2)| \leq 2$ for all embeddings σ of K_d.*

Remark 1 For $d = -1$ and -2, we consider the admissible pair $\{0, 2\}$. Then, by taking norms, there are infinitely many rational primes of the form $p_1 = a^2 + db^2, p_2 = m^2 +$

$dn^2, p_3 = a_1^2 + db_1^2, p_4 = m_1^2 + dn_1^2$ with $p_1p_2 = (a^2 + db^2)(m^2 + dn^2)$ and $p_3p_4 = (a^2 + db^2)(m^2 + dn^2) + 2h \begin{vmatrix} a & \sqrt{d}b \\ \sqrt{d}n & m \end{vmatrix} + h^2$ with $a, b, m, n, a_1, b_1, m_1, n_1 \in \mathbb{Z}$ and $|h| \le 2$.

Similarly for $d = -3, -7, -11, -19, -43, -67$ and -163 considering the admissible pair $\{0, 2\}$ we get infinitely many rational primes of the form $p_1 = (a^2 + db^2)/4, p_2 = (m^2 + dn^2)/4, p_3 = (a_1^2 + db_1^2)/4, p_4 = (m_1^2 + dn_1^2)/4$ with $p_1p_2 = (a^2 + db^2)(m^2 + dn^2)/16$ and $p_3p_4 = \frac{1}{16}(a^2+db^2)(m^2+dn^2)+\frac{1}{16}\left(2h \begin{vmatrix} a & \sqrt{d}b \\ \sqrt{d}n & m \end{vmatrix} + h^2\right)$ with $a, b, m, n, a_1, b_1, m_1, n_1 \in \mathbb{Z}$ and $|h| \le 2$.

Corollary 2 *Let $K_d := \mathbb{Q}\left(\sqrt{d}\right)$ be an imaginary quadratic field with class number two. There exist infinitely many $G_2^{K_d}$-numbers α_1, α_2 such that $|\sigma(\alpha_1 - \alpha_2)| \le 8$ for all embeddings σ of K_d.*

This article is organized as follows. In Sect. 2 we provide the necessary preliminaries to prove Theorem 1. In Sect. 3 we prove a variant of Bombieri–Vinogradov theorem for G_2^K-numbers. In Sect. 4 we explain the method of the proof. Section 2 is devoted to prove Proposition 2. In Sect. 5 we will prove some preparatory lemmas which are essential for the proof. In Sect. 6 we will choose the appropriate weights. In Sect. 8 we will conclude the proofs of Theorem 1, Corollary 1 and Corollary 2.

2 Notations and preliminaries

Here and in what follows, K denotes an imaginary quadratic field unless otherwise mentioned. For much of this article, we follow the notations of Hinz [6] and Castillo et al. [1]. Being an imaginary quadratic field K has no real embeddings and it has exactly two complex embeddings, namely σ_0 (the identity) and σ (complex conjugation). We observe that for any non-zero $\alpha \in \mathcal{O}_K$, $|\sigma(\alpha)| \ge 1$. For $N > 1$, let

$$A^0(N) = \{\alpha \in \mathcal{O}_K : 1 \le |\sigma(\alpha)| \le N\} \text{ and } \mathcal{P}^0(N) = \mathcal{P} \cap A^0(N).$$

Further, for $N_1 < N_2$, we define

$$A(N_1, N_2) = A^0(N_2) \setminus A^0(N_1), \quad \mathcal{P}(N_1, N_2) = A(N_1, N_2) \cap \mathcal{P}.$$

We would also use $A(N)$ and $\mathcal{P}(N)$ for $A(2N, N)$ and $\mathcal{P}(2N, N)$ respectively. For a set S, $|S|$ denotes its cardinality, for an element $\alpha \in K$ and an ideal $\mathfrak{q}$ of $\mathcal{O}_K$, $|\alpha|$ and $|\mathfrak{q}|$ denote the respective norms.

Remark 2 A clarification about the notations is much called for at this point. For an element $\alpha \in \mathcal{O}_K$, $|\alpha|$ denotes its norm whereas $|\sigma(\alpha)|$ denotes absolute value as a complex number. For imaginary quadratic fields, they are related by

$$|\alpha| = \sigma_0(\alpha)\sigma(\alpha) = |\sigma(\alpha)|^2.$$

Hence $A^0(N)$ as defined above can also be described as

$$A^0(N) = \{\alpha \in \mathcal{O}_K : 1 \le |\alpha| \le N^2\}.$$

These usages will be clear from the context as we proceed.

For elements $a, b \in \mathcal{O}_K$ and an ideal $\mathfrak{q}$ of $\mathcal{O}_K$, we write $a \equiv b \mod \mathfrak{q}$ to mean that the ideal generated by $a - b$ is contained in $\mathfrak{q}$, i.e $(a - b) \subset \mathfrak{q}$. Moreover, if the ideal (a) generated by $a \in \mathcal{O}_K$ does not have any common factor with $\mathfrak{q}$ then we write $(a, \mathfrak{q}) = 1$.
Given a non-zero ideal $\mathfrak{q} \subseteq \mathcal{O}_K$, we define analogues of three classical multiplicative functions, namely the norm $|\mathfrak{q}| := |\mathcal{O}_K/\mathfrak{q}|$, the Euler *phi-function* $\varphi(\mathfrak{q}) := |(\mathcal{O}_K/\mathfrak{q})^\times|$ and the Möbius function $\mu(\mathfrak{q}) := (-1)^r$ if $\mathfrak{q} = \mathfrak{p}_1 \dots \mathfrak{p}_r$ for distinct prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and $\mu(\mathfrak{q}) = 0$ otherwise. We use $\tau_k(\mathfrak{q})$ to denote the number of ways of writing $\mathfrak{q}$ as a product of k factors and $\omega(\mathfrak{q})$ to denote the number of distinct prime ideals containing $\mathfrak{q}$. For ideals $\mathfrak{a}, \mathfrak{b}$, we use $[\mathfrak{a}, \mathfrak{b}]$ and $(\mathfrak{a}, \mathfrak{b})$ to denote LCM and GCD of $\mathfrak{a}, \mathfrak{b}$.
The k-tuple $(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ with $\mathfrak{a}_j \in \mathcal{O}_K$ for all j $(1 \le j \le k)$ is denoted by $\underline{\mathfrak{a}}$. We use w_1, w_2 to denote prime elements of $\mathcal{O}_K$. For any $R \in \mathbb{R}$, $|\underline{\mathfrak{a}}| \le R$ is to be interpreted as $\prod_{j=1}^k |\mathfrak{a}_j| \le R$. The notion of divisibility among k-tuples is defined componentwise, i.e,

$$\underline{\mathfrak{a}} | \underline{\mathfrak{b}} \Leftrightarrow \mathfrak{a}_j | \mathfrak{b}_j \quad \forall 1 \le j \le k.$$

For any integral ideal $\mathfrak{q}$ of $\mathcal{O}_K$, $\underline{\mathfrak{a}} | \mathfrak{q} \Leftrightarrow \prod_{j=1}^k \mathfrak{a}_j | \mathfrak{q}$. We use the notation $[\underline{\mathfrak{a}}, \underline{\mathfrak{b}}]$ to denote the product of the component-wise least common multiples, i.e. $[\underline{\mathfrak{a}}, \underline{\mathfrak{b}}] = \prod_{j=1}^k [\mathfrak{a}_j, \mathfrak{b}_j]$ and $(\mathfrak{a}, \mathfrak{b}) = 1$ to mean that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, where 1 is the trivial ideal.
For $\mathcal{R}e(s) > 1$, the Dedekind zeta function of K is defined by

$$\zeta_K(s) := \sum_{\mathfrak{q} \subseteq \mathcal{O}_K} |\mathfrak{q}|^{-s}$$

where the sum is over all non-zero ideals of $\mathcal{O}_K$. This function admits meromorphic continuation to the whole complex plane with a pole at $s = 1$. Let c_K denote its residue at $s = 1$.
Now we note that [1, page 4] the number of elements $\alpha \in A(N)$ satisfying a congruence condition $\alpha \equiv \alpha_0 \pmod{\mathfrak{q}}$ is given by

$$\frac{|A(N)|}{|\mathfrak{q}|} + O(|\partial A(N, \mathfrak{q})|),$$

where

$$|\partial A(N, \mathfrak{q})| \ll 1 + \left(\frac{|A(N)|}{|\mathfrak{q}|} \right)^{\frac{1}{2}}. \tag{2.1}$$

The following lemma is central in estimation of the sums that arise in Selberg's higher dimensional sieve.

Lemma 1 *(Lemma 2.5, [1]) Suppose γ is a multiplicative function on the non zero ideals of $\mathcal{O}_K$ such that there are constants $\kappa > 0, A_1 > 0, A_2 \ge 1$, and $L \ge 1$ satisfying*

$$0 \le \frac{\gamma(\mathfrak{p})}{|\mathfrak{p}|} \le 1 - A_1,$$

and

$$-L \le \sum_{w \le |\mathfrak{p}| < z} \frac{\gamma(\mathfrak{p}) \log |\mathfrak{p}|}{|\mathfrak{p}|} - \kappa \log(z/w) \le A_2,$$

for any $2 < w \le z$. Let h be the completely multiplicative function defined on prime ideals by $h(\mathfrak{p}) = \gamma(\mathfrak{p})/(|\mathfrak{p}| - \gamma(\mathfrak{p}))$. Let $G\colon [0, 1] \to \mathbb{R}$ be a piecewise differentiable function and let $G_{max} = \sup_{t \in [0,1]}(|G(t)| + |G'(t)|)$. Then

$$\sum_{|\mathfrak{d}|<z} \mu(\mathfrak{d})^2 h(\mathfrak{d}) G\left(\frac{\log|\mathfrak{d}|}{\log z}\right) = \mathfrak{S}\frac{c_K^{\kappa}(\log z)^{\kappa}}{\Gamma(\kappa)} \int_0^1 G(x)x^{\kappa-1}dx + O_{K,A_1,A_2,\kappa}\left(LG_{max}(\log z)^{\kappa-1}\right),$$

where $c_K := \mathrm{Res}_{s=1}\, \zeta_K(s)$ *and the singular series*

$$\mathfrak{S} = \prod_{\mathfrak{p}} \left(1 - \frac{\gamma(\mathfrak{p})}{|\mathfrak{p}|}\right)^{-1} \left(1 - \frac{1}{|\mathfrak{p}|}\right)^{\kappa}.$$

The following lemma is a consequence of Minkowski's lattice point theorem (see [1, page 12]).

Lemma 2 *Let* $A^0(N)$ *and* $A(N)$ *be defined as above. We have*

$$|A^0(N)| = (1+o(1))\frac{2\pi N^2}{\sqrt{|D_K|}} \text{ and } |A(N)| = (1+o(1))\frac{6\pi N^2}{\sqrt{|D_K|}}$$

where D_K *is the discriminant of* K.

Let ω_K be the number of roots of unity contained in K and h_K be the class number of K. The following lemma is a special case of Mitsui's generalized Prime number theorem [10].

Lemma 3 *Let* $\mathcal{P}^0(N)$ *be defined as above. We have*

$$|\mathcal{P}^0(N)| = \frac{\omega_K}{h_K R_K} \int_2^{N^2} \frac{du}{\log u} + O_K(N^2 \exp(-c\sqrt{\log N}))$$

where c *is a non-zero positive real number.*

We denote $m_K := \frac{\omega_K}{h_K R_K}$ as Mitsui's constant. As a direct consequence of Lemma 3 we get

Lemma 4 *Let* $\mathcal{P}^0(N)$ *be defined as above. Then we have*

$$|\mathcal{P}^0(N)| = \frac{\omega_K}{h_K}\frac{N^2}{\log(N^2)}\left(1 + O\left(\frac{1}{\log N}\right)\right).$$

We shall also use Dedekind's class number formula.

Lemma 5 ([12], Corollary 5.11) *Let* c_K, ω_K *and* h_K *be defined as above. We have*

$$c_K = \frac{2\pi h_K}{\omega_K\sqrt{|D_K|}}.$$

Lemma 6 *Let* K *be an algebraic number field. For any natural number* $R \geq 2$, *we have*

$$\sum_{\substack{\mathfrak{u}\subseteq\mathcal{O}_K \\ |\mathfrak{u}|\leq R}} \frac{1}{|\mathfrak{u}|} \ll_K \log R \quad \text{and} \quad \sum_{\substack{\mathfrak{p}\in\mathcal{P} \\ |\mathfrak{p}|\leq R}} \frac{1}{|\mathfrak{p}|} \ll_K \log\log R,$$

where first sum is over all non-zero integral ideals of $\mathcal{O}_K$ *whose norm is less than or equal to* R.

3 A generalization of the Bombieri–Vinogradov theorem

A subset S of $\mathcal{O}_K$ is said to have *level of distribution* ϑ for $0 < \vartheta \leq 1$ if for any $C > 0$ there exists a constant $B = B(C)$ such that

$$\sum_{|\mathfrak{q}| \leq \frac{|A^0(N)|^\vartheta}{(\log |A^0(N)|)^B}} \max_{M \leq N} \max_{\substack{a \\ (a,\mathfrak{q})=1}} \left| \sum_{\substack{w \in S \cap A^0(M) \\ w \equiv a (\bmod \mathfrak{q})}} 1 - \frac{|S \cap A^0(M)|}{\varphi(\mathfrak{q})} \right| \ll_{A,K} \frac{|A^0(N)|}{(\log N)^C}. \tag{3.1}$$

Most important case is when $\mathcal{S} = \mathcal{P}$. In this case, an analog of Elliott–Halberstam conjecture for number fields predicts that the inequality (3.1) holds with any ϑ in $0 < \vartheta \leq 1$. Hinz [6] showed that primes have level of distribution $\frac{1}{2}$ in totally real algebraic number fields. Huxley [7] obtained level of distribution $\frac{1}{2}$ for a weighted version of (3.1). The G_2^K-numbers for $K = \mathbb{Q}$ was shown by Motohashi to have level of distribution $\frac{1}{2}$.
For our purposes, it is convenient to define the following related quantities.

$$\pi^\flat(N) = |\mathcal{P}^0(N)|, \qquad \pi^\flat(N; \mathfrak{q}, \mathfrak{a}) = \sum_{\substack{w \in \mathcal{P}^0(N) \\ w \equiv \mathfrak{a} (\bmod \mathfrak{q})}} 1,$$

$$\varepsilon(N; \mathfrak{q}, \alpha) = \pi^\flat(N; \mathfrak{q}, \alpha) - \frac{1}{\varphi(\mathfrak{q})}\pi^\flat(N), \quad \varepsilon^*(N; \mathfrak{q}) = \max_{M \leq N} \max_{\alpha; (\alpha,\mathfrak{q})=1} |\varepsilon(M; \mathfrak{q}, \alpha)|.$$

Using a theorem of [7] and following the argument in Lemma 10.2 of [15], we prove the following generalization of the Bombieri–Vinodradov theorem.

Proposition 1 *Let K be an imaginary quadratic number field. Then* (3.1) *holds for any $\vartheta \leq \frac{1}{2}$ when $\mathcal{S} = \mathcal{P}$.*

Proof Let $\mathfrak{q}$ be an ideal in $\mathcal{O}_K$. We denote the ray class group $(\bmod\, \mathfrak{q})$ by $\mathcal{C}_\mathfrak{q}$ and a ray class by $\mathcal{L}_\mathfrak{q}$. Let $\pi(x, K)$ be the number of prime ideals in $\mathcal{O}_K$ of norm $\leq x$ and $\chi_\mathbb{P}$ be the characteristic function of the prime ideals in $\mathcal{O}_K$. We define

$$E(x, \mathfrak{q}, \mathcal{L}_\mathfrak{q}) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ |\mathfrak{a}| \leq x \\ \mathfrak{a} \in \mathcal{L}_\mathfrak{q}}} \chi_\mathbb{P}(\mathfrak{a}) - \frac{\pi(x, K)}{h(\mathfrak{q})},$$

where $h(\mathfrak{q})$ denotes the cardinality of the ray class group $\mathcal{C}_\mathfrak{q}$. We will now use the following lemma.

Lemma 7 *(Huxley [7]) Using the notations as above, for any $A > 0$, there exists a real number $B > 0$ such that for any $\vartheta \leq \frac{1}{2}$ we have*

$$\sum_{|\mathfrak{q}| \leq \frac{x^\vartheta}{(\log x)^B}} \frac{h(\mathfrak{q})}{\varphi(\mathfrak{q})} \max_{\mathcal{L}_\mathfrak{q} \in \mathcal{C}_\mathfrak{q}} \max_{y \leq x} \left| E_\mathbb{P}(y, \mathfrak{q}, \mathcal{L}_\mathfrak{q}) \right| \ll \frac{x}{(\log x)^A}.$$

We have the following relation between the number of ray classes and the class number ([15]):

$$h(\mathfrak{q}) = \frac{\varphi(\mathfrak{q}) h_K}{[U : U_{\mathfrak{q},1}]}$$

where U is the unit group of $\mathcal{O}_K$, $U_{\mathfrak{q},1} = \{\alpha \in U : \alpha \equiv 1 \, (\bmod\, \mathfrak{q}), \alpha \succ 0\}$ and h_K is the class number of K where $\alpha \succ 0$ means all the real conjugates (if any) of α are positive.

Now we will estimate the index set $[U : U_{\mathfrak{q},1}]$. To do that we define the following homomorphism

$$\psi : U \to (\mathcal{O}_K/\mathfrak{q})^*$$

by $\psi(u) = u \,(\mathrm{mod}\, \mathfrak{q})$. Then the kernel of ψ is $U_{\mathfrak{q},1}$ and image of ψ is the residue classes $(\mathrm{mod}\, \mathfrak{q})$ that contain a unit. Let $T_\mathfrak{q} = \mathrm{Im}(\psi)$. Then $|T_\mathfrak{q}| = [U : U_{\mathfrak{q},1}]$ and $\frac{h(\mathfrak{q})}{\varphi(\mathfrak{q})} = \frac{h}{|T_\mathfrak{q}|}$. Since number of units in a imaginary quadratic number field is 2, 4 or 6, so if $\mathfrak{u}_1, \mathfrak{u}_2 \in U$ satisfies $\mathfrak{u}_1 \equiv \mathfrak{u}_2 \,(\mathrm{mod}\, \mathfrak{q})$ then $|\mathfrak{q}|$ must divide $|\mathfrak{u}_1 - \mathfrak{u}_2|$, which is atmost 4. Thus for $|\mathfrak{q}| > 4$ we see that $T_\mathfrak{q} = |U|$, which only depends only on K and not on $\mathfrak{q}$. Therefore using these estimates, from Lemma 7 we obtain the following.

Lemma 8 *Using the notation as in Lemma 7, for any $A > 0$ there exists a positive real number B such that for any $0 < \vartheta \le \frac{1}{2}$, we have*

$$\sum_{4<|\mathfrak{q}|\le \frac{x^\vartheta}{(\log x)^B}} \max_{\mathcal{L}_\mathfrak{q}\in\mathcal{C}_\mathfrak{q}} \max_{y\le x} \left|E_\mathbb{P}(y, \mathfrak{q}, \mathcal{L}_\mathfrak{q})\right| \ll_K \frac{x}{(\log x)^A}. \tag{3.2}$$

Proof Let $\mathfrak{a} \in \mathcal{O}_K$, $(\mathfrak{a}, \mathfrak{q}) = 1$ and $\mathcal{L}_\mathfrak{q}(\mathfrak{a})$ be the ray class containing $(\mathfrak{a})$. Then from (3.2) we get

$$\sum_{4<|\mathfrak{q}|\le \frac{x^\vartheta}{(\log x)^B}} \max_{(\mathfrak{a},\mathfrak{q})=1} \max_{y\le x} \left|E_\mathbb{P}(y, \mathfrak{q}, \mathcal{L}_\mathfrak{q}(\mathfrak{a}))\right| \ll \frac{x}{(\log x)^A}. \tag{3.3}$$

It is easy to see that all integral ideals belonging to $\mathcal{L}_\mathfrak{q}(\mathfrak{a})$ are principal. Therefore we obtain

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ |\mathfrak{a}|\le y\\ \mathfrak{a}\in\mathcal{L}_\mathfrak{q}}} \chi_\mathbb{P}(\mathfrak{a}) = \sum_{\substack{\eta\in\mathcal{O}_K\\ |\eta|\le y\\ (\eta)\in\mathcal{L}_\mathfrak{q}(\mathfrak{a})}} \chi_\mathbb{P}(\eta).$$

We also observe that there is an one to many correspondence between

$$\{\eta \in \mathcal{P}, |\eta| \le x, (\eta) \in \mathcal{L}_\mathfrak{q}(\mathfrak{a})\} \quad \text{and} \quad \{w \in \mathcal{P}, |w| \le x, w \equiv \mathfrak{a}(\mathfrak{q})\}$$

depending on the number of units in $\mathcal{O}_K$ (see [[15], Sect. 10] for more details). More precisely, we have

$$\sum_{\substack{\eta\in\mathcal{O}_K\\ |\eta|\le y\\ (\eta)\in\mathcal{L}_\mathfrak{q}(\mathfrak{a})}} \chi_\mathbb{P}(\eta) = |U| \sum_{\substack{w\in\mathcal{O}_K\\ |w|\le y\\ w\equiv\mathfrak{a}(\mathfrak{q})}} \chi_\mathbb{P}(w).$$

For $|\mathfrak{q}| > 4$, we recall that $h(\mathfrak{q}) = \frac{h_K\varphi(\mathfrak{q})}{|U|}$. So from (3.3), we get

$$\sum_{4<|\mathfrak{q}|\le \frac{x^\vartheta}{(\log x)^B}} \max_{(\mathfrak{a},\mathfrak{q})=1} \max_{y\le x} \left| |U| \sum_{\substack{w\in\mathcal{O}_K\\ |w|\le y\\ w\equiv\mathfrak{a}(\mathfrak{q})}} \chi_\mathbb{P}(w) - \frac{|U|\pi(y,K)}{h_K\varphi(\mathfrak{q})} \right| \ll \frac{x}{(\log x)^A} \tag{3.4}$$

for any $\vartheta \le \frac{1}{2}$ and for any $A > 0$. Now Prime ideal theorem tells us

$$\pi(y,K) \sim \frac{y}{\log y} \tag{3.5}$$

Also from Lemma 4 and using $\omega_K = |U|$, we get

$$|\mathcal{P}^0(y^{1/2})| \sim \frac{|U|}{h_K} \frac{y}{\log y}. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\pi(y, K) \sim \frac{h_K}{|U|} |\mathcal{P}^0(y^{1/2})|. \tag{3.7}$$

Also note that

$$|U| \sum_{\substack{|w| \le y \\ w \equiv \mathfrak{a}(\mathfrak{q})}} \chi_{\mathbb{P}}(w) = \sum_{\substack{w \in \mathcal{P}^0(y^{1/2}) \\ w \equiv \mathfrak{a}(\mathfrak{q})}} 1.$$

From (3.7) and (3.4) we complete proof of the proposition. □

We would use the above result in the following form which can be easily deduced by partial summation.

Lemma 9 *Let K be an imaginary quadratic number field. For any ϑ, $0 < \vartheta \le \frac{1}{2}$, any $B > 0$ and a fixed integer $h \ge 0$, there exists $C = C(B, h)$ such that if $Q \le |A(N)|^{\vartheta} (\log N)^{-C}$, then*

$$\sum_{|\mathfrak{q}| \le Q} \mu^2(\mathfrak{q}) h^{\omega(\mathfrak{q})} \varepsilon^*(N; \mathfrak{q}) \ll_{B,K} |A(N)| (\log N)^{-B}.$$

For $0 < \frac{\vartheta}{2} < b \le \frac{1}{2}$, and for $1 \le Y' \le N^b$ ($Y' := N^\eta$ with $\eta \le \frac{\vartheta}{2}$ to be made precise later) we define a function β on $\mathcal{O}_K$ by

$$\beta(\alpha) = \begin{cases} 1 & \text{if } \alpha = w_1 w_2, w_1 \in \mathcal{P}(Y', N^b), w_2 \in \mathcal{P}(N^b, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

For the function β, we define

$$\pi_\beta(N) = \sum_{\alpha \in A(N)} \beta(\alpha), \quad \pi_{\beta,\mathfrak{q}}(N) = \sum_{\substack{\alpha \in A(N) \\ (\alpha, \mathfrak{q}) = 1}} \beta(\alpha), \quad \pi_\beta(N; \mathfrak{q}, \gamma) = \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \gamma (\mathrm{mod}\ \mathfrak{q})}} \beta(\alpha)$$

$$\varepsilon_\beta(N; \mathfrak{q}, \gamma) = \pi_\beta(N; \mathfrak{q}, \gamma) - \frac{1}{\varphi(\mathfrak{q})} \pi_{\beta,\mathfrak{q}}(N), \quad \varepsilon^*_\beta(N; \mathfrak{q}) = \max_{M \le N} \max_{\gamma; (\gamma, \mathfrak{q}) = 1} |\varepsilon_\beta(M; \mathfrak{q}, \gamma)|.$$

An arithmetic function f is said to have *level of distribution* ϑ for $0 < \vartheta \le 1$ if for any $A > 0$ there exists a constant $B = B(A)$ such that

$$\sum_{q \le \frac{N^\vartheta}{(\log N)^B}} \max_{M \le N} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{n \le M \\ n \equiv a (\mathrm{mod}\ q)}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \le M \\ (n,q)=1}} f(n) \right| \ll_A \frac{N}{(\log N)^A}. \tag{3.8}$$

Let $\tau(n)$ be the number of divisors of a natural number n. A complex valued arithmetic function f is said to satisfy Siegel–Walfisz condition if there exist positive constant C such that

$$f(n) = O\left(\tau(n)^C\right) \quad \text{and} \quad \sum_{n \le x} f(n) \chi(n) = O\left(\frac{x}{(\log x)^{3D}}\right), \tag{3.9}$$

holds for all $D > 0$ and for any non-principal Dirichlet character $\chi \pmod q$ with $q \ll (\log x)^D$.

If arithmetic functions f and g both satisfy (3.9) and have level of distribution $\frac{1}{2}$ then Motohashi [11] obtained that the Dirichlet convolution $f * g$ also does so. In [2], we extend Motohashi's [11] result to arithmetic functions on imaginary quadratic number fields. As the proof can be carried forward for any level of distribution $0 < \vartheta \leq \frac{1}{2}$, viewing β as a Dirichlet convolution of characteristic functions of $\mathcal{P}(Y', N^b)$ and $\mathcal{P}(N^b, \infty)$, we get the following lemma. More precisely, it is a direct application of Cauchy–Schwarz inequality and Corollary 1.5 of [2].

Lemma 10 *Let K be an imaginary quadratic number field. For $0 < \vartheta \leq \frac{1}{2}$, $B > 0$ and fixed integer $h \geq 0$, there exists $C = C(B, h)$ such that if $Q \leq |A(N)|^{\vartheta} (\log N)^{-C}$, then*

$$\sum_{|\mathfrak{q}| \leq Q} \mu^2(\mathfrak{q}) h^{\omega(\mathfrak{q})} \varepsilon_{\beta}^*(N; \mathfrak{q}) \ll_{B,K} |A(N)| (\log N)^{-B}. \tag{3.10}$$

4 Method

Now we will describe the method of proof which is a combination of methods of [5] and [9].

Recall that a tuple $(\mathfrak{h}_1, \ldots, \mathfrak{h}_k) \in \mathcal{O}_K^k$ is admissible if it does not cover all residue classes modulo $\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. Let $D_0 = \log \log \log N$, $\mathfrak{m} := \prod_{|\mathfrak{p}| < D_0} \mathfrak{p}$. Since $(\mathfrak{h}_1, \ldots, \mathfrak{h}_k) \in \mathcal{O}_K^k$ is admissible, there exists ν_0 modulo $\mathfrak{m}$ such that each $\alpha + \mathfrak{h}_i$ lies in $(\mathcal{O}_K/\mathfrak{m})^{\times}$ for all $j = 1, \cdots, k$. The main objects of consideration are the sums

$$S_1 := \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \nu_0 (\mathrm{mod}\, \mathfrak{m})}} \left(\sum_{\substack{\mathfrak{d}_1, \ldots, \mathfrak{d}_k: \\ \mathfrak{d}_i | (\alpha + \mathfrak{h}_i) \, \forall i}} \lambda_{\mathfrak{d}_1, \ldots, \mathfrak{d}_k} \right)^2$$

and

$$S_2 := \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \nu_0 (\mathrm{mod}\, \mathfrak{m})}} \left(\sum_{i=1}^{k} \beta(\alpha + \mathfrak{h}_i) \right) \left(\sum_{\substack{\mathfrak{d}_1, \ldots, \mathfrak{d}_k: \\ \mathfrak{d}_i | (\alpha + \mathfrak{h}_i) \, \forall i}} \lambda_{\mathfrak{d}_1, \ldots, \mathfrak{d}_k} \right)^2, \tag{4.1}$$

where the inner sum is a k-fold sum over integral ideals and $\lambda_{\mathfrak{d}_1, \ldots, \mathfrak{d}_k}$ are suitably chosen weights to be made explicit later.

Since each summand is non-negative, if we can show that $S_2 > \rho S_1$ for some positive ρ, then there must be at least one $\alpha \in A(N)$ such that among $\alpha + \mathfrak{h}_1, \ldots, \alpha + \mathfrak{h}_k$ atleast $[\rho] + 1$ are G_2^K-numbers. We choose the weights $\lambda_{\mathfrak{d}_1, \ldots, \mathfrak{d}_k}$ in such a way that $\lambda_{\mathfrak{d}_1, \ldots, \mathfrak{d}_k} = 0$ unless $(\mathfrak{d}_i, \mathfrak{m}) = 1$, $\mathfrak{d}_i$ is square-free, and $|\mathfrak{d}_1 \cdots \mathfrak{d}_k| \leq R$ for each $i = 1, \cdots, k$, where R will be chosen later to be a small power of N. The main result of this section is the following.

Proposition 2 *Let K be an imaginary quadratic number field. Suppose that the primes $\mathcal{P}$ and G_2^K-numbers have a common level of distribution $0 < \vartheta \leq 1$, and set $R = N^{\vartheta} (\log N)^{-C}$ for some constant $C > 0$. For a given a piecewise differentiable function $F \colon [0, 1]^k \to \mathbb{R}$ supported on the simplex $\mathcal{R}_k := \{(x_1, \ldots, x_k) \in [0, 1]^k : x_1 + \cdots + x_k \leq 1\}$, we set*

$$\lambda_{\mathfrak{d}_1, \ldots, \mathfrak{d}_k} := \left(\prod_{i=1}^{k} \mu(\mathfrak{d}_i) |\mathfrak{d}_i| \right) \sum_{\substack{\mathfrak{r}_1, \ldots, \mathfrak{r}_k \\ \mathfrak{d}_i | \mathfrak{r}_i \, \forall i \\ (\mathfrak{r}_i, \mathfrak{m}) = 1 \, \forall i}} \frac{\mu(\mathfrak{r}_1 \ldots \mathfrak{r}_k)^2}{\prod_{i=1}^{k} \varphi(\mathfrak{r}_i)} F\left(\frac{\log |\mathfrak{r}_1|}{\log R}, \ldots, \frac{\log |\mathfrak{r}_k|}{\log R} \right)$$

whenever $|\mathfrak{d}_1 \dots \mathfrak{d}_k| < R$ *and* $(\mathfrak{d}_1 \dots \mathfrak{d}_k, \mathfrak{m}) = 1$, *and* $\lambda_{\mathfrak{d}_1,\dots,\mathfrak{d}_k} = 0$ *otherwise.*

Then

$$S_1 = (1 + o(1))\frac{\varphi(\mathfrak{m})^k |A(N)|(c_K \log R)^k}{|\mathfrak{m}|^{k+1}}\widetilde{I}_{1k}(F)$$

and

$$S_2 = (1 + o(1))\frac{m_K \varphi(\mathfrak{m})^k |P(N)|(c_K \log R)^{k+1}}{|\mathfrak{m}|^{k+1}}\sum_{m=1}^{k}\left(\widetilde{I}_{2k}^{(m)}(F) + \widetilde{I}_{3k}^{(m)}(F)\right)$$

where $0 < \eta \le \frac{\vartheta}{2}$, $m_K = \frac{\omega_K}{h_K}$ *is Mitsui's constant,*

$$\widetilde{I}_{1k}(F) := \int \cdots \int_{\mathcal{R}_k} F(x_1, \dots, x_k)^2 \, dx_1 \dots dx_k,$$

$$\widetilde{I}_{2k}^{(m)}(F) :=$$

$$\left(\int_1^{B/2} \frac{B}{y(B-y)} dy\right)\left(\int \cdots \int_{\mathcal{R}_{k-1}} \left(\int_0^{T_m} F(x_1, \dots, x_k)\, dx_m\right)^2 dx_1 \dots dx_{m-1} dx_{m+1} \dots dx_k\right)$$

and

$$\widetilde{I}_{3k}^{(m)}(F) = \int_{B\eta}^{1} \frac{B}{y(B-y)} \int \cdots \int_{\mathcal{R}_{k-1}} \left(\int_0^{T_m(y)} F(x_1, \dots, x_k)\, dx_m\right)^2 dx_1 \dots dx_{m-1} \dots dx_k dy$$

with $B = 2/\vartheta$, $T_m = 1 - x_1 - \dots - x_{m-1} - x_{m+1} - \dots - x_k$ *and* $T_m(y) = \min(y, T_m)$.

5 Preparations

The sum S_1 has been calculated in [1, Proposition 2.1]. So we would only work with S_2. By squaring innermost sum and interchanging summation from Eq. (4.1) we can write S_2 as

$$S_2 := \sum_{m=1}^{k} S_{2m} = \sum_{m=1}^{k}\sum_{\underline{\mathfrak{a}},\underline{\mathfrak{b}}} \lambda_{\underline{\mathfrak{a}}}\lambda_{\underline{\mathfrak{b}}} \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \nu_0(\mathfrak{m}) \\ [\mathfrak{a}_j, \mathfrak{b}_j] | (\alpha + \mathfrak{h}_j) \forall j}} \beta(\alpha + \mathfrak{h}_m). \tag{5.1}$$

We note that $[\mathfrak{a}_i, \mathfrak{b}_i]$ and $[\mathfrak{a}_j, \mathfrak{b}_j]$ are relatively coprime for $i \ne j$ since the primes dividing $\mathfrak{h}_i - \mathfrak{h}_j$ also divides $\mathfrak{m}$.

If $\beta(\alpha + \mathfrak{h}_m) = 1$ then $\alpha + \mathfrak{h}_m = w_1 w_2$ with $w_1 \in \mathcal{P}(Y', N^b)$, $w_2 \in \mathcal{P}(N^b, \infty)$ where Y' and N^b are as in the definition of β. So the norm of w_2, $|w_2| = |\sigma(w_2)|^2 > N^{2b} > N^{\vartheta} > R$ by our choice of R and b. Hence $\alpha + \mathfrak{h}_m$ has exactly one prime divisor w_1 with $|w_1| \le N^{2b}$. Since $|\underline{\mathfrak{a}}| \le R$, $|\underline{\mathfrak{b}}| \le R$ and $\underline{\mathfrak{a}}, \underline{\mathfrak{b}}$ are square-free, so all prime divisors of $[\underline{\mathfrak{a}}, \underline{\mathfrak{b}}]$ have norm $\le R$.

Hence we conclude that either $[\mathfrak{a}_m, \mathfrak{b}_m] = 1$ or $[\mathfrak{a}_m, \mathfrak{b}_m] = (w_1)$. Before discussing either of these cases we need the following lemma.

Lemma 11 *For any function* $f : \mathcal{O}_K \to \mathbb{C}$ *with* $|f| \le 1$,

$$\sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \alpha_0(\mathfrak{q})}} f(\alpha + \mathfrak{h}) = \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv (\alpha_0 + \mathfrak{h})(\mathfrak{q})}} f(\alpha) + O\left(1 + \left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2}\right).$$

Proof Putting $\alpha' = \alpha + \mathfrak{h}$ and $\alpha'_0 = \alpha_0 + \mathfrak{h}$ in the L.H.S, we get

$$\sum_{\substack{\alpha' \in A(N)+\mathfrak{h} \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} f(\alpha').$$

Since $|f| \le 1$, we get

$$\sum_{\substack{\alpha' \in A(N)+\mathfrak{h} \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} f(\alpha') = \sum_{\substack{\alpha' \in A(N) \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} f(\alpha') + O\Bigg(\sum_{\substack{\alpha' \in A(N)+\mathfrak{h} \setminus A(N) \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} 1 \Bigg).$$

Now using (2.1), the O-term is

$$\sum_{\substack{\alpha' \in A(N)+\mathfrak{h} \setminus A(N) \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} 1 = \sum_{\substack{\alpha' \in A(N)+\mathfrak{h} \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} 1 - \sum_{\substack{\alpha' \in A(N) \\ \alpha' \equiv \alpha'_0 (\mathfrak{q})}} 1 = \frac{|A(N)+\mathfrak{h}|}{|\mathfrak{q}|} - \frac{|A(N)|}{|\mathfrak{q}|}$$
$$+ O\left(\partial(A(N)+\mathfrak{h}, \mathfrak{q})\right) + O\left(\partial(A(N), \mathfrak{q})\right) \ll 1 + \left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2}.$$

5.1 The case $[\mathfrak{a}_m, \mathfrak{b}_m] = 1$

Replacing $\alpha + \mathfrak{h}_m$ by α, the condition $[\mathfrak{a}_j, \mathfrak{b}_j] | (\alpha + \mathfrak{h}_j)$ of the inner sum becomes $\alpha \equiv (\mathfrak{h}_m - \mathfrak{h}_j)$ modulo $[\mathfrak{a}_j, \mathfrak{b}_j]$ for all $j \neq m$. Since $[\mathfrak{a}_j, \mathfrak{b}_j]$ is coprime of $\mathfrak{m}$ for all j, by Chinese remainder theorem, these $k-1$ congruence equations have a common solution $\alpha_0 \left(\bmod\ \mathfrak{m} \prod_{j=1}^{k} [\mathfrak{a}_j, \mathfrak{b}_j]\right)$ where the last product remains unchanged by excluding or including the index $j = m$ (as $[\mathfrak{a}_m, \mathfrak{b}_m] = 1$). Using Lemma 11 with $f = \beta$, we get

$$\sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \alpha_0 (\mathfrak{q})}} \beta(\alpha + \mathfrak{h}_m) = \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv (\alpha_0 + \mathfrak{h}_m)(\mathfrak{q})}} \beta(\alpha) + O\left(\left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2} \right)$$

where $\mathfrak{q} = \mathfrak{m} \prod_{j=1}^{k} [\mathfrak{a}_j, \mathfrak{b}_j]$. Using this we have

$$\sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \nu_0 (\mathfrak{m}) \\ [\mathfrak{a}_j, \mathfrak{b}_j] | (\alpha + \mathfrak{h}_j) \forall j}} \beta(\alpha + \mathfrak{h}_m) = \sum_{\substack{\alpha \in A(N) \\ \alpha \equiv \alpha'_0 (\mathfrak{q})}} \beta(\alpha) + O\left(\left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2} \right)$$
$$= \frac{1}{\varphi(\mathfrak{q})} \sum_{\substack{\alpha \in A(N) \\ (\alpha, \mathfrak{q}) = 1}} \beta(\alpha) + \mathcal{E}_\beta(N, \mathfrak{q}, \alpha'_0) + O\left(\left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2} \right)$$

where $\alpha_0^{'} = \alpha_0 + \mathfrak{h}_m$.

5.2 The case $[\mathfrak{a}_m, \mathfrak{b}_m] = (w_1)$

In this case $w_1 \in \mathcal{P}(Y', R')$ with $R' = R^{1/2}$ because of the support of $\lambda_{\underline{\mathfrak{a}}}$ and $\lambda_{\underline{\mathfrak{b}}}$. Let $\widetilde{w_1}$ be the inverse of $w_1 \ (\bmod\ \mathfrak{q}/(w_1))$. Similarly as above

$$\sum_{\substack{\alpha\in A(N)\\ \alpha\equiv\nu_0(\mathfrak{m})\\ [\mathfrak{a}_j,\mathfrak{b}_j]|(\alpha+\mathfrak{h}_j)\forall j}} \beta(\alpha+\mathfrak{h}_m) = \sum_{\substack{\alpha-\mathfrak{h}_m\in A(N)\\ \alpha\equiv\alpha_o(\mathfrak{q}/(w_1))}} \beta(\alpha) = \sum_{\substack{\alpha\in A(N)\\ \alpha\equiv\alpha_o(\mathfrak{q}/(w_1))}} \beta(\alpha) - \sum_{\substack{\alpha\in(A(N)+\mathfrak{h}_m)\backslash A(N)\\ \alpha\equiv\alpha_o(\mathfrak{q}/(w_1))}} \beta(\alpha)$$
$$= \sum_{\substack{\alpha\in A(N)\\ \alpha\equiv\alpha_o(\mathfrak{q}/(w_1))}} \beta(\alpha) + O\left(\left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2}\right).$$

Now $\alpha\in A(N)$ and $\alpha = w_1w_2$. So we separate the above sum with respect to primes w_1 and w_2. We note that $w_1w_2\in A(N)$ if and only if $w_2\in A(N/|w_1|^{1/2})$. Therefore in this case, we have

$$\sum_{\substack{\alpha\in A(N)\\ \alpha\equiv\alpha_o(\mathfrak{q}/(w_1))}} \beta(\alpha) = \sum_{\substack{w_2\in A\left(\frac{N}{|w_1|^{1/2}}\right)\cap\mathcal{P}\\ w_2\equiv\alpha_o\widetilde{w_1}(\mathfrak{q}/(w_1))}} 1 + O\left(\left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2}\right)$$
$$= \frac{\pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right)}{\varphi\left(\mathfrak{q}/(w_1)\right)} + \varepsilon\left(\frac{N}{|w_1|^{1/2}}, \mathfrak{q}/(w_1), \alpha_o\widetilde{w_1}\right) + O\left(\left(\frac{|A(N)|}{|\mathfrak{q}|}\right)^{1/2}\right).$$

For each $\mathfrak{q}$, the number of ways of choosing $\mathfrak{a}_1,\ldots,\mathfrak{a}_k$ and $\mathfrak{b}_1,\ldots,\mathfrak{b}_k$ so that

$$\mathfrak{m}\prod_{j=1}^{k}[\mathfrak{a}_j,\mathfrak{b}_j] = \mathfrak{q}$$

is at most $\tau_{3k}(\mathfrak{q})$. Therefore for each $1\le m\le k$, from Eq. (5.1), the sum S_{2m} can be written as

$$S_{2m} = \sum_{w_1\in\mathcal{P}_1^0(Y',N^b)} \pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right) \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}\\ [\mathfrak{a}_m,\mathfrak{b}_m]=1\\ (w_1,\mathfrak{q})=1}} \frac{\lambda_{\underline{\mathfrak{a}}}\lambda_{\underline{\mathfrak{b}}}}{\varphi\left(\mathfrak{m}\prod_{j\neq m}[\mathfrak{a}_j,\mathfrak{b}_j]\right)}$$
$$+ \sum_{w_1\in\mathcal{P}_1^0(Y',R')} \pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right) \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}}\\ [\mathfrak{a}_m,\mathfrak{b}_m]=(w_1)}} \frac{\lambda_{\underline{\mathfrak{a}}}\lambda_{\underline{\mathfrak{b}}}}{\varphi\left(\mathfrak{m}\prod_{j\neq m}[\mathfrak{a}_j,\mathfrak{b}_j]\right)}$$
$$+ O\left(\lambda_{\max}^2\left(\frac{|A(N)|}{|\mathfrak{m}|}\right)^{1/2} \sum_{\substack{\mathfrak{a}_1,\ldots,\mathfrak{a}_k\\ \mathfrak{b}_1,\ldots,\mathfrak{b}_k}} \frac{1}{\prod_{j=1}^k |[\mathfrak{a}_j,\mathfrak{b}_j]|^{\frac{1}{2}}}\right) + O\left(\lambda_{\max}^2 \sum_{|\mathfrak{q}|<|\mathfrak{m}|R^2} \mu^2(\mathfrak{q})\tau_{3k}(\mathfrak{q})\varepsilon_\beta^*(N,\mathfrak{q})\right)$$
$$+ O\left(\lambda_{\max}^2 \sum_{|\mathfrak{q}|<|\mathfrak{m}|R^2} \mu^2(\mathfrak{q})\tau_{3k}(\mathfrak{q}) \sum_{w_1|\mathfrak{q};w_1\in\mathcal{P}_1^0(Y',R')} \varepsilon^*\left(\frac{N}{|w_1|^{1/2}}, \mathfrak{q}/(w_1)\right)\right)$$

where $\lambda_{\max} = \sup_{\underline{\mathfrak{a}}}|\lambda_{\underline{\mathfrak{a}}}|$.
Using Lemma 6, it can seen that the first error term of the above expression of S_{2m} is bounded above by

$$\lambda_{\max}^2|A(N)|^{1/2}. \sum_{|\mathfrak{q}|\le R^2\log\log N} \frac{\mu^2(\mathfrak{q})\tau_{3k}(\mathfrak{q})}{|\mathfrak{q}|^{1/2}}$$
$$\le \lambda_{\max}^2|A(N)|^{1/2}\cdot R(\log\log N)^{1/2} \prod_{|\mathfrak{p}|\le R^2\log\log N}\left(1+\frac{3k}{|\mathfrak{p}|}\right)$$
$$\ll \lambda_{\max}^2|A(N)|^{1/2}\cdot R(\log R)^{3k}.$$

Lemma 10 gives that the second error term of S_{2m} is bounded above by $\lambda_{\max}^2 \frac{|A(N)|}{(\log N)^B}$ for any $B > 0$.

Lemma 9 gives that the third error term of S_{2m} is bounded above by

$$\begin{aligned}
&\ll \lambda_{\max}^2 (\log R)^{2k} \sum_{|\mathfrak{q}| \le |\mathfrak{m}| R^2} \mu(\mathfrak{q})^2 \tau_{3k}(\mathfrak{q}) \sum_{\substack{w_1 | \mathfrak{q} \\ w_1 \in \mathcal{P}_1^0(Y', R')}} \varepsilon^* \left(\frac{N}{|w_1|^{1/2}}, \mathfrak{q}/(w_1) \right) \\
&\ll \lambda_{\max}^2 (\log R)^{2k} \sum_{|w_1| \le R} \tau_{3k}(w_1) \sum_{|\mathfrak{s}| \le \frac{|\mathfrak{m}| R^2}{|w_1|}} \mu(\mathfrak{s})^2 \tau_{3k}(\mathfrak{s}) \varepsilon^* \left(\frac{N}{|w_1|^{1/2}}, \mathfrak{s} \right) \\
&\ll \lambda_{\max}^2 \sum_{|w_1| \le R} \frac{|A(N)|}{|w_1| \log (N/|(w_1)|)} \ll \lambda_{\max}^2 |A(N)|.
\end{aligned}$$

Combining these estimations of error terms we get the following lemma.

Lemma 12 *Let S_{2m} be defined as in (5.1). Then with the hypothesis of Proposition 2 we have*

$$\begin{aligned}
S_{2m} = &\sum_{w_1 \in \mathcal{P}_1^0(Y', N^b)} \pi^\flat \left(\frac{N}{|w_1|^{1/2}} \right) \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{b}} \\ [\mathfrak{a}_m, \mathfrak{b}_m] = 1 \\ \left(w_1, \mathfrak{m} \prod_{j=1}^k [\mathfrak{a}_j, \mathfrak{b}_j]\right) = 1}} \frac{\lambda_{\underline{\mathfrak{a}}} \lambda_{\underline{\mathfrak{b}}}}{\varphi \left(\mathfrak{m} \prod_{j \ne m} [\mathfrak{a}_j, \mathfrak{b}_j] \right)} \\
&+ \sum_{w_1 \in \mathcal{P}_1^0(Y', R')} \pi^\flat \left(\frac{N}{|w_1|^{1/2}} \right) \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{b}} \\ [\mathfrak{a}_m, \mathfrak{b}_m] = (w_1)}} \frac{\lambda_{\underline{\mathfrak{a}}} \lambda_{\underline{\mathfrak{b}}}}{\varphi \left(\mathfrak{m} \prod_{j \ne m} [\mathfrak{a}_j, \mathfrak{b}_j] \right)} + O\left(\lambda_{\max}^2 |A(N)|\right).
\end{aligned}$$

We define

$$S_{2m}(w_1) = \sum_{\substack{\underline{\mathfrak{a}}, \underline{\mathfrak{b}} \\ [\mathfrak{a}_m, \mathfrak{b}_m] | (w_1)}} \frac{\lambda_{\underline{\mathfrak{a}}} \lambda_{\underline{\mathfrak{b}}}}{\varphi \left(\mathfrak{m} \prod_{j \ne m} [\mathfrak{a}_j, \mathfrak{b}_j] \right)}. \tag{5.2}$$

The sum $S_{2m}(w_1)$ is estimated in the following lemma.

Lemma 13 *Let $S_{2m}(w_1)$ be defined as in (13). For ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_k$ of $\mathcal{O}_K$, we define*

$$y^{(m)}_{\mathfrak{r}_1, \ldots, \mathfrak{r}_k}(w_1) = \prod_{j \ne m} \mu(\mathfrak{r}_j) g(\mathfrak{r}_j) \sum_{\substack{\underline{\mathfrak{a}} \\ \mathfrak{r}_j | \mathfrak{a}_j \forall j \\ \mathfrak{a}_m | (w_1)}} \frac{\lambda_{\underline{\mathfrak{a}}}}{\prod_{j \ne m} \varphi(\mathfrak{a}_j)} \tag{5.3}$$

where g is the multiplicative function defined by $g(\mathfrak{p}) = |\mathfrak{p}| - 2$ for all prime ideals $\mathfrak{p}$ of A. Let $y^{(m)}_{\max}(w_1) = \sup_{\mathfrak{r}_1, \ldots, \mathfrak{r}_k} |y^{(m)}_{\mathfrak{r}_1, \ldots, \mathfrak{r}_k}(w_1)|$. Then we have

$$S_{2m}(w_1) = \frac{1}{\varphi(\mathfrak{m})} \sum_{\substack{\underline{\mathfrak{u}} \\ \mathfrak{u}_m = 1}} \prod_{j \ne m} \frac{\mu^2(\mathfrak{u}_j)}{g(\mathfrak{u}_j)} \left(y^{(m)}_{\underline{\mathfrak{u}}}(w_1) \right)^2 + O\left((y^{(m)}_{\max}(w_1))^2 (\log R)^{k-1} \frac{(\varphi(\mathfrak{m}))^{k-2}}{|\mathfrak{m}|^{k-1}} \frac{1}{D_0} \right).$$

Proof From the definition of g it follows that

$$\frac{1}{\varphi([\mathfrak{a}_i, \mathfrak{b}_i])} = \frac{1}{\varphi(\mathfrak{a}_i) \varphi(\mathfrak{b}_i)} \sum_{\mathfrak{u}_i | (\mathfrak{a}_i, \mathfrak{b}_i)} g(\mathfrak{u}_i).$$

From Eq. (5.2) we get,

$$S_{2m}(w_1) = \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}} \\ [\mathfrak{a}_m,\mathfrak{b}_m]|(w_1)}} \frac{\lambda_{\underline{\mathfrak{a}}}\lambda_{\underline{\mathfrak{b}}}}{\varphi(\mathfrak{m})} \prod_{j\neq m} \frac{1}{\varphi(\mathfrak{a}_j)\varphi(\mathfrak{b}_j)} \sum_{\mathfrak{u}_j|(\mathfrak{a}_j,\mathfrak{b}_j)} g(\mathfrak{u}_j)$$

$$= \frac{1}{\varphi(\mathfrak{m})} \sum_{\substack{\underline{\mathfrak{u}} \\ \mathfrak{u}_m=1}} \prod_{j\neq m} g(\mathfrak{u}_j) \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}} \\ [\mathfrak{a}_m,\mathfrak{b}_m]|(w_1) \\ \mathfrak{u}_j|(\mathfrak{a}_j,\mathfrak{b}_j)\forall j}} \frac{\lambda_{\underline{\mathfrak{a}}}\lambda_{\underline{\mathfrak{b}}}}{\prod_{j\neq m}\varphi(\mathfrak{a}_j)\varphi(\mathfrak{b}_j)}.$$

Note that $\lambda_{\mathfrak{a}_1,\ldots,\mathfrak{a}_k}$ is supported on ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_k$ with $(\mathfrak{a}_i,\mathfrak{m})=1$ for each i and $(\mathfrak{a}_i,\mathfrak{a}_j)=1\,\forall i\neq j$. Thus we may drop the requirement that $\mathfrak{m}$ is coprime to each of the $[\mathfrak{a}_i,\mathfrak{b}_i]$ from the summation, since these terms have no contribution. Thus the only remaining restriction is that $(\mathfrak{a}_i,\mathfrak{b}_j)=1\,\forall i\neq j$. So we can remove this coprimality condition by Möbius inversion to get

$$S_{2m}(w_1) = \frac{1}{\varphi(\mathfrak{m})} \sum_{\substack{\underline{\mathfrak{u}} \\ \mathfrak{u}_m=1}} \prod_{j\neq m} g(\mathfrak{u}_j) \sum_{\mathfrak{s}_{1,2},\ldots,\mathfrak{s}_{k,k-1}} \prod_{\substack{1\leq i,j\leq k \\ i\neq j}} \mu(\mathfrak{s}_{i,j}) \sum_{\substack{\underline{\mathfrak{a}},\underline{\mathfrak{b}} \\ [\mathfrak{a}_m,\mathfrak{b}_m]|(w_1) \\ \mathfrak{u}_j|(\mathfrak{a}_j,\mathfrak{b}_j)\forall j \\ \mathfrak{s}_{i,j}|(\mathfrak{a}_i,\mathfrak{b}_j),\forall i\neq j}} \frac{\lambda_{\underline{\mathfrak{a}}}\lambda_{\underline{\mathfrak{b}}}}{\prod_{j\neq m}\varphi(\mathfrak{a}_j)\varphi(\mathfrak{b}_j)}.$$

Now we make the following change of variables:

$$\mathfrak{c}_j = \mathfrak{u}_j \prod_{i\neq j} \mathfrak{s}_{i,j} \text{ and } \mathfrak{d}_j = \mathfrak{u}_j \prod_{i\neq j} \mathfrak{s}_{j,i}.$$

By using Eq. (5.3) we can rewrite $S_{2m}(w_1)$ as

$$S_{2m}(w_1) = \frac{1}{\varphi(\mathfrak{m})} \sum_{\substack{\underline{\mathfrak{u}} \\ \mathfrak{u}_m=1}} \prod_{j\neq m} \frac{\mu^2(\mathfrak{u}_j)}{g(\mathfrak{u}_j)} \sum_{\mathfrak{s}_{1,2},\ldots,\mathfrak{s}_{k,k-1}} \prod_{i\neq j} \frac{\mu(\mathfrak{s}_{i,j})}{g(\mathfrak{s}_{i,j})^2} y^{(m)}_{\mathfrak{c}_1,\ldots,\mathfrak{c}_k}(w_1) y^{(m)}_{\mathfrak{d}_1,\ldots,\mathfrak{d}_k}(w_1).$$

In the above sum $\mathfrak{s}_{i,j} \mid ([\mathfrak{a}_i,\mathfrak{b}_i],[\mathfrak{a}_j,\mathfrak{b}_j])$, hence $\mathfrak{s}_{i,j}$ is coprime to $\mathfrak{m}$ for all $i\neq j$. Then either $\mathfrak{s}_{i,j}=1$ or $|\mathfrak{s}_{i,j}|>D_0$. For a fixed i and j, the total contribution from the terms with $|\mathfrak{s}_{i,j}|>D_o$ is bounded above by

$$\frac{(y^{(m)}_{\max}(w_1))^2}{\varphi(\mathfrak{m})} \Bigg(\sum_{\substack{|\mathfrak{u}|<R \\ (\mathfrak{u},\mathfrak{m})=1}} \frac{\mu^2(\mathfrak{u})}{g(\mathfrak{u})} \Bigg)^{k-1} \Bigg(\sum_{\mathfrak{s}} \frac{\mu(\mathfrak{s})^2}{g(\mathfrak{s})^2} \Bigg)^{k(k-1)-1} \Bigg(\sum_{|\mathfrak{s}_{i,j}|>D_0} \frac{\mu(\mathfrak{s}_{i,j})^2}{g(\mathfrak{s}_{i,j})^2} \Bigg).$$

From Lemma 6 the above quantity is bounded above by

$$\frac{(y^{(m)}_{\max}(w_1))^2}{\varphi(\mathfrak{m})} (\log R)^{k-1} \left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\right)^{k-1} \frac{1}{D_0}.$$

The main term of $S_{2m}(w_1)$ is obtained from $\mathfrak{s}_{i,j}=1$ for all $i\neq j$ which completes the proof.

For ideals $\mathfrak{r}_1,\ldots,\mathfrak{r}_k$, we define

$$y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k} = \left(\prod_{i=1}^k \mu(\mathfrak{r}_i)\varphi(\mathfrak{r}_i)\right) \sum_{\substack{\mathfrak{a}_1,\ldots,\mathfrak{a}_k \\ \mathfrak{r}_i|\mathfrak{a}_i\,\forall i}} \frac{\lambda_{\mathfrak{a}_1,\ldots,\mathfrak{a}_k}}{\prod_{i=1}^k |\mathfrak{a}_i|}$$

and $y_{\max} = \sup_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k} |y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k}|$.

We recall the inversion formula from [9, Eq. (5.8)] that

$$\lambda_{\mathfrak{a}_1,\ldots,\mathfrak{a}_k} = \left(\prod_{i=1}^{k} \mu(\mathfrak{a}_i)|\mathfrak{a}_i|\right) \sum_{\substack{\mathfrak{r}_1,\ldots,\mathfrak{r}_k \\ \mathfrak{a}_i|\mathfrak{r}_i\,\forall i}} \frac{y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k}}{\prod_{i=1}^{k}\varphi(\mathfrak{r}_i)}. \tag{5.4}$$

Therefore $\lambda_{\max} \ll y_{\max}(\log R)^k$.
The following lemma gives a relation between the quantities $y^{(m)}_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k}(w_1)$ and $y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k}$.

Lemma 14 *If* $\mathfrak{u}_m = 1$ *(trivial ideal* $\mathcal{O}_K$ *), then*

$$y^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1) = \sum_{\substack{\mathfrak{r}_m \\ |\mathfrak{r}_m|\le R}} \frac{y_{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},\mathfrak{r}_m,\mathfrak{u}_{m+1},\ldots,\mathfrak{u}_k}}{\varphi(\mathfrak{r}_m)} - \frac{|(w_1)|}{\varphi((w_1))} \sum_{\substack{\mathfrak{s}_m \\ |\mathfrak{s}_m|\le R/|w_1|}} \frac{y_{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},(w_1)\mathfrak{s}_m,\ldots,\mathfrak{u}_k}}{\varphi(\mathfrak{s}_m)} + O\left(y_{max}\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\frac{\log(R/D_0)}{D_0}\right).$$

Proof Using (5.4), we get

$$\begin{aligned}
y^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1) &= \prod_{j\ne m}\mu(\mathfrak{u}_j)g(\mathfrak{u}_j) \sum_{\substack{\underline{\mathfrak{a}} \\ \mathfrak{u}_j|\mathfrak{a}_j\forall j \\ \mathfrak{a}_m|(w_1)}} \frac{1}{\prod_{j\ne m}\varphi(\mathfrak{a}_j)} \prod_{i=1}^{k}|\mathfrak{a}_i|\mu(\mathfrak{a}_i) \sum_{\substack{\underline{\mathfrak{r}} \\ \mathfrak{a}_i|\mathfrak{r}_i\forall i}} \frac{y_{\underline{\mathfrak{r}}}}{\prod_{i=1}^{k}\varphi(\mathfrak{r}_i)} \\
&= \prod_{j\ne m}\mu(\mathfrak{u}_j)g(\mathfrak{u}_j) \sum_{\substack{\underline{\mathfrak{r}} \\ \mathfrak{u}_j|\mathfrak{r}_j\forall j}} \frac{y_{\underline{\mathfrak{r}}}}{\prod_{i=1}^{k}\varphi(\mathfrak{r}_j)} \sum_{\substack{\underline{\mathfrak{a}}|\underline{\mathfrak{r}} \\ \mathfrak{u}_j|\mathfrak{a}_j\forall j \\ \mathfrak{a}_m|(w_1)}} \frac{\prod_{j=1}^{k}|\mathfrak{a}_j|\mu(\mathfrak{a}_j)}{\prod_{j\ne m}\varphi(\mathfrak{a}_j)} \\
&= \prod_{j\ne m}\mu(\mathfrak{u}_j)g(\mathfrak{u}_j) \sum_{\substack{\underline{\mathfrak{r}} \\ \mathfrak{u}_j|\mathfrak{r}_j\forall j}} \frac{y_{\underline{\mathfrak{r}}}}{\prod_{j=1}^{k}\varphi(\mathfrak{r}_j)} \prod_{j\ne m}\frac{|\mathfrak{u}_j|\mu(\mathfrak{r}_j)}{\varphi(\mathfrak{r}_j)}\left(1-|(w_1)|\mathbb{1}_{(w_1)|\mathfrak{r}_m}\right) \\
&= \prod_{j\ne m}\mu(\mathfrak{u}_j)g(\mathfrak{u}_j)\left(\sum_{\substack{\underline{\mathfrak{r}};\mathfrak{u}_j|\mathfrak{r}_j \\ \mathfrak{u}_m=1}} \frac{y_{\underline{\mathfrak{r}}}}{\prod_{i=1}^{k}\varphi(\mathfrak{r}_j)} \prod_{j\ne m}\frac{|\mathfrak{u}_j|\mu(\mathfrak{r}_j)}{\varphi(\mathfrak{r}_j)} - |(w_1)| \sum_{\substack{\underline{\mathfrak{r}};\mathfrak{u}_j|\mathfrak{r}_j \\ (w_1)|\mathfrak{r}_m}} \frac{y_{\underline{\mathfrak{r}}}}{\prod_j\varphi(\mathfrak{r}_j)} \prod_{j\ne m}\frac{|\mathfrak{u}_j|\mu(\mathfrak{r}_j)}{\varphi(\mathfrak{r}_j)}\right)
\end{aligned}$$

where $\mathbb{1}_{(w_1)|\mathfrak{r}_m}$ is the indicator function which takes value 1, if $(w_1) \mid \mathfrak{r}_m$ and 0 otherwise. We see from the support of $y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k}$ that we may restrict the summation over $\mathfrak{r}_j$ to $(\mathfrak{r}_j,\mathfrak{m}) = 1$. The main term is given by $\mathfrak{r}_j = \mathfrak{u}_j\forall j$, for all other terms there exists $j \ne m$ such that $|\mathfrak{r}_j| > D_0|\mathfrak{u}_j|$. Therefore the error term is bounded above by

$$y_{\max}\left(\prod_{j\ne m}|\mathfrak{u}_j|g(\mathfrak{u}_j)\right)\left(\sum_{\substack{\mathfrak{r}_j>D_0\mathfrak{u}_j \\ \mathfrak{u}_j|\mathfrak{r}_j}}\frac{\mu^2(\mathfrak{r}_j)}{\varphi(\mathfrak{r}_j)^2}\right)\left(|(w_1)|\sum_{\substack{|\mathfrak{r}_m|<R;(w_1)|\mathfrak{r}_m \\ (\mathfrak{r}_m,\mathfrak{m})=1}}\frac{\mu(\mathfrak{r}_m)^2}{\varphi(\mathfrak{r}_m)}\right)\prod_{\substack{1\le i\le k \\ i\ne j,m}}\left(\sum_{\mathfrak{u}_i|\mathfrak{r}_i}\frac{\mu(\mathfrak{r}_i)^2}{\varphi(\mathfrak{r}_i)^2}\right) \ll y_{\max}\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\frac{\log(R/D_0)}{D_0}.$$

The main term given by $\mathfrak{r}_j = \mathfrak{u}_j\forall j \ne m$ is

$$y^{(m)}_{\substack{\mathfrak{u}_1,\ldots,\mathfrak{u}_k \\ \mathfrak{u}_m=1}}(w_1) = \prod_{j\ne m}\frac{|\mathfrak{u}_j|g(\mathfrak{u}_j)}{\varphi(\mathfrak{u}_j)^2}\left(\sum_{\mathfrak{r}_m}\frac{y_{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},\mathfrak{r}_m,\mathfrak{u}_{m+1},\ldots,\mathfrak{u}_k}}{\varphi(\mathfrak{r}_m)} - |(w_1)|\sum_{\substack{\mathfrak{r}_m \\ (w_1)|\mathfrak{r}_m}}\frac{y_{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},\mathfrak{r}_m,\ldots,\mathfrak{u}_k}}{\varphi(\mathfrak{r}_m)}\right).$$

Now the proof can be completed by noting that $\frac{g(\mathfrak{p})|\mathfrak{p}|}{\varphi(\mathfrak{p})^2} = 1 + O(|\mathfrak{p}|^{-2})$ and $\mathfrak{r}_m = (w_1)\mathfrak{s}_m$.

6 Choosing the weights

For a real valued piecewise differentiable function F on $\mathcal{R}_k$ as in Proposition 2 we define

$$y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k} := \mu\left(\prod_{i=1}^{k}\mathfrak{r}_i\right)^2 F\left(\frac{\log|\mathfrak{r}_1|}{\log R},\ldots,\frac{\log|\mathfrak{r}_k|}{\log R}\right).$$

Note that, $y_{\mathfrak{r}_1,\ldots,\mathfrak{r}_k}$ is supported on square-free $\mathfrak{r}=\prod_{i=1}^{k}\mathfrak{r}_i$ such that $(\mathfrak{r},\mathfrak{m})=1$. Hence

$$\begin{aligned} y^{(m)}_{\substack{\mathfrak{u}_1,\ldots,\mathfrak{u}_k\\ \mathfrak{u}_m=1}}(w_1) = & \sum_{\substack{|\mathfrak{r}|\le R\\ \left(\mathfrak{r},\mathfrak{m}\prod_{j\neq m}\mathfrak{u}_j\right)=1}} \frac{\mu(\mathfrak{r})^2}{\varphi(\mathfrak{r})}F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,\frac{\log|\mathfrak{r}|}{\log R},\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)\\ & -\frac{|(w_1)|}{\varphi((w_1))}\sum_{\substack{|\mathfrak{s}|\le R/|w_1|\\ \left(\mathfrak{s},\mathfrak{m}(w_1)\prod_{j\neq m}\mathfrak{u}_j\right)=1}}\frac{\mu(\mathfrak{s})^2}{\varphi(\mathfrak{s})}F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,\frac{\log|\mathfrak{s}(w_1)|}{\log R},\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)\\ & + O\left(y_{max}\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\frac{\log R}{D_0}\right) =: S_1' + S_2' + O\left(y_{max}\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\frac{\log R}{D_0}\right). \end{aligned}$$

Estimation of S_1'. To use Lemma 1 we set

$$\gamma(\mathfrak{p}) = \begin{cases} 1 & \text{if } (\mathfrak{p},\mathfrak{m}\prod_{j\neq m}\mathfrak{u}_j)=1\\ 0 & \text{otherwise}\end{cases} \quad\text{and } h(\mathfrak{p}) = \frac{\gamma(\mathfrak{p})}{|\mathfrak{p}|-\gamma(\mathfrak{p})}.$$

and

$$F_{\max} = \sup_{t\in[0,1]}(|F(t)|+|F'(t)|).$$

The singular series in Lemma 1 is easily computed to be

$$\mathfrak{S} = \frac{\varphi(\mathfrak{m}\prod_{j\neq m}\mathfrak{u}_j)}{|\ \mathfrak{m}\prod_{j\neq m}\mathfrak{u}_j\ |}$$

and also $L \ll \log\log R$. Thus we get

$$\begin{aligned} S_1' = & \frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)}{|\mathfrak{u}_j|}c_K(\log R)\int_0^1 F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,t_m,\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)dt_m\\ & + O\left(F_{\max}\log\log R\right) \end{aligned}$$

where $c_K = Res_{s=1}\zeta_K(s)$.

Estimation of S_2'. Observe that

$$\frac{\log(|(w_1)\mathfrak{s}|)}{\log(R/|w_1|)} = \frac{\log R}{\log(R/|w_1|)}\left(\frac{\log|\mathfrak{s}|}{\log R}+\frac{\log|(w_1)|}{\log R}\right).$$

Therefore by Lemma 1, we get

$$\begin{aligned} S_2' = & -c_K\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)}{|\mathfrak{u}_j|}(\log R)\int_{\frac{\log|w_1|}{\log R}}^1 F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,s_m,\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)ds_m\\ & + O\left(F_{\max}\log\log R\right). \end{aligned}$$

Putting S_1' and S_2' together, we get

$$\begin{aligned} y^{(m)}_{\substack{\mathfrak{u}_1,\ldots,\mathfrak{u}_k\\ \mathfrak{u}_m=1}}(w_1) = & \frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}c_K\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)}{|\mathfrak{u}_j|}(\log R)\left(F^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k} - F^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1)\right)\\ & + O\left(F_{\max}\log\log R\right) + O\left(F_{\max}\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\frac{\log R}{D_0}\right) \end{aligned} \tag{6.1}$$

where

$$F^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k} = \int_0^1 F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,t_m,\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)dt_m$$

and

$$F^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1) = \int_{\frac{\log|w_1|}{\log R}}^1 F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,s_m,\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)ds_m.$$

7 Proof of Proposition 2

Using the value of $y^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1)$ given by Eq. (6.1) in Lemma 13 we get

$$S_{2m}(w_1) = \frac{1}{\varphi(\mathfrak{m})}\sum_{\substack{\underline{\mathfrak{u}}\\ \mathfrak{u}_m=1}}\prod_{j\neq m}\frac{\mu(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)}\left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}c_K\log R\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)}{|\mathfrak{u}_j|}\left(F^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}-F^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1)\right)\right)^2 + O\left((F_{\max})^2(\log R)^{k+1}\frac{(\varphi(\mathfrak{m}))^k}{|\mathfrak{m}|^{k+1}}\frac{1}{D_0}\right).$$

Setting $Y' := Y^{1/2}$ and the above equation in Lemma 12, we have

$$\begin{aligned}S_{2m} = {}& \frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|^2}c_K^2(\log R)^2\sum_{\substack{w_1\\ Y\le|w_1|\le R}}\pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right)\sum_{\substack{\underline{\mathfrak{u}}\\ \mathfrak{u}_m=1}}\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2}\left(\widetilde{F}^{(m)}_{\underline{\mathfrak{u}}}(w_1)\right)^2\\ &+\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|^2}c_K^2(\log R)^2\sum_{\substack{w_1\\ R<|w_1|\le N^{2b}}}\pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right)\sum_{\substack{\underline{\mathfrak{u}}\\ \mathfrak{u}_m=1}}\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2}\left(F^{(m)}_{\underline{\mathfrak{u}}}\right)^2\\ &+O\left((F_{\max})^2(\log R)^{k+1}\frac{(\varphi(\mathfrak{m}))^k}{|\mathfrak{m}|^{k+1}}\frac{1}{D_0}\sum_{w_1\in\mathcal{P}_1^0(Y',N^b)}\pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right)\right)+O\left(F_{\max}^2|A(N)|\right)\end{aligned}$$

where

$$\widetilde{F}^{(m)}_{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}(w_1) = \int_0^{\frac{\log|w_1|}{\log R}} F\left(\frac{\log|\mathfrak{u}_1|}{\log R},\ldots,t_m,\ldots,\frac{\log|\mathfrak{u}_k|}{\log R}\right)dt_m.$$

Using Lemma 4 we can say that $\pi^\flat\left(\frac{N}{|w_1|^{1/2}}\right) = |\mathcal{P}(N)|\frac{\alpha(|w_1|)}{|w_1|} + O_K\left(\frac{N^2}{|w_1|(\log N)^2}\right)$, where $\alpha(u) := \frac{\log(N^2)}{\log(N^2/u)}$.

Using this the main term of S_{2m} becomes

$$\begin{aligned}&= \frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|^2}c_K^2(\log R)^2|\mathcal{P}(N)|\sum_{\substack{w_1\\ Y\le|w_1|\le R}}\frac{\alpha(|w_1|)}{|w_1|}\sum_{\substack{\underline{\mathfrak{u}}\\ \mathfrak{u}_m=1}}\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2}\left(\widetilde{F}^{(m)}_{\underline{\mathfrak{u}}}(w_1)\right)^2\\ &+\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|^2}c_K^2(\log R)^2|\mathcal{P}(N)|\sum_{\substack{w_1\\ R<|w_1|\le N^{2b}}}\frac{\alpha(|w_1|)}{|w_1|}\sum_{\substack{\underline{\mathfrak{u}}\\ \mathfrak{u}_m=1}}\prod_{j\neq m}\frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2}\left(F^{(m)}_{\underline{\mathfrak{u}}}\right)^2\\ &=: \frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|^2}c_K^2(\log R)^2|\mathcal{P}(N)|\left(\sum_{\substack{w_1\\ Y\le|w_1|\le R}}\frac{\alpha(|w_1|)}{|w_1|}S_4+\sum_{\substack{w_1\\ R<|w_1|\le N^{2b}}}\frac{\alpha(|w_1|)}{|w_1|}S_5\right).\end{aligned}$$

Estimation of S_4 and S_5. To calculate both S_4 and S_5 we use Lemma 1 with

$$h(\mathfrak{p}) = \frac{\gamma(\mathfrak{p})}{|\mathfrak{p}|-\gamma(\mathfrak{p})} \text{ and } \gamma(\mathfrak{p}) = \begin{cases}1-\frac{|\mathfrak{p}|^2-3|\mathfrak{p}|+1}{|\mathfrak{p}|^3-|\mathfrak{p}|^2-2|\mathfrak{p}|+1} & \text{if } (\mathfrak{p},\mathfrak{m})=1\\ 0 & \text{otherwise.}\end{cases}$$

The singular series can be easily computed to be $\mathfrak{S} = \frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|} + O\left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|D_0}\right)$ and also $L \ll \log D_0$. Recalling the coprimality conditions S_4 can be written as

$$S_4 = \sum_{\substack{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},\mathfrak{u}_{m+1},\ldots,\mathfrak{u}_k \\ (\mathfrak{u}_i,\mathfrak{u}_j)=1\forall i\neq j \\ (\mathfrak{u}_j,\mathfrak{m})=1}} \prod_{j\neq m} \frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2} \left(\widetilde{F}^{(m)}_{\underline{\mathfrak{u}}}(w_1)\right)^2.$$

We note that, two ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $(\mathfrak{a},\mathfrak{m}) = (\mathfrak{b},\mathfrak{m}) = 1$ but $(\mathfrak{a},\mathfrak{b}) \neq 1$ must have a common prime factor with norm greater than D_0. Thus we can drop the requirement that $(\mathfrak{u}_i,\mathfrak{u}_j) = 1$, at the cost of an error of size

$$\ll F_{\max}^2 \sum_{|\mathfrak{p}|>D_0} \sum_{\substack{|\mathfrak{u}_1|,\ldots,|\mathfrak{u}_{m-1}|,|\mathfrak{u}_{m+1}|,\ldots,|\mathfrak{u}_k|<R \\ \mathfrak{p}|\mathfrak{u}_i,\mathfrak{u}_j\forall i\neq j \\ (\mathfrak{u}_j,\mathfrak{m})=1}} \prod_{j\neq m} \frac{\mu(\mathfrak{u}_j)^2\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2}$$

$$\ll F_{\max}^2 \Bigg(\sum_{|\mathfrak{p}|>D_0} \frac{\varphi(\mathfrak{p})^4}{g(\mathfrak{p})^2|\mathfrak{p}|^4}\Bigg)\Bigg(\sum_{\substack{|\mathfrak{r}|<R \\ (\mathfrak{r},\mathfrak{m})=1}} \frac{\mu(\mathfrak{r})^2\varphi(\mathfrak{r})^2}{g(\mathfrak{r})|\mathfrak{r}|^2}\Bigg)^{k-1} \ll F_{\max}^2 \left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\right)^{k-1} \frac{(\log R)^{k-1}}{D_0}.$$

Thus it is enough to evaluate the following sums

$$\sum_{\substack{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},\mathfrak{u}_{m+1},\ldots,\mathfrak{u}_k \\ (\mathfrak{u}_j,\mathfrak{m})=1}} \prod_{j\neq m} \frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2} \left(\widetilde{F}^{(m)}_{\underline{\mathfrak{u}}}(w_1)\right)^2 \text{ and } \sum_{\substack{\mathfrak{u}_1,\ldots,\mathfrak{u}_{m-1},\mathfrak{u}_{m+1},\ldots,\mathfrak{u}_k \\ (\mathfrak{u}_j,\mathfrak{m})=1}} \prod_{j\neq m} \frac{\varphi(\mathfrak{u}_j)^2}{g(\mathfrak{u}_j)|\mathfrak{u}_j|^2} \left(F^{(m)}_{\underline{\mathfrak{u}}}\right)^2.$$

Using Lemma 1 we have,

$$S_4 = \left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\right)^{k-1} c_K^{k-1}(\log R)^{k-1} I^{(m)}_{3k}(F(w_1)) + O\left(F_{\max}^2 \left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\right)^{k-1} \frac{1}{D_0}(\log R)^{k-1}\right)$$

and

$$S_5 = \left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\right)^{k-1} c_K^{k-1}(\log R)^{k-1} I^{(m)}_{2k}(F) + O\left(F_{\max}^2 \left(\frac{\varphi(\mathfrak{m})}{|\mathfrak{m}|}\right)^{k-1} \frac{1}{D_0}(\log R)^{k-1}\right)$$

where

$$I^{(m)}_{3k}(F(w_1)) = \int\cdots\int_{\mathcal{R}_{k-1}} \left(\int_0^{T_m\left(\frac{\log|w_1|}{\log R}\right)} F(x_1,\ldots,x_k)\,dx_m\right)^2 dx_1\ldots dx_{m-1}dx_{m+1}\ldots dx_k$$

and

$$I^{(m)}_{2k}(F) = \int\cdots\int_{\mathcal{R}_{k-1}} \left(\int_0^{T_m} F(x_1,\ldots,x_k)\,dx_m\right)^2 dx_1\ldots dx_{m-1}dx_{m+1}\ldots dx_k$$

where T_m and $T_m(y)$ are as defined in the statement of Proposition 2. We note that the integral $I^{(m)}_{2k}(F)$ is independent of prime element w_1 of $\mathcal{O}_K$. Using the estimations of the sums S_4 and S_5 the term S_{2m} becomes

$$S_{2m} = \frac{\varphi(\mathfrak{m})^k}{|\mathfrak{m}|^{k+1}}(c_K \log R)^{k+1}|\mathcal{P}(N)|\left(\sum_{\substack{w_1 \\ Y \le |w_1| \le R}} \frac{\alpha(|w_1|)}{|w_1|} I_{3k}^{(m)}(F(w_1)) + I_{2k}^{(m)}(F) \sum_{\substack{w_1 \\ R < |w_1| \le N^{2b}}} \frac{\alpha(|w_1|)}{|w_1|}\right) + O\left((F_{\max})^2 (\log R)^{k+1} \frac{\varphi(\mathfrak{m})^k}{|\mathfrak{m}|^{k+1}} \frac{1}{D_0} \sum_{Y < |w_1| \le N^{2b}} \pi^{\flat}\left(\frac{N}{|w_1|^{1/2}}\right)\right).$$

Finally it remains to calculate the following sums

$$S_6 := \sum_{\substack{w_1 \\ R < |w_1| \le N^{2b}}} \frac{\alpha(|w_1|)}{|w_1|} \text{ and } S_7 := \sum_{\substack{w_1 \\ Y \le |w_1| \le R}} \frac{\alpha(|w_1|)}{|w_1|} \left(V^{(m)}\left(\frac{\log |w_1|}{\log R}\right)\right)^2$$

where $V^{(m)}(y) := \int_0^y F(x_1, \ldots, x_k) dx_m$.
Using Lemma 3, we have

$$\sum_{w \in \mathcal{P}^0(u^{1/2})} \log |w| = m_K u + E(u) \text{ where } E(u) = O_K\left(\frac{u}{\log u}\right)$$

where $m_K = \frac{\omega_K}{h_K}$. From the above estimations, we get

$$S_7 = m_K \int_Y^R \alpha(u) \left(V^{(m)}\left(\frac{\log u}{\log R}\right)\right)^2 \frac{du}{u \log u} + \int_Y^R \alpha(u) \left(V^{(m)}\left(\frac{\log u}{\log R}\right)\right)^2 \frac{dE(u)}{u \log u}$$
$$:= S_8 + S_9.$$

Putting $u = R^y$, first integral S_8 gives main term for S_7 which is

$$S_8 = m_K \int_{B'\eta}^1 \frac{B'}{y(B' - y)} \left(V^{(m)}(y)\right)^2 dy$$

where

$$\eta = \frac{\log Y}{\log(N^2)}, \quad B' = \frac{\log(N^2)}{\log R}.$$

Second integral S_9 giving error term of S_7 can be estimated as

$$S_9 \ll (\log R)^{-1}.$$

Since $R = N^{\vartheta}(\log N)^{-C}$ we observe that

$$B' = B + O\left(\frac{\log \log N}{\log N}\right)$$

where $B = \frac{2}{\vartheta}$ as defined in Proposition 2. Therefore combining these estimations we have

$$S_7 = m_K \int_{B\eta}^1 \frac{B}{y(B - y)} \left(V^{(m)}(y)\right)^2 dy + O\left(\frac{\log \log N}{\log N}\right).$$

By using the same method we have

$$S_6 = m_K \int_1^{B/2} \frac{B}{y(B - y)} dy + O\left(\frac{\log \log N}{\log N}\right).$$

Therefore we conclude that

$$S_{2m} = m_K \frac{\varphi(\mathfrak{m})^k}{|\mathfrak{m}|^{k+1}} (c_K \log R)^{k+1} |\mathcal{P}(N)| \left(\tilde{I}_{2k}^{(m)}(F) + \tilde{I}_{3k}^{(m)}(F) \right) + O\left(F_{\max}^2 (\log R)^{k+1} \frac{(\varphi(\mathfrak{m}))^k}{|\mathfrak{m}|^{k+1}} \frac{1}{D_0} |\mathcal{P}(N)| \right). \tag{7.1}$$

Recall that

$$S_2 = \sum_{1 \le m \le k} S_{2m}.$$

Therefore Proposition 2 follows from 7.1.

8 proof of theorem 1, corollary 1 and corollary 2

We start with the following corollary of the Proposition 2.

Corollary 3 *Let K be an imaginary quadratic number field and m_K be its Mitsui constant. Suppose that the primes $\mathcal{P}$ and G_2^K-numbers have a common level of distribution $0 < \vartheta \le 1$. Let $(\mathfrak{h}_1, \ldots, \mathfrak{h}_k) \in \mathcal{O}_K^k$ be an admissible tuple. Let $B, \tilde{I}_{1k}(F), \tilde{I}_{2k}^{(m)}(F)$ and $\tilde{I}_{3k}^{(m)}(F)$, $1 \le m \le k$ be defined as in the statement of Proposition 2. Let $\mathcal{S}_k$ denote the set of piecewise differentiable functions $F : [0,1] \to \mathbb{R}$ supported on $\mathcal{R}_k$ such that $\tilde{I}_{1k}(F), \tilde{I}_{2k}^{(m)}(F)$ and $\tilde{I}_{3k}^{(m)}(F)$, are non-zero for all m in $1 \le m \le k$. Let*

$$\tilde{M}_k := \sup_{F \in \mathcal{S}_k} \frac{\sum_{m=1}^k (\tilde{I}_{2k}^{(m)}(F) + \tilde{I}_{3k}^{(m)}(F))}{\tilde{I}_{1k}(F)} \quad \text{and} \quad \tilde{r}_k := \left\lceil \frac{m_K \tilde{M}_k}{B} \right\rceil$$

Then there are infinitely many $\alpha \in \mathcal{O}_K$ such that at least $\tilde{r}_k$ of the $\alpha + \mathfrak{h}_1, \ldots, \alpha + \mathfrak{h}_k$ are G_2^K-numbers.

Proof Since each summand is non-negative, if $S := S_2 - \rho S_1 > 0$ for some positive ρ, then there is an $\alpha \in A(N)$ such that $\alpha + \mathfrak{h}_1, \ldots, \alpha + \mathfrak{h}_k$ contains atleast $[\rho] + 1$ G_2^K-numbers. Therefore it is enough to show that $S > 0$ for all sufficiently large N.

Fix a $\delta > 0$ and $0 < \epsilon < \frac{\delta B}{m_K}$, then choose $\tilde{F} \in \mathcal{S}_K$ so that

$$\sum_{m=1}^{k} (\tilde{I}_{2k}^{(m)}(\tilde{F}) + \tilde{I}_{3k}^{(m)}(\tilde{F})) > (\tilde{M}_k - \epsilon) \tilde{I}_{1k}(\tilde{F}).$$

Using Proposition 2, we obtain

$$\begin{aligned} S &= (1 + o(1)) \frac{(\varphi(\mathfrak{m})^k |A(N)| (c_K \log R)^k}{|\mathfrak{m}|^{k+1}} \left(\frac{m_K}{B} \sum_{m=1}^{k} (\tilde{I}_{2k}^{(m)}(F) + \tilde{I}_{3k}^{(m)}(F)) - \rho \tilde{I}_{1k}(F) \right) \\ &\ge \frac{(\varphi(\mathfrak{m})^k |A(N)| (c_K \log R)^k}{|\mathfrak{m}|^{k+1}} \tilde{I}_{1k}(\tilde{F}) \left(\frac{m_K}{B} (\tilde{M}_k - \epsilon) - \rho \right). \end{aligned}$$

If $\rho = \frac{m_K \tilde{M}_k}{B} - \delta$, then $S > 0$ for large N. Since δ is arbitrary, there are infinitely many $\alpha \in \mathcal{O}_K$ such that at least $\left\lceil \frac{m_K \tilde{M}_K}{B} \right\rceil$ of the $\alpha + \mathfrak{h}_1, \ldots, \alpha + \mathfrak{h}_k$ are G_2^K-numbers.

To complete the proof of the Theorem 1 it is enough to show that $\tilde{r}_k \to \infty$ as $k \to \infty$. Since the integrals $\tilde{I}_{3k}^{(m)}(F)$ are positive

$$\tilde{M}_k \geq \sup_{F \in \mathcal{S}_k} \frac{\sum_{m=1}^{k} \tilde{I}_{2k}^{(m)}(F)}{\tilde{I}_{1k}(F)}$$

It follows from Sect. 7 of [9] that

$$\sup_{F \in \mathcal{S}_k} \frac{\sum_{m=1}^{k} \tilde{I}_{2k}^{(m)}(F)}{\tilde{I}_{1k}(F)} > c \log k$$

for sufficiently large k and an absolute constant $c > 0$ (note that $\tilde{I}_{2k}^{(m)}(F)$ is a positive constant multiple of $J_k^{(m)}(F)$ in [9]). This completes the proof as $\tilde{r}_k$ is directly proportional to $\tilde{M}_k$.

Remark 3 Comparing the integral $\tilde{I}_{2k}^{(m)}(F)$ with $J_k^{(m)}(F)$ as in [9], we can show that $\tilde{M}_k \geq (1.0986)M_k$ where M_k is as in [9]. Since $\omega_K = 2$ for fields with class number more than 2, From Proposition 4.3 of [9], it follows that

$$\tilde{r}_k > \frac{1.0986}{2h_K}(\log k - 2\log\log k - 2)$$

for sufficiently large k. We conclude that there exist infinitely many $\alpha \in \mathcal{O}_K$ such that for any admissible k-tuple $(\mathfrak{h}_1, \cdots, \mathfrak{h}_k)$ there is atleast one G_2^K-number among $\alpha + \mathfrak{h}_1, \cdots, \alpha + \mathfrak{h}_k$ (i.e $\tilde{r}_k \geq 1$) provided

$$\log k - 2\log\log k \geq (1.82)h_K + 2$$

and in that case the gap is bounded above by $\mathfrak{h}_k - \mathfrak{h}_1$ where $(\mathfrak{h}_1, \cdots, \mathfrak{h}_k)$ is an admissible k-tuple. Therefore gaps between G_2^K-numbers are bounded in terms of class numbers.

To prove the Corollary 1 stated in Sect. 1, we need following lemmas .

Lemma 15 *(Proposition 3.1, [1]) Suppose that $\mathcal{H}$ is an admissible tuple in $\mathbb{Z}$. Then $\mathcal{H}$ is also an admissible tuple in $\mathcal{O}_K$ for every number field K.*

Using Proposition 1, we obtain the following lemma.

Lemma 16 *(Corollary 1.4, [2]) Let K be an imaginary quadratic field. Then product of two primes in $\mathcal{O}_K$ have level of distribution $\frac{1}{2}$.*

Proof of Corollary 1 Recall that

$$S = S_2 - \rho S_1 \tag{8.1}$$

where S_2 and S_1 are defined as in Proposition 2. We choose $F(t_1, \ldots, t_k)$ to be a symmetric polynomials in $t_1, \ldots, t_k$. By Proposition 2, we see that

$$S = (1 + o(1))\frac{(\varphi(\mathfrak{m})^k |A(N)| (c_K \log R)^k}{|\mathfrak{m}|^{k+1}} \widetilde{I}$$

where

$$\widetilde{I} = \frac{m_K k}{B}\left(\widetilde{I}_{2k}^{(1)}(F) + \widetilde{I}_{3k}^{(1)}(F)\right) - \rho \widetilde{I}_{1k}(F).$$

We know that

$$\omega_{K_d} = \begin{cases} 4 & \text{if } d = -1 \\ 6 & \text{if } d = -3 \\ 2 & \text{otherwise.} \end{cases}$$

Therefore Mitsui's constant for imaginary quadratic number fields of class number one are listed below:

$$m_{K_d} = \begin{cases} 4 & \text{if } d = -1 \\ 6 & \text{if } d = -3 \\ 2 & \text{otherwise.} \end{cases}$$

For Corollary 1, we take $k = 2, \vartheta = \frac{1}{2}, \rho = 1, \eta = \frac{1}{200}, F(t_1, t_2) = 1 - F_1(t_1, t_2) + F_2(t_1, t_2)$, where $F_1(t_1, t_2) = t_1 + t_2$ and $F_2(t_1, t_2) = t_1^2 + t_2^2$.
Using SageMath we obtain

$$\widetilde{I}_{12}(F) = 0.227778, \widetilde{I}_{22}^{(1)}(F) = 0.169151, \widetilde{I}_{32}^{(1)}(F) = 0.150712.$$

Therefore we have $\widetilde{I} > 0$. Hence Corollary 1 follows from Lemma 15 considering the admissible set $\{0, 2\}$ and invoking Lemma 16.

Proof of Corollary 2 For imaginary quadratic number field K_d of class number two

$$\omega_{K_d} = 2 \quad \text{and} \quad m_{K_d} = 1.$$

For Corollary 2, we choose $k = 4, \vartheta = \frac{1}{2}, \rho = 1, \eta = \frac{1}{200}, F(t_1, t_2, t_3, t_4) = 1 - F_1(t_1, t_2, t_3, t_4) + F_2(t_1, t_2, t_3, t_4)$, where $F_1(t_1, t_2, t_3, t_4) = t_1 + t_2 + t_3 + t_4$ and $F_2(t_1, t_2, t_3, t_4) = t_1^2 + t_2^2 + t_3^2 + t_4^2$.
Using SageMath we obtain

$$\widetilde{I}_{14}(F) = 0.0095238, \widetilde{I}_{24}^{(1)}(F) = 0.0044928, \widetilde{I}_{34}^{(1)}(F) = 0.0059492.$$

Therefore we have $\widetilde{I} > 0$. Hence Corollary 2 follows from Lemma 15 considering the admissible set $\{0, 2, 6, 8\}$ and using Lemma 16.

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