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Eiolf Kaspersen

On the Thom Morphism for Lie Groups

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Outline of the thesis

This thesis consists of an introductory chapter and the following papers. The introduction provides background and context for the papers and a summary of their contents.

Paper I
On the cokernel of the Thom morphism for compact Lie groups
Eiolf Kaspersen and Gereon Quick
To appear in *Homology, Homotopy and Applications*

Paper II
Geometric cohomology classes for the Lie groups Spin(7) and Spin(8)
Eiolf Kaspersen and Gereon Quick
Preprint

Paper III
A note on the Thom morphism for the classifying space of certain Lie groups and gauge groups
Eiolf Kaspersen and Gereon Quick
Preprint

Introduction

Algebraic Topology and Steenrod's Problem

A fundamental goal in algebraic topology is to study geometric/topological objects using methods from algebra. Some basic examples include singular homology and cohomology, the fundamental group and higher homotopy groups, both stable and unstable. These methods all allow us to associate an algebraic structure to a topological space, thus enabling us to use tools from algebra to understand the properties of these spaces.

A classical question which displays the relationship between the objects involved in such methods is Steenrod's problem [12]: Which elements in the homology of a space X can be represented by manifolds? More concretely, let $\alpha \in H_n(X)$. We say that α is represented by a manifold if there exists some compact, connected, oriented n-manifold M and a continuous map $f\colon M \longrightarrow X$ such that $f_*(\iota) = \alpha$, where $\iota \in H_n(M)$ is the fundamental class of M. Constructing a homology class using a manifold in this way can lead to an increased understanding of the underlying geometry of various phenomena in homology, so one could hope that this were possible for all homology classes. While it has been shown that this is not the case, there are several known cases where it is possible. For example, Thom showed that all elements in $H_n(X)$ can be represented by manifolds for $n \leq 6$, but that nonrepresentable elements exist in all other degrees [24].

We may ask a more specific question by imposing restrictions on which manifolds M we can use. Such a subclass is the *stably almost complex manifolds*, meaning smooth, oriented manifolds M such that the stable tangent bundle $T(M) \oplus \mathbb{R}^N$ has a complex structure. By Quillen's work in [22], the complex bordism of a space X can be defined as the free abelian group over the set of continuous maps $M \longrightarrow X$ modulo an equivalence relation, where M is stably almost complex. One can then ask which elements of $H_*(X;\mathbb{Z})$ can be represented by such maps.

Generalised Cohomology and the Thom Morphism

Some of the most important tools for studying topological spaces are homology and cohomology. Originally defined in their current form by Eilenberg in [11], these theories produce a sequence of groups (or in the case of singular cohomology, a ring) for every topological space. In [13], Eilenberg and Steenrod presented a list of axioms for (co)homology theories. For CW-complexes, the only homology theories which satisfy all of these axioms are singular homology with different coefficient groups. However, by removing the dimension axiom from this list, one is able to define a wide range of (co)homology theories. Throughout the latter half of the 20th century, major advances were made on such generalised homology theories. Some examples are the work on K-theory by Atiyah, Hirzebruch and Bott in [3], [4] and [6], as well as the development of the various bordism theories in [24], [1] and [14].

From now on, we focus on cohomology. Once we can apply several different cohomology theories to a space X, a natural question to ask is what relationships the various cohomology groups of X have to each other. Thus, it is useful to study maps between these cohomology groups. Early examples of this go back to Thom's work in [24] and Conner and Floyd's work on a morphism from cobordism to K-theory [8]. This leads to the construction of the Thom morphism, which is the main topic of study in this thesis.

The Thom morphism is a map between complex cobordism and singular cohomology, or more precisely, for every CW complex X it is a ring homomorphism $\tau\colon MU^*(X) \longrightarrow H^*(X;\mathbb{Z})$, where we sometimes replace $\mathbb{Z}$ by $\mathbb{Z}/p$ for some prime p. There are multiple equivalent ways of defining this morphism. Firstly, using Brown's representability theorem [7], it can be constructed by a map of spectra $MU \longrightarrow H\mathbb{Z}$. Alternatively, it can be defined as the edge map for the top row of the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(X; MU^q(\mathrm{pt})) \Rightarrow MU^{p+q}(X),$$

see [25]. With this definition, the Thom morphism can be studied from an algebraic point of view by examining the differentials in the spectral sequence. Finally, using Quillen's description of complex cobordism, we have a geometric interpretation of the Thom morphism. Suppose that X is a compact, oriented manifold of dimension n. Then $MU^k(X)$ is defined as the group of cobordism classes of complex oriented maps $M \longrightarrow X$, where M is a manifold of dimension $n-k$. Let f represent an element of $MU^k(X)$. Then, as explained previously, f induces a homology class $\alpha_f \in H_{n-k}(X;\mathbb{Z})$. We can then define $\tau([f])$ to be the element of $H^k(X;\mathbb{Z})$ which corresponds to α_f under the Poincaré duality isomorphism.

The Thom morphism is a particularly important map in the study of complex oriented cohomology theories due to how singular cohomology and complex cobordism relate to other such cohomology theories. One way to see this is by using the fact that every complex oriented cohomology theory is characterised by its associated formal group law. While singular cohomology corresponds to the *additive formal group law* (which can informally be called the simplest formal group law), MU corresponds to the *universal formal group law*. Therefore, the Thom morphism can be said to connect the opposite ends of the world of complex oriented cohomology theories.

In general, the Thom morphism has a large kernel. In fact, there is an ideal denoted by $MU^{*<0} \cdot MU^*(X)$ which is always contained in $\operatorname{Ker}\tau$. However, even after dividing out by this ideal, the Thom morphism is not necessarily injective. Furthermore, it is not surjective in general, as evidenced by Thom's negative result to Steenrod's problem. Thus, both the kernel and cokernel of the Thom morphism are interesting objects to study, see for example [9], [10], [17], [23], [25].

A consistent goal in this thesis is to use algebraic obstructions to find cases where the Thom morphism is not surjective. Then, we show that certain cohomology classes are in the image of the Thom morphism by constructing elements of $MU^*(X)$ using manifolds. In this way, we combine methods from algebra and topology to gain a deeper insight into the relationship between different cohomology theories.

Differential Refinement of the Thom Morphism

While the Thom morphism in classical stable homotopy theory is a well studied object, much less is known for its differential refinement induced by the work of Hopkins and Singer in [15]. The idea for differential refinements of topological invariants goes at least back to the work of Chern–Simons and Deligne. Such refinements often allow us to define non-trivial secondary invariants on manifolds that are not detected by a purely topological theory. Roughly speaking, the idea for the construction is based on the fact that, on a smooth manifold X, the de Rham isomorphism provides an interpretation of classes in singular cohomology with coefficients in $\mathbb{R}$ via differential forms. The latter contain information about the geometry of X. In cases where a singular cohomology class vanishes, this geometric information may still be detected. Oftentimes when a class in $H^q(X;\mathbb{Z})$ vanishes, differential cohomology induces a non-trivial secondary invariant detected in the group $H^{q-1}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$.

In [15] Hopkins and Singer extended the construction of differential refinements to generalized cohomology theories via homotopy theory. For a rationally even spectrum E and a smooth manifold X, let $\check{E}(q)^q(X)$ denote the differential E-cohomology group of X in degree q. Since the natural homomorphism $\check{E}(q)^q(X) \to E^q(X)$ is surjective, the Thom morphism $\tau\colon MU \to H\mathbb{Z}$ induces a commutative diagram

$$\begin{tikzcd} \check{MU}(q)^q(X) \arrow[r] \arrow[d, "\check{\tau}"'] & MU^q(X) \arrow[d, "\tau"] \\ \check{H}(q)^q(X) \arrow[r] & H^q(X;\mathbb{Z}) \end{tikzcd}$$

where both horizontal maps are *surjective*. It follows that if τ is not surjective, then $\check{\tau}$ fails to be surjective as well. On the other hand, the Thom morphism induces a commutative diagram

$$\begin{tikzcd} MU^{q-1}(X) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \arrow[r] \arrow[d, "\tau_{\mathbb{R}/\mathbb{Z}}"] & \check{MU}(q)^q(X) \arrow[d, "\check{\tau}"'] \\ H^{q-1}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \arrow[r] & \check{H}(q)^q(X) \end{tikzcd}$$

in which the horizontal maps are *injective*. Since the kernel of the Thom morphism always contains the ideal $MU^{*<0} \cdot MU^*(X)$, we say that an element in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ or $\check{\tau}$ is *non-trivial* if it is not contained in the respective ideal generated by $MU^{*<0}$. Roughly speaking, the non-trivial kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ detects classes which are not seen by differential cohomology. Providing concrete examples of such classes is therefore highly desirable. It turns out that the failure of the surjectivity of τ also allows us to find non-trivial elements in the kernel of $\check{\tau}$.

Lie Groups and Gauge Groups

There are many different classes of spaces that we may want to apply these methods to. Since differential cohomology theories require smooth manifolds as input, the Lie groups are a natural class to study. While every method mentioned so far can be applied to these spaces, the Lie groups can also themselves be said to exist within the intersection of topology and algebra.

A Lie group is defined as a smooth manifold G with a group structure such that the multiplication map $G \times G \longrightarrow G$ and the inverse map $g \longmapsto g^{-1}$ for $g \in G$ are smooth. These groups can be used to describe symmetries in several important spaces, e.g. $O(n)$ for real Euclidean space and G_2 for the octonions. These connections between the Lie groups and symmetries make the Lie groups highly applicable within various areas of mathematics such as Riemannian geometry, see for example [18]. Furthermore, they have many applications within mathematical physics. An example is how the spin groups serve as a model for the fermions [16].

The properties of a Lie group G are strongly linked to those of its associated Lie algebra. The Lie algebra can be defined as the tangent space of G at the identity element, while the exponential map is a map from the Lie algebra back to G. Therefore, a deep understanding of the various Lie algebras is useful for studying the Lie groups.

However, classifying the Lie groups using a classification of Lie algebras is not as straight forward as one might think. For example, the symplectic group $\mathrm{Sp}(n)$ corresponds to the simple Lie algebra $\mathfrak{c}_\mathfrak{n}$. Since the centre of $\mathrm{Sp}(n)$ is isomorphic to $\mathbb{Z}/2$ and thus nontrivial, we can quotient out by this subgroup to obtain the group $\mathrm{PSp}(n)$, known as the projective symplectic group. This also corresponds to the Lie algebra $\mathfrak{c}_\mathfrak{n}$. In fact, any quotient by a subgroup of the centre of a Lie group yields a quotient group with the same Lie algebra. See [21] for more information on this phenomenon. Since the quotient by a discrete subgroup can have a significant impact on the amount of torsion in the homology of a topological group, it follows that Lie groups with the same Lie algebra can have different topological properties. It is therefore interesting to determine when the Thom morphism acts differently on Lie groups with the same Lie algebra.

Since a Lie group G is defined as both an object in algebra and topology, it can be studied using methods from both disciplines. Considering its structure as a smooth manifold, it can be endowed with additional topological structures such as a tangent bundle. Furthermore, any homology or cohomology theory can be applied to it to study its properties. However, since G is also a group, we can construct a classifying space BG, which can again be studied as a topological space. Not only do these properties make the Lie groups into interesting objects of study, they also demonstrate the power of combining methods from topology and algebra. This leads to many interesting questions, such as which properties the cohomology of G and BG have in common. The singular cohomology of these spaces were studied extensively in [19].

The classifying space BG is also useful for classifying the possible principal G-bundles. More concretely, let ξ be the fibre bundle

$$G \hookrightarrow P \xrightarrow{\pi} X.$$

Then there exists a unique (up to homotopy) map $f\colon X \longrightarrow BG$ such that ξ is the pullback bundle

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi"] & EG \arrow[d] \\ X \arrow[r, "f"] & BG. \end{tikzcd}$$

Principal G-bundles have a wide range of applications, such as studying the properties of the space P using the spaces G and X. Therefore, one may ask further questions such as which automorphisms exist for a principal G-bundle ξ. In fact, the composition of automorphisms makes the set

$$\mathcal{G}(\xi) = \{\varphi \in \mathrm{Aut}_G(P) \mid \pi \circ \varphi = \pi\}$$

into a group, known as the *gauge group of* ξ. Being a group, $\mathcal{G}(\xi)$ itself admits a classifying space $B\mathcal{G}(\xi)$, which is homotopy equivalent to the moduli space of connections on the bundle ξ [20]. Thus, the gauge group is a useful tool for studying principal G-bundles, hence leading to further applications within topology and other areas of mathematics, see for example [2]. In addition to the Lie groups and their classifying spaces, the properties of the Thom morphism for the spaces $B\mathcal{G}(\xi)$ will therefore be studied in this thesis.

Paper I: On the Cokernel of the Thom Morphism for Compact Lie Groups

We determine whether the integral Thom morphism is surjective for all compact, connected Lie groups which correspond to a simple Lie algebra. This includes both classical and exceptional Lie groups. In the cases where the Thom morphism is not surjective, we provide an element which is not in $\mathrm{Im}\,\tau$ in the lowest possible cohomological degree. The method we use to show that a class $\alpha \in H^*(X;\mathbb{Z})$ is not in the image of τ can be summarised as follows:

1. Choose a prime p (usually $p = 2$).
2. Examine the mod p Bockstein homomorphisms to determine the image of α under $r\colon H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}/p)$
3. Use Steenrod's power operations on $r(\alpha)$ as an obstruction to lifting α to $MU^*(X)$.

For some of the groups, the Thom morphism is surjective. In those cases, we show that all the differentials in the Atiyah–Hirzebruch spectral sequence are trivial. By the interpretation of the Thom morphism as an edge map in the spectral sequence above, it then follows that the Thom morphism is surjective. In the torsion free cases $SU(n)$ and $Sp(n)$, this follows straight forwardly from the fact that all differentials in the Atiyah–Hirzebruch spectral sequence are torsion. However, it turns out that there exist quotients of these groups which have torsion in their cohomology, but where the Thom morphism is still surjective. For these groups, we once again choose a prime p and determine the image of the map r. We then check that all possible mod p cohomology operations are trivial for $\operatorname{Im} r$.

While partial results exist for classifying spaces of Lie groups [25], [5] as well as for the Lie groups themselves in mod p-cohomology [26], [27], [28], the image of the *integral* Thom morphism has not been studied this comprehensively before for Lie groups.

We also examine the differential Thom morphism

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}}\colon MU^*(X) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow H^*(X, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}.$$

We prove that if a nontorsion cohomology class α is not in the image of τ, but an integer multiple of α is in the image, then the differential Thom morphism has a nontrivial element in its kernel. Thus, we get examples of such elements in all the previous cases where the Thom morphism was not surjective.

Finally, we focus on a particular case, namely the special orthogonal groups $SO(n)$, and we construct elements of their complex cobordism geometrically. We construct a complex oriented map

$$g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow SO(5)$$

which represents an element of $MU^3(SO(5))$. Next, we prove that the Thom morphism maps this element to 2 times the generator $e_3 \in H^3(SO(5); \mathbb{Z}) \cong \mathbb{Z}$. Since we have earlier shown that e_3 itself is not in the image of the Thom morphism, it follows that g serves as a geometric model for a nontrivial element in the kernel of the differential Thom morphism. Finally, we generalise the construction of g to $SO(n)$ where $n > 5$ and to other cohomological degrees.

Paper II: Geometric cohomology classes for the Lie groups Spin(7) and Spin(8)

In this paper, we continue the endeavour to geometrically construct elements of the complex cobordism of compact Lie groups, similarly to the constructions for $SO(n)$ in Paper I. The original goal for the work that lead to this paper was to construct an element of $MU^3(G_2)$ which is mapped to 2 times a generator for $H^3(G_2; \mathbb{Z}) \cong \mathbb{Z}$. The motivation for this was two-fold. Firstly, it would provide a nontrivial element in the kernel of the differential Thom morphism for the Hopkins–Singer theories, as in the case studied in Paper I. Secondly, such a construction would complete the geometric construction of *all* cohomology classes for G_2, since more obvious constructions are already known for the elements in all other cohomological degrees.

We successfully constructed a stably almost complex manifold M^{10} and a well-defined map $\Psi\colon M^{10} \times S^1 \longrightarrow G_2$ where, apart from a low-dimensional skeleton, every point in $\operatorname{Im}\Psi$ had exactly two points in its preimage. However, it turned out that the nontrivial map $\epsilon : G_2 \longrightarrow G_2$ such that $\tau \circ \epsilon = \tau$ (where τ denotes the Thom morphism) reverses the orientation of $M^{10} \times S^1$ rather than preserving it. Thus, the element of $MU^3(G_2)$ represented by Ψ is mapped to the trivial element of $H^3(G_2;\mathbb{Z})$ by τ. Thus, our attempt was unsuccessful.

Instead, the paper provides a construction of geometric cobordism classes for the groups Spin(7) and Spin(8). We construct maps

$$\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow \mathrm{Spin}(7)$$
$$\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1 \times \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \times S^7 \longrightarrow \mathrm{Spin}(8).$$

The first map represents an element of $MU^3(\mathrm{Spin}(7))$ which is mapped to 8 times a generator of $H^3(\mathrm{Spin}(7);\mathbb{Z})$, and the second map an element of $MU^3(\mathrm{Spin}(8))$ which is mapped to 8 times a generator of $H^3(\mathrm{Spin}(8);\mathbb{Z})$. As with $SO(n)$, these constructions provide additional insight into why the Thom morphism is nonsurjective for these groups from a geometric point of view. Furthermore, this leads to a geometric construction of a nontrivial element in the kernel of the differential Thom morphism $\bar{\tau}_{\mathbb{R}/\mathbb{Z}}$.

Paper III: A note on the Thom morphism for the classifying space of certain Lie groups and gauge groups

This paper comprises three related topics. Firstly, we study the image of the Thom morphism for classifying spaces of compact Lie groups. Using the same method as we used in Paper I for Lie groups G, we examine which elements of the cohomology of BG can be lifted to complex cobordism when G is a special orthogonal group or an exceptional group. In the cases G_2, F_4, E_6 and E_7 we give a complete answer to which nontorsion generators are in the image of the Thom morphism, using obstructions at $p = 2$. However, for E_8 we only give a partial result, since the mod 2 cohomology of BE_8 is not known.

Secondly, we study the properties of the Thom morphism for a different type of spaces, namely the classifying spaces of gauge groups. Our main result is that for every principal E_7-bundle ξ over S^4, the Thom morphism $MU^4(B\mathcal{G}(\xi)) \longrightarrow H^4(B\mathcal{G}(\xi);\mathbb{Z})$ is not surjective. This result relies on the insight into the cohomology of BE_7 gained in the first part of the paper. Furthermore, we attempt to determine whether the same is true for a principal $SO(6)$-bundle over S^2, but are unable to reach a conclusive result. To our knowledge, the question of the surjectivity of the Thom morphism for classifying spaces of gauge groups has not been studied before, although recent results on torsion in their cohomology exist.

In [25], Totaro established a method for constructing nontrivial elements in the kernel of the Thom morphism, thereby showing that the *reduced Thom morphism*

$$MU^*(X) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^*(X;\mathbb{Z})$$

is noninjective. In the final section of this paper, we show that this method can be applied to a larger class of examples than those studied in [25]. Thus, we show that the reduced Thom morphism can be noninjective both for the spaces $BG \times B\mathbb{Z}/p$ and $G \times B\mathbb{Z}/p$, and we provide results on for which groups G and for which values of p this happens, as well as in which cohomological degrees.

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Paper I

On the cokernel of the Thom morphism for compact Lie groups

Eiolf Kaspersen and Gereon Quick

To appear in *Homology, Homotopy and Applications*

ON THE COKERNEL OF THE THOM MORPHISM FOR COMPACT LIE GROUPS

EIOLF KASPERSEN AND GEREON QUICK

Abstract. We give a complete description of the potential failure of the surjectivity of the Thom morphism from complex cobordism to integral cohomology for compact Lie groups via a detailed study of the Atiyah–Hirzebruch spectral sequence and the action of the Steenrod algebra. We show how the failure of the surjectivity of the topological Thom morphism can be used to find examples of non-trivial elements in the kernel of the induced differential Thom morphism from differential cobordism of Hopkins and Singer to differential cohomology. These arguments are based on the particular algebraic structure and interplay of the torsion and non-torsion parts of the cohomology and cobordism rings of a given compact Lie group. We then use the geometry of special orthogonal groups to construct concrete cobordism classes in the non-trivial part of the kernel of the differential Thom morphism.

Contents

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1. Introduction

The Thom morphism

$$\tau\colon MU \longrightarrow H\mathbb{Z}$$

from complex cobordism to integral singular cohomology is of fundamental importance for the study of the stable homotopy category. A special feature of τ is that it encodes both deep algebraic and geometric structures. This is a common theme of the present paper and is reflected in the following two ways τ may be described. On the one hand, τ interpolates between two extreme ends of the spectrum of oriented cohomology theories which may be classified by their formal group laws, as τ corresponds the unique morphism from the universal formal group law to the additive one (see [1, II Example (4.7)]). On the other hand, τ may be described geometrically in the following way. Let X be a smooth manifold. By Quillen's work in [19], classes in $MU^*(X)$ can be represented by proper complex-oriented maps $g\colon M \to X$. The Thom morphism sends the class $[g]$ to $g_*[M]$ where $[M]$ denotes the Poincaré dual of the fundamental class of M. Thus, roughly speaking, a cohomology class is in the image of τ if it can be expressed by a fundamental class of an almost-complex manifold. Hence the question whether τ is surjective or not is directly connected to concrete geometric phenomena, which is also why Thom introduced τ to solve Steenrod's problem in [21]. In cohomological degrees $i = 0, 1, 2$, the Thom morphism is surjective for all spaces, since the Eilenberg–MacLane spaces $K(\mathbb{Z}, i)$ are torsion-free for $i = 0, 1, 2$. In cohomological degrees $i \geq 3$, however, τ may fail to be surjective, even though the coefficient ring of MU is much larger than the one of $H\mathbb{Z}$. It is well-known that the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(X; MU^q) \implies MU^{p+q}(X)$$

both provides a way to show that τ may be surjective and that its differentials may yield obstructions to the surjectivity of τ (see [2]). However, the image of the Thom morphism has not been studied for many types of spaces.

The purpose of the present paper is to give a complete description of the potential failure of the surjectivity of the Thom morphism for compact connected Lie groups which provide an important class of examples of smooth manifolds. Our first main result is the following:

Theorem 1.1. *Let G be a compact connected Lie group with simple Lie algebra. Then table* (1) *shows the minimal cohomological degree q for which the Thom morphism $\tau\colon MU^q(G) \longrightarrow H^q(G;\mathbb{Z})$ fails to be surjective.*

In fact, for each minimal cohomological degree where τ fails to be surjective, we provide concrete non-torsion classes in $H^k(G;\mathbb{Z})$ which are not in the image of τ. The methods to prove theorem 1.1 are described in sections 2.1 and 2.2, and the study of the individual types of Lie groups occupies section 3. We note that generalised cohomology groups for some types of compact Lie groups are well-known, for example for complex K-theory from [10], for exceptional Lie groups and Morava K-theory form [13, 17], and in Brown–Peterson cohomology from [24, 25, 26]. Some of our computations could have been deduced from these papers. However, in order to give a unified and self-contained picture we provide direct proofs for all groups we consider.

Table 1. Summary of the results of theorem 1.1

Lie Algebra	Lie Group	Surjective	Min. degree where surjectivity fails
$\mathfrak{a}_n$	$SU(n)$ - Special unitary group	yes	–
	$SU(n)/\Gamma_l$ - Quotient of special unitary group	not for $4 \mid n$ and $l \equiv 2 \pmod 4$, yes otherwise	$2^r - 1$ where $r \in \mathbb{Z}$ is max. st. $2^r \mid n$
$\mathfrak{c}_n$	$Sp(n)$ - Symplectic group	yes	–
	$PSp(n)$ - Projective symplectic group	not for n even, yes for n odd	$2^{r+1} - 1$ where $r \in \mathbb{Z}$ is max. st. $2^r \mid n$
$\mathfrak{b}_n$, $\mathfrak{d}_n$	$Spin(n)$ - Spin group	not for $n \geq 7$	3
	$SO(n)$ - Special orthogonal group	not for $n \geq 5$	3
	$Ss(n)$ - Semi-spin group	not for $n \geq 4$	3 if $8 \mid n$, 7 otherwise
	$PSO(n)$ - Projective special orthog. group	not for $n \geq 8$	3 if $8 \mid n$, 7 otherwise
$\mathfrak{g}_2$	G_2	no	3
$\mathfrak{f}_4$	F_4	no	3
$\mathfrak{e}_6$	E_6, simply-connected	no	3
	E_6/Γ_3, centerless	no	3
$\mathfrak{e}_7$	E_7, simply-connected	no	3
	E_7/Γ_2, centerless	no	3
$\mathfrak{e}_8$	E_8	no	3

Remark 1.2. We recall in section 2.1 why τ is surjective whenever $H^*(G;\mathbb{Z})$ is torsion-free. However, we point out that this argument is not sufficient to explain the cases in table (1) where τ is surjective. The pattern we observe in table (1) indicates that Lie groups of type $\mathfrak{a}_n$ and $\mathfrak{c}_n$ tend to have a surjective Thom morphism, while groups of type $\mathfrak{b}_n$ and $\mathfrak{d}_n$ do not have a surjective Thom morphism in sufficiently high dimensions. The exceptional Lie groups on the other hand show a clear pattern. We note, however, that the behaviors of E_7 and E_8 are slightly different from the one of the other groups (see section 3.4). We do not know of a general geometric explanation for why τ is surjective or not surjective for a given Lie group. In section 4, however, we use the geometry and cell structure of special orthogonal groups to construct concrete geometric elements in $MU^*(SO(n))$.

Remark 1.3. We note that in the cases where τ fails to be surjective in cohomological degree 3, the generator $e_3 \in H^3(G;\mathbb{Z})$ which is not hit by τ is not in the image of the homomorphism

$$ku^3(G) \to H^3(G;\mathbb{Z})$$

from connective complex K-theory ku either. This is due to the fact that the Milnor operation Q_1 and the Steenrod operation Sq^3 provide obstructions which are differentials in the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(G; ku^q) \implies ku^{p+q}(G).$$

This applies to several of the groups of type $\mathfrak{b}_\mathfrak{n}$ and $\mathfrak{d}_\mathfrak{n}$ and to all exceptional Lie groups (see table (1) for the specific groups). In the other cases, however, surjectivity may not fail for ku but only on a higher stage in the tower of cohomology theories $MU \to \cdots \to MU\langle 2\rangle \to MU\langle 1\rangle = ku \to MU\langle 0\rangle = H\mathbb{Z}$.

A concrete motivation for our study of the Thom morphism arises from the theory of generalised differential cohomology theories for smooth manifolds developed by Hopkins and Singer in [12]. For a rationally even spectrum E and a smooth manifold X, the differential E-cohomology groups are denoted by $\check{E}(q)^n(X)$. The most interesting choice of degrees is $n = q$. The group $\check{E}(q)^q(X)$ then sits in several short exact sequences as described in [12, diagram (4.57)]. In particular, the natural homomorphism $\check{E}(q)^q(X) \to E^q(X)$ is surjective. Hence the Thom morphism $\tau\colon MU \to H\mathbb{Z}$ induces a commutative diagram

$$\begin{tikzcd}
\check{MU}(q)^q(X) \arrow[r] \arrow[d, "\check{\tau}"'] & MU^q(X) \arrow[d, "\tau"] \\
\check{H}(q)^q(X) \arrow[r] & H^q(X;\mathbb{Z})
\end{tikzcd}$$

in which the horizontal maps are surjective. Thus, if τ is not surjective, then $\check{\tau}$ fails to be surjective as well. We note that Grady and Sati study in [7] the surjectivity of the differential analog of the map from complex K-theory to cohomology using a differential version of the Atiyah–Hirzebruch spectral sequence.

However, the failure of the surjectivity of τ also allows us to find non-trivial elements in the kernel of $\check{\tau}$. For every rationally even spectrum E, $\check{E}(q)^q(X)$ sits in a short exact sequence of the form

$$0 \to E^{q-1}(X) \otimes \mathbb{R}/\mathbb{Z} \to \check{E}(q)^q(X) \to A_E^q(X) \to 0$$

where the group $A_E^q(X)$ is defined by the following pullback square, in which $\Omega^*(X; \pi_*E \otimes \mathbb{R})^q_{\mathrm{cl}}$ denotes closed forms on X of total degree q:

$$\begin{tikzcd}
A_E^q(X) \arrow[r] \arrow[d] & \Omega^*(X; \pi_*E \otimes \mathbb{R})^q_{\mathrm{cl}} \arrow[d] \\
E^q(X) \arrow[r] & H^q(X; \pi_*E \otimes \mathbb{R}).
\end{tikzcd}$$

The Thom morphism $\tau\colon MU \to H\mathbb{Z}$ induces a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & MU^{q-1}(X) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \arrow[r] \arrow[d, "\tau_{\mathbb{R}/\mathbb{Z}}"] & \check{MU}(q)^q(X) \arrow[r] \arrow[d, "\check{\tau}"] & A^q_{MU}(X) \arrow[r] \arrow[d, "\tau_A"] & 0 \\ 0 \arrow[r] & H^{q-1}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \arrow[r] & \check{H}(q)^q(X) \arrow[r] & A^q_H(X) \arrow[r] & 0. \end{tikzcd}$$

Recall that the kernel of the Thom morphism always contains the ideal $MU^{*<0} \cdot MU^*(M)$ of $MU^*(X)$, since τ is a natural transformation of oriented cohomology theories. We therefore use the following terminology:

Definition 1.4. We say that an element in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ or $\check{\tau}$ is *non-trivial* if it is not contained in the respective ideal generated by $MU^{*<0}$.

We will explain in section 2.3 how the failure of τ to be surjective enables us to find non-trivial elements in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$. This leads to the following result, for which we emphasise that the assumption applies to a large class of compact Lie groups by theorem 1.1:

Theorem 1.5. *Let G be a compact Lie group G and q an integer such that the Thom morphism $\tau\colon MU^{q-1}(G) \longrightarrow H^{q-1}(G;\mathbb{Z})$ fails to be surjective on a non-torsion class. Then the kernel of the differential Thom morphism*

$$\check{\tau}\colon \check{MU}(q)^q(G) \to \check{H}(q)^q(G)$$

is non-trivial in the sense of definition 1.4.

The significance of theorem 1.5 is that, together with theorem 1.1, it provides important examples of classes on smooth manifolds which can be studied using differential cobordism but not using differential cohomology. We thus demonstrate by concrete examples that the generalized differential invariants of [12] are stronger than invariants that can be obtained by just using differential cohomology. In section 2.4 we explain how we can use the Atiyah–Hirzebruch spectral sequence to find non-trivial elements in the kernel of $\tau_{\mathbb{R}/\mathbb{Z}}$ and $\check{\tau}$ whenever τ is not surjective.

In section 4 we switch perspectives and give a concrete and geometric construction of a non-trivial element in the kernel of $\check{\tau}$ for special orthogonal groups. From proposition 3.1 we know that the generator $e_3 \in H^3(SO(5);\mathbb{Z})$ is not hit by τ. In section 4.2 we show that the class $2e_3$, however, is in the image of τ by constructing a proper complex-oriented smooth map

$$g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow SO(5)$$

such that $\tau([g]) = 2e_3$ where $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ denotes the Grassmannian of *oriented* 2-planes in $\mathbb{R}^5$. In section 4.2 we prove the following result which we generalise in section 4.3 to higher dimensional $SO(n)$:

Theorem 1.6. *The class $\frac{1}{2}[g]$ is a non-trivial element in the kernel of*

$$\check{\tau}\colon \check{MU}(4)^4(SO(5)) \longrightarrow \check{H}(4)^4(SO(5)).$$

As in remark 1.3, we could have formulated theorem 1.5 for ku instead of MU as well, and the corresponding assumption would apply to the groups where surjectivity fails for ku already. The geometric construction of theorem 1.6, however, and its generalisation to higher $SO(n)$ are particular to MU. Moreover, since the Thom morphism allows for a unified picture, we formulate our findings for τ.

Finally, we note that the phenomenon the example of theorem 1.6 detects bears a certain similarity with the example used in [12, §2.7] to explain the behavior of a certain partition function in mathematical physics. We refer for example to [7, Example 48] for other interesting phenomena in mathematical physics related to the study of the morphisms between generalised differential cohomology theories. We do not know of a potential similar application of theorem 1.6 yet. We hope that the techniques to prove theorem 1.6 will be useful to shed new light on the Abel–Jacobi invariant for complex cobordism of [9] and [11].

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2. Obstructions and detecting elements in the kernel

In this section we explain the techniques that we use in section 3 to study the cokernel of τ. We assume that X is a finite CW-complex for simplicity.

2.1. **The Thom morphism is an edge map.** A key tool in our study of the Thom homomorphism is the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(X; MU^q) \implies MU^{p+q}(X).$$

Since $MU^* \cong \mathbb{Z}[x_{-2}, x_{-4}, \ldots]$, this spectral sequence is concentrated in the fourth quadrant. Since the top row of the E_2-page is the integral cohomology of X, there is a well-defined edge map such that the composition

$$MU^p(X) \longrightarrow E_\infty^{p,0} \longrightarrow E_2^{p,0} \cong H^p(X; \mathbb{Z})$$

can be identified with the Thom morphism.

It then follows from the general theory of spectral sequences that the Thom morphism is surjective if and only if all the differentials starting in the top row of the spectral sequence are trivial. Since all the differentials are torsion, the Thom morphism is surjective whenever $H^*(X;\mathbb{Z})$ has no torsion.

If $H^*(X;\mathbb{Z})$ has torsion, the Thom morphism may still be surjective. Since the construction of the spectral sequence is functorial, the differentials starting in the top row of the E_2-page are cohomology operations of the form $d\colon H^*(X;\mathbb{Z}) \to H^*(X;A)$ where A is a finitely generated free abelian group. If a differential d is p-torsion, then so is the composition $\rho \circ d$, where ρ is the map induced by the reduction modulo p homomorphism of A. Thus we can describe all differentials using cohomology operations of type $(\mathbb{Z}, m; \mathbb{Z}/p, n)$. These operations correspond to the elements in the cohomology group $H^n(K(\mathbb{Z}, m); \mathbb{Z}/p)$.

For $p = 2$, the cohomology ring $H^*(K(\mathbb{Z}, m); \mathbb{Z}/2)$ is a polynomial ring over generators of the form $\mathrm{Sq}^I(\iota_m)$, where I is an admissible sequence where the last term is different from 1, and ι_m is the fundamental class of $K(\mathbb{Z}, m)$ as explained in [16, Chapter 9, Theorem 3]. Thus, in order to prove that there are no non-trivial differentials that are 2-torsion, it suffices to check that all Steenrod operations of odd degree are trivial (except Sq^1, since a non-trivial differential increases the cohomological degree by at least 3).

For odd primes p, the cohomology operations we have to study can all be described using the reduced power operations P^k combined with Bocksteins β (see [5] for a complete description). In order to prove surjectivity it therefore suffices to show that all sequences of reduced power operations and Bocksteins that increase the cohomological degree by an odd number greater than 1 must be trivial. The fact that this also works in cases where we have torsion of the form $\mathbb{Z}/p^k$ with $k > 1$ can be deduced by considering short exact sequences of the form $\mathbb{Z}/p \to \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1}$.

2.2. **Obstructions and Bockstein cohomology.** Now we explain how we can find cohomology classes which are not in the image of the Thom homomorphism. From the description of τ as an edge map we know that an element $x \in H^n(X;\mathbb{Z})$ is not in the image of τ if there is at least one differential d on the E_2-page of the Atiyah–Hirzebruch spectral sequence with $(\rho \circ d)(x) \neq 0$ where

$$\rho\colon H^*(X;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}/p)$$

is the homomorphism induced by reduction mod p. Suppose now we know how the Steenrod algebra acts on $H^*(X;\mathbb{Z}/p)$. In fact, all Steenrod operations of odd degree vanish on the image of $MU^*(X)$ in $H^*(X;\mathbb{Z}/p)$ for all prime numbers p (see for example [22, page 468], [4, Proposition 3.6], [6]). Then it remains to understand how ρ acts. The tool we use to find the concrete element in $H^*(X;\mathbb{Z}/p)$ a given $x \in H^*(X;\mathbb{Z})$ maps to is *Bockstein cohomology*, the definition of which we now recall from [8, Chapter 3E]:

The Bockstein homomorphism $\beta\colon H^n(X;\mathbb{Z}/p) \longrightarrow H^{n+1}(X;\mathbb{Z}/p)$ is the connecting homomorphism in the long exact sequence induced in cohomology by the short exact sequence $0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$. It satisfies $\beta^2 = 0$, and thus defines a chain complex

$$\cdots \xrightarrow{\beta_{n-1}} H^n(X;\mathbb{Z}/p) \xrightarrow{\beta_n} H^{n+1}(X;\mathbb{Z}/p) \xrightarrow{\beta_{n+1}} \cdots .$$

The *nth Bockstein cohomology* of X is defined to be

$$BH^n(X;\mathbb{Z}/p) := \frac{\operatorname{Ker}\beta_n}{\operatorname{Im}\beta_{n-1}}.$$

We compute the groups $BH^n(X;\mathbb{Z}/p)$ by providing concrete descriptions of the Bockstein complex. Since X is assumed to be a finite CW-complex, all cohomology groups of X are finitely generated. By [8, Proposition 3E.3] the relationship between $H^*(X;\mathbb{Z})$ and $BH^*(X;\mathbb{Z}/p)$ is then given as follows :

- Each $\mathbb{Z}$-summand of $H^n(X;\mathbb{Z})$ contributes one $\mathbb{Z}/p$-summand to $BH^n(X;\mathbb{Z}/p)$.
- Each $\mathbb{Z}/p$-summand of $H^n(X;\mathbb{Z})$ contributes nothing to $BH^n(X;\mathbb{Z}/p)$.
- Each $\mathbb{Z}/p^k$-summand (with $k \geq 2$) of $H^n(X;\mathbb{Z})$ contributes one $\mathbb{Z}/p$-summand to $BH^{n-1}(X;\mathbb{Z}/p)$ and one $\mathbb{Z}/p$-summand to $BH^n(X;\mathbb{Z}/p)$.

Finally, for an odd prime p, we will also use the following obstruction.

Lemma 2.1. *Let $Q_1\colon H^*(X;\mathbb{Z}/p) \to H^{*+2p-1}(X;\mathbb{Z}/p)$ be the first Milnor operation and let $x \in H^i(X;\mathbb{Z})$ be a non-torsion class. If $Q_1(\rho(x)) \neq 0$, then x is not in the image of $ku^i(X) \to H^i(X;\mathbb{Z})$ and hence not in the image of the Thom morphism.*

Proof. By [23, Proposition 1.7] (see also [20, Proposition 4-4]), there is a commutative diagram

$$\begin{tikzcd} ku^i_{(p)}(X) \arrow[r, "\tau_{ku_{(p)}}"] & H^i(X;\mathbb{Z}_{(p)}) \arrow[r, "\cdot v_1"] \arrow[d] & ku^{i+2p-1}_{(p)}(X) \arrow[d] \\ & H^i(X;\mathbb{Z}/p) \arrow[r, "\pm Q_1"'] & H^{i+2p-1}(X;\mathbb{Z}/p) \end{tikzcd}$$

in which the top row is exact, where $ku^i_{(p)}(X)$ denotes p-local connective complex K-theory and the map $\tau_{ku_{(p)}}$ is the map which factors the canonical morphism $\tau_{BP}\colon BP \to H\mathbb{Z}_{(p)}$ for Brown–Peterson theory. Thus, if $Q_1(\rho(x)) \neq 0$, then the image of x in $H^i(X;\mathbb{Z}_{(p)})$ cannot be lifted to $ku^i_{(p)}(X)$. This implies that x cannot be lifted to $ku^i(X)$ either, and hence the assertion. □

2.3. **The kernel of the differential Thom morphism.** We will now explain how the failure of τ to be surjective enables us to find non-trivial elements in the kernel of $\check{\tau}$. We write $MU^{*<0} \cdot MU^k(X)$ for the subgroup of $MU^k(X)$ consisting of elements of the form $\gamma \cdot \mu$ where $\gamma \in MU^{k-s}$ and $\mu \in MU^s(X)$ with $s > k$. The sum over all k defines an ideal in $MU^*(X)$ which we denote by $MU^{*<0} \cdot MU^*(X)$. Since $\tau(MU^{*<0}) = 0$, we get that τ induces a well-defined homomorphism

$$\tau\colon MU^*(X)/(MU^{*<0} \cdot MU^*(X)) \to H^*(X;\mathbb{Z}),$$

which we also denote by τ. Consider the homomorphism $MU^* \to \mathbb{Z}$ which sends $n \cdot 1 \in MU^0$ to $n \in \mathbb{Z}$ and $\gamma \in MU^{*<0}$ to 0. Then there is an isomorphism of rings

$$MU^*(X)/(MU^{*<0} \cdot MU^*(X)) \cong MU^*(X) \otimes_{MU^*} \mathbb{Z}.$$

By slight abuse of notation, we then also write $MU^k(X) \otimes_{MU^*} \mathbb{Z}$ for the group $MU^k(X)/(\oplus_s MU^s \cdot MU^{k-s}(X))$. Now we let MU^* act on $\mathbb{R}/\mathbb{Z}$ by the map $MU^* \otimes \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$ defined by

$$\begin{aligned} n \otimes a &\longmapsto na, \text{ for } n \in MU^0 \cong \mathbb{Z} \\ \gamma \otimes a &\longmapsto 0, \text{ for } \gamma \in MU^{*<0}. \end{aligned}$$

Then we get a canonical isomorphism

$$(MU^*(X) \otimes_{MU^*} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \xrightarrow{\cong} MU^*(X) \otimes_{MU^*} \mathbb{R}/\mathbb{Z}.$$

We will now explain how the information on the cokernel of τ helps to understand the kernel of the induced Thom homomorphism

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}}\colon MU^*(X) \otimes_{MU^*} \mathbb{R}/\mathbb{Z} \longrightarrow H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$$

in differential cobordism.

Lemma 2.2. *Let $\alpha \in H^k(X;\mathbb{Z})$ be a non-torsion class. Assume that the image of the Thom morphism*

$$\tau\colon MU^k(X) \longrightarrow H^k(X;\mathbb{Z})$$

contains $n\alpha$ for some integer $n > 1$, but not α itself or an element of the form $\alpha + y$, where $n \cdot y = 0$. Let $\mu \in MU^k(X)$ be an element such that $\tau(\mu) = n\alpha$. Then

$$\mu \otimes \frac{1}{n} \in MU^k(X) \otimes_{MU^*} \mathbb{R}/\mathbb{Z}$$

is a non-trivial element in the kernel of the induced map

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}} \colon MU^k(X) \otimes_{MU^*} \mathbb{R}/\mathbb{Z} \longrightarrow H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Proof. The element $\mu \otimes \frac{1}{n}$ maps to 0 under $\bar{\tau}_{\mathbb{R}/\mathbb{Z}}$ since $n\alpha \otimes \frac{1}{n} = \alpha \otimes 1 = 0$. However, $\mu \otimes \frac{1}{n}$ cannot be 0 in $MU^k(X) \otimes_{MU^*} \mathbb{R}/\mathbb{Z}$, since if μ had been of the form $n\gamma$, then γ would map to α or $\alpha + y$. □

2.4. **Detecting elements in the kernel of the differential Thom morphism.** We now describe a procedure to find an element μ as in lemma 2.2 using the Atiyah–Hirzebruch spectral sequence. In section 4 we will give a geometric construction for special orthogonal groups. For the other cases, we can proceed as follows. Assume that we have a non-torsion cohomology class $\alpha \in H^k(X;\mathbb{Z})$ which is not in the image of the Thom morphism, while an integer multiple $k\alpha$ is in the image. We will now explain how we can then find a cobordism class which maps to $n\alpha$. Since α is not in the image of the Thom morphism, there must be at least one non-trivial differential starting at $H^k(X;\mathbb{Z})$. If this differential is, say, m-torsion, then $m\alpha$ is in the kernel of the differential and survives to the next page of the spectral sequence. Since X is assumed to be finite dimensional, the spectral sequence is bounded on the right, and there can only be finitely many non-trivial differentials starting at any one position. By counting how much torsion there is in cohomological degrees greater than n, we can then determine an integer n for which $n\alpha$ must be in the image of the Thom morphism. Once we have reached the E_∞-page of the spectral sequence, the position $(k, 0)$ contains the desired cobordism class.

3. The cokernel for compact Lie groups

The goal of this section is to determine whether or not the Thom morphism is surjective for a given compact, connected, simple Lie group. Such a Lie group has a simple Lie algebra. Given a simple Lie algebra $\mathfrak{g}$, we find the associated Lie groups using the following method based on [18, 10.7.2, Theorem 4]. We first determine the unique (up to isomorphism) compact, simply-connected Lie group G with Lie algebra $\mathfrak{g}$. The center $Z(G)$ is always finite. The other compact, connected, simple Lie groups with the same Lie algebra are of the form G/K, where K is a subgroup of $Z(G)$. Organising our analysis by the associated Lie algebra is justified by the following observation. Given a Lie group G, we denote by $H^*_{\text{free}}(G;\mathbb{Z})$ the non-torsion part of the cohomology $H^*(G;\mathbb{Z})$. Then there is an isomorphism $H^*_{\text{free}}(G;\mathbb{Z}) \cong H^*_{\text{free}}(H;\mathbb{Z})$ if G and H are Lie groups with the same Lie algebra. We will therefore recall the non-torsion cohomology part only once in the section for a given Lie algebra.

Unless otherwise stated, the computation of the cohomology rings can be found in one of the following two sources: The cohomology of the groups $SU(n)$, $Sp(n)$, $\mathrm{Spin}(n)$, $SO(n)$ as well as all the exceptional Lie groups and classifying spaces can be found in [15], while the cohomology of $Ss(n)$, $PSO(n)$, $PSp(n)$ and the quotients of $SU(n)$ can be found in [3]. Finally, given a ring R we write $\Lambda_R(x_{i_1}, \ldots, x_{i_n}) := R[x_{i_1}, \ldots, x_{i_n}]/(x_{i_1}^2, \ldots, x_{i_n}^2)$, and unless otherwise stated x_{i_j} is an element of degree i_j. When the choice of ring is clear from the context, we omit R from the notation.

3.1. **Groups with Lie algebra $\mathfrak{b}_\mathfrak{n}$ and $\mathfrak{d}_\mathfrak{n}$.** The simply-connected Lie groups that correspond to the Lie algebras of type $\mathfrak{b}_\mathfrak{n}$ and $\mathfrak{d}_\mathfrak{n}$ are the *spin groups* $\mathrm{Spin}(2n+1)$ and $\mathrm{Spin}(2n)$, respectively. We will consider both types of spin groups together, since their cohomology rings are similar. However, the possible quotients are different in the odd and even cases. The center of $\mathrm{Spin}(2n+1)$ is isomorphic to $\mathbb{Z}/2$, which gives us only one possible quotient, the *odd special orthogonal group*, denoted $SO(2n+1)$.

For the even case, we know by [15, Chapter II, Theorem 4.14] that the centers are given by $Z(\mathrm{Spin}(4n+2)) \cong \mathbb{Z}/4$ and $Z(\mathrm{Spin}(4n)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. For $\mathrm{Spin}(4n+2)$, taking the quotient by the subgroup of order 2 yields the *even special orthogonal group* $SO(4n+2)$, while taking the quotient by the whole center gives the *projective special orthogonal group* $PSO(4n+2)$. For $\mathrm{Spin}(4n)$, the center $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ has three subgroups of order 2. Taking the quotient by the whole center once again produces a projective special orthogonal group $PSO(4n)$. One of the subgroups of order 2 will again give us a special orthogonal group $SO(4n)$. The remaining two subgroups produce isomorphic quotient groups, known as the *semi-spin group* $Ss(4n)$ (see [15, Chapter II, Theorem 4.15]). In total we have to consider four different types of groups.

3.1.1. *Special orthogonal groups.* The non-torsion cohomology of the special orthogonal groups is given by

$$H^*_{\mathrm{free}}(SO(n);\mathbb{Z}) \cong \begin{cases} \Lambda(e_3, e_7, \dots, e_{2n-3}), & n \text{ odd} \\ \Lambda(e_3, e_7, \dots, e_{2n-5}, y_{n-1}), & n \text{ even.} \end{cases}$$

Proposition 3.1. *For $n \geq 5$, the generator $e_3 \in H^3(SO(n);\mathbb{Z})$ is not in the image of the Thom homomorphism.*

Proof. The $\mathbb{Z}/2$-cohomology of $SO(n)$ is given by

$$H^*(SO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[u_1, u_3, \dots, u_{2m-1}]/(u_1^{k_1}, u_3^{k_3}, \dots, u_{2m-1}^{k_{2m-1}}), \tag{1}$$

where $m = \lfloor \frac{n}{2} \rfloor$ and k_i is the least power of 2 such that $|u_i^{k_i}| \geq n$. In order to find the image of e_3 in $H^3(SO(n);\mathbb{Z}/2)$ under ρ, we need to analyse the Bockstein homomorphism β. From [8], we have

$$\beta(u_{2i-1}) = u_{2i} \text{ and } \beta(u_{2i}) = 0,$$

where we interpret u_{2i} as u_i^2 and iterate if necessary. Assuming $n \geq 5$, we have the following table for the generators for $H^*(SO(n);\mathbb{Z}/2)$ in low degrees, where the arrows denote the non-trivial Bockstein homomorphisms.

$$\begin{tikzcd}
\text{Degree:} & 1 & 2 & 3 & 4 \\
\text{Generators:} & u_1 \arrow[r] & u_1^2 & u_1^3 \arrow[r] & u_1^4 \\
 & & & u_3 \arrow[ur] & u_1u_3
\end{tikzcd} \tag{2}$$

We see that $BH^3(SO(n);\mathbb{Z}/2)$ is generated by $u_1^3 + u_3$. It follows that the reduction map to $\mathbb{Z}/2$-cohomology maps e_3 to $u_1^3 + u_3$. We can then deduce that e_3 is not in the image of the Thom homomorphism, since

$$\mathrm{Sq}^3(u_1^3 + u_3) = u_1^6 + u_3^2 \neq 0.$$

Note that $u_3^2 = 0$ if $n = 5$, but u_1^6 is nonzero. $\square$

Remark 3.2. Using the same methods, we can show that any given generator $e_{4k+3} \in H^{4k+3}(SO(n);\mathbb{Z})$ is not in the image of the Thom morphism for sufficiently large n. However, we do not know of an efficient way to determine a minimal n for each generator e_{4k+3}, apart from analysing the Bockstein diagrams on a case by case basis. We return to this question in section 4.3.

Proposition 3.3. *For $SO(n)$ with $n \leq 4$, the Thom morphism is surjective in all degrees.*

Proof. We have the homeomorphisms

$$SO(1) \cong \mathrm{pt},\ SO(2) \cong S^1,\ SO(3) \cong \mathbb{RP}^3, \text{ and } SO(4) \cong \mathbb{RP}^3 \times S^3.$$

For $SO(1)$ and $SO(2)$, the surjectivity follows from the fact that the Thom morphism is surjective in degrees ≤ 2. For $SO(3)$, we use the same fact for degrees ≤ 2. Moreover, there is no nontrivial differential in the Atiyah–Hirzebruch spectral sequence starting in cohomological degree 3, since $\mathbb{RP}^3$ is 3-dimensional. This shows that the Thom morphism is also surjective in all degrees for $SO(3)$. Finally, the integral cohomology of $SO(4)$ is given by

$$H^k(SO(4);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 6 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 3 \\ \mathbb{Z}/2, & k = 2, 5 \\ 0, & \text{else.} \end{cases}$$

Again, there are no differentials which start in degree ≥ 3, increase the cohomological degree by at least 3, and which end in torsion. Thus, the Thom morphism is surjective for $SO(4)$. □

3.1.2. *Spin groups.* For the rest of this section, we use the following notation. Given $n \in \mathbb{N}$, we let q be the greatest power of 2 such that $q|n$, and let t be the least power of 2 such that $n \leq t$. The cohomology of the spin groups is given by

$$H^*(\mathrm{Spin}(n);\mathbb{Z}/2) \cong \Lambda(z) \otimes \mathbb{Z}/2[u_3, u_5, \ldots, u_{2m-1}]/(u_3^{k_3}, \ldots, u_{2m-1}^{k_{2m-1}}),$$

where $|z| = t - 1$ and where m and the k_i's are as in (1).

Proposition 3.4. *For $n \geq 7$, the generator $e_3 \in H^3(\mathrm{Spin}(n);\mathbb{Z})$ is not in the image of the Thom morphism. For $n \leq 6$, the Thom morphism is surjective in all degrees.*

Proof. For $n \leq 7$, there is no torsion in the integral cohomology of $\mathrm{Spin}(n)$, and it follows that the Thom morphism is surjective. However, for $n \geq 7$, the generator $e_3 \in H^3(\mathrm{Spin}(n);\mathbb{Z})$ maps to $u_3 \in H^3(\mathrm{Spin}(n);\mathbb{Z}/2)$, for which

$$\mathrm{Sq}^3 u_3 = u_3^2 \neq 0.$$

Thus, e_3 is not in the image of the Thom morphism for $n \geq 7$. □

3.1.3. *Semi-spin groups.* The cohomology of the semi-spin groups with coefficients in $\mathbb{Z}/2$ is given by

$$H^*(Ss(n);\mathbb{Z}/2)$$
$$\cong \mathbb{Z}/2[v]/(v^q) \otimes \Lambda(z) \otimes \mathbb{Z}/2[u_3, u_5, \ldots, \widehat{u}_{q-1}, \ldots, u_{n-1}, u_{2q-2}]/(u_3^{k_3}, \ldots, u_{n-1}^{k_{n-1}}, u_{2q-2}^{k_{2q-2}}),$$

where $|v| = 1$ and $|z| = t - 1$. The Steenrod operations are given by

$$\mathrm{Sq}^j(u_k) = \binom{k}{j} u_{k+j}$$

wherever it makes sense, with the exception that

$$\mathrm{Sq}^1(u_k) = v^{k+1} \tag{3}$$

if $q \geq 8$ and $k = \frac{q}{2} - 1$. Recall that since we are dealing with semi-spin groups, n must be a multiple of 4. Hence we do not need separate cases for even and odd n. Note also that the class u_{2q-2} will only be included if $2q - 2 < n$.

Proposition 3.5. *For $Ss(4)$, the Thom morphism is surjective in all degrees. For $k \geq 2$, we have: If $8 \mid n$, then the generator $e_3 \in H^3(Ss(4k); \mathbb{Z})$ is not in the image of τ. If $8 \nmid n$, then the generator $e_7 \in H^7(Ss(4k); \mathbb{Z})$ is not in the image of τ.*

Proof. For $n = 4$, the cohomology ring together with its Steenrod operations of $Ss(4)$ is isomorphic to the cohomology ring with Steenrod operations of $SO(4)$. It then follows from proposition 3.3 that the Thom morphism is surjective for $Ss(4)$.

Now we assume $n = 4k$ and $k \geq 2$. There are three cases to consider:

Case 1: $\mathbf{n \equiv 8}$ (mod 16)

In this case, $q = 8$, and the $\mathbb{Z}/2$-cohomology ring is given by

$$H^*(Ss(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^8) \otimes \Lambda(z) \otimes \mathbb{Z}/2[u_3, \ldots, \widehat{u}_7, \ldots, u_{n-1}, u_{14}]/(u_3^{k_3}, \ldots),$$

where $|z| \geq 7$ since t is at least 8. Since $q = 8$, we get that

$$\mathrm{Sq}^1(u_3) = v^4$$

by equation (3). The other Sq^1s are easy to work out, which leads to the following Bockstein diagram in low degrees:

$$\begin{array}{lcccc}
\text{Degree:} & 1 & 2 & 3 & 4 \\
 & & & & \\
\text{Generators:} & v & \longrightarrow v^2 & v^3 & \longrightarrow v^4 \\
 & & & & \nearrow \\
 & & & u_3 & vu_3
\end{array}$$

The similarity to diagram (2) is not coincidental, since there is an isomorphism $Ss(8) \cong SO(8)$, and the other semi-spin groups of this form look similar in low degrees. The non-torsion class $e_3 \in H^3(Ss(n); \mathbb{Z})$ maps to $v^3 + u_3 \in H^3(Ss(n); \mathbb{Z}/2)$, and we can check that Sq^3 does not act trivially on this class:

$$\mathrm{Sq}^3(v^3 + u_3) = v^6 + u_3^2 \neq 0.$$

Case 2: $\mathbf{n \equiv 0}$ (mod 16)

In this case, since q is now greater than 8, the Bockstein homomorphism acts trivially on u_3.

$$\begin{tikzcd}[row sep=small]
\text{Degree:} & 1 & 2 & 3 & 4 \\
\text{Generators:} & v \arrow[r] & v^2 & v^3 \arrow[r] & v^4 \\
 & & & u_3 & vu_3
\end{tikzcd}$$

Moreover, the Bockstein cohomology in degree 3 is generated by u_3 alone. This implies that e_3 is sent to u_3, and we see that u_3 is not in the image of the Thom morphism.

Case 3: $\mathbf{n \equiv 4 \ (mod\ 8)}$

We have $q = 4$ and $n \geq 12$, which means that the $\mathbb{Z}/2$-cohomology is given by

$$H^*(Ss(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^4) \otimes \Lambda(z) \otimes \mathbb{Z}/2[u_5, u_6, u_7, u_9, \ldots, u_{n-1}]/(u_5^{k_5}, \ldots),$$

where we see that $|z| \geq 15$. We get the Bockstein diagram

$$\begin{tikzcd}[row sep=small]
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
v \arrow[r] & v^2 & v^3 & & & & & \\
 & & & & u_5 \arrow[dr] & vu_5 \arrow[r] \arrow[dr] & v^2u_5 \arrow[dr] & v^3u_5 \\
 & & & & & u_6 & vu_6 \arrow[r] & v^2u_6 \\
 & & & & & & u_7 & vu_7
\end{tikzcd}$$

where the two arrows starting in vu_5 indicate that the Bockstein is given by a sum, i.e., $\beta(vu_5) = v^2u_5 + vu_6$. In this case, e_3 is in the image of the Thom morphism, since its reduction v^3 does not survive any Steenrod operations. However, we can find a suitable class in degree 7. From the Bockstein diagram we see that e_7 is sent to either u_7 or $u_7 + v^2u_5 + vu_6$, and we can check that both classes survive Sq^3:

$$\mathrm{Sq}^3(u_7) = \binom{7}{3} u_{10} = u_5^2 \neq 0$$
$$\mathrm{Sq}^3(u_7 + v^2u_5 + vu_6) = u_5^2 \neq 0. \qquad \square$$

3.1.4. *Projective special orthogonal groups.* The $\mathbb{Z}/2$-cohomology of the projective special orthogonal groups is given by

$$\begin{aligned} &H^*(PSO(n); \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2[v]/(v^q) \otimes \mathbb{Z}/2[u_1, u_3, \ldots, \widehat{u}_{q-1}, \ldots, u_{n-1}, u_{2q-2}]/(u_1^{k_1}, \ldots, u_{n-1}^{k_{n-1}}, u_{2q-2}^{k_{2q-2}}), \end{aligned}$$

where $|v| = 1$, with the Steenrod operations acting by

$$\mathrm{Sq}^j u_k = \binom{k}{j} u_{k+j},$$

whenever it makes sense, except when $j = 1,\ k = \frac{q}{2} - 1$ and $q \geq 8$, in which case

$$\mathrm{Sq}^1(u_k) = u_{k+1} + v^{k+1}.$$

Note that u_{2q-2} is only be included if $2q - 2 < n$, as for the semi-spin groups. If n is odd, then $PSO(n) = SO(n)$. Hence we will focus on the case that n is even.

Proposition 3.6. *Let $n \geq 8$ be even. If $8 \mid n$, then the generator $e_3 \in H^3(PSO(n);\mathbb{Z})$ is not in the image of τ. If $8 \nmid n$, then the generator $e_7 \in H^7(PSO(n);\mathbb{Z})$ is not in the image of τ.*

Proof. We have to consider the following cases:

Case 1: $\mathbf{n \equiv 8 \pmod{16}}$

We have $q = 8$ and note that $\mathrm{Sq}^1(u_3) = u_1^4 + v^4$. We get the following diagram of Bockstein homomorphisms:

$$\begin{tikzcd}
\text{Degree:} & 1 & 2 & 3 & 4 \\
\text{Generators:} & v \arrow[r] & v^2 & v^3 \arrow[r] & v^4 \\
 & & & u_3 \arrow[ur] \arrow[dr] & vu_3 \\
 & u_1 \arrow[r] & u_1^2 & u_1^3 \arrow[r] & u_1^4 \\
 & & & & u_1u_3 \\
 & & vu_1 \arrow[r] \arrow[dr] & v^2u_1 \arrow[dr] & v^3u_1 \\
 & & & vu_1^2 \arrow[r] & v^2u_1^2 \\
 & & & & vu_1^3
\end{tikzcd}$$

The non-torsion class $e_3 \in H^3(PSO(n);\mathbb{Z})$ maps to either $v^3 + u_1^3 + u_3$ or $v^3 + u_1^3 + u_3 + v^2u_1 + vu_1^2$ in $H^3(PSO(n);\mathbb{Z}/2)$, and we have

$$\mathrm{Sq}^3(v^3 + u_1^3 + u_3) = v^6 + u_1^6 + u_3^2 \neq 0$$
$$\mathrm{Sq}^3(v^3 + u_1^3 + u_3v^2u_1 + vu_1^2) = v^6 + u_1^6 + u_3^2 + v^4u_1^2 + v^2u_1^4 \neq 0.$$

Case 2: $\mathbf{n \equiv 0 \pmod{16}}$

In low degrees, this is almost the same as the previous case, with the exception that $\mathrm{Sq}^1(u_3) = u_1^4$ since $q \geq 16$. This gives us the diagram

$$\begin{tikzcd}
\text{Degree:} & 1 & 2 & 3 & 4 \\
\text{Generators:} & v \arrow[r] & v^2 & v^3 \arrow[r] & v^4 \\
 & & & u_3 \arrow[dr] & vu_3 \\
 & u_1 \arrow[r] & u_1^2 & u_1^3 \arrow[r] & u_1^4 \\
 & & & & u_1u_3 \\
 & & vu_1 \arrow[r] \arrow[dr] & v^2u_1 \arrow[dr] & v^3u_1 \\
 & & & vu_1^2 \arrow[r] & v^2u_1^2 \\
 & & & & vu_1^3.
\end{tikzcd}$$

Then $e_3 \in H^3(PSO(n);\mathbb{Z})$ maps to $u_1^3 + u_3$ or $u_1^3 + u_3 + v^2u_1 + vu_1^2$, for which

$$\mathrm{Sq}^3(u_1^3 + u_3) = u_1^6 + u_3^2 \neq 0$$

$$\mathrm{Sq}^3(u_1^3 + u_3 + v^2u_1 + vu_1^2) = u_1^6 + u_3^2 + v^4u_1^2 + v^2u_1^4 \neq 0.$$

Case 3: $\mathbf{n \equiv 4 \ (mod\ 8)}$

Assume $n \geq 12$. In this case $q = 4$, and consequently the class u_3 does not exist. The Bockstein diagram is

$$\begin{tikzcd}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
v \arrow[r] & v^2 & v^3 & & & & & \\
u_1 \arrow[r] & u_1^2 & u_1^3 \arrow[r] & u_1^4 & u_1^5 \arrow[r] & u_1^6 & u_1^7 \arrow[r] & u_1^8 \\
 & & & & & & u_7 \arrow[ur] & vu_7 \\
 & & & & u_5 \arrow[dr] & u_1u_5 \arrow[r] \arrow[dr] & u_1^2u_5 \arrow[dr] & u_1^3u_5 \\
 & & & & & u_6 & u_1u_6 \arrow[r] & u_1^2u_6 \\
 & & & & & vu_5 \arrow[dr] \arrow[ddr] & vu_1u_5 \arrow[r] \arrow[dr] \arrow[ddr] & vu_1^2u_5 \\
 & & & & & & v^2u_5 \arrow[ddr] & v^2u_1u_5 \\
 & & & & & & vu_6 \arrow[dr] & vu_1u_6 \\
 & & & & & & & v^2u_6 \\
 & & & & & & & v^3u_5 \\
 & & & & & & & u_1u_7 \\
 & vu_1 \arrow[r] \arrow[dr] & vu_1^2 \arrow[dr] & vu_1^3 \arrow[r] \arrow[dr] & vu_1^4 \arrow[dr] & vu_1^5 \arrow[r] \arrow[dr] & vu_1^6 \arrow[dr] & vu_1^7 \\
 & & v^2u_1 \arrow[r] & v^2u_1^2 & v^2u_1^3 \arrow[r] & v^2u_1^4 & v^2u_1^5 \arrow[r] & v^2u_1^6 \\
 & & & v^3u_1 \arrow[r] & v^3u_1^2 & v^3u_1^3 \arrow[r] & v^3u_1^4 & v^3u_1^5.
\end{tikzcd}$$

We see that the class e_7 can reduce to several different classes in $H^7(PSO(n);\mathbb{Z}/2)$ depending on our choice of isomorphism $H^7(PSO(n);\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2^4$. We have that

e_7 maps to $u_7 + u_1^7$ or $u_7 + u_1^7$ plus any of the classes $u_1^2 u_5 + u_1 u_6$, $v^2 u_5 + vu_6$, $vu_1^6 + v^2 u_1^5$, $v^3 u_1^5$. We have

$$\mathrm{Sq}^3(u_7 + u_1^7) = u_{10} + u_1^{10} \neq 0.$$

We then note that applying Sq^3 to any of the torsion classes $u_1^2 u_5 + u_1 u_6$, $v^2 u_5 + vu_6$, $vu_1^6 + v^2 u_1^5$, $v^3 u_1^5$ cannot yield u_{10}. Hence they cannot cancel out the nonzero contribution we got from $\mathrm{Sq}^3(u_7 + u_1^7)$. This proves that e_7 and e_7 plus torsion are not in the image of the Thom morphism.

Case 4: $\mathbf{n \equiv 2}$ (mod 4)
Assume $n \geq 10$. Since $q = 2$, we there is no class u_1, but instead a class u_2. The cohomology and Bockstein diagrams are therefore

$$H^*(PSO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^2) \otimes \mathbb{Z}/2[u_2, u_3, u_5, \ldots, u_{n-1}]/(u_2^{k_2}, \ldots)$$

$$\begin{tikzcd}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
v & u_2 & vu_2 & u_2^2 & vu_2^2 & u_2^3 & vu_2^3 & u_2^4 \\
 & & u_3 \arrow[ur] & vu_3 \arrow[ur] & u_2u_3 \arrow[ur] & vu_2u_3 \arrow[ur] & u_2^2u_3 \arrow[ur] & vu_2^2u_3 \\
 & & & & & & u_7 \arrow[uur] & vu_7 \\
 & & & & & u_3^2 & vu_3^2 & u_2u_3^2 \\
 & & & & u_5 \arrow[ur] & vu_5 \arrow[ur] & u_2u_5 \arrow[ur] & vu_2u_5 \\
 & & & & & & & u_3u_5.
\end{tikzcd}$$

The class e_7 is sent to either $u_7 + u_2^2 u_3$, or $u_7 + u_2^2 u_3$ plus one or both of the classes vu_2^3, vu_3^2. As above, we may use Sq^3 as an obstruction, but the computation is easier if we use Sq^7. We get

$$\begin{aligned}
&\mathrm{Sq}^7(u_7 + u_2^2 u_3) = u_7^2 + u_2^4 u_3^2 \neq 0 \\
&\mathrm{Sq}^7(vu_2^3) = v^2 u_2^6 = 0 \\
&\mathrm{Sq}^7(vu_3^2) = v^2 u_3^4 = 0.
\end{aligned}$$

Hence e_7 is not be in the image of the Thom morphism. Note that $u_7^2 = 0$ if $n \leq 14$, but $u_2^4 u_3^2$ is nonzero. □

Proposition 3.7. *For $PSO(2)$, $PSO(4)$ and $PSO(6)$, the Thom morphism is surjective in all degrees.*

Proof. The isomorphism $PSO(2) \cong SO(2)$ implies that the Thom morphism is surjective for $PSO(2)$, by proposition 3.3. For $PSO(4)$, the relevant cohomology rings are given by

$$
\begin{aligned}
&H^*_{\mathrm{free}}(PSO(4);\mathbb{Z}) \cong \Lambda(e_3, y_3)\\
&H^*(PSO(4);\mathbb{Z}/2) \cong \mathbb{Z}[v]/(v^4) \otimes \mathbb{Z}/2[u_1]/(u_1^4).
\end{aligned}
$$

Moreover, we have the Bockstein diagram

$$
\begin{tikzcd}
\text{Degree:} & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{Generators:} & v \arrow[r] & v^2 & v^3 & & & \\
& u_1 \arrow[dr] & vu_1 \arrow[r] \arrow[dr] & v^2u_1 \arrow[dr] & v^3u_1 \arrow[dr] & & \\
& & u_1^2 & vu_1^2 \arrow[r] & v^2u_1^2 & v^3u_1^2 & \\
& & & u_1^3 & vu_1^3 \arrow[r] & v^2u_1^3 & v^3u_1^3.
\end{tikzcd}
$$

It follows that the integral cohomology of $PSO(4)$ is given by

$$
H^k(PSO(4);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 6 \\ \mathbb{Z}/2, & k = 4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & k = 2, 5 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, & k = 3 \\ 0, & \text{else.} \end{cases}
$$

Since the differentials in the Atiyah–Hirzebruch spectral sequence increase the cohomological degree by at least 3, the only differentials which start at a non-trivial cohomology group to a group with torsion are

$$
\begin{aligned}
&d_3\colon H^0(PSO(4);\mathbb{Z}) \longrightarrow H^3(PSO(4);\mathbb{Z})\\
&d_3\colon H^2(PSO(4);\mathbb{Z}) \longrightarrow H^5(PSO(4);\mathbb{Z}).
\end{aligned}
$$

However, since the Thom morphism is surjective in degrees ≤ 2, these differentials must also be trivial. Thus we can conclude that the Thom morphism is surjective for $PSO(4)$.

For the group $PSO(6)$, we also need to study the full cohomology ring. Since

$$
\begin{aligned}
&H^*_{\mathrm{free}}(PSO(6);\mathbb{Z}) \cong \Lambda(e_3, e_7, y_5)\\
&H^*(PSO(6);\mathbb{Z}/2) \cong \mathbb{Z}[v]/(v^2) \otimes \mathbb{Z}/2[u_2, u_3, u_5]/(u_2^4, u_3^2, u_5^2),
\end{aligned}
$$

we get the Bockstein diagram

$$\begin{tikzcd}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
v \arrow[r] & u_2 & vu_2 & u_2^2 & vu_2^2 & u_2^3 & vu_2^3 \\
 & & u_3 \arrow[ur] & vu_3 \arrow[ur] & u_2u_3 \arrow[ur] & vu_2u_3 \arrow[ur] & u_2^2u_3 \\
 & & & & u_5 & vu_5 & u_2u_5
\end{tikzcd}$$

$$\begin{tikzcd}
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
vu_2^2u_3 & u_2^3u_3 & vu_2^3u_3 & & & & & \\
vu_2u_5 & u_2^2u_5 & vu_2^2u_5 & u_2^3u_5 & vu_2^3u_5 & & & \\
u_3u_5 \arrow[ur] & vu_3u_5 \arrow[ur] & u_2u_3u_5 \arrow[ur] & vu_2u_3u_5 \arrow[ur] & u_2^2u_3u_5 & vu_2^2u_3u_5 & u_2^3u_3u_5 & vu_2^3u_3u_5
\end{tikzcd}.$$

The elements that correspond to non-torsion in $H^*(PSO(6);\mathbb{Z})$ have been circled. This gives us the cohomology groups

$$H^k(PSO(6);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 3, 8, 15 \\ \mathbb{Z}/4, & k = 2, 14 \\ \mathbb{Z}/2, & k = 4, 6, 11 \\ \mathbb{Z} \oplus \mathbb{Z}/2, & k = 5, 10, 12 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2, & k = 9 \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2, & k = 7 \\ 0, & \text{else.} \end{cases} \tag{4}$$

Since isomorphism (4) gives us all the elements in the image of the reduction homomorphism $H^*(PSO(6);\mathbb{Z}) \to H^*(PSO(6);\mathbb{Z}_2)$, it is straight-forward to check that all Steenrod operations of odd degree greater than 1 are trivial for these elements. This shows that all differentials in the Atiyah–Hirzebruch spectral sequence are trivial. Thus the Thom morphism is surjective. □

3.2. **Groups with Lie algebra $\mathfrak{c}_\mathfrak{n}$.** The integral cohomology of the simply-connected Lie group $Sp(n)$ is given by

$$H^*(Sp(n);\mathbb{Z}) \cong \Lambda(e_3, e_7, \ldots, e_{4n-1}).$$

Since the cohomology is torsion-free, we can conclude:

Proposition 3.8. *The Thom morphism is surjective in all cohomological degrees for $Sp(n)$.* □

The center of $\mathrm{Sp}(n)$ is isomorphic to $\mathbb{Z}/2$, consisting of the positive and negative of the identity matrix. It follows that there is only one other compact Lie group with the same Lie algebra which we obtain by dividing out by $Z(\mathrm{Sp}(n))$ (see [3]). This group is known as *the projective symplectic group*, denoted by $PSp(n)$.

The $\mathbb{Z}/2$-cohomology of $PSp(n)$ is given by

$$H^*(PSp(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^{4q}) \otimes \Lambda(b_3, b_7, \ldots, \widehat{b}_{4q-1}, \ldots, b_{4n-1}),$$

where q is the largest power of 2 dividing n. The Steenrod squares are given by

$$\mathrm{Sq}^{4j}(b_{4k+3}) = \binom{k}{j} b_{4k+4j+3}$$

with all other Steenrod squares trivial, except

$$\mathrm{Sq}^1(b_{2q-1}) = v^{2q} \tag{5}$$

when n is even. Equation (5) makes the cases of even and odd n different. We start with the even case.

Proposition 3.9. *For all even $n \geq 2$, the generator $e_{2q-1} \in H^{2q-1}(PSp(n);\mathbb{Z})$ is not in the image of the Thom morphism.*

Proof. Assume that $n \geq 2$ is even. We may consider the following part of the Bockstein diagram

$$\begin{tikzcd}
2q-2 & 2q-1 & 2q \\
v^{2q-2} & v^{2q-1} \arrow[r] & v^{2q} \\
 & b_{2q-1}. \arrow[ur] &
\end{tikzcd}$$

It follows that the generator e_{2q-1} is mapped to $v^{2q-1} + b_{2q-1}$ plus torsion. We have $\mathrm{Sq}^{2q-1}(v^{2q-1} + b_{2q-1}) = v^{4q-2} \neq 0$, and hence the assertion. ☐

Proposition 3.10. *For all odd n, the Thom morphism is surjective for $PSp(n)$ in all cohomological degrees.*

Proof. Assume that n is odd. We have the cohomology ring

$$H^*(PSp(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[v]/(v^4) \otimes \Lambda(b_7, b_{11}, \ldots, b_{4n-1}).$$

The only non-trivial Bocksteins are the ones that go from a term containing v to a term containing v^2. The Bockstein diagram then takes the form

$$\begin{array}{ccccccccccc}
1 & & 2 & 3 & \cdots & 7 & 8 & & 9 & 10 & \cdots \\
 & & & & & & & & & & \\
v & \longrightarrow & v^2 & v^3 & & b_7 & vb_7 & \longrightarrow & v^2 b_7 & v^3 b_7 & \cdots
\end{array}$$

The non-torsion elements of $H^*_{\mathrm{free}}(PSp(n);\mathbb{Z})$ map to elements of the form $b_{i_1} \cdots b_{i_k}$ or $v^3 b_{i_1} \cdots b_{i_k}$ in $H^*(PSp(n);\mathbb{Z}/2)$, while the torsion elements are sent to elements of the form $v^2 b_{i_1} \cdots b_{i_k}$. We now claim that none of these elements can survive an odd-dimensional Steenrod square.

Assume that $\alpha \in H^*(PSp(n);\mathbb{Z}/2)$ is such that $\mathrm{Sq}^{2n+1}(\alpha) \neq 0$. Then

$$\mathrm{Sq}^1 \mathrm{Sq}^{2n} \alpha \neq 0,$$

which implies that $\mathrm{Sq}^{2n}\alpha$ is of the form $vb_{i_1}\cdots b_{i_k}$. Since the number of v's cannot be changed by a Steenrod square of even degree, the class α is a product of b_i's and precisely one v. However, as seen in the Bockstein diagram, such elements do not correspond to elements of the integral cohomology of $PSp(n)$. This implies the assertion. □

3.3. **Groups with Lie algebra $\mathfrak{a}_\mathfrak{n}$.** The integral cohomology of the simply-connected Lie group $SU(n)$ is given by

$$H^*(SU(n);\mathbb{Z}) \cong \Lambda(e_3, e_5, \ldots, e_{2n-1}),$$

and is torsion-free. Hence we can conclude:

Proposition 3.11. *The Thom morphism is surjective in all cohomological degrees for $SU(n)$.* □

The center of $SU(n)$ is isomorphic to $\mathbb{Z}/n$. Hence, depending on n, we can take several different quotients of this group. The case where we divide out be the entire center is known as the *projective special unitary group $PSU(n)$*. The cohomology groups of the various quotients are as follows. Let l be a natural number dividing n. Let Γ_l be the subgroup of $Z(SU(n))$ of order l, and set $G(n,l) := SU(n)/\Gamma_l$. Suppose p is an odd prime dividing l and p^r is the largest power of p dividing n. Then

$$H^*(G(n,l);\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^{p^r}) \otimes \Lambda(z_1, z_3, \ldots, \widehat{z}_{2p^r-1}, \ldots, z_{2n-1}),$$

where $|y| = 2$. The power operations are given by

$$P^k(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k(p-1)}, \text{ and } \beta(z_{2p^{r-1}-1}) = y^{p^{r-1}}.$$

Similarly, if $p = 2$ and $4 \mid l$, then

$$H^*(G(n,l);\mathbb{Z}/2) \cong \mathbb{Z}/2[y]/(y^{2^r}) \otimes \Lambda(z_1, z_3, \ldots, \widehat{z}_{2^{r+1}-1}, \ldots, z_{2n-1}),$$

with

$$\mathrm{Sq}^{2k}(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k}, \text{ and } \mathrm{Sq}^1(z_{2^r-1}) = y^{2^{r-1}},$$

and all other odd-degree Steenrod operations are trivial. Finally, if $l \equiv 2 \pmod 4$, then

$$H^*(G(n,l);\mathbb{Z}/2) \cong \mathbb{Z}/2[z_1]/({z_1}^{2^{r+1}}) \otimes \Lambda(z_3, z_5, \ldots, \widehat{z}_{2^{r+1}-1}, \ldots, z_{2n-1}), \tag{6}$$

with the Steenrod operations

$$\mathrm{Sq}^{2k}(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k} \text{ and } \mathrm{Sq}^1(z_{2^r-1}) = z_1^{2^r}.$$

Although there is significant torsion in the cohomology of $G(n,l)$, the Thom morphism is only non-surjective in specific cases.

Proposition 3.12. *Let $n, l \geq 1$ be integers with $4 \mid n$ and $l \equiv 2 \pmod 4$. Then the generator $e_{2^r-1} \in H^{2^r-1}(G(n,l);\mathbb{Z})$ is not in the image of the Thom morphism.*

Proof. The cohomology of $G(n,l)$ is as in (6), with $r \geq 2$. The relevant part of the Bockstein diagram is

$$\begin{tikzcd}
\text{Degree:} & 2^{r-1} & 2^r - 1 & 2^r \\
\text{Generators:} & z_1^{2^{r-1}} & {z_1}^{2^r-1} \arrow[r] & {z_1}^{2^r} \\
 & & z_{2^r-1} \arrow[ur] &
\end{tikzcd}$$

It follows that $e_{2^r-1} \in H^{2^r-1}(G(n,l);\mathbb{Z})$ maps to ${z_1}^{2^r-1} + z_{2^r-1}$ (possibly plus torsion), for which

$$\mathrm{Sq}^{2^r-1}({z_1}^{2^r-1} + z_{2^r-1}) = {z_1}^{2^{r+1}-2} \neq 0.$$

Thus, e_{2^r-1} is not in the image of the Thom morphism. □

Proposition 3.13. *Let n, l be positive integers such that $4 \nmid n$ or $l \not\equiv 2 \pmod 4$. Then the Thom morphism is surjective in all cohomological degrees for $G(n,l)$.*

Proof. We will show that all the differentials starting in the top row in the Atiyah–Hirzebruch spectral sequence for $G(n,l)$ must be trivial. Suppose a sequence of power operations and Bocksteins of odd degree ≥ 3 is non-trivial when evaluated on an element of $H^*(G(n,l);\mathbb{Z}/p)$, where p is an odd prime. Since the only non-trivial Bockstein is

$$\beta(z_{2p^{r-1}-1}) = y^{p^{r-1}},$$

this can only occur if there is some z_{2i-1} such that

$$P^k(z_{2i-1}) = \binom{i-1}{k} z_{2i-1+2k(p-1)} = z_{2p^{r-1}-1} \tag{7}$$

for some k. We will now show that this is impossible by showing that for any i, k, r satisfying (7), the binomial coefficient is divisible by p.

To simplify the notation, let $j = i - 1$. The binomial coefficient $\binom{j}{k}$ can be computed using Lucas' theorem, which says that if

$$\begin{aligned} j &= j_0 + j_1 p + \dots j_m p^m \\ k &= k_0 + k_1 p + \dots k_m p^m \end{aligned}$$

are the base p expansions of j and k, then

$$\binom{j}{k} \equiv \prod_{t=0}^{m} \binom{j_t}{k_t} \pmod p.$$

Here we use the convention that $\binom{j}{k} = 0$ if $j < k$. From equation (7) we see that

$$j = p^{r-1} + k - kp - 1. \tag{8}$$

We can assume that $r > 1$, since otherwise the only non-trivial Bockstein is $\beta(z_1) = y$, which leads to a surjective Thom morphism. Now, let s be the smallest natural number such that $p^{s-1} \mid k$, but $p^s \nmid k$. From equation (8) we see that $s < r$, since otherwise j would be negative. We then get

$$j \equiv k - 1 \pmod{p^s}.$$

Since $k \not\equiv 0 \pmod{p^s}$, we see that $j_v < k_v$ for some $v < s$, and it follows from Lucas' theorem that $\binom{j}{k} \equiv 0 \pmod{p}$. This proves that there are no non-trivial differentials of odd torsion.

It remains to show that there are no nontrivial differentials of 2-torsion in the remaining cases. There are two such cases: when n and l are both multiples of 4 and when $n \equiv 2 \pmod 4$. In the former case, the only non-trivial Bockstein is

$$\mathrm{Sq}^1(z_{2^r-1}) = y^{2^{r-1}},$$

and the surjectivity of the Thom morphism follows from the same argument as with the odd torsion, by setting $P^k = \mathrm{Sq}^{2k}$. In the latter case, if $n \equiv 2 \pmod 4$ and $l \equiv 2 \pmod 4$, the cohomology ring is

$$H^*(G(n,l);\mathbb{Z}/2) \cong \mathbb{Z}/2[z_1]/(z_1^4) \otimes \Lambda(z_5, z_7, \dots, z_{2n-1}),$$

and the only Bockstein is $\mathrm{Sq}^1(z_1) = z_1^2$. The surjectivity of the Thom morphism then follows from the same argument as in the case $PSp(n)$ with n odd. □

3.4. **Groups with exceptional Lie algebras.** We will now consider Lie groups with exceptional Lie algebras. It turns out that the cases $\mathfrak{g}_2$, $mathfrakf_4$ and $\mathfrak{e}_6$ follow the same pattern, while we can say a bit more on $\mathfrak{e}_7$ and $\mathfrak{e}_8$. We will therefore split our analysis into three subsections.

3.4.1. *Groups with Lie algebra $\mathfrak{g}_2$, $\mathfrak{f}_4$ and $\mathfrak{e}_6$.* The free cohomologies of the exceptional Lie groups G_2, F_4 and E_6 are given by

$$\begin{aligned} H^*_{\mathrm{free}}(G_2;\mathbb{Z}) &\cong \Lambda(e_3, e_{11}), \\ H^*_{\mathrm{free}}(F_4;\mathbb{Z}) &\cong \Lambda(e_3, e_{11}, e_{15}, e_{23}), \\ H^*_{\mathrm{free}}(E_6;\mathbb{Z}) &\cong \Lambda(e_3, e_9, e_{11}, e_{15}, e_{17}, e_{23}). \end{aligned}$$

The center of E_6 is isomorphic to $\mathbb{Z}/3$ (see [15]). Hence there is another group which has Lie algebra $\mathfrak{e}_6$. We will refer to this group as the *centerless* E_6 and denote it by E_6/Γ_3.

Proposition 3.14. *For G_2, F_4, E_6 and E_6/Γ_3, the generator e_3 in integral cohomology is not in the image of the Thom morphism.*

Proof. The $\mathbb{Z}/2$-cohomologies of G_2, F_4 and E_6 are given by

$$\begin{aligned} H^*(G_2;\mathbb{Z}/2) &\cong \mathbb{Z}/2[x_3]/(x_3^4) \otimes \Lambda(x_5), \\ H^*(F_4;\mathbb{Z}/2) &\cong \mathbb{Z}/2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_{15}, x_{23}), \\ H^*(E_6;\mathbb{Z}/2) &\cong \mathbb{Z}/2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_9, x_{15}, x_{17}, x_{23}). \end{aligned}$$

In each case, the generator e_3 in integral cohomology reduces to x_3 in $\mathbb{Z}/2$-cohomology and $\mathrm{Sq}^3(x_3) = x_3^2 \neq 0$. Hence e_3 is not in the image of the Thom morphism. Since neither the free cohomology nor the cohomology with coefficients in $\mathbb{Z}/2$ are altered by dividing out by a subgroup of order 3, the same argument as for E_6 applies to E_6/Γ_3. □

Remark 3.15. We note that the other generators in the integral cohomology groups of G_2, F_4, E_6 and E_6/Γ_3 are in the image the Thom morphism. As we will see in proposition 3.17 and 3.18, the behaviors of the cohomology of the groups corresponding to the Lie algebras $\mathfrak{e}_7$ and $\mathfrak{e}_8$ are different.

Remark 3.16. The integral cohomology of the exceptional groups, except for G_2, also have 3-torsion, and we could have used a computation at $p = 3$ to show non-surjectivity.

3.4.2. *Groups with Lie algebra* $\mathfrak{e}_7$. The free cohomology of the group E_7 is given by

$$H^*_{\text{free}}(E_7; \mathbb{Z}) \cong \Lambda(e_3, e_{11}, e_{15}, e_{19}, e_{23}, e_{27}, e_{35}).$$

The center of E_7 is isomorphic to $\mathbb{Z}/2$ (see [15]), and hence there is another group which has the Lie algebra E_7. We will refer to this group as the *centerless* E_7 and denote it by E_7/Γ_2.

Proposition 3.17. *For E_7 and E_7/Γ_2, the generators e_3 and e_{15} in integral cohomology are not in the image of the Thom morphism.*

Proof. While there is significant 2-torsion in the cohomology of E_7, it is more convenient to show the non-surjectivity of the Thom morphism in degree 3 using 3-torsion. The relevant cohomology ring is

$$\begin{gathered} H^*(E_7; \mathbb{Z}/3) \cong \mathbb{Z}/3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}), \\ \text{with } P^1 x_3 = x_7,\ P^3 x_7 = x_{19},\ \beta x_7 = x_8. \end{gathered}$$

The reduction homomorphism $\rho\colon H^3(E_7; \mathbb{Z}) \to H^3(E_7; \mathbb{Z}/3)$ sends e_3 to x_3. Let Q_1 denote the first Milnor operation. We then have

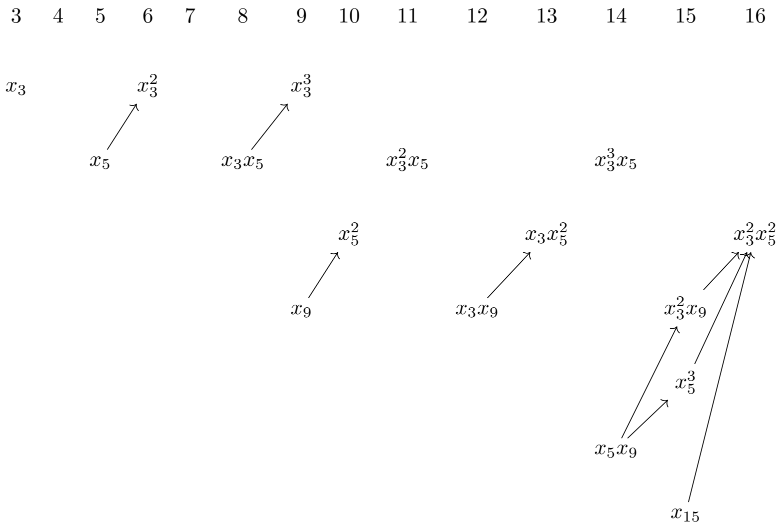
$$Q_1(x_3) = P^1\beta(x_3) - \beta P^1(x_3) = P^1(0) - \beta(x_7) = -x_8,$$

and we conclude that e_3 is not in the image of τ by lemma 2.1. Since dividing out by a subgroup of order 2 changes neither the free cohomology nor the $\mathbb{Z}/3$-cohomology, we deduce that $e_3 \in H^3(E_7/\Gamma_2; \mathbb{Z})$ is not in the image of τ either.

To see that the generator e_{15} is not in the image of τ we will use 2-torsion. The $\mathbb{Z}/2$-cohomology and Steenrod operations of E_7 and E_7/Γ_2 are given by

$$\begin{aligned} &H^*(E_7; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27}), \\ &H^*(E_7/\Gamma_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4) \otimes \Lambda(x_6, x_{15}, x_{17}, x_{23}, x_{27}), \\ &\quad \text{where } \mathrm{Sq}^1 x_1 = x_1^2,\ \mathrm{Sq}^1 x_5 = x_3^2,\ \mathrm{Sq}^1 x_9 = x_5^2,\ \mathrm{Sq}^1 x_{15} = x_3^2 x_5^2, \\ &\quad \mathrm{Sq}^1 x_{17} = x_9^2,\ \mathrm{Sq}^1 x_{23} = x_3^2 x_9^2,\ \mathrm{Sq}^1 x_{27} = x_5^2 x_9^2, \\ &\quad \mathrm{Sq}^2 x_3 = x_5,\ \mathrm{Sq}^2 x_{15} = x_{17}, \\ &\quad \mathrm{Sq}^4 x_5 = x_9,\ \mathrm{Sq}^4 x_{23} = x_{27}, \\ &\quad \mathrm{Sq}^8 x_9 = x_{17},\ \mathrm{Sq}^8 x_{15} = x_{23}, \end{aligned} \tag{9}$$

with all other $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8$ trivial and where x_3^2 is interpreted as x_6 in $H^*(E_7/\Gamma_2; \mathbb{Z}/2)$. In low degrees, we get the following Bockstein diagram for E_7:

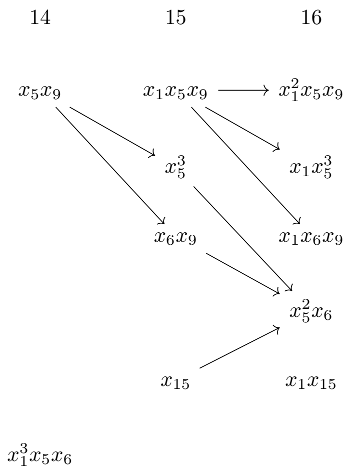


The generator $e_{15} \in H^{15}(E_7, \mathbb{Z})$ reduces to $x_{15} + x_3^2x_9$ or $x_{15} + x_5^3$ in $H^{15}(E_7; \mathbb{Z}_2)$, and we compute

$$\mathrm{Sq}^3(x_{15} + x_3^2x_9) = \mathrm{Sq}^3(x_{15} + x_5^3) = x_9^2 \neq 0$$

to see that $e_{15} \in H^{15}(E_7; \mathbb{Z})$ is not in the image of the Thom morphism.

For E_7/Γ_2, we have the following Bockstein diagram:



It follows that $e_{15} \in H^{15}(E_7/\Gamma_2;\mathbb{Z})$ reduces to either $x_{15}+x_6x_9$ or $x_{15}+x_5^3$ in $H^{15}(E_7/\Gamma_2;\mathbb{Z}/2)$. Since

$$\mathrm{Sq}^3(x_{15}+x_6x_9) = \mathrm{Sq}^3(x_{15}+x_5^3) = x_9^2 \neq 0,$$

we conclude that $e_{15} \in H^{15}(E_7/\Gamma_2;\mathbb{Z})$ and hence the assertion. □

3.4.3. *The group E_8.* The free cohomology of the group E_8 is given by

$$H^*_{\mathrm{free}}(E_8;\mathbb{Z}) \cong \Lambda(e_3, e_{15}, e_{23}, e_{27}, e_{35}, e_{39}, e_{47}, e_{59}).$$

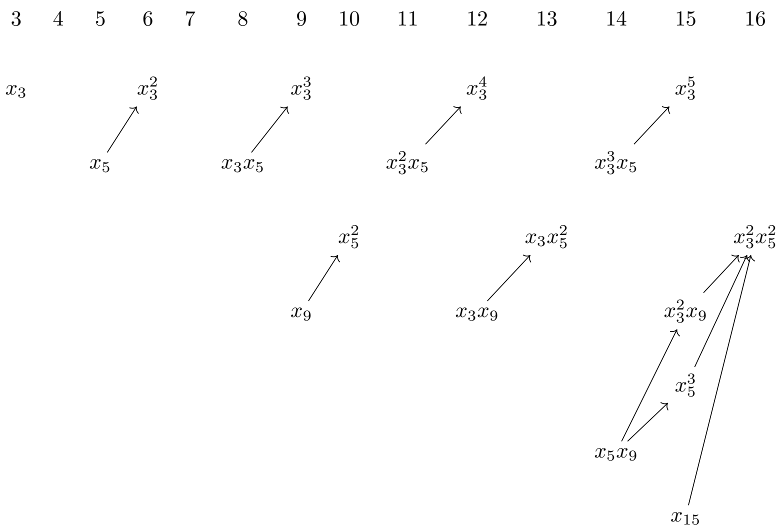
Proposition 3.18. *The generators $e_3, e_{15}, e_{23}, e_{27} \in H^*_{free}(E_8;\mathbb{Z})$ as well as the sum of any of these generators with a torsion class in the same degree are not in the image of the Thom morphism.*

Proof. The $\mathbb{Z}/2$-cohomology of E_8 and the Steenrod operations are given by

$$\begin{aligned} (10)\quad & H^*(E_8;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_3,x_5,x_9,x_{15}]/(x_3^{16},x_5^8,x_9^4,x_{15}^4)\otimes\Lambda(x_{17},x_{23},x_{27},x_{29}),\\ & \text{where } \mathrm{Sq}^1x_5 = x_3^2,\ \mathrm{Sq}^1x_9 = x_5^2,\ \mathrm{Sq}^1x_{15} = x_3^2x_5^2,\ \mathrm{Sq}^1x_{17} = x_9^2,\\ & \mathrm{Sq}^1x_{23} = x_3^2x_9^2,\ \mathrm{Sq}^1x_{27} = x_5^2x_9^2,\ \mathrm{Sq}^1x_{29} = x_{15}^2,\\ & \mathrm{Sq}^2x_3 = x_5,\ \mathrm{Sq}^2x_{15} = x_{17},\ \mathrm{Sq}^2x_{27} = x_{29},\\ & \mathrm{Sq}^4x_5 = x_9,\ \mathrm{Sq}^4x_{23} = x_{27},\\ & \mathrm{Sq}^8x_9 = x_{17},\ \mathrm{Sq}^8x_{15} = x_{23}, \end{aligned}$$

with all other $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8$ trivial.

In low degrees, we get the following Bockstein diagram for $p=2$:



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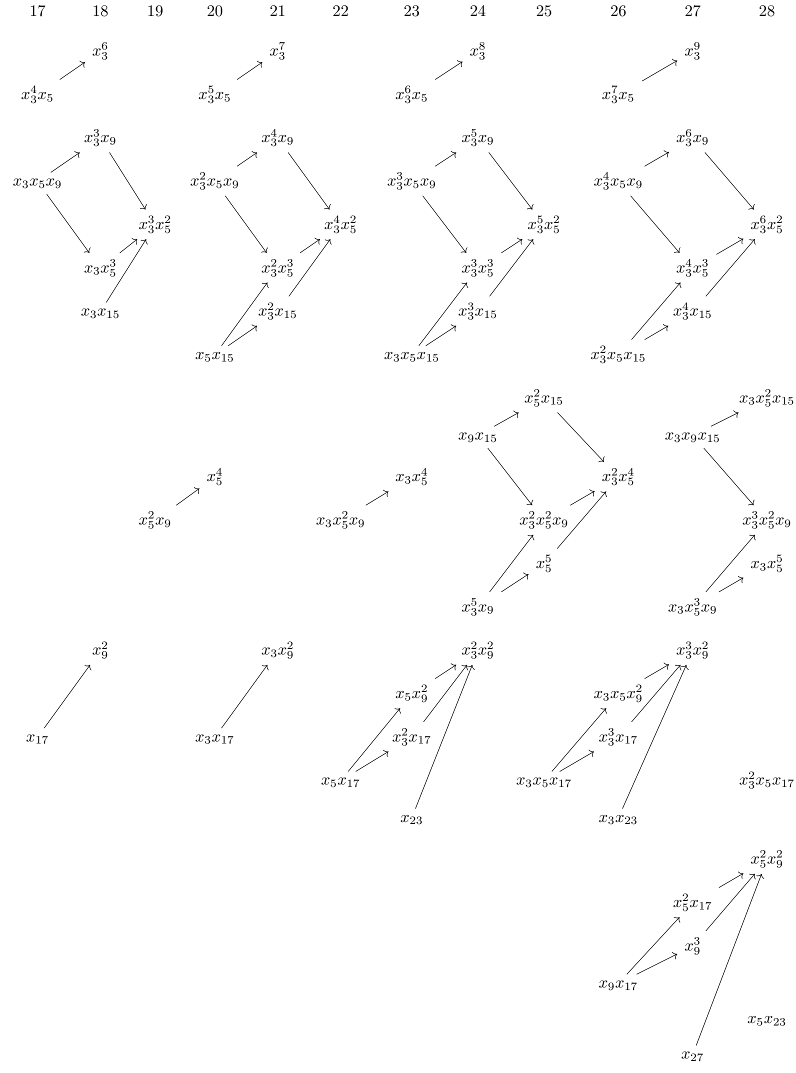


The class $e_3 \in H^3_{\text{free}}(E_8;\mathbb{Z})$ reduces to the class $x_3 \in H^3(E_8;\mathbb{Z}/2)$, for which Sq^3 is non-trivial. For the other degrees, we start in degree 27 and work our way down.

From diagram (11), we see that under the reduction map $H^{27}(E_8;\mathbb{Z}) \to H^{27}(E_8;\mathbb{Z}/2)$, the class e_{27} + (torsion) is sent to either

$$x_{27} + x_5^2 x_{17} + L \quad \text{or} \quad x_{27} + x_9^3 + L, \tag{12}$$

where L is some linear combination of the classes

$$x_3^3 x_9^2,\ x_3^9,\ x_3^6 x_9 + x_3^4 x_5^3, \text{ and } x_3^4 x_{15} + x_3^4 x_5^3.$$

We observe that

$$\mathrm{Sq}^3(x_{27}) = \mathrm{Sq}^1\mathrm{Sq}^2(x_{27}) = \mathrm{Sq}^1(x_{29}) = x_{15}^2.$$

Then, using (10), we see that none of the other terms in (12) can yield an x_{15}^2-term when applying Sq^3. Thus, the term x_{15}^2 cannot get cancelled out, and it follows that e_7 and e_7 plus torsion are not in the image of the Thom morphism.

For the class e_{23}, we get four alternatives for its reduction to $\mathbb{Z}/2$-cohomology:

$$x_{23} + x_5 x_9^2,\ x_{23} + x_5 x_9^2 + x_3 x_5^4,\ x_{23} + x_3^2 x_{17}, \text{ and } x_{23} + x_3^2 x_{17} + x_3 x_5^4. \tag{13}$$

We evaluate Sq^4 on each of these classes and get

$$\begin{aligned}
\mathrm{Sq}^4(x_{23} + x_5 x_9^2) &= x_{27} + x_9^3 \\
\mathrm{Sq}^4(x_{23} + x_5 x_9^2 + x_3 x_5^4) &= x_{27} + x_9^3 + x_3^9 \\
\mathrm{Sq}^4(x_{23} + x_3^2 x_{17}) &= x_{27} + x_5^2 x_{17} \\
\mathrm{Sq}^4(x_{23} + x_3^2 x_{17} + x_3 x_5^4) &= x_{27} + x_5^2 x_{17} + x_3^9.
\end{aligned}$$

We see that each cohomology class in (13) is mapped to a class in (12) by Sq^4. It follows that e_{23} reduces to a class for which $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4$ is nonzero. This shows that e_{23} plus any torsion is not in the image of the Thom morphism.

Finally, there are four possibilities for the mod-2 reduction of e_{15} plus torsion, given by

$$x_{15} + x_3^2 x_9,\ x_{15} + x_3^2 x_9 + x_3^5,\ x_{15} + x_5^3, \text{ and } x_{15} + x_5^3 + x_3^5. \tag{14}$$

The operation Sq^8 acts on these classes by

$$\begin{aligned}
\mathrm{Sq}^8(x_{15} + x_3^2 x_9) &= x_{23} + x_3^2 x_{17} \\
\mathrm{Sq}^8(x_{15} + x_3^2 x_9 + x_3^5) &= x_{23} + x_3^2 x_{17} + x_3 x_5^4 \\
\mathrm{Sq}^8(x_{15} + x_5^3) &= x_{23} + x_5 x_9^2 \\
\mathrm{Sq}^8(x_{15} + x_5^3 + x_3^5) &= x_{23} + x_5 x_9^2 + x_3 x_5^4.
\end{aligned}$$

Hence all classes in (14) have a nonzero image under $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^8$, proving that e_{15} plus torsion is not in the image of the Thom morphism. □

4. Geometric examples for special orthogonal groups

Recall from proposition 3.1 that the generator $e_3 \in H^*_{\text{free}}(SO(5);\mathbb{Z}) \cong \Lambda(e_3, e_7)$ is not in the image of τ. However, we will now show that the element $2e_3$ is in the image of the Thom morphism. We will prove this by geometrically constructing an element of $MU^3(SO(5))$ which is mapped to $2e_3 \in H^3(SO(5);\mathbb{Z})$ under τ.

To do so we make use of the fact that $SO(5)$ is a 10-dimensional compact manifold. Let $2\tilde{e}_3 \in H_7(SO(5);\mathbb{Z})$ denote the image of $2e_3$ under the isomorphism $H^3(SO(5);\mathbb{Z}) \cong H_7(SO(5);\mathbb{Z})$ defined by Poincaré duality. By [19, Proposition 1.2], elements in $MU^3(SO(5))$ can be represented by proper complex-oriented maps of the form $M \to SO(5)$ where M is a 7-dimensional manifold. Thus, in order to show that $2e_3$ is in the image of τ, it suffices to find a proper complex-oriented map $g\colon M \to SO(5)$ such that $g_*[M] = 2\tilde{e}_3$ where $[M]$ denotes the fundamental class of M in $H_7(SO(5);\mathbb{Z})$.

We will therefore now compute the homology group $H_7(SO(5);\mathbb{Z})$. We will do this using an explicit cell structure of $SO(5)$.

4.1. **The cell structure of special orthogonal groups.** We recall the cell structure of special orthogonal groups using maps from products of real projective spaces from [8, Proposition 3D.1]. Let v be a nonzero vector in $\mathbb{R}^n$. We define the linear transformation $r(v)\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the reflection across the orthogonal complement of v. We may use this map to define an embedding from $\mathbb{RP}^{k-1}$ to $SO(n)$ for $k \leq n$ as follows. Representing elements of $\mathbb{RP}^{k-1}$ by vectors in $\mathbb{R}^k$ and embedding into $\mathbb{R}^n$ in the canonical way if $k < n$, we define

$$\begin{aligned} \mathbb{RP}^{k-1} &\longrightarrow SO(n) \\ [v] &\longmapsto r(v)\cdot r(e_1) \end{aligned}$$

where $e_1, \ldots, e_n$ are the standard basis vectors. We extend this to a map defined on products of real projective spaces by taking compositions, i.e.,

$$\begin{aligned} f_{i_1,\ldots,i_m}\colon \mathbb{RP}^{i_1} \times \ldots \times \mathbb{RP}^{i_m} &\longrightarrow SO(n) \\ ([v_1],\ldots,[v_m]) &\longmapsto r(v_1)\cdot r(e_1)\cdots r(v_m)\cdot r(e_1). \end{aligned}$$

For $SO(n)$, there is a k-cell for each sequence $(i_1,\ldots,i_m)$ which satisfies both $n > i_1 > \ldots > i_m > 0$ and $i_1 + \ldots + i_m = k$. The characteristic map is given by

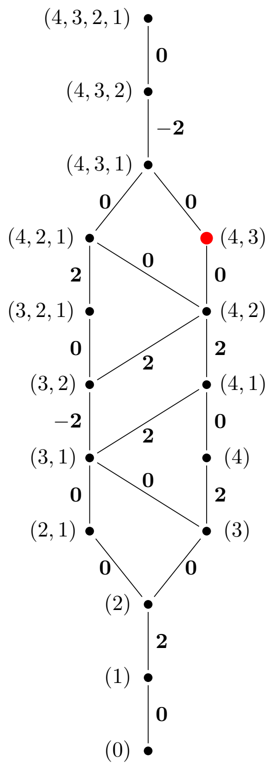
$$D^k \xrightarrow{\cong} D^{i_1} \times \ldots \times D^{i_m} \longrightarrow \mathbb{RP}^{i_1} \times \ldots \times \mathbb{RP}^{i_m} \longrightarrow SO(n),$$

where the second map is the product of the characteristic maps for the top cells of each real projective space. There is a single 0-cell, namely the identity of $SO(n)$.

This gives us all the information we need to construct the cellular chain complex of $SO(5)$. The differentials are determined by the differentials in the cellular chain complexes of real projective spaces, as well as the product formula

$$d(e^i \times e^j) = d(e^i) \times e^j + (-1)^i e^i \times d(e^j).$$

This provides a complete description of the cellular chain complex of $SO(5)$. In the following diagram each node is a cell, where for example $(2,1)$ is the cell given by the map $f_{2,1}\colon \mathbb{RP}^2 \times \mathbb{RP}^1 \to SO(5)$. The line segments indicate when a cell is contained in another cell.



(15)

4.2. **A geometric cobordism class on $SO(5)$.** From the structure of the complex displayed in (15), we see that $H_7(SO(5);\mathbb{Z}) \cong \mathbb{Z}$, where the generator comes from the cell $(4,3)$, coloured red in the diagram. However, since $\mathbb{RP}^4$ is not orientable, the map $f_{4,3}\colon \mathbb{RP}^4 \times \mathbb{RP}^3 \to SO(5)$ does not represent a bordism class. In fact, we know from the algebraic obstruction that the image of e_3 under $H_7(SO(5))$ cannot be hit by a bordism class on $SO(5)$. We will now show that we can replace $\mathbb{RP}^4 \times \mathbb{RP}^3$ with an orientable smooth manifold M and the map $f_{4,3}$ by a smooth map $g\colon M \to SO(5)$ with the same image as $f_{4,3}$.

To do so, we first observe that the cell decomposition implies that every element of $SO(5)$ can be expressed as a composition of reflections in $\mathbb{R}^5$, where every pair of reflections leaves a 3-dimensional subspace fixed and performs a rotation in the remaining 2-plane. Let $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ be the Grassmann manifold of *oriented* 2-dimensional planes in $\mathbb{R}^5$. We will write elements of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5)$ in the form (L,σ), where L is a plane and σ is an orientation of L. We then define

$$g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow SO(5)$$

to be the map that sends $((L,\sigma),e^{it})$ to the element of $SO(5)$ which rotates the plane L by the angle t according to the orientation σ. More precisely, given a point $((L,\sigma),e^{it}) \in \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$, let $r_{L,\sigma,t}$ be the rotation of L by the angle t along σ. Let $L^\perp$ denote the orthogonal complement of L in $\mathbb{R}^5$ with the respect to the standard inner product. Then we can write $v \in \mathbb{R}^5$ in a unique way as $v = v_1 + v_2$ such that $v_1 \in L$ and $v_2 \in L^\perp$. The transformation $g((L,\sigma),e^{it}) \in SO(5)$ is then defined by

$$g((L,\sigma),e^{it})(v) = r_{L,\sigma,t}(v_1) + v_2.$$

The transformation $g((L,\sigma),e^{it}) \in SO(5)$ varies smoothly with (L,σ) and t in $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$.

Lemma 4.1. *The map g admits a complex orientation. In particular, g is a proper complex-oriented smooth map and represents an element in $MU^3(SO(5))$.*

Proof. The fact that g admits a complex orientation follows from the facts that S^1 is stably almost complex, $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \cong SO(5)/(SO(2) \times SO(3))$ is almost complex, and $SO(5)$ is a compact Lie group. □

Lemma 4.2. *The images of the maps $f_{4,3}$ and g in $SO(5)$ are equal, i.e., the image of the map $g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \to SO(5)$ is the cell $(4,3)$.*

Proof. To simplify the notation we write $f = f_{4,3}$. We begin with showing that $\mathrm{Im}\, f \subseteq \mathrm{Im}\, g$. We let $(u,v) \in \mathbb{RP}^4 \times \mathbb{RP}^3$, and we will show that there exist two elements in $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$ which map to the element $f(u,v) \in SO(5)$. When showing this we will assume that u and v are both different from $\pm e_1$, since otherwise the argument is similar.

The 4-planes $e_1^\perp$ and $u^\perp$ intersect on a 3-dimensional subspace of $\mathbb{R}^5$ which remains fixed under the map $r(u) \cdot r(e_1)$. We will call this subspace M_u. Likewise, we let M_v denote the 3-dimensional subspace fixed by $r(v) \cdot r(e_1)$. One can observe that both M_u and M_v are contained in $\mathrm{Span}\{e_2, e_3, e_4, e_5\}$.

We first deal with the case where $u = v$. Then $M_u = M_v$, and it follows that $f(u,v)$ is a rotation in the plane $L = M_u^\perp$. For each of the two possible orientations of L, there is precisely one angle in S^1 which gives the rotation corresponding to $f(u,v)$, which shows that there are two elements of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$ which map to $f(u,v)$.

On the other hand, if $u \neq v$, we get that $M_u \cap M_v$ is a 2-dimensional subspace of $\mathrm{Span}\{e_2, e_3, e_4, e_5\}$. Let $N = (M_u \cap M_v)^\perp$. We observe that $f(u,v) \in SO(5)$ maps N to N. Since N is 3-dimensional, a 1-dimensional subspace of N is left fixed by $f(u,v)$, and we call this line T. We have now seen that $f(u,v)$ leaves $T \oplus (M_u \cap M_v)$ fixed. The remaining 2-dimensional subspace of N is then our choice of L, in other words

$$L := (T \oplus (M_u \cap M_v))^\perp.$$

Having found the plane where the rotation takes place, we may combine orientations σ and elements of S^1 as in the case $u = v$ to get the desired element of $SO(5)$. This proves that $\mathrm{Im}\, f \subseteq \mathrm{Im}\, g$.

We now show that $\mathrm{Im}\, g \subseteq \mathrm{Im}\, f$. Let

$$((L,\sigma),e^{it}) \in \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1.$$

Our goal is to find vectors u and v such that $f(u,v) = g((L,\sigma), e^{it})$. We can first observe that $L \cap \mathbb{R}^4$ is at least 1-dimensional. We then choose v' to be any unit vector in this intersection and note that v' represents a point in $\mathbb{RP}^3$. Next, we want to find a suitable $u \in \mathbb{RP}^4$ such that

$$r(u) \cdot r(v') = g((L,\sigma), e^{it}).$$

Clearly, u must be in L, since then $L^\perp$ is fixed by both $r(u)$ and $r(v')$. Furthermore, the angle between u and v' is uniquely determined by t to yield the desired rotation. (In fact, the angle must be $t/2$ or $t/2 + \pi$, depending on which representative we choose for the point in $\mathbb{RP}^4$.) With the exception of the cases $t = 0$ and $t = \pi$, this leaves two options for u, which we choose between by making sure the rotation $r(u) \cdot r(v')$ goes in the right direction according to the orientation σ. The composition $r(v') \cdot r(e_1)$ has a unique inverse, which is given by $r(v) \cdot r(e_1)$ for some vector v in $\mathbb{RP}^3$. We then have

$$\begin{aligned} f(u,v) &= r(u) \cdot r(e_1) \cdot r(v) \cdot r(e_1) \\ &= r(u) \cdot r(e_1) \cdot [r(v') \cdot r(e_1)]^{-1} \\ &= r(u) \cdot r(e_1) \cdot r(e_1) \cdot r(v') \\ &= r(u) \cdot r(v') \\ &= g((L,\sigma), e^{it}). \end{aligned}$$

This proves that $\operatorname{Im} f = \operatorname{Im} g$. $\square$

We can now show the main result of this section.

Theorem 4.3. *The cobordism class represented by g maps to $2e_3 \in H^3(SO(5);\mathbb{Z})$ under the Thom morphism.*

Proof. By Poincaré duality it suffices to show that the fundamental class of $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1$ in homology is mapped to two times the generator $\tilde{e}_3 \in H_7(SO(5);\mathbb{Z})$. By lemma 4.2, the image of g is the cell $(4,3)$. Hence we can consider g as a map

$$\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \longrightarrow (4,3).$$

Let

$$q\colon (4,3) \longrightarrow (4,3)/(4,3)_6$$

be the map that collapses the 6-skeleton of the cell $(4,3)$. We then get the following commutative diagram in homology

$$\begin{tikzcd} H_7(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1;\mathbb{Z}) \arrow[r, "g_*"] \arrow[dr, "(q\circ g)_*"'] & H_7((4,3);\mathbb{Z}) \arrow[d, "\cong"', "q_*"] \\ & H_7((4,3)/(4,3)_6;\mathbb{Z}). \end{tikzcd}$$

Using the homology long exact sequence of the pair $((4,3)_6, (4,3))$, it is straightforward to see that the map q_* is an isomorphism. The quotient $(4,3)/(4,3)_6$ is homeomorphic to S^7. Hence by choosing an orientation we can assume that $q \circ g$ is a map between compact, oriented topological manifolds. Proving that g_* is a multiplication by ± 2 is hence reduced to the claim that the map $(q \circ g)_*$ has degree

± 2. We will show this claim by computing the local degree of $q \circ g$ at two points in $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \times S^1$.

Let $y \in SO(5)$ be the point corresponding to the matrix

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then y defines a rotation by the angle π in the plane $\mathrm{Span}\{\mathrm{e}_4, \mathrm{e}_5\}$. If we set $L = \mathrm{Span}\{\mathrm{e}_4, \mathrm{e}_5\}$, and let $\pm\sigma$ be the two possible orientations of L, we get

$$g((L,\sigma), e^{i\pi}) = g((L,-\sigma), e^{i\pi}) = y,$$

and these are the only two points that are sent to y under g. Since $\mathrm{Im}\, g = \mathrm{Im}\, f$ by lemma 4.2, it follows that $y \in (4,3)$. We now define open neighborhoods of the points $((L, \pm\sigma), e^{i\pi})$ that are mapped homeomorphically to an open neighborhood of y. Let

$$\mathcal{U} = \{(M,\sigma) \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \mid M \cap \mathrm{Span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\} = 0\}.$$

Since $\mathcal{U}$ consists of two open path-components, so does the product

$$\mathcal{U} \times (S^1 \setminus \{e^0\}).$$

We denote these two path-components of $\mathcal{U} \times (S^1 \setminus \{e^0\})$ by $\mathcal{U}^+$ and $\mathcal{U}^-$. We then have $((L,\sigma), e^{i\pi}) \in \mathcal{U}^+$ and $((L,-\sigma), e^{i\pi}) \in \mathcal{U}^-$. We let $\mathcal{V} \subset (4,3)/(4,3)_6$ be the interior of the cell $(4,3)$. We observe that $\mathcal{V}$ consists of the rotations of $\mathbb{R}^5$ that do not leave any nonzero vector in $\mathrm{Span}\{\mathrm{e}_4, \mathrm{e}_5\}$ fixed, which corresponds to the planes in $\mathcal{U}$. From the proof of lemma 4.2 we deduce that g maps precisely two points of $\mathcal{U} \times (S^1 \setminus \{e^0\})$ to every point in $\mathcal{V}$. This implies that g sends $\mathcal{U}^+$ and $\mathcal{U}^-$ homeomorphically to $\mathcal{V}$. Thus, at each of the points $((L, \pm\sigma), e^{i\pi})$, the map $q \circ g$ has local degree $+1$ or -1.

Now we observe that the points

$$((L,\sigma), e^{it}) \text{ and } ((L,-\sigma), e^{-it})$$

have the same image in $SO(5)$ under g. The map $S^1 \to S^1$, $e^{it} \mapsto e^{-it}$, reverses the orientation. We recall that $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^5)$ is a double cover of the unoriented Grassmannian $\mathrm{Gr}_2(\mathbb{R}^5)$, and that $\mathrm{Gr}_2(\mathbb{R}^5)$ is not orientable. It then follows from [14, Theorem 15.36] that the map

$$\begin{aligned} \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) &\longrightarrow \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \\ (L,\sigma) &\longmapsto (L,-\sigma) \end{aligned}$$

is not orientation-preserving, since this map is the only non-trivial automorphism of $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^5)$ compatible with the projection to $\mathrm{Gr}_2(\mathbb{R}^5)$. Hence, the map

$$\begin{aligned} \varepsilon\colon \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \times S^1 &\longrightarrow \widetilde{\mathrm{Gr}}_2(\mathbb{R}^5) \times S^1 \\ ((L,\sigma), e^{it}) &\longmapsto ((L,-\sigma), e^{-it}) \end{aligned}$$

is a product of two maps which reverse the orientation. Thus, ε preserves the orientation. By the construction of ε, the diagram

$$\begin{tikzcd} \mathcal{U}^+ \arrow[d, leftrightarrow, "\varepsilon"'] \arrow[dr, "(q\circ g)_{|\mathcal{U}^+}"] & \\ & \mathcal{V} \\ \mathcal{U}^- \arrow[ur, "(q\circ g)_{|\mathcal{U}^-}"'] & \end{tikzcd}$$

commutes. Since ε preserves the orientation, we have that either both $(q \circ g)_{|\mathcal{U}^+}$ and $(q \circ g)_{|\mathcal{U}^-}$ preserve the orientation, or they both reverse it. Thus, $q \circ g$ has the same local degree at the points $((L, \pm\sigma), e^{i\pi})$, and we conclude that g_* is a multiplication by ± 2, which completes the proof. $\square$

By lemma 2.2, theorem 4.3 implies the following result:

Corollary 4.4. *The class* $[g] \otimes \frac{1}{2}$ *is a non-trivial element in the kernel of*

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}} \colon MU^3(SO(5)) \otimes_{MU^*} \mathbb{R}/\mathbb{Z} \longrightarrow H^3(SO(5);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

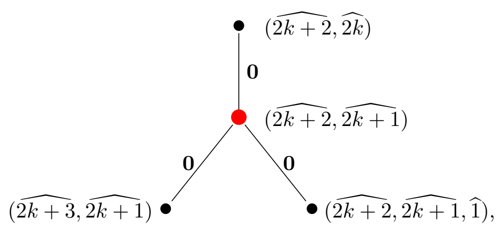
In the next section we explain how the method to prove theorem 4.3 can be generalised to special orthogonal groups of higher dimensions.

4.3. **Generalization to higher dimensions.** We first show which cells provide the generators for cohomology groups of $SO(n)$ and then make a generalised geometric construction.

Let $k \geq 0$ and $n \geq 2k+3$. Then there is a non-torsion generator e_{4k+3} in $H^{4k+3}(SO(n);\mathbb{Z})$. Using the cell structure of $SO(n)$, we can now determine a small part of the cellular chain complex of $SO(n)$. We use the following notation. Let $(i_1, \dots, i_m)$ be a sequence of integers with $n-1 \geq i_1 > i_2 > \dots > i_m \geq 1$. We let $(\widehat{i_1}, \dots, \widehat{i_m})$ denote the image of the map

$$f_{j_1,\dots,j_s} \colon \mathbb{R}P^{j_1} \times \dots \times \mathbb{R}P^{j_s} \longrightarrow SO(n),$$

where $(j_1, \dots, j_s)$ is the sequence obtained by removing the numbers $i_1, \dots, i_m$ from the sequence $(n-1, n-2, \dots, 1)$. We then have the diagram



from which we can see that e_{4k+3} comes from the cell $(\widehat{2k+2}, \widehat{2k+1})$.

Using the methods in section 3.1.1 we can show that for every k, the generator e^{4k+3} is not in the image of the Thom morphism for sufficiently large n. Determining a minimal such n is more difficult, and we have been unable to find a more efficient method than to study the Bockstein diagrams on a case by case basis. However, we will now show how a multiple of e_{4k+3} can always be constructed geometrically, whether or not e_{4k+3} itself is in the image of the Thom morphism.

Let $n \geq 3$ be odd. We can then define the map

$$g_n\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n) \times S^1 \longrightarrow SO(n)$$

in the same way as the map $g\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1 \to SO(5)$ in section 4.2. For $m > n$, we will also denote by g_n the composition $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^n) \times S^1 \to SO(n) \hookrightarrow SO(m)$ with the canonical embedding of $SO(n)$ into $SO(m)$.

Lemma 4.5. *For every $n \geq 3$ odd and every $m \geq n$, the map $g_n\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n) \times S^1 \to SO(m)$ admits a complex orientation. In particular, g_n is a proper complex-oriented smooth map and represents an element in $MU^*(SO(m))$.*

Proof. This follows again from the facts that S^1 is stably almost complex, $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^n)$ is almost complex, and $SO(m)$ is a compact Lie group. □

We define the map

$$\begin{aligned} \varepsilon_n\colon \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n) \times S^1 &\longrightarrow \widetilde{\mathrm{Gr}_2}(\mathbb{R}^n) \times S^1 \\ ((L,\sigma), e^{it}) &\longmapsto ((L,-\sigma), e^{-it}). \end{aligned}$$

Lemma 4.6. *The maps τ_n, g_n and $f_{i_1,\dots,i_m}$ have the properties*
(i) $\mathrm{Im}\, g_n = \mathrm{Im}\, f_{n-1,n-2}$
(ii) $g_n \circ \varepsilon_n = g_n$
(iii) ε_n *is orientation-preserving.*

Proof. This follows from similar arguments as in the proofs of lemma 4.2 and theorem 4.3. □

We can now construct the cobordism class which maps to a multiple of $e_{4k+3} \in H^{4k+3}(SO(m))$. The construction depends on whether m is even or odd, and we start with $m = 2n+1$ odd. For k and m with $2k+1 < m$, let i denote the canonical embedding $SO(2k+1) \to SO(m)$. We can now define the map

$$\begin{aligned} &h_{2n+1,k} := g_{2n+1} \times g_{2n-1} \times \cdots \times g_{2k+5} \times i\colon \\ &\prod_{l=k+2}^{n} \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^{2l+1}) \times S^1\right) \times SO(2k+1) \longrightarrow SO(2n+1). \end{aligned} \tag{16}$$

It follows from lemma 4.6 that the image of this map is the cell

$$(2n, 2n-1, \dots, 2k+3, 2k, \dots, 2, 1) = (\widehat{2k+2}, \widehat{2k+1}).$$

If $m = 2n$ is even, then we need to define one more map. Given the map $f_k\colon \mathbb{RP}^k \to SO(m)$, let f'_k be the composite map

$$S^k \longrightarrow \mathbb{RP}^k \xrightarrow{f_k} SO(m),$$

where the first map is the canonical double cover. We can then define

$$\begin{aligned} &h_{2n,k} := f'_{2n-1} \times g_{2n-1} \times g_{2n-3} \times \cdots \times g_{2k+5} \times i\colon \\ &S^{2n-1} \times \prod_{l=k+2}^{n-1} \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^{2l+1}) \times S^1\right) \times SO(2k+1) \longrightarrow SO(2n). \end{aligned} \tag{17}$$

Again, it follows from lemma 4.6 that the image of this map is the cell

$$(2n-1, 2n-2, \dots, 2k+3, 2k, \dots, 2, 1) = (\widehat{2k+2}, \widehat{2k+1}).$$

Theorem 4.7. *Let $k \geq 0$ and $n \geq k+1$. Then the Thom morphism sends the cobordism class represented by the map $h_{2n+1,k}$ in $MU^{4k+3}(SO(2n+1))$ to $2^{n-k-1}e_{4k+3} \in H^{4k+3}(SO(2n+1);\mathbb{Z})$. If $n \geq k+2$, then the Thom morphism sends the cobordism class represented by the map $h_{2n,k}$ in $MU^{4k+3}(SO(2n))$ to $2^{n-k-1}e_{4k+3} \in H^{4k+3}(SO(2n);\mathbb{Z})$.*

Proof. The assertion follows as in the proof of theorem 4.3 from the fact that each factor $\widetilde{\mathrm{Gr}_2}(\mathbb{R}^{2l+1}) \times S^1$ is wrapped twice around the cell $(2l, 2l-1)$. □

Corollary 4.8. *For every $k \geq 0$, there is a sufficiently large integer m such that the class $[h_{m,k}] \otimes \frac{1}{2^{n-k-1}}$ with $n = \lfloor \frac{m}{2} \rfloor$ is a non-trivial element in the kernel of*

$$\bar{\tau}_{\mathbb{R}/\mathbb{Z}} \colon MU^*(SO(m)) \otimes_{MU^*} \mathbb{R}/\mathbb{Z} \longrightarrow H^*(SO(m);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Remark 4.9. There is a notable case where the construction of map (17) can be simplified. For $SO(8)$, the factor S^7 can be replaced by $\mathbb{RP}^7$, since this space is parallelisable. Thus, the map

$$f_7 \times g_7 \times g_5 \colon \mathbb{RP}^7 \times \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^7) \times S^1\right) \times \left(\widetilde{\mathrm{Gr}_2}(\mathbb{R}^5) \times S^1\right) \longrightarrow SO(8)$$

represents an element of $MU^3(SO(8))$ which is mapped to $4e_3 \in H^3(SO(8);\mathbb{Z})$.

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Paper II

Geometric cohomology classes for the Lie groups Spin(7) and Spin(8)

Eiolf Kaspersen and Gereon Quick

Preprint

This paper is awaiting publication and is not included in NTNU Open

Paper III

A note on the Thom morphism for the classifying space of certain Lie groups and gauge groups

Eiolf Kaspersen and Gereon Quick

Preprint
Available at arXiv:2405.12717

A NOTE ON THE THOM MORPHISM FOR THE CLASSIFYING SPACE OF CERTAIN LIE GROUPS AND GAUGE GROUPS

EIOLF KASPERSEN AND GEREON QUICK

Abstract. We give a complete description of which non-torsion generators are not in the image of the Thom morphism from complex cobordism to integral cohomology for the classifying space of exceptional Lie groups except for E_8. We then show that the Thom morphism is not surjective for the classifying space of the gauge group of a principal E_7-bundle over the four-dimensional sphere. We use the results to detect nontrivial elements in the kernel of the reduced Thom morphism for Lie groups and their classifying spaces.

1. Introduction

Let G be a compact, connected Lie group, and let ξ denote a principal G-bundle

$$G \hookrightarrow P \xrightarrow{\pi} X$$

over a paracompact space X. Recall that the *gauge group* $\mathcal{G}(\xi)$ of ξ is defined to be the group of automorphisms of ξ, i.e.,

$$\mathcal{G}(\xi) = \{\phi \in \mathrm{Aut}_G(P) \mid \pi \circ \phi = \pi\}.$$

Gauge groups play an important role in geometry, topology and mathematical physics. Moreover, the classifying space $B\mathcal{G}(\xi)$ is homotopy equivalent to the moduli space of connections on ξ. Integral singular cohomology $H^*(B\mathcal{G}(\xi);\mathbb{Z})$ is a fundamental invariant of $B\mathcal{G}(\xi)$, and it is an important question which elements in $H^*(B\mathcal{G}(\xi);\mathbb{Z})$ are the fundamental class of a smooth manifold $M \to B\mathcal{G}(\xi)$. This question is closely related to the Thom morphism

$$\tau\colon MU^*(B\mathcal{G}(\xi)) \longrightarrow H^*(B\mathcal{G}(\xi);\mathbb{Z})$$

from complex cobordism to singular cohomology. The Thom morphism plays a key role in algebraic and geometric topology. The purpose of the present paper is to show that the Thom morphism for the classifying space of certain gauge groups is not surjective and thereby to show that there is a restriction for how non-torsion classes can be represented by fundamental classes.

The cohomology of $B\mathcal{G}(\xi)$ is closely related to that of BG, the classifying space of the Lie group G. The torsion in the cohomology of $B\mathcal{G}(\xi)$ has been studied in [9], [18] and [21], see also [12]. When the cohomology has torsion, the Thom morphism can be non-surjective. The question of when the Thom morphism is surjective for BG has previously been studied in for example [20] and [3] where the non-surjectivity of the Thom morphism for classifying spaces of certain Lie groups is used to detect new phenomena of the cycle map and Deligne cohomology in algebraic geometry.

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In section 2, we determine the image of the Thom morphism for the classifying spaces of $SO(2n)$ as well as the exceptional Lie groups, with the exception of E_8 where we only provide a partial result. We note that partial results were already known for $SO(4)$ and G_2. Therefore, our results significantly extend the known cases by providing a complete list of which non-torsion generators are in the image for G_2, F_4, E_6 and E_7. It turns out that the classifying space of E_7 behaves in a slightly different way compared to the other exceptional Lie groups. We do not know of a geometric explanation of this phenomenon. We note that some of our results could also have been deduced from the computations on BP-cohomology of Kono and Yagita in [14, Theorem 5.5]. In section 3 we study the image of the Thom morphism for the classifying spaces of the gauge groups of principal $SO(6)$- and E_7-bundles over spheres. Our main result is the following:

Theorem 1.1. *Let ξ be a principal E_7-bundle over S^4. Then there is a non-torsion generator in $H^4(B\mathcal{G}(\xi);\mathbb{Z})$ which is not in the image of the Thom morphism.*

As explained in [20], the non-surjectivity of τ can be used to construct examples where the reduced Thom morphism

$$\bar{\tau}\colon MU^*(X \times B\mathbb{Z}/p) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^*(X \times B\mathbb{Z}/p;\mathbb{Z})$$

is not injective. The non-injectivity of the reduced Thom morphism is crucial for the applications in algebraic geometry in [20]. In section 4, we show that $\bar{\tau}$ is not injective when X is the classifying space of $SO(n)$ with n even or an exceptional Lie group and $p = 2$. We note that the work of Kono and Yagita in [14] implies a stronger statement on the integral reduced Thom morphism for the classifying spaces of F_4 and E_6. In Theorem 4.5, we provide a complete list of the simplest cases of non-injectivity of $\bar{\tau}$ for X a compact connected Lie group with simple Lie algebra.

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2. Classifying spaces of Lie groups

Let X denote a finite CW-complex. It is a well-known fact that all Steenrod operations of odd degree vanish on the image of $MU^*(X)$ in $H^*(X;\mathbb{Z}/p)$ for all prime numbers p (see for example [20, page 468], [3, Proposition 3.6], [4]). Hence, in order to show that an element $x \in H^*(X;\mathbb{Z})$ is not in the image of the Thom morphism, it suffices to find a Steenrod operation of odd degree which does not vanish on $r(x)$ where $r\colon H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}/p)$ denotes the reduction map. We will now apply this observation to the cases where X is $BSO(n)$ or the classifying space of an exceptional Lie group. We let $H^*_{\text{free}}(X;\mathbb{Z})$ denote the quotient of $H^*(X;\mathbb{Z})$ by the torsion subgroup. When there is no risk of confusion we often use the same notation for an element in $H^*(X;\mathbb{Z})$ and its image in $H^*_{\text{free}}(X;\mathbb{Z})$.

2.1. **Special Orthogonal Groups.** Recall from [17, Theorem III.5.16] that the free cohomology of $BSO(n)$ is given by

$$H^*_{\text{free}}(BSO(n);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}[e_4, e_8, \dots, e_{2n-2}], & n \text{ odd} \\ \mathbb{Z}[e_4, e_8, \dots, e_{2n-4}, \chi_n], & n \text{ even.} \end{cases}$$

Theorem 2.1. *Let $n \geq 4$ be even and let $z \in H^n(BSO(n);\mathbb{Z})$ be a torsion class (or 0). Then $(\chi_n + z) \in H^n(BSO(n);\mathbb{Z})$ is not in the image of the Thom morphism.*

Proof. From [17, Theorems III.3.19 and III.5.12] we know that the mod 2 cohomology of $BSO(n)$ is given by

$$H^*(BSO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[y_2, y_3, \ldots, y_n],$$

with

$$\mathrm{Sq}^j(y_k) = \sum_{i=0}^{j} \binom{k-i-1}{j-i} y_{k+j-i} y_i. \tag{1}$$

Let r denote the mod 2-reduction map $H^*(BSO(n);\mathbb{Z}) \to H^*(BSO(n);\mathbb{Z}/2)$. Then

$$\begin{aligned} r(e_{4k}) &= y_{2k}^2 + (\text{other terms}) \\ r(\chi_n) &= y_n + (\text{other terms}), \end{aligned}$$

since y_{2k}^2 and y_n are neither the image nor the source of a nontrivial Bockstein homomorphism. Furthermore, we have

$$\mathrm{Sq}^3(y_n) = \sum_{i=0}^{3} \binom{n-1-i}{3-i} y_{n+3-i} y_i = y_3 y_n \neq 0.$$

We will now show that no other term in $r(\chi_n + z)$ is mapped to $y_3 y_n$ by Sq^3. It follows from equation (1) that the only other element of $H^{n-3}(BSO(n);\mathbb{Z}/2)$ which can map to $y_3 y_n$ under Sq^3 is $y_2 y_{n-2}$. For this element, the relevant part of the Bockstein diagram for $BSO(n)$ is the following:

$$\begin{tikzcd}
\text{Degree:} & n & n+1 & n+2 \\
\text{Generators:} & y_2 y_{n-2} \arrow[r] \arrow[dr] & y_3 y_{n-2} \arrow[dr] & \\
 & & y_2 y_{n-1} \arrow[r] & y_3 y_{n-1}
\end{tikzcd}$$

where there are no other nontrivial Bocksteins going into or out of any of these elements (see for example [7, Chapter 3E] and [11, Section 2.2] for the use of Bockstein cohomology). It follows that $y_2 y_{n-2}$ does not generate a nontrivial element of the Bockstein cohomology of $BSO(n)$, and thus it is not one of the summands in $r(\chi_n + z)$. Thus, we have $\mathrm{Sq}^3(r(\chi_n + z)) \neq 0$. The statement then follows from the fact that all odd-degree elements of the Steenrod algebra vanish on the image of the Thom morphism. □

2.2. **Exceptional Groups.** Recall from [17, Theorems VI.5.5 and VI.5.10] that the free cohomologies of the classifying spaces of the simply connected exceptional Lie groups are given by

$$\begin{aligned}
H^*_{\text{free}}(BG_2;\mathbb{Z}) &\cong \mathbb{Z}[e_4, e_{12}] \\
H^*_{\text{free}}(BF_4;\mathbb{Z}) &\cong \mathbb{Z}[e_4, e_{12}, e_{16}, e_{24}] \\
H^*_{\text{free}}(BE_6;\mathbb{Z}) &\cong \mathbb{Z}[e_4, e_{10}, e_{12}, e_{16}, e_{18}, e_{24}] \\
H^*_{\text{free}}(BE_7;\mathbb{Z}) &\cong \mathbb{Z}[e_4, e_{12}, e_{16}, e_{20}, e_{24}, e_{28}, e_{36}] \\
H^*_{\text{free}}(BE_8;\mathbb{Z}) &\cong \mathbb{Z}[e_4, e_{16}, e_{24}, e_{28}, e_{36}, e_{40}, e_{48}, e_{60}].
\end{aligned}$$

Theorem 2.2. *Let $G = G_2$, F_4 or E_6. Then the generator $e_4 \in H^*(BG;\mathbb{Z})$ is not in the image of the Thom morphism, while all other non-torsion generators are in the image.*

Proof. The mod 2 cohomology of each of the classifying spaces is given by

$$\begin{aligned} H^*(BG_2;\mathbb{Z}/2) &\cong \mathbb{Z}/2[y_4, y_6, y_7, y_{10}] \\ H^*(BF_4;\mathbb{Z}/2) &\cong \mathbb{Z}/2[y_4, y_6, y_7, y_{16}, y_{24}] \\ H^*(BE_6;\mathbb{Z}/2) &\cong \mathbb{Z}/2[y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{34}, y_{48}]/I, \end{aligned}$$

where I denotes the ideal given by

$$I = \langle y_7y_{10}, y_7y_{18}, y_7y_{34}, y_{34}^2 + y_{18}^2y_{32} + y_{10}^2y_{48} + y_6y_{10}y_{18}y_{34} + y_4y_{10}y_{18}^3 + y_4y_{10}^3y_{34}\rangle$$

see [17, Corollary VII.6.3 and Theorem VII.6.6] and [10, Theorem 1.1 and Proposition 5.1]. In each of the four cases, the reduction map $H^*(BG;\mathbb{Z}) \to H^*(BG;\mathbb{Z}/2)$ sends e_4 to y_4, and in each case $\mathrm{Sq}^3(y_4) = y_7$. This proves the first claim since Sq^3 vanishes on the image of the Thom morphism. For the other generators one can check that all differentials in the Atiyah–Hirzebruch spectral sequence vanish, which proves the second claim as explained in for example [11, Section 2.1] or [20, page 471]. □

As the following theorem shows, the situation is a bit different for the classifying space BE_7. While in the previous cases only the generator in degree 4 is not hit by the Thom morphism, for E_7 there are several generators which are not in the image of τ.

Theorem 2.3. *The generators $e_4, e_{16}, e_{24}, e_{28} \in H^*(BE_7;\mathbb{Z})$ are not in the image of the Thom morphism, and nor are the sums of any of these generators with a 2-torsion element in the same degree. For the generator $e_{36} \in H^{36}_{free}(BE_7;\mathbb{Z})$, there exist lifts to $H^{36}(BE_7;\mathbb{Z})$ which are not in the image of the Thom morphism, while other lifts are in the image. The non-torsion generators e_{12} and e_{20} are in the image of the Thom morphism.*

Proof. The mod 2 cohomology of BE_7 is given by

$$\begin{aligned} H^*(BE_7;\mathbb{Z}/2) \cong \mathbb{Z}/2[&y_4, y_6, y_7, y_{10}, y_{11}, y_{18}, y_{19}, \\ &y_{34}, y_{35}, y_{64}, y_{66}, y_{67}, y_{96}, y_{112}]/J, \end{aligned}$$

where J denotes the ideal given by

$$\begin{aligned} J =\langle &y_6y_{11} + y_7y_{10}, y_6y_{19} + y_7y_{18}, y_{10}y_{19} + y_{11}y_{18}, y_{11}^3 + y_7^2y_{19}, y_{11}y_{19}^2, y_{19}^3, \\ &y_7y_{34} + y_6x_{35} + y_{11}^2y_{19}, y_{11}y_{34} + y_{10}y_{35} + y_7y_{19}^2, y_{19}y_{34} + y_{18}y_{35}, \\ &y_{11}y_{35}^2 + y_7^2y_{67}, y_{19}y_{35}^2 + y_{11}^2y_{67}, y_{19}^2y_{67}, y_{34}^4 + y_{18}^4y_{64} + y_{10}^4y_{96} + y_6^4y_{112}, \\ &y_{35}^4 + y_{19}^4y_{64} + y_{11}^4y_{96} + y_7^4y_{112}, y_{66}^2 + y_{10}^2y_{112} + y_{18}^2y_{96}, y_{67}^2 + y_{11}^2y_{112} + y_{19}^2y_{96}\rangle, \end{aligned}$$

see [13, Theorem 2.8]. The action of the Steenrod algebra on these generators can be found in [13, page 276 and Corollary 6.9]. We then find that

$$\begin{aligned} \mathrm{Sq}^3(r(e_4)) &= \mathrm{Sq}^3(y_4) = y_7 \\ \mathrm{Sq}^{15}(r(e_{16})) &= \mathrm{Sq}^{15}(y_6y_{10}) = y_6^2y_{19} + y_{10}^2y_{11} + y_4y_7y_{10}^2 \\ \mathrm{Sq}^3(r(e_{24})) &= \mathrm{Sq}^3(y_6y_{18}) = y_7y_{10}^2, \end{aligned}$$

which implies that these elements are not in the image of the Thom morphism. By analysing the Bockstein homomorphisms, we see that an element of $H^{28}(BE_7;\mathbb{Z})$ which maps to the generator $e_{28} \in H^{28}_{\text{free}}(BE_7;\mathbb{Z})$ can be mapped to either $y_{10}y_{18}$ or $y_{10}y_{18} + y_7^4$ by r. Both of these elements are mapped to $y_{10}^2 y_{11}$ by Sq^3, so e_{28} plus torsion is not in the image of the Thom morphism either.

Finally, we have the following Bockstein diagram in degree 36:

$$
\begin{tikzcd}
35 & 36 & 37 \\
y_4y_6y_7y_{18} \arrow[r] \arrow[dr] & y_4y_6y_7y_{19} \arrow[r] & y_4y_7^2y_{19} \\
y_4^6y_{11} & y_4y_7^2y_{18} \arrow[ur] & y_4^3y_6^3y_7 \\
y_4^3y_6y_7y_{10} \arrow[r] \arrow[dr] & y_4^3y_7^2y_{10} \arrow[r] & y_4^3y_7^2y_{11} \\
y_4^2y_7y_{10}^2 & y_4^3y_6y_7y_{11} \arrow[ur] & y_6^5y_7 \\
y_6^3y_7y_{10} \arrow[r] \arrow[dr] & y_6^2y_7^2y_{10} \arrow[r] & y_6^2y_7^2y_{11} \\
y_4^3y_6^2y_{11} & y_6^3y_7y_{11} \arrow[ur] & y_4^6y_6y_7 \\
y_7y_{10}y_{18} \arrow[r] \arrow[dr] & y_7y_{11}y_{18} \arrow[r] & y_7y_{11}y_{19} \\
y_6y_{10}y_{19} \arrow[r] \arrow[dr] & y_7y_{10}y_{19} \arrow[ur] & y_4^5y_6y_{11} \\
y_6y_{11}y_{18} \arrow[r] \arrow[uur] & y_6y_{11}y_{19} \arrow[uur] & y_4^5y_7y_{10} \\
y_4^4y_{19} & y_4^5y_6y_{10} \arrow[uur] \arrow[ur] & y_4y_6^2y_{10}y_{11} \\
y_4^2y_6y_{10}y_{11} \arrow[r] \arrow[dr] & y_4^2y_7y_{10}y_{11} \arrow[r] & y_4^2y_7y_{11}^2 \\
y_4y_6^2y_{19} & y_4^2y_6y_{11}^2 \arrow[ur] & y_4^4y_{10}y_{11} \\
y_4y_7^3y_{10} \arrow[r] & y_4y_7^3y_{11} & y_4^2y_6^3y_{11} \\
y_4y_6y_7^2y_{11} \arrow[ur] & y_4^2y_6^3y_{10} \arrow[ur] \arrow[r] & y_4^2y_6^2y_7y_{10} \\
y_6y_7y_{11}^2 \arrow[r] & y_7^2y_{11}^2 & y_7y_{10}^3 \\
y_7^2y_{10}y_{11} \arrow[ur] & y_6y_{10}^3 \arrow[ur] \arrow[r] & y_6y_{10}^2y_{11} \\
y_4^2y_6y_7^3 \arrow[r] & y_4^2y_7^4 & y_4^2y_{11}y_{18} \\
y_4y_{10}^2y_{11} & y_4^2y_{10}y_{18} \arrow[ur] \arrow[r] & y_4^2y_{10}y_{19} \\
y_6^4y_{11} & y_4^3y_6y_{18} \arrow[r] \arrow[dr] & y_4^3y_7y_{18} \\
y_4^7y_7 & y_6^6 & y_4^3y_6y_{19} \\
y_4^4y_6^2y_7 & y_6^3y_{18} \arrow[r] \arrow[dr] & y_6^3y_{19} \\
y_4y_6^4y_7 & y_4^3y_6^4 & y_6^2y_7y_{18} \\
y_{35} & y_4y_{10}y_{11}^2 \arrow[r] & y_4y_{11}^3 \\
 & y_4y_6^3y_7^2 \arrow[r] & y_4y_6^2y_7^3 \\
 & y_4^4y_6y_7^2 \arrow[r] & y_4^4y_7^3 \\
 & y_4y_6^2y_{10}^2 & y_4y_6y_7y_{10}^2 \\
 & y_4^6y_6^2 & y_6y_7^3y_{10} \\
 & y_4^4y_{10}^2 & y_{18}y_{19} \\
 & y_4^9 & \\
 & y_{18}^2 &
\end{tikzcd}
$$

The diagram shows that an element of $H^{36}(BE_7;\mathbb{Z})$ corresponding to the generator $e_{36} \in H^{36}_{\mathrm{free}}(BE_7;\mathbb{Z})$ is mapped to $y_{18}^2 + L$ by r, where L is some linear combination of the elements

$$y_4 y_7^3 y_{11},\ y_7^2 y_{11}^2,\ y_4^2 y_7^4.$$

While all odd-degree elements of the Steenrod algebra act trivially on y_{18}^2, $y_7^2 y_{11}^2$ and $y_4^2 y_7^4$, we have $\mathrm{Sq}^3(y_4 y_7^3 y_{11}) = y_7^4 y_{11}$. Thus, any lift of e_{36} which contains the term $y_4 y_7^3 y_{11}$ is not in the image of the Thom morphism, while any lift which does not contain that term is in the image. Finally, for e_{12} and e_{20} one can again check that all differentials in the Atiyah–Hirzebruch spectral sequence vanish. □

Remark 2.4. As for the other exceptional Lie groups, the generator $e_4 \in H^4(BE_8;\mathbb{Z})$ is not in the image of the Thom morphism. This can for example be shown using the fact that BE_8 and the Eilenberg–MacLane space $K(\mathbb{Z},4)$ have homotopy equivalent 15-skeletons [8, page 185]. However, since the mod 2 cohomology of BE_8 is not known, we cannot give a complete answer to which other generators are in the image of the Thom morphism.

Remark 2.5. We note that some of our results could have been deduced from the computations of BP-cohomology of Kono and Yagita. Moreover, it follows from [14, Theorem 5.5] that $2e_4$ in $H^4(BG;\mathbb{Z})$ is not in the image of τ for $G = SO(4)$.

3. Gauge Groups

Let X be a paracompact space, and let ξ be a principal G-bundle over X. The bundle ξ is classified by a map $f\colon X \to BG$. Let $\mathrm{Map}_f(X, BG)$ denote the path component of $\mathrm{Map}(X, BG)$ which contains the map f. By [2, Proposition 2.4], $B\mathcal{G}(\xi)$ equals $\mathrm{Map}_f(X, BG)$ in homotopy theory, and we will consider $\mathrm{Map}_f(X, BG)$ as a model for $B\mathcal{G}(\xi)$. For $X = S^n$, there is a fibre sequence

$$\Omega^n_f(BG) \hookrightarrow \mathrm{Map}_f(S^n, BG) \xrightarrow{\mathrm{ev}} BG, \tag{2}$$

where $\Omega^n_f(BG)$ denotes the path component of $\Omega^n(BG)$ which contains f and ev denotes the evaluation map at the basepoint of S^n. Note that maps in $\mathrm{Map}(S^n, BG)$ are not required to be pointed, while maps in $\Omega^n(BG)$ are. We further note that $\Omega BG \cong G$ and that all path-components of $\Omega^{n-1}(G)$ are homotopy equivalent. Therefore, sequence (2) simplifies to

$$\Omega^{n-1}_0(G) \hookrightarrow \mathrm{Map}_f(S^n, BG) \xrightarrow{\mathrm{ev}} BG, \tag{3}$$

where $\Omega^{n-1}_0(G)$ denotes the path-component of $\Omega^{n-1}(G)$ which contains the constant map. If f is homotopic to a constant map, then the Serre spectral sequence of (3) collapses at the E_2-page, and consequently the Thom morphism is easily seen to be non-surjective for $\mathrm{Map}_f(S^n, BG)$ if it is non-surjective for BG. When f is not homotopic to a constant map, however, then the question of surjectivity of the Thom morphism is more interesting.

Lemma 3.1. *The cohomology group $\widetilde{H}^k(\Omega^3_0(E_7);\mathbb{Z}/2)$ is trivial for $k \le 7$.*

Proof. Let Q_i denote the ith Dyer–Lashof operation [1, Definition 4.1]

$$Q_i\colon H_k(\Omega^n X;\mathbb{Z}/2) \longrightarrow H_{2k+i}(\Omega^n X;\mathbb{Z}/2)$$

for $0 \leq i \leq n-1$, and let $\beta\colon H_k(X,\mathbb{Z}/2) \to H_{k-1}(X;\mathbb{Z}/2)$ denote the Bockstein homomorphism. By [6, Theorem 3.15] there is an isomorphism

$$H_*(\Omega_0^3(E_7);\mathbb{Z}/2) \cong \mathbb{Z}/2[Q_1^a\beta(u_{30})] \otimes \mathbb{Z}/2[Q_1^aQ_2^b(u_{30})] \otimes \left(\bigotimes_{k\in J} H_*(\Omega^3(S^k);\mathbb{Z}/2)\right),$$

where $J = \{11, 15, 19, 23, 27, 35\}$ and where a and b range over all non-negative integers. Since the reduced homology of the triple loop space of a sphere of dimension at least 11 is concentrated in degrees ≥ 8 (see [5, page 74]), we get that $\widetilde{H}_k(\Omega_0^3(E_7);\mathbb{Z}/2)$ is trivial for $k \leq 7$. By the universal coefficient theorem the same holds for $\widetilde{H}^k(\Omega_0^3(E_7);\mathbb{Z}/2)$. □

We now prove our main result.

Theorem 3.2. *Let ξ be a principal E_7-bundle over S^4. Then the image of* $\mathrm{ev}^*(e_4)$ *in $H^4(B\mathcal{G}(\xi);\mathbb{Z})$ is not in the image of the Thom morphism.*

Proof. Due to the homotopy equivalence $B\mathcal{G}(\xi) \simeq \mathrm{Map}_f(S^4, BE_7)$, it suffices to show that $\mathrm{ev}^*(e_4) \in H^4(\mathrm{Map}_f(S^4, BE_7);\mathbb{Z})$ is not in the image of the Thom morphism. We consider the mod 2 cohomology Serre spectral sequence of the fibre sequence

$$\Omega_0^3(E_7) \hookrightarrow \mathrm{Map}_f(S^4, BE_7) \xrightarrow{\mathrm{ev}} BE_7.$$

The E_2-page is given by

$$E_2^{p,q} = H^p(BE_7;\mathbb{Z}/2) \otimes H^q(\Omega_0^3(E_7);\mathbb{Z}/2).$$

By Lemma 3.1, the element $y_7 \in H^7(BE_7;\mathbb{Z}/2) \cong E_2^{7,0}$ is not in the image of any nontrivial differentials for degree reasons. Thus, $\mathrm{ev}^*(y_7) \in H^7(\mathrm{Map}_f(S^4, BE_7);\mathbb{Z}/2)$ is nonzero. Furthermore, the proof of Theorem 2.3 shows that $\mathrm{Sq}^3(r(e_4)) = y_7$. The commutative diagram

$$\begin{tikzcd}
H^4(BE_7;\mathbb{Z}) \arrow[r, "\mathrm{ev}^*"] \arrow[d, "r"] & H^4(\mathrm{Map}_f(S^4, BE_7);\mathbb{Z}) \arrow[d, "r"] \\
H^4(BE_7;\mathbb{Z}/2) \arrow[r, "\mathrm{ev}^*"] \arrow[d, "\mathrm{Sq}^3"] & H^4(\mathrm{Map}_f(S^4, BE_7);\mathbb{Z}/2) \arrow[d, "\mathrm{Sq}^3"] \\
H^7(BE_7;\mathbb{Z}/2) \arrow[r, "\mathrm{ev}^*"] & H^7(\mathrm{Map}_f(S^4, BE_7);\mathbb{Z}/2)
\end{tikzcd}$$

then shows that $\mathrm{Sq}^3 \circ r$ acts non-trivially on $\mathrm{ev}^*(e_4)$, which completes the proof. □

Remark 3.3. We can similarly examine whether the Thom morphism is surjective for principal $SO(n)$-bundles. Let ξ denote a principal $SO(n)$-bundle over S^2 where the classifying map f is not homotopic to the constant map. It was shown in [18, Theorem 1.2] that $B\mathcal{G}(\xi)$ has torsion if and only if $n \geq 5$. Since we found in Theorem 2.1 that the Thom morphism is non-surjective for $BSO(n)$ when n is even, a natural candidate for a gauge group whose classifying space has a non-surjective

Thom morphism is ξ as above when $n = 6$. Since $\Omega^2_f(BSO(6)) \cong \Omega_0(SO(6)) \cong \Omega(\mathrm{Spin}(6))$, there is a fibre sequence

$$\Omega(\mathrm{Spin}(6)) \hookrightarrow \mathrm{Map}_f(S^2, BSO(6)) \xrightarrow{\mathrm{ev}} BSO(6). \tag{4}$$

By [5, Lemma 2.2], we know that

$$H^*(\Omega(\mathrm{Spin}(6)); \mathbb{Z}/2) \cong \mathbb{Z}/2[a_2] \otimes \Gamma[c_6] \otimes \left(\bigotimes_{i=0}^{\infty} \mathbb{Z}/2[\gamma_{2^i}(b_4)]/(\gamma_{2^i}(b_4)^4) \right),$$

where $\Gamma[x]$ denotes the divided power algebra generated by $\{\gamma_0(x), \gamma_1(x), \dots\}$. It is then possible that the differential $d_9 \colon H^8(\Omega(\mathrm{Spin}(6)); \mathbb{Z}/2) \longrightarrow H^9(BSO(6); \mathbb{Z}/2)$ in the Serre spectral sequence of sequence (4) maps $\gamma_2(b_4)$ to y_3y_6, which would imply that the element y_3y_6 does not survive to the E_∞-page. However, we were unable to determine whether this differential is trivial or not and we therefore do not know whether the Thom morphism is surjective in this case.

4. Non-injectivity of the integral Thom morphism

Let X be a CW-complex and let X_k denote the k-skeleton. Recall that $MU^*(X)$ is said to *satisfy the Mittag–Leffler condition* if for all $n \geq 0$ there exists some $m \geq n$ such that $\mathrm{Im}\,(MU^*(X) \to MU^*(X_n)) = \mathrm{Im}\,(MU^*(X_m) \to MU^*(X_n))$. It follows from the Milnor short exact sequence that if $MU^*(X)$ satisfies the Mittag–Leffler condition, then $MU^*(X) = \varprojlim MU^*(X_n)$. Letting MU^* act on $\mathbb{Z}$ (or $\mathbb{Z}/p$) by having all generators in nonzero degree act trivially on $\mathbb{Z}$, we get a tensor product $MU^*(X) \otimes_{MU^*} \mathbb{Z}$. This is isomorphic to $MU^*(X)$ modulo an ideal contained in the kernel of the Thom morphism. Thus, the reduced Thom morphism

$$MU^*(X) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^*(X; \mathbb{Z})$$

is well-defined (see also [20, page 470]). The following lemma is a general version of [20, Corollary 5.3] which is formulated for classifying spaces of compact Lie groups only. The proof, however, is the same as in [20]. For completeness, we provide the reader with the full argument here since we will apply the assertion later in Theorem 4.5.

Lemma 4.1. *Let X be a CW-complex of finite type such that $MU^*(X)$ satisfies the Mittag–Leffler condition. Let p be a prime, k an integer and $x \in H^k(X; \mathbb{Z})$. Suppose that the image of the Thom morphism $MU^k(X) \to H^k(X; \mathbb{Z})$ contains px but no element y such that $py = px$. Then the Thom morphism*

$$MU^{k+2}(X \times B\mathbb{Z}/p) \otimes_{MU^*} \mathbb{Z} \to H^{k+2}(X \times B\mathbb{Z}/p; \mathbb{Z})$$

is not injective.

Proof. We first show that the map $MU^k(X) \otimes_{MU^*} \mathbb{Z}/p \to H^k(X; \mathbb{Z}/p)$ is not injective. Suppose that $\alpha \in MU^k(X)$ maps to px under the Thom morphism. If the element $(\alpha \otimes 1) \in MU^k(X) \otimes_{MU^*} \mathbb{Z}/p$ is zero, then $\alpha = p\beta$ for some $\beta \in MU^k(X)$. It follows that the Thom morphism maps β to some y with $py = px$, which contradicts the assumption. Therefore, $\alpha \otimes 1$ is nonzero. However, the Thom morphism maps $\alpha \otimes 1$ to $0 \in H^k(X; \mathbb{Z}/p)$. Thus $MU^k(X) \otimes_{MU^*} \mathbb{Z}/p \to H^k(X; \mathbb{Z}/p)$ is not injective.

For all finite CW-complexes Y, there is an isomorphism

$$MU^*(Y \times B\mathbb{Z}/p) \cong MU^*(Y) \otimes_{MU^*} MU^*(B\mathbb{Z}/p)$$

by [15, Theorem 2']. This implies

$$\begin{aligned} MU^*(X \times B\mathbb{Z}/p) &\cong \varprojlim MU^*(X_n \times B\mathbb{Z}/p) \cong \varprojlim (MU^*(X_n) \otimes_{MU^*} MU^*(B\mathbb{Z}/p)) \\ &\cong MU^*(X) \otimes_{MU^*} MU^*(B\mathbb{Z}/p). \end{aligned}$$

It then follows that

$$\begin{aligned} MU^*(X \times B\mathbb{Z}/p) \otimes_{MU^*} \mathbb{Z} &\cong (MU^*(X) \otimes_{MU^*} MU^*(B\mathbb{Z}/p)) \otimes_{MU^*} \mathbb{Z} \\ &\cong (MU^*(X) \otimes_{MU^*} \mathbb{Z}) \otimes_{MU^*} (MU^*(B\mathbb{Z}/p) \otimes_{\mathbb{Z}} \mathbb{Z}) \\ &\cong (MU^*(X) \otimes_{MU^*} \mathbb{Z}) \otimes_{\mathbb{Z}} (MU^*(B\mathbb{Z}/p) \otimes_{MU^*} \mathbb{Z}) \\ &\cong (MU^*(X) \otimes_{MU^*} \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}[c]/(pc)), \end{aligned}$$

where $|c| = 2$ and where the final isomorphism comes from the fact that the Thom morphism

$$MU^*(B\mathbb{Z}/p) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^*(B\mathbb{Z}/p; \mathbb{Z}) \cong \mathbb{Z}[c]/(pc)$$

is an isomorphism. Moreover, we get

$$\begin{aligned} &(MU^*(X) \otimes_{MU^*} \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}[c]/(pc)) \cong (MU^*(X) \otimes_{MU^*} \mathbb{Z})[c]/(pc) \\ &\cong (MU^*(X) \otimes_{MU^*} \mathbb{Z}) \oplus \prod_{i=1}^{\infty} (MU^*(X) \otimes_{MU^*} \mathbb{Z})\, c^i/(pc^i) \\ &\cong (MU^*(X) \otimes_{MU^*} \mathbb{Z}) \oplus \prod_{i=1}^{\infty} (MU^*(X) \otimes_{MU^*} \mathbb{Z}/p)\, c^i. \end{aligned}$$

Thus, the integral Thom morphism for the space $X \times B\mathbb{Z}/p$ is given by the composite map

$$\begin{tikzcd} (MU^*(X) \otimes_{MU^*} \mathbb{Z}) \oplus \prod_{i=1}^{\infty} (MU^*(X) \otimes_{MU^*} \mathbb{Z}/p)\, c^i \arrow[d] & \\ H^*(X; \mathbb{Z}) \oplus \prod_{i=1}^{\infty} H^*(X; \mathbb{Z}/p) c^i \arrow[r, "\cong"] & H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(B\mathbb{Z}/p; \mathbb{Z}) \arrow[d, hook] \\ & H^*(X \times B\mathbb{Z}/p; \mathbb{Z}). \end{tikzcd}$$

Since $MU^k(X) \otimes_{MU^*} \mathbb{Z}/p \to H^k(X; \mathbb{Z}/p)$ is not injective, we get that the subgroup $\big(MU^k(X) \otimes_{MU^*} \mathbb{Z}/p\big)c$ maps noninjectively to $H^{k+2}(X \times B\mathbb{Z}/p)$, and the statement follows. ☐

By combining the previous lemma with the results in section 2, we get multiple examples of when the Thom morphism is not injective.

Theorem 4.2. *Let $G = G_2$, F_4, E_6, E_7 or $SO(n)$ with $n \geq 4$ even. Then the Thom morphism*

$$MU^6(BG \times B\mathbb{Z}/2) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^6(BG \times B\mathbb{Z}/2; \mathbb{Z})$$

is not injective.

Proof. We first note that $H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$, where e_4 denotes the canonical generator. By Theorems 2.1, 2.2 and 2.3 e_4 is not in the image of the Thom morphism. However, by [16, Theorem 1], $MU^*(BG)$ satisfies the Mittag–Leffler condition, from which it follows that some integer multiple ne_4 is in the image of the Thom morphism. Assume that n is minimal. Since Sq^3 acts non-trivially on e_4, it follows that n is a multiple of 2. Setting $p = 2$, we then see that the conditions of Lemma

4.1 are satisfied, and the statement follows. In fact, assuming that $\alpha \in MU^4(BG)$ maps to ne_4, we see that the element

$$(\alpha \otimes 1)c \in (MU^4(BG) \otimes_{MU^*} \mathbb{Z}/2)c \subseteq MU^6(BG \times B\mathbb{Z}/2)$$

is mapped to 0 in $H^k(BG \times B\mathbb{Z}/2; \mathbb{Z})$. □

Remark 4.3. For E_7, we note that also other generators can be used to construct non-trivial elements in the kernel by Theorem 2.3.

Remark 4.4. We note that, for some of the groups, the work of Kono and Yagita allows for a stronger statement. One can deduce from [14, pages 795-796] that the reduced Thom morphism $MU^*(BG) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^*(BG; \mathbb{Z})$ is not injective for $G = F_4$ and $G = E_6$.

If X is a Lie group, and not the classifying space, the Thom morphism is in many cases not injective. In the following theorem we give a complete list for the simplest cases where injectivity fails.

Theorem 4.5. *Let G be a compact connected Lie group with simple Lie algebra. For an integer $n \geq 1$, let r denote the smallest natural number such that $2^r \mid n$. The Thom morphism*

$$MU^k(G \times B\mathbb{Z}/p) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^k(G \times B\mathbb{Z}/p; \mathbb{Z})$$

is not injective in the following cases:

Group	n	k	p
Spin(n)	$n \geq 7$	5	2
$SO(n)$	$n \geq 5$	5	2
$Ss(n)$	$8 \mid n$	5	2
$Ss(n)$	$8 \nmid n$	9	2
$PSO(n)$	$8 \mid n$	5	2
$PSO(n)$	$8 \nmid n,\ n \geq 10$	9	2
$PSp(n)$	n *even*	$2^{r+1} + 1$	2
$SU(n)/\Gamma_l$	$4 \mid n$, $l \equiv 2 \pmod 4$	$2^r + 1$	2
G_2		5	2
F_4		5	2 *or* 3
E_6		5	2 *or* 3
E_6/Γ_3		5	2
E_7		$5, 17$	2
E_7/Γ_2		5	3
E_7/Γ_2		17	2
E_8		$5, 17, 25, 29$	2

Proof. This follows from [11, Theorem 1.1] and Lemma 4.1. □

Remark 4.6. If $MU^*(B\mathcal{G}(\xi))$ satisfies the Mittag–Leffler condition for the bundle ξ of Theorem 3.2, then the Thom morphism

$$MU^6(B\mathcal{G}(\xi) \times B\mathbb{Z}/2) \otimes_{MU^*} \mathbb{Z} \longrightarrow H^6(B\mathcal{G}(\xi) \times B\mathbb{Z}/2; \mathbb{Z})$$

is not injective. However, we were unable to determine whether this is the case.

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