



On the existence of generalized solutions to a spatio-temporal predator–prey system with prey-taxis

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Abstract. In this paper we consider a pair of coupled nonlinear partial differential equations describing the interaction of a predator–prey pair including random movement as well as prey-taxis. We introduce a concept of generalized solutions and show the existence of such solutions in all space dimensions with the aid of a regularizing term. Additionally, we prove the weak–strong uniqueness of these generalized solutions and the existence of strong solutions at least locally in time for space dimension two and three.

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1. Introduction

The modelling approaches for predator–prey interactions are as diverse as the wildlife itself.

There are many different effects to be thought of, leading to different response functions as well as movements, see [33] for a derivation of a quite general class of models. The subject of the present work is the analysis of the prey-taxis model stated below. For Ω being a bounded C^2 -domain in \mathbb{R}^d for some $d \in \mathbb{N}$ with $d \geq 2$ and $T > 0$, we are going to consider the following model

$$\{\partial_t u - \nu \Delta u + \kappa \nabla \cdot (u \nabla w) = (\alpha w - \beta)u \quad \text{in } (0, T) \times \Omega, \quad (1.1a)$$

$$\partial_t w - \mu \Delta w = (\gamma - \delta u)w \quad \text{in } (0, T) \times \Omega, \quad (1.1b)$$

$$u = u_0 \quad \text{on } \{0\} \times \Omega, \quad (1.1c)$$

$$w = w_0 \quad \text{on } \{0\} \times \Omega, \quad (1.1d)$$

$$\nabla u \cdot n = 0 \quad \text{on } [0, T] \times \partial \Omega, \quad (1.1e)$$

$$\nabla w \cdot n = 0 \quad \text{on } [0, T] \times \partial \Omega. \quad (1.1f)$$

The unknown $u : [0, T] \times \Omega \rightarrow \mathbb{R}$ represents the density of predators, the unknown $w : [0, T] \times \Omega \rightarrow \mathbb{R}$ the density of prey and n denotes the outer normal vector of Ω . All other appearing parameters are positive constants.

The system considered is inspired by an application in biological pest control. In the production of ornamental plants, as for example roses, it is desirable to reduce the use of chemical pesticides. This can be achieved by releasing natural enemies of the pest involved, which do not have a damaging effect on the plants. A typical example of such a predator–prey pair is the two-spotted spider mite (*Tetranychus urticae*) and the predatory mite (*Phytoseiulus persimilis*), see [54] for a detailed discussion of this predator–prey pair.

In order to describe the interaction of these two populations over a bounded domain and on a finite time horizon, we consider a typical Lotka–Volterra system coupled with diffusive movement of both populations over the whole domain, modelling the random movement of the mite populations. The Lotka–Volterra model was first introduced in form of a system of ordinary differential equations in 1920 by Alfred J. Lotka [46] and in 1926 by Vito Volterra [58] describing the evolution of the number of predators and prey in time. In this model the prey population is assumed to grow exponentially with rate γ if there are no predators present and decline by predation with a rate proportional to the number of predators $-\delta u$. The predator population is assumed to decline exponentially in the absence of prey with rate $-\beta$ and has a natality rate proportional to the number of prey available αw . Even though the model has some drawbacks, as the lack of capturing saturation effects, it is a good starting point for the investigation of population dynamics, see [50, Ch. 3.1].

The considered model (1.1) is very close to the model introduced in [10] and identical to it except for the nonlinear higher-order coupling term $\kappa \nabla \cdot (u \nabla w)$, we include

this cross-diffusion term in order to model the predator's hunting behaviour as some directed movement towards higher concentrations of prey. This term was first introduced by Evelyn F. Keller and Lee A. Segel in 1970 and 1971, see [34] and [35], where the movement of one-celled organisms under the influence of some chemical attractant was considered. The classical Keller–Segel model

$$\begin{aligned}\partial_t u &= \Delta u - \nabla \cdot (u \nabla w), \\ \partial_t w &= \Delta w - w + u,\end{aligned}$$

where u denotes the density of the one-celled organisms and w the concentration of the chemical attractant, has been of large mathematical interest, see for example [28] for an extensive survey and the references therein. Of particular interest is the question whether finite time blow-up occurs or not. In 1992 blow-up was proven to exist in two spatial dimensions in a simplified setting where the second equation is only elliptic, see [30]. In 2001 this result was extended to the classical Keller–Segel model, see [29], where the existence of unbounded solutions was shown. The survey [40] provides a nice overview of the mathematical challenges of chemotaxis models due to blow-up. Even though for the model at hand the existence of finite time blow-up is, to the best of the authors knowledge, still an open question. The existing results on blow-up behaviour of models including a cross-diffusion term like $\kappa \nabla \cdot (u \nabla w)$ justify the usage of generalized solution concepts as we will introduce in this work.

Relying on field observations made on the behaviour of the ladybug beetle and the goldenrod aphid and on ideas from Keller and Segel, Peter Kareiva and Garrett Odell derived a model similar to the Keller–Segel model for the behavior of a predator–prey pair in 1987, see [33]. This indicates that the chemotaxis term $\nabla \cdot (u \nabla w)$ from the Keller–Segel model is also appropriate to model prey-taxis.

A significant contribution to the analysis of model (1.1) was made in 2017 by Michael Winkler, see [62]. Within this work the existence of global weak solution was proven for convex domains with space dimension $d \leq 5$.

Similar models to the one above have been of interest to researchers in the mathematical and numerical analysis of partial differential equations over the last decades [6]. Replacing the taxis coefficient κ by a function $\chi(u)$ dependent on u and presuming various conditions on this function, the existence of weak or even classical solutions to models similar to (1.1) is known. Assuming that $\chi(u)$ vanishes for large values of u and considering a different response function on the right-hand side, classical solutions are known to exist in dimensions $d = 1, 2, 3$, see [48, 57] and weak solutions exist in all space dimensions [7]. Presuming some smallness condition for the taxis coefficient κ , the existence of classical solution is also proven in all space dimensions in [64] and in [60], relying on a milder growth of either the prey or predator density. In [31] global existence of classical solutions for a prey-taxis model is shown with no restriction on the taxis coefficient but with dampened growth conditions of the predator and the prey, which prevent blow-up. These result were extended in [32] to include a prey-density dependence of the predators mobility. Renormalized solutions

were considered in [59]. Here, the equation constitutive for the solution is a weak formulation for some smooth function of the solution u . In [59] the global existence of these solutions was shown for a chemo-taxis model including the nonlinear coupling term $\kappa \nabla \cdot (u \nabla w)$ with the taxis coefficient $\kappa = 1$ in all space dimension $d \geq 4$.

Another generalized solution concept for a Keller–Segel model was considered in [39], with a weak formulation for the prey and some weak inequalities for the coupled quantity $u^p w^q$ for some $p, q \in (0, 1)$. Here it was still necessary to impose some smallness conditions on the taxis coefficient.

To the best of the authors knowledge model (1.1) was not yet considered in this generality, with no constraints, apart from the positivity, on the taxis coefficient κ .

Our definition of generalized solution consists of a weak formulation for the prey equation for w and two inequalities for the predator u , see Definition 2.1 below. These two inequalities bear some resemblance to mass conservation (in)equalities and entropy inequalities, as they include the total number of predators $\int_{\Omega} u \, dx$ and the term $-\int_{\Omega} \ln u \, dx$ commonly associated to the entropy of a physical system. Since reasonable *a priori* estimates, that is estimates leading to the existence of weak solutions, seem to be out of reach for the function u itself, we rather formulate the solvability concept for a nonlinear function of the predator variable, namely $\ln u$. With this, we follow the landmark paper [16], where renormalized solutions were introduced for the first time in the context of Boltzmann equations. Similar concepts of generalized solutions have been considered for different versions of the Keller–Segel model [14, 23, 38, 39, 61, 63].

Our main motivation to make these inequalities constitutive in our definition of generalized solutions is purely mathematical. With these inequalities we are able to prove a relative energy inequality, an estimate for the so-called relative energy,

$$\mathcal{R}(u, w | \tilde{u}, \tilde{w}) = \int_{\Omega} u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) + \frac{\kappa^2}{\mu\nu} \tilde{u} |w - \tilde{w}|^2 \, dx$$

for u and \tilde{u} two different predator and w and \tilde{w} two different prey populations, which is inspired by the Kullback–Leibler divergence, see [36]. This distance measure has many names, one of which is relative entropy, and many application areas. In its original form it measures the difference of two probability densities and is commonly used in information theory but also finds its application in biological systems as the Lotka–Volterra model, see [3]. In the context of the incompressible Navier–Stokes equations, *i.e.*, for a quadratic energy, the relative energy approach was already used by Leray in his seminal work [45] and later on by Serrin [56] to prove weak–strong uniqueness. For more general energy or entropy functionals this approach can be traced back to Dafermos, see [11, 15], in the context of conservation laws. This technique has since been generalized in different direction, for instance to the case of renormalized solutions [22] or non-convex energies [43]. The relative energy serves nowadays as a general tool in the analysis of PDEs and is used to consider aside from the weak–strong uniqueness of solutions [9, 22, 27] also

long-time behaviour [42], singular limits [20], convergence of numerical schemes [4] or comparison with reduced models [21] and even optimal control [41].

The deviation from the solution concept of the commonly used weak solutions allowed us to tackle the higher-order nonlinear coupling introduced by the Keller–Segel taxis term in (1.1). The meaningfulness of this solution concept is further supported by the fact that weak–strong uniqueness holds, which is a consequence of the above mentioned relative energy inequality. Furthermore, the proposed relative energy approach has numerous other applications as explained above and the relative energy inequality should be investigated in these directions in the future. In the numerical simulations, we performed to visualize the influence of the prey-taxis, the bias of the random motion of the predator u , modelled by the diffusion, towards higher concentration of prey is clearly visible. This suggests that model (1.1) is suitable to model the populations of a predator–prey pair which includes some hunting behaviour.

Plan of the paper: The paper is structured as follows. In the next section, Sect. 2, we collect the main results and highlight some aspects of the proofs. In Sect. 3, we present some numerical simulations motivating the chosen model and illustrating the effect of the prey-taxis term. In Sect. 4, we prove the existence of generalized solutions using the special regularization of adding a term modelling overcrowding. The weak–strong uniqueness proof is conducted in Sect. 5, whereas the existence of strong solutions locally in time in dimension two and three is proven in Sect. 6. The “Appendix” contains certain technical lemmata.

Notation: Before we begin with the main part of this work, we make some remarks on our notation. By $\Omega \subseteq \mathbb{R}^d$, we denote a bounded C^2 -domain with $d \geq 2$. The variable $T \in (0, \infty)$ denotes the finite time horizon. By n , we denote the outer normal vector of the domain Ω . For any Banach space V we denote the dual pairing between V^* and V by $\langle \cdot, \cdot \rangle_V$. In the remainder of this paper we will drop the subscript V for the sake of readability as it will be clear from the context which space is meant. Additionally, we will sometimes use the shorthand notations $L^r(L^q)$ for the Bochner space $L^r(0, T; L^q(\Omega))$ and $L^r(W^{k,q})$ for the Bochner space $L^r(0, T; W^{k,q}(\Omega))$ for $k \in \mathbb{N}$ and $r, q \geq 1$. Furthermore, we denote the space of abstract functions of bounded variation with values in V by $BV([0, T]; V)$. The space of weakly continuous functions with values in V is denoted by $C_w([0, T]; V)$ and the space of V -valued regular measures on $[0, T]$ by $\mathcal{M}(0, T; V)$, see for example [13] for an introduction.

We generally use $C > 0$ for constant upper bounds, where the exact value of C may change throughout a calculation without this being indicated in the notation. Throughout this paper we take $T, \alpha, \beta, \gamma, \delta, \kappa, \nu, \mu > 0$ arbitrary but fixed. Further, we take the dimension $d \in \mathbb{N}$ with $d \geq 2$, the C^2 -domain $\Omega \subseteq \mathbb{R}^d$ and $p > \max\{4, d\}$ to be fixed.

2. Main results

We start off by defining the appropriate spaces for our solutions. We first define the regularity space of the solution \mathcal{X} . We say $(u, w) \in \mathcal{X}$ if

$$\begin{aligned} u &\in L^\infty(0, T; L^1(\Omega)), \\ \ln u &\in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^1(\Omega)) \cap \text{BV}([0, T]; W^{1,p}(\Omega)^*), \\ u &> 0 \text{ a.e. in } (0, T) \times \Omega, \\ w &\in L^\infty((0, T) \times \Omega) \cap L^2(0, T; W^{1,2}(\Omega)), \\ \partial_t w &\in L^2(0, T; W^{1,2}(\Omega)^* + L^1(\Omega)), \\ w &\geq 0 \text{ a.e. in } (0, T) \times \Omega, \end{aligned}$$

where $p > d$ and the sum $X = X_1 + X_0$ of two Banach spaces X_0 and X_1 continuously embedded into a Hausdorff topological vector space \mathcal{H} is the set of all elements $x \in \mathcal{H}$ such that there are $x_0 \in X_0$ and $x_1 \in X_1$ with $x = x_0 + x_1$, see [8, p. 97]. The here given sum of Banach spaces is well-defined since both $L^1(\Omega)$ and $W^{1,2}(\Omega)^*$ are continuously embedded into the space of distributions $\mathcal{D}'(\Omega)$. We now define the generalized solutions as follows.

Definition 2.1. (*Generalized solution*) We say $(u, w) \in \mathcal{X}$ is a generalized solution to (1.1) for the initial data $(u_0, w_0) \in L^1(\Omega) \times L^2(\Omega)$ if the population inequality for the predator

$$\int_\Omega u(t) \, dx + \beta \int_0^t \int_\Omega u \, dx \, ds \leq \alpha \int_0^t \int_\Omega wu \, dx \, ds + \int_\Omega u_0 \, dx \tag{2.1}$$

and the logarithmic inequality for the predator

$$\begin{aligned} & - \int_\Omega \ln u(t) \vartheta(t) \, dx + \int_0^t \int_\Omega v |\nabla \ln u|^2 \vartheta - v \nabla \ln u \cdot \nabla \vartheta \, dx \, ds \\ & + \int_0^t \int_\Omega -\kappa \vartheta \nabla w \cdot \nabla \ln u + \kappa \nabla w \cdot \nabla \vartheta \, dx \, ds \\ & \leq \int_0^t \int_\Omega (\beta - \alpha w) \vartheta - \ln u \, \partial_t \vartheta \, dx \, ds - \int_\Omega \ln u_0 \vartheta(0) \, dx \end{aligned} \tag{2.2}$$

hold for all non-negative test function $\vartheta \in C^1([0, T]; L^\infty(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$ and all $t \in [0, T]$. Additionally, the prey equation is fulfilled in the weak sense, that is

$$\begin{aligned} & \int_\Omega w(t) \varphi(t) \, d\vec{x} - \int_0^t \int_\Omega w \partial_t \varphi - \mu \nabla w \cdot \nabla \varphi - \delta u w \varphi \, d\vec{x} \, ds \\ & = \int_0^t \int_\Omega \gamma w \varphi \, d\vec{x} \, ds + \int_\Omega w_0 \varphi(0) \, d\vec{x} \end{aligned} \tag{2.3}$$

holds for all $\varphi \in C^1([0, T]; L^\infty(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$ and all $t \in [0, T]$.

Remark 2.2. The energy inequality (2.1) is formally derived by testing the predator equation (1.1a) with the test function $\varphi \equiv 1$ and relaxing the equality to an inequality. The logarithmic inequality (2.2) is formally derived by testing the predator equation (1.1a) with $\varphi = -\frac{\vartheta}{u}$ and again relaxing the equality to an inequality.

Remark 2.3. The values of the solution (u, w) at time zero are well-defined in the given spaces as along as the initial conditions (u_0, w_0) live in the appropriate spaces, i.e. $\ln u_0, u_0 \in L^1(\Omega)$ and $w_0 \in L^2(\Omega)$, since by the definition of \mathcal{X} we have $\ln u \in \text{BV}([0, T]; W^{1,p}(\Omega)^*)$ and

$$w \in L^\infty(0, T; L^2(\Omega)) \cap C_w([0, T]; W^{1,p}(\Omega)^*) \leftrightarrow C_w([0, T]; L^2(\Omega)).$$

Note that the inequalities (2.1)–(2.2) do hold, unlike in many similar cases, for all $t \in [0, T]$. This follows from an application of an abstract version of Helly’s selection principle, see for example [5]. More details can be found in the proof of Theorem 2.4.

The main result of this work is the proof, that such solutions exist under certain assumptions on the initial data, which are formulated in the following theorem.

Theorem 2.4. (Existence of generalized solutions) *Let $\Omega \subseteq \mathbb{R}^d$ be a smooth and bounded domain with $d \in \mathbb{N}, d \geq 2$. Additionally, assume $u_0 \in L^1(\Omega)$ with $u_0 > 0$ almost everywhere in Ω as well as $\ln u_0 \in L^1(\Omega)$ and $w_0 \in L^\infty(\Omega)$ with $w_0 \geq 0$ almost everywhere in Ω . Then there exists a generalized solution $(u, w) \in \mathcal{X}$ to (1.1) in the sense of Definition 2.1.*

To see that our generalized solution concept is meaningful, we show weak–strong uniqueness. To do that we need the notion of a strong solution. We define strong solutions in the following way.

Definition 2.5. (Strong solution) We call the pair (\tilde{u}, \tilde{w}) a strong solution to the system (1.1) on $[0, \tilde{T}]$ for some $\tilde{T} > 0$ to the initial data $u_0, w_0 \in C^3(\overline{\Omega})$ non-negative, if

$$\tilde{u}, \tilde{w} \in C^1([0, \tilde{T}] \times \overline{\Omega}) \text{ and } \Delta \tilde{u}, \Delta \tilde{w} \in C([0, \tilde{T}] \times \overline{\Omega}),$$

\tilde{w} and \tilde{u} are non-negative and the equations (1.1a)–(1.1f) are fulfilled pointwise.

Theorem 2.6. (Weak–strong uniqueness) *Let (\tilde{u}, \tilde{w}) be a strong solution according to Definition 2.5 for the initial conditions $u_0, w_0 \in C^3(\overline{\Omega})$ non-negative, with u_0 bounded away from zero. Then every generalized solution $(u, w) \in \mathcal{X}$ emanating from the same initial values coincides with the strong solution and thus the generalized solution is unique.*

In the final part of this paper we show that under stronger assumption on the initial conditions, we indeed have at least local-in-time existence of strong solutions.

Theorem 2.7. (Local existence of weak solutions) *For $d \in \{2, 3\}, u_0 \in W^{1,2}(\Omega)$ and $w_0 \in W^{2,6}(\Omega)$ both non-negative and fulfilling zero Neumann boundary conditions there is a $T^* > 0$ such that (1.1) has a weak solution (u, w) with*

$$u \in W^{1,2}(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; W^{2,2}(\Omega)) \cap L^\infty(0, T^*; W^{1,2}(\Omega)),$$

$$w \in W^{1,6}(0, T^*; L^6(\Omega)) \cap L^6(0, T^*; W^{2,6}(\Omega)) \cap L^\infty(0, T^*; W^{2,2}(\Omega)).$$

The necessary *a priori* estimates for this result are given at the end of the paper. Given smoother initial data we can deduce even more regularity of the local solution so that we obtain a strong solution.

Proposition 2.8. (Local existence of strong solutions) *For $d \in \{2, 3\}$ and non-negative $w_0, u_0 \in C^3(\overline{\Omega})$ both fulfilling zero Neumann boundary conditions, we find that the solutions from Theorem 2.7 are strong solutions.*

Main contributions: In this section we will briefly present the main structure of the proofs of the existence theorem, cf. Theorem 2.4, the weak–strong uniqueness result, cf. Theorem 2.6 and the local existence result, cf. Theorem 2.7 and highlight the main ideas.

The existence of generalized solutions is proven via showing the existence of weak solutions to a regularized system, deriving *a priori* estimates and extracting convergent subsequences to send the regularization to zero.

Major contributions to the analysis of a system like (1.1) were made in [62], where the existence of weak solutions was shown, but there the *a priori* estimates crucially depended on the convexity of the domain Ω , which we do not assume to hold here. Similar generalized solution concepts to the one in Definition 2.1 based on replacing the weak formulation of (1.1a) by two variational inequalities were considered in [38, 39]. The challenge in proving the existence of such generalized solutions lies in deriving strong enough *a priori* estimates to pass to the limit in the nonlinear terms $\nabla \cdot (u \nabla w)$ and uw due to the low regularity of u , which is a common challenge when analysing taxis models, see [40]. If superlinear damping is present in the first equation (1.1a), additional L^p -estimates could help to deduce the desired equi-integrability (cf. [38]). In [39] highly nonlinear *a priori* estimates helped to deduce equi-integrability for approximations of u under certain conditions on the taxis coefficient.

Besides rather classical energy-like estimates, the new idea of the article at hand lies in deriving an $L^1(L^1)$ estimate for $uw \ln(uw + 1)$, which gives the equi-integrability of the product uw needed to extract convergent subsequences in $L^1(L^1)$ independent of the coefficients and without superlinear damping. This estimate is obtained by testing (1.1a) by $\frac{-w}{u+1}$ and adding (1.1b) tested by $\ln(u + 1)$. In the resulting relation, the product stemming from the last term in (1.1b) generates the desired estimate, since all other terms can be bounded appropriately due to the classical energy-like estimates. We can then use the weak convergence of the product uw to deduce the strong convergence of ∇w , which then suffices (with the weak convergence of u) to pass to the limit in the taxis term. This *a priori* estimate is, to the best of the authors' knowledge, the first of this kind in this general taxis system.

The other rather new technique in the context of Keller–Segel models presented, is the relative energy, our tool of choice to prove the weak–strong uniqueness, mentioned above in the introduction. Together with the theorem of local existence of a strong solution this provides a full self-contained analysis of the system comprising global

existence of generalized solutions, local existence of classical solutions, and weak-strong uniqueness that is standard nowadays (cf. [43]).

3. Numerical simulations

With the application of model (1.1) in biological pest control in mind, we performed some numerical simulations illustrating the influence of the prey-taxis term in the predator equation, using the finite element method and the python package FEniCS. The illustrations were produced with the python library Plotly. The nonlinear, higher-order coupling prey-taxis term made the existence proof of solutions quite challenging, but it also made the modelling of a certain hunting behaviour possible as can be seen, when keeping the diffusion coefficients constant and increasing the coefficient of the prey-taxis term.

Simulations of another Keller–Segel model were conducted in [25]. As described there, blow-up solution are known to exist for certain initial values and as we have seen in Sect. 1 also the value of the prey-taxis coefficient played a crucial role in various existence proofs. With these difficulties in mind, we would like to point out that the simulations performed here are only used as a visualization tool and we do not claim any accuracy.

Numerical simulations were performed over the domain $\overline{\Omega} = [-1, 1]^2$, discretized by a regular triangulation with $n_x = n_y = 200$, choosing the final time $T = 10$ and the time step $dt = 0.01$ and continuous $P1$ -Lagrange elements for u and continuous $P2$ -Lagrange elements for w . Additionally, we chose the coefficients of system (1.1) to be

$$\nu = 0.1, \mu = 0.01, \alpha = 2.0, \beta = 0.8, \gamma = 0.8, \delta = 2.0, \kappa \in \{0, 1, 2, 3\}$$

and the initial conditions

$$\begin{aligned} u_0 &= 4 \exp(-30(x + 0.6)^2 - 30(y - 0.6)^2), \\ w_0 &= 2 \exp(-9(x + 0.4)^2 - 9(y + 0.5)^2) + 2 \exp(-9(x - 0.5)^2 - 9(y - 0.4)^2). \end{aligned}$$

Note that the case $\kappa = 0$ is not covered by our analysis, but was part of our numerical simulations for illustration purposes. In this case the difficult prey-taxis term in (1.1a) vanishes and the existence of weak solution should follow by standard means. The initial conditions are visualized in Fig. 1.

For $\kappa = 0$ the densities at time $t = 0.4$ are shown in Fig. 2. We chose this rather short time to illustrate the difference of the model for $\kappa = 0$ and $\kappa = 1$. At later times the prey density is distributed more evenly over the whole domain due to the diffusion and the difference gets less pronounced. The initial concentration of predators and prey in Gaussians has already spread out slightly as one would expect from diffusive movement. Here no predators have accumulated at the peaks of the prey density yet.

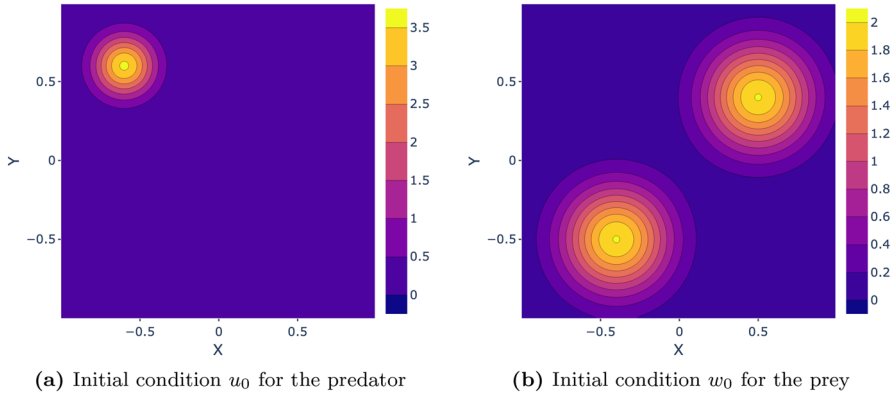


Figure 1. Initial conditions

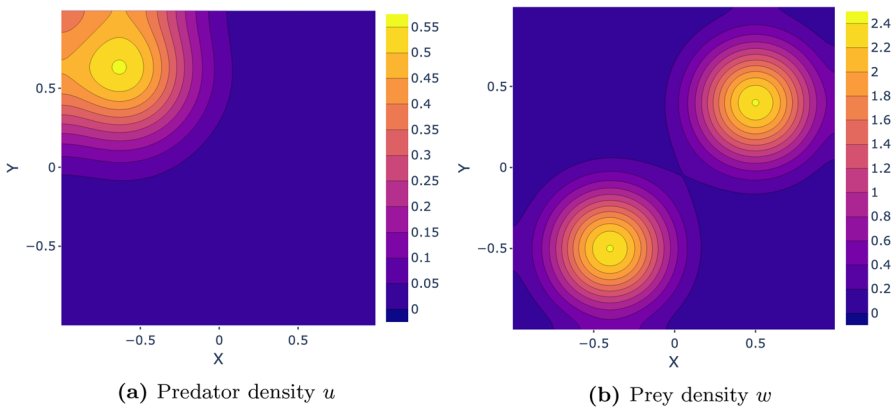


Figure 2. Mite densities for $\kappa = 0$ at $t = 0.4$

In contrast, for $\kappa = 1$ and $t = 0.4$ we have a higher concentration of predators where prey is plenty, see Fig. 3. Here, the biased movement towards higher concentration of prey, modelled by the prey-taxis term, is nicely visible.

4. Existence of generalized solutions

4.1. The regularized system

Now we introduce the regularizing term $-\varepsilon u|u|^{p-1}$ on the right-hand side of the predator equation (1.1a), where $\varepsilon > 0$ is the regularizing coefficient, which we will take to zero in the proof of the existence of generalized solutions, and the exponent $p > \max\{d, 4\}$. So the system we are considering in this section is identical to the

system (1.1) with (1.1a) replaced by

$$\partial_t u - \nu \Delta u + \kappa \nabla \cdot (u \nabla w) = (\alpha w - \beta)u - \varepsilon u|u|^{p-1} \text{ in } (0, T) \times \Omega. \tag{4.1a}$$

4.2. Existence of weak solutions to the regularized system

The chosen regularization term allows for an easy attainable estimate of u in $L^{p+1}(L^{p+1})$ and with the use of maximal L^p -regularity in the prey equation (1.1b) we get rather strong estimates for ∇w , which we need to estimate the prey-taxis term in (4.1a), which is the term with the prefactor κ .

Let's start by defining our notion of weak solutions. We first define our solution space. We say $w \in \mathbb{W}$ if

$$w \in L^2(0, T; W^{1,2}(\Omega)) \text{ and } \partial_t w \in L^2(0, T; W^{1,2}(\Omega)^*)$$

and we say that $u \in \mathbb{U}$ if

$$u \in L^2(0, T; W^{1,2}(\Omega)) \cap L^{p+1}(0, T; L^{p+1}(\Omega)) =: \mathcal{U},$$

$$\partial_t u \in L^2(0, T; W^{1,2}(\Omega)^*) + L^q(0, T; L^q(\Omega)) = \mathcal{U}^*.$$

Here, $q > 1$ denotes the conjugate exponent of $p + 1$.

The time derivative of w and u are to be understood in the distributional sense, that is $u_t \in \mathcal{U}^*$ is the weak time derivative of $u \in \mathcal{U}$ if

$$\int_0^T \langle u(t), \varphi \rangle \phi'(t) dt = - \int_0^T \langle u_t(t), \varphi \rangle \phi(t) dt \tag{4.2}$$

holds for all $\varphi \in L^{p+1}(\Omega) \cap W^{1,2}(\Omega)$ and $\phi \in C_0^\infty(0, T)$. We then write $\partial_t u = u_t$. The time derivative of w is defined similarly with test functions $\varphi \in W^{1,2}(\Omega)$.

Definition 4.1. (*Weak solution to the regularized system*) We say that a pair $(u, w) \in \mathbb{U} \times \mathbb{W}$ is a weak solution to problem (4.1), if

$$u, w \geq 0, \quad w \in L^\infty((0, T) \times \Omega), \quad \nabla w \in L^p(0, T; L^\infty(\Omega)),$$

the integral equalities

$$\int_0^t \langle \partial_t u, \psi \rangle ds + \int_0^t \int_\Omega \nu \nabla u \cdot \nabla \psi - \kappa u \nabla w \cdot \nabla \psi - (\alpha w - \beta)u \psi \, dx \, ds$$

$$+ \int_0^t \int_\Omega \varepsilon u|u|^{p-1} \psi \, dx \, ds = 0, \tag{4.3}$$

$$\int_0^t \langle \partial_t w, \varphi \rangle ds + \int_0^t \int_\Omega \mu \nabla w \cdot \nabla \varphi - (\gamma - \delta u)w \varphi \, dx \, ds = 0 \tag{4.4}$$

are fulfilled for all $\varphi \in L^2(0, T; W^{1,2}(\Omega))$ and all $\psi \in \mathcal{U}$ and almost all $t \in (0, T)$ and the initial conditions are fulfilled in the sense of traces in $L^2(\Omega)$.

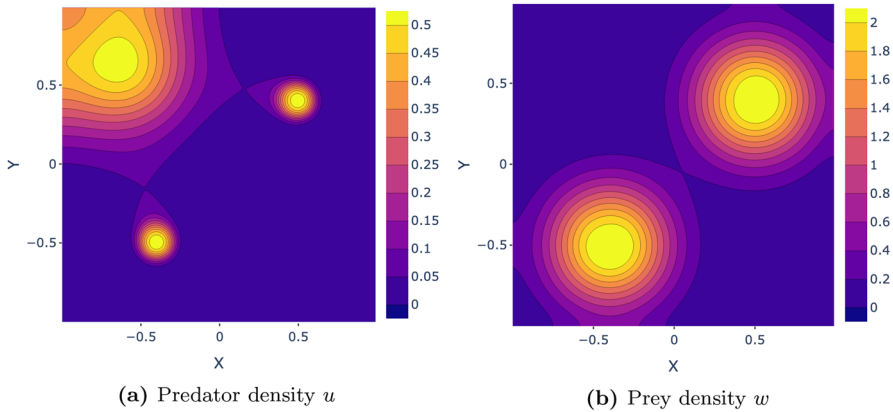


Figure 3. Mite densities for $\kappa = 1$ at $t = 0.4$

Theorem 4.2. (Existence of weak solutions to the regularized system) *For every pair of initial data (u_0, w_0) with non-negative $u_0 \in L^p(\Omega)$ and non-negative and bounded*

$$w_0 \in \left\{ w \in W^{2,p}(\Omega) \mid \nabla w \cdot n = 0 \text{ on } \partial\Omega \right\}$$

there exists a weak solution according to Definition 4.1 to the regularized system (4.1).

Proof. We only give a rough outline of the proof since it mainly follows the standard procedure of decoupling the system, using a Galerkin approximation to tackle the single equations and Schauder’s fixed point theorem to show the existence of a solution to the coupled equations. A detailed proof for $d = 2$ can be found in [51].

The existence and uniqueness of a weak solution w to the prey equation (1.1b) for fixed non-negative $\bar{u} \in L^p(L^p)$ follows from [53, Thm. 8.30 and Thm. 8.34]. The non-negativity and boundedness of w follows from a comparison principle, see Lemma 4.3 below. The additional regularity of w follows from the maximal L^p -regularity of the Laplace operator, see [12, Thm. 8.2]. Here the assumptions that Ω has a C^2 -boundary and the initial condition w_0 fulfills some compatibility condition are needed.

For fixed $\bar{w} \in \mathbb{W}$ non-negative and bounded with $\nabla \bar{w} \in L^p(L^\infty)$, the existence of a weak solution $u \in \mathbb{U}$ to (4.1a) with w replaced by \bar{w} can be shown via a standard Galerkin discretization using the Gelfand triple $W^{1,2}(\Omega) \cap L^{p+1}(\Omega) = V \subseteq L^2(\Omega) \subseteq V^*$ and Minty’s trick to handle the monotone regularization term. Here we needed the integration by parts rule for elements of \mathbb{U} and the continuous embedding of \mathbb{U} into $C([0, T]; L^2(\Omega))$, which can be proven analogously to the well-known versions of these results for the space \mathbb{W} . We obtained the non-negativity of u by testing equation (4.1a) with u^- and applying Gronwall’s inequality. Additionally we obtained a constant $L^p(L^p)$ -bound for u by using 1 as a test function.

We define the solution operator $\mathcal{T} : L^p(L^p) \rightarrow L^p(L^p)$, which maps a non-negative $\bar{u} \in L^p(L^p)$ to the solution of (4.1a) with $w = \bar{w}$, where \bar{w} is the solution of (1.1b) with u replaced by \bar{u} . Using the above mentioned existence results and *a priori*

estimates for the decoupled equations, we find that \mathcal{T} is a well-defined self-map on a bounded, convex and closed subset of the Banach space $L^p(L^p)$. The continuity can be proven by testing appropriately, using L^p -interpolation inequalities and maximal L^p -regularity of the heat equation. For any bounded sequence $(\bar{u}_n)_n \subseteq L^p(L^p)$ the sequence $(\mathcal{T}(\bar{u}_n))_n$ is bounded in $L^{p+1}(L^{p+1})$, where the bound depends on ε , and the strong convergence of a subsequence of $(\mathcal{T}(\bar{u}_n))_n$ in $L^p(L^p)$ follows by Vitali's theorem, see [18, Thm. 5.6]. This implies the compactness of \mathcal{T} and Schauder's fixed point theorem then implies the existence of a weak solution to the regularized system (4.1). \square

4.3. *A priori* estimates

In order to show the existence of generalized solutions according to Definition 2.1, we show some *a priori* estimates for solutions $(u_\varepsilon, w_\varepsilon)$ of the regularized system. These *a priori* estimates will allow us to extract a convergent subsequence, whose limit fulfills our notion of generalized solutions.

At first, we prove the following comparison principle for the prey variable.

Lemma 4.3. (Comparison principle for w) *Let $u \in L^1(0, T; L^1(\Omega))$ be non-negative and assume that \underline{w} and \bar{w} are a sub- and a super-solution of (1.1b), i.e. \underline{w} and \bar{w} fulfill equation (1.1b) in the weak sense with the equality sign replaced by \leq and \geq , respectively, and*

$$\begin{aligned} \underline{w}, \bar{w} &\in L^2(0, T; L^\infty(\Omega) \cap W^{1,2}(\Omega)), \\ \partial_t \underline{w}, \partial_t \bar{w} &\in L^2(0, T; L^1(\Omega) + W^{1,2}(\Omega)^*), \\ 0 &\leq \text{ess inf}_{(0,T) \times \Omega} \underline{w} \leq \text{ess sup}_{(0,T) \times \Omega} \underline{w} < \infty, \\ 0 &\leq \text{ess inf}_{(0,T) \times \Omega} \bar{w} \leq \text{ess sup}_{(0,T) \times \Omega} \bar{w} < \infty \end{aligned}$$

hold as well as $\underline{w}(0, \mathbf{x}) \leq \bar{w}(0, \mathbf{x})$ a.e. in Ω . Then

$$\underline{w}(t, \mathbf{x}) \leq \bar{w}(t, \mathbf{x}) \text{ a.e. in } (0, T) \times \Omega \tag{4.5}$$

holds.

Proof. Subtracting the equation for the sub-solution \underline{w} and the super-solution \bar{w} and testing the resulting inequality with $(\underline{w} - \bar{w})^+ := \max\{0, \underline{w} - \bar{w}\}$, we obtain

$$\begin{aligned} \frac{1}{2} \|(\underline{w}(t) - \bar{w}(t))^+\|_{L^2(\Omega)}^2 &+ \int_0^t \int_\Omega \mu |\nabla(\underline{w} - \bar{w})^+|^2 + \delta u |(\underline{w} - \bar{w})^+|^2 \, \mathbf{d}\mathbf{x} \, \mathbf{d}s \\ &= \frac{1}{2} \|(\underline{w}(0) - \bar{w}(0))^+\|_{L^2(\Omega)}^2 + \gamma \int_0^t \int_\Omega |(\underline{w} - \bar{w})^+|^2 \, \mathbf{d}\mathbf{x} \, \mathbf{d}s \end{aligned}$$

for almost all $t \in (0, T)$, where we integrated the first term by parts. The standard integration by parts rule can be generalized to this case by an approximation with

smooth functions, cf. Corollary 7.4 in the “Appendix”. An application of Gronwall’s inequality yields

$$\|(\underline{w}(t) - \overline{w}(t))^+\|_{L^2(\Omega)}^2 \leq \|(\underline{w}(0) - \overline{w}(0))^+\|_{L^2(\Omega)}^2 e^{2\gamma t} = 0$$

for almost all $t \in (0, T)$, by the assumption on the initial values and our proof is complete. \square

Corollary 4.4. (Boundedness of w) *Let $u \in L^1(0, T; L^1(\Omega))$ and $w_0 \in L^\infty(\Omega)$ both be non-negative. Then the solution w of (1.1b) fulfills*

$$0 \leq w \leq \text{ess sup}_{x \in \Omega} w_0(x) e^{\gamma t} \text{ a.e. in } (0, T) \times \Omega. \tag{4.6}$$

Proof. One easily checks that 0 is a sub- and $\text{ess sup}_{x \in \Omega} w_0(x) e^{\gamma t}$ is a super-solution to (1.1b). Then the assertion follows from Lemma 4.3. \square

Proposition 4.5. (A priori estimates) *Assume that the initial values can be bounded independently of ε . That is $\ln u_{0\varepsilon}$ and $u_{0\varepsilon}$ are bounded in $L^1(\Omega)$ independently of ε and $w_{0\varepsilon}$ is bounded in $L^\infty(\Omega)$ independently of ε . For a solution $(u_\varepsilon, w_\varepsilon)$ to the approximate system (4.1) we have the following a priori estimates*

$$\|u_\varepsilon\|_{L^\infty(L^1)} + \varepsilon^{\frac{1}{p}} \|u_\varepsilon\|_{L^p(L^p)} + \|\ln u_\varepsilon\|_{L^\infty(L^1)} + \|\nabla \ln u_\varepsilon\|_{L^2(L^2)} + \|\partial_t \ln u_\varepsilon\|_{L^1(W^{1,p}(\Omega)^*)} \leq C, \tag{4.7a}$$

$$\|w_\varepsilon\|_{L^\infty((0,T) \times \Omega)} + \|\nabla w_\varepsilon\|_{L^2(L^2)} + \|w_\varepsilon u_\varepsilon \ln(u_\varepsilon w_\varepsilon + 1)\|_{L^1(L^1)} + \|\partial_t w_\varepsilon\|_{L^2(W^{1,p}(\Omega)^*)} \leq C, \tag{4.7b}$$

where the constant C is independent of ε .

Remark 4.6. A sequence of initial values fulfilling the boundedness assumptions and the appropriate regularity criteria is constructed in the proof of Theorem 2.4 below.

Proof. The $L^\infty((0, T) \times \Omega)$ -bound of w_ε follows from Corollary 4.4. Note that this bound is independent of ε as long as $w_{0\varepsilon}$ is bounded independently of ε . Testing equation (1.1b) with w_ε , we get

$$\begin{aligned} & \frac{1}{2} \|w_\varepsilon(T)\|_{L^2(\Omega)}^2 + \mu \|\nabla w_\varepsilon\|_{L^2(L^2)}^2 + \delta \int_0^T \int_\Omega u_\varepsilon w_\varepsilon^2 \, dx \, dt \\ & \leq \gamma \|w_\varepsilon\|_{L^\infty((0,T) \times \Omega)}^2 |\Omega| T + \frac{1}{2} \|w_{0\varepsilon}\|_{L^2(\Omega)}^2, \end{aligned}$$

which gives the $L^2(L^2)$ -bound for ∇w_ε , since u_ε is non-negative. Using $\varphi \equiv 1$ as a test function in the predator equation (4.1a), we get

$$\int_0^t \langle \partial_t u_\varepsilon(s), 1 \rangle \, ds + \int_0^t \int_\Omega \varepsilon u_\varepsilon^p + \beta u_\varepsilon \, dx \, ds = \alpha \int_0^t \int_\Omega w_\varepsilon u_\varepsilon \, dx \, ds \tag{4.8}$$

for almost all $t \in (0, T)$. All following equations throughout this proof, which depend on t , also hold for almost all $t \in (0, T)$. By the non-negativity of u_ε , we obtain

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} \leq \|u_{0\varepsilon}\|_{L^1(\Omega)} + \alpha \|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \int_0^t \|u_\varepsilon(s)\|_{L^1(\Omega)} \, ds.$$

An application of Gronwall's inequality then yields

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} \leq e^{\alpha\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}} \|u_{0\varepsilon}\|_{L^1(\Omega)}. \tag{4.9}$$

Additionally, (4.8) implies

$$\varepsilon \|u_\varepsilon\|_{L^p(L^p)}^p \leq \alpha \|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)} T e^{\alpha\|w_\varepsilon\|_{L^\infty((0,T)\times\Omega)}} \|u_{0\varepsilon}\|_{L^1(\Omega)}.$$

Now we have shown that the first two terms of (4.7a) and (4.7b) have constant upper bounds independent of ε . To derive an estimate for the third and fourth term in (4.7a) we use $-\frac{1}{u_\varepsilon + \lambda}$ as a test function in the predator equation (4.1a) for some $\lambda \in (0, 1)$. This yields

$$\begin{aligned} & \int_0^t \left\langle \partial_t u_\varepsilon(s), \frac{-1}{u_\varepsilon(s) + \lambda} \right\rangle ds + \int_0^t \int_\Omega \nu \nabla u_\varepsilon \nabla \left(\frac{-1}{u_\varepsilon + \lambda} \right) \, dx \, ds \\ &= \int_0^t \int_\Omega \kappa u_\varepsilon \nabla w_\varepsilon \nabla \left(\frac{-1}{u_\varepsilon + \lambda} \right) + \varepsilon u_\varepsilon^{p-1} \frac{u_\varepsilon}{u_\varepsilon + \lambda} + (\alpha w_\varepsilon - \beta) \frac{-u_\varepsilon}{u_\varepsilon + \lambda} \, dx \, ds. \end{aligned}$$

We choose this test function, as we have only shown that $u_\varepsilon \geq 0$ holds and therefore we are not allowed to test with $1/u_\varepsilon$. Using an integration by parts and Young's inequality we can estimate

$$\begin{aligned} & - \int_\Omega \ln(u_\varepsilon + \lambda) \Big|_0^t \, d\vec{x} + \frac{\nu}{2} \int_0^t \int_\Omega |\nabla \ln(u_\varepsilon + \lambda)|^2 \, d\vec{x} \, ds \\ & \leq \frac{\kappa^2}{2\nu} \int_0^t \int_\Omega |\nabla w_\varepsilon|^2 \, d\vec{x} \, ds + \varepsilon \int_0^t \int_\Omega \frac{p-1}{p} u_\varepsilon^p + \frac{1}{p} \, d\vec{x} \, ds + \beta T |\Omega|. \end{aligned} \tag{4.10}$$

By the already derived estimates for the first two terms of (4.7a) and (4.7b), this gives a constant upper bound for $-\int_\Omega \ln(u_\varepsilon(t) + \lambda) \, dx$ as long as $\int_\Omega \ln(u_{0\varepsilon} + \lambda) \, dx$ is bounded, which is the case since $u_{0\varepsilon}$ is assumed to be bounded in $L^1(\Omega)$, as can be seen from the following estimate

$$\begin{aligned} \int_\Omega \ln(u_\varepsilon(t) + \lambda) \, d\vec{x} & \leq \int_\Omega u_\varepsilon(t) + \lambda \, d\vec{x} \leq \|u_\varepsilon(t)\|_{L^1(\Omega)} + \lambda |\Omega| \\ & \leq \|u_\varepsilon(t)\|_{L^1(\Omega)} + |\Omega| \leq C \end{aligned} \tag{4.11}$$

for some $C > 0$, where we used $\ln(s) \leq s$ for all $s > 0$. This constant C , like all following C 's in this proof, is independent of λ and ε using that λ is upper bounded by 1. The constant upper bound follows from the fact, that we already know that u_ε is bounded in the $L^\infty(L^1)$ -norm, cf. (4.9). We can now derive a constant upper bound for the $L^1(\Omega)$ -norm of $\ln u_\varepsilon(t)$. We estimate

$$\begin{aligned}
 & \| \ln(u_\varepsilon(t) + \lambda) \|_{L^1(\Omega)} \\
 &= - \int_{\{u_\varepsilon(t) + \lambda < 1\}} \ln(u_\varepsilon(t) + \lambda) d\vec{x} + \int_{\{u_\varepsilon(t) + \lambda \geq 1\}} \ln(u_\varepsilon(t) + \lambda) d\vec{x} \\
 &= - \int_{\Omega} \ln(u_\varepsilon(t) + \lambda) d\vec{x} + 2 \int_{\{u_\varepsilon(t) + \lambda \geq 1\}} \ln(u_\varepsilon(t) + \lambda) d\vec{x} \leq C,
 \end{aligned}
 \tag{4.12}$$

where we use (4.10) and (4.11). To get the $L^\infty(L^1)$ -bound for $\ln u_\varepsilon$, we take λ to zero from above. Since the logarithm is continuous, we have

$$\liminf_{\lambda \downarrow 0} | \ln(u_\varepsilon(t, \mathbf{x}) + \lambda) | = | \ln(u_\varepsilon(t, \mathbf{x})) | \text{ for almost all } (t, \mathbf{x}) \in (0, T) \times \Omega,$$

where we define $\ln(0) := -\infty$. By Fatou’s lemma we can estimate

$$\begin{aligned}
 \int_{\Omega} | \ln u_\varepsilon(t) | d\mathbf{x} &= \int_{\Omega} \liminf_{\lambda \downarrow 0} | \ln(u_\varepsilon(t) + \lambda) | d\mathbf{x} \\
 &\leq \liminf_{\lambda \downarrow 0} \int_{\Omega} | \ln(u_\varepsilon(t) + \lambda) | d\mathbf{x} \leq C,
 \end{aligned}$$

where the constant upper bound follows from (4.12). Thus we get the $L^\infty(L^1)$ -bound of $\ln u_\varepsilon$ and we have $u_\varepsilon(t) > 0$ almost everywhere. Similarly, we get the bound for $\nabla \ln u_\varepsilon$. We already have that $\nabla \ln(u_\varepsilon + \lambda)$ is bounded in $L^2(L^2)$ independently of λ and ε by (4.10). Again using Fatou’s lemma we get

$$\begin{aligned}
 \| \nabla \ln u_\varepsilon \|_{L^2(L^2)}^2 &= \int_0^T \int_{\Omega} \frac{1}{u_\varepsilon^2} | \nabla u_\varepsilon |^2 d\mathbf{x} dt \\
 &= \int_0^T \int_{\Omega} \liminf_{\lambda \downarrow 0} \frac{1}{(u_\varepsilon + \lambda)^2} | \nabla u_\varepsilon |^2 d\mathbf{x} dt = \int_0^T \int_{\Omega} \liminf_{\lambda \downarrow 0} | \nabla \ln(u_\varepsilon + \lambda) |^2 d\mathbf{x} dt \\
 &\leq \liminf_{\lambda \downarrow 0} \int_0^T \int_{\Omega} | \nabla \ln(u_\varepsilon + \lambda) |^2 d\mathbf{x} dt \leq C.
 \end{aligned}$$

In order to show the $L^1(L^1)$ -bound of $u_\varepsilon w_\varepsilon \ln(u_\varepsilon w_\varepsilon + 1)$, we note that

$$\begin{aligned}
 0 &\leq u_\varepsilon w_\varepsilon \ln(u_\varepsilon w_\varepsilon + 1) \leq u_\varepsilon w_\varepsilon \ln((u_\varepsilon + 1)(w_\varepsilon + 1)) \\
 &= u_\varepsilon w_\varepsilon \ln(u_\varepsilon + 1) + u_\varepsilon w_\varepsilon \ln(w_\varepsilon + 1)
 \end{aligned}$$

holds pointwise and it suffices to show a $L^1(L^1)$ -bound for $u_\varepsilon w_\varepsilon \ln(u_\varepsilon + 1)$ and $u_\varepsilon w_\varepsilon \ln(w_\varepsilon + 1)$ separately. The upper bounds for u_ε and $\nabla \ln u_\varepsilon$ from (4.7a) can be transferred to $\ln(u_\varepsilon + 1)$ and $\nabla \ln(u_\varepsilon + 1)$. We test the prey equation (1.1b) by $\ln(u_\varepsilon + 1)$. It can be shown by an approximation with smooth functions, see [65, Ex. 21.3c] that this is indeed an admissible test function since $u_\varepsilon \in L^2(W^{1,2})$ and $u_\varepsilon + 1$ is bounded away from zero by 1. We then have

$$\nabla \ln(u_\varepsilon + 1) = \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon.$$

Using $\partial_t w_\varepsilon \ln(u_\varepsilon + 1) = -\partial_t \ln(u_\varepsilon + 1) w_\varepsilon + \partial_t (\ln(u_\varepsilon + 1) w_\varepsilon)$ and integrating by parts we get

$$\begin{aligned}
 & \delta \int_0^T \int_{\Omega} w_{\varepsilon} u_{\varepsilon} \ln(u_{\varepsilon} + 1) d\vec{x} dt \\
 & \leq \int_0^T \int_{\Omega} \partial_t \ln(u_{\varepsilon} + 1) w_{\varepsilon} d\vec{x} dt + \mu \|\nabla w_{\varepsilon}\|_{L^2(L^2)} \|\nabla \ln(u_{\varepsilon} + 1)\|_{L^2(L^2)} \\
 & \quad + (1 + \gamma T) \|w_{\varepsilon}\|_{L^{\infty}((0,T) \times \Omega)} \|\ln(u_{\varepsilon} + 1)\|_{L^{\infty}(L^1)} \\
 & \quad + \|w_{\varepsilon}\|_{L^{\infty}((0,T) \times \Omega)} \|\ln(u_{0\varepsilon} + 1)\|_{L^1(\Omega)}.
 \end{aligned} \tag{4.13}$$

All terms on the right-hand side except the first are already known to be bounded by a constant. To upper bound the first term on the right-hand side, we test (4.1a) with $\frac{w_{\varepsilon}}{u_{\varepsilon} + 1}$, which is an admissible test function, since w_{ε} and u_{ε} are in $L^2(W^{1,2})$ and w_{ε} is bounded,

$$\begin{aligned}
 & \int_0^T \int_{\Omega} \partial_t u_{\varepsilon} \frac{w_{\varepsilon}}{u_{\varepsilon} + 1} d\vec{x} dt \\
 & \quad + \int_0^T \int_{\Omega} \nu \nabla u_{\varepsilon} \cdot \nabla \left(\frac{w_{\varepsilon}}{u_{\varepsilon} + 1} \right) - \kappa u_{\varepsilon} \nabla w_{\varepsilon} \cdot \nabla \left(\frac{w_{\varepsilon}}{u_{\varepsilon} + 1} \right) d\vec{x} dt \\
 & = \int_0^T \int_{\Omega} (\alpha w_{\varepsilon} - \beta) w_{\varepsilon} \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} - \varepsilon u_{\varepsilon}^{p-1} w_{\varepsilon} \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} d\vec{x} dt.
 \end{aligned}$$

Rearranging the terms and calculating the gradients yields

$$\begin{aligned}
 & \int_0^T \int_{\Omega} \partial_t u_{\varepsilon} \frac{w_{\varepsilon}}{u_{\varepsilon} + 1} d\vec{x} dt \\
 & = - \int_0^T \int_{\Omega} \nu \nabla \ln(u_{\varepsilon} + 1) \cdot \nabla w_{\varepsilon} - \nu w_{\varepsilon} |\nabla \ln(u_{\varepsilon} + 1)|^2 d\vec{x} dt \\
 & \quad + \int_0^T \int_{\Omega} \kappa \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} |\nabla w_{\varepsilon}|^2 - \kappa w_{\varepsilon} \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} \nabla w_{\varepsilon} \cdot \nabla \ln(u_{\varepsilon} + 1) d\vec{x} dt \\
 & \quad + \int_0^T \int_{\Omega} (\alpha w_{\varepsilon} - \beta) w_{\varepsilon} \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} - \varepsilon u_{\varepsilon}^{p-1} w_{\varepsilon} \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} d\vec{x} dt.
 \end{aligned}$$

Using $\partial_t \ln(u_{\varepsilon} + 1) w_{\varepsilon} = \frac{w_{\varepsilon}}{u_{\varepsilon} + 1} \partial_t u_{\varepsilon}$, we find a constant upper bound for the first term on the right-hand side of (4.13), by estimating $\frac{u_{\varepsilon}}{u_{\varepsilon} + 1}$ by one and using the already derived *a priori* estimates for the second and fourth term in (4.7a) and the first two terms in (4.7b). Now all terms on the right-hand side of (4.13) are known to be bounded and we find that $\|w_{\varepsilon} u_{\varepsilon} \ln(u_{\varepsilon} + 1)\|_{L^1(L^1)}$ is bounded, since $u_{\varepsilon} w_{\varepsilon} \ln(u_{\varepsilon} + 1) \geq 0$. The estimate for $u_{\varepsilon} w_{\varepsilon} \ln(w_{\varepsilon} + 1)$ now follows directly by

$$\begin{aligned}
 & \int_0^T \int_{\Omega} u_{\varepsilon} w_{\varepsilon} \ln(w_{\varepsilon} + 1) dx dt \\
 & \leq \|w_{\varepsilon}\|_{L^{\infty}((0,T) \times \Omega)} \ln(\|w_{\varepsilon}\|_{L^{\infty}((0,T) \times \Omega)} + 1) \int_0^T \int_{\Omega} u_{\varepsilon} dx dt \leq C.
 \end{aligned}$$

This shows the upper bound for the third term in (4.7b). Using equation (1.1b) tested with $\varphi \in W^{1,p}(\Omega)$, yields

$$\begin{aligned} \left| \int_{\Omega} \partial_t w_{\varepsilon} \varphi \, d\mathbf{x} \right| &= \left| \int_{\Omega} (\gamma - \delta u_{\varepsilon}) w_{\varepsilon} \varphi - \mu \nabla w_{\varepsilon} \cdot \nabla \varphi \, d\mathbf{x} \right| \\ &\leq C \|\varphi\|_{W^{1,p}(\Omega)} \left(\|w_{\varepsilon}\|_{L^{\infty}((0,T)\times\Omega)} (\gamma + \delta \|u_{\varepsilon}\|_{L^1(\Omega)}) + \mu \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \right), \end{aligned}$$

where we used $p > d$ such that $W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$. Noting that the right-hand side is bounded in $L^2(0, T)$ independently of ε , we obtain the upper bound for $\partial_t w_{\varepsilon}$ in $L^2(0, T; W^{1,p}(\Omega)^*)$.

To obtain the upper bound for $\partial_t \ln u_{\varepsilon}$ in $L^1(0, T; W^{1,p}(\Omega)^*)$, we first note that a regularized form of the logarithmic inequality (2.2) holds with an equality sign on the approximate level. To see that we use $-\frac{\vartheta}{u_{\varepsilon} + \lambda}$ as a test function in (4.1a) for an arbitrary $\vartheta \in C^1([0, T]; L^{\infty}(\Omega)) \cap L^2(W^{1,2})$ and $\lambda \in (0, 1)$. This yields,

$$\begin{aligned} & - \int_{\Omega} \partial_t \ln(u_{\varepsilon} + \lambda) \vartheta \, d\mathbf{x} + \int_{\Omega} v |\nabla \ln(u_{\varepsilon} + \lambda)|^2 \vartheta - v \nabla \ln(u_{\varepsilon} + \lambda) \cdot \nabla \vartheta \, d\mathbf{x} \\ & + \int_{\Omega} \kappa \frac{u_{\varepsilon}}{u_{\varepsilon} + \lambda} \nabla w_{\varepsilon} \cdot \nabla \vartheta - \kappa \frac{u_{\varepsilon}}{u_{\varepsilon} + \lambda} \vartheta \nabla w_{\varepsilon} \cdot \nabla \ln(u_{\varepsilon} + \lambda) \, d\mathbf{x} \\ & = \int_{\Omega} (\beta - \alpha w_{\varepsilon}) \frac{u_{\varepsilon}}{u_{\varepsilon} + \lambda} \vartheta + \varepsilon |u_{\varepsilon}|^{p-1} \frac{u_{\varepsilon}}{u_{\varepsilon} + \lambda} \vartheta \, d\mathbf{x}, \end{aligned}$$

where we omitted the argument t for the sake of readability and will continue to do so for the remainder of the proof. We take $\lambda \downarrow 0$ and pull the limit into the integral, which we are allowed to do by the following reasoning. Note that $-\partial_t \ln(u_{\varepsilon} + \lambda)$ is monotonically increasing for $\lambda \downarrow 0$, thus, by the monotone convergence theorem, we can pull the limit into the integral in the first term. For the other integrals we note that we have pointwise convergence in λ and that we can find integrable dominating functions by the estimates from the first three terms of (4.7a) and the first two terms of (4.7b). The dominating functions of the terms including λ are given by

$$|\nabla \ln(u_{\varepsilon} + \lambda)|^2 = \frac{1}{(u_{\varepsilon} + \lambda)^2} |\nabla u_{\varepsilon}|^2 \leq \frac{1}{u_{\varepsilon}^2} |\nabla u_{\varepsilon}|^2 = |\nabla \ln u_{\varepsilon}| \quad \text{and} \quad \frac{u_{\varepsilon}}{u_{\varepsilon} + \lambda} \leq 1.$$

Thus by Lebesgue’s dominated convergence theorem we can also pull the limit $\lambda \downarrow 0$ into the remaining integrals with equality and we get

$$\begin{aligned} & - \int_{\Omega} \partial_t \ln u_{\varepsilon} \vartheta \, d\mathbf{x} + \int_{\Omega} v |\nabla \ln u_{\varepsilon}|^2 \vartheta - v \nabla \ln u_{\varepsilon} \cdot \nabla \vartheta + \kappa \nabla w_{\varepsilon} \cdot \nabla \vartheta \, d\mathbf{x} \\ & - \int_{\Omega} \kappa \vartheta \nabla w_{\varepsilon} \cdot \nabla \ln u_{\varepsilon} \, d\mathbf{x} = \int_{\Omega} (\beta - \alpha w_{\varepsilon}) \vartheta + \varepsilon |u_{\varepsilon}|^{p-1} \vartheta \, d\mathbf{x}. \quad (4.14) \end{aligned}$$

Here we used the convergence $\frac{u_{\varepsilon}}{u_{\varepsilon} + \lambda} \rightarrow 1$ for $\lambda \downarrow 0$ pointwise almost everywhere as we have $u_{\varepsilon} > 0$ almost everywhere due to the fact that $\ln u_{\varepsilon}$ is bounded in $L^{\infty}(L^1)$, cf. (4.9). Using the test function $\vartheta = -\varphi$ for some $\varphi \in W^{1,p}(\Omega)$ we can derive an estimate for $\partial_t \ln u_{\varepsilon}$ in $L^1(W^{1,p}(\Omega)^*)$, by estimating

$$\left| \int_{\Omega} \partial_t \ln u_{\varepsilon} \varphi \, d\mathbf{x} \right| = \left| \int_{\Omega} v |\nabla \ln u_{\varepsilon}|^2 \varphi - v \nabla \ln u_{\varepsilon} \cdot \nabla \varphi + \kappa \nabla w_{\varepsilon} \cdot \nabla \varphi \, d\mathbf{x} \right|$$

$$\begin{aligned}
 & - \int_{\Omega} \kappa \varphi \nabla w_{\varepsilon} \cdot \nabla \ln u_{\varepsilon} - (\beta - \alpha w_{\varepsilon}) \varphi - \varepsilon |u_{\varepsilon}|^{p-1} \varphi \, dx \Big| \\
 & \leq C \|\varphi\|_{W^{1,p}(\Omega)} \left(1 + \|\nabla \ln u_{\varepsilon}\|_{L^2(\Omega)}^2 + \|\nabla w_{\varepsilon}\|_{L^2(\Omega)}^2 \right. \\
 & \quad \left. + \|w_{\varepsilon}\|_{L^{\infty}((0,T) \times \Omega)} + \varepsilon \|u_{\varepsilon}\|_{L^p(\Omega)}^{p-1} \right).
 \end{aligned}$$

Note that this time the right-hand side is only in $L^1(0, T)$ and thus we get the weaker bound for $\partial_t \ln u_{\varepsilon}$. □

4.4. Convergence of approximate solutions

We now tend the regularization coefficient to zero and show the existence of generalized solutions for vanishing regularization.

Proposition 4.7. (Convergence of solutions) *Let $\{(u_{0\varepsilon}, w_{0\varepsilon})\}$ be as in Proposition 4.5 and let $\{(u_{\varepsilon}, w_{\varepsilon})\}$ be a sequence of solutions to the approximate system (4.1). Then there exists a pair $(u, w) \in \mathcal{X}$, such that the following convergences hold for a subsequence $\varepsilon \downarrow 0$,*

$$w_{\varepsilon} \rightharpoonup w \quad \text{in } L^2(0, T; W^{1,2}(\Omega)), \tag{4.15}$$

$$w_{\varepsilon} \rightarrow w \quad \text{in } L^2(0, T; L^2(\Omega)), \tag{4.16}$$

$$w_{\varepsilon} \rightarrow w \quad \text{pointwise a.e. in } (0, T) \times \Omega, \tag{4.17}$$

$$w_{\varepsilon} \rightharpoonup^* w \quad \text{in } L^{\infty}((0, T) \times \Omega) \tag{4.18}$$

$$\ln u_{\varepsilon} \rightharpoonup \ln u \quad \text{in } L^2(0, T; W^{1,2}(\Omega)), \tag{4.19}$$

$$\ln u_{\varepsilon} \rightarrow \ln u \quad \text{in } L^2(0, T; L^2(\Omega)), \tag{4.20}$$

$$\ln u_{\varepsilon} \rightarrow \ln u \quad \text{pointwise a.e. in } (0, T) \times \Omega, \tag{4.21}$$

$$u_{\varepsilon} \rightarrow u \quad \text{pointwise a.e. in } (0, T) \times \Omega, \tag{4.22}$$

$$w_{\varepsilon} u_{\varepsilon} \rightarrow w u \quad \text{in } L^1(0, T; L^1(\Omega)), \tag{4.23}$$

$$\ln u_{\varepsilon} \rightharpoonup^* \ln u \quad \text{in } \text{BV}([0, T]; W^{1,p}(\Omega)^*), \tag{4.24}$$

$$\partial_t w_{\varepsilon} \rightharpoonup \partial_t w \quad \text{in } L^2(0, T; W^{1,p}(\Omega)^*), \tag{4.25}$$

$$w_{\varepsilon} \rightarrow w \quad \text{in } C_w([0, T]; L^2(\Omega)). \tag{4.26}$$

Proof. We will not relabel subsequences throughout this proof. The boundedness of $\{w_{\varepsilon}\}$ and $\{\nabla w_{\varepsilon}\}$ in $L^2(L^2)$ from Proposition 4.5 imply that there exists a subsequence weakly convergent in $L^2(W^{1,2})$ to some $w \in L^2(W^{1,2})$, since $L^2(W^{1,2})$ is reflexive. Additionally, due to the boundedness of $\{\partial_t w_{\varepsilon}\}$ in $L^2(W^{1,p}(\Omega)^*)$ and Aubin–Lions Lemma, see for example [53, Lem. 7.7], there exists a subsequence $\{w_{\varepsilon}\}$ strongly convergent to w in $L^2(L^2)$ and thus we find another subsequence which converges pointwise almost everywhere to w in $(0, T) \times \Omega$. The uniform boundedness of $\{w_{\varepsilon}\}$ in $L^{\infty}((0, T) \times \Omega)$ additionally implies the weak*-convergence of a subsequence to w in $L^{\infty}((0, T) \times \Omega)$, see for example [19, Thm. A.2.18].

Since $\{\nabla \ln u_\varepsilon\}$ is bounded in $L^2(L^2)$ there is a subsequence, which is weakly convergent in $L^2(L^2)$. By Poincaré’s inequality, see [44, Thm. 13.27, p. 432], we have

$$\|\ln u_\varepsilon(t) - (\ln u_\varepsilon(t))^{\text{ave}}\|_{L^2(\Omega)} \leq C \|\nabla \ln u_\varepsilon(t)\|_{L^2(\Omega)} \tag{4.27}$$

for almost all $t \in (0, T)$, where for $g \in L^1(\Omega)$ we define g^{ave} by

$$g^{\text{ave}} := \frac{1}{|\Omega|} \int_{\Omega} g(\mathbf{x}) \, d\mathbf{x}.$$

We find the constant upper bound for the $L^2(L^2)$ -norm of $\ln u_\varepsilon$ by the *a priori* estimates from Proposition 4.5. Using the $L^\infty(L^1)$ *a priori* bound for $\ln u_\varepsilon$, cf. the third term of (4.7a), we can estimate

$$(\ln u_\varepsilon(t))^{\text{ave}} = \frac{1}{|\Omega|} \int_{\Omega} \ln u_\varepsilon(t) \, d\mathbf{x} \leq \frac{1}{|\Omega|} \|\ln u_\varepsilon\|_{L^\infty(L^1)} \leq \frac{1}{|\Omega|} C$$

for almost all $t \in (0, T)$. Plugging this into (4.27) and using the reverse triangle inequality of the norm, we obtain

$$\int_0^T \|\ln u_\varepsilon(t)\|_{L^2(\Omega)}^2 \, dt \leq \int_0^T C^2 (\|\nabla \ln u_\varepsilon(t)\|_{L^2(\Omega)} + 1)^2 \, dt \leq C,$$

by the *a priori* estimate from the fourth term of (4.7a). Again we find a subsequence $\{\ln u_\varepsilon\}$, which is weakly convergent to some $\xi \in L^2(W^{1,2})$. We define $u := e^\xi$. This implies the convergence from (4.19). Using the *a priori* bound on $\{\partial_t \ln u_\varepsilon\}$ and the compact embedding from Aubin–Lions Lemma, see [53, Lem. 7.7], we find, as above, another subsequence $\{\ln u_\varepsilon\}$ converging strongly in $L^2(L^2)$ and pointwise almost everywhere to $\ln u$. This implies that also $u_\varepsilon \rightarrow u$ pointwise almost everywhere and we have shown the convergences from (4.19)–(4.22).

The product $\{u_\varepsilon w_\varepsilon\}$ is bounded in $L^1(L^1)$ and equi-integrable by Proposition 4.5. The equi-integrability follows from the fact that $u_\varepsilon w_\varepsilon \ln(u_\varepsilon w_\varepsilon + 1)$ is bounded in $L^1(L^1)$ and by an application of the de la Vallée Poussin theorem, see for example [44, p. 675], which is applicable since $G : [0, \infty) \rightarrow [0, \infty]$ with $x \mapsto x \ln(x + 1)$ is an increasing, convex function and fulfills $\frac{G(|x|)}{x} \rightarrow \infty$ as $x \rightarrow \infty$. By Vitali’s theorem, see [18, Ch. VI, Thm. 5.6], and the pointwise convergence of $\{u_\varepsilon w_\varepsilon\}$ to uw , cf. convergences (4.17) and (4.22), we can deduce the strong convergence of $\{u_\varepsilon w_\varepsilon\}$ to uw in $L^1(L^1)$.

By the reflexivity of $L^2(W^{1,p}(\Omega)^*)$ we find a weakly convergent subsequence of $\{\partial_t w_\varepsilon\}$ convergent to some $\eta \in L^2(W^{1,p}(\Omega)^*)$. By the convergence of $\{w_\varepsilon\}$ to w in $L^2(L^2)$ we find $\eta = \partial_t w$. We can infer the convergence of a subsequence $\{w_\varepsilon\}$ in $C_w([0, T]; L^2(\Omega))$ as a consequence of the boundedness of $\{w_\varepsilon\}$ in $L^\infty(L^2)$ and of $\{\partial_t w_\varepsilon\}$ in $L^2(0, T; W^{1,p}(\Omega)^*)$, see [17, Prop. 4.9]. This gives the convergence in (4.26).

For the convergence of $\{\partial_t \ln u_\varepsilon\}$, we consider $\{\ln u_\varepsilon\}$ as abstract functions of bounded variation, more precisely in the space $BV([0, T]; W^{1,p}(\Omega)^*)$, for a definition see [5, Sec. 1.3] or [26]. We first show that $\{\ln u_\varepsilon\}$ is bounded in that space, since boundedness implies relative compactness with respect to the weak* topology [26, Cor. 3.11]. A characterization of weak* convergence in $BV([0, T]; W^{1,p}(\Omega)^*)$ can be found in [5, Thm. 1.126]. We start by showing that $\{\ln u_\varepsilon\}$ is bounded in $W^{1,1}(0, T; W^{1,p}(\Omega)^*)$. The $L^1(0, T; W^{1,p}(\Omega)^*)$ -norm of $\{\ln u_\varepsilon\}$ is bounded since

$$L^2(L^2) \hookrightarrow L^2(0, T; W^{1,p}(\Omega)^*) \hookrightarrow L^1(0, T; W^{1,p}(\Omega)^*)$$

holds and the $L^1(0, T; W^{1,p}(\Omega)^*)$ -norm of $\{\partial_t \ln u_\varepsilon\}$ is bounded by the *a priori* estimates from (4.7a). From the continuous embedding

$$W^{1,1}(0, T; W^{1,p}(\Omega)^*) \hookrightarrow BV([0, T]; W^{1,p}(\Omega)^*),$$

which follows from [5, Thm. 1.129], we get $\|\ln u_\varepsilon\|_{BV} \leq C$ for some $C > 0$ independent of ε and we can find a subsequence, which converges weak* in this space to some l . By the convergence of $\ln u_\varepsilon \rightarrow \ln u$ in $L^2(L^2)$ and the uniqueness of weak limits we can deduce $l = \ln u$ by possibly redefining u on a set of measure zero, since both $L^2(L^2)$ and $BV([0, T]; W^{1,p}(\Omega)^*)$ are embedded into $L^1(0, T; W^{1,p}(\Omega)^*)$. □

Proposition 4.8. *Let $\{(u_\varepsilon, w_\varepsilon)\}$ be the sequence of approximate solutions and (u, w) the limit from Proposition 4.7. Under the assumption that $w_\varepsilon(0) \rightarrow w_0$ strongly in $L^2(\Omega)$ as $\varepsilon \downarrow 0$, we have $\nabla w_\varepsilon \rightarrow \nabla w$ in $L^2(0, T; L^2(\Omega))$ as $\varepsilon \downarrow 0$.*

Proof. For fixed $\varepsilon, \tilde{\varepsilon} > 0$ we can subtract the equations for w_ε and $w_{\tilde{\varepsilon}}$ and test the equation with $(w_\varepsilon - w_{\tilde{\varepsilon}})$.

This yields

$$\begin{aligned} \frac{1}{2} \int_\Omega (w_\varepsilon - w_{\tilde{\varepsilon}})^2 d\vec{x} \Big|_0^t + \int_0^t \int_\Omega \mu |\nabla(w_\varepsilon - w_{\tilde{\varepsilon}})|^2 d\vec{x} ds \\ = \int_0^t \int_\Omega \gamma (w_\varepsilon - w_{\tilde{\varepsilon}})^2 - \delta(u_\varepsilon w_\varepsilon - u_{\tilde{\varepsilon}} w_{\tilde{\varepsilon}})(w_\varepsilon - w_{\tilde{\varepsilon}}) d\vec{x} ds \end{aligned}$$

for almost all $t \in (0, T)$, which in turn implies

$$\begin{aligned} \mu \int_0^t \int_\Omega |\nabla(w_\varepsilon - w_{\tilde{\varepsilon}})|^2 d\vec{x} ds \leq \frac{1}{2} \int_\Omega (w_\varepsilon - w_{\tilde{\varepsilon}})^2(0) d\vec{x} + \int_0^t \int_\Omega \gamma (w_\varepsilon - w_{\tilde{\varepsilon}})^2 \\ - \delta(u_\varepsilon w_\varepsilon - u_{\tilde{\varepsilon}} w_{\tilde{\varepsilon}})(w_\varepsilon - w_{\tilde{\varepsilon}}) d\vec{x} ds \end{aligned}$$

for almost all $t \in (0, T)$.

As the product of an equi-integrable and a uniformly bounded family of functions the product $\{u_\varepsilon w_\varepsilon^2\}$ is also equi-integrable and again applying Vitali’s theorem [18, Ch. VI, Thm. 5.6], we can deduce the strong convergence to uw^2 in $L^1(L^1)$ from the pointwise almost everywhere convergence, cf. (4.17) and (4.22).

Using this and the weak lower semicontinuity of the norm, similarly to the procedure in the proof of [63, Lem. 4.5], and the weak convergence of $\{\nabla w_\varepsilon\}$ to ∇w in $L^2(L^2)$, cf. (4.15), we obtain

$$\begin{aligned} \mu \int_0^t \int_\Omega |\nabla(w_\varepsilon - w)|^2 d\vec{x} ds &\leq \liminf_{\varepsilon \searrow 0} \mu \int_0^t \int_\Omega |\nabla(w_\varepsilon - w_{\bar{\varepsilon}})|^2 d\vec{x} ds \\ &\leq \liminf_{\varepsilon \searrow 0} \frac{1}{2} \int_\Omega (w_\varepsilon - w_{\bar{\varepsilon}})^2(0) d\vec{x} + \liminf_{\varepsilon \searrow 0} \int_0^t \int_\Omega \gamma(w_\varepsilon - w_{\bar{\varepsilon}})^2 d\vec{x} ds \\ &\quad + \liminf_{\varepsilon \searrow 0} \int_0^t \int_\Omega -\delta(u_\varepsilon w_\varepsilon - u_{\bar{\varepsilon}} w_{\bar{\varepsilon}})(w_\varepsilon - w_{\bar{\varepsilon}}) d\vec{x} ds \\ &= \int_\Omega (w_\varepsilon(0) - w_0)^2 d\vec{x} + \int_0^t \int_\Omega \gamma(w_\varepsilon - w)^2 d\vec{x} ds \\ &\quad + \int_0^t \int_\Omega -\delta(w_\varepsilon u_\varepsilon - uw)(w_\varepsilon - w) d\vec{x} ds \end{aligned} \tag{4.28}$$

for almost all $t \in (0, T)$, where the equality follows from the assumed strong convergence of the initial values and the strong convergence of $\{w_\varepsilon\}$ in $L^2(L^2)$, cf. (4.16). Again using the weak lower semicontinuity of the norm we can also take ε to zero. Then the right-hand side of (4.28) vanishes and we obtain the strong convergence of $\{\nabla w_\varepsilon\}$ in $L^2(L^2)$. □

Proof of Theorem 2.4. For every (u_0, w_0) , fulfilling the conditions from Theorem 2.4, we now show that we can construct a sequence $\{(u_{0\varepsilon}, w_{0\varepsilon})\}$ of initial values fulfilling the assumptions from Theorem 4.2 and Proposition 4.5, such that $u_{0\varepsilon} \rightharpoonup u_0$ in $L^1(\Omega)$ and $w_{0\varepsilon} \rightarrow w_0$ in $L^2(\Omega)$. Using mollifiers we find a sequence $\{w_{0\varepsilon}\} \subseteq C_0^\infty(\Omega)$ such that $w_{0\varepsilon} \geq 0$, $w_{0\varepsilon} \rightharpoonup^* w_0$ in $L^\infty(\Omega)$ and

$$\|w_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)}$$

holds. Thus $\{w_{0\varepsilon}\}$ is uniformly bounded in ε and fulfills all the required assumptions. Applying an L^p -interpolation theorem, we get the strong convergence $w_{0\varepsilon} \rightarrow w_0$ in $L^r(\Omega)$ for all $r \in [1, \infty)$. Defining

$$u_{0\varepsilon} := \begin{cases} u_0 & \text{if } u_0 \leq n_\varepsilon, \\ n_\varepsilon & \text{if } u_0 > n_\varepsilon, \end{cases}$$

for a sequence $\{n_\varepsilon\} \subseteq \mathbb{N}$ with $n_\varepsilon \rightarrow \infty$ as $\varepsilon \downarrow 0$, we have constructed a sequence of approximate initial conditions for the predator fulfilling the required assumptions.

The weak convergence (4.15) of $\{w_\varepsilon\}$ from Proposition 4.7 implies

$$\int_0^t \int_\Omega w_\varepsilon g \, dx \, dt \rightarrow \int_0^t \int_\Omega wg \, dx \, ds$$

for all $g \in L^2(L^2)$ and all $t \in [0, T]$. All remaining convergences throughout this proof for which it is not stated otherwise, also hold for all $t \in [0, T]$. Since we have $\vartheta \in L^2(L^2)$, $\nabla \vartheta \in L^2(L^2)$ and $\partial_t \vartheta \in L^2(L^2)$, we get

$$\int_0^t \int_{\Omega} w_{\varepsilon} \partial_t \vartheta - \mu \nabla w_{\varepsilon} \cdot \nabla \vartheta \, dx \, ds \rightarrow \int_0^t \int_{\Omega} w \partial_t \vartheta - \mu \nabla w \cdot \nabla \vartheta \, dx \, ds,$$

$$\int_0^t \int_{\Omega} \gamma w_{\varepsilon} \vartheta \, dx \, ds \rightarrow \int_0^t \int_{\Omega} \gamma w \vartheta \, dx \, ds$$

for all $\vartheta \in C^1([0, T]; L^{\infty}(\Omega)) \cap L^2(W^{1,2})$. The strong convergence of $\{w_{\varepsilon} u_{\varepsilon}\}$ in $L^1(L^1)$ implies

$$\int_0^t \int_{\Omega} \delta u_{\varepsilon} w_{\varepsilon} \vartheta \, dx \, ds \rightarrow \int_0^t \int_{\Omega} \delta u w \vartheta \, dx \, ds,$$

since $\vartheta \in C^1([0, T]; L^{\infty}(\Omega)) \hookrightarrow L^{\infty}((0, T) \times \Omega)$. Making use of the convergence of $\{w_{\varepsilon}\}$ in $C_w([0, T]; L^2(\Omega))$, cf. (4.26), we find

$$\int_{\Omega} w_{\varepsilon}(t) \vartheta(t) \, dx \rightarrow \int_{\Omega} w(t) \vartheta(t) \, dx$$

for all $\vartheta \in C^1([0, T]; L^{\infty}(\Omega))$. By the convergence of the initial data, we have

$$\int_{\Omega} w_{\varepsilon}(0) \, dx = \int_{\Omega} w_{0\varepsilon} \, dx \rightarrow \int_{\Omega} w_0 \, dx.$$

This shows that the limit identified in Proposition 4.7 fulfills the variational formulation (2.3). The equality of $w(0) = w_0$ in $L^2(\Omega)$ follows by the convergence of the initial data $\{w_{0\varepsilon}\}$ in $L^2(\Omega)$ and the convergence of $\{w_{\varepsilon}\}$ in $C_w([0, T]; L^2(\Omega))$, cf. (4.26). Since $\{w_{\varepsilon}\}$ is pointwise almost everywhere convergent and uniformly bounded in $L^{\infty}((0, T) \times \Omega)$ this bound transfers to the limit and we also get $w \in L^{\infty}((0, T) \times \Omega)$.

The Laplace is a bounded linear map between $W^{1,2}(\Omega)$ and its dual $W^{1,2}(\Omega)^*$. Since we know w to be in $L^2(W^{1,2})$, we find $\Delta w \in L^2(0, T; W^{1,2}(\Omega)^*)$ and equation (1.1b) now implies

$$\partial_t w \in L^2(0, T; W^{1,2}(\Omega)^* + L^1(\Omega)).$$

Next, we show that the population and the logarithmic inequality for the predator, cf. equations (2.1) and (2.2), respectively, are fulfilled. On the approximate level the population inequality (2.1) holds with equality, cf. equation (4.8). By the strong convergence of $\{u_{\varepsilon} w_{\varepsilon}\}$ in $L^1(L^1)$ we get

$$\alpha \int_0^t \int_{\Omega} u_{\varepsilon} w_{\varepsilon} \, dx \, ds \rightarrow \alpha \int_0^t \int_{\Omega} u w \, dx \, ds.$$

Using the weak lower semicontinuity of the convex functional $x \mapsto e^x$ and the weak convergence of $\{\ln u_{\varepsilon}\}$ we get

$$\beta \int_0^t \int_{\Omega} u \, dx \, ds = \beta \int_0^t \int_{\Omega} e^{\ln u} \, dx \, ds \leq \liminf_{\varepsilon \downarrow 0} \beta \int_0^t \int_{\Omega} e^{\ln u_{\varepsilon}} \, dx \, ds.$$

Using the boundedness of $\{\ln u_\varepsilon\}$ in $L^2(L^2) \cap BV([0, T]; W^{1,p}(\Omega)^*)$ and applying [52, Thm. A.5], we find a subsequence of $\{\ln u_\varepsilon\}$ and some l in $L^2(L^2) \cap L^\infty(0, T; W^{1,p}(\Omega)^*)$, such that

$$\ln u_\varepsilon \rightharpoonup^* l \text{ in } L^2(L^2) \cap L^\infty(0, T; W^{1,p}(\Omega)^*), \tag{4.29}$$

$$\ln u_\varepsilon(t) \rightharpoonup l(t) \text{ in } L^2(\Omega) \text{ for almost all } t \in (0, T). \tag{4.30}$$

Since, we already have the strong convergence of $\{\ln u_\varepsilon\}$ in $L^2(L^2)$ to $\ln u$, we can identify the limit l with $\ln u$. The uniform boundedness of $\{\ln u_{0\varepsilon}\}$ in $W^{1,p}(\Omega)^*$, which is needed to apply Thm. A.5, follows from the boundedness of $\{\ln u_{0\varepsilon}\}$ in $L^1(\Omega)$ and the continuous embedding of $L^1(\Omega)$ into $W^{1,p}(\Omega)^*$. Using [49, Lem. 7.2], we find that

$$\ln u_\varepsilon(t) \rightharpoonup^* \bar{l}(t) \text{ in } W^{1,p}(\Omega)^* \text{ for all } t \in [0, T], \tag{4.31}$$

for some $\bar{l} \in BV([0, T]; W^{1,p}(\Omega)^*)$. We find $\ln u = \bar{l}$ everywhere in $[0, T]$, after possibly redefining u on a set of measure zero. Again using the weak lower semicontinuity of $x \mapsto e^x$, we get

$$\int_\Omega u(t) \, dx = \int_\Omega e^{\ln u(t)} \, dx \leq \liminf_{\varepsilon \downarrow 0} \int_\Omega e^{\ln u_\varepsilon(t)} \, dx = \liminf_{\varepsilon \downarrow 0} \int_\Omega u_\varepsilon(t) \, dx.$$

The weak convergence of the initial values implies

$$\int_\Omega u_0 \, dx = \lim_{\varepsilon \downarrow 0} \int_\Omega u_{0\varepsilon} \, dx.$$

Putting these results together we get

$$\begin{aligned} \int_\Omega u \, dx \Big|_0^t + \beta \int_0^t \int_\Omega u \, dx \, ds &\leq \liminf_{\varepsilon \downarrow 0} \int_\Omega u_\varepsilon(t) \, dx \\ &\quad - \lim_{\varepsilon \downarrow 0} \int_\Omega u_{0\varepsilon} \, dx + \liminf_{\varepsilon \downarrow 0} \beta \int_0^t \int_\Omega u_\varepsilon \, dx \, ds + \liminf_{\varepsilon \downarrow 0} \int_0^t \int_\Omega \varepsilon u_\varepsilon |u_\varepsilon|^{p-1} \, dx \, ds \\ &= \liminf_{\varepsilon \downarrow 0} \alpha \int_0^t \int_\Omega w_\varepsilon u_\varepsilon \, dx \, ds = \alpha \int_0^t \int_\Omega wu \, dx \, ds. \end{aligned}$$

This shows, that the population inequality (2.1) holds.

To derive the logarithmic inequality (2.2), we use the regularized logarithmic equality from (4.14). To take the limit $\varepsilon \downarrow 0$ we note the following. By the weak lower semicontinuity of the norm and the weak convergence (4.19), we get

$$\int_0^t \int_\Omega \vartheta |\nabla \ln u|^2 \, dx \, ds \leq \liminf_{\varepsilon \downarrow 0} \int_0^t \int_\Omega \vartheta |\nabla \ln u_\varepsilon|^2 \, dx \, ds,$$

where we used $\vartheta \geq 0$ to pull it out of the absolute value. Using the strong convergence of the sequence $\{\nabla w_\varepsilon\}$ in $L^2(L^2)$ to ∇w , which is shown in Proposition 4.8, we can deduce that

$$\nabla \ln u_\varepsilon \cdot \nabla w_\varepsilon \rightharpoonup \nabla \ln u \cdot \nabla w \text{ in } L^1(L^1).$$

By the convergence of the initial data, we get

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega} \ln u_{0\varepsilon}(\mathbf{x}) \vartheta(0, \mathbf{x}) \, d\mathbf{x} = \int_{\Omega} \ln u_0 \vartheta(0, \mathbf{x}) \, d\mathbf{x},$$

for all $\vartheta \in C([0, T]; L^\infty(\Omega))$. Putting all this together, we see that the shown convergences are sufficient to pass to the limit with $\varepsilon \downarrow 0$ in the regularized logarithmic equality, where the equality becomes an inequality,

$$\begin{aligned} & - \int_{\Omega} \ln u \vartheta \, d\mathbf{x} \Big|_0^t \\ & + \int_0^t \int_{\Omega} \nu |\nabla \ln u|^2 \vartheta - \nu \nabla \ln u \cdot \nabla \vartheta - \kappa \vartheta \nabla w \cdot \nabla \ln u + \kappa \nabla w \cdot \nabla \vartheta \, d\mathbf{x} \, ds \\ & \leq \liminf_{\varepsilon \downarrow 0} \left[- \int_{\Omega} \ln u_{\varepsilon} \vartheta \, d\mathbf{x} \Big|_0^t \right] + \liminf_{\varepsilon \downarrow 0} \int_0^t \int_{\Omega} \nu |\nabla \ln u_{\varepsilon}|^2 \vartheta \, d\mathbf{x} \, ds \\ & + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\Omega} -\nu \nabla \ln u_{\varepsilon} \cdot \nabla \vartheta - \kappa \vartheta \nabla w_{\varepsilon} \cdot \nabla \ln u_{\varepsilon} + \kappa \nabla w_{\varepsilon} \cdot \nabla \vartheta \, d\mathbf{x} \, ds \\ & = \liminf_{\varepsilon \downarrow 0} \int_0^t \int_{\Omega} (\beta - \alpha w_{\varepsilon}) \vartheta - \ln u_{\varepsilon} \partial_t \vartheta + \varepsilon |u_{\varepsilon}|^{p-1} \vartheta \, d\mathbf{x} \, ds \\ & = \int_0^t \int_{\Omega} (\beta - \alpha w) \vartheta - \ln u \partial_t \vartheta \, d\mathbf{x} \, ds, \end{aligned}$$

where the regularizing term vanishes as ε goes to zero due to the boundedness of $\{\varepsilon^{1/p} u_{\varepsilon}\}$ in $L^p(L^p)$ from Proposition 4.5. The fact that u fulfills the initial condition u_0 in $L^1(\Omega)$ follows from the convergence for all $t \in [0, T]$ in $W^{1,p}(\Omega)^*$, cf. (4.31), and the convergence of the initial data. For all $\varphi \in W^{1,p}(\Omega)$ we have

$$\int_{\Omega} \ln u_0 \varphi \, d\mathbf{x} = \lim_{\varepsilon \downarrow 0} \int_{\Omega} \ln u_{0\varepsilon} \varphi \, d\mathbf{x} = \int_{\Omega} \ln u(0) \varphi \, d\mathbf{x}.$$

By a density argument we find $\ln u(0) = \ln u_0$ in $L^1(\Omega)$ and thus $u(0) = u_0$ in $L^1(\Omega)$. Using the population inequality, we find that u is bounded in $L^\infty(L^1)$, where we used the $L^1(L^1)$ integrability of uw and the $L^1(\Omega)$ integrability of u_0 . Again using the fact, that $\ln u_{\varepsilon}(t) \rightharpoonup \ln u(t)$ in $L^2(\Omega)$ for almost all $t \in (0, T)$, we can transfer the $L^\infty(L^1)$ -bound from $\{\ln u_{\varepsilon}\}$ to $\ln u$ by using the weak lower semicontinuity of the norm. Thus we have shown that $\ln u \in L^\infty(L^1)$, which implies $u > 0$ almost everywhere. The non-negativity of w follows from the comparison principle, cf. Lemma 4.3. \square

5. Weak–strong uniqueness

In order to justify that our solution concept is meaningful, we prove that it fulfills the weak–strong uniqueness property. That is, if there exists a strong solution to the system (1.1) for some initial data, then all generalized solutions emanating from the same initial data coincide with the strong solution and thus this solution is unique as long as it exists. In order to be able to prove such a property for the generalized solutions, some additional properties of strong solutions are needed.

5.1. Properties of strong solution

Later on, in the proof of the weak–strong uniqueness, we would like to test with $1/\tilde{u}$, for a strong solution \tilde{u} to (1.1a). In order to justify that this is possible, we prove the following lemma.

Lemma 5.1. *Let the initial condition $u_0 \in C^3(\overline{\Omega})$ be bounded away from zero by some $\underline{l} > 0$ and $\tilde{w} \in C^1([0, T] \times \overline{\Omega})$ with $\Delta\tilde{w} \in C([0, T] \times \overline{\Omega})$. Then a strong solution \tilde{u} to (1.1a) with w replaced by \tilde{w} is bounded away from zero.*

To be able to prove this lemma, we first show the following comparison principle.

Proposition 5.2. (Comparison principle for \tilde{u}) *Let $\tilde{w} \in C^1([0, T] \times \overline{\Omega})$ be such that $\Delta\tilde{w} \in C([0, T] \times \overline{\Omega})$. Assume that there exist a strong sub-solution \underline{u} and a strong super-solution \bar{u} to (1.1a) with $w = \tilde{w}$, fulfilling the non-negative initial data $\underline{u}_0, \bar{u}_0 \in C^3(\overline{\Omega})$ respectively. That is \underline{u} and \bar{u} fulfill*

$$\underline{u}, \bar{u} \in C^1([0, T] \times \overline{\Omega}) \text{ and } \Delta\underline{u}, \Delta\bar{u} \in C([0, T] \times \overline{\Omega}),$$

are non-negative and it holds,

$$\underline{u}_t - \nu \Delta\underline{u} + \kappa \nabla \cdot (\underline{u} \nabla \tilde{w}) \leq (\alpha \tilde{w} - \beta) \underline{u} \quad \text{in } (0, T) \times \Omega, \tag{5.1}$$

$$\bar{u}_t - \nu \Delta\bar{u} + \kappa \nabla \cdot (\bar{u} \nabla \tilde{w}) \geq (\alpha \tilde{w} - \beta) \bar{u} \quad \text{in } (0, T) \times \Omega, \tag{5.2}$$

$$\underline{u} \leq \underline{u}_0 \quad \text{on } \{0\} \times \Omega, \tag{5.3}$$

$$\bar{u} \geq \bar{u}_0 \quad \text{on } \{0\} \times \Omega, \tag{5.4}$$

$$\nabla \underline{u} \cdot n = 0 \quad \text{on } [0, T] \times \partial\Omega, \tag{5.5}$$

$$\nabla \bar{u} \cdot n = 0 \quad \text{on } [0, T] \times \partial\Omega. \tag{5.6}$$

If additionally $\underline{u}_0 \leq \bar{u}_0$ holds, then $\underline{u}(t, \mathbf{x}) \leq \bar{u}(t, \mathbf{x})$ holds everywhere in $[0, T] \times \overline{\Omega}$.

Proof. Subtracting (5.2) from (5.1) and testing the resulting inequality with the difference $(\underline{u} - \bar{u})^+$, we find

$$\begin{aligned} & \int_0^t \int_{\Omega} \partial_t(\underline{u} - \bar{u})(\underline{u} - \bar{u})^+ + \nu |\nabla(\underline{u} - \bar{u})^+|^2 - \kappa(\underline{u} - \bar{u})^+ \nabla \tilde{w} \cdot \nabla(\underline{u} - \bar{u})^+ \, dx \, ds \\ & + \int_0^t \int_{\Omega} \beta ((\underline{u} - \bar{u})^+)^2 \, dx \, ds = \int_0^t \int_{\Omega} \alpha \tilde{w} ((\underline{u} - \bar{u})^+)^2 \, dx \, ds \end{aligned} \tag{5.7}$$

for all $t \in [0, T]$. Using Young’s inequality, collecting alike terms and integrating by parts we get

$$\begin{aligned} & \frac{1}{2} \|(\underline{u} - \bar{u})^+(t)\|_{L^2(\Omega)}^2 \\ & \leq \int_0^t \left(\alpha \|\tilde{w}\|_{L^\infty([0, T] \times \Omega)} + \frac{\kappa^2}{2\nu} \|\nabla \tilde{w}(s)\|_{L^\infty(\Omega)}^2 \right) \|(\underline{u} - \bar{u})^+(s)\|_{L^2(\Omega)}^2 \, ds \end{aligned}$$

for all $t \in [0, T]$. An application of Gronwall’s inequality yields $(\underline{u} - \bar{u})^+ = 0$ almost everywhere and with the continuity of \underline{u} and \bar{u} this equality extends to the whole domain $[0, T] \times \bar{\Omega}$. Thus we have $\underline{u}(t, \mathbf{x}) \leq \bar{u}(t, \mathbf{x})$ everywhere in $[0, T] \times \bar{\Omega}$ as claimed. \square

Now we prove Lemma 5.1.

Proof of Lemma 5.1. We will construct a sub-solution \underline{u}^* which is bounded away from zero and constant in space. The predator equation for such a sub-solution then reads

$$\partial_t \underline{u} + (\kappa \Delta \tilde{w} - \alpha \tilde{w} + \beta) \underline{u} \leq 0,$$

since all space derivatives of \underline{u} vanish. We first consider the ordinary differential equation

$$\partial_t \underline{u} + (\kappa \|\Delta \tilde{w}\|_{L^\infty((0,T) \times \Omega)} + \alpha \|\tilde{w}\|_{L^\infty((0,T) \times \Omega)} + \beta) \underline{u} = 0$$

with the initial value $u(0) \equiv \underline{l}$. This ordinary differential equation can be solved explicitly by

$$\underline{u}^*(t) = \underline{l} \exp\left(-(\kappa \|\Delta \tilde{w}\|_{L^\infty((0,T) \times \Omega)} + \alpha \|\tilde{w}\|_{L^\infty((0,T) \times \Omega)} + \beta) t\right)$$

for $t \in [0, T]$. The solution \underline{u}^* is monotonically decreasing and continuous, thus it is bounded from below on $[0, T]$ by $\underline{u}^*(T)$ and $\underline{u}^*(T) > 0$ holds. Now,

$$\begin{aligned} \partial_t \underline{u}^* + (\kappa \Delta \tilde{w} - \alpha \tilde{w} + \beta) \underline{u}^* \\ \leq \partial_t \underline{u}^* + (\kappa \|\Delta \tilde{w}\|_{L^\infty((0,T) \times \Omega)} + \alpha \|\tilde{w}\|_{L^\infty((0,T) \times \Omega)} + \beta) \underline{u}^* = 0 \end{aligned}$$

holds everywhere in $[0, T] \times \bar{\Omega}$, since \underline{u}^* is non-negative, and thus \underline{u}^* is a sub-solution to (1.1a) and by the comparison principle from Proposition 5.2 we conclude that \tilde{u} is bounded away from zero. \square

Now we have finished the technical groundwork for our proof of weak–strong uniqueness.

5.2. Relative energy estimates

Using the integration by parts formula from the “Appendix”, cf. Lemma 7.3, we can prove a relative energy inequality, which serves as a strong tool when proving weak–strong uniqueness of generalized solutions. In order to formulate this inequality we need the following definitions. We say $(\tilde{u}, \tilde{w}) \in \mathcal{Y}$ if

$$\begin{aligned} \tilde{u}, \tilde{w} \in C^1([0, T] \times \bar{\Omega}) \text{ and } \Delta \tilde{u}, \Delta \tilde{w} \in C([0, T] \times \bar{\Omega}), \\ \tilde{w}, \tilde{u} \geq 0 \text{ and } \tilde{u} \geq l > 0 \text{ for some } l > 0 \end{aligned}$$

and \tilde{u} and \tilde{w} fulfill zero Neumann boundary conditions.

Definition 5.3. For $(u, w) \in \mathcal{X}$ and a test function $(\tilde{u}, \tilde{w}) \in \mathcal{Y}$, we define the relative energy $\mathcal{R} : \mathcal{X} \times \mathcal{Y} \rightarrow L^\infty(0, T)$ by

$$\mathcal{R}(u, w | \tilde{u}, \tilde{w}) = \int_{\Omega} u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) + \frac{\kappa^2}{\mu\nu} \tilde{u} |w - \tilde{w}|^2 \, dx, \tag{5.8}$$

the relative dissipation $\mathcal{W} : \mathcal{X} \times \mathcal{Y} \rightarrow L^1(0, T)$

$$\begin{aligned} \mathcal{W}(u, w | \tilde{u}, \tilde{w}) &= \int_{\Omega} (\beta - \alpha \tilde{w})(u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})) \, d\vec{x} \\ &\quad + \frac{2\kappa^2}{\mu\nu} \int_{\Omega} (\delta \tilde{u} - \gamma) \tilde{u} |w - \tilde{w}|^2 \, d\vec{x} + \frac{\nu}{2} \int_{\Omega} \tilde{u} |\nabla \ln u - \nabla \ln \tilde{u}|^2 \, d\vec{x} \\ &\quad + \frac{\kappa^2}{2\nu} \int_{\Omega} \tilde{u} |\nabla w - \nabla \tilde{w}|^2 \, d\vec{x}, \end{aligned} \tag{5.9}$$

the regularity weight $\mathcal{K} : \mathcal{Y} \rightarrow L^\infty(0, T)$

$$\begin{aligned} \mathcal{K}(\tilde{u}, \tilde{w}) &= \|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \mu \|\nabla \ln \tilde{u}\|_{L^\infty(\Omega)}^2 + \max\{\alpha \|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\} \\ &\quad + \max\left\{ \frac{1}{\|w\|_{L^\infty(\Omega)} + \|\tilde{w}\|_{L^\infty(\Omega)} + 1}, \|w\|_{L^\infty(\Omega)} + \|\tilde{w}\|_{L^\infty(\Omega)} + 1 \right\} \\ &\quad \times \left(\frac{\alpha\mu\nu}{\kappa^2} + \alpha + 2\delta \|\tilde{u}\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)} \left(\frac{\kappa^2}{\mu\nu} + 1 \right) + \frac{\kappa^2\delta}{2\mu\nu} \|\tilde{u}\|_{L^\infty(\Omega)} \right), \end{aligned} \tag{5.10}$$

and the system operator $\mathcal{A} : \mathcal{Y} \rightarrow (L^\infty((0, T) \times \Omega))^2$

$$\mathcal{A}(\tilde{u}, \tilde{w}) = \begin{pmatrix} \partial_t \tilde{u} - \nu \Delta \tilde{u} + \kappa \nabla \cdot (\tilde{u} \nabla \tilde{w}) + (\beta - \alpha \tilde{w}) \tilde{u} \\ \partial_t \tilde{w} - \mu \Delta \tilde{w} + \delta \tilde{u} \tilde{w} - \gamma \tilde{w} \end{pmatrix}. \tag{5.11}$$

Note, that the regularity weight \mathcal{K} can be considered as independent of w for fixed and bounded initial data, since the $L^\infty((0, T) \times \Omega)$ -bound of w only depends on the initial value w_0 and the final time T , cf. Lemma 4.3.

Remark 5.4. The relative energy \mathcal{R} defined above is non-negative and

$$\mathcal{R}(u, w | \tilde{u}, \tilde{w}) = 0$$

if and only if $u = \tilde{u}$ and $w = \tilde{w}$ almost everywhere. For $u, \tilde{u} > 0$, which holds almost everywhere by the definition of \mathcal{X} and \mathcal{Y} , we can rewrite the first term of \mathcal{R} pointwise almost everywhere with $u = e^z$ and $\tilde{u} = e^{\tilde{z}}$ as

$$e^z - e^{\tilde{z}} - e^{\tilde{z}}(z - \tilde{z}) \geq 0,$$

where the inequality follows from the convexity of the exponential and since this convexity is strict this inequality is strict whenever $z \neq \tilde{z}$ and thus $e^z - e^{\tilde{z}} - e^{\tilde{z}}(z - \tilde{z}) = 0$ implies $u = \tilde{u}$.

Lemma 5.5. (Relative energy inequality) *Let $(u, w) \in \mathcal{X}$ be a generalized solution to system (1.1) according to Definition 2.1 and $(\tilde{u}, \tilde{w}) \in \mathcal{Y}$ be a test function. Then, the following relative energy inequality holds,*

$$\begin{aligned} & \mathcal{R}(u, w|\tilde{u}, \tilde{w})(t) \\ & + \int_0^t (\mathcal{W}(u, w|\tilde{u}, \tilde{w}) + \max\{\alpha \|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\} \mathcal{R}(u, w|\tilde{u}, \tilde{w})) e^{\int_s^t \mathcal{K}(\tilde{u}, \tilde{w}) \, d\tau} \, ds \\ & \leq \mathcal{R}(u, w|\tilde{u}, \tilde{w})(0) e^{\int_0^t \mathcal{K}(\tilde{u}, \tilde{w}) \, ds} \\ & - \int_0^t \int_\Omega \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ \frac{2\kappa}{\mu\nu} \tilde{u}(w - \tilde{w}) \end{pmatrix} e^{\int_s^t \mathcal{K}(\tilde{u}, \tilde{w}) \, d\tau} \, dx \, ds \end{aligned} \tag{5.12}$$

for almost all $t \in (0, T)$, where $\mathcal{R}, \mathcal{W}, \mathcal{A}$ and \mathcal{K} are defined in Definition 5.3.

Proof. Adding the population inequality (2.1) for u and the logarithmic inequality (2.2) for u tested with \tilde{u} and adding and subtracting the system operator $\mathcal{A}(\tilde{u}, \tilde{w})$ tested with $(\ln u - \ln \tilde{u}, 0)^T$ yields

$$\begin{aligned} & \int_\Omega u - \tilde{u} \ln u \Big|_0^t \, dx + \beta \int_0^t \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \, dx \, ds \\ & + \int_0^t \int_\Omega \nu \tilde{u} |\nabla \ln u|^2 - \nu \nabla \ln u \cdot \nabla \tilde{u} + \nu \Delta \tilde{u} (\ln u - \ln \tilde{u}) \, dx \, ds \\ & + \int_0^t \int_\Omega \kappa \nabla w \cdot \nabla \tilde{u} - \kappa \tilde{u} \nabla w \cdot \nabla \ln u - \kappa \nabla \cdot (\tilde{u} \nabla \tilde{w})(\ln u - \ln \tilde{u}) \, dx \, ds \\ & \leq \alpha \int_0^t \int_\Omega u w - \tilde{u} \tilde{w} - \tilde{u}(w - \tilde{w}) - \tilde{w} \tilde{u} (\ln u - \ln \tilde{u}) \, dx \, ds \\ & - \int_0^t \int_\Omega \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ 0 \end{pmatrix} + (\partial_t \tilde{u}) \ln \tilde{u} \, dx \, ds \end{aligned} \tag{5.13}$$

for almost all $t \in (0, T)$. All following equations in this proof will also hold for almost all $t \in (0, T)$. Using $\partial_t(\tilde{u} \ln \tilde{u}) - \partial_t \tilde{u} = (\partial_t \tilde{u}) \ln \tilde{u}$ and integrating by parts using the zero Neumann boundary conditions, equation (5.13) can be rewritten as

$$\begin{aligned} & \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \Big|_0^t \, dx + \beta \int_0^t \int_\Omega u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \, dx \, ds \\ & + \nu \int_0^t \int_\Omega \tilde{u} |\nabla \ln u|^2 - \nabla \ln u \cdot \nabla \tilde{u} - \nabla \tilde{u} \cdot \nabla (\ln u - \ln \tilde{u}) \, dx \, ds \\ & + \kappa \int_0^t \int_\Omega \nabla w \cdot \nabla \tilde{u} - \tilde{u} \nabla w \cdot \nabla \ln u + \tilde{u} \nabla \tilde{w} \cdot \nabla (\ln u - \ln \tilde{u}) \, dx \, ds \\ & \leq \alpha \int_0^t \int_\Omega u w - \tilde{u} \tilde{w} - \tilde{u}(w - \tilde{w}) - \tilde{w} \tilde{u} (\ln u - \ln \tilde{u}) \, dx \, ds \\ & - \int_0^t \int_\Omega \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ 0 \end{pmatrix} \, dx \, ds. \end{aligned} \tag{5.14}$$

We observe

$$\begin{aligned} \nu \int_{\Omega} \tilde{u} |\nabla \ln u|^2 - \nabla \ln u \cdot \nabla \tilde{u} - \nabla \tilde{u} \cdot \nabla (\ln u - \ln \tilde{u}) d\vec{x} \\ = \int_{\Omega} \nu \tilde{u} |\nabla \ln u - \nabla \ln \tilde{u}|^2 d\vec{x}, \end{aligned}$$

which is a reformulation of the terms in the second line of (5.14). Using similar transformations for terms of the third line of (5.14), yields

$$\begin{aligned} \kappa \int_{\Omega} \nabla w \cdot \nabla \tilde{u} - \tilde{u} \nabla w \cdot \nabla \ln u + \tilde{u} \nabla \tilde{w} \cdot \nabla (\ln u - \ln \tilde{u}) d\vec{x} \\ = \int_{\Omega} -\kappa \tilde{u} (\nabla \ln u - \nabla \ln \tilde{u}) \cdot (\nabla w - \nabla \tilde{w}) d\vec{x}. \end{aligned}$$

Now, inequality (5.14) may be rewritten as

$$\begin{aligned} \int_{\Omega} u - \tilde{u} - \tilde{u} (\ln u - \ln \tilde{u}) \Big|_0^t dx + \beta \int_0^t \int_{\Omega} u - \tilde{u} - \tilde{u} (\ln u - \ln \tilde{u}) dx ds \\ + \int_0^t \int_{\Omega} \nu \tilde{u} |\nabla \ln u - \nabla \ln \tilde{u}|^2 + \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} \ln u - \ln \tilde{u} \\ 0 \end{pmatrix} dx ds \\ \leq \alpha \int_0^t \int_{\Omega} uw - \tilde{u}\tilde{w} - \tilde{u}(w - \tilde{w}) - \tilde{w}\tilde{u} (\ln u - \ln \tilde{u}) dx ds \\ + \int_0^t \int_{\Omega} \kappa \tilde{u} (\nabla w - \nabla \tilde{w}) \cdot (\nabla \ln u - \nabla \ln \tilde{u}) dx ds \\ \leq \frac{\nu}{2} \int_0^t \int_{\Omega} \tilde{u} |\nabla \ln u - \nabla \ln \tilde{u}|^2 dx ds + \frac{\kappa^2}{2\nu} \int_0^t \int_{\Omega} \tilde{u} |\nabla w - \nabla \tilde{w}|^2 dx ds \\ + \alpha \int_0^t \int_{\Omega} \tilde{w} (u - \tilde{u} - \tilde{u} (\ln u - \ln \tilde{u})) + (w - \tilde{w})(u - \tilde{u}) dx ds, \end{aligned} \tag{5.15}$$

where we have used Young’s inequality. Note, that the first term on the right-hand side can be absorbed into the left. Now we test the weak formulation for the prey (2.3) with $\tilde{u}(w - \tilde{w})$. We are allowed to do so, since the weak formulation holds in particular for all $\vartheta \in C^1([0, T]; L^\infty(\Omega) \cap W^{1,2}(\Omega))$ and this space is dense in the solution space of the prey w , cf. Lemma 7.5 in the “Appendix”.

Setting $\tilde{w} = (w - \tilde{w})$ and adding and subtracting the system operator tested with $(0, \tilde{u}\tilde{w})^T$, we obtain

$$\begin{aligned} \int_0^t \langle \partial_t \tilde{w}, \tilde{u}\tilde{w} \rangle ds + \int_0^t \int_{\Omega} \mu \nabla w \cdot \nabla (\tilde{u}\tilde{w}) + \delta u w \tilde{u}\tilde{w} + \mu \Delta \tilde{u}\tilde{w} - \delta \tilde{u}\tilde{w} \tilde{u}\tilde{w} dx ds \\ = \int_0^t \int_{\Omega} \gamma w \tilde{u}\tilde{w} - \gamma \tilde{w} \tilde{u}\tilde{w} - \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \begin{pmatrix} 0 \\ \tilde{u}\tilde{w} \end{pmatrix} dx ds. \end{aligned} \tag{5.16}$$

Using the integration by parts rule from Lemma 7.3 in the “Appendix”, we get

$$\begin{aligned} \int_0^t \langle \partial_t \bar{w}, \tilde{u} \bar{w} \rangle \, ds &= - \int_0^t \langle \partial_t (\tilde{u} \bar{w}), \bar{w} \rangle \, ds + \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \Big|_0^t \\ &= - \int_0^t \int_{\Omega} \partial_t \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \, ds - \int_0^t \langle \partial_t \bar{w}, \tilde{u} \bar{w} \rangle \, ds + \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \Big|_0^t, \end{aligned}$$

where we also used the product rule from Lemma 7.6 in the ‘‘Appendix’’. Pulling the second term on the right-hand side into the left-hand side and dividing by 2, we obtain

$$\int_0^t \langle \partial_t \bar{w}, \tilde{u} \bar{w} \rangle \, ds = \frac{1}{2} \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \Big|_0^t - \frac{1}{2} \int_0^t \int_{\Omega} \partial_t \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \, ds.$$

Plugging this into (5.16), gives

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \Big|_0^t + \int_0^t \int_{\Omega} \mu \bar{w} \nabla \bar{w} \cdot \nabla \tilde{u} + \mu \tilde{u} |\nabla \bar{w}|^2 + \delta \tilde{u} (u \bar{w} - \tilde{u} \bar{w}) \bar{w} \, d\mathbf{x} \, ds \\ = \int_0^t \int_{\Omega} \gamma \tilde{u} |\bar{w}|^2 + \frac{1}{2} \partial_t \tilde{u} |\bar{w}|^2 - \mathcal{A}(\tilde{u}, \bar{w}) \cdot \begin{pmatrix} 0 \\ \tilde{u} \bar{w} \end{pmatrix} \, d\mathbf{x} \, ds. \end{aligned}$$

Adding $\delta \tilde{u}^2 |\bar{w}|^2$ on both sides in the integral and using the equality,

$$\delta \tilde{u}^2 |\bar{w}|^2 - \delta \tilde{u} (u \bar{w} - \tilde{u} \bar{w}) \bar{w} = -\delta \tilde{u} w (u - \tilde{u}) \bar{w},$$

some reordering of the terms yields

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \Big|_0^t + \int_0^t \int_{\Omega} \mu \tilde{u} |\nabla \bar{w}|^2 + (\delta \tilde{u} - \gamma) \tilde{u} |\bar{w}|^2 + \mathcal{A}(\tilde{u}, \bar{w}) \cdot \begin{pmatrix} 0 \\ \tilde{u} \bar{w} \end{pmatrix} \, d\mathbf{x} \, ds \\ = \int_0^t \int_{\Omega} -\delta \tilde{u} w (u - \tilde{u}) \bar{w} + \frac{1}{2} \partial_t \tilde{u} |\bar{w}|^2 - \mu \bar{w} \nabla \bar{w} \cdot \nabla \tilde{u} \, d\mathbf{x} \, ds. \end{aligned}$$

Further estimating the right-hand side of this equality by an application of Young’s inequality, we arrive at

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \Big|_0^t + \int_0^t \int_{\Omega} \mu \tilde{u} |\nabla \bar{w}|^2 + (\delta \tilde{u} - \gamma) \tilde{u} |\bar{w}|^2 + \mathcal{A}(\tilde{u}, \bar{w}) \cdot \begin{pmatrix} 0 \\ \tilde{u} \bar{w} \end{pmatrix} \, d\mathbf{x} \, ds \\ \leq \int_0^t \left(\left(\frac{1}{2} \|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \frac{\mu}{2} \|\nabla \ln \tilde{u}\|_{L^\infty(\Omega)}^2 \right) \int_{\Omega} \tilde{u} |\bar{w}|^2 \, d\mathbf{x} \right) \, ds \\ + \int_0^t \int_{\Omega} \frac{\mu}{2} \tilde{u} |\nabla \bar{w}|^2 - \delta \tilde{u} w (u - \tilde{u}) \bar{w} \, d\mathbf{x} \, ds. \end{aligned}$$

Absorbing the second to last term from the right into the left-hand side, multiplying by $\frac{2\kappa^2}{\mu\nu}$ and adding (5.15) leaves us with

$$\begin{aligned}
 & \int_{\Omega} u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) + \frac{\kappa^2}{\mu\nu} \tilde{u}|w - \tilde{w}|^2 d\tilde{x} \Big|_0^t \\
 & + \int_0^t \int_{\Omega} (\beta - \alpha\tilde{w})(u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u})) + \frac{2\kappa^2}{\mu\nu} (\delta\tilde{u} - \gamma)\tilde{u}|w - \tilde{w}|^2 d\tilde{x} ds \\
 & + \int_0^t \int_{\Omega} \frac{\kappa^2}{2\nu} \tilde{u}|\nabla w - \nabla \tilde{w}|^2 + \frac{\nu}{2} \tilde{u}|\nabla \ln u - \nabla \ln \tilde{u}|^2 \\
 & + \int_0^t \int_{\Omega} \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \left(\frac{\ln u - \ln \tilde{u}}{\frac{2\kappa^2}{\mu\nu} \tilde{u}(w - \tilde{w})} \right) d\tilde{x} ds \\
 & \leq \int_0^t \int_{\Omega} \alpha(w - \tilde{w})(u - \tilde{u}) + \frac{2\kappa^2}{\mu\nu} \delta\tilde{u}w(\tilde{w} - w)(u - \tilde{u}) d\tilde{x} ds \\
 & + \int_0^t \left(\left(\frac{\kappa^2}{\mu\nu} \|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \frac{\kappa^2}{\nu} \|\nabla \ln \tilde{u}\|_{L^\infty(\Omega)}^2 \right) \int_{\Omega} \tilde{u}|w - \tilde{w}|^2 d\tilde{x} \right) ds.
 \end{aligned} \tag{5.17}$$

Since w and \tilde{w} are non-negative and bounded by $K := \|w\|_{L^\infty(\Omega)} + \|\tilde{w}\|_{L^\infty(\Omega)} + 1$ and $\tilde{u}, u > 0$ almost everywhere, we can apply Lemma 7.7 from the ‘‘Appendix’’, which is an application of the Fenchel–Young inequality, to get the pointwise almost everywhere estimate,

$$(w - \tilde{w})(u - \tilde{u}) \leq \max \left\{ \frac{1}{4K}, 4K \right\} \left(\tilde{u}|w - \tilde{w}|^2 + u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \right).$$

Note that this estimate also holds for exchanged roles of w and \tilde{w} and that we know that $\tilde{u}, w \geq 0$ holds, so that the inequality remains true when multiplied with the product $\tilde{u}w$. Inserting this estimate into (5.17) and introducing the operators defined in Definition 5.3, yields

$$\begin{aligned}
 & \mathcal{R}(u, w|\tilde{u}, \tilde{w}) \Big|_0^t + \int_0^t \left(\mathcal{W}(u, w|\tilde{u}, \tilde{w}) + \int_{\Omega} \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \left(\frac{\ln u - \ln \tilde{u}}{\frac{2\kappa^2}{\mu\nu} \tilde{u}(w - \tilde{w})} \right) \right) dx ds \\
 & \leq \int_0^t \int_{\Omega} \left(\alpha + \frac{2\kappa^2}{\mu\nu} \delta\tilde{u}w \right) \max \left\{ \frac{1}{4K}, 4K \right\} \\
 & \quad \times \left(\tilde{u}|w - \tilde{w}|^2 + u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \right) dx ds \\
 & + \int_0^t \left(\left(\frac{\kappa^2}{\mu\nu} \|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \frac{\kappa^2}{\nu} \|\nabla \ln \tilde{u}\|_{L^\infty(\Omega)}^2 \right) \int_{\Omega} \tilde{u}|w - \tilde{w}|^2 dx \right) ds \\
 & \leq \int_0^t \left(\left(\|\partial_t \ln \tilde{u}\|_{L^\infty(\Omega)} + \mu \|\nabla \ln \tilde{u}\|_{L^\infty(\Omega)}^2 + \max \left\{ \frac{1}{4K}, 4K \right\} \right. \right. \\
 & \quad \times \left. \left. \left(\frac{\alpha\mu\nu}{\kappa^2} + \alpha + 2\delta \|\tilde{u}\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)} + \frac{2\kappa^2\delta}{\mu\nu} \|\tilde{u}\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)} \right) \right) \\
 & \quad \times \int_{\Omega} u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) + \frac{\kappa^2}{\mu\nu} \tilde{u}|w - \tilde{w}|^2 dx \Big) ds.
 \end{aligned}$$

Adding $\max\{\alpha \|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}(u, w|\tilde{u}, \tilde{w})$ on both sides of the inequality we find, with the definition of the regularity weight \mathcal{K} , cf. Definition 5.3,

$$\begin{aligned} & \mathcal{R}(u, w|\tilde{u}, \tilde{w}) \Big|_0^t + \int_0^t \mathcal{W}(u, w|\tilde{u}, \tilde{w}) + \max\{\alpha \|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}(u, w|\tilde{u}, \tilde{w}) \, ds \\ & + \int_0^t \int_\Omega \mathcal{A}(\tilde{u}, \tilde{w}) \cdot \left(\frac{\ln u - \ln \tilde{u}}{\frac{2\kappa^2}{\mu\nu} \tilde{u}(w - \tilde{w})} \right) \, d\vec{x} \, ds \leq \int_0^t \mathcal{K}(\tilde{u}, \tilde{w})\mathcal{R}(u, w|\tilde{u}, \tilde{w}) \, ds. \end{aligned} \tag{5.18}$$

Applying Gronwall’s inequality, see for example Lemma 7.3.1 in [19, p. 180], the relative energy inequality (5.12) follows and our proof of the lemma is finished. \square

Using the relative energy inequality from the previous lemma, makes the proof of the weak–strong uniqueness, cf. Theorem 2.6, quite simple.

Proof of Theorem 2.6. The relative energy inequality (5.12) from Lemma 5.5 holds for the generalized solution $(u, w) \in \mathcal{X}$ and the strong solution (\tilde{u}, \tilde{w}) . We indeed have $\tilde{u}, \tilde{w} \in \mathcal{Y}$, since the initial value u_0 is bounded away from zero and thus also \tilde{u} is bounded away from zero, by Lemma 5.1. Since (\tilde{u}, \tilde{w}) is a strong solution to (1.1) the term including the system operator \mathcal{A} vanishes. By the non-negativity of $\mathcal{W} + \max\{\alpha \|\tilde{w}\|_{L^\infty(\Omega)}, 2\gamma\}\mathcal{R}$, we estimate

$$\mathcal{R}(u, w|\tilde{u}, \tilde{w})(t) \leq \mathcal{R}(u, w|\tilde{u}, \tilde{w})(0) e^{\int_0^t \mathcal{K}(\tilde{u}, \tilde{w}) \, ds} = 0$$

for almost all $t \in (0, T)$ and thus we have $w = \tilde{w}$ and $u = \tilde{u}$ almost everywhere in $(0, T) \times \Omega$ by the definition of the relative energy \mathcal{R} , see (5.8), where we used that $x - y - y(\ln x - \ln y) = 0$ implies $x = y$ for all $x, y > 0$, see Remark 5.4 above. \square

6. Local existence of strong solution

We only show the *a priori* estimates needed for the local-in-time existence of strong solutions for system (1.1) in a formal way, working directly with the equations from (1.1). To make this rigorous one would need to choose an appropriate Galerkin basis of eigenfunctions, which fulfill $\nabla(\Delta\varphi) \cdot n = 0$ on $\partial\Omega$ and perform these estimates on the discrete level. The structure of the proof follows the proof of [43, Thm 2.5].

Remark 6.1. The local existence also follows directly from the abstract existence theory based on semigroup theory, see for example [2, Thm. 14.4]. However, there, only the existence of strong solution for some $t^+ > 0$ follows, while with the approach given here we can construct a lower bound on the time horizon of local existence and we will do so below.

For all functions $\varphi \in W^{2,2}(\Omega)$ fulfilling zero Neumann boundary conditions we can consider the norm

$$\|\varphi\|_{W^{2,2}(\Omega)}^2 := \|\varphi\|_{L^2(\Omega)}^2 + \|\Delta\varphi\|_{L^2(\Omega)}^2,$$

which defines an equivalent norm to the standard $W^{2,2}(\Omega)$ -norm.

Proof of Theorem 2.7. We show this by deriving appropriate *a priori* estimates and using an ODE comparison principle for

$$\xi(t) := \frac{1}{2} \left(2 + \|w(t)\|_{\overline{W}^{2,2}}^2 + \|u(t)\|_{W^{1,2}}^2 \right),$$

where we write L^2 instead of $L^2(\Omega)$ etc. throughout the proof. Note that for $d \leq 3$ we have $W^{1,2} \hookrightarrow L^6$ and $W^{2,2} \hookrightarrow L^\infty$ by the Sobolev embedding theorem. Testing the prey equation (1.1b) with w itself we get

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|w\|_{L^2}^2 + \mu \|\nabla w\|_{L^2}^2 &\leq \left(\gamma + \frac{1}{2} \right) \|w\|_{L^2}^2 + \frac{\delta^2}{2} \|u\|_{L^2}^2 \\ &\leq C \left(\|u\|_{W^{1,2}}^2 + \|w\|_{\overline{W}^{2,2}}^2 \right) \end{aligned} \tag{6.1}$$

and testing (1.1b) with the bi-Laplacian $\Delta^2 w$ and applying Young’s inequality yields

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\Delta w\|_{L^2}^2 + \mu \|\nabla(\Delta w)\|_{L^2}^2 &= \int_{\Omega} \gamma |\Delta w|^2 + \delta u \nabla w \cdot \nabla(\Delta w) + \delta w \nabla u \cdot \nabla(\Delta w) \, dx \\ &\leq \gamma \|\Delta w\|_{L^2}^2 + \delta \|u\|_{L^4} \|\nabla w\|_{L^4} \|\nabla(\Delta w)\|_{L^2} + \delta \|w\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla(\Delta w)\|_{L^2} \\ &\leq \gamma \|\Delta w\|_{L^2}^2 + \frac{\mu}{2} \|\nabla(\Delta w)\|_{L^2}^2 + \frac{\delta^2}{2\mu} \left(\|u\|_{L^4}^4 + \|\nabla w\|_{L^4}^4 + \|w\|_{L^\infty}^4 + \|\nabla u\|_{L^2}^4 \right), \end{aligned}$$

where the boundary terms vanished since $\nabla(\Delta w) \cdot n = 0$ on $\partial\Omega$ holds by our choice of Galerkin basis. Absorbing the term including the third partial derivatives of w from the right into the left-hand side and using the above mentioned Sobolev embeddings, we obtain

$$\frac{d}{dt} \frac{1}{2} \|\Delta w\|_{L^2}^2 + \frac{\mu}{2} \|\nabla(\Delta w)\|_{L^2}^2 \leq C \left(1 + \|u\|_{W^{1,2}}^4 + \|w\|_{\overline{W}^{2,2}}^4 \right). \tag{6.2}$$

Testing the predator equation (1.1a) with u and applying Young’s inequality gives

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 + \beta \|u\|_{L^2}^2 &= \int_{\Omega} \alpha w u^2 + \kappa u \nabla w \cdot \nabla u \, dx \\ &\leq \frac{\alpha}{2} \|w\|_{L^\infty}^2 + \frac{\alpha}{2} \|u\|_{L^2}^4 + \frac{\kappa^2}{2\nu} \|\nabla w\|_{L^4}^2 \|u\|_{L^4}^2 + \frac{\nu}{2} \|\nabla u\|_{L^2}^2. \end{aligned}$$

Absorbing the last term on the right-hand side into the left-hand side and again using the above mentioned Sobolev embeddings, yields

$$\frac{d}{dt} \frac{1}{2} \|u\|_{L^2}^2 \leq C \left(1 + \|u\|_{W^{1,2}}^4 + \|w\|_{\overline{W}^{2,2}}^4 \right). \tag{6.3}$$

Finally testing (1.1a) with $-\Delta u$ and integrating by parts we obtain

$$\frac{d}{dt} \frac{1}{2} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 = \int_{\Omega} \kappa u \Delta w \Delta u + \kappa \Delta u \nabla u \cdot \nabla w + (\beta - \alpha) u \Delta u \, dx$$

$$\begin{aligned}
 &\leq - \int_{\Omega} \kappa \Delta w |\nabla u|^2 + \kappa u \nabla(\Delta w) \cdot \nabla u \, dx - \int_{\Omega} \beta |\nabla u|^2 \, dx \\
 &\quad + \kappa \|\nabla u\|_{L^3} \|\nabla w\|_{L^6} \|\Delta u\|_{L^2} + \alpha \|w\|_{L^\infty} \|u\|_{L^2} \|\Delta u\|_{L^2} \\
 &\leq \kappa \|\Delta w\|_{L^2} \|\nabla u\|_{L^4}^2 + \kappa \|u\|_{L^6} \|\nabla u\|_{L^3} \|\nabla(\Delta w)\|_{L^2} \\
 &\quad + \kappa \|\nabla u\|_{L^3} \|\nabla w\|_{L^6} \|\Delta u\|_{L^2} + \alpha \|w\|_{L^\infty} \|u\|_{L^2} \|\Delta u\|_{L^2}. \tag{6.4}
 \end{aligned}$$

For $u \in W^{2,2}$ fulfilling zero Neumann boundary conditions we can estimate $\|\nabla u\|_{L^6} \leq C \|\Delta u\|_{L^2}$ and we can deduce the following interpolation inequalities

$$\begin{aligned}
 \|\nabla u\|_{L^3} &\leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}, \\
 \|\nabla u\|_{L^4} &\leq C \|\nabla u\|_{L^2}^{\frac{1}{4}} \|\Delta u\|_{L^2}^{\frac{3}{4}}.
 \end{aligned}$$

Using these inequalities and Young’s inequality, we can further estimate (6.4) by

$$\begin{aligned}
 &\frac{d}{dt} \frac{1}{2} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \\
 &\leq C \|\Delta w\|_{L^2} \|\Delta u\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} + C \|u\|_{L^6} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\nabla(\Delta w)\|_{L^2} \\
 &\quad + C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^6} \|\Delta u\|_{L^2}^{\frac{3}{2}} + \alpha \|w\|_{L^\infty} \|u\|_{L^2} \|\Delta u\|_{L^2} \\
 &\leq \frac{\nu}{2} \|\Delta u\|_{L^2}^2 + C \left(\|\Delta w\|_{L^2}^6 + \|\nabla u\|_{L^2}^6 \right) + \frac{\mu}{2} \|\nabla(\Delta w)\|_{L^2}^2 + C \left(\|\nabla u\|_{L^2}^6 + \|u\|_{L^6}^6 \right) \\
 &\quad + \frac{\nu}{2} \|\Delta u\|_{L^2}^2 + C \left(\|\nabla w\|_{L^6}^6 + \|\nabla u\|_{L^2}^6 \right) + \frac{\alpha^2}{2\nu} \left(\|w\|_{L^\infty}^4 + \|u\|_{L^2}^4 \right).
 \end{aligned}$$

Absorbing the Δu terms from the right into the left-hand side, we can deduce

$$\frac{d}{dt} \frac{1}{2} \|\nabla u\|_{L^2}^2 \leq C \left(1 + \|u\|_{W^{1,2}}^6 + \|w\|_{W^{2,2}}^6 \right) + \frac{\mu}{2} \|\nabla(\Delta w)\|_{L^2}^2. \tag{6.5}$$

Adding equations (6.1), (6.2), (6.3) and (6.5) we get

$$\frac{d}{dt} \xi(t) \leq C \xi(t)^3. \tag{6.6}$$

For the initial values given in Theorem 2.7 the initial value $\xi(0)$ is bounded and thus we find a $T^* > 0$ and some $C^* > 0$ such that (6.6) has a solution on $[0, T^*)$ and

$$\|\xi\|_{L^\infty(0, T^*)} \leq C^*$$

holds. The constant $C = C(\Omega, d, \nu, \mu, \alpha, \gamma, \delta)$ from (6.6) is dependent on the domain Ω and the dimension d through the Sobolev embedding constant. By solving the ordinary differential equation $y' = Cy^3$ explicitly, we can give a lower bound on the time T^* in terms of C and the initial value, $T^* = 1/2C\xi(0)^2$. This shows that $u \in L^\infty(0, T^*; W^{1,2})$ and $w \in L^\infty(0, T^*; W^{2,2})$. By the maximal L^p -regularity of the heat equation, see [12, Thm. 8.2] we obtain the required regularity in time, where we proceeded as follows. First we find

$$w \in W^{1,6}(0, T^*; L^6) \cap L^6(0, T^*; W^{2,6}) \hookrightarrow C^1([0, T^*] \times \overline{\Omega}),$$

by the embedding from [37, Lem. 3.3] and maximal L^p -regularity for (1.1b). With this additional regularity for w and maximal L^p -regularity for (1.1a) we find that $u \in W^{1,2}(0, T^*; L^2) \cap L^2(0, T^*; W^{2,2})$. \square

Finally, we infer the additional regularity of strong solutions asserted in Proposition 2.8.

Proof of Proposition 2.8. The proof is performed via a bootstrap argument. We apply the maximal L^p -regularity of the heat equation, see [12, Thm. 8.2], multiple times and use the following embeddings for parabolic Sobolev spaces

$$W^{1,q}(0, T^*; L^q(\Omega)) \cap L^q(0, T^*; W^{2,q}(\Omega)) \hookrightarrow L^p(0, T^*; W^{k,p}(\Omega)) \tag{6.7}$$

for $k = 0, 1$ with $p \geq q$ and $2 - k - (1/q - 1/p)(n + 2) \geq 0$ and

$$W^{1,q}(0, T^*; L^q(\Omega)) \cap L^q(0, T^*; W^{2,q}(\Omega)) \hookrightarrow C^{(k+\alpha)/2, k+\alpha}([0, T^*] \times \overline{\Omega}) \tag{6.8}$$

for $q > (n + 2)/(2 - k)$, $k = 0, 1$ and $0 \leq \alpha < 2 - k - (n + 2)/q$, see [37, Lem. 3.3]. The space $C^{\alpha/2, \alpha}([0, T^*] \times \overline{\Omega})$ is the parabolic Hölder space, see [47, p. 177] for a definition. We make use of the Hölder continuity of the right-hand side to deduce the regularity up to $t = 0$. Roughly speaking, we can say that when the right-hand side and initial condition of the heat equation are Hölder continuous, this continuity transfers to the time derivative and the Laplace of the solution and can be extended to $t = 0$, see Section 5.1.2 in [47]. This will be done more precisely later in the proof. We begin by noting that by the embedding (6.8) we find

$$w \in W^{1,6}(0, T^*; L^6(\Omega)) \cap L^6(0, T^*; W^{2,6}(\Omega)) \hookrightarrow C^{(1+\alpha)/2, 1+\alpha}([0, T^*] \times \overline{\Omega}) \tag{6.9}$$

for all $\alpha \in (0, \frac{1}{6})$. Using the embedding from (6.7) we find

$$\begin{aligned} u &\in W^{1,2}(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; W^{2,2}(\Omega)) \hookrightarrow L^{10}(0, T^*; L^{10}(\Omega)), \\ u &\in W^{1,2}(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; W^{2,2}(\Omega)) \hookrightarrow L^{\frac{10}{3}}(0, T^*; W^{1, \frac{10}{3}}(\Omega)). \end{aligned}$$

Using this additional regularity and considering equation (1.1a)

$$\partial_t u - \nu \Delta u = -\kappa u \Delta w - \kappa \nabla u \cdot \nabla w + (\alpha w - \beta)u, \tag{6.10}$$

we notice that the right-hand side of (6.10) is in $L^{\frac{10}{3}}(0, T^*; L^{\frac{10}{3}}(\Omega))$ and thus by the maximal L^p -regularity of the heat equation and again using embedding (6.7), we find

$$u \in W^{1, \frac{10}{3}}(0, T^*; L^{\frac{10}{3}}(\Omega)) \cap L^{\frac{10}{3}}(0, T^*; W^{2, \frac{10}{3}}(\Omega)) \hookrightarrow L^{10}(0, T^*; W^{1, 10}(\Omega)).$$

An application of (6.8) yields

$$u \in W^{1, \frac{10}{3}}(0, T^*; L^{\frac{10}{3}}(\Omega)) \cap L^{\frac{10}{3}}(0, T^*; W^{2, \frac{10}{3}}(\Omega)) \hookrightarrow C^{\tilde{\alpha}/2, \tilde{\alpha}}([0, T^*] \times \overline{\Omega}),$$

for all $\tilde{\alpha} \in (0, \frac{1}{2})$. Now, we can deduce that the right-hand side of (6.10) is in $L^6(0, T^*; L^6(\Omega))$ and again applying maximal L^p -regularity we obtain

$$u \in W^{1,6}(0, T^*; L^6(\Omega)) \cap L^6(0, T^*; W^{2,6}(\Omega)) \hookrightarrow C^{(1+\alpha)/2, 1+\alpha}([0, T^*] \times \overline{\Omega}) \quad (6.11)$$

for all $\alpha \in (0, \frac{1}{6})$ by embedding (6.8). Now we have that the right-hand side of (1.1b)

$$\partial_t w - \mu \Delta w = (\gamma - \delta u)w$$

is in the parabolic Hölder space $C^{(1+\alpha)/2, 1+\alpha}([0, T^*] \times \overline{\Omega})$ for all $\alpha \in (0, \frac{1}{6})$, since the product of two Hölder continuous functions with Hölder exponents $\lambda_1 > 0$ and $\lambda_2 > 0$ is again Hölder continuous with Hölder exponent $\lambda = \min\{\lambda_1, \lambda_2\}$, see [24, Ch. 4.1]. Additionally, we have that $w_0 \in C^3(\overline{\Omega}) \hookrightarrow C^{2+2\alpha}(\overline{\Omega})$ holds. The embedding holds by the Sobolev embedding theorem since Ω has a smooth enough boundary, see for example [1, Thm. 5.4], and w_0 fulfills the boundary conditions. Thus, by [47, Thm. 5.1.18iii)], we find

$$w_t \in C^{\alpha, 2\alpha}([0, T^*] \times \overline{\Omega}) \quad \text{and} \quad \Delta w \in C^{\alpha, 0}([0, T^*] \times \overline{\Omega}).$$

For the regularity of u we find that the right-hand side of (6.10) is in $C^{\alpha, 0}([0, T^*] \times \overline{\Omega})$ and with the properties of u_0 , namely that $u_0 \in C^3(\overline{\Omega}) \hookrightarrow C^{2+2\alpha}(\overline{\Omega})$ and that it fulfills the zero Neumann boundary conditions, we find, again applying [47, Thm. 5.1.18iii)], that

$$u_t \in C^{\alpha, 2\alpha}([0, T^*] \times \overline{\Omega}) \quad \text{and} \quad \Delta u \in C^{\alpha, 0}([0, T^*] \times \overline{\Omega}).$$

Combining these continuity results of the time derivatives and the Laplacians with the continuity of the first derivatives in space, cf. (6.9) and (6.11), we obtain

$$u, w \in C^1([0, T^*] \times \overline{\Omega}) \quad \text{and} \quad \Delta u, \Delta w \in C([0, T^*] \times \overline{\Omega}).$$

The non-negativity of the initial values u_0 and w_0 can be transferred to the solutions u and w by first applying the comparison principle for strong solutions of the predator equation, cf. Proposition 5.2 and then the comparison principle for the prey w , cf. Lemma 4.3. This completes our proof. □

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Conflict of interest The authors declare that there is no conflict of interest.

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7. Appendix

7.1. A generalized integration by parts and product rule

By the definition of the solution space \mathcal{X} , cf. Sect.2, we know w to be in $L^2(0, T; W^{1,2}(\Omega) \cap L^\infty(\Omega))$ with $\partial_t w \in L^2(0, T; L^1(\Omega) + W^{1,2}(\Omega)^*)$. In this section we prove that the standard integration by parts and product rules are valid in this space.

We define $V_1 = W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $V_2 = L^1(\Omega) + W^{1,2}(\Omega)^*$. Both $L^1(\Omega)$ and $W^{1,2}(\Omega)^*$ are continuous embedded in $W^{1,p}(\Omega)^*$ for $p > d$ and thus the sum of these spaces is well-defined. Equipped with the norms

$$\begin{aligned} \|v_1\|_{V_1} &:= \max \left\{ \|v_1\|_{W^{1,2}(\Omega)} + \|v_1\|_{L^\infty(\Omega)} \right\} \\ \|v_2\|_{V_2} &:= \inf \left\{ \|v_2^1\|_{L^1(\Omega)} + \|v_2^2\|_{W^{1,2}(\Omega)^*} : v_2^1 \in L^1(\Omega), v_2^2 \in W^{1,2}(\Omega)^* \right. \\ &\quad \left. s.t. v_2^1 + v_2^2 = v_2 \right\} \end{aligned}$$

for $v_1 \in V_1$ and $v_2 \in V_2$ these spaces are Banach, see [8, Thm. 1.3]. We will continue to denote the $L^1(\Omega)$ part of $v \in V_2$ by an upper index 1 and the $W^{1,2}(\Omega)^*$ by an upper index 2 throughout this section. Next we define the following space.

Definition 7.1. We define $\mathbb{Y} := \{w \in L^2(V_1) \mid \partial_t w \in L^2(V_2)\}$, where the distributional time derivative is given as usual via

$$-\int_0^T \int_\Omega w(t, \mathbf{x}) v(\mathbf{x}) \, dx \, \phi'(t) \, dt = \int_0^T \phi(t) \langle \partial_t w(t), v \rangle \, dt \tag{7.1}$$

for all $\phi \in C_0^\infty(0, T)$ and all $v \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$.

Remark 7.2. The space \mathbb{Y} equipped with the norm

$$\|w\|_{\mathbb{Y}} := \|w\|_{L^2(V_1)} + \|\partial_t w\|_{L^2(V_2)}$$

is a Banach space.

We now state the generalized integration by parts rule for the space \mathbb{Y} .

Lemma 7.3. *The space \mathbb{Y} from Definition 7.1 is continuously embedded into the space $C([0, T]; L^2(\Omega))$ and for arbitrary $w, v \in \mathbb{Y}$, we have*

$$(w(t), v(t))_{L^2} - (w(s), v(s))_{L^2} = \int_s^t \langle \partial_t w(\tau), v(\tau) \rangle + \langle \partial_t v(\tau), w(\tau) \rangle \, d\tau \quad (7.2)$$

for all $0 \leq s \leq t \leq T$.

Before we turn to the proof, we note the following easy consequence of this lemma.

Corollary 7.4. *For $w \in \mathbb{Y}$ we have $w^+ := \max\{0, w\} \in L^2(V_1)$ and*

$$\frac{d}{dt} \frac{1}{2} \|w^+(t)\|_{L^2(\Omega)}^2 = \langle \partial_t w(t), w^+(t) \rangle \quad (7.3)$$

holds for almost all $t \in (0, T)$.

Proof. This can be shown by first taking $w \in C^1([0, T]; V_1)$. We then have $w^+ \in \mathbb{Y}$ and we can apply the integration by parts rule from Lemma 7.3 to obtain (7.3). For arbitrary $w \in \mathbb{Y}$ we find an approximating sequence in $C^1([0, T]; V_1)$ by Lemma 7.5, see below, and deduce that (7.3) remains true in the limit. \square

In order to prove Lemma 7.3 we follow the standard procedure by moving to a dense subset of smooth functions.

Lemma 7.5. *The space $C^1([0, T]; V_1)$ is dense in \mathbb{Y} .*

Proof. This can be shown by the use of mollifiers as it is done in [53, Lem. 7.2]. \square

The following proof is conducted along the lines of [53, Lem. 7.3]. One cannot apply this Lemma directly, since $V_1 \hookrightarrow L^2(\Omega) \hookrightarrow V_2$ does not define an evolution triple, since $V_1^* \neq V_2$.

Proof of Lemma 7.3. We start by proving the embedding into $C([0, T]; L^2(\Omega))$, using the density from Lemma 7.5 to show that the embedding via the identity $i : C^1([0, T]; V_1) \subseteq \mathbb{Y} \rightarrow C([0, T]; L^2(\Omega))$, can be extended to the whole space \mathbb{Y} . First, we note that for arbitrary $w, v \in C^1([0, T]; V_1)$ we have

$$\frac{d}{dt} (w(t), v(t))_{L^2(\Omega)} = \langle \partial_t w(t), v(t) \rangle + \langle \partial_t v(t), w(t) \rangle \quad (7.4)$$

for all $t \in (0, T)$. Using the mean value theorem, we find $t_1 \in [0, T]$ such that $w(t_1) = \frac{1}{T} \int_0^T w(s) ds$ holds. Using this identity and Young's inequality we get

$$\begin{aligned} \|w(t)\|_{L^2(\Omega)}^2 &= \|w(t_1)\|_{L^2(\Omega)}^2 + \left(\|w(t)\|_{L^2(\Omega)}^2 - \|w(t_1)\|_{L^2(\Omega)}^2 \right) \\ &\leq \frac{1}{T} \int_0^T \|w(s)\|_{L^2(\Omega)}^2 ds + 2 \int_{t_1}^t \langle \partial_t w(s), w(s) \rangle ds \\ &= \frac{1}{T} \int_0^T \|w(s)\|_{L^2(\Omega)}^2 ds + 2 \int_{t_1}^t \langle \partial_t w(s)^1, w(s) \rangle + \langle \partial_t w(s)^2, w(s) \rangle ds \\ &\leq C \left(\|w\|_{L^2(V_1)} + \left\| \left\| \partial_t w^1 \right\|_{L^1(\Omega)} + \left\| \partial_t w^2 \right\|_{W^{1,2}(\mathcal{S})^*} \right\|_{L^2(0,T)} \right)^2 \end{aligned}$$

for some $C > 0$ and all $t \in [0, T]$. Since this inequality holds for all decompositions of $\partial_t w(s)$ into its $L^1(\Omega)$ part $\partial_t w(s)^1$ and its $W^{1,2}(\mathcal{S})^*$ part $\partial_t w(s)^2$, the inequality also holds for the infimum over all these decompositions and we obtain

$$\|w\|_{C([0,T];L^2(\Omega))} \leq C \|w\|_{\mathbb{Y}}.$$

We can extend this densely defined linear operator i to the whole space \mathbb{Y} by the extension principle, see [65, Prop.18.29].

The integration by parts formula for arbitrary $w, v \in \mathbb{Y}$ follows by a density argument. We take approximating sequences $(v_n)_n, (w_n)_n \subseteq C^1([0, T]; V_1)$ such that $v_n \rightarrow v$ and $w_n \rightarrow w$ in \mathbb{Y} holds. We then have that the identity (7.4) holds for w and v replaced by w_n and v_n respectively. Integrating this identity over (s, t) we find that (7.2) holds, again with w and v replaced by w_n and v_n . Finally we can conclude that the equality remains true in the limit. \square

Next we show a generalized product rule.

Lemma 7.6. (Product rule) *For $w \in \mathbb{Y}$ and $v \in W^{1,2}(0, T; V_1)$ the product rule $\partial_t(wv) = v\partial_t w + w\partial_t v$ holds, where*

$$W^{1,2}(0, T; V_1) := \left\{ v \in L^2(0, T; V_1) \mid \partial_t v \in L^2(0, T; V_1) \right\}.$$

Proof. The proof is analogous to the proof of the product rule for weak derivatives. First we show that it holds for $w \in C^1([0, T]; V_1)$ and $v \in C_0^\infty([0, T]) \otimes V_1$. In this case v has enough regularity and can be split up to join the test functions φ and ϕ from the definition of the generalized time derivative, cf. Definition 7.1. By Lemma 5.12 in [55, p. 70], we have the density of $C_0^\infty([0, T]) \otimes V_1$ in $W^{1,2}(0, T; V_1)$. With this density we can deduce the product rule for all $u \in C^1([0, T]; V_1)$ and $v \in W^{1,2}(0, T; V_1)$, where we use the continuous embedding of $W^{1,2}(0, T; V_1)$ into $C([0, T]; V_1)$ to pass to the limit. With the density of $C^1([0, T]; V)$ in \mathbb{Y} , cf. Lemma 7.5, we finally get the product rule as stated in the lemma. \square

7.2. Auxiliary results

We reprove a nice pointwise inequality resulting from the Fenchel–Young inequality. The proof is taken from Lemma 26 in [27].

Lemma 7.7. (An application of the Fenchel–Young inequality) *Let $r > 0$ and $u, \tilde{u}, w, \tilde{w} \in \mathbb{R}$ be non-negative, such that $w, \tilde{w} \in B_r(0)$ and $\tilde{u}, u > 0$ holds. We then have*

$$(w - \tilde{w})(u - \tilde{u}) \leq \max \left\{ \frac{1}{4r}, 4r \right\} \left(\tilde{u} |w - \tilde{w}|^2 + u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \right). \tag{7.5}$$

Proof. First, we define the proper convex function $g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ via

$$g(x) := \begin{cases} x - \ln(x + 1) & \text{for } x \in (-1, \infty), \\ +\infty & \text{otherwise.} \end{cases}$$

The convex conjugate, see Section 2.1.4 in [5, p. 75], is easily computed and given by $g^* : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ with

$$g^*(y) = \begin{cases} -\ln(1 - y) - y & \text{for } y \in (-\infty, 1), \\ +\infty & \text{otherwise.} \end{cases}$$

Choosing $y = \frac{w - \tilde{w}}{4r}$ such that $|y| < \frac{1}{2}$ and $x = \left(\frac{u}{\tilde{u}} - 1\right)$, we find that

$$g(x) = \frac{u}{\tilde{u}} - 1 - \ln\left(\frac{u}{\tilde{u}}\right) = \frac{u}{\tilde{u}} - 1 - (\ln u - \ln \tilde{u}).$$

We estimate $g^*(y)$ by writing it via the Taylor expansion using the integral form of the remainder term,

$$\begin{aligned} g^*(y) &= g^*(0) + (g^*)'(0)y + \int_0^1 (1 - s)(g^*)''(sy) \, ds \, y^2 = \int_0^1 \frac{1 - s}{(1 - sy)^2} \, ds \, y^2 \\ &\leq \int_0^1 \frac{1 - s}{\left(1 - \frac{s}{2}\right)^2} \, ds \, y^2 \leq 2 \int_0^1 (1 - s) \, ds \, y^2 = y^2, \end{aligned}$$

since $|y| \leq \frac{1}{2}$ by our choice of y . An application of the Fenchel–Young inequality, again see Section 2.1.4 in [5], yields

$$\begin{aligned} (w - \tilde{w})(u - \tilde{u}) &= (4r\tilde{u}) \frac{w - \tilde{w}}{4r} \left(\frac{u}{\tilde{u}} - 1\right) = (4r\tilde{u})xy \leq 4r\tilde{u} (g^*(y) + g(x)) \\ &\leq 4r\tilde{u} \left(\frac{|w - \tilde{w}|^2}{16r^2} + \frac{u}{\tilde{u}} - 1 - (\ln u - \ln \tilde{u}) \right) \\ &\leq \max \left\{ \frac{1}{4r}, 4r \right\} \left(\tilde{u} |w - \tilde{w}|^2 + u - \tilde{u} - \tilde{u}(\ln u - \ln \tilde{u}) \right). \end{aligned}$$

This concludes our proof of the inequality (7.5). □

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