Master's thesis

NTNU
Norwegian University of Science and Technology
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A Numerical Method for Fractional Mean Field Games

Analysis and Simulations of a Convergent Numerical Method for Mean Field Games with Symmetric α-stable Lévy Diffusion

Master's thesis in Industrial Mathematics
Supervisor: Espen R. Jakobsen
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PREFACE

This thesis completes my master's degree in industrial mathematics at NTNU. I would first like to thank my supervisor Espen R. Jakobsen for his support and guidance. His excellent grasp of the topic has been invaluable in achieving the goals of this project. I would further like to thank both my family and my enduring girlfriend, the latter of whom who by now must be one of very few child welfare workers who recognize the symbol $(-\Delta)^{\frac{\alpha}{2}}$ with awe. Finally, I would like to thank my fellow $\int$-boys for making these last five years truly (un)rememberable.

$\mathcal{L}[1](s)^{-1} \sum_{n=0}^{\infty} \left(\frac{k-1}{k}\right)^n [10^{-10}\mathrm{m}] \int_0^\infty x^\ell e^{-x} \mathrm{d}x, \qquad k > 0, \ell \in \mathbb{N}.$

Trondheim 2024,
Ulrik Unneberg

ABSTRACT

Mean Field Games describe the limiting behavior of stochastic differential games as the number of players tends to ∞. In this master's thesis, we develop a new convergent numerical scheme for solving fractional Mean Field Games, a coupled forward-backward system of nonlinear integro-differential equations where the diffusion is given by the fractional Laplacian. The method is based on finite differences and powers of the discrete Laplacian. We derive the scheme, prove its convergence, and validate our results with numerical experiments. The thesis concludes with suggestions for potential improvements and directions for further research.

SAMMENDRAG

Mean Field Games beskriver grenseatferden til stokastiske differensialspill når antall spillere tenderer mot ∞. I denne masteroppgaven utvikler vi et nytt konvergent numerisk skjema for å løse fraksjonelle Mean Field Games, et koblet fremover-bakover system av ikke-lineære integro-differensialligninger der diffusjonen er gitt av den fraksjonelle Laplace-operatoren. Metoden er basert på endelige differanser og potenser av den diskrete Laplace-operatoren. Vi utleder skjemaet, beviser dets konvergens, og validerer våre resultater med numeriske eksperimenter. Oppgaven avsluttes med forslag til potensielle forbedringer og retninger for videre forskning.

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ABBREVIATIONS

- **FPK** Fokker-Planck-Kolmogorov
- **HJB** Hamilton-Jacobi-Bellman
- **MFGs** Mean Field Games
- **PDF** Probability Density Function
- **PDL** Powers of the Discrete Laplacian
- **SDE** Stochastic Differential Equation

CHAPTER

ONE

INTRODUCTION

1.1 Background and general problem

Mean Field Games is a formalized theory for stochastic differential games with infinitely many indistinguishable, rational, non-cooperative players, all seeking to minimize their own cost functional. The system is considered so large that no single player influences any other player or the total, but where every player is influenced by how the masses behave. For such large populations, it is unrealistic for a player to collect detailed state information about all the other players, and therefore only use information about the expected mean field, which is the distribution of the other players. This is a realistic scenario in applications ranging from asset pricing to crowd dynamics.

The field was simultaneously developed by both Lasry and Lions [1, 2, 3, 4], and by Huang *et al.* [5] for which the motivation was to study the dynamics of N-player differential games as $N \to \infty$ [6]. They introduced a novel framework that combines methods from game theory and partial differential equations.

Mean Field Games are in general described by a nonlinear PDE system, consisting of a Hamilton-Jacobi-Bellman equation (HJB), and a Fokker-Planck-Kolmogorov equation (FPK), together with initial and terminal conditions. The system takes the form

$$\begin{cases} \partial_t u - \nu\mathcal{L}u + H(x, \mathrm{D}u) = F[m] & \text{in} \quad \Omega \times (0,T) \\ -\partial_t m - \nu\mathcal{L}^* m - \operatorname{div}(m\nabla_p H(x, \mathrm{D}u)) = 0 & \text{in} \quad \Omega \times (0,T) \\ m(\cdot, T) = m_T, \qquad u(\cdot, 0) = G[m(\cdot, 0)] & \text{in} \quad \Omega \\ \int_\Omega m \mathrm{d}x = 1, \qquad m \geq 0 & \text{in} \quad \Omega \times (0,T), \end{cases} \tag{1.1}$$

defined on some spatial domain Ω, with the diffusion parameter $\nu > 0$, and with $\mathcal{L}$ as the generator of the stochastic process describing the motion of an arbitrary agent. We will in this paper study fractional Mean Field Games, where $\mathcal{L}$ is a non-local integro-differential operator, which for $\Omega = \mathbb{R}^d$ is of the form

$$\mathcal{L}\phi(x) = \int_{\mathbb{R}^d} \big(\phi(x+z) - \phi(x) - \mathrm{D}\phi(x) \cdot z\mathbb{1}_{\{|z|<1\}}\big)\mathrm{d}\mu(z). \tag{1.2}$$

Here, D denotes the gradient, and $\mathbb{1}$ is the indicator function. Specifically, we will let $\mathrm{d}\mu(z) = \frac{c_{d,\alpha}\mathrm{d}z}{|z|^{\alpha+d}}$, $\alpha \in (0,2)$ throughout the paper, where $c_{d,\alpha}$ is a constant dependent on the dimension d and α. Such a measure results in the generator being the (negative) fractional Laplacian $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$.

1.2 Main contribution

A brief list of our main contributions is given below.

1.2.1 Main results

1. Derivation of a novel finite difference method for multidimensional fractional Mean Field Games, using the (fractional) powers of the discrete Laplacian (PDL).

2. Numerical analysis: Existence, uniqueness, and convergence to the continuous solution.

3. High-performing algorithms with full code base including validation tests publicly available on GitHub.

1.2.2 Secondary results

1. Existence and uniqueness results of discrete fractional HJB and FPK equations, and validated numerical solvers for these equations.

2. Derivation of the PDL on the torus, and proved several properties including self-adjointness, consistency estimates for both $\mathcal{C}_b^4$ and $\mathcal{C}_b^2$ functions, and degenerate ellipticity.

3. Developed a fast and precise implementation of the discrete fractional Laplacian on the torus, by constructing a PDL matrix. We combined well-known asymptotics of the ratio of gamma functions with a trick involving the Riemann zeta function. The latter effort gave impressing accuracy, particularly in the challenging limiting case $\alpha \to 0$.

The main contribution of this paper is to derive a new numerical method for solving fractional Mean Field Games, using a finite difference approach. To the best of our knowledge, this has not at the time of writing been developed before. We further show that under standard assumptions, the scheme has a unique uniformly bounded solution, which converges to the classical solution of the continuous system. We contribute with novel results within both numerical analysis and computation.

As mentioned, we only consider fractional Mean Field Games with symmetric α-diffusion for $\alpha \in (0, 2)$, which has the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ as its generator. However, it should be fairly straightforward to extend our scheme to other non-local diffusion operators, as long as degenerate ellipticity and self-adjointness of the discretization is preserved. We also confine ourselves to systems with nonlocal coupling, but we demonstrate our solvers tackling local couplings as well.

The general discretization framework is inspired by Achdou et al.'s finite difference approach for second order MFGs [7], and we have in this paper generalized their numerical analysis for local diffusion to also hold in the nonlocal case. In particular, the discretization choice of the Hamiltonian and the divergence term in the FPK equation is made to ensure monotonicity. We chose the powers of discrete Laplacian (PDL) as discretization for the fractional Laplacian, given its second order consistency given some regularity on the function, and since it has a closed form expression in one dimension.

While we followed Achdou et al.'s general approach for proving convergence of our numerical method, every proof required generalization to hold for nonlocal diffusion. Furthermore, they do not prove convergence of m in [7]. They do provide an L^2-convergence result in [8] for a Hamiltonian of a certain form, where they require positivity of the terminal density m_T. In this paper, we prove L^1-convergence of m without any further assumptions on m, given a consistency assumption of the approximation of the divergence term. The proof uses M-matrix theory, Neumann-series expansion with theory of degenerate elliptic schemes, and has to the best of the author's knowledge not been used in this context before.

While working, we realized that many technicalities regarding fractional calculus on the torus were not too available, as also mentioned in [9]. In particular, while the problem itself is defined on the torus, the fractional Laplacian is a nonlocal operator with dependence on the whole space $\mathbb{R}^d$. Therefore, we will switch back and forth between working on the whole space and on the torus, in order to properly define our problem on the torus. The concept of periodic extension will serve as our main tool to bridge the gap between the two domains. In order to discretize the problem, we first have to derive the PDL, together with its explicit kernel in one dimension, on the torus. We also discovered a both fast and accurate way of computing the PDL in one dimension, particularly showcasing its strengths in the limiting case $\alpha \to 0$, with a trick involving the Riemann zeta function. We combine this trick with linearity of PDL, to build a discrete fractional Laplacian matrix for high-performance vectorized computation of the fractional Laplacian on the torus. This implementation can be extended to the whole space, as explained in Appendix A.

Outline of paper

We begin in Chapter 2 by heuristically deriving how the fractional Mean Field Game system (1.1) arises, starting from a continuum of agents all seeking to optimize their own cost functional. We further address the existence and uniqueness of classical solutions of the continuous system. In Chapter 3, we derive the aforementioned theory of the discretization of the fractional Laplacian on the torus, which we will use when solving the MFG system. Here, we prove several properties that we will need later on to prove existence, uniqueness, and convergence. We also provide some simulations, to validate the discretization. Using our results from the previous chapter, we derive in Chapter 4 the discretization of the MFG system. We then treat the HJB and FPK separately, and prove existence and uniqueness of both. We then prove existence, uniqueness and convergence of the full MFG system to classical solutions. In Chapter 5, we derive the algorithms for computing the coupled MFG system, using the general scheme that we developed.

We phrase all algorithms in pseudocode, making it simple to implement in any programming language. The code used by the author, including examples and test cases, is written in Julia and uploaded to his GitHub, see Appendix B. We demonstrate correctness of our implementations using test cases along the way, mostly in Chapter 5. We conclude the project by demonstrating a potential application, namely by modeling the Astroworld crowd crush tragedy in 2019, causing the life of ten individuals.

1.3 Contribution to Sustainability

Mean Field Games can significantly contribute to the United Nations' Sustainable Development Goals (SDGs) by solving complex global challenges. As mentioned among potential applications, MFGs can optimize transportation systems and energy distribution, leading to more sustainable cities (SDG 11). In the energy sector, they can enhance the efficiency and reliability of smart grids, promoting affordable and clean energy (SDG 7). Additionally, as shown in Chapter 5, it can be applied to maintaining safer large-scale events, potentially saving lives. Overall, MFGs offer a powerful tool for advancing sustainable development across various domains.

CHAPTER
TWO

FRACTIONAL MEAN FIELD GAMES

We will in this chapter give the reader an introduction to Mean Field Games, in particular with nonlocal diffusion. After defining some notation we will use throughout the paper, we will begin by heuristically showing how the PDE system (1.1) arises from the stochastic differential game formulation defining the Mean Field Game. We will then constrain our nonlocal diffusion to be symmetric and α-stable, for $\alpha \in (0, 2)$, which has the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ as its infinitesimal generator. The remaining part of the chapter addresses existence and uniqueness of classical solutions of the system.

2.1 Notation

When using the bracket notation around an integer, it should be read as $[k] := \{z \in \mathbb{N}^+ : z \leq k\}$, and for $a, b \in \mathbb{Z}$, $a \leq b$, $[a, b]_{\mathbb{Z}} := \{z \in \mathbb{Z} : a \leq z \leq b\}$. Let I_d be the identity matrix of dimension d. Let $\mathbf{1}$ denote the vector of all ones, and let $\top$ be the transposing operator. Moreover, we will often use C as some generic constant. Letting 2^X denote the power set of a set X, define the closure operator $\mathrm{cl}(\cdot) : 2^X \to 2^X$ as a function taking in a subset of X and returns the subset with its limit points. We will for the standard vector dot product write either $(\cdot, \cdot)_2$ or simply with the $\cdot$ dot.

Derivatives might be given by either ∂_y for a scalar-valued y, or the gradient $\nabla_p := [\partial_{p_1}, \partial_{p_2}, \dots, \partial_{p_d}]^\top$ for a vector-valued argument $p \in \mathbb{R}^d$. D will indicate the gradient with respect to the spatial variable x, $\mathrm{D} := \nabla_x$. D^2 operating on a scalar valued function will hence produce the Hessian matrix.

$|\cdot|$ will most often be used as a vector 2-norm (absolute value), but might also indicate the number of elements in a set. $\|\cdot\|$ will be used both as a function norm and a vector norm, where a subscript will indicate which norm. Examples of norms include the uniform (sup) norm of a function $f : X \to Y$, given by $\|f\|_{\mathcal{C}_b} := \sup\{|f(x)| : x \in X\}$ The L^1 norm on a continuous domain given by $\|f\|_{L^1} := \int_X |f(x)| \mathrm{d}x$. The L^∞-norm is on a continuous domain given by the essential supremum norm $\|f\|_{L^\infty(X)} := \operatorname{ess\,sup}_{x \in X} |f(x)|$. Let $\mathcal{C}_b^k(X) : \{f : X \to \mathbb{R} \mid \sum_{i=0}^k \|\mathrm{D}^k f\|_{\mathcal{C}_b} < \infty\}$, and let $\mathcal{C}_b^{l,m}(X \times Y)$ have l and m bounded derivatives in the first and second argument respectively. The Sobolev space $W^{n,m}(X)$ consist of functions $f : X \to \mathbb{R}$ with n $L^m(X)$-bounded derivatives. Define the d-dimensional torus $\mathbb{T}^d$ as

$$\mathbb{T}^d := \mathbb{S}^d = \mathbb{R}^d/\mathbb{Z}^d, \tag{2.1}$$

where $\mathbb{S}$ is the unit circle. Defining a function on the torus $f : \mathbb{T}^d \to \mathbb{R}$ can equivalently be seen as a function on $\mathbb{R}^d$ which is 1-periodic in every direction. Hence,

$$f(x) = f(\{x\}_d), \quad \forall x \in \mathbb{R}^d,$$

where

$$\{x\}_d = (x_i - \lfloor x_i \rfloor)_{i=1}^d, \quad \forall x \in \mathbb{R}^d,$$

is the element-wise fractional part of a vector. When we refer to the periodic extension $f_{\mathbb{R}^d}$ of a function $f : \mathbb{T}^d \to \mathbb{R}$ defined on the torus, it should be understood as the function $f_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}$ such that $f_{\mathbb{R}^d}(x) = f(\{x\}_d), \forall x \in \mathbb{R}^d$.

Let $Q_T = \mathbb{T}^d \times (0, T)$, and let $*$ denote the convolution operator. Let $B_r(x)$ be an open ball of radius r centered at $x \in \mathbb{R}^d$, and $\mathcal{P}(\mathbb{T}^d)$ be the space of Borel probability measures on $\mathbb{T}^d$, endowed with the Kantorovich-Rubinstein metric

$$d_0(\mu_1, \mu_2) := \sup_{f \in \mathrm{Lip}_{1,1}(\mathbb{T}^d)} \left\{ \int_{\mathbb{T}^d} f(x) d(\mu_1 - \mu_2)(x) \right\}, \tag{2.2}$$

where $\mathrm{Lip}_{1,1}(\mathbb{T}^d) := \{f : f \text{ is Lipschitz continuous on } \mathbb{T}^d \text{ and } \|f\|_{L^\infty}, \|\mathrm{D}f\|_{L^\infty} \leq 1\}$ [10]. For an operator $\mathcal{L}$ of the form (1.2), its adjoint $\mathcal{L}^*$ is defined as

$$\mathcal{L}^*\phi(x) = \int_{\mathbb{R}^d} \big(\phi(x+z) - \phi(x) - \mathrm{D}\phi(x) \cdot z \mathbb{1}_{\{|z|<1\}}\big) \mathrm{d}\mu^*(z), \tag{2.3}$$

with $\mu^*(A) = \mu(-A)$, and $-A := \{-a : a \in A\}$, for all Borel subsets $A \subset \mathbb{R}^d$.

2.2 Derivation of the MFG system

Starting at the game-theoretic formulation, we will here heuristically demonstrate how the MFG system (1.1) arise. Consider a continuum of agents (players) distributed across the torus $\Omega = \mathbb{T}^d$ according to some distribution $m(\cdot, t) \in \mathcal{P}(\mathbb{T}^d)$ at time $t \in [0, T]$. For any agent located at $(x, t) \in \mathbb{T}^d \times [0, T]$, its motion is governed by the stochastic differential equation (SDE)

$$\mathrm{d}X_s = v_s \mathrm{d}t + \sqrt{2\nu}\mathrm{d}L_s, \quad X_t = x, \tag{2.4}$$

where L_s is some Lévy process [10]. The drift term v_s is called the control, also referred to as the action, and is the velocity contribution the agent is in control over. Each agent is rational, and seeks to minimize its cost functional

$$\mathbb{E}\left[\int_t^T \Big(L(X_s, v_s) + F[m(\cdot, s)](X_s)\Big) \mathrm{d}s + G[m(\cdot, T)](X_T)\right], \tag{2.5}$$

with respect to the control. The cost functional consists of two running cost terms, L and F, and one terminal cost G. L is the running cost dependent on the action v_s of the agent. L might be a quadratic in v, and can hence be thought of as a kinetic energy used by the agent. F is the running cost incurred by the agent's interaction with the rest of the population m [11]. The control can freely be chosen from some set of admissible controls, $\mathcal{A}_t$. We define the value function $u(x,t)$ as the optimal cost for an agent located at (x,t), namely

$$u(x,t) = \inf_{v\in\mathcal{A}_t} \mathbb{E}\Bigg[\int_t^T \Big(L(X_s,v_s) + F[m(\cdot,s)](X_s)\Big)\mathrm{d}s + G[m(\cdot,T)](X_T)\Bigg]. \tag{2.6}$$

It can be shown using the Dynamic Programming Principle that under Nash equilibrium, u satisfies the forward Hamilton-Jacobi-Bellman equation,

$$\begin{cases} \partial_t u - \nu\mathcal{L}u + H(x,\mathrm{D}u) = F[m] & \text{in} \quad Q_T \\ u(\cdot,0) = G[m(\cdot,0)] & \text{in} \quad \mathbb{T}^d, \end{cases} \tag{2.7}$$

where $\mathcal{L}$ is the infinitesimal generator of the Lévy process L_t [10].[1] H is called the Hamiltonian, and is the Fenchel conjugate [12] of L, explicitly defined as

$$H(x,p) = \sup_{q\in\mathbb{R}^d}\Big\{p\cdot q - L(x,q)\Big\}. \tag{2.8}$$

The author has given a heuristic derivation of this latter part in the first order case (no diffusion) in his project thesis [13]. Finally, we will describe the time evolution of the density $m(\cdot,t)\in\mathcal{P}(\mathbb{T}^d)$, $t\in[0,T]$. Define $v(x,t)$ as the control field, which is the action of an agent at (x,t). It can be showed that any density with terminal density m_T where each point x moves according to (2.4), with drift term equal to the velocity field $v(x,t)$, is governed by the backward Fokker-Planck-Kolmogorov equations

$$\begin{cases} -\partial_t m - \nu\mathcal{L}^* m + \operatorname{div}(mv(x,t)) = 0 & \text{in} \quad Q_T \\ m(\cdot,T) = m_T, & \text{in} \quad \mathbb{T}^d \\ \int_{\mathbb{T}^d} m\mathrm{d}x = 1, \qquad m\geq 0 & \text{in} \quad Q_T. \end{cases} \tag{2.9}$$

Here, $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ as defined in (2.3). [2] Details of the derivation of the fractional FPK equation can be found in [14]. It can further be shown that the optimal feedback control is given by

$$v(x,t) = -\mathrm{D}_p H(x,\mathrm{D}u). \tag{2.10}$$

Combining these results yields in (1.1). Note that one can easily reverse the time direction, by letting $t\to T-t$, which yields

$$\begin{cases} -\partial_t u - \nu\mathcal{L}u + H(x,\mathrm{D}u) = F[m] & \text{in} \quad Q_T \\ \partial_t m - \nu\mathcal{L}^* m - \operatorname{div}(m\nabla_p H(x,\mathrm{D}u)) = 0 & \text{in} \quad Q_T \\ m(\cdot,0) = m_0, \qquad u(\cdot,T) = G[m(\cdot,T)] & \text{in} \quad \mathbb{T}^d \\ \int_{\mathbb{T}^d} m\mathrm{d}x = 1, \qquad m\geq 0 & \text{in} \quad Q_T. \end{cases}$$

Note that the systems are equivalent, and we will in this paper use the time direction in 1.1.

[1] We will come back to how $\mathcal{L}$ is defined on the torus in the next section.

[2] Assuming for a moment we have an analogue definition of the adjoint on $\mathbb{T}^d$.

2.3 Fractional Laplacian

The diffusion we will consider is isotropic α-stable Lévy processes L_t, with the fractional Laplacian as its generator $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ [15, page 2406]. The fractional Laplacian is an integro-differential operator, which can be derived from a random walk with arbitrarily long jumps [16]. Unlike the Laplacian $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$, which is the generator of Brownian motion, the fractional Laplacian is a nonlocal operator defined through an integral. While Brownian motion is the most widely used model for stochastic processes due to its simplicity, it fails to capture discontinuous jumps, which arise quite often in nature. Examples include fluid dynamics [17], biological systems [18, 19] and financial modelling [20].

Definition 1 (Fractional Laplacian)**.** Let $\phi \in \mathcal{C}^2 \cap \mathcal{C}_b(\mathbb{R}^d)$. Then, the fractional Laplacian can be defined in three equivalent ways,

$$\begin{aligned} -(-\Delta)^{\frac{\alpha}{2}}\phi(x) &:= \int_{\mathbb{R}^d} \big(\phi(x+z) - \phi(x) - \mathrm{D}\phi(x)\cdot z \mathbb{1}_{\{|z|<1\}}\big) \frac{c_{d,\alpha}\mathrm{d}z}{|z|^{\alpha+d}} \\ &= \mathcal{F}^{-1}\big(|\xi|^{\alpha}\mathcal{F}(\phi)(\xi)\big)(x) \\ &= \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^{\infty} \bigg(e^{t\Delta}\phi(x) - \phi(x)\bigg)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}. \end{aligned} \tag{2.11}$$

The first definition is from the general definition of nonlocal diffusion operators, which we saw in (1.2), with the measure [10]

$$\mathrm{d}\mu(z) := \frac{c_{d,\alpha}\mathrm{d}z}{|z|^{\alpha+d}}, \quad \alpha \in (0,2). \tag{2.12}$$

The second definition is the definition given through the Fourier transform $\mathcal{F}$. [16, Lemma 2.1] Finally, we have the semigroup definition[16, Lemma 2.2], which we will refer to most frequently. Here, $e^{t\Delta}\phi(x)$ is the solution of the heat equation with initial condition ϕ, defined by

$$e^{t\Delta}\phi(x) := \int_{\mathbb{R}^d} \phi(x-y)G_c(y,t)\mathrm{d}y = \int_{\mathbb{R}^d} \phi(y)G_c(x-y,t)\mathrm{d}y, \tag{2.13}$$

where

$$G_c(x,t) := \frac{1}{(4\pi t)^{d/2}} \exp\bigg(-\frac{|x|^2}{4t}\bigg) = \mathcal{N}(x; \mathbf{0}, 2tI_d)$$

is a d-variate zero-mean Gaussian probability density function. See Lemma 38 in Appendix A for more details on the heat equation.

Lemma 1 (Self-adjointness of the fractional Laplacian)**.** *The fractional Laplacian is self-adjoint (Hermitian), meaning that* $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ *implies* $\mathcal{L}^* = -(-\Delta)^{\frac{\alpha}{2}}$.

Proof. By the definition of the adjoint (2.3), we immediately see that

$$\mu^*(A) = \mu(-A) = \int_{-A} \mathrm{d}\mu(x) = \int_{-A} \frac{c_{d,\alpha}\mathrm{d}z}{|z|^{\alpha+d}} = \int_{A} \frac{c_{d,\alpha}\mathrm{d}z}{|-z|^{\alpha+d}} = \mu(A).$$

□

Proposition 1. The adjoint satisfies L^2-adjointness, $\int_{\mathbb{R}^d} \big(\mathcal{L}(f)g - f\mathcal{L}^*(g)\mathrm{d}x = 0,\ \forall f, g \in (\mathcal{C}^2 \cap L^2)(\mathbb{R}^d)$. [10]

Before we can discuss existence and uniqueness of fractional Mean Field Games, we need to state some necessary properties of the fractional Laplacian. Most of what follows are also given in [10]. First, we define the heat kernel of an elliptic operator $\mathcal{L}$.

Definition 2 (Fractional Heat Kernel)**.** A heat kernel of an elliptic operator $\mathcal{L}$ is the fundamental solution of $\partial_t u = \mathcal{L}u$, given by $u = \mathcal{F}^{-1}(e^{t\hat{\mathcal{L}}})$. $\mathcal{F}^{-1}$ is the inverse Fourier transform, and $\hat{\mathcal{L}}$ is the Fourier multiplier given by

$$\mathcal{F}(\mathcal{L}u) = \hat{\mathcal{L}}\mathcal{F}(u).$$

The mentioned Fourier multiplier is called the Lévy symbol of the pseudo-differential operator, which here is the fractional Laplacian. It is the exponent of the characteristic function [21, Theorem 3.3.3],

$$\exp(t\hat{\mathcal{L}}(\xi)).$$

Using the Fourier definition of the fractional Laplacian in (2.11),

$$\mathcal{F}((-\Delta)^{\frac{\alpha}{2}}\phi) = |\xi|^\alpha \mathcal{F}(\phi)(\xi), \quad \forall \phi \in (\mathcal{C}^2 \cap \mathcal{C}_b)(\mathbb{R}^d),$$

we infer that the Lévy symbol is $\hat{\mathcal{L}}(\xi) = |\xi|^\alpha$. Hence, we define the fractional Laplacian's heat kernel, denoted as the fractional heat kernel, as

$$K_h(t,x) := \mathcal{F}^{-1}(e^{t|\xi|^\alpha})(x). \tag{2.14}$$

We can use the Lévy-Khintchine theorem (Theorem 1.2.14 in [21]), since

$$|\xi|^\alpha = \int_{\mathbb{R}^d} (e^{iz\cdot\xi} - 1 - i\xi \cdot z \mathbb{1}_{|z|<1})\mathrm{d}\mu(z),$$

and hence the characteristic function $\exp(t|\xi|^\alpha)$ satisfies the conditions of a characteristic function of a probability measure on $\mathbb{R}^d$. [3] Therefore, as also noted in section 4 in [10], K_h is a probability measure for all $t > 0$ and thus

$$\|K_h(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1 \quad K_h \geq 0 \tag{2.15}$$

Next, we state some properties of the fractional Laplacian.

Lemma 2 (Properties of the fractional Laplacian)**.** *The fractional Laplacian* $(-\Delta)^{\frac{\alpha}{2}}$ *with its measure* μ *given by* (2.12) *satisfies the following properties.*

1. $\mu \geq 0$ *is a Radon measure satisfying* $\int_{\mathbb{R}^d} 1 \ \wedge |z|^2 d\mu(z) < +\infty$.

[3] While the theorem requires a positive definite covariance matrix A, a zero covariance matrix is also satisfactory, and explicitly mentioned as an example of the Poisson case in Notes (4), page 29 in [21].

2. There exists $\sigma \in [1,2), c > 0$ such that

$$r^{\sigma} \int_{|z|<1} \frac{|z|^2}{r^2} \wedge 1 \, d\mu(z) \leq c \quad \forall r \in (0,1),$$

and there exists a $C > 0$ such that the fractional heat kernel K_h satisfy

$$\| D^{\beta} K_h(t,\cdot) \|_{L^p(\mathbb{R}^d)} \leq C t^{-\frac{1}{\sigma}(|\beta| + (1-\frac{1}{p})d)}$$

for any $t \in (0,T)$ and any $p \in [1,\infty)$ and $\beta \in (\mathbb{N} \cup \{0\})^d$.

Proof. The proof of *1* follows from computing an integral. Splitting the integral and changing to polar coordinates yields for all $\alpha \in (0,2)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} 1 \ \wedge |z|^2 \mathrm{d}\mu(z) &= c_{d,\alpha} \int_{|z|<1} \frac{|z|^2 \mathrm{d}z}{|z|^{\alpha+d}} + c_{d,\alpha} \int_{|z|\geq 1} \frac{\mathrm{d}z}{|z|^{\alpha+d}} \\
&= c_{d,\alpha} \int_{\mathbb{S}^{d-1}} \int_0^1 r^{2-(d+\alpha)} r^{d-1} \mathrm{d}r \mathrm{d}\theta + c_{d,\alpha} \int_{\mathbb{S}^{d-1}} \int_1^{\infty} r^{-(d+\alpha)} r^{d-1} \mathrm{d}r \mathrm{d}\theta \\
&= c_{d,\alpha} \mathrm{Area}(\mathbb{S}^{d-1}) \int_0^1 r^{1-\alpha} \mathrm{d}r + c_{d,\alpha} \mathrm{Area}(\mathbb{S}^{d-1}) \int_1^{\infty} r^{-(1+\alpha)} \mathrm{d}r < \infty,
\end{aligned}$$

where $\mathrm{Area}(\mathbb{S}^{d-1})$ is the surface area of a d-dimensional sphere. Property *2* is stated in Example 4.4 in [10] using (2.15), and we will not prove it here. □

We complete this section by showing well-posedness of the fractional Laplacian, before defining it on the torus.

Lemma 3 (Well-posedness of fractional Laplacian)**.** *Let $\phi \in (\mathcal{C}_b \cap \mathcal{C}^2)(\mathbb{R}^d)$. Then, the fractional Laplacian is well-defined.*

Proof. Using the first definition of the fractional Laplacian, splitting up the integral, and Taylor expanding ϕ yields

$$\begin{aligned}
|(-\Delta)^{\frac{\alpha}{2}} \phi(x)| = &\Bigg| \int_{|z|<1} \big(\phi(x+z) - \phi(x) - \mathrm{D}\phi(x) \cdot z\big) \mathrm{d}\mu(z) \\
&+ \int_{|z|\geq 1} \big(\phi(x+z) - \phi(x)\big) \mathrm{d}\mu(z) \Bigg| \\
\leq & \frac{1}{2} \|\mathrm{D}^2 \phi\|_{\mathcal{C}_b(B_1(x))} \int_{|z|<1} |z|^2 \mathrm{d}\mu(z) + 2\|\phi\|_{\mathcal{C}_b} \int_{|z|\geq 1} \mathrm{d}\mu(z) < \infty,
\end{aligned}$$

where we used property *1* in Lemma 2 for the last two integrals. The proof is also given in [10]. □

Definition 3 (Fractional Laplacian on the torus)**.** Let $f \in \mathcal{C}^2(\mathbb{T}^d)$, and let $f_{\mathbb{R}^d}$ be its periodic extension. Then we define the fractional Laplacian on the torus simply as

$$(-\Delta)^{\frac{\alpha}{2}} f := (-\Delta)^{\frac{\alpha}{2}} f_{\mathbb{R}^d},$$

without changing any notation. Since $f_{\mathbb{R}^d} \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\forall f \in \mathcal{C}^2(\mathbb{T}^d)$, it is well-defined by Lemma 3.

Lemma 4 (Self-adjointness on the torus)**.** *The fractional Laplacian on the torus satisfies L^2-adjointness,*

$$\int_{\mathbb{T}^d} \big((-\Delta)^{\frac{\alpha}{2}}[f]g - f(-\Delta)^{\frac{\alpha}{2}}[g]\big)\, dx = 0,\ \forall f, g \in \mathcal{C}^2(\mathbb{T}^d).$$

Proof. Let $f, g \in \mathcal{C}^2(\mathbb{T}^d)$, and thus the fractional Laplacian is well-defined. We can therefore use Fubini's theorem [22] and periodic extension of f and g to find that

$$\begin{aligned}
&\int_{\mathbb{T}^d} \big((-\Delta)^{\frac{\alpha}{2}}[f]g - f(-\Delta)^{\frac{\alpha}{2}}[g]\big)\mathrm{d}x \\
&= \frac{1}{|\Gamma(-\frac{\alpha}{2})} \int_{\mathbb{T}^d}\int_0^\infty \big((e^{t\Delta}f)g - f(e^{t\Delta}g)\big)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}\mathrm{d}x \\
&= \frac{1}{|\Gamma(-\frac{\alpha}{2})} \int_{\mathbb{T}^d}\int_0^\infty \left(\int_{\mathbb{R}^d} \big(f(y)G_c(x-y,t)g(x) - g(y)G_c(x-y,t)\big)\mathrm{d}y f(x)\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}\mathrm{d}x \\
&= \frac{1}{|\Gamma(-\frac{\alpha}{2})} \int_0^\infty \left(\sum_{\nu\in\mathbb{Z}^d}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \Big[f(y)g(x) - g(y)f(x)\Big]G_c(x-y+\nu,t)\mathrm{d}x\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\
&= 0,
\end{aligned}$$

since the Gaussian kernel is symmetric about zero. We can write out the inner two integrals as a limit of a double Riemann sum. For any $(x_i, y_j) \in \mathbb{T}^d$ and $\nu_0 \in \mathbb{Z}^d$, the contribution from $f(x_i)g(y_j)G_c(y_j - x_i + \nu_0, t)\Delta x\Delta y$ will cancel with the contribution from $g(y_j)f(x_i)G_c(x_i - y_j - \nu_0, t)\Delta x\Delta y$. $\square$

2.4 Existence and uniqueness of fractional MFGs

While well-posedness of the continuous system is not the main objective of this project, we will in this section state our assumptions, and address existence and uniqueness. As we fix $\mathcal{L} = \mathcal{L}^* = -(-\Delta)^{\frac{\alpha}{2}}$, we redefine the system as

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\frac{\alpha}{2}} u + H(x, \mathrm{D}u) = F[m] & \text{in} \quad Q_T \\ -\partial_t m + \nu(-\Delta)^{\frac{\alpha}{2}} m - \mathrm{div}(m \nabla_p H(x, t, \mathrm{D}u)) = 0 & \text{in} \quad Q_T \\ m(\cdot, T) = m_T, \qquad u(\cdot, 0) = G[m(\cdot, 0)] & \text{in} \quad \mathbb{T}^d \\ \int_{\mathbb{T}^d} m \mathrm{d}x = 1, \qquad m \geq 0 & \text{in} \quad Q_T. \end{cases} \tag{MFG}$$

Definition 4. By a classical solution of (MFG), we mean a pair (u, m) solving (MFG) point-wise such that

1. $u, m \in \mathcal{C}(\mathbb{T}^d \times [0, T])$.
2. $m \in \mathcal{C}([0, T]; \mathcal{P}(\mathbb{T}^d))$
3. $\mathrm{D}u, \mathrm{D}^2 u, (-\Delta)^{\frac{\alpha}{2}} u, u_t, \mathrm{D}m, (-\Delta)^{\frac{\alpha}{2}} m, m_t \in \mathcal{C}(\mathbb{T}^d \times [0, T])$.

In Mean Field Games, one typically classify the coupling $F[m](x) = F(x, m)$ into either being local or nonlocal. Local coupling means $F : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$ is of the form $F[m](x) = f(x, m(x))$, evaluating m locally at x. In contrast, nonlocal (smoothing), coupling $F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is of the form $F[m](x) = f(x, (\phi * m)(x))$, for some smoothing kernel ϕ. Although we will only consider nonlocal couplings in the following, it's important to note that one can (at least with some regularity on m), get arbitrarily close in $\mathcal{C}_b$ to a local evaluation of m by choosing a mollifying kernel. One example is the Gaussian mollifier $\phi_\epsilon = \mathcal{N}(\cdot; \mathbf{0}, \epsilon^{-2} I_d)$, and letting ϵ vanish. In fact, this approximation technique have in the literature been used when extending results for nonlocal couplings to local ones (see e.g. [10, 3]).

To the best of our knowledge, existence and uniqueness of classical solutions of non-stationary fractional Mean Field Games of the form (MFG) defined on the torus has in general not yet been proven for all $\alpha \in (0, 2)$. The authors of [9] have shown existence of classical solutions of fractional Mean Field Games with nonlocal coupling on the torus for $\alpha \in (1, 2)$, and weak solutions for $\alpha \in (0, 1]$, using the vanishing viscosity method. Furthermore, Ersland et al. have in [10] shown existence and uniqueness of fractional Mean Field Games (with both local and nonlocal coupling) defined on the whole space, with more general nonlocal operators (which include the fractional Laplacian). Ersland's paper will be our main resource regarding existence and uniqueness theory of the continuous problem. We will use the same assumptions, and aim to heuristically argue how one might go about proving existence and uniqueness on the torus. We emphasize again that rigorously proving these results is outside the scope of this project, but we require regular classical solutions when proving convergence of our numerical method.

Assumption 1. Let $F, G \in \mathcal{C}(\mathcal{P}(\mathbb{T}^d); \mathcal{C}_b^2(\mathbb{T}^d))$. There exist constants $C_F, C_G > 0$, such that

$$\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \|F[m]\|_{\mathcal{C}_b^2(\mathbb{T}^d)} \leq C_F \quad \text{and} \quad \sup_{m \in \mathcal{P}(\mathbb{T}^d)} \|G[m]\|_{\mathcal{C}_b^2(\mathbb{T}^d)} \leq C_G.$$

Assumption 2. There exists a $C_0 > 0$ such that for all $(x_1, m_1), (x_2, m_2) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$,

$$|F(m_1)[(x_1)] - F[m_2](x_2)| + |G[m_1](x_1) - G[m_2](x_2)| \leq C_0(|x_1 - x_2| + d_0(m_1, m_2))$$

where d_0 is defined in (2.2).

Assumption 3. H is $\mathcal{C}^3(\mathbb{T}^d \times \mathbb{R}^d)$, and for every $R > 0$ there is $C_R > 0$ such that for all $x \in \mathbb{T}^d,\ p \in B_R, \alpha \in \mathbb{N}_0^{2d}$, such that $|\alpha| \leq 3$,

$$|D^\alpha_{x,p} H(x,p)| \leq C_R.$$

Assumption 4. For every $R > 0$ there is $C_R > 0$ such that for any $x, y \in \mathbb{T}^d,\ p \in \mathbb{R}^d$,

$$|H(x,p) - H(y,p)| \leq C_R(|p| + 1)|x - y|.$$

Assumption 5. $m_T \in (W^{2,\infty} \cap \mathcal{P})(\mathbb{T}^d)$.

Assumption 6. F and G satisfy monotonicity conditions:

$$\int_{\mathbb{R}^d} (F[m_1](x,) - F[m_2](x)) \mathrm{d}(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d), \quad \forall t \in [0,T]$$

$$\int_{\mathbb{R}^d} (G[m_1](x) - G[m_2](x)) \mathrm{d}(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d), \quad \forall t \in [0,T]$$

Assumption 7. The Hamiltonian $H = H(x,p)$ is uniformly convex with respect to p:

$$\exists C > 0, \quad \frac{1}{C} I_d \leq \mathrm{D}^2_{pp} H(x,p) \leq C I_d, \quad \forall t \in [0,T].$$

The assumptions we make are analogue to those in [10], but defined on the torus rather than the whole space. Summarized, we let F, G be of nonlocal type, mapping to continuous and bounded functions on the torus, and uniformly bounded independent of $m \in \mathcal{P}(\mathbb{T}^d)$. We further let the Hamiltonian H be $\mathcal{C}^3$ in all arguments. The monotonicity and convexity of F and G are required for uniqueness. Hence, Assumption 1-5 are needed for existence of classical solutions, and Assumption 6-7 are needed for uniqueness.

2.4.1 Fractional HJB and FPK equations

The existence proof of the fractional Mean Field Game system in [10] consists of first tackling the fractional HJB and FPK equations separately, before deriving existence for the coupled system. Uniqueness follows quite trivially once existence is shown. As mentioned, we will not go into the details, but rather paint a picture of how the arguments used in their paper might also hold in our case. That is, when the system is defined on the torus. The strategy is to let the data be periodic, show that existence and uniqueness hold, and finally show that the solution is periodic. Regarding the fractional HJB equation [10, Theorem 5.5 in], the assumptions *(L1-L2, A3-A5)* are satisfied with Lemma 2 and assumptions 1-5, and we can indeed create a periodic right-hand side and a periodic initial condition which satisfies *B1-B4* (boundedness, Lipschitzness and regularity on the initial condition). Therefore, there exists a unique classical solution. To see that the solution is periodic, consider the solution given by the Duhamel formula, equation (17). By Banach fixed point theorem, they proved the solution is the unique fixed point of

$$v(x,t) = K_h(t,\cdot) * v_0(x) - \int_0^t K_h(t-s,\cdot) * (H(\cdot, \mathrm{D}v(\cdot,s)) - f(\cdot,s))(x)\mathrm{d}s, \quad (2.16)$$

where K_h is the fractional heat kernel defined in (2.14). Given periodicity with period P of v_0, H and f, we see that

$$\begin{aligned} v(x+P,t) &= K_h(t,\cdot) * v_0(x+P) - \int_0^t \int_{\mathbb{R}^d} K_h(t-s,y)(H(x+P-y, \mathrm{D}v(x+P-y,s)) \\ &\quad - f(x+P,s))(x)\mathrm{d}y\mathrm{d}s \\ &= K_h(t,\cdot) * v_0(x) - \int_0^t \int_{\mathbb{R}^d} \big(K_h(t-s,y)(H(x-y, \mathrm{D}v(x+P-y,s)) \\ &\quad - f(x,s))(x)\big)\mathrm{d}y\mathrm{d}s \end{aligned}$$

is satisfied only for a periodic v. This is because $(v(x+P,t), \mathrm{D}v(x+P,t))$ solves the same equation (with a unique fixed point) as $(v(x,t), \mathrm{D}v(x,t))$, and this holds for any $x \in \mathbb{R}^d$ and any $t > 0$.

The strategy is not that straight-forward for the fractional FPK equation. The reason is that the solution m requires global L^1 integrability, which is incompatible with periodic non-zero solutions. As we operate on the torus, we only require continuous boundedness and periodicity on $\mathbb{R}^d$. The solution will then have a bounded integral on the torus. The strategy is therefore to relax the L^1 requirements on the initial data. Since we don't require integrability, the tightness arguments can be omitted. In particular, we will in Proposition 6.8 keep the assumptions on b and boundedness of m_0, but omit the L^1 requirement on m_0. The proof is similar to that of HJB, as it also applies a Banach fixed point argument on the Duhamel formula. As the authors point out themselves, they only require the boundedness assumptions to conclude the fixed point argument, and to find boundedness of the derivatives of m. Only after they conclude existence and uniqueness of a classical solution, do they use the L^1-assumtion on the initial data (and on the solution itself) to give integrability results for all $t > 0$, but this is necessary in our setting. To summarize, existence and uniqueness results of the fractional FPK equations on the torus should not require more than removing the L^1 assumption on m_0. The Duhamel formula gives the solution implicit through

$$m(x,t) = K_h(\cdot,t) * m_0(\cdot)(x) - \sum_{i=1}^d \int_0^t \partial_{x_i} K_h(\cdot, t-s) * (b_i m)(\cdot,s)(x)\mathrm{d}s, \quad (2.17)$$

and we can do the exact same argument as for HJB to show periodicity of solutions. Mass conservation follows from fundamental properties of the FPK equation. [14]

2.4.2 Coupled MFG system

To prove existence of classical solutions of MFG systems with nonlocal coupling, Ersland et al. used a Schauder fixed point argument (which we in fact will use later for proving existence of the discrete HJB equation). They begin by creating a convex, closed and compact subset $\mathcal{C}$ of time-dependent probability measures on $\mathbb{R}^d$. Then, they create a mapping $S : \mathcal{C} \to \mathcal{C}$, such that $S(\mu)$ is the solution of the second equation (FPK) in (MFG), given that u is the solution of the first equation (HJB) in (MFG), with $\mu \in \mathcal{C}$ inserted for m in HJB. To clarify,

$$\partial_t u + \nu(-\Delta)^{\frac{\alpha}{2}} u + H(x, \mathrm{D}u) = F[\mu] \quad \text{in} \quad Q_T \tag{2.18}$$

$$u(\cdot, 0) = G[\mu(\cdot, 0)] \quad \text{in} \quad \mathbb{T}^d \tag{2.19}$$

$$-\partial_t m + \nu(-\Delta)^{\frac{\alpha}{2}} m - \operatorname{div}(m\nabla_p H(x, t, \mathrm{D}v)) = 0 \quad \text{in} \quad Q_T \tag{2.20}$$

$$m(\cdot, T) = m_T, \quad \int_{\mathbb{T}^d} m\mathrm{d}x = 1, \qquad m \geq 0 \quad \text{in} \quad Q_T, \tag{2.21}$$

and we define

$$\begin{aligned} S_1(v_0) &:= \{m : \ (2.20) - (2.21) \text{ with } v = v_0\} \\ S_2(\mu_0) &:= \{u : \text{Solves } (2.18) - (2.19) \text{ with } \mu = \mu_0\}, \end{aligned}$$

together with initial and terminal conditions. We then let $S(\mu) = S_1 \circ S_2(\mu)$. They show the mapping is well-defined, before they show continuity, and conclude therefore with existence and uniqueness of a fixed point by Schauder's fixed point theorem. By the results of the HJB and FPK equations, the mappings S_1, S_2 are each well-defined, and returns bounded continuous functions on the torus. In particular, the uniform boundedness assumption (Assumption 1) guarantees that the solution of S_2 and its derivatives are uniformly bounded independent of μ. From there, we suspect there is little additional effort to show that S indeed maps $\mathcal{C}$ to itself. The continuity result also holds on the torus. Thus, if all details work out, existence of classical solutions to the fractional Mean Field Games on the torus is proved. We summarize the analogous theorem from [10] (Theorem 3.4) in the following proposition, which we will assume holds true from here on.

Proposition 2. Let Assumption 1-5 hold. Then, there exists a classical solution (u, m) of (MFG) such that $u \in \mathcal{C}_b^{1,3}(\mathbb{T}^d \times (0, T))$, and $m \in \mathcal{C}_b^{1,2}(\mathbb{T}^d \times (0, T)) \cap \mathcal{C}([0, T]; \mathcal{P}(\mathbb{T}^d))$.

For completeness, we also provide the uniqueness proof, which is essentially the same as in that of Lasry and Lions [3] with inspiration from [10].

Theorem 5. *Let Assumption 1-7 hold. Then there exists at most one solution to the system* (MFG).

Proof. Assume we have two solutions (u_1, m_1), (u_2, m_2) to the MFG system (MFG). Let $\tilde{u} = u_1 - u_2$ and $\tilde{m} = m_1 - m_2$. Subtract the HJB equations for the first and second solution pair (u_1, m_1) and (u_2, m_2), and multiply with $\tilde{m}$, integrate over Q_T, and obtain

$$\begin{aligned} &- \int_{Q_T} \big(F[m_1](x) - F[m_2](x)\big)\tilde{m}\mathrm{d}q + \int_{Q_T} (\partial_t \tilde{u})\tilde{m}\mathrm{d}q + \int_{Q_T} \nu((-\Delta)^{\frac{\alpha}{2}} u)\tilde{m}\mathrm{d}q \\ &+ \int_{Q_T} (H(x, \mathrm{D}u_1) - H(x, \mathrm{D}u_2))\tilde{m}\mathrm{d}q = 0. \end{aligned} \tag{2.22}$$

Integration by parts on the second term, using periodicity and that $\tilde{m}(x,T) = 0$, gives

$$\int_{Q_T} (\partial_t \tilde{u})\tilde{m}\mathrm{d}q = -\int_{\mathbb{T}^d} \big(G[m_1(\cdot,0)](x) - G[m_2(\cdot,0)](x)\big)\big)\tilde{m}(x,0) - \int_{Q_T} (\partial_t \tilde{m})\tilde{u}\mathrm{d}q.$$

Similarly, subtracting the FPK equations for the two solution pairs, multiplying with $\tilde{u}$ and integrating gives

$$0 = -\int_{Q_T} (\partial_t \tilde{m})\tilde{u}\mathrm{d}q - \int_{Q_T} \nu((-\Delta)^{\frac{\alpha}{2}}\tilde{m})\tilde{u}\mathrm{d}q + \int_{Q_T} \Big(\mathrm{D}\tilde{u}\cdot\big(m_1\mathrm{D}_pH(x,\mathrm{D}u_1) - m_2\mathrm{D}_pH(x,\mathrm{D}u_2)\big)\Big)\mathrm{d}q.$$

For the last integral, we performed integration by parts on the divergence term, and used periodicity. Adding the two equations together, and using self-adjointness $\int_{\mathbb{T}^d}(m(-\Delta)^{\frac{\alpha}{2}}u - u(-\Delta)^{\frac{\alpha}{2}}m)\mathrm{d}x = 0$ from Lemma 4, we get

$$\begin{aligned}
0 = &\int_{Q_T} \big(F[m_1](x) - F[m_2](x)\big)\tilde{m}\mathrm{d}q + \int_{Q_T} (G[m_1(\cdot,T)](x) - G[m_2(\cdot,T)](x)))\tilde{m}(x,T)\mathrm{d}q \\
&+ \int_{Q_T} m_1\Big(H(x,\mathrm{D}u_2) - H(x,\mathrm{D}u_1) - \mathrm{D}_pH(x,\mathrm{D}u_1)\cdot\mathrm{D}(u_2 - u_1)\Big)\mathrm{d}q \\
&+ \int_{Q_T} m_2\Big(H(x,\mathrm{D}u_1) - H(x,\mathrm{D}u_2) - \mathrm{D}_pH(x,\mathrm{D}u_2)\cdot\mathrm{D}(u_1 - u_2)\Big)\mathrm{d}q
\end{aligned}$$

Given monotonicity (6) and convexity (7), all terms are non-negative and must therefore be zero. Given the strict convexity in H from (7), we know that $\mathrm{D}u_1 = \mathrm{D}u_2$ on the set $\{m_1 > 0\} \cup \{m_2 > 0\}$. Hence, $m_1 = m_2$ follows from uniqueness of the FPK equation, as the divergence terms are now equal. $u_1 = u_2$ follows from uniqueness of the HJB equation. □

CHAPTER
THREE

DISCRETIZATION OF FRACTIONAL LAPLACIAN

In this chapter, we will discretize the fractional Laplacian, defined in the previous chapter. We will use the (fractional) powers of the discrete Laplacian (PDL), which is a powerful second order discretization of the fractional Laplacian, for sufficiently smooth functions [23]. Furthermore, it can be computed very fast in one dimension, as we here have a closed-form kernel estimate from [24]. As the fractional Laplacian is an integro-differential operator with global dependence, it's mostly treated on the whole space in the literature. Since our system is defined on the torus, we must therefore ensure that all required theory on the fractional Laplacian and its discretization holds on the torus. We begin by providing some notation that we need for the following chapter, before we define and show well-posedness of the PDL in section 3.2. We then show second order consistency in section 3.3. Next, we define and show well-posedness of the PDL on the torus in section 3.4, before we in 3.5 derive a computational trick for obtaining a fast and precise approximation of the PDL on the torus. In section 3.6, we validate the approximation from section 3.6 and the second order consistency from section 3.3 with numerical simulations. Finally, in section 3.7, we show self-adjointness of the PDL on the torus, a key result which we will apply in several of our proofs in Chapter 4.

3.1 Notation

Define the index set

$$\mathcal{I}_h = \{z \in \mathbb{Z} : 0 \leq z_j < N_h\}, \tag{3.1}$$

where h is a step-size such that $N_h = 1/h \in \mathbb{N}$. Let $\mathcal{I}_h^d := \overbrace{\mathcal{I}_h \times \cdots \times \mathcal{I}_h}^{d \text{ times}}$. We further define the equispaced grid $\mathbb{T}_h^d := \{hp : p \in \mathcal{I}_h^d\} \subset \mathbb{T}^d$. A grid function U is a function defined on the grid $U : \mathbb{T}_h^d \to \mathbb{R}$. For any $h > 0$, a grid function can equivalently be defined as a vector $U \in \mathbb{R}^{|\mathcal{I}_h^d|}$ with

$$U(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) = U_{i_1,i_2,\ldots,i_d},$$

where $(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) = h(i_1, i_2, \ldots, i_d) \in \mathbb{T}_h^d$, and $p = (i_1, i_2, \ldots, i_d) \in \mathcal{I}_h^d$. We might also for convenience use the same subscript notation $u_{i_1,i_2,\ldots,i_d} = u(x_{i_1}, x_{i_2}, \ldots, x_{i_d})$ for functions u defined on the continuous space $\mathbb{T}^d$. With some abuse of notation, we might let vector operators act on functions u, and it should be interpreted as acting on the vector $(u_p)_{p\in\mathcal{I}_h^d}$. We define the periodic extension similarly as we did for the continuous space, but we will omit the $\mathbb{R}^d$-subscript to avoid clutter. Define the L^∞-norm of a grid function $\|f\|_{L^\infty(\mathbb{T}_h^d)} := \sup_{x\in X}|f(x)|$. For a grid function defined on $\mathbb{T}_h^d$, we might use both $\|\cdot\|_{L^\infty(\mathbb{T}_h^d)}$ and the vector infinity-norm $\|\cdot\|_\infty$ interchangeably. We simply extend the domain by defining

$$U_k = U_{k \ \otimes\mathrm{mod} N_h}, \quad \forall k \in \mathbb{Z}^d,$$

where $\otimes$mod is the element-wise modulo operator. Writing U still only refers to the vector in $\mathbb{R}^{|\mathcal{I}_h^d|}$. Define the discrete Laplacian Δ_h in d dimensions as

$$\Delta_h\phi(x) = \frac{1}{h^2}\sum_{i=1}^{d}\left[\phi(x+e_ih)+\phi(x-e_ih)-2\phi(x)\right], \tag{3.2}$$

where $e_i = (\mathbb{1}_{\{j=i\}})_{j=1}^d$.

3.2 Powers of discrete Laplacian

When discretizing the fractional Laplacian, speed is an important factor when determining the discretization framework. We have chosen to use the (fractional) powers of discrete Laplacian (PDL), which for $\mathcal{C}_b^4$-functions are second order consistent in h, and has an explicit formula in one dimension. Since our problem is defined on the torus while the fractional Laplacian has global dependence, we will switch back and forth between working on the torus and the whole space. We will begin by defining the PDL on $\mathbb{R}^d$, before we derive its definition on $\mathbb{T}^d$.

Consider the semigroup definition of the fractional Laplacian (2.11),

$$-(-\Delta)^{\frac{\alpha}{2}}\phi(x) = \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty \left(e^{t\Delta}\phi(x)-\phi(x)\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}, \quad \forall \phi \in (\mathcal{C}^2\cap\mathcal{C}_b)(\mathbb{R}^d).$$

The essential idea is to replace the Laplacian Δ with the discrete Laplacian Δ_h from (3.2).

Definition 5 (Powers of Discrete Laplacian (PDL))**.** Let $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. Then, we define the Powers of Discrete Laplacian operator as

$$-(-\Delta_h)^{\frac{\alpha}{2}}\phi(x) := \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty \left(e^{t\Delta_h}\phi(x)-\phi(x)\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}. \tag{3.3}$$

Remark. We here define the PDL for functions defined on the continuous space, to properly study consistency in the next section. Continuity is not necessary for the PDL to be well-defined, but since it is required for the fractional Laplacian, we assume it for convenience. In general, we only require boundedness on the grid $h\mathbb{Z}^d := \{hz : z \in \mathbb{Z}^d\}$.

Analogous to the heat kernel defined in (2.13), we define the semi-discrete heat kernel $e^{t\Delta_h}\phi(x)$ as the solution of the semi-discrete heat equation,

$$e^{t\Delta_h}\phi(x) = \sum_{\beta\in\mathbb{Z}^d} \phi(x-h\beta)G_d(\beta, \frac{t}{h^2}). \tag{3.4}$$

We here perform a discrete convolution with the Bessel kernel

$$G_d(\beta,t) := e^{-2td}\prod_{i=1}^{d} I_{|\beta_i|}(2t) \geq 0 \quad \forall\beta\in\mathbb{Z}^d, \forall t\geq 0, \tag{3.5}$$

with

$$\sum_{\beta\in\mathbb{Z}^d} G_d(\beta,t) = 1. \tag{3.6}$$

See Lemma 39 in Appendix A for more details. To avoid computing an integral for each $x\in\mathbb{R}^d$, we define the following series representation of the PDL.

Lemma 6 (Series representation of PDL)**.** *Let $\phi\in\mathcal{C}_b(\mathbb{R}^d)$, $h>0$, $\alpha\in(0,2)$, and $d\geq 1$. Then, the PDL is equivalently defined as*

$$-(-\Delta_h)^{\frac{\alpha}{2}}\phi(x) = \frac{1}{h^\alpha}\sum_{\beta\in\mathbb{Z}^d}\big(\phi(x+h\beta)-\phi(x)\big)\overline{K}_\alpha(\beta), \tag{3.7}$$

where

$$\overline{K}_\alpha(\beta) = \begin{cases} \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty G_d(\beta,t)\frac{dt}{t^{1+\frac{\alpha}{2}}}, & \beta\neq 0 \\ 0, & \beta = 0. \end{cases} \tag{3.8}$$

Before we prove Lemma 6, we need two more lemmas.

Lemma 7 (Boundedness of kernel sum)**.** *Let $\overline{K}_\alpha(\beta)$ be defined as in Lemma 6. Then,*

$$\sum_{\beta\in\mathbb{Z}^d}\overline{K}_\alpha(\beta) < +\infty.$$

Proof. The proof is the same as in the proof of Lemma 4.22 in [25], and we give it here for completeness. Recall that $\overline{K}_\alpha(0)=0$. We define

$$C_2 := \sum_{\beta\neq 0}\int_1^\infty \frac{G_d(\beta,t)\,dt}{t^{1+\frac{\alpha}{2}}} = \int_1^\infty\sum_{\beta\neq 0}\frac{G_d(\beta,t)\,dt}{t^{1+\frac{\alpha}{2}}} \leq \int_1^\infty \frac{dt}{t^{1+\frac{\alpha}{2}}} = \frac{2}{\alpha}$$

$$C_1 := \sum_{\beta\neq 0}\int_0^1 \frac{G_d(\beta,t)\,dt}{t^{1+\frac{\alpha}{2}}} = \int_0^1\sum_{\beta\neq 0}\frac{G_d(\beta,t)\,dt}{t^{1+\frac{\alpha}{2}}} = \int_0^1 \frac{(1-G_d(0,t))\,dt}{t^{1+\frac{\alpha}{2}}}.$$

G_d is t-differentiable for $t\in[0,1]$ since G_d is given by a product of an exponential with Bessel functions, which are differentiable, as they solve the Bessel differential equations [26]. Let then $C=\max_{\xi\in[0,t]}\{\partial_t G(0,t)\}$, and since $G(0,0)=1$ from (A.12),

$$C_1 \leq \int_0^1 |1-G(0,t)|\frac{\mathrm{d}t}{t^{1+\alpha}} = \int_0^1 |G(0,0)-G(0,t)|\frac{\mathrm{d}t}{t^{1+\alpha}} \leq \int_0^1 C\frac{\mathrm{d}t}{t^\alpha} = C\frac{2}{2-\alpha}.$$

We conclude that,

$$\sum_{\beta\in\mathbb{Z}^d} \overline{K}_\alpha(\beta) = \frac{1}{|\Gamma(-\frac{\alpha}{2})|}(C_1 + C_2) < +\infty.$$

□

Lemma 8 (A Fubini Lemma)**.** *Let* $\phi \in \mathcal{C}_b(\mathbb{R}^d)$*. Then*

$$\int_0^\infty \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}} (\phi(x-h\beta)-\phi(x))\, G_d\left(\beta, \frac{t}{h^2}\right)\frac{dt}{t^{1+\frac{\alpha}{2}}}$$
$$= \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}} \int_0^\infty (\phi(x-h\beta)-\phi(x))\, G_d\left(\beta, \frac{t}{h^2}\right)\frac{dt}{t^{1+\frac{\alpha}{2}}}.$$

Proof. We will use the Monotone Convergence Theorem (MCT) and Dominated Convergence Theorem (DCT) to switch order of summation and integration. First, define

$$f_k(t) := \sum_{\substack{\beta\in\mathbb{Z}^d\setminus\{0\}:\\ |\beta|_\infty\le k}} (\phi(x-h\beta)-\phi(x))\, G_d\left(\beta, \frac{t}{h^2}\right)\frac{1}{t^{1+\frac{\alpha}{2}}}, \quad f(t) := \lim_{k\to\infty} f_k(t),$$

and

$$g_k(t) := \sum_{\substack{\beta\in\mathbb{Z}^d\setminus\{0\}:\\ |\beta|_\infty\le k}} 2\|\phi\|_{C_b} G_d\left(\beta, \frac{t}{h^2}\right)\frac{1}{t^{1+\frac{\alpha}{2}}}, \quad g(t) := \lim_{k\to\infty} g_k(t).$$

We will first use the MCT show that $g(t)$ is well-defined for almost every $t \ge 0$, and from there use the DCT to show that

$$\lim_{k\to\infty}\int_0^\infty f_k(t)\mathrm{d}t = \int_0^\infty f(t)\mathrm{d}t$$

Since $G_d \ge 0$ and $t \ge 0$, we have that g_k is a non-decreasing sequence. We further find that

$$\begin{aligned}
\lim_{k\to\infty}\int_0^\infty g_k(t)\mathrm{d}t &= \lim_{k\to\infty}\int_0^\infty \sum_{\substack{\beta\in\mathbb{Z}^d\setminus\{0\}:\\ |\beta|_\infty\le k}} 2\|\phi\|_{C_b} G_d\left(\beta, \frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\
&= \lim_{k\to\infty}\sum_{\substack{\beta\in\mathbb{Z}^d\setminus\{0\}:\\ |\beta|_\infty\le k}} \int_0^\infty 2\|\phi\|_{C_b} G_d\left(\beta, \frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\
&= \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}} \int_0^\infty 2\|\phi\|_{C_b} G_d\left(\beta, \frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\
&= 2\|\phi\|_{C_b}\sum_{\beta\in\mathbb{Z}^d\setminus\{0\}} \int_0^\infty G_d\left(\beta, \frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} < +\infty
\end{aligned}$$

by Lemma 7. Thus, by MCT (Lemma 47), it follows that $g(t)$ is well-defined for almost every t and, since $g(t) = |g(t)|,\ \forall t \in \mathbb{R}_+$,

$$\int_0^\infty |g(t)|\mathrm{d}t < +\infty. \tag{3.9}$$

Notice further that

$$\begin{aligned}|f_k(t)| &= \left| \sum_{\substack{\beta\in\mathbb{Z}^d\setminus\{0\}:\\ |\beta|_\infty\le k}} (\phi(x-h\beta)-\phi(x))\, G_d\left(\beta,\frac{t}{h^2}\right)\frac{1}{t^{1+\frac{\alpha}{2}}}\right| \\ &= \sum_{\substack{\beta\in\mathbb{Z}^d\setminus\{0\}:\\ |\beta|_\infty\le k}} \Big|\phi(x-h\beta)-\phi(x)\Big| G_d\left(\beta,\frac{t}{h^2}\right)\frac{1}{t^{1+\frac{\alpha}{2}}} \\ &\le g_k(t)\le g(t),\end{aligned}$$

for all t and k. Combining this with (3.9), we can use DCT (Lemma 48) to find that

$$\lim_{k\to\infty}\int_0^\infty f_k(t)\mathrm{d}t = \int_0^\infty f(t)\mathrm{d}t,$$

or equivalently,

$$\begin{aligned}&\int_0^\infty \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}} (\phi(x-h\beta)-\phi(x))\, G_d\left(\beta,\frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\ &= \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}}\int_0^\infty (\phi(x-h\beta)-\phi(x))\, G_d\left(\beta,\frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}.\end{aligned}$$

□

We are finally ready to prove Lemma 6.

Proof of Lemma 6. By the definition of the PDL (3.3), our Fubini lemma (Lemma 8), and (3.6), we get

$$\begin{aligned}-(-\Delta_h)^{\frac{\alpha}{2}}\phi(x) &= \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty\left(\sum_{\beta\in\mathbb{Z}^d}\phi(x-h\beta)G_d\left(\beta,\frac{t}{h^2}\right)-\phi(x)\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\ &= \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}}(\phi(x-h\beta)-\phi(x))\, G_d\left(\beta,\frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\ &= \sum_{\beta\in\mathbb{Z}^d\setminus\{0\}}(\phi(x-h\beta)-\phi(x))\,\frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty G_d\left(\beta,\frac{t}{h^2}\right)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\ &= \frac{1}{h^\alpha}\sum_{\beta\in\mathbb{Z}^d}(\phi(x-h\beta)-\phi(x))\,\overline{K}_\alpha(\beta),\end{aligned}$$

where $\overline{K}_\alpha(\beta)$ is defined in the lemma. □

Lemma 9 (Well-posedness of PDL)**.** *Let* $\phi\in\mathcal{C}_b(\mathbb{R}^d)$*. Then,*

$$|(-\Delta_h)^{\frac{\alpha}{2}}\phi(x)| < +\infty, \quad \forall x\in\mathbb{R}^d, \quad \forall\alpha\in(0,2).$$

Proof. The proof follows easily from our derived results, as we observe that

$$|(-\Delta_h)^{\frac{\alpha}{2}}\phi(x)| \le \frac{1}{h^\alpha}2\|\phi\|_{\mathcal{C}_b}\sum_{\beta\in\mathbb{Z}^d}\overline{K}_\alpha(\beta) < +\infty,$$

by Lemma 7. □

Lemma 10 (PDL is well-defined for bounded grid functions). *Let $\phi \in L^\infty(h\mathbb{Z}^d)$, $h > 0$, $\alpha \in (0,2)$, $d \geq 0$. The PDL is well-defined and given by*

$$-(-\Delta_h)^{\frac{\alpha}{2}}\phi_j = \frac{1}{h^\alpha}\sum_{\beta\in\mathbb{Z}^d}\big(\phi_{j+\beta} - \phi_j\big)\overline{K}_\alpha(\beta), \quad \forall j \in \mathbb{Z}^d.$$

Proof. The proof is identical to that of Lemma 9 with changed norm. □

Lemma 11 (One-dimensional PDL kernel on the whole space). *In one dimension, we have that*

$$\overline{K}_\alpha(m) = \begin{cases} \frac{2^\alpha\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}|\Gamma(-\frac{\alpha}{2})|}\frac{\Gamma(|m|-\frac{\alpha}{2})}{\Gamma(|m|+1+\frac{\alpha}{2})}, & m \neq 0 \\ 0, & m = 0. \end{cases} \tag{3.10}$$

Proof. Using (A.14) with $c = 2$, $k = |m|$ (since $G(m,t) = G(-m,t), m \in \mathbb{Z}$, which follows from (A.11)), and $\gamma = -\frac{\alpha}{2}$, we get (3.10). □

3.3 Consistency of PDL

In order to demonstrate convergence of the MFG system, we need a consistency result for our discretization of the fractional Laplacian.

Lemma 12 (Second order consistency of PDL for $\mathcal{C}_b^4$-functions). *Let $\psi \in \mathcal{C}_b^4(\mathbb{R}^d)$. For any $\alpha \in (0,2)$ and $d \geq 1$, the PDL (3.7) is an approximation of $-(-\Delta)^{\frac{\alpha}{2}}$ with local truncation error*

$$\big\|(-\Delta_h)^{\frac{\alpha}{2}}\psi - (-\Delta)^{\frac{\alpha}{2}}\psi\big\|_{L^\infty(\mathbb{R}^d)} = C(\max_{i\in[d]}\|\partial_{x_i}^4\psi\|_{L^\infty} + \|\psi\|_{L^\infty})h^2 \tag{3.11}$$

Proof. The theorem is essentially the same as Theorem 4.22 in [25], where they prove the second order consistency in $L^p(\mathbb{R}^d), p \in \{1,\infty\}$, and where $\psi \in \mathcal{C}_c^\infty$. As we later will operate on the torus, we need to relax the requirement of compact support on ψ, but restrict ourselves to the $L^\infty(\mathbb{R}^d)$ norm. Since $\psi \in \mathcal{C}_b^2$, the fractional Laplacian and the PDL is well-defined. Using the semigroup definition in (2.11) and (3.3), we have that

$$(-\Delta_h)^{\frac{\alpha}{2}}\psi(x) - (-\Delta)^{\frac{\alpha}{2}}\psi(x) = \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty \big(e^{t\Delta_h}\psi(x) - e^{t\Delta}\psi(x)\big)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}.$$

Define now

$$\tau(x,t) := \partial_t e^{t\Delta}\psi(x) + \Delta_h e^{t\Delta}\psi(x),$$

where we recall $e^{t\Delta}\psi(x)$ from (2.13). By the heat equation in Lemma 38,

$$\partial_t e^{t\Delta}\psi(x) = \Delta e^{t\Delta}\psi(x) = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}e^{t\Delta}\psi(x).$$

Recall the discrete Laplacian,

$$\Delta_h e^{t\Delta}\psi(x) = \frac{1}{h^2}\sum_{i=1}^d\left[e^{t\Delta}\psi(x)(x+e_ih) + e^{t\Delta}\psi(x)(x-e_ih) - 2e^{t\Delta}\psi(x)\right].$$

Since $e^{t\Delta}\psi(x)$ is a smooth function, Taylor-expanding $e^{t\Delta}\psi(x)$ in any direction $e_i = (\delta_{i,j})_{j=1}^d$, gives[1]

$$\begin{aligned} &e^{t\Delta}\psi(x)(x+e_ih) \\ &= e^{t\Delta}\psi(x) + h\partial_{x_i}e^{t\Delta}\psi(x) + \frac{h^2}{2}\partial_{x_i}^2 e^{t\Delta}\psi(x) + \frac{h^3}{6}\partial_{x_i}^3 e^{t\Delta}\psi(x) + \frac{h^4}{24}\partial_{x_i}^4 e^{t\Delta}\psi(\xi_i^+) \end{aligned}$$

for a $\xi_i^+ \in (x, x+e_ih)$. Similarly,

$$\begin{aligned} &e^{t\Delta}\psi(x)(x-e_ih) \\ &= e^{t\Delta}\psi(x) - h\partial_{x_i}e^{t\Delta}\psi(x) + \frac{h^2}{2}\partial_{x_i}^2 e^{t\Delta}\psi(x) - \frac{h^3}{6}\partial_{x_i}^3 e^{t\Delta}\psi(x) + \frac{h^4}{24}\partial_{x_i}^4 e^{t\Delta}\psi(\xi_i^-) \end{aligned}$$

for a $\xi_i^- \in (x, x-e_ih)$. Adding the two equations gives

$$e^{t\Delta}\psi(x)(x+e_ih) + e^{t\Delta}\psi(x)(x-e_ih) = 2e^{t\Delta}\psi(x) + h^2\partial_{x_i}^2 e^{t\Delta}\psi(x) + \frac{h^4}{12}\partial_{x_i}^4 e^{t\Delta}\psi(\xi_i)$$

for a $\xi_i \in (x-e_ih, x+e_ih)$, by the mean value theorem. Therefore,

$$\begin{aligned} \tau(x,t) &= \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} e^{t\Delta}\psi(x) - \frac{1}{h^2}\sum_{i=1}^d \left[e^{t\Delta}\psi(x)(x+e_ih) + e^{t\Delta}\psi(x)(x-e_ih) - 2e^{t\Delta}\psi(x)\right] \\ &= -\frac{h^2}{12}\sum_{i=1}^d \partial_{x_i}^4 e^{t\Delta}\psi(\xi_i) \end{aligned}$$

making

$$\|\tau(\cdot,t)\|_{L^\infty} \leq Ch^2 \max_{i\in[d]}\|\partial_{x_i}^4 e^{t\Delta}\psi\|_{L^\infty},$$

for a $C > 0$. Now, since

$$\partial_{x_i}^4 e^{t\Delta}\psi(x) = \int_{\mathbb{R}^d}(\partial_{x_i}^4\psi)(x-y)G_c(y,t)\mathrm{d}y = \int_{\mathbb{R}^d}\psi(y)(\partial_{x_i}^4 G_c)(x-y,t)\mathrm{d}y, \quad (3.12)$$

we have two equivalent expressions of $\partial_{x_i}^4 e^{t\Delta}\psi(x)$. Using the former when $t \leq 1$, we compute

$$\begin{aligned} \max_{i\in[d]}\|\partial_{x_i}^4 e^{t\Delta}\psi\|_{L^\infty} &= \max_{i\in[d]}\|\int_{\mathbb{R}^d}(\partial_{x_i}^4\psi)(x-y)G_c(y,t)\mathrm{d}y\|_{L^\infty} \\ &\leq \max_{i\in[d]}\|\int_{\mathbb{R}^d}|(\partial_{x_i}^4\psi)(x-y)||G_c(y,t)|\mathrm{d}y\|_{L^\infty} \\ &\leq \max_{i\in[d]}\|\partial_{x_i}^4\psi\|_{L^\infty}\int_{\mathbb{R}^d}|G_c(y,t)|\mathrm{d}y = \max_{i\in[d]}\|\partial_{x_i}^4\psi\|_{L^\infty}. \end{aligned}$$

[1] $\delta_{i,j} := \mathbb{1}_{i=j}$ is the Kronecker delta.

Using the latter expression when $t > 1$, we find

$$\begin{aligned}
&\max_{i\in[d]}\|\partial_{x_i}^4 e^{t\Delta}\psi\|_{L^\infty}\\
&= \max_{i\in[d]}\|\Big(\int_{\mathbb{R}^d}\psi(y)(\mathrm{D}_s^4 G_c)(x-y,t)\mathrm{d}y\Big)(x)\|_{L^\infty}\\
&\leq \max_{i\in[d]}\|\Big(\int_{\mathbb{R}^d}|\psi(y)||(\partial_{x_i}^4 G_c)(x-y,t)|\mathrm{d}y\Big)(x)\|_{L^\infty}\\
&\leq \|\psi\|_{L^\infty}\max_{i\in[d]}\int_{\mathbb{R}^d}|(\partial_{x_i}^4 G_c)(x-y,t)|\mathrm{d}y\\
&= \|\psi\|_{L^\infty}\max_{i\in[d]}\int_{\mathbb{R}^d}|(\partial_{x_i}^4 G_c)(x,t)|\mathrm{d}x \qquad \text{(changing variables } x-y\to x)\\
&\leq \|\psi\|_{L^\infty}\max_{i\in[d]}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\left|\frac{(x_i^4-6x_i^2(2t)+3(2t)^2)}{\sqrt{2\pi}(2t)^{\frac{9}{2}}}e^{-\frac{x_i^2}{4t}}\right|\mathrm{d}x_i\prod_{j\neq i}\frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{x_j^2}{4t}\Big)\mathrm{d}x_j\\
&= \|\psi\|_{L^\infty}\max_{i\in[d]}\int_{\mathbb{R}}\left|\frac{(x_i^4-6x_i^2(2t)+3(2t)^2)}{\sqrt{2\pi}(2t)^{\frac{9}{2}}}e^{-\frac{x_i^2}{4t}}\right|\mathrm{d}x_i\prod_{j\neq i}\underbrace{\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{x_j^2}{4t}\Big)\mathrm{d}x_j}_{=1}\\
&\leq \|\psi\|_{L^\infty}\int_{\mathbb{R}}\frac{(x^4+3(2t)^2)}{\sqrt{2\pi}(2t)^{\frac{9}{2}}}e^{-\frac{x^2}{4t}}\mathrm{d}x\\
&= \|\psi\|_{L^\infty}\frac{1}{\sqrt{2\pi}(2t)^{\frac{9}{2}}}\Big(3\sqrt{2\pi}(2t)^{\frac{5}{2}}+3(2t)^2\sqrt{2\pi}(2t)^{\frac{1}{2}}\Big)\\
&= \|\psi\|_{L^\infty}\frac{C_2}{t^2}.
\end{aligned}$$

Hence,

$$\|\tau(\cdot,t)\|_{L^\infty}\leq\begin{cases} C_1\max_{i\in[d]}\|\partial_{x_i}^4\psi\|_{L^\infty}h^2 & \text{when } 0\leq t\leq 1\\ \frac{C_2\|\psi\|_{L^\infty}h^2}{t^2} & \text{when } t>1.\end{cases}$$

Defining now the quantity

$$E(x,t) := e^{t\Delta_h}\psi(x) - e^{t\Delta}\psi(x),$$

we observe that $\partial_t E(x,t) = \Delta_h E(x,t) - \tau(x,t)$, and $E(x,0) = 0$. This is a perturbed semi-discrete heat equation (39), and we propose the following solution, which follows from Duhamel's principle [27]

$$E(x,t) = -\int_0^t e^{(t-s)\Delta_h}\tau(x,s)\mathrm{d}s.$$

See Appendix A for a demonstration that the above indeed solves the perturbed equation. Next, since $\|e^{t\Delta_h}f(\cdot)\|_{L^\infty}\leq\|f(\cdot)\|_{L^\infty}$, as $e^{t\Delta_h}f$ is a discrete convolution with a kernel summing to unity, we can conclude that

$$\|E(\cdot,t)\|_{L^\infty}\leq\int_0^t\|\tau(\cdot,s)\|_{L^\infty}\mathrm{d}s.$$

Let first $t\leq 1$. Then, we have

$$\begin{aligned}
\|E(\cdot,t)\|_{L^\infty} &\leq \int_0^t\|\tau(\cdot,s)\|_{L^\infty}\mathrm{d}s\\
&\leq \int_0^t C_1\max_{i\in[d]}\|\partial_{x_i}^4\psi\|_{L^\infty}h^2\mathrm{d}s = C_1\max_{i\in[d]}\|\partial_{x_i}^4\psi\|_{L^\infty}h^2 t.
\end{aligned}$$

Now, let $t > 1$. Then,

$$\begin{aligned}\|E(\cdot,t)\|_{L^\infty} \leq \int_0^t \|\tau(\cdot,s)\|_{L^\infty} \mathrm{d}s &= \int_0^1 \|\tau(\cdot,s)\|_{L^\infty} \mathrm{d}s + \int_1^t \|\tau(\cdot,s)\|_{L^\infty} \mathrm{d}s \\ &\leq \int_0^1 C_1 \max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} h^2 \mathrm{d}s + \int_1^t C_2 \|\psi\|_{L^\infty} \frac{h^2}{s^2} \mathrm{d}s \\ &= C_1 \max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} h^2 - C_2 \|\psi\|_{L^\infty} \frac{h^2}{t} + C_2 \|\psi\|_{L^\infty} h^2 \\ &\leq C_3 (\max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} + \|\psi\|_{L^\infty}) h^2 - C_2 \|\psi\|_{L^\infty} \frac{h^2}{t},\end{aligned}$$

where $C_3 = \max(C_1, C_2) \geq C_2$, so that $C_3 - \frac{C_2}{t} > 0, \forall t > 0$. To summarize,

$$\|e^{t\Delta_h}\psi - e^{t\Delta}\psi\|_{L^\infty} \leq \begin{cases} C_1 \max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} h^2 t & \text{if } 0 \leq t \leq 1 \\ C_3 (\max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} + \|\psi\|_{L^\infty}) h^2 - C_2 \|\psi\|_{L^\infty} \frac{h^2}{t} & \text{if } t > 1 \end{cases}$$

With this upper bound, we compute

$$\begin{aligned}\|(-\Delta_h)^{\frac{\alpha}{2}}\psi - (-\Delta)^{\frac{\alpha}{2}}\psi\|_{L^\infty} &\leq \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty \|e^{t\Delta_h}\psi - e^{t\Delta}\psi\|_{L^\infty} \frac{dt}{t^{1+\frac{\alpha}{2}}} \\ &\leq C_1 \max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} h^2 \int_0^1 \frac{\mathrm{d}t}{t^{\frac{\alpha}{2}}} \\ &\quad + \int_1^\infty \left(C_3 (\max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} + \|\psi\|_{L^\infty}) h^2 - C_2 \|\psi\|_{L^\infty} \frac{h^2}{t} \right) \frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \\ &= \frac{C_1 \max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} h^2}{1 - \frac{\alpha}{2}} \\ &\quad + \frac{2C_3 (\max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} + \|\psi\|_{L^\infty}) h^2}{\alpha} - \frac{C_2 \|\psi\|_{L^\infty} h^2}{1 + \frac{\alpha}{2}} \\ &\leq C (\max_{i\in[d]} \|\partial_{x_i}^4 \psi\|_{L^\infty} + \|\psi\|_{L^\infty}) h^2.\end{aligned}$$

□

We have shown second order consistency for functions in $\mathcal{C}_b^4(\mathbb{R}^d)$. However, our assumptions on the continuous system will only guarantee classical solutions (u, m) in $\mathcal{C}_b^2(\mathbb{R}^d)$ for all $t \in (0, T)$, from Proposition 2. We therefore have to show that the PDL is also consistent for these functions, albeit not in second order. We prove this in the following lemma.

Lemma 13 (Consistency of PDL for $\mathcal{C}_b^2$-functions)**.** *Let $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$. Then, we have that*

$$\lim_{h\to 0} \left\|(-\Delta_h)^{\frac{\alpha}{2}}\psi - (-\Delta)^{\frac{\alpha}{2}}\psi\right\|_{L^\infty(\mathbb{R}^d)} = 0. \tag{3.13}$$

Proof. Define the Gaussian mollifier as

$$\phi_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp(-\frac{|x|^2}{2\epsilon^2}).$$

We define

$$\psi_\epsilon := \phi_\epsilon * \psi(x).$$

Since $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$, it follows that $\psi_\epsilon \xrightarrow{\epsilon\to 0} \psi$ in $\mathcal{C}_b^2(\mathbb{R}^d)$. To see why, note that $\phi_\epsilon \xrightarrow{\epsilon\to 0} \delta$ in distribution[2], and that $\partial_{x_i}^2 \psi_\epsilon(x) = (\phi_\epsilon * \partial_{x_i}^2 \psi)(x),\ \forall i \in [d]$. To get a precise estimate of $\|\psi_\epsilon - \psi\|_{\mathcal{C}_b}$, we Taylor expand ψ and get

$$\psi(y) = \psi(x) + \mathrm{D}\psi(x)^\top (y-x) + \frac{1}{2}(y-x)^\top \mathrm{D}^2\psi(\xi_{x,y})(y-x),$$

for some $\xi_{x,y}$ on the line segment between x and y. We now insert this Taylor expansion in the convolution,

$$\begin{aligned}
\psi_\epsilon(x) &= \int_{\mathbb{R}^d} \psi(y)\phi_\epsilon(x-y)\mathrm{d}y \\
&= \psi(x)\int_{\mathbb{R}^d} \phi_\epsilon(x-y)\mathrm{d}y + \int_{\mathbb{R}^d} \mathrm{D}\psi(x)^\top (y-x)\phi_\epsilon(x-y)\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2}(y-x)^\top \mathrm{D}^2\psi(\xi_{x,y})(y-x)\phi_\epsilon(x-y)\mathrm{d}y \\
&= \psi(x) + \int_{\mathbb{R}^d} \frac{1}{2}(y-x)^\top \mathrm{D}^2\psi(\xi_{x,y})(y-x)\phi_\epsilon(x-y)\mathrm{d}y,
\end{aligned}$$

where we used that ϕ_ϵ is a Gaussian probability density function (PDF) with diagonal covariance matrix. Hence, the second integral can be split up into a sum of d integrals, where each integral ends up being of the form $\mathrm{D}\psi(x)_i \int_{\mathbb{R}}(y_i - x_i)\mathcal{N}(y_i; x_i, \epsilon^2)\mathrm{d}y_i = 0$, where $\mathcal{N}(x;\mu,\sigma^2)$ is the PDF of a one-dimensional Gaussian with mean μ and variance σ^2. Define the last integral as

$$g_\epsilon(x) := \int_{\mathbb{R}^d} \frac{1}{2}(y-x)^\top \mathrm{D}^2\psi(\xi_{x,y})(y-x)\phi_\epsilon(x-y)\mathrm{d}y$$

Define further $\mathbf{1}\mathbf{1}^\top$ as the matrix of all ones. Since $\psi \in \mathcal{C}_b^2$, there exists an $M \geq 0$ with $\max_{i,j}\|(\mathrm{D}^2\psi)_{i,j}\|_{L^\infty} = M$, which means

$$\begin{aligned}
|g_\epsilon(x)| &\leq \int_{\mathbb{R}^d} \frac{1}{2}|(y-x)^\top \mathrm{D}^2\psi(\xi_{x,y})(y-x)|\phi_\epsilon(x-y)\mathrm{d}y \\
&\leq \int_{\mathbb{R}^d} M|(y-x)^\top \mathbf{1}\mathbf{1}^\top (y-x)|\phi_\epsilon(x-y)\mathrm{d}y \\
&\leq M\sum_{i=1}^d \int_{\mathbb{R}} (y_i - x_i)^2 \mathcal{N}(y_i; x_i, \epsilon^2)\mathrm{d}y_i \\
&\leq C\epsilon^2,
\end{aligned}$$

for a C dependent on M and d, as ϵ^2 is the variance of the PDF given by ϕ_ϵ. Here, we used that the covariance $\int_{\mathbb{R}^d}(y_i - x_i)(y_j - x_j)\phi_\epsilon(x-y)\mathrm{d}y = 0$, for all $i \neq j$, since the covariance matrix of ϕ_ϵ is diagonal. Therefore, only the diagonal elements (corresponding to the variance) in the second to last inequality survives.

Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be some function with $\omega(0) = 0$. By the triangle inequality,

[2] δ is the Dirac Delta function [28]. Proof given in [29].

Lemma 9 and Lemma 3, we have that

$$\begin{aligned}
|\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi(x)| \leq\ & |(-\Delta)^{\frac{\alpha}{2}}(\psi - \psi_\epsilon)(x)| + |(-\Delta_h)^{\frac{\alpha}{2}}(\psi - \psi_\epsilon)(x)| \\
& + |\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi_\epsilon(x)| \\
\leq\ & C_1\|\psi - \psi_\epsilon\|_{\mathcal{C}_b^2} + \frac{C_2}{h^\alpha}\|\psi - \psi_\epsilon\|_{\mathcal{C}_b} \\
& + |\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi_\epsilon(x)| \\
\leq\ & \omega(\epsilon) + \frac{C_2\epsilon^2}{h^\alpha} + \|\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi_\epsilon\|_{L^\infty},
\end{aligned}$$

It remains to find an expression for the last term. Now, since ψ_ϵ is smooth for any $\epsilon > 0$, we can use Lemma 12, and we have that

$$\|\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi_\epsilon\|_{L^\infty} \leq C(\max_{i\in[d]}\|\partial_{x_i}^4\psi_\epsilon\|_{L^\infty} + \|\psi_\epsilon\|_{L^\infty})h^2.$$

We compute

$$\begin{aligned}
\max_{i\in[d]}\|\partial_{x_i}^4\psi_\epsilon\|_{L^\infty} &\leq \max_{i\in[d]}\|\partial_{x_i}^2\psi * \partial_{x_i}^2\phi_\epsilon(x)\|_{L^\infty} \\
&\leq \max_{i\in[d]}\|\partial_{x_i}^2\psi\|_{L^\infty}\|\partial_{x_i}^2\phi_\epsilon(x)\|_{L^1} \\
&\leq \|\mathrm{D}^2\psi\|_{L^\infty}\frac{C_3}{\epsilon^2}.
\end{aligned}$$

The last inequality is a matter of computation. For a one-dimensional zero-mean Gaussian PDF $p(x)$ with variance σ^2, we have that $p''(x) = \frac{-\sigma^2+x^2}{\sigma^4}p(x)$, and thus $\int_{\mathbb{R}}|p''(x)|\mathrm{d}x = \frac{1}{\sigma^2} + \frac{1}{\sigma^2} = \frac{2}{\sigma^2}$. Combining our results yields

$$\|\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi\|_{L^\infty} \leq \omega(\epsilon) + C\left(\frac{\epsilon^2}{h^\alpha} + h^2\big(\|\psi\|_{L^\infty} + \frac{\|\mathrm{D}^2\psi\|_{L^\infty}}{\epsilon^2}\big)\right).$$

Letting $\epsilon = \mathcal{O}(h^s)$ for any s such that $\frac{\alpha}{2} < s < 1$, it follows that

$$\lim_{h\to 0}\|\big((-\Delta)^{\frac{\alpha}{2}} - (-\Delta_h)^{\frac{\alpha}{2}}\big)\psi\|_{L^\infty(\mathbb{R}^d)} = 0.$$

□

3.4 PDL on Torus

As mentioned in the introduction, most results on fractional calculus is defined on the whole space, and we therefore had to derive analogue properties on the torus. In particular, we will require certain properties of the fractional Laplacian and its discretization on the torus. What follows is inspired by the work of Roncal et al. in [30], where they derive the continuous fractional Laplacian on the torus. We will instead derive the powers of the discrete Laplacian on the torus, and will utilize our results from Section 3.2. We summarize the result in the following lemma.

Lemma 14. *Let $\phi \in L^\infty(\mathbb{T}_h^d)$ and $h > 0$. Then, the PDL on the torus is well-defined, and given by*

$$-(-\Delta_h)^{\frac{\alpha}{2}}\phi_\gamma = \frac{1}{h^\alpha}\sum_{\beta\in\mathcal{I}_h^d}\big(\phi_\beta - \phi_\gamma\big)K_\alpha(\gamma-\beta), \tag{3.14}$$

where

$$K_\alpha(\beta) := \begin{cases} \frac{1}{|\Gamma(-\frac{\alpha}{2})|}\sum_{\nu\in\mathbb{Z}^d}\int_0^\infty G_d(\beta - N_h\nu, t)\frac{dt}{t^{1+\frac{\alpha}{2}}} & \text{if } \beta \neq 0 \\ 0 & \text{if } \beta = 0. \end{cases} \tag{3.15}$$

Furthermore, $\sum_{\beta\in\mathcal{I}_h^d} K_\alpha(\beta) < +\infty$.

Proof of Lemma 14. The proof is somewhat heuristic, and aims to give the reader some intuition along the way. Before we proceed, it's easy to check that

$$\sum_{i\in\mathbb{Z}^d} f(i) = \sum_{\nu\in\mathbb{Z}^d}\sum_{i\in\mathcal{I}_h^d} f(i - N_h\nu). \tag{3.16}$$

The intuition is that we divide the space $\mathbb{Z}^d$ into a space of hypercubes with side lengths of $N_h - 1$, and sum each of them separately before adding them together. A visualization in two dimensions can be seen in Figure 3.4.1.

Let $\phi \in L^\infty(\mathbb{T}_h^d)$ we have that $\phi_{\gamma+kN_h} = \phi_\gamma$, $\forall\gamma \in \mathcal{I}_h^d, \forall k \in \mathbb{Z}^d$. Since ϕ is bounded on the discrete torus, it's bounded on the whole space, and thus the periodic extension of ϕ is in $L^\infty(h\mathbb{Z}^d)$, and therefore the PDL is well-defined by Lemma 10. By the definition of the PDL (3.7), we find that

$$\begin{aligned}
-(-\Delta_h)^{\frac{\alpha}{2}}\phi_\gamma &= \sum_{\beta\in\mathbb{Z}^d}\big(\phi_{\gamma-\beta} - \phi_\gamma\big)\left(\frac{\mathbb{1}_{\beta\neq0}}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty G_d(\beta, \frac{t}{h^2})\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}\right) \\
&= \sum_{\nu\in\mathbb{Z}^d}\sum_{\beta\in\mathcal{I}_h^d}\big(\phi_{\gamma-\beta+N_h\nu} - \phi_\gamma\big)\left(\frac{\mathbb{1}_{\beta-N_h\nu\neq0}}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty G_d(\beta - N_h\nu, \frac{t}{h^2})\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}\right) \\
&= \sum_{\beta\in\mathcal{I}_h^d}\sum_{\nu\in\mathbb{Z}^d}\big(\phi_{\gamma-\beta} - \phi_\gamma\big)\left(\frac{\mathbb{1}_{\beta\neq0}}{|\Gamma(-\frac{\alpha}{2})|}\int_0^\infty G_d(\beta - N_h\nu, \frac{t}{h^2})\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}\right) \\
&= \sum_{\beta\in\mathcal{I}_h^d}\big(\phi_{\gamma-\beta} - \phi_\gamma\big)\left(\frac{\mathbb{1}_{\beta\neq0}}{|\Gamma(-\frac{\alpha}{2})|}\sum_{\nu\in\mathbb{Z}^d}\int_0^\infty G_d(\beta - N_h\nu, t)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}}\right) \\
&= \frac{1}{h^\alpha}\sum_{\beta\in\mathcal{I}_h^d}\big(\phi_{\gamma-\beta} - \phi_\gamma\big)K_\alpha(\beta) \\
&= \frac{1}{h^\alpha}\sum_{\beta\in\mathcal{I}_h^d}\big(\phi_\beta - \phi_\gamma\big)K_\alpha(\gamma-\beta),
\end{aligned}$$

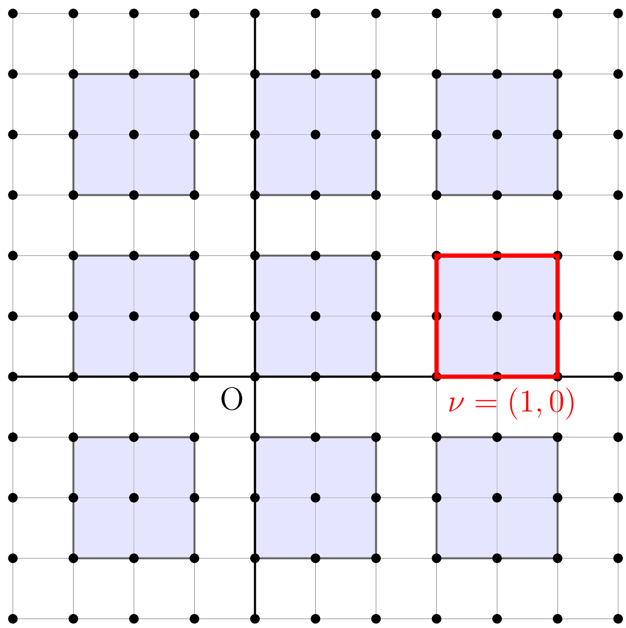


Figure 3.4.1: Visualization of (3.16) in two dimensions with $N_h = 3$.

where we defined

$$K_\alpha(\beta) := \begin{cases} \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \sum_{\nu\in\mathbb{Z}^d} \int_0^\infty G_d(\beta - N_h\nu, t)\frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} & \text{if } \beta \neq 0 \\ 0, & \beta = 0. \end{cases} \tag{3.17}$$

In the second equality, we used (3.16), and in the third equality, we used periodicity of ϕ, which gives a zero term when $\beta = 0$, $\forall \nu \in \mathbb{Z}^d$. K_α is well-defined, since for $\beta \neq 0$,

$$\int_0^\infty \sum_{\nu\in\mathbb{Z}^d} |G_d(\beta - N_h\nu, t)| \frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} \leq \int_0^\infty \sum_{\substack{\nu\in\mathbb{Z}^d \\ \nu\neq 0}} |G_d(\nu, t)| \frac{\mathrm{d}t}{t^{1+\frac{\alpha}{2}}} < +\infty, \quad \forall \alpha \in (0,2),$$

by Lemma 7. It follows that $\sum_{\beta\in\mathcal{I}_h^d} K_\alpha(\beta) < +\infty$. □

Lemma 15 (Kernel in 1D on Torus)**.** *In one dimension, we have*

$$K_\alpha(\beta) = \begin{cases} \sum_{\nu\in\mathbb{Z}} \frac{2^\alpha \Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}|\Gamma(-\frac{\alpha}{2})|} \frac{\Gamma(|\gamma-\beta-N_h\nu| - \frac{\alpha}{2})}{\Gamma(|\gamma-\beta-N_h\nu|+1+\frac{\alpha}{2})}, & \beta \neq 0 \\ 0, \quad \beta = 0. \end{cases} \tag{3.18}$$

Proof. Using (3.10) with (3.15) we get (3.18). □

3.5 A fast and accurate approximation of the PDL

The formula (3.14) is an exact formula for the discretized fractional Laplacian using discrete powers. To compute the PDL in practice, we must truncate the infinite sum, and therefore we can only achieve an approximation of the PDL. As done in [24], we can use (A.15) and write

$$\begin{aligned}
-(-\Delta_h)^{\frac{\alpha}{2}}u_\gamma &= \frac{1}{h^\alpha}\sum_{\beta\in\mathcal{I}_h}(u_\beta-u_\gamma)K_\alpha(\beta-\gamma)\\
&= \frac{c_\alpha}{h^\alpha}\sum_{\beta\in\mathcal{I}_h}\sum_{\nu\in\mathbb{Z}}(u_\beta-u_\gamma)\widetilde{K}_\alpha(\beta-\gamma-N_h\nu)\\
&= \frac{c_\alpha}{h^\alpha}\sum_{\beta\in\mathcal{I}_h}\left(\sum_{|\nu|\le R}+\sum_{|\nu|>R}\right)(u_\beta-u_\gamma)\widetilde{K}_\alpha(\beta-\gamma-N_h\nu)
\end{aligned}$$

where

$$K_\alpha(m) = \sum_{\nu\in\mathbb{Z}} c_\alpha\widetilde{K}_\alpha(m-N_h\nu), \tag{3.19}$$

$$c_\alpha = \frac{2^\alpha\Gamma(\frac{1+\alpha}{2})}{\sqrt{\pi}|\Gamma(-\frac{\alpha}{2})|}, \tag{3.20}$$

$$\widetilde{K}_\alpha(m) = \begin{cases}\frac{\Gamma(|m|-\frac{\alpha}{2})}{\Gamma(|m|+1+\frac{\alpha}{2})} & m\neq 0\\ 0 & m=0\end{cases} \tag{3.21}$$

Define

$$F_{1,\gamma} := \frac{c_\alpha}{h^\alpha}\sum_{\beta\in\mathcal{I}_h}\sum_{|\nu|\le R}(u_\beta-u_\gamma)\widetilde{K}_\alpha(\beta-\gamma-N_h\nu).$$

Without being too formal, we follow the approach in [24] and approximate

$$\begin{aligned}
-(-\Delta_h)^{\frac{\alpha}{2}}u_\gamma - F_{1,\gamma} &= \frac{c_\alpha}{h^\alpha}\sum_{\beta\in\mathcal{I}_h}\sum_{|\nu|>R}(u_\beta-u_\gamma)\widetilde{K}_\alpha(\beta-\gamma-N_h\nu)\\
&\approx -\frac{c_\alpha}{h^\alpha}u_\gamma\sum_{\beta\in\mathcal{I}_h}\sum_{|\nu|>R}\widetilde{K}_\alpha(\beta-\gamma-N_h\nu),
\end{aligned}$$

where we neglected the second sum. Furthermore, we can for m large enough use the approximation formula for the ratio of gamma functions (A.15) and approximate $\widetilde{K}_\alpha$ to be

$$\widetilde{K}_\alpha(m) \xrightarrow{m\to\infty} \widetilde{\widetilde{K}}_\alpha(m) := \begin{cases}\frac{1}{|m|^{1+\alpha}} & m\neq 0\\ 0 & m=0.\end{cases} \tag{3.22}$$

Hence, for a large enough R, we can use the approximation above and get

$$\begin{aligned}
&-\frac{c_\alpha}{h^\alpha}u_\gamma\sum_{\beta\in\mathcal{I}_h}\sum_{|\nu|>R}\widetilde{K}_\alpha(\beta-\gamma-N_h\nu) && (3.23)\\
\approx\ &-\frac{c_\alpha}{h^\alpha}u_\gamma\sum_{\beta\in\mathcal{I}_h}\sum_{|\nu|>R}\widetilde{\widetilde{K}}_\alpha(\beta-\gamma-N_h\nu) && (3.24)\\
=\ &-\frac{c_\alpha}{h^\alpha}u_\gamma\sum_{\beta\in\mathcal{I}_h}\left(\sum_{\nu\in\mathbb{Z}}-\sum_{|\nu|\le R}\right)\widetilde{\widetilde{K}}_\alpha(\beta-\gamma-N_h\nu). && (3.25)
\end{aligned}$$

Now, consider the first sum. From (3.16), we can write

$$\begin{aligned}
\frac{c_\alpha}{h^\alpha}u_\gamma \sum_{\beta\in\mathcal{I}_h}\sum_{\nu\in\mathbb{Z}} \widetilde{\widetilde{K}}_\alpha(\beta-\gamma-N_h\nu) &= \frac{c_\alpha}{h^\alpha}u_\gamma \sum_{\beta\in\mathbb{Z}} \widetilde{\widetilde{K}}_\alpha(\beta-\gamma) \\
&= \frac{c_\alpha}{h^\alpha}u_\gamma \sum_{m\in\mathbb{Z}} \widetilde{\widetilde{K}}_\alpha(m) \\
&= \frac{c_\alpha}{h^\alpha}u_\gamma \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{|m|^{1+\alpha}} \\
&= 2\frac{c_\alpha}{h^\alpha}u_\gamma \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} \\
&= 2\frac{c_\alpha}{h^\alpha}u_\gamma \zeta(1+\alpha)
\end{aligned}$$

where ζ is the Riemann zeta function (see e.g. [31]) defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3.26}$$

Define now

$$F_{2,\gamma} := -\frac{c_\alpha}{h^\alpha}u_\gamma \left(2\zeta(1+\alpha) - \sum_{\beta\in\mathcal{I}_h}\sum_{|\nu|\le R} \widetilde{\widetilde{K}}_\alpha(\beta-\gamma-N_h\nu) \right).$$

We will approximate

$$-(-\Delta_h)^{\frac{\alpha}{2}}u_\gamma \approx F_{1,\gamma} + F_{2,\gamma}. \tag{3.27}$$

This approximation easily generalizes to the PDL on the whole space, as can be seen in Appendix A. The trick of using the Riemann zeta function gave a substantial improvement compared to simply approximating (3.24) by a truncated sum of the form

$$F_{2,\gamma,\neg\zeta} = -\frac{c_\alpha}{h^\alpha}u_\gamma \sum_{\beta\in\mathcal{I}_h}\sum_{R_2>|\nu|>R} \widetilde{\widetilde{K}}_\alpha(\beta-\gamma-N_h\nu),$$

for some $R_2 \gg R \gg 0$. As discussed in the next section, using the ζ function had a significant effect for small α. This is probably due to that $\zeta(1+\alpha) \xrightarrow{\alpha\to 0} +\infty$, making the truncation error grow large. Using a lookup table embedded in the `SpecialFunctions.jl` package in Julia [32], we get an exact expression for $F_{2,\gamma}$. This trick has not to the best of the author's knowledge been used for approximating the discrete fractional Laplacian before. The approximation can further be implemented as a matrix-vector product by defining a PDL matrix, as we will see in the implementation derivation in Chapter 5, making it very fast while remaining high accuracy. Furthermore, this trick can also be applied for the PDL defined on the whole space, which is described in Appendix A.

3.6 Simulations of the PDL

In this section we will perform simulations to test the tractability of our discretization of the fractional Laplacian. The examples are on functions defined on the whole space $\mathbb{R}$, but the approximation formula derived in the previous section has a simple analogue on the whole space (see Appendix A for this derivation). We will later validate the PDL matrix defined on the torus with the implementation we validate here.

Example 1: *Comparison with analytical fractional Laplacian* We first perform compare the discrete fractional Laplacian with an analytical fractional Laplacian.

Consider the function

$$u(x) = (1+x^2)^{-(\frac{1}{2}-\frac{\alpha}{2})},\ x \in \mathbb{R}.$$

It can be shown that u has the analytical fractional Laplacian

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{2^\alpha \Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1-\alpha}{2})}(1+x^2)^{-(\frac{1+\alpha}{2})},$$

(See [24], formula (6.7)). Figure 3.6.1 compares the analytical fractional Laplacian with our implementation.

Example 2: *Effect of the ζ-trick*

We demonstrate the effect of using the ζ-function provides as α gets smaller. Let

$$u(x) = \exp(-x^2).$$

Define now the temporary notation $(-\Delta_h)^{\alpha/2}_{\zeta}$ as the PDL using the ζ-trick defined in (3.27). Let then $(-\Delta_h)^{\alpha/2}_{\neg\zeta}$ be defined almost as (3.27), but rather approximate $F_{2,\gamma}$ by the truncated sum[3]

$$F_{2,\gamma,\neg\zeta} = -\frac{c_\alpha}{h^\alpha} u_\gamma \sum_{\beta \in \mathcal{I}_h} \sum_{R_2 > |\nu| > R} \widetilde{\widetilde{K}}_\alpha(\beta - \gamma - N_h \nu),$$

for some large $R_2 \gg R$. Here, R_2 serves as the truncation cutoff. To be clear, whenever there is no subscript, it should be interpreted as $(-\Delta_h)^{\alpha/2}_{\zeta}$. We let $R = 10^4$ and $R_2 = 10^8$. Figure 3.6.2 (a-c) shows how the approximations increasingly differs as α decreases. Define further the sums

$$S_\zeta = 2\zeta(1+\alpha) - \sum_{\beta \in \mathcal{I}_h} \sum_{|\nu| \le R} \widetilde{\widetilde{K}}_\alpha(\beta - \gamma - N_h \nu,$$

$$S_{\neg\zeta} = \sum_{\beta \in \mathcal{I}_h} \sum_{R_2 > |\nu| > R} \widetilde{\widetilde{K}}_\alpha(\beta - \gamma - N_h \nu)$$

used in $(-\Delta_h)^{\alpha/2}_{\zeta}$ and $(-\Delta_h)^{\alpha/2}_{\neg\zeta}$ respectively. Figure 3.6.2 (d) shows how these quantities grow for small α.

[3] This is almost precise. The example is here defined on the whole space $\mathbb{R}$. Hence, we rather truncate the sum defined in the derivation of the PDL on the whole space, see Appendix A. All details regarding the experiments can be found in the notebooks located in the GitHub repo.

Example 3: *Limiting cases*
We also verify the limiting cases of the fractional Laplacian given in Lemma 40. That is,

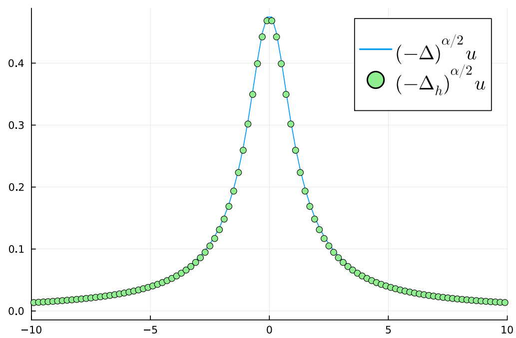
$$\lim_{\alpha \to 2} (-\Delta)^{\frac{\alpha}{2}} u = -\Delta u$$
$$\lim_{\alpha \to 0} (-\Delta)^{\frac{\alpha}{2}} u = u.$$

We will compute

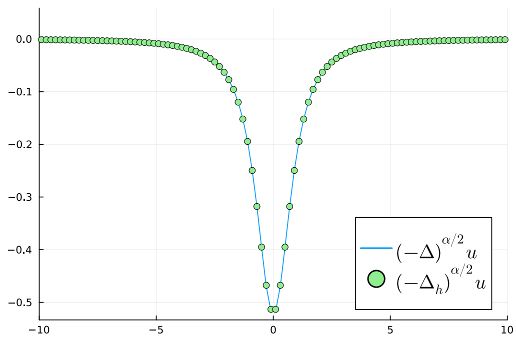
$$\lim_{\alpha \to k} (-\Delta)^{\frac{\alpha}{2}} \exp(-x^2), \quad k \in \{0, 2\}.$$

The results are shown in Figure 3.6.3, and confirms our expectations. We also plot $(-\Delta_h)^{\alpha/2}_{\neg\zeta} u$ in both cases. We observe that it has a significant effect in the case $\alpha \to 0$, but no noticeble effect for $\alpha \to 2$.

Example 4: *Consistency of discrete fractional Laplacian*
We demonstrate in Figure 3.6.4 the second order local truncation error of the PDL (Lemma 12), for $\exp(-x^2) \in \mathcal{C}_b^4(\mathbb{R})$.

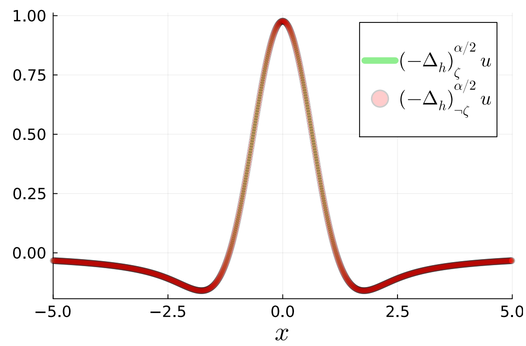


(a) $\alpha = 0.5$

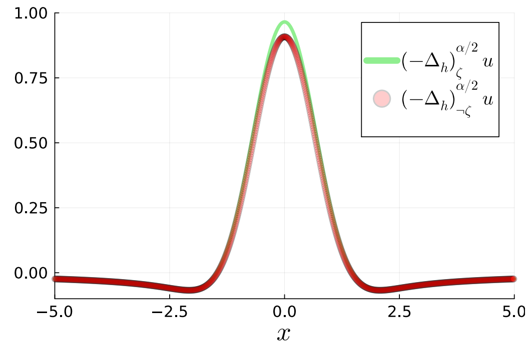


(b) $\alpha = 1.5$

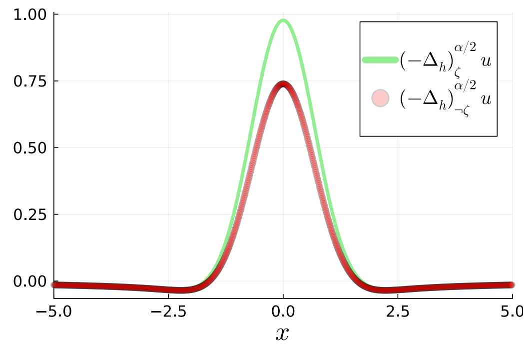
Figure 3.6.1: Comparing the PDL with an analytical fractional Laplacian.



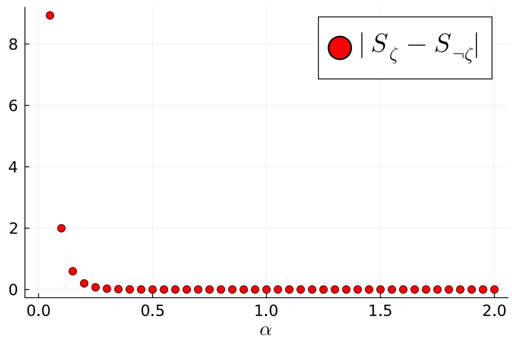
(a) $\alpha = 0.5$



(b) $\alpha = 0.2$

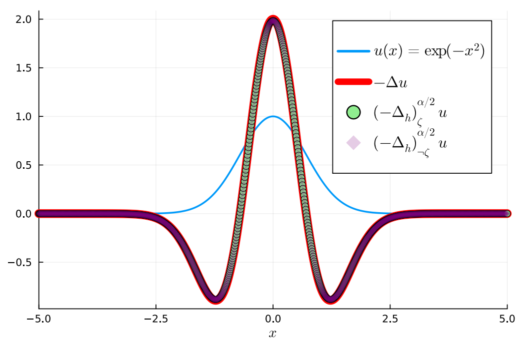


(c) $\alpha = 0.1$

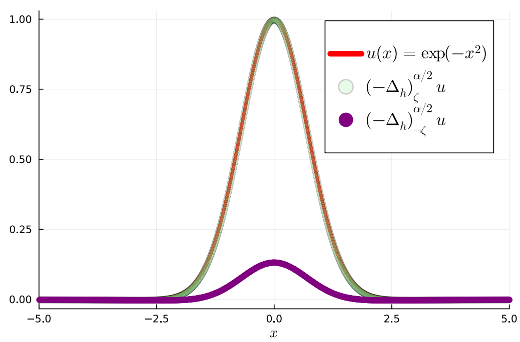


(d) $|S_{\zeta} - S_{\neg\zeta}|$ plotted against α.

Figure 3.6.2: Demonstrating the effect of the Riemann zeta trick.



(a) $\alpha = 1.99$



(b) $\alpha = 0.01$

Figure 3.6.3: Comparison in the limiting cases, $\alpha \to 2$ and $\alpha \to 0$.

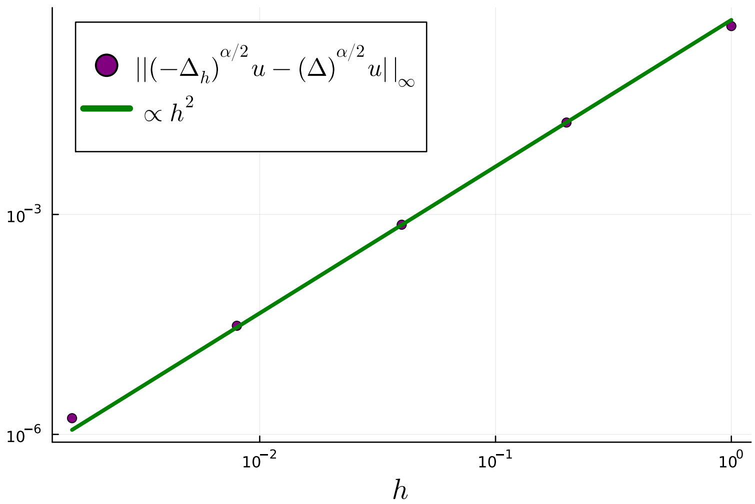


Figure 3.6.4: Second-order consistency of the PDL.

3.7 Self-adjointness of PDL

The final property of the PDL we need to derive is self-adjointness on the torus, which we use repeatedly throughout Chapter 4 and is therefore a key result. There exists to the best of the author's knowledge few results regarding the PDL on the torus, and this proof is therefore also original work.

Theorem 16 (Self-adjointness of discrete fractional Laplacian)**.** *Let $\phi \in L^\infty(\mathbb{T}_h^d)$. Then, we have that*

$$\sum_{\gamma \in \mathcal{I}_h^d} \psi_\gamma (-\Delta_h)^{\frac{\alpha}{2}} \phi_\gamma = \sum_{\gamma \in \mathcal{I}_h^d} \phi_\gamma (-\Delta_h)^{\frac{\alpha}{2}} \psi_\gamma. \tag{3.28}$$

Proof. Direct calculation reveals

$$\begin{aligned}
&\sum_{\gamma \in \mathcal{I}_h^d} \psi_\gamma (-\Delta_h)^{\frac{\alpha}{2}} \phi_\gamma - \sum_{\gamma \in \mathcal{I}_h^d} \phi_\gamma (-\Delta_h)^{\frac{\alpha}{2}} \psi_\gamma \\
=& \frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \psi_\gamma \Big(\sum_{\beta \in \mathcal{I}_h^d} \big(\phi_{\gamma+\beta} - \phi_\gamma\big) K_\alpha(\beta) \Big) \\
&- \frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \phi_\gamma \Big(\sum_{\beta \in \mathcal{I}_h^d} \big(\psi_{\gamma+\beta} - \psi_\gamma\big) K_\alpha(\beta) \Big) \\
=& \frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \psi_\gamma \sum_{\beta \in \mathcal{I}_h^d} \phi_{\gamma+\beta} K_\alpha(\beta) - \cancel{\frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \psi_\gamma \phi_\gamma \sum_{\beta \in \mathcal{I}_h^d} K_\alpha(\beta)} \\
&- \Big(\frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \phi_\gamma \sum_{\beta \in \mathcal{I}_h^d} \psi_{\gamma+\beta} K_\alpha(\beta) - \cancel{\frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \phi_\gamma \psi_\gamma \sum_{\beta \in \mathcal{I}_h^d} K_\alpha(\beta)} \Big) \\
=& \frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \sum_{\beta \in \mathcal{I}_h^d} \phi_\gamma \psi_{\gamma+\beta} K_\alpha(\beta) - \frac{1}{h^\alpha} \sum_{\gamma \in \mathcal{I}_h^d} \sum_{\beta \in \mathcal{I}_h^d} \psi_\gamma \phi_{\gamma+\beta} K_\alpha(\beta).
\end{aligned}$$

To conclude the proof, we must show that the above is zero. Define $\gamma' = \gamma + \beta$, and $\beta' = -\beta$. We can write

$$\sum_{\gamma \in \mathcal{I}_h^d} \sum_{\beta \in \mathcal{I}_h^d} \phi_\gamma \psi_{\gamma+\beta} K_\alpha(\beta) = \sum_{\beta \in \mathcal{I}_h^d} K_\alpha(\beta) \sum_{\substack{\gamma' \in \mathbb{Z}^d: \\ \gamma' - \beta \in \mathcal{I}_h^d}} \phi_{\gamma'-\beta} \psi_{\gamma'}$$

Now, because of periodicity of ψ and ϕ, we have that $x \to \phi(x - \beta h)\psi(x)$ is also periodic for any $\beta \in \mathcal{I}_h^d$, since $\phi(x - \beta h + k)\psi(x + k) = \phi(x - \beta h)\psi(x),\ \forall k \in \mathbb{Z}^d$. Periodicity means we can shift the summation index and get the same result. Therefore, for any $k \in \mathbb{Z}^d$,

$$\sum_{\substack{\gamma' \in \mathbb{Z}^d: \\ \gamma' + k \in \mathcal{I}_h^d}} \phi_{\gamma'-\beta} \psi_{\gamma'} = \sum_{\gamma' \in \mathcal{I}_h^d} \phi_{\gamma'-\beta} \psi_{\gamma'}. \tag{3.29}$$

We therefore omit the shift of β in the summation index, as we sum over all indices on the torus and summation order is irrelevant. We obtain

$$\begin{aligned}\sum_{\gamma\in\mathcal{I}_h^d}\sum_{\beta\in\mathcal{I}_h^d}\phi_\gamma\psi_{\gamma+\beta}K_\alpha(\beta) &= \sum_{\beta\in\mathcal{I}_h^d}K_\alpha(\beta)\sum_{\gamma'\in\mathcal{I}_h^d}\phi_{\gamma'-\beta}\psi_{\gamma'}\\ &= \sum_{\substack{\beta'\in\mathbb{Z}^d:\\ -\beta'\in\mathcal{I}_h^d}}\sum_{\gamma'\in\mathcal{I}_h^d}\phi_{\gamma'+\beta'}\psi_{\gamma'}K_\alpha(-\beta')\\ &= \sum_{\beta'\in\mathcal{I}_h^d}\sum_{\gamma'\in\mathcal{I}_h^d}\phi_{\gamma'+\beta'}\psi_{\gamma'}K_\alpha(\beta')\\ &= \sum_{\beta\in\mathcal{I}_h^d}\sum_{\gamma\in\mathcal{I}_h^d}\phi_{\gamma+\beta}\psi(\gamma h)K_\alpha(\beta),\end{aligned}$$

where we in the fourth equality used, similarly as in (3.29), that for any $\gamma\in\mathcal{I}_h^d$,

$$\sum_{\substack{\beta'\in\mathbb{Z}^d:\\ -\beta'\in\mathcal{I}_h^d}}\phi_{\gamma'+\beta'}\psi_{\gamma'}K_\alpha(-\beta') = \sum_{\beta'\in\mathcal{I}_h^d}\phi_{\gamma'+\beta'}\psi_{\gamma'}K_\alpha(-\beta')$$

and that K_α is symmetric about zero. Since

$$\sum_{\gamma\in\mathcal{I}_h^d}\sum_{\beta\in\mathcal{I}_h^d}\phi_\gamma\psi_{\gamma+\beta}K_\alpha(\beta) = \sum_{\beta\in\mathcal{I}_h^d}\sum_{\gamma\in\mathcal{I}_h^d}\phi_{\gamma+\beta}\psi(\gamma h)K_\alpha(\beta),$$

it follows that

$$\sum_{\gamma\in\mathcal{I}_h^d}\psi_\gamma(-\Delta_h)^{\frac{\alpha}{2}}\phi_\gamma - \sum_{\gamma\in\mathcal{I}_h^d}\phi_\gamma(-\Delta_h)^{\frac{\alpha}{2}}\psi_\gamma = 0.$$

□

CHAPTER

FOUR

DISCRETIZATION OF THE SYSTEM

Now that we have derived a discretization of the fractional Laplacian, we are ready to discretize the full system. For simplicity, we will derive the method in $d = 2$ dimensions, but the framework also holds for $d \geq 2$. First, we will in section 4.1 introduce the standard discretization framework of second order PDEs, with some relevant properties, before we give some section specific notation. Next, we will in 4.2 derive the discretization of the MFG system (MFG). We give some necessary technical results in section 4.3 before we show existence and uniqueness for the uncoupled HJB equation and FPK equation in section 4.4-4.5, each separately. Finally, we show existence, uniqueness and convergence of the coupled MFG system in section 4.6-4.8, which is this project's main contribution.

The scheme and some proofs are inspired by Achdou et al.'s numerical method for local diffusion MFGs [7]. However, as we operate with nonlocal diffusion, most of the proofs required generalization to account for the fractional Laplacian, some of which were derived in the previous chapter. In particular, our existence and uniqueness proofs on the HJB equations is a trivial generalization of the proofs Achdou et al., using our results from Chapter 3. For the isolated FPK equations, existence and uniqueness was merely treated in [7], but rather mentioned as a remark (Remark 7). We generalize their sketch of proof to encapsulate nonlocal diffusion, by for instance using self-adjointness proved in the previous section. When proving existence of the coupled MFG system, we also here used a similar approach as Achdou et al., using a Brouwer fixed point argument. However, we use a more direct approach, by directly defining the fixed point function as the composition of the solution mappings of the HJB and FPK equations, whereas they define a perturbed fixed point equation for FPK. Our approached followed more naturally in our case, since we first proved well-posedness of the two uncoupled equations separately. Finally, we prove convergence for both u and m, whereas the convergence proof for m is untreated in [7]. The proof architecture of the latter is therefore also an original contribution from our end.

4.1 Discretizing a second order PDE

We will begin by introducing the reader to the discretization framework we will operate with when discretizing a second order partial differential equation. Furthermore, we will define certain properties we will need in order to prove existence, uniqueness and convergence. Among these properties are monotonicity and non-expansivity, which are properties which follows from *degenerate elliptic schemes*, which we soon will define. The description we provide here is also given in [33, section 2.3]. Consider a general second order PDE,

$$\mathcal{F}(x, u, \mathrm{D}u, \mathrm{D}^2 u) = 0, \quad \text{in } \Omega \subset \mathbb{R}^d. \tag{4.1}$$

We recall that D is the gradient with respect to the spatial variable x, and D^2 is the Hessian operator producing the Hessian matrix with entries $(\mathrm{D}^2 u)_{i,j} = \partial_{x_i}\partial_{x_j} u,\ \forall i, j \in [d]$.

Definition 6 (Degenerate elliptic equation)**.** We say that (4.1) is degenerate elliptic if

$$\mathcal{F}(x, u, p, X) \leq \mathcal{F}(x, v, p, Y) \quad \text{whenever } u \leq v \text{ and } Y \leq X, \tag{4.2}$$

where $Y \leq X$ signifies that $Y - X$ is a non-negative definite matrix.

When solving systems like (4.1) numerically, we discretize the space into some grid $\mathcal{G} \subset \Omega$ consisting of grid points $x_p \in \Omega$, $p \in [|\mathcal{G}|]$. Define the neighborhood $N(p)$ of $x_p \in \mathcal{G}$ as a set of neighboring points necessary to compute the discretization of the equation in question. Let $U : \mathcal{G} \to \mathbb{R}$ be a grid function, which serves as an approximation of u. A finite difference scheme for (4.1) are equations which at each grid point x_p describes the grid function,

$$\mathcal{F}^p_{\mathcal{G}}(x_p, U_p, \{U_q\}_{q \in N(p)}) = 0, \tag{4.3}$$

and approximates the PDE at that particular grid point. At the boundary $\partial\mathcal{G}$, we need a boundary condition g such that $\mathcal{F}^p_{\mathcal{G}} = U_p - g(x_p) = 0$. Most differential operators, and indeed those we will consider in this paper, consists of linear combination of local differences with the point in question, $U_p - U_q|_{q \in N(p)}$. Hence, we can redefine $F_{\mathcal{G}}$ to be of the form

$$\mathcal{F}^p_{\mathcal{G}}[U] := \mathcal{F}^p_{\mathcal{G}}(x_p, U_p, \{U_p - U_q\}_{q \in N(p)}) = 0. \tag{4.4}$$

Definition 7 (Degenerate elliptic scheme)**.** A scheme

$$\mathcal{F}^p_{\mathcal{G}}(x_p, U_p, \{U_p - U_q\}_{q \in N(p)}) = 0$$

is called degenerate elliptic if $\mathcal{F}^p_{\mathcal{G}}$ is non-decreasing in $(U_p, \{U_p - U_q\}_{q \in N(p)})$. Likewise, we say that a finite difference operator $[\mathcal{L}_h U]_p = \mathcal{F}^p_{\mathcal{G}}(x_p, U_p, \{U_p - U_q\}_{q \in N(p)})$ satisfying the above is a degenerate elliptic operator.

Remark. We acknowledge that Definition 7 is an unconventional definition of degenerate ellipticity, and that one usually defines a degenerate elliptic scheme as a function of the form (4.3) which is monotone (non-decreasing in U_p and non-increasing in $\{U_q\}_{q \in N(p)}$) and non-expansive in the maximum-norm. Equivalence of the two definitions for finite difference schemes are discussed in section 2.3 in [33].

Lemma 17 (Maximum principle for degenerate elliptic schemes)**.** *Let $F_{\mathcal{G}}$ be a degenerate elliptic scheme, and U, V be grid functions. Let x_p be a point where $U - V$ attains a non-negative maximum, then $\mathcal{F}^p_{\mathcal{G}}[U] \geq \mathcal{F}^p_{\mathcal{G}}[V]$*

Proof. Let $U_p - V_p = \max_{q\in[|\mathcal{G}|]} \big(U_q - V_q\big) \geq 0$. Then, $U_p - U_q \geq V_p - V_q$, $\forall q \in [|\mathcal{G}|]$. It follows by Definition 7 that

$$\mathcal{F}^p_{\mathcal{G}}[U] := \mathcal{F}^p_{\mathcal{G}}(x_p, U_p, \{U_p - U_q\}_{q\in N(p)}) \geq \mathcal{F}^p_{\mathcal{G}}(x_p, V_p, \{V_p - V_q\}_{q\in N(p)}) =: \mathcal{F}^p_{\mathcal{G}}[V]$$

The proof is also given in [33, Lemma 3]. □

Remark. Note that a linear combination of degenerate elliptic operators is also degenerate elliptic.

4.1.1 Notation and spaces

From here on, we will work in two dimensions ($d = 2$) on the torus defined in (2.1), and grid functions will from here on be of the sort $U \in \mathbb{R}^{N_h^2}$, and will be indexed as $U_{i,j} = U(ih, jh)$, for $(i,j) \in \mathcal{I}_h^2$. We will use grid function and vector somewhat interchangeably depending on context. To see that $U_{i,j}, (i,j) \in \mathcal{I}_h^2$ defines a vector in $\mathbb{R}^{N_h^2}$, we can for instance stack the rows on top of each other of the grid function to create a global index $p(i,j) = N_h i + j + 1 \in [N_h^2],\ \forall (i,j) \in \mathcal{I}_h^2$. As the system is time-dependent, let U^n indicate the grid function at time step $n \in [0, N_T]_{\mathbb{Z}}$. We introduce the shorthand $\sum_{i,j}$ for $\sum_{(i,j)\in\mathcal{I}_h^2}$. Define the space of discrete probability density functions

$$\mathcal{P}_h := \left\{ M \in \mathbb{R}^{N_h^2} : h^2 \sum_{i,j} M_{i,j} = 1, M_{i,j} \geq 0,\ \forall (i,j) \in \mathcal{I}_h^2 \right\}, \tag{4.5}$$

and $\mathcal{P}_h^{N_T} := \{(M^j)_{j=1}^{N_T} : M^j \in \mathcal{P}_h\} \subset \mathbb{R}^{N_h^2 N_T}$. Let further $\square_{h,(i,j)} := \{x \in \mathbb{T}^2 : |x - x_{i,j}| \leq \frac{h}{2}\}$. We define the piecewise constant interpolation $\mathcal{J}_h[V] : \mathbb{T}^2 \to \mathbb{R}$ of a grid function $V \in \mathbb{T}_h^2 \to \mathbb{R}$ as

$$\mathcal{J}_h[V](x) := V_{i,j} \quad \text{for} \quad x \in \square_{h,(i,j)},\ \forall (i,j) \in \mathcal{I}_h^2. \tag{4.6}$$

We define further $\mathcal{H}_h[v] : \mathbb{T}_h^2 \to \mathbb{R}$ as the mean-discretization operator acting on a function $v : \mathbb{T}^2 \to \mathbb{R}$, given by

$$\mathcal{H}_h[v]_{i,j} := h^{-2} \int_{x:|x-x_{i,j}|\leq h/2} v(x)\mathrm{d}x, \quad \forall (i,j) \in \mathcal{I}_h^2. \tag{4.7}$$

Define the $L^1(\mathbb{T}_h^2)$-norm for a grid function $f : \mathbb{T}_h^2 \to \mathbb{R}$, $\|f\|_{L^1(\mathbb{T}_h^2)} := h^2 \sum_{(i,j)\in\mathcal{I}_h^2} |f_{i,j}|$.

4.1.2 Assumptions on approximating operators

For later reference and clarity, we here list all assumptions on the approximating operators we soon will define. Many of the assumptions are of similar form as those in [7]. We will let Assumption 8 and Assumption 14-18 hold throughout the paper. Assumption 9 are needed for proving uniqueness of the discrete MFG system, and Assumption 10-12 are needed for proving convergence of the discrete MFG system.

Assumption 8 (Continuity and uniform boundedness of F_h and G_h)**.** Let

$$F_h \in \mathcal{C}(\mathcal{P}_h; \mathcal{C}^2(\mathbb{T}^2)),$$
$$G_h \in \mathcal{C}(\mathcal{P}_h; \mathcal{C}^2(\mathbb{T}^2)).$$

There exist constants $C_F, C_G > 0$, such that

$$\sup_{M \in \mathcal{P}_h} \|F_h[M]\|_{\mathcal{C}_b^2(\mathbb{T}^2)} \le C_F \quad \text{and} \quad \sup_{M \in \mathcal{P}_h} \|G_h[M]\|_{\mathcal{C}_b^2(\mathbb{T}^2)} \le C_G.$$

Assumption 9 (Monotonicity of F_h and G_h)**.** F_h, G_h are strictly monotone. That is, for all $M, \widetilde{M} \in \mathcal{P}_h$,

$$\left(\mathbf{F}_h[M] - \mathbf{F}_h[\widetilde{M}], M - \widetilde{M}\right)_2 \le 0 \implies \mathbf{F}_h[M] = \mathbf{F}_h[\widetilde{M}],$$

$$\left(\mathbf{G}_h[M] - \mathbf{G}_h[\widetilde{M}], M - \widetilde{M}\right)_2 \le 0 \implies \mathbf{G}_h[M] = \mathbf{G}_h[\widetilde{M}],$$

Assumption 10 (Consistency of F_h and G_h)**.** Let there exist a function $b : \mathbb{R}_+ \to R_+$, with $b(0) = 0$, such that for all $m \in \mathcal{P}(\mathbb{T}^2)$, and all sequences $(M_h)_h$, where $M_h \in \mathcal{P}_h$,

$$\|F[M] - F_h[M_h]\|_{L^\infty(\mathbb{T}_h^2)} \le b(\|m - \mathcal{J}_h M_h\|_{L^1(\mathbb{T}^2)})$$
$$\|G[M] - G_h[M_h]\|_{L^\infty(\mathbb{T}_h^2)} \le b(\|m - \mathcal{J}_h M_h\|_{L^1(\mathbb{T}^2)}).$$

Assumption 11 (Stricter monotonicity of F_h and G_h.)**.** There exists constants $p, c > 0$ such that for all $M, \widetilde{M} \in \mathcal{P}_h$ and $h < 1$,

$$h^2 \left(\mathbf{F}_h[M] - \mathbf{F}_h[\widetilde{M}], M - \widetilde{M}\right)_2 \ge c\|\mathbf{F}_h[M] - \mathbf{F}_h[\widetilde{M}]\|^p_{L^\infty(\mathbb{T}_h^2)}, \tag{4.8}$$

$$h^2 \left(\mathbf{G}_h[M] - \mathbf{G}_h[\widetilde{M}], M - \widetilde{M}\right)_2 \ge c\|\mathbf{G}_h[M] - \mathbf{G}_h[\widetilde{M}]\|^p_{L^\infty(\mathbb{T}_h^2)}. \tag{4.9}$$

Assumption 12 (Consistency of transport operator to divergence term)**.** For every $u, m \in \mathcal{C}^1(\mathbb{T}^2)$, there are constant $C > 0, r > 0$ such that for all $h < 1$, and all $(i, j) \in \mathcal{I}_h^2$,

$$\left|\mathcal{T}_{i,j}(\mathcal{H}_h u, \mathcal{H}_h m) - \operatorname{div}\big(m \mathrm{D}_p H(x, \mathrm{D}u)(x_{i,j})\big)\right| \le Ch^r.$$

Assumption 13 (Consistency of transport operator in U.)**.**

$$\sup_{i,j} \left|\mathcal{T}_{i,j}(\widetilde{U}, M) - \mathcal{T}_{i,j}(U, M)\right| = \omega(\|U - \widetilde{U}\|_{L^\infty(\mathbb{T}_h^2)})$$

for some function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$.

Assumption 14 (Monotonicity of g)**.** $(q_1, q_2, q_3, q_4) \to g(x, q_1, q_2, q_3, q_4)$ is nonincreasing with respect to q_1 and q_3 and non-decreasing with respect to q_2 and q_4. In other words,

$$\partial_{q_1} g \le 0$$
$$\partial_{q_2} g \ge 0$$
$$\partial_{q_3} g \le 0$$
$$\partial_{q_4} g \ge 0, \quad \forall (x, q) \in \mathbb{T}^2 \times \mathbb{R}^4.$$

Assumption 15 (Consistency of g)**.**

$$g(x, q_1, q_1, q_2, q_2) = H(x, q), \quad \forall x \in \mathbb{T}^2, \quad \forall q = (q_1, q_2) \in \mathbb{R}^2. \tag{4.10}$$

Assumption 16 (Differentiability of g)**.** $g \in \mathcal{C}^1(\mathbb{T}^2 \times \mathbb{R}^4)$.

Assumption 17 (Convexity of g)**.** The function $(q_1, q_2, q_3, q_4) \to g(x, q_1, q_2, q_3, q_4)$ is convex.

Assumption 18 (Sublinearity of $\partial_x g$ in q)**.** There exists a constant C_g such that

$$\left|\frac{\partial g}{\partial x}(x, q_1, q_2, q_3, q_4)\right| \leq C_g(1 + |q_1| + |q_2| + |q_3| + |q_4|), \quad \forall (x, q) \in \mathbb{T}^2 \times \mathbb{R}^4 \tag{4.11}$$

4.2 Derivation of the discrete system

In this section, we will derive the discretization of the fractional MFG system (MFG). Starting out with the continuous system, we will discretize each term separately. Recall our fractional MFG system,

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\frac{\alpha}{2}} u + H(x, \mathrm{D}u) = F[m] & \text{in} \quad \mathbb{T}^2 \times (0, T) \\ \partial_t m(x, t) - \nu(-\Delta)^{\frac{\alpha}{2}} m + \mathrm{div}(m \nabla_p H(x, \mathrm{D}u)) = 0 & \text{in} \quad \mathbb{T}^2 \times (0, T) \\ m(x, T) = m_T(x) \in \mathcal{P}(\mathbb{T}^2), \qquad u(\cdot, 0) = G[m(\cdot, 0)] & \text{in} \quad \mathbb{T}^2 \\ \int_{\mathbb{T}^2} m \mathrm{d}x = 1, \qquad m \geq 0 & \text{in} \quad \mathbb{T}^2 \times (0, T). \end{cases} \tag{4.12}$$

Let (u, m) be the unique classical solution of (4.12) given by Proposition 2. Hence, $u \in \mathcal{C}_b^{1,3}(\mathbb{T}^2 \times (0, T))$, and $m \in \mathcal{C}_b^{1,2}(\mathbb{T}^2 \times (0, T)) \cap \mathcal{C}([0, T]; \mathcal{P}(\mathbb{T}^2))$. We will be using the notation $u_{i,j}^n := u(x_{i,j}, n\Delta t)$. Standard arguments involving Taylor expansion gives

$$\partial_t u_{i,j}^n = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t), \quad \partial_t m_{i,j}^n = \frac{m_{i,j}^{n+1} - m_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t)$$
$$\partial_{x_k} u_{i,j}^n = D_k^{\pm} u_{i,j}^n + \mathcal{O}(h), \quad \partial_{x_k} m_{i,j}^n = D_k^{\pm} m_{i,j}^n + \mathcal{O}(h) \ k \in \ 1, 2.$$

Since $u(\cdot, t), m(\cdot, t) \in \mathcal{C}_b^2(\mathbb{T}^2)$, $\forall t \in (0, T)$, we can apply Lemma 13 and find that

$$(-\Delta)^{\frac{\alpha}{2}} u_{i,j}^n = (-\Delta_h)^{\frac{\alpha}{2}} u_{i,j}^n + o(1)$$
$$(-\Delta)^{\frac{\alpha}{2}} m_{i,j}^n = (-\Delta_h)^{\frac{\alpha}{2}} m_{i,j}^n + o(1)$$

where $o(1)$ is a quantity going to zero as $h, \Delta t \to 0$. Recall from Lemma 12 that smoother solutions (u, m) gives higher consistency order of the PDL, but consistency is in itself satisfactory for what follows. We define the difference vector

$$[D_h u]_{i,j} = \left((D_1^+ u)_{i,j}, (D_1^- u)_{i,j}, (D_2^+ u)_{i,j}, (D_2^- u)_{i,j}\right)^\top \in \mathbb{R}^4,$$

where

$$(D_1^+ u)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}, \quad (D_2^+ u)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h},$$

$$(D_1^- u)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h} \quad \text{and} \quad (D_2^- u)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{h}.$$

We further define the numerical Hamiltonian as

$$g(x_{i,j}, [D_h u]_{i,j}),$$

where g satisfies Assumption 14 - 18. Consistency of numerical Hamiltonian (Assumption 15) yields

$$\begin{aligned} H(x_{i,j}, \mathrm{D}u^n_{i,j}) &= H(x_{i,j}, [\partial_{x_1} u^n_{i,j}, \partial_{x_2} u^n_{i,j}]^\top) = g(x_{i,j}, [\partial_{x_1} u^n_{i,j}, \partial_{x_1} u^n_{i,j}, \partial_{x_2} u^n_{i,j}, \partial_{x_2} u^n_{i,j}]^\top) \\ &= g(x_{i,j}, [D_1^+ u^n_{i,j}, D_1^- u^n_{i,j}, D_2^+ u^n_{i,j}, D_2^- u^n_{i,j}]^\top + \mathbf{1}\mathcal{O}(h)) \\ &= g(x_{i,j}, [D_1^+ u^n_{i,j}, D_1^- u^n_{i,j}, D_2^+ u^n_{i,j}, D_2^- u^n_{i,j}]^\top) + \mathbf{1}\mathcal{O}(h), \end{aligned}$$

where $\mathbf{1}$ is the vector of ones. The last equality follows from Taylor expansion since g is $\mathcal{C}^1$ in q, by Assumption 16. We will now approximate $\mathrm{div}(m\nabla_p H(x, \mathrm{D}u))$. Multiplying with a smooth test function w, integrating by parts, and using periodicity gives

$$\int_{\mathbb{T}^2} \mathrm{div}(m\nabla_p H(x, \mathrm{D}u)) w \mathrm{d}x = -\int_{\mathbb{T}^2} m\mathrm{D}_p H(x, \mathrm{D}u) \cdot \mathrm{D}w \mathrm{d}x.$$

Without being too precise for a moment, we approximate the integral with

$$\begin{aligned} -\int_{\mathbb{T}^2} m\mathrm{D}_p H(x, \mathrm{D}u) \cdot \mathrm{D}w \mathrm{d}x &\approx -h^2 \sum_{i,j} (H_h m_{i,j}) \nabla_q g(x_{i,j}, [D_h \mathcal{H}_h u]_{i,j}) \cdot [D_h w]_{i,j} \\ &= h^2 \sum_{i,j} \mathcal{T}_{i,j}(\mathcal{H}_h u, \mathcal{H}_h m) w_{i,j}, \end{aligned} \tag{4.13}$$

where we defined the transport operator

$$\begin{aligned} \mathcal{T}_{i,j}(u, m) := \frac{1}{h}\Bigg(& m_{i,j} \frac{\partial g}{\partial q_1}(x_{i,j}, [D_h u]_{i,j}) - m_{i-1,j} \frac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h u]_{i-1,j}) \\ & + m_{i+1,j} \frac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h u]_{i+1,j}) - m_{i,j} \frac{\partial g}{\partial q_2}(x_{i,j}, [D_h u]_{i,j}) \\ & + m_{i,j} \frac{\partial g}{\partial q_3}(x_{i,j}, [D_h u]_{i,j}) - m_{i,j-1} \frac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h u]_{i,j-1}) \\ & + m_{i,j+1} \frac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h u]_{i,j+1}) - m_{i,j} \frac{\partial g}{\partial q_4}(x_{i,j}, [D_h u]_{i,j}) \Bigg). \end{aligned} \tag{4.14}$$

It's a simple matter to check that the second equality in (4.13) holds, by for instance collecting the terms involving $M_{i,j}$ for an arbitrary $(i, j) \in \mathcal{I}_h^2$ (see also e.g. [8, (2.12)]). Consistency of the first approximation in (4.13) follows from Assumption 12, as we assume

$$[\mathrm{div}(m\nabla_p H(x, \mathrm{D}u))]_{i,j} = \mathcal{T}_{i,j}(\mathcal{H}_h u, \mathcal{H}_h m) + \mathcal{O}(h^r)$$

for some $r > 0$. A similar assumption is also given in [8, (H5)]. We approximate $F[m](x)$, $G[m](x)$ by the functions

$$\begin{aligned} F_h &: \mathcal{P}_h \to C_b^2(\mathbb{T}^2) \\ G_h &: \mathcal{P}_h \to C_b^2(\mathbb{T}^2), \end{aligned}$$

satisfying Assumption 8-11. Let $F_h[M]_{i,j}$ mean $F_h[M](x_{i,j})$, and equally for G_h. We let the terminal condition be

$$M_T := \mathcal{H}_h m_T,$$

where $\mathcal{H}_h$ is defined in (4.7). We will use capital letters U, M for the approximated grid functions defined on $\mathbb{T}_h^2$, which equivalently can be viewed as vectors in $\mathbb{R}^{N_h^2}$. To summarize, we end up with the semi-implicit scheme, for $(i,j) \in \mathcal{I}_h^2$ and $n \in [0, N_T - 1]_{\mathbb{Z}}$.

$$\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i,j}^{n+1} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = F_h[M^n]_{i,j}, \tag{4.15}$$

$$\frac{M_{i,j}^{n+1} - M_{i,j}^n}{\Delta t} - \nu(-\Delta_h)^{\frac{\alpha}{2}} M_{i,j}^n + \mathcal{T}_{i,j}(U^{n+1}, M^n) = 0, \tag{4.16}$$

$$M_{i,j}^n \geq 0, \tag{4.17}$$

$$M^{N_T} = M_T \tag{4.18}$$

$$h^2(M^n, 1)_2 = 1, \ \forall n \in [0, N_T - 1] \tag{4.19}$$

$$U_{i,j}^0 = G_h[M^0]_{i,j}. \tag{4.20}$$

Next, we will first prove existence and uniqueness for the discrete HJB and FPK equations separately, before we do it for the coupled discrete MFG system. Then, we will prove convergence for the discrete MFG system.

Remark. Note that when an operator originally defined index-wise, as the ones defined above, is written without any indices, it should be interpreted as the vector containing all elements $(i,j) \in \mathcal{I}_h^2$. For instance, $\mathcal{T}(U, M) := (\mathcal{T}_{i,j}(U, M))_{(i,j) \in \mathcal{I}_h^2}$. Since $F_h[M], G_h[M]$ are functions, we will use $\mathbf{F}_h[M^n] := (F_h[M^n]_{i,j})_{(i,j) \in \mathcal{I}_h^2}$ and $\mathbf{G}_h[M^n] := (G_h[M^n]_{i,j})_{i,j}$.

Remark. One possible approximation of F_h, G_h is

$$\begin{aligned} F_h[M]_{i,j} &:= F[\mathcal{J}_h M](x_{i,j}), \\ G_h[M]_{i,j} &:= G[\mathcal{J}_h M](x_{i,j}), \end{aligned} \tag{4.21}$$

where $\mathcal{J}_h$ is the piecewise constant interpolation (4.6). It's cleat that $\mathcal{J}_h M \in \mathcal{P}(\mathbb{T}^2)$, $\forall M \in \mathcal{P}_h$. Strict monotonicity and uniform boundedness of F and G, both standard assumptions (see e.g. [10] or [1, Theorem 2.4]), immediately implies Assumption 8 and 9. See Lemma 43 in Appendix A for a proof of this.

4.3 Technical prerequisites

We will here state some useful lemmas we will use in the forthcoming proofs.

Lemma 18 (Degenerate ellipticity of numerical Hamiltonian and PDL)**.** *The discrete finite difference operators g and $(-\Delta_h)^{\frac{\alpha}{2}}$ are both degenerate elliptic operators for all $h > 0$.*

Proof. By the monotonicity of g (in particular, that $\mathbb{R}^4 \ni q \to g(x, q)$ is nonincreasing function in the first and third argument, and nondecreasing in the second and fourth, Assumption 14) and since

$$[D_h U]_{i,j} = [D_1^+ U_{i,j}, D_1^- U_{i,j}, D_2^+ U_{i,j}, D_2^- U_{i,j}]^\top,$$

we agree that g is a non-decreasing function in $\{U_p - U_q\}_{q\in N(p)}$ (the difference between the point indexed by $p=(i,j)$ and its neighbors). The same goes for the PDL,

$$(-\Delta_h)^{\frac{\alpha}{2}} U_{i,j} = \frac{1}{h^\alpha}\sum_{l,m}\big(U_{i,j} - U_{i+l,j+m}\big)K_\alpha(\beta),$$

since $K_\alpha \geq 0$. Therefore, g and $(-\Delta_h)^{\frac{\alpha}{2}}$ satisfies the definition of degenerate ellipticity, Definition 7, for all $h>0$. □

Lemma 19 (Stability with respect to the right-hand side of HJB)**.** *Let $a,b\geq 0$, and let grid functions U,V satisfy*

$$U_{i,j} - V_{i,j} + a(-\Delta_h)^{\frac{\alpha}{2}}(U_{i,j}-V_{i,j}) + b\big(g(x_{i,j},[D_hU]_{i,j}) - g(x_{i,j},[D_hV]_{i,j})\big) = B_{i,j},$$

for all $(i,j)\in\mathcal{I}_h^2$. Then,

$$\|U-V\|_\infty \leq \|B\|_\infty, \quad \forall h>0.$$

Proof. Assume for a moment that the maximum of $|U_{i,j}-V_{i,j}|$ is equal to the maximum of $U_{i,j}-V_{i,j}$, with maximizing indices (i_0,j_0). Then, since g and $(-\Delta_h)^{\frac{\alpha}{2}}$ are degenerate elliptic operators by Lemma 18, we have by Lemma 17 that

$$(-\Delta_h)^{\frac{\alpha}{2}}(U_{i_0,j_0}-V_{i_0,j_0}) \geq 0,$$
$$g(x_{i_0,j_0},[D_hU]_{i_0,j_0}) - g(x_{i_0,j_0},[D_hV]_{i_0,j_0}) \geq 0.$$

Hence,

$$\begin{aligned}\|U-V\|_\infty =& \ U_{i_0,j_0}-V_{i_0,j_0}\\ \leq& \ U_{i_0,j_0}-V_{i_0,j_0} + a(-\Delta_h)^{\frac{\alpha}{2}}(U_{i_0,j_0}-V_{i_0,j_0})\\ &+ b\big(g(x_{i_0,j_0},[D_hU]_{i_0,j_0}) - g(x_{i_0,j_0},[D_hV]_{i_0,j_0})\big)\\ =& \ B_{i_0,j_0}\\ \leq& \ \|B\|_\infty.\end{aligned}$$

If instead the maximum of $|U_{i,j}-V_{i,j}|$ is equal to the maximum of $V_{i,j}-U_{i,j}$, an identical argument can be made, by changing variables $U-V\to V-U$ and $B\to -B$. We made no assumptions on h, and it holds for all $h>0$.

□

Lemma 20 (Leray-Schauder fixed point theorem)**.** *Let X be an open, bounded set in $\mathbb{R}^n$ containing the origin and $G: X\subseteq\mathbb{R}^n\to\mathbb{R}^n$ a continuous mapping. If $G(x)\neq\lambda x$ whenever $\lambda>1$ and $x\in\partial X$, then G has a fixed point in the closure of X (denoted cl(X)).*

Proof. The proof is found in [34], pages 162-163. □

Lemma 21 (Brouwer fixed point theorem)**.** *Let X be a non-empty, compact and convex subset in $\mathbb{R}^d$, and let $G: X\to X$ be a continuous mapping. Then G has a fixed point in X.*

Proof. The proof is found in [35], page 7. □

Definition 8 (Diagonally dominant matrix)**.** A matrix $A \in \mathbb{R}^{d\times d}$ is (weakly) row diagonally dominant if

$$A_{i,i} \geq \sum_j |A_{i,j}|, \quad \forall i \in [d].$$

A is strictly row diagonally dominant if

$$A_{i,i} > \sum_j |A_{i,j}|, \quad \forall i \in [d].$$

For column diagonally dominance, let $(i, j) \to (j, i)$ and the definitions are equivalent.

Remark. When using diagonally dominant without any prefix, it should be interpreted as weakly row diagonally dominant.

Lemma 22 (Invertibility of strictly row diagonally dominant matrices)**.** *Let $A \in \mathbb{C}^{d\times d}$, and let*

$$|A_{i,i}| > \sum_{\substack{j\in[d]\\ j\neq i}} |A_{i,j}|, \quad \forall i \in [d].$$

Then, A is non-singular.

Proof. It can be proved by Gershgorin's circle theorem (Satz II in [36]), since all eigenvalues λ_i of A will be contained within circles centered at $A_{i,i} \in \mathbb{C}$ with radius $\sum_{\substack{j\in[d]\\ j\neq i}} |A_{i,j}|$ and thus can't be zero. □

The following definitions are found in pages 113-116 in [37].

Definition 9 (Z-matrix)**.** A matrix $A \in \mathbb{R}^{d\times d}$ is a Z-matrix if $A_{i,j} \leq 0, \ \forall i \neq j$.

Definition 10 (M-matrix)**.** Let $A \in \mathbb{R}^{d\times d}$ be a Z-matrix. If A has positive diagonal, and AD is strictly row diagonally dominant for some positive diagonal matrix D, then A is an M-matrix.

Equivalently, a Z-matrix $A \in \mathbb{R}^{d\times d}$ is an M-matrix if it can be expressed as

$$A = sI - P,$$

where $P_{i,j} \geq 0, \forall (i, j) \in [d]^2$, and where $\rho(P) < s$, where $\rho(P)$ is the spectral radius of P.

Lemma 23. *If $A \in \mathbb{R}^{d\times d}$ is an M-matrix, then so is its transpose $A^\top$.*

Proof. This is easily seen by the second definition of an M-matrix. Transposing a real matrix does not change its eigenvalues, and hence not its spectral radius, and the properties of an M-matrix is thus conserved under transposing. □

Lemma 24 (Invertibility of M-matrices)**.** *Every M-matrix $A \in \mathbb{R}^{d\times d}$ is nonsingular, and $A^{-1}_{i,j} \geq 0, \forall i, j \in [d]$.*

Proof. We show invertibility. If AD is strictly row diagonally dominant, then we can use Lemma 22 and conclude that $D^{-1}A^{-1}$ exists. Since D^{-1} exists, as it's a positive diagonal matrix, we must have that A^{-1} exists. Non-negativity of A^{-1} is proved in 2.5.3.17 in [37]. □

4.4 Existence and Uniqueness for the Discrete HJB equation

4.4.1 Existence and uniqueness of a general discrete HJB equation

In the following section, we will define and show existence and uniqueness of the discrete HJB equation. We begin by defining a general discrete HJB equation as

$$\rho U_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i,j} = B_{i,j}, \tag{4.22}$$

where $\rho > 0$ is a parameter, and $B_{i,j}$ is some grid function, and g is the numerical Hamiltonian which satisfies the assumptions A14-A18. We have removed the M dependence, to work with an uncoupled equation.

Lemma 25 (Existence and uniqueness of the discrete general HJB equation). *There exists a unique grid function U satisfying* (4.22).

Proof. The proof is essentially the same as for Lemma 1 in [7], generalized to hold for the PDL.

Existence

We will prove existence of (4.22) by using a Leray-Schauder fixed point argument (Lemma 20). Let's define a mapping $\Lambda : \mathbb{R}^{N_h^2} \to \mathbb{R}^{N_h^2}$, such that

$$(\Lambda(U))_{i,j} = \frac{1}{\rho}\left(-\nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i,j} - g(x_{i,j}, [D_h U]_{i,j}) + B_{i,j}\right). \tag{4.23}$$

Define further $r = \max_{(i,j)} |B_{i,j} - H(x_{i,j}, 0)|/\rho \geq 0$. $\Lambda(U)$ is a continuous function. To see why, note that g is a continuous function (Assumption 16) of $[D_h U]$, which in turn is a linear combination of components of U. Furthermore, the PDL is also simply a linear combination of components of U, and B is constant. Hence, it's also a continuous from the open and bounded set (which also contains the origin) $B_r = \{U \in \mathbb{R}^{N_h^2} : \|U\|_\infty < r\} \subset \mathbb{R}^{N_h^2}$ to $\mathbb{R}^{N_h^2}$.

Assume now that $U \in \partial B_r$. Then, there must exist at least one pair of indices (i_0, j_0) such that $U_{i_0,j_0} = \pm r$. Assume first that $U_{i_0,j_0} = r$. Note that we then have

$$-\nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i_0,j_0} - g(x_{i_0,j_0}, [D_h U]_{i_0,j_0}) \leq -H(x_{i_0,j_0}, 0). \tag{4.24}$$

To see why, note that $U \in \partial B_r$, meaning r is a non-negative maximum of U. By degenerate ellipticity of the PDL (Lemma 18), and letting $V = 0$ in Lemma 17, we see that $-\nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i_0,j_0} \leq 0$, as (i_0, j_0) attains a non-negative maximum value of U. By the same argument, we have that

$$g(x_{i_0,j_0}, [D_h U]_{i_0,j_0}) \geq g(x_{i_0,j_0}, 0) = H(x_{i_0,j_0}, 0) \tag{4.25}$$

where we in the first inequality used degenerate ellipticity of g (Lemma 18) with Lemma 17 (with $V = 0$), and in the second equality used consistency of g (Assumption 15). Multiplying (4.25) with -1 and adding $-\nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i_0,j_0} \leq 0$ to the left-hand side gives (4.24).

Adding B_{i_0,j_0} to (4.24) and dividing by ρ gives

$$\begin{aligned}(\Lambda(U))_{i_0,j_0} &= \frac{1}{\rho}\left(-\nu(-\Delta_h)^{\frac{\alpha}{2}}U_{i_0,j_0} - g(x_{i_0,j_0},[D_hU]_{i_0,j_0}) + B_{i_0,j_0}\right)\\ &\leq \frac{1}{\rho}\left(-H(x_{i_0,j_0},0) + B_{i_0,j_0}\right)\\ &\leq \frac{1}{\rho}\max_{(i,j)}|B_{i,j} - H(x_{i,j},0)|\\ &= r = U_{i_0,j_0}.\end{aligned} \tag{4.26}$$

Since $(\Lambda(U))_{i_0,j_0} \leq U_{i_0,j_0}$, we have that $(\Lambda(U))_{i_0,j_0} \neq \lambda U_{i_0,j_0}$ whenever $\lambda > 1$. Since $r \geq 0$, we either have $r = 0$ or $r > 0$. If $r = 0$, $U_{i_0,j_0} = r = -r$, and the previous argument holds. If we instead assume $U_{i_0,j_0} = -r$ and $r > 0$, then the degenerate elliptic argument for $-U$ holds, and we get

$$\begin{aligned}(\Lambda(U))_{i_0,j_0} &= \frac{1}{\rho}\left(-\nu(-\Delta_h)^{\frac{\alpha}{2}}U_{i_0,j_0} - g(x_{i_0,j_0},[D_hU]_{i_0,j_0}) + B_{i_0,j_0}\right)\\ &\geq \frac{1}{\rho}\left(-H(x_{i_0,j_0},0) + B_{i_0,j_0}\right)\\ &\geq -\frac{1}{\rho}\max_{(i,j)}|B_{i,j} - H(x_{i,j},0)|\\ &= -r.\end{aligned} \tag{4.27}$$

Assume ad absurdum that there exists a $\lambda > 1$ such that

$$(\Lambda(U))_{i_0,j_0} = \lambda U_{i_0,j_0}.$$

Then,

$$(\Lambda(U))_{i_0,j_0} = \lambda U_{i_0,j_0} = \lambda(-r) < -r,$$

contradicting (4.27). Hence, there must exist at least one pair of indices (i_0, j_0) such that $(\Lambda(U))_{i_0,j_0} \neq \lambda U_{i_0,j_0}$ whenever $\lambda > 1$. Therefore, $\Lambda(U) \neq \lambda U$ whenever $\lambda > 1$ and $U \in \partial B_r$. Leray-Schauder fixed point theorem (Lemma 20) can be used. There exists indeed a solution of (4.22) in $\mathrm{cl}(B_r)$.

Uniqueness

Uniqueness follows from degenerate ellipticity. Let U and $\widetilde{U}$ solve (4.22). Subtracting their equations yields

$$\rho(U_{i,j} - \widetilde{U}_{i,j}) + g(x_{i,j},[D_hU]_{i,j}) - g(x_{i,j},[D_h\widetilde{U}]_{i,j}) + \nu(-\Delta_h)^{\frac{\alpha}{2}}(U - \widetilde{U})_{i,j} = 0.$$

Using Lemma 19 with $a = b = \frac{1}{\rho}$, we get

$$\|U - \widetilde{U}\|_\infty = 0.$$

□

4.4.2 A solution mapping for the discrete time-dependent HJB equation

We have now shown existence and uniqueness for a discrete time-independent HJB equation. Next, we will use this result to show existence and uniqueness for (4.15), explicitly given by

$$\begin{aligned}\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{\Delta t}+\nu(-\Delta_h)^{\frac{\alpha}{2}}U_{i,j}^{n+1}+g(x_{i,j},[D_hU^{n+1}]_{i,j}) &= F_h[M^n]_{i,j}, \quad \forall n\in[0,N_T-1]_{\mathbb{Z}},\\ U^0 &= G_h(M^0)\end{aligned} \tag{4.28}$$

assuming now that $(M^n)_{n=0}^{N_T-1}$ is given.

Theorem 26 (Existence and uniqueness of discrete HJB equation)**.** *The discrete HJB equation* (4.28) *has a unique solution.*

Proof. We will for each time step $n+1\in[N_T]$ use Lemma 25 with $\rho=\frac{1}{\Delta t}$ and $B_{i,j}=U_{i,j}^n/\Delta t+F_h[M^n]_{i,j}$, since U^n is given from the previous time step. This ensures existence and uniqueness for $(U^n)_{n=0}^{N_T}$. □

Definition 11 (Solution mapping of the discrete HJB equations)**.** We define the solution mapping $\Phi_U:\mathcal{P}_h^{N_T}\to\mathbb{R}^{N_h^2N_T}$

$$\Phi_U((M^n)_{n=0}^{N_T-1})=(U^n)_{n=1}^{N_T}, \quad \text{such that (4.28).} \tag{4.29}$$

Note, we omit including U^0 in the solution, as it is known given M^0 from (4.20).

Lemma 27 (Boundedness and continuity of Φ_U)**.** *Assume Φ_U is defined as in* (4.29)*. Then, there exists $h_0,\Delta t>0$ such that Φ_U is a bounded and continuous mapping for all $h<h_0$ and $\Delta t<\Delta t_0$.*

Proof. We begin by showing boundedness, before we show continuity.

Boundedness

Returning to the function Λ we defined in (4.23) for the general discrete HJB equation, we found that there exist a fixed point $U=\Lambda(U)$ for a $U\in\mathrm{cl}(B_r)$. Hence,

$$\|U\|_\infty\le r=\frac{1}{\rho}\max_{(i,j)}|B_{i,j}-H(x_{i,j},0)|.$$

Inserting the ρ and $B_{i,j}$ we used in the proof of Theorem 26, we get

$$\|U^{n+1}\|_\infty\le\max_{(i,j)}\left|\Delta t\ (F_h[M^n]_{i,j}-H(x_{i,j},0))+U_{i,j}^n\right|.$$

We have that $F_h[M]$ is uniformly bounded by C_F on $\mathbb{T}_h^2$ independent of $M\in\mathcal{P}_h$ (Assumption 1). We also have that $H(\cdot,0)$ is bounded by C_R on the torus (Assumption 3), yielding

$$\|U^{n+1}\|_\infty\le\Delta t(C_F+C_R)+\|U^n\|_\infty.$$

Using the discrete Gronwall lemma (Lemma 44), there exists a constant C depending on C_F, C_G, C_R and $\|U^0\|_\infty$ such that

$$\|U^n\|_\infty \leq C(1+T), \quad \forall n \ \in [N_T]. \tag{4.30}$$

Finally, from Assumption 1, $G_h[M^0]$ is uniformly bounded independent of M^0 by C_G. It hence follows that $\|U^0\|_\infty \leq C_G$. Therefore, C depend only on C_F, C_G and C_R. We made no assumptions on $h, \Delta t$, and it certainly holds as $h, \Delta t \to 0$. Therefore, Φ_U maps $\mathcal{P}_h^{N_T}$ to a bounded subset of $\mathbb{R}^{N_h^2 N_T}$.

Continuity

Since Φ_U is a mapping between subspaces of $\mathbb{R}^{N_h^2 N_T}$, we need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|(M^n)_{n=0}^{N_T-1} - (\widetilde{M}^n)_{n=0}^{N_T-1}\|_\infty < \delta \implies \|\Phi_U((M^n)_{n=0}^{N_T-1}) - \Phi_U((\widetilde{M}^n)_{n=0}^{N_T-1})\|_\infty < \epsilon.$$

Now, we have that since $M \to \mathbf{F}_h[M] \in \mathbb{R}^{N_h^2}$, $M \to \mathbf{G}_h[M] \in \mathbb{R}^{N_h^2}$ are continuous mappings (Assumption 8), it's enough to show that

$$\begin{aligned}\|\Phi_U((M^n)_{n=0}^{N_T-1}) - \Phi_U((\widetilde{M}^n)_{n=0}^{N_T-1})\|_\infty \leq C\big(&\|(\mathbf{F}_h[M^n])_{n=0}^{N_T-1} - (\mathbf{F}_h[\widetilde{M}^n])_{n=0}^{N_T-1}\|_\infty \\ &+\|\mathbf{G}_h[M^0] - \mathbf{G}_h[\widetilde{M}^0]\|_\infty\big)\end{aligned} \tag{4.31}$$

for a constant $C > 0$. We will show that (4.31) holds. Consider the two equations

$$\begin{aligned}\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i,j}^{n+1} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) &= F_h[M^n]_{i,j} \\ \frac{\widetilde{U}_{i,j}^{n+1} - \widetilde{U}_{i,j}^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}} \widetilde{U}_{i,j}^{n+1} + g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j}) &= (F_h[\widetilde{M}^n])_{i,j}.\end{aligned}$$

Subtracting them yields

$$\begin{aligned}&U_{i,j}^{n+1} - \widetilde{U}_{i,j}^{n+1} + \Delta t\Bigg(\nu(-\Delta_h)^{\frac{\alpha}{2}} U_{i,j}^{n+1} - \nu(-\Delta_h)^{\frac{\alpha}{2}} \widetilde{U}_{i,j}^{n+1} \\ &+g(x_{i,j}, [D_h U^{n+1}]_{i,j}) - g(x_{i,j}, [D_h \widetilde{U}^{n+1}]_{i,j})\Bigg) = U_{i,j}^n - \widetilde{U}_{i,j}^n + \Delta t\Bigg(F_h[M^n]_{i,j} - (F_h[\widetilde{M}^n])_{i,j}\Bigg)\end{aligned}$$

Using stability with respect to the right-hand side (Lemma 19),

$$\|U^{n+1} - \widetilde{U}^{n+1}\|_\infty \leq \|U^n - \widetilde{U}^n\|_\infty + \Delta t\|\mathbf{F}_h[M^n] - \mathbf{F}_h[\widetilde{M}^n]\|_\infty$$

which implies

$$\begin{aligned}\|(U^n)_{n=1}^{N_T} - (\widetilde{U}^n)_{n=1}^{N_T}\|_\infty &\leq \sum_{k=1}^{N_T} \Delta t\|\mathbf{F}_h[M^k] - \mathbf{F}_h[\widetilde{M}^k]\|_\infty + \|\mathbf{G}_h[M^0] - \mathbf{G}_h[\widetilde{M}^0]\|_\infty \\ &\leq C\big(\|(\mathbf{F}_h[M^n])_{n=0}^{N_T-1} - (\mathbf{F}_h[\widetilde{M}^n])_{n=0}^{N_T-1}\|_\infty + \|\mathbf{G}_h[M^0] - \mathbf{G}_h[\widetilde{M}^0]\|_\infty\big),\end{aligned}$$

where $C > 0$ is a constant. Since both $\mathbf{F}_h$ and $\mathbf{G}_h$ are continuous uniformly as $h, \Delta t \to 0$ by Assumption 8, it follows that Φ_U also is continuous. Hence, Φ_U is a bounded and continuous mapping from $\mathcal{P}_h^{N_T}$ to $\mathbb{R}^{N_h^2 N_T}$ as $h, \Delta t \to 0$. □

Next, we want to show Lipschitz continuity of Φ_U. We begin by defining the solution map of the discrete HJB equation for each time-step.

Definition 12. The solution of (4.28) given M^n and U^n is defined as

$$U^{n+1} := \Psi(U^n, M^n). \tag{4.32}$$

Before we proceed, we need to show non-expansivity of Ψ, which we do in the following lemma.

Lemma 28 (Non-expansivity of the HJB scheme)**.** *Let Ψ be a defined as* (4.32)*. For all $M \in \mathcal{P}_h$, $U, W \in \mathbb{R}^{N_h^2}$,*

$$\|\Psi(U,M) - \Psi(W,M)\|_\infty \leq \|U - W\|_\infty \tag{4.33}$$

Proof. Subtracting (4.28) substituted with U and W yields

$$\begin{aligned} U_{i,j}^n - W_{i,j}^n = U_{i,j}^{n+1} - W_{i,j}^{n+1} + \Delta t\big(&\nu(-\Delta_h)^{\frac{\alpha}{2}}(U^{n+1} - W^{n+1})_{i,j} \\ &+ g(x_{i,j}, [D_h U^{n+1}]_{i,j}) - g(x_{i,j}, [D_h W^{n+1}]_{i,j})\big). \end{aligned}$$

We then use Lemma 19 with $B_{i,j} = U_{i,j}^n - W_{i,j}^n$. □

Lemma 29 (Discrete Lipschitz estimate of U)**.** *There exists a constant L independent of h and Δt such that*

$$\sup_{n \in [N_T]} \|D_h U^n\|_\infty \leq L \tag{4.34}$$

Proof. The following proof is also given in [7]. For $(\ell, m) \in \mathbb{Z}^2$, call $\tau_{\ell,m}U$ the shifting function defined by

$$(\tau_{\ell,m}U)_{i,j} = U_{\ell+i,m+j}.$$

It is easy to verify that

$$\begin{aligned} &\frac{(\tau_{\ell,m}U)_{i,j}^{n+1} - (\tau_{\ell,m}U)_{i,j}^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}}(\tau_{\ell,m}U^{n+1})_{i,j} + g(x_{i,j}, [D_h(\tau_{\ell,m}U^{n+1})]_{i,j}) \\ =&F_h[M^n]_{i,j} + F_h[M^n]_{i+\ell,j+m} - F_h[M^n]_{i,j} \\ &- g(x_{i+\ell,j+m}, [D_h(\tau_{\ell,m}U^{n+1})]_{i,j}) + g(x_{i,j}, [D_h(\tau_{\ell,m}U^{n+1})]_{i,j}), \end{aligned}$$

and therefore

$$\tau_{\ell,m}U^{n+1} = \Psi(\tau_{\ell,m}U^n + \Delta t E, M^n),$$

where

$$\begin{aligned} E_{i,j} :=&F_h[M^n]_{i+\ell,j+m} - (F_h[M^{n+1}])_{i,j} \\ &- g(x_{i+\ell,j+m}, [D_h(\tau_{\ell,m}U^{n+1})]_{i,j}) + g(x_{i,j}, [D_h(\tau_{\ell,m}U^{n+1})]_{i,j}). \end{aligned}$$

Since $F_h[M]$ is uniformly $\mathcal{C}_b^2$-bounded with respect to M (Assumption 8), it follows that $F_h[M^n]_{i+\ell,j+m}^n - (F_h[M^{n+1}])_{i,j}^{n+1} \leq C_1 h\sqrt{\ell^2 + m^2}$ for a $C_1 > 0$ depending on the $\mathcal{C}_b^2$-bound C_F. Since also $\partial_x g$ is sublinear in q (Assumption 18), we have that

$$\begin{aligned} \|E\|_\infty &\leq C_1 h\sqrt{\ell^2 + m^2} + C_2 h\sqrt{\ell^2 + m^2}\sup_{i,j}|\partial_x g(x_{i,j}, [D_h(\tau_{\ell,m}U^{n+1})]_{i,j})| \\ &\leq C_1 h\sqrt{\ell^2 + m^2} + C_2 h\sqrt{\ell^2 + m^2}C_g(1 + \underbrace{\max\{\|D_1^+U\|_\infty, \|D_2^+U\|_\infty\}}_{=\|D_h U^{n+1}\|_\infty}) \\ &\leq C(1 + \|D_h U^{n+1}\|_\infty)h\sqrt{\ell^2 + m^2}, \end{aligned}$$

where C is a constant depending on C_F and C_g, and where $h\sqrt{\ell^2+m^2}$ is the (maximum, given periodicity) distance between the grid points indexed by (i,j) and $(i+l, j+m)$. From (4.33), we get that

$$\begin{aligned}\|\tau_{\ell,m}U^{n+1} - U^{n+1}\|_\infty &= \|\Psi(\tau_{\ell,m}U^n + \Delta t E, M^n) - \Psi(U^n, M^n)\|_\infty \\ &\leq \|\tau_{\ell,m}U^n + \Delta t E - U^n\|_\infty \\ &\leq \|\tau_{\ell,m}U^n - U^n\|_\infty + Ch\Delta t\sqrt{\ell^2+m^2}(1+\|D_hU^{n+1}\|_\infty),\end{aligned}$$

where we used the triangle inequality in the last inequality. As this result holds for arbitrary (l,m), we know that it holds for $(l,m)\in\{(1,0),(0,1)\}$, where, for instance,

$$\|\tau_{1,0}U^{n+1} - U^{n+1}\|_\infty = h\|D_1^+U^{n+1}\|_\infty.$$

It follows that

$$\|D_hU^{n+1}\|_\infty h \leq \|D_hU^n\|_\infty h + Ch\Delta t(1+\|D_hU^{n+1}\|_\infty),$$

and hence

$$(1-C\Delta t)\|D_hU^{n+1}\|_\infty \leq \|D_hU^n\|_\infty + C\Delta t.$$

The discrete version of Gronwall's lemma (Lemma 44) yields that for all n,

$$\begin{aligned}\|D_hU^{n+1}\|_\infty &\leq \frac{1}{(1-C\Delta t)^n}\left(\|D_hU^0\|_\infty + \frac{C\Delta t}{1-C\Delta t}\sum_{j=1}^{n}\frac{1}{(1-C\Delta t)^{-j}}\right) \\ &\leq \frac{1}{(1-C\Delta t)^{N_T}}\left(\|D_hU^0\|_\infty + \frac{C\Delta t}{1-C\Delta t}\sum_{j=1}^{N_T}\frac{1}{(1-C\Delta t)^{-j}}\right) \\ &\leq L,\end{aligned}$$

where L is a constant depend on T, C_g (Assumption 18), C_F, and C_G defined in Assumption 8. The last constant follows from the initial condition $\|D_hU^0\|_\infty$. Since $U^0 = \mathbf{G}_h[M^0]$, and G_h is $\mathcal{C}_b^2$-bounded uniformly in M by Assumption 8. □

The following corollary can be used to show regularity of the transport operator $\mathcal{T}_{i,j}$, which we demonstrate in Lemma 46 in Appendix A.

Corollary 1 (Uniform Lipschitzness of g in q)**.** From Lemma 29, $\sup_n\|D_hU^n\|$ is bounded by a constant only dependent on T, C_g, C_F and C_G. Therefore, since $g(x,[D_hU])$ is $\mathcal{C}^1(\mathbb{T}^2\times\mathbb{R}^4)$, we have that

$$\max_{\ell\in[4]}\sup_{(i,j)\in\mathcal{I}_h^2}\left|\frac{\partial g}{\partial q_\ell}(x_{i,j},[D_hU^n]_{i,j})\right| \leq C, \quad \forall n\in[N_T], \tag{4.35}$$

where C depends on T, C_g, C_F and C_G.

4.5 Existence and Uniqueness for the Discrete FPK Equation

We will now give existence and uniqueness for the discrete FPK equation (4.16), explicitly given by

$$
\begin{aligned}
\frac{M_{i,j}^{n+1}-M_{i,j}^{n}}{\Delta t} - \nu(-\Delta_h)^{\frac{\alpha}{2}} M_{i,j}^n + \mathcal{T}_{i,j}(U^{n+1}, M^n) &= 0, \quad \forall n \in [0, N_T-1]_{\mathbb{Z}} \\
M_{i,j}^n &\geq 0, \\
M^{N_T} &= m_T \in \mathcal{P}_h \\
h^2(M^n, 1)_2 &= 1, \ \forall n \in [0, N_T-1],
\end{aligned} \tag{4.36}
$$

where we now assume $(U^n)_{n=1}^{N_T} \in \mathbb{R}^{N_h^2 N_T}$ given. Taking a look at the transport operator (4.14), we notice that it is a linear operator on the vector $M^n \in \mathbb{R}^{N_h^2}$. We can rewrite (4.36) more compactly by multiplying with -1 and rearranging,

$$
M^n + \Delta t A^n M^n = M^{n+1}, \quad \forall n \in [0, N_h - 1]_{\mathbb{Z}}. \tag{4.37}
$$

$A^n \in \mathbb{R}^{N_h^2 \times N_h^2}$ is a linear operator dependent on U^{n+1} satisfying

$$
(A^n M)_{i,j} = \nu(-\Delta_h)^{\frac{\alpha}{2}} M_{i,j} - \mathcal{T}_{i,j}(U^{n+1}, M), \quad \forall (i,j) \in \mathcal{I}_h^2. \tag{4.38}
$$

We will prove existence and uniqueness of (4.36) by showing that the linear system (4.37) has a unique solution.

Lemma 30 (Uniform boundedness of $\mathcal{T}$)**.** *For any $h > 0$, any $(M)_{n=0}^{N_T-1} \in \mathcal{P}_h^{N_T}$ and any $(U)_{n=1}^{N_T} \in \mathbb{R}^{N_h^2 N_T}$,*

$$
\sup_{i,j} \mathcal{T}_{i,j}(U^{n+1}, M^n) < +\infty, \quad \forall n \in [0, N_T - 1].
$$

Proof. $\mathcal{T}_{i,j}$ is a linear operator in M^n, which is finite, and $\partial_q g(x_{i,j}, [D_h U^n]_{i,j})$. $\partial_q g(x_{i,j}, [D_h U^n]_{i,j})$ is continuous in the last argument, and $[D_h U^n]$ is finite for every $(U)_{n=1}^{N_T} \in \mathbb{R}^{N_h^2 N_T}$ and $h > 0$. The lemma follows. □

Lemma 31. *Let $A^n \in \mathbb{R}^{N_h^2 \times N_h^2}$ be defined as in* (4.38)*. Then, the adjoint A_*^n is a weakly diagonally dominant Z-matrix with positive diagonal.*

Proof. First, A^n is a well-defined linear operator by Lemma 30 and Lemma 14. The adjoint of A^n is the operator A_*^n satisfying

$$
(A_*^n V)_{i,j} = \nu(-\Delta_h)^{\frac{\alpha}{2}} V_{i,j} + \nabla_q g(x_{i,j}, [D_h U^{n+1}]_{i,j}) \cdot [D_h V]_{i,j}, \quad \forall (i,j) \in \mathcal{I}_h^2.
$$

From Lemma 16, the PDL is self-adjoint, so we only need to show that the adjoint of $M \to -\mathcal{T}(U, M) \in \mathbb{R}^{N_h^2}$ is $M \to \nabla_q g(x, [D_h U]) \cdot [D_h M]$. By definition, the adjoint of a matrix A is the matrix A^* such that

$$
(AM, V)_2 = (M, A^* V)_2.
$$

In this case, we would need to show that

$$
-\sum_{i,j} \mathcal{T}_{i,j}(U, M) V_{i,j} = \sum_{i,j} M_{i,j} \big(\nabla_q g(x_{i,j}, [D_h U]_{i,j}) \cdot [D_h V]_{i,j} \big),
$$

which follows directly from the definition of the transport operator, see the last equality in (4.13). In fact, the adjoint A_*^n is the linearization of the discrete HJB equation (see Remark 1 in [7]).

Writing out the entry $(A_*^n V)_{i,j}$ more explicitly gives

$$(A_*^n V)_{i,j} = \frac{\nu}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} (V_{i,j} - V_{(i,j)+\beta}) K_\alpha(\beta) + \sum_{k=1}^{4} \partial_{q_k} g(x_{i,j}, [D_h U^{n+1}]_{i,j}) \Big([D_h V]_{i,j} \Big)_k, \tag{4.39}$$

$\forall (i,j) \in \mathcal{I}_h^2$, where we recall that

$$\begin{aligned} [D_h V]_{i,j} &= \Big[(D_1^+ V)_{i,j}, (D_1^- V)_{i,j}, (D_2^+ V)_{i,j}, (D_2^- V)_{i,j} \Big]^\top \\ &= \frac{1}{h} \Big[V_{i+1,j} - V_{i,j}, V_{i,j} - V_{i-1,j}, V_{i,j+1} - V_{i,j}, V_{i,j} - V_{i,j-1} \Big]^\top. \end{aligned}$$

The diagonal elements of any matrix is the coefficients operating linearly on the entry on the same index in the input as the output. Thus, the diagonal elements of A_*^n is the coefficients operating on $V_{i,j}$ in (4.39), and the off-diagonal elements are the remaining coefficients. Combining all coefficients working on $V_{i,j}$, we get

$$\frac{\nu}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\beta) + \frac{1}{h} \sum_{k=1}^{4} (-1)^k \partial_{q_k} g(x_{i,j}, [D_h U^{n+1}]_{i,j}) \tag{4.40}$$

which is the diagonal element at to row $p(i,j) = N_h i + j + 1 \in [N_h^2]$ (using the global indexing mentioned in the introduction of the section) in A_*^n. Notice by the monotonicity of g (Assumption 14), all the four terms in the last summation of (4.40) is non-negative, meaning we can write

$$\frac{\nu}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\beta) + \frac{1}{h} \sum_{k=1}^{4} |\partial_{q_k} g(x_{i,j}, [D_h U^{n+1}]_{i,j})|.$$

We also have $K_\alpha \geq 0$ so all terms are non-negative. Now, consider the off-diagonal elements at row p. This would be the remaining coefficients operating on $V_{p'}$, $p' \neq p$. Summing up the off-diagonal elements yields

$$-\frac{\nu}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\beta) + \frac{1}{h} \sum_{k=1}^{4} (-1)^{k+1} \partial_{q_k} g(x_{i,j}, [D_h U^{n+1}]_{i,j}). \tag{4.41}$$

Notice by the same arguments as above that all off-diagonal terms are non-positive, by the monotonicity of g. Defining the neighborhood of the index pair $(i,j) \in \mathcal{I}_h^2$ as $N(i,j) = \{(i+1,j), (i-1,j), (i,j+1), (i,j-1)\}$, we can write out the sum of the absolute value of the off-diagonal terms,

$$\begin{aligned} &\frac{\nu}{h^\alpha} \sum_{\substack{\beta \in \mathcal{I}_h^2 \\ \beta \notin N(i,j)}} |K_\alpha(\beta)| + \sum_{k=1}^{4} \Big| - \frac{\nu}{h^\alpha} K_\alpha(1) - \frac{1}{h} |\partial_{q_k} g(x_{i,j}, [D_h U^{n+1}]_{i,j})| \Big| \\ =&\frac{\nu}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\beta) + \frac{1}{h} \sum_{k=1}^{4} |\partial_{q_k} g(x_{i,j}, [D_h U^{n+1}]_{i,j})|. \end{aligned}$$

Which is the same quantity as the diagonal element at row $p(i,j)$. Hence, the sum of the absolute value of the off-diagonal elements in A_*^n is equal to the diagonal element. It follows that A_*^n is a weakly diagonally dominant Z-matrix with positive diagonal. Positivity of diagonal follows from that $K_\alpha > 0$. □

Lemma 32. *Let $A^n \in \mathbb{R}^{N_h^2 \times N_h^2}$ be defined as in (4.38). Then, $I_{N_h^2} + \Delta t A^n$ is an M-matrix for all $\Delta t > 0$*

Proof. By Lemma 31, A_*^n is a weakly diagonally dominant Z-matrix. Since the diagonal is positive, $I_{N_h^2} + \Delta t A_*^n$, creates a strictly diagonally dominant matrix for all $\Delta t > 0$. The diagonal elements are positive, and all off-diagonal elements are non-positive, and the matrix is thus a Z-matrix. Since $(I_{N_h^2} + \Delta t A_*^n)$ is a strictly diagonally dominant Z-matrix, it's also an M-matrix by Definition 10 (using $D = I_{N_h^2}$). Since M-matrices are closed under transposing by Lemma 23, we conclude that $I_{N_h^2} + \Delta t A^n$ is an M-matrix for all $\Delta t > 0$. □

Theorem 33. *There exists a unique solution $(M)_{n=0}^{N_T} \in \mathcal{P}_h^{N_T - 1}$ of (4.36).*

Proof. From Lemma 32, and the Lemma 24 existence and uniqueness of M^n follows, given M^{n+1}. As we are given a terminal condition m_T, there exists a unique solution $(M^n)_{n=0}^{N_T-1}$ of (4.36). Furthermore, non-negativity of M^n given non-negativity of M^{n+1} follows also immediately from Lemma 24. Finally, we would like to show that

$$h^2(M^n, 1)_2 = 1, \ \forall n \in [0, N_T - 1]$$

Consider, for any two grid functions W and Z,

$$\begin{aligned}(A^n W, Z)_2 &= \nu \sum_{i,j} ((-\Delta_h)^{\frac{\alpha}{2}} W)_{i,j} Z_{i,j} - \sum_{i,j} \mathcal{T}_{i,j}(U^{n+1}, W) Z_{i,j} \\ &= \nu \sum_{i,j} W_{i,j} ((-\Delta_h)^{\frac{\alpha}{2}} Z)_{i,j} + \sum_{i,j} W_{i,j} [D_h Z]_{i,j} \nabla_q g(x_{i,j}, [D_h U]_{i,j}),\end{aligned}$$

for any n, where we used self-adjointness of the PDL (Theorem 16), and the definition of the transport operator. We observe that $(A^n W, 1)_2 = 0$ for all grid functions W. Hence, we get by taking the inner product of (4.37) with 1,

$$\begin{aligned} 0 &= \left(M^n + \Delta t A^n M^n - M^{n+1}, 1 \right)_2 \\ 0 &= \left(M^n - M^{n+1}, 1 \right)_2 \\ \left(M^n, 1 \right)_2 &= \left(M^{n+1}, 1 \right)_2, \end{aligned} \tag{4.42}$$

which means that given $M^{N_T} \in \mathcal{P}_h$, inductively,

$$h^2(M^n, 1)_2 = 1, \ \forall n \in [0, N_T - 1].$$

This shows that there exists a unique $(M)_{n=0}^{N_T} \in \mathcal{P}_h^{N_T+1}$ of (4.36). □

As we did for the discrete HJB equations, we define a solution mapping.

Definition 13 (Solution mapping of the discrete FPK equation)**.** Define the solution mapping $\Phi_M : \mathbb{R}^{N_h^2 N_T} \to \mathcal{P}_h^{N_T}$ as

$$\Phi_M((U^n)_{n=1}^{N_h}) := (M^n)_{n=0}^{N_T-1}, \quad \text{such that (4.36).} \tag{4.43}$$

We will use this definition in the next part, where we will show existence for the discrete coupled MFG system.

Lemma 34 (Continuity of Φ_M)**.** *Let $\Phi_M : \mathbb{R}^{N_h^2 N_T} \to \mathcal{P}_h^{N_T}$ be defined as in* (4.43)*. Then Φ_M is continuous.*

Proof. As the system is linear, it's enough to verify that $\mathcal{T}(U, M)$ is a continuous function of U to show continuity of Φ_M. Since g is $\mathcal{C}^1(\mathbb{T}^2 \times \mathbb{R}^4)$ (Assumption 16), $\partial_{q_k} g$ is continuous for all $k \in \{1, 2, 3, 4\}$. Since $U \to [D_h U]$ is a linear and hence continuous operator for all $h > 0$, $\partial_{q_k} g(x_{i,j}, [D_h U]_{i,j})$ is continuous in U. We have shown that Φ_M is continuous. □

4.6 Existence of the discrete MFG system

$$\begin{aligned}
\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{\Delta t}+\nu(-\Delta_h)^{\frac{\alpha}{2}}U_{i,j}^{n+1}+g(x_{i,j},[D_hU^{n+1}]_{i,j}) &= F_h[M^n]_{i,j},\\
\frac{M_{i,j}^{n+1}-M_{i,j}^{n}}{\Delta t}-\nu(-\Delta_h)^{\frac{\alpha}{2}}M_{i,j}^{n}+\mathcal{T}_{i,j}(U^{n+1},M^n) &= 0,\\
M_{i,j}^n &\geq 0,\\
h^2(M^n,1)_2 &= 1,\ \forall n\in[0,N_T-1]_{\mathbb{Z}}\\
M^{N_T}=M_T,\quad h^2(m_T,1)_2 &= 1\\
U_{i,j}^0 &= (G_h(M^0))_{i,j}.
\end{aligned}$$

Theorem 35 (Existence of the discrete MFG system)**.** *If $h>0, M_T\in\mathcal{P}_h$, then (4.15)-(4.20) has a solution $(U,M)_{n=0}^{N_T}$. Furthermore, there exists an $h_0>0$ such that for all $h<h_0$,*

$$\sup_n\|U^n\|_{L^\infty(\mathbb{T}_h^2)}+\sup_n\|[D_hU^n]\|_{L^\infty(\mathbb{T}_h^2)}\leq C$$

for a $C>0$ depending only on T, C_g, C_R, C_F and C_G.

Proof. The proof strategy is to use a Brouwer fixed point argument, as done in [7], but with a different mapping. We are going to create a bounded continuous mapping from $\mathcal{P}_h^{N_T}$ to itself, and use a Brouwer fixed point argument (Lemma 21) to show that there exists a solution to (4.15)-(4.20). Note that

$$\mathcal{P}_h^{N_T}=\left\{M\in\mathbb{R}^{N_h^2}: h^2\sum_{i,j}M_{i,j}=1, M_{i,j}\geq 0,\ \forall(i,j)\in\mathcal{I}_h^2\right\}^{N_T}$$

is a compact and convex subset of $\mathbb{R}^{N_h^2N_T}$. Compactness follows from the Heine-Borel theorem [38], since for any fixed $h>0$, it is a closed and bounded subset of a Euclidean space. The latter seen by

$$\max_{(i,j),(\ell,m)}|M_{i,j}-M_{\ell,m}|<h^{-2}+1.$$

Convexity follows from that for any two $M,M'\in P_h$, and any $\lambda\in[0,1]$,

$$\lambda M+(1-\lambda)M'\in\mathcal{P}_h,$$

and therefore also holds for all $(M)_{n=0}^{N_T},(M')_{n=0}^{N_T}\in P_h$. Define $\chi:\mathcal{P}_h^{N_T}\to\mathcal{P}_h^{N_T}$ as

$$\chi:=\Phi_M\circ\Phi_U, \tag{4.44}$$

where Φ_U,Φ_M is defined in (4.29) and (4.43) respectively. Since we have already shown boundedness and continuity of Φ_U and continuity of Φ_M from Lemma 27 and Lemma 34 respectively. A composition of two continuous mappings is also continuous, and hence χ is continuous. Therefore, we can apply Lemma 21 and obtain that χ has a fixed point.

Boundedness and uniform Lipschitzness of $(U)_{n=0}^{N_T}$ follows directly from Lemma 27 and Lemma 29. □

Remark. Since $(M)_{n=0}^{N_T}\in\mathcal{P}_h^{N_T}$, M^n is bounded in $L^1(\mathbb{T}_h^2)$ for all n as $h\to 0$. However, we have not shown an $L^\infty(\mathbb{T}_h^2)$-bound uniformly for all $h<0$. One can, using some assumptions on Δt and h show an L^∞-bound uniformly as $h,\Delta t\to 0$. See Lemma 46 in Appendix A for more details.

4.7 Uniqueness of the discrete MFG system

Theorem 36 (Uniqueness of the discrete MFG system)**.** *Let Assumption 9 hold. Then* (4.15)-(4.20) *has a unique solution* (U, M).

Proof. The following uniqueness proof is also a trivial generalization of the uniqueness proof in [7]. Let $(U^n, M^n)_{n=0}^{N_T}$ and $(\widetilde{U}^n, \widetilde{M}^n)_{n=0}^{N_T}$ be two solutions of (4.15)-(4.20). We will now perform a cross-multiplication argument. Subtract the two HJB equations (4.15) satisfied by the solution pairs $(U^n_{i,j}, M^n_{i,j})$ and $(\widetilde{U}^n_{i,j}, \widetilde{M}^n_{i,j})$, and multiply with $M^n_{i,j} - \widetilde{M}^n_{i,j}$. Then, sum over all $n \in [0, N_T - 1]_{\mathbb{Z}}$ and all (i, j). We obtain

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{n=0}^{N_T-1}\Big((U^{n+1}-U^n)-(\widetilde{U}^{n+1}-\widetilde{U}^n), M^n-\widetilde{M}^n\Big)_2\\
&+\sum_{n=0}^{N_T-1}\sum_{i,j}\Big(g(x_{i,j},[D_hU^{n+1}]_{i,j})-g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j})\Big)(M^n_{i,j}-\widetilde{M}^n_{i,j})\\
&+\nu\sum_{n=0}^{N_T-1}\big((-\Delta_h)^{\frac{\alpha}{2}}(U^{n+1}-\widetilde{U}^{n+1}), M^n-\widetilde{M}^n\big)_2=\sum_{n=0}^{N_T-1}\Big(\mathbf{F}_h[M^n]-\mathbf{F}_h[\widetilde{M}^n], M^n-\widetilde{M}^n\Big)_2.
\end{aligned}
\tag{4.45}
$$

Then, we perform a similar operation on (4.16), where we subtract the equations for the two solution pairs $(U^n_{i,j}, M^n_{i,j})$ and $(\widetilde{U}^n_{i,j}, \widetilde{M}^n_{i,j})$, multiply with $U^{n+1}_{i,j} - \widetilde{U}^{n+1}_{i,j}$ before summing over all i, j, n. This yields

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{n=0}^{N_T-1}\Big((M^{n+1}-M^n)-(\widetilde{M}^{n+1}-\widetilde{M}^n), U^{n+1}-\widetilde{U}^{n+1}\Big)_2\\
&-\nu\sum_{n=0}^{N_T-1}\Big((-\Delta_h)^{\frac{\alpha}{2}}(M^n-\widetilde{M}^n), U^{n+1}-\widetilde{U}^{n+1}\Big)_2\\
&-\sum_{n=0}^{N_T-1}\underbrace{\sum_{i,j}M^n_{i,j}\Big[D_h(U^{n+1}-\widetilde{U}^{n+1})\Big]_{i,j}\cdot\nabla_q g\left(x_{i,j},[D_hU^{n+1}]_{i,j}\right)}_{-\sum_{i,j}\mathcal{T}_{i,j}(U^{n+1},M^n)(U^{n+1}-\widetilde{U}^{n+1})}\\
&+\sum_{n=0}^{N_T-1}\underbrace{\sum_{i,j}\widetilde{M}^n_{i,j}\Big[D_h(U^{n+1}-\widetilde{U}^{n+1})\Big]_{i,j}\cdot\nabla_q g\Big(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j}\Big)}_{-\sum_{i,j}\mathcal{T}_{i,j}(\widetilde{U}^{n+1},\widetilde{M}^n)(U^{n+1}-\widetilde{U}^{n+1})}=0.
\end{aligned}
\tag{4.46}
$$

We here used the definition of the transport operator

$$
h^2\sum_{i,j}\mathcal{T}_{i,j}(U,M)W_{i,j}=-h^2\sum_{i,j}M_{i,j}\nabla_q g(x_{i,j},[D_hU]]_{i,j})[D_hW]_{i,j},
\tag{4.47}
$$

where $W = U^{n+1} - \widetilde{U}^{n+1}$ in this case. By the initial and terminal conditions,

$$
\begin{aligned}
(U^0-\widetilde{U}^0, M^0-\widetilde{M}^0)_2 &= (\mathbf{G}_h[M^0]-\mathbf{G}_h[\widetilde{M}^0], M^0-\widetilde{M}^0)_2\\
(U^{N_T}-\widetilde{U}^{N_T}, \underbrace{M^{N_T}-\widetilde{M}^{N_T}}_{m_T-m_T})_2 &= 0,
\end{aligned}
$$

and we therefore have the following cancellation,

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{n=0}^{N_T-1}\Big((U^{n+1}-U^n)-(\widetilde{U}^{n+1}-\widetilde{U}^n), M^n-\widetilde{M}^n\Big)_2\\
+&\frac{1}{\Delta t}\sum_{n=0}^{N_T-1}\Bigg((M^{n+1}-M^n)-(\widetilde{M}^{n+1}-\widetilde{M}^n), U^{n+1}-\widetilde{U}^{n+1}\Bigg)_2\\
=&\frac{1}{\Delta t}(\mathbf{G}_h[M^0]-\mathbf{G}_h[\widetilde{M}^0], M^0-\widetilde{M}^0)_2,
\end{aligned}
$$

when subtracting the first sum in (4.45) from that in (4.46).

We further have by self-adjointness of the PDL (Theorem 16) that

$$
\nu\sum_{n=0}^{N_T-1}\Bigg((-\Delta_h)^{\frac{\alpha}{2}}(M^n-\widetilde{M}^n), U^{n+1}-\widetilde{U}^{n+1}\Bigg)_2 = \nu\sum_{n=0}^{N_T-1}((-\Delta_h)^{\frac{\alpha}{2}}(U^{n+1}-\widetilde{U}^{n+1}), M^n-\widetilde{M}^n)_2.
$$

By these two results, adding (4.45) and (4.46) results in

$$
\begin{aligned}
&\sum_{n=0}^{N_T-1}\sum_{i,j}M_{i,j}^n\Bigg(g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j})-g(x_{i,j},[D_hU^{n+1}]_{i,j})\Bigg)\\
&-[D_h(\widetilde{U}^{n+1}-U^{n+1})]_{i,j}\cdot\nabla_q g(x_{i,j},[D_hU^n]_{i,j})\\
&+\sum_{n=0}^{N_T-1}\sum_{i,j}\widetilde{M}_{i,j}^n\Bigg(g(x_{i,j},[D_hU^{n+1}]_{i,j})-g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j})\\
&-[D_h(U^{n+1}-\widetilde{U}^{n+1})]_{i,j}\cdot\nabla_q g(x_{i,j},[D_h\widetilde{U}^n]_{i,j})\Bigg)\\
&+\sum_{n=0}^{N_T-1}\Big(\mathbf{F}_h[M^n]-\mathbf{F}_h[\widetilde{M}^n], M^n-\widetilde{M}^n\Big)_2\\
&+\frac{1}{\Delta t}\Big(\mathbf{G}_h[M^0]-\mathbf{G}_h[\widetilde{M}^0], M^0-\widetilde{M}^0\Big)_2=0.
\end{aligned}
\tag{4.48}
$$

From convexity of g (Assumption 17), from $M_{i,j}\geq 0$, and from the monotonicity of F_h and G_h, all the terms are non-negative, and must therefore be zero for the equality to hold. Strict monotonicity of F_h and G_h (Assumption 9) implies that $F_h[M^n]=F_h[\widetilde{M}^n]$ for all $n\in[0,N_T]_{\mathbb{Z}}$ and $G_h[M^0]=G_h[\widetilde{M}^0]$. Hence, $G_h[M^0]=U^0=\widetilde{U}^0=G_h[\widetilde{M}^0]$. From this, we have by existence and uniqueness of the HJB equation (Theorem 26) that $(U^n)_{n=0}^{N_T}=(\widetilde{U}^n)_{n=0}^{N_T}$. Given $(U^n)_{n=0}^{N_T}$, there exists a unique solution $(M^n)_{n=0}^{N_T}$, by Theorem 33. □

4.8 Convergence of the discrete MFG system to the continuous system

We will now prove L^∞-convergence of U and L^1-convergence of M to the classical solution of (MFG) given by Proposition 2.

Theorem 37. *Let Assumption 10- 11 and 12 hold. Assume further that Proposition 2 holds, and let thus (u, m) be the classical solution of* (4.12). *Let (U^n, M^n) be the solution of the discrete problem* (4.15)-(4.20). *Assume $\Delta t = \mathcal{O}(h^s), s \in (0, 2)$. We then have*

$$\lim_{h,\Delta t\to 0} \sup_{i,j,n} \left|U^n_{i,j} - u(x_{i,j}, t_n)\right| = 0,$$

and

$$\lim_{h,\Delta t\to 0} \sup_{n} h^2 \sum_{(i,j)\in\mathcal{I}_h^2} |M^n_{i,j} - m(x_{i,j}, t_n)| = 0.$$

Proof. The following proof is inspired by [7] when proving convergence for U. Convergence for M is original work, and uses M-matrix theory with Neumann series. We will first show that the classical solution solves the discrete equation up to a consistency error. Then, we will use a cross-multiplication argument similar to that used in the uniqueness proof. We will use degenerate ellipticity of the operators in HJB to show L^∞ convergence for U. Since we don't have this luxury for the FPK, as $\mathcal{T}$ is not a degenerate elliptic operator in general, we must use properties of the adjoint together with some M-matrix theory to prove L^1 convergence for M.

Deriving the discrete equation for the exact solution

Let (u, m) be the classical solutions of

$$\begin{aligned}
&\partial_t u + \nu(-\Delta)^{\frac{\alpha}{2}} u + H(x, \mathrm{D}u) = F[m] \\
&-\partial_t m + \nu(-\Delta)^{\frac{\alpha}{2}} m - \operatorname{div}(m\nabla_p H(x, \mathrm{D}u)) = 0 \\
&m(x, T) = m_T(x), \qquad u(x, 0) = G(x, m(0)) \\
&\int_{\mathbb{T}^2} m(x, t)\mathrm{d}x = 1, \qquad m \geq 0.
\end{aligned}$$

By Proposition 2, solutions are of the form $u \in \mathcal{C}^{1,3}_b(\mathbb{T}^2 \times (0, T))$, and $m \in \mathcal{C}^{1,2}_b(\mathbb{T}^2 \times (0, T)) \cap \mathcal{C}([0, T]; \mathcal{P}(\mathbb{T}^2))$. We recall from Section 4.2 the following consistency estimates,

$$\begin{aligned}
\partial_t u^n_{i,j} &= \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} + \mathcal{O}(\Delta t), \quad \partial_t m^n_{i,j} = \frac{m^{n+1}_{i,j} - m^n_{i,j}}{\Delta t} + \mathcal{O}(\Delta t) \\
\partial_{x_k} u^n_{i,j} &= D^\pm_k u^n_{i,j} + \mathcal{O}(h), \quad \partial_{x_k} m^n_{i,j} = D^\pm_k m^n_{i,j} + \mathcal{O}(h)\ k \in\ 1, 2 \\
(-\Delta)^{\frac{\alpha}{2}} u^n_{i,j} &= (-\Delta_h)^{\frac{\alpha}{2}} u^n_{i,j} + o(1), \quad (-\Delta)^{\frac{\alpha}{2}} m^n_{i,j} = (-\Delta_h)^{\frac{\alpha}{2}} m^n_{i,j} + o(1) \\
H(x_{i,j}, \mathrm{D}u^n_{i,j}) &= g(x_{i,j}, [D^+_1 u^n_{i,j}, D^-_1 u^n_{i,j}, D^+_2 u^n_{i,j}, D^-_2 u^n_{i,j}]^\top) + \mathcal{O}(h) \\
[\operatorname{div}(m\nabla_p H(x, \mathrm{D}u))]_{i,j} &= \mathcal{T}_{i,j}(\mathcal{H}_h u, \mathcal{H}_h m) + \mathcal{O}(h^r).
\end{aligned} \tag{4.49}$$

Let $\widetilde{U}$ and $\widetilde{M}$ be the grid functions such that

$$\widetilde{U}_{i,j}^n = h^{-2} \int_{\square_{h,(i,j)}} u(x, n\Delta t)\mathrm{d}x$$
$$\widetilde{M}_{i,j}^n = h^{-2} \int_{\square_{h,(i,j)}} m(x, n\Delta t)\mathrm{d}x$$

We will now show $L^1(\mathbb{T}^2)$-consistency of $\widetilde{M}$ to m. That is, that

$$\lim_{h\to 0} \sup_n \|m(\cdot, t_n) - \mathcal{J}_h \widetilde{M}^n\|_{L^1(\mathbb{T}^2)} = 0, \tag{4.50}$$

where $\mathcal{J}_h$ is the piecewise constant interpolation defined in (4.6). By the mean value theorem, for each $(i, j) \in \mathcal{I}_h^2$, there is a point $y_{i,j} \in \square_{h,(i,j)}$ such that $m(y_{i,j}, t_n) = \widetilde{M}_{i,j}^n$, for all n. Furthermore, for any $(i, j) \in \mathcal{I}_h^2$, we have by continuity of $m(\cdot, t_n)$ that there for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \implies |m(x) - m(y)| < \epsilon,$$

for all $x, y \in \mathbb{T}^2$. Fix $\epsilon > 0$, and let δ satisfy the above. Let $\delta_0(h) = 2h$, and let us control δ_0 by controlling h. Let h_0 be such that $\delta_0(h_0) = 2h_0 < \delta$. We have that $\sup_{i,j}\{\sup_{x\in\square_{h,(i,j)}} |x - x_{i,j}|\} < \delta_0(h) < \delta$ for all $h < h_0$. Hence, we have that

$$\begin{aligned}
\|m(x, t_n) - \mathcal{J}_h \widetilde{M}^n(x)\|_{L^1(\mathbb{T}^2)} &= \int_{\mathbb{T}^2} |m(x, t_n) - \mathcal{J}_h \widetilde{M}^n(x)|\mathrm{d}x \\
&= \sum_{(i,j)\in\mathcal{I}_h^2} \int_{\square_{h,(i,j)}} |m(x, t_n) - (\mathcal{J}_h \widetilde{M}^n)(x)|\mathrm{d}x \\
&= \sum_{(i,j)\in\mathcal{I}_h^2} \int_{\square_{h,(i,j)}} |m(x, t_n) - \widetilde{M}^n(x_{i,j})|\mathrm{d}x \\
&= \sum_{(i,j)\in\mathcal{I}_h^2} \int_{\square_{h,(i,j)}} |m(x, t_n) - m(y_{i,j}, t_n)|\mathrm{d}x \\
&\le \sum_{(i,j)\in\mathcal{I}_h^2} \int_{\square_{h,(i,j)}} \epsilon \mathrm{d}x \\
&= \frac{1}{h^2} h^2 \epsilon \\
&= \epsilon.
\end{aligned} \tag{4.51}$$

As $\epsilon > 0$ was arbitrary, we can get arbitrarily close to zero by letting $h \to 0$. Hence, (4.50) holds. Using (4.50) with consistency of F_h (Assumption 10), we get

$$\lim_{h\to 0} \sup_{m\in\mathcal{P}(\mathbb{T}^2)} \|F[m(\cdot, t_n)] - F_h[\widetilde{M}^n]\|_{L^\infty(\mathbb{T}_h^2)} = 0,$$

or, equivalently,

$$F[m(\cdot, t_n)] = F_h[\widetilde{M}^n] + o(1). \tag{4.52}$$

Next, we want to show uniform consistency of $\widetilde{U}$. Let us Taylor expand $\widetilde{U}_{i,j}$ in the first component, letting $x^{(k)}, k \in \{1, 2\}$ denote the components in $x = (x^{(1)}, x^{(2)}) \in$

$\mathbb{T}^2$.

$$\begin{aligned}
\widetilde{U}^n_{i,j} &= h^{-2}\int_{\square_{h,(i,j)}} u(x,t_n)\mathrm{d}x \\
&= h^{-2}\int_{\square_{h,(i,j)}} \big(u^n_{i,j} + (x^{(1)} - x_{i,j})(\partial_{x^{(1)}}u)^n_{i,j} + (x^{(2)} - x_{i,j})(\partial_{x^{(2)}}u)^n_{i,j} + e(x)\big)\mathrm{d}x \\
&= u^n_{i,j} + \int_{\square_{h,(i,j)}} e(x)\mathrm{d}x \\
&= u^n_{i,j} + h^{-2}\int_{\square_{h,(i,j)}} e(x)\mathrm{d}x \\
&= u^n_{i,j} + \mathcal{O}(h^2),
\end{aligned}$$

since $\int_{\square_{h,(i,j)}} e(x)\mathrm{d}x = \mathcal{O}(h^4)$, as $|e(x)| \leq \|\mathrm{D}^2 u\|_{L^\infty(\mathbb{T}^2)} h^2$ is a bounded second order polynomial (plus higher order terms), and the domain of integration is of size h^2. We also used that the first order terms is odd over the domain, resulting in a zero integral. Hence,

$$\sup_n \|\widetilde{U}^n - u^n\|_{L^\infty(\mathbb{T}^2_h)} = \mathcal{O}(h^2). \tag{4.53}$$

We will now use L^∞-consistency of $\widetilde{U}$ to find a consistency expression for g. Using this result, we Taylor inspect the consistency properties of a forward difference operator in the spatial dimension. We show it in the $x^{(1)}$-direction, but it holds for both directions.

$$\begin{aligned}
& \frac{u^n_{i+1,j} - u^n_{i,j}}{h} - \frac{\widetilde{U}^n_{i+1,j} - \widetilde{U}^n_{i,j}}{h} \\
= & \frac{u^n_{i+1,j} - u^n_{i,j}}{h} - \frac{u^n_{i+1,j} - u^n_{i,j} + \mathcal{O}(h^2)}{h} \\
= & \mathcal{O}(h).
\end{aligned}$$

Hence, the forward (and backward) difference is consistent, and we have

$$[D_h u]_{i,j} = [D_1^+ u_{i,j}, D_1^- u_{i,j}, D_2^+ u_{i,j}, D_2^- u_{i,j}]^\top = [D_1^+ \widetilde{U}_{i,j}, D_1^- \widetilde{U}_{i,j}, D_2^+ \widetilde{U}_{i,j}, D_2^- \widetilde{U}_{i,j}]^\top + \mathbf{1}\mathcal{O}(h),$$

for all $(i,j) \in \mathcal{I}_h^2$. Since g is C^1, we have

$$\begin{aligned}
g(x_{i,j}, [D_h u]_{i,j}) &= g\big(x_{i,j}, [D_h \widetilde{U}]_{i,j} + \mathbf{1}\mathcal{O}(h)\big) \\
&= g\big(x_{i,j}, [D_h \widetilde{U}]_{i,j}\big) + \mathcal{O}(h).
\end{aligned} \tag{4.54}$$

Next, we inspect the consistency of the spatial derivative. Using the $\widetilde{U}$ is second order in h, we find that

$$\begin{aligned}
& \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - \frac{\widetilde{U}^{n+1}_{i,j} - \widetilde{U}^n_{i,j}}{\Delta t} \\
= & \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} - \frac{u^{n+1}_{i,j} - u^n_{i,j} + \mathcal{O}(h^2)}{\Delta t} \\
= & \mathcal{O}\left(\frac{h^2}{\Delta t}\right) = o(1),
\end{aligned} \tag{4.55}$$

assuming $\Delta t = \mathcal{O}(h^s), s \in (0,2)$. Finally, we consider the PDL operator. We have for any $\gamma \in \mathcal{I}_h^2$,

$$\begin{aligned}(-\Delta_h)^{\frac{\alpha}{2}} u_\gamma &= \frac{1}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\gamma - \beta)(u_\gamma - u_\beta) \\ &= \frac{1}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\gamma - \beta)(\widetilde{U}_\gamma - \widetilde{U}_\beta + \mathcal{O}(h^2)) \\ &= (-\Delta_h)^{\frac{\alpha}{2}} \widetilde{U}_\gamma + \mathcal{O}(h^{2-\alpha}),\end{aligned}$$

which is consistent for every $\alpha \in (0,2)$. It can be shown that when $\alpha \to 2$, the PDL reduces to the discrete Laplacian (3.2) (see e.g. [24, Theorem 1.2]), which also can be proved consistent by simple Taylor expansion.

We are now ready to substitute the approximate operators into the exact equation for the solution. We begin with the HJB-equation, before we tackle the FPK-equation. The strategy is to substitute in our consistent discrete operators (4.49), and use our derived consistency results (4.52) - (4.55). On the left-hand side, we have

$$\begin{aligned}&\partial_t u(x_{i,j}, t_n) + \nu(-\Delta)^{\frac{\alpha}{2}} u(x_{i,j}, t_n) + H(x, \mathrm{D}u)(x_{i,j}) \\ =\;& \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t) + \nu(-\Delta_h)^{\frac{\alpha}{2}} u_{i,j}^n + o(1) + g(x_{i,j}, [D_h u]_{i,j}) + \mathcal{O}(h) \\ =\;& \frac{\widetilde{U}_{i,j}^{n+1} - \widetilde{U}_{i,j}^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}} \widetilde{U}_{i,j}^n + g(x_{i,j}, [D_h \widetilde{U}]_{i,j}) + o(1),\end{aligned}$$

where we let $o(1)$ be a quantity going to zero as $h, \Delta t$ goes to zero at proportional rates, meaning $h = C\Delta t$ for a constant $C > 0$. Here, it accounts for $\mathcal{O}(\frac{h^2}{\Delta t}) + \mathcal{O}(\Delta t) + \mathcal{O}(h) + \mathcal{O}(h^{2-\alpha})$. On the left-hand side, we have (4.52). Moving on to the FPK equation, we have on the left-hand side, using the consistency of the transport operator (Assumption 12),

$$\begin{aligned}&- \partial_t m(x_{i,j}, t_n) + \nu(-\Delta)^{\frac{\alpha}{2}} m(x_{i,j}, t_n) - \mathrm{div}(m \nabla_p H(x, \mathrm{D}u))(x_{i,j}) \\ =& - \frac{m_{i,j}^{n+1} - m_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t) + \nu(-\Delta_h)^{\frac{\alpha}{2}} m_{i,j}^n + o(1) - \mathcal{T}_{i,j}(\widetilde{U}^{n+1}, \widetilde{M}^n) + \mathcal{O}(h^r), \\ =& - \frac{\widetilde{M}_{i,j}^{n+1} - \widetilde{M}_{i,j}^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}} \widetilde{M}_{i,j}^n - \mathcal{T}_{i,j}(\widetilde{U}^{n+1}, \widetilde{M}^n) + o(1),\end{aligned}$$

where r is defined in Assumption 12, and where the right-hand side is zero. The terminal condition of $\widetilde{M}^{N_T}$ is $M_T = \mathcal{H}_h m_T$. The initial condition of $\widetilde{U}$ is found using Assumption 10 with (4.50), since $u_{i,j}^0 = G[m(\cdot,0)]$. Thus, using (4.50), we get

$$\lim_{h \to 0} \sup_{m \in \mathcal{P}(\mathbb{T}^2)} \|G[m(\cdot,0)] - G_h[\widetilde{M}^0]\|_{L^\infty(\mathbb{T}_h^2)} = 0.$$

Therefore,

$$\begin{aligned}\widetilde{U}_{i,j}^0 &= u_{i,j}^0 + \mathcal{O}(h^2) \\ &= G[m(\cdot,0)] + \mathcal{O}(h^2) \\ &= G_h[\widetilde{M}^0] + \mathcal{O}(h) \\ &= G_h[\widetilde{M}^0] + o(1).\end{aligned}$$

We end up with the following discrete equation for $(\widetilde{U}, \widetilde{M})$ at the point $(x_{i,j}, t_n)$:

$$\begin{aligned}
\frac{\widetilde{U}_{i,j}^{n+1} - \widetilde{U}_{i,j}^{n}}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}}\widetilde{U}_{i,j}^{n+1} + g(x_{i,j}, [D_h\widetilde{U}^{n+1}]_{i,j}) &= F_h[\widetilde{M}^n]_{i,j} + o(1)\\
\frac{\widetilde{M}_{i,j}^{n+1} - \widetilde{M}_{i,j}^{n}}{\Delta t} - \nu(-\Delta_h)^{\frac{\alpha}{2}}\widetilde{M}_{i,j}^{n} + \mathcal{T}_{i,j}(\widetilde{U}^{n+1}, \widetilde{M}^n) &= o(1)\\
\widetilde{M}_{i,j}^{n} &\geq 0,\\
\|\widetilde{M}^n\|_{L^1(\mathbb{T}_h^2)} &= 1, \quad \forall n \in [0, N_T]_{\mathbb{Z}},\\
\widetilde{M}^{N_T} &= M_T,\\
\widetilde{U}_{i,j}^{0} &= (G_h(\widetilde{M}^0))_{i,j} + o(1).
\end{aligned}$$

In other words, have that the discrete scheme (4.15)-(4.20) is satisfied by $\widetilde{U}^n$ and $\widetilde{M}^n$ up to a consistency error. We will do a cross multiplication, then summing over all indices in time and space, which is an argument similar to the one done in the uniqueness proof, see equation (4.48). The difference is that when we sum $o(1)$ over $N_T N_h^2$ terms, we get $o(1)\frac{T}{\Delta t}\frac{1}{h^2} = o(h^{-2}\Delta t^{-1})$. We get

$$\begin{aligned}
&\sum_{n=0}^{N_T-1}\sum_{i,j} M_{i,j}^n\Big(g(x_{i,j}, [D_h\widetilde{U}^{n+1}]_{i,j}) - g(x_{i,j}, [D_hU^{n+1}]_{i,j})\\
&-[D_h(\widetilde{U}^{n+1} - U^{n+1})]_{i,j}\cdot\nabla_q g(x_{i,j}, [D_hU^n]_{i,j})\Big)\\
&+\sum_{n=0}^{N_T-1}\sum_{i,j} \widetilde{M}_{i,j}^n\Big(g(x_{i,j}, [D_hU^{n+1}]_{i,j}) - g(x_{i,j}, [D_h\widetilde{U}^{n+1}]_{i,j})\\
&-[D_h(U^{n+1} - \widetilde{U}^{n+1})]_{i,j}\cdot\nabla_q g(x_{i,j}, [D_h\widetilde{U}^n]_{i,j})\Big)\\
&+\sum_{n=0}^{N_T-1}\left(\mathbf{F}_h[M^n] - \mathbf{F}_h[\widetilde{M}^n], M^n - \widetilde{M}^n\right)_2\\
&+\frac{1}{\Delta t}\left(\mathbf{G}_h[M^0] - \mathbf{G}_h[\widetilde{M}^0], M^0 - \widetilde{M}^0\right)_2 = o(h^{-2}\Delta t^{-1}),
\end{aligned}$$

Convexity of g and strict monotonicity of F_h and G_h (Assumption 9) gives that all four terms on the left-hand side is non-negative, and must therefore be $o(h^{-2}\Delta t^{-1})$. Writing this out for the last two terms,

$$\begin{aligned}
\lim_{h,\Delta t\to 0} h^2\sum_{n=0}^{N_T-1}\Delta t(\mathbf{F}_h[M^n] - \mathbf{F}_h[\widetilde{M}^n], M^n - \widetilde{M}^n)_2 &= 0\\
\lim_{h,\Delta t\to 0} h^2(\mathbf{G}_h[M^0] - \mathbf{G}_h[\widetilde{M}^0], M^0 - \widetilde{M}^0)_2 &= 0,
\end{aligned}$$

From (4.8), we have

$$\lim_{h,\Delta t\to 0}\sum_{n=0}^{N_T-1}\Delta t\|\mathbf{F}_h[M^n] - \mathbf{F}_h[\widetilde{M}^n]\|_{L^\infty(\mathbb{T}_h^2)}^p = 0, \tag{4.56}$$

for a $p > 0$. Similarly, for G_h through (4.9), we get

$$\lim_{h,\Delta t\to 0}\|\mathbf{G}_h[M^0]-\mathbf{G}_h[\widetilde{M}^0]\|^p_{L^\infty(\mathbb{T}^2_h)}=0. \tag{4.57}$$

Since all terms in the sum in non-negative, we must have

$$\sup_n \Delta t\|\mathbf{F}_h[M^n]-\mathbf{F}_h[\widetilde{M}^n]\|_{L^\infty(\mathbb{T}^2_h)} = o(1), \tag{4.58}$$

$$\|\mathbf{G}_h[M^0]-\mathbf{G}_h[\widetilde{M}^0]\|_{L^\infty(\mathbb{T}^2_h)} = o(1), \tag{4.59}$$

We will now first prove convergence for U in L^∞ and for M in L^1.

Convergence of U to u

Consider the equations

$$\frac{U^{n+1}_{i,j}-U^n_{i,j}}{\Delta t}+\nu(-\Delta_h)^{\frac{\alpha}{2}}U^{n+1}_{i,j}+g(x_{i,j},[D_hU^{n+1}]_{i,j}) = F_h[M^n]_{i,j}$$
$$\frac{\widetilde{U}^{n+1}_{i,j}-\widetilde{U}^n_{i,j}}{\Delta t}+\nu(-\Delta_h)^{\frac{\alpha}{2}}\widetilde{U}^{n+1}_{i,j}+g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j}) = F_h[\widetilde{M}^n]_{i,j}+o(1).$$

Subtracting the first from the second yields

$$\widetilde{U}^{n+1}_{i,j}-U^{n+1}_{i,j}+\Delta t\bigg(\nu(-\Delta_h)^{\frac{\alpha}{2}}(\widetilde{U}^{n+1}-U^{n+1})_{i,j}+g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j})-g(x_{i,j},[D_hU^{n+1}]_{i,j})\bigg)$$
$$= \widetilde{U}^n_{i,j}-U^n_{i,j}+\Delta t\bigg(F_h[\widetilde{M}^n]_{i,j}-(F_h[M^n])_{i,j}\bigg)+o(1).$$

From (4.58),

$$\widetilde{U}^{n+1}_{i,j}-U^{n+1}_{i,j}+\Delta t\bigg(\nu(-\Delta_h)^{\frac{\alpha}{2}}(\widetilde{U}^{n+1}-U^{n+1})_{i,j}+g(x_{i,j},[D_h\widetilde{U}^{n+1}]_{i,j})-g(x_{i,j},[D_hU^{n+1}]_{i,j})\bigg)$$
$$= \widetilde{U}^n_{i,j}-U^n_{i,j}+o(1).$$

Consider now the case for $n=0$:

$$\widetilde{U}^1_{i,j}-U^1_{i,j}+\Delta t\bigg(\nu(-\Delta_h)^{\frac{\alpha}{2}}(\widetilde{U}^1-U^1)_{i,j}+g(x_{i,j},[D_h\widetilde{U}^1]_{i,j})-g(x_{i,j},[D_hU^1]_{i,j})\bigg)$$
$$= \widetilde{U}^0_{i,j}-U^0_{i,j}+o(1).$$

Inserting the initial condition gives

$$\widetilde{U}^1_{i,j}-U^1_{i,j}+\Delta t\bigg(\nu(-\Delta_h)^{\frac{\alpha}{2}}(\widetilde{U}^1-U^1)_{i,j}+g(x_{i,j},[D_h\widetilde{U}^1]_{i,j})-g(x_{i,j},[D_hU^1]_{i,j})\bigg)$$
$$= G_h[\widetilde{M}^0]_{i,j}-G_h[M^0]_{i,j}+o(1)=o(1),$$

by (4.59). By degenerate ellipticity, we use Lemma 19, and find

$$\|\widetilde{U}^1-U^1\|_\infty\le o(1).$$

We can perform the same argument inductively for all $n \in [N_T]$, and we find that

$$\|\widetilde{U}^n - U^n\|_\infty \leq o(1), \quad \forall n \in [0, N_T]_\mathbb{Z}.$$

In other words,

$$\lim_{h,\Delta t \to 0} \sup_n \|\widetilde{U}^n - U^n\|_\infty = 0. \tag{4.60}$$

Therefore, using this with (4.53) and the triangle inequality, we get that

$$\begin{aligned}\sup_{i,j,n} \left|U^n_{i,j} - u(x_{i,j}, t_n)\right| &\leq \sup_{i,j,n} \left|U^n_{i,j} - \widetilde{U}^n_{i,j}\right| + \sup_{i,j,n} \left|\widetilde{U}^n_{i,j} - u(x_{i,j}, t_n)\right| \\ &\leq o(1) + o(1) = o(1)\end{aligned}$$

We conclude that

$$\lim_{h,\Delta t \to 0} \sup_{i,j,n} \left|U^n_{i,j} - u(x_{i,j}, t_n)\right| = 0.$$

Convergence of M to m

Consider now

$$\begin{aligned}&\frac{M^{n+1}_{i,j} - M^n_{i,j}}{\Delta t} - \nu(-\Delta_h)^{\frac{\alpha}{2}} M^n_{i,j} + \mathcal{T}_{i,j}(U^{n+1}, M^n) = 0 \\ &\frac{\widetilde{M}^{n+1}_{i,j} - \widetilde{M}^n_{i,j}}{\Delta t} - \nu(-\Delta_h)^{\frac{\alpha}{2}} \widetilde{M}^n_{i,j} + \mathcal{T}_{i,j}(\widetilde{U}^{n+1}, \widetilde{M}^n) = o(1).\end{aligned}$$

The strategy is to use the consistency assumption of $\mathcal{T}$ in U (Assumption 13). In other words,

$$\mathcal{T}_{i,j}(\widetilde{U}^n, M) = \mathcal{T}_{i,j}(U^n, M) + o(1) \quad \forall (i,j,n) \in \mathcal{I}^2_h \times [0, N_T]_\mathbb{Z}.$$

Then, we subtract the equations and multiply with Δt, yielding

$$\begin{aligned}&\widetilde{M}^n_{i,j} - M^n_{i,j} + \Delta t \Big(\nu(-\Delta_h)^{\frac{\alpha}{2}} \big(\widetilde{M}^n_{i,j} - M^n_{i,j}\big) - \big(\mathcal{T}_{i,j}(\widetilde{U}^{n+1}, \widetilde{M}^n) - \mathcal{T}_{i,j}(U^{n+1}, M^n)\big)\Big) \\ =&\widetilde{M}^{n+1}_{i,j} - M^{n+1}_{i,j} + o(1).\end{aligned}$$

Using our definition of A^n from (4.38), we write the system in matrix form,

$$(I_{N_h^2} + \Delta t A^n)(\widetilde{M}^n - M^n) = (\widetilde{M}^{n+1} - M^{n+1}) + \mathbf{1} o(1),$$

where $\mathbf{1} \in \mathbb{R}^{N_h^2}$ is the vector of ones. Now, $(I_{N_h^2} + \Delta t A^n)^{-1}$ is non-expansive in the induced matrix 1-norm for all $\Delta t > 0$, by Lemma 45. That is, $\|(I_{N_h^2} + \Delta t A^n)^{-1}\|_1 \leq 1$. By the definition of any induced matrix p-norm $\|\cdot\|_p$, we have that for any vector $y \in \mathbb{R}^d \backslash \{0\}$, and matrix A that [39, Definition 4.7]

$$\|A\|_p = \sup_{x \in \mathbb{R}^d \backslash \{0\}} \frac{\|Ax\|_p}{\|x\|_p} \geq \frac{\|Ay\|_p}{\|y\|_p} \implies \|Ay\|_p \leq \|A\|_p \|y\|_p.$$

Recall that $\|\cdot\|_{L^1(\mathbb{T}^2_h)}$ is previously discussed h^2 times the vector 1-norm,

$$\|M\|_{L^1(\mathbb{T}^2_h)} = h^2 \big(M, 1\big)_2, \quad \forall M \in \mathcal{P}_h.$$

From there, we have that

$$\begin{aligned}
\|\widetilde{M}^n - M^n\|_{L^1(\mathbb{T}_h^2)} &= \left\| (I_{N_h^2} + \Delta t A^n)^{-1} \left(\widetilde{M}^{n+1} - M^{n+1} + \mathbf{1}o(1) \right) \right\|_{L^1(\mathbb{T}_h^2)} \\
&\leq \|(I_{N_h^2} + \Delta t A^n)^{-1}\|_1 \left(\|\widetilde{M}^{n+1} - M^{n+1}\|_{L^1(\mathbb{T}_h^2)} + \|\mathbf{1}o(1)\|_{L^1(\mathbb{T}_h^2)} \right) \\
&\leq \|\widetilde{M}^{n+1} - M^{n+1}\|_{L^1(\mathbb{T}_h^2)} + \|\mathbf{1}o(1)\|_{L^1(\mathbb{T}_h^2)} \\
&= \|\widetilde{M}^{n+1} - M^{n+1}\|_{L^1(\mathbb{T}_h^2)} + o(1),
\end{aligned}$$

where we used that $\|\mathbf{1}o(1)\|_{L^1(\mathbb{T}_h^2)} = h^2 \sum_{(i,j)\in\mathcal{I}_h^2} o(1) = o(1)$. To prove convergence, we consider $n = N_T - 1$. Since $\widetilde{M}_{i,j}^{N_T} = M_{i,j}^{N_T} = (M_T)_{i,j},\ \forall (i,j) \in \mathcal{I}_h^2$, we have that $\|\widetilde{M}^{N_T} - M^{N_T}\|_{L^1(\mathbb{T}_h^2)} = 0$, for all $h > 0$.

$$\begin{aligned}
\|\widetilde{M}^{N_T-1} - M^{N_T-1}\|_{L^1(\mathbb{T}_h^2)} &\leq \|\widetilde{M}^{N_T} - M^{N_T}\|_{L^1(\mathbb{T}_h^2)} + o(1) \\
&= o(1).
\end{aligned}$$

We use the same argument inductively, as non-expansivity holds for all $n \in [0, N_T - 1]_{\mathbb{Z}}$. Hence,

$$\sup_n \|\widetilde{M}^n - M^n\|_{L^1(\mathbb{T}_h^2)} = o(1).$$

Using this with (4.50) and the triangle inequality, we get

$$\begin{aligned}
\sup_n \left(h^2 \sum_{i,j} |M_{i,j}^n - m(x_{i,j}, t_n)| \right) &\leq \sup_n \left(h^2 \sum_{i,j} |M_{i,j}^n - \widetilde{M}_{i,j}^n| \right) \\
&\quad + \sup_n \left(h^2 \sum_{i,j} |\widetilde{M}_{i,j}^n - m(x_{i,j}, t_n)| \right) \\
&= o(1) + o(1) = o(1).
\end{aligned}$$

Finally then,

$$\lim_{h,\Delta t \to 0} \sup_n h^2 \sum_{(i,j)\in\mathcal{I}_h^2} |M_{i,j}^n - m(x_{i,j}, t_n)| = 0,$$

and the theorem follows. □

Remark. The assumptions on $h, \Delta t$ for L^∞-boundedness of M given in Lemma 46 are compatible with convergence. We will however not discuss this any further.

CHAPTER

FIVE

SIMULATIONS

In this chapter, we are going to demonstrate the method that we developed in the preceding sections. First, we are going to provide details on how to compute the different operators efficiently, before we tackle how to solve the full system. From here on, we are going to work in one spatial dimension, as we here have a closed-form formula for the coefficients K_α (3.18) used when calculating the PDL. However, given computational resources, there is nothing in the way of extending the following code to encapsulate higher spatial dimensions.

The HJB equation is a nonlinear implicit equation in U^{n+1}, and we will here for each time-step use Newton's method. For this, we needed an explicit Jacobian matrix, which in turn requires the Jacobian of the numerical Hamiltonian. The FPK equation is linear, and we solve this linear system for M^n backwards in time. In both equations, we need an efficient way of computing the discrete fractional Laplacian (PDL). We solve this by designing a PDL matrix, which is symmetric Toeplitz ($A_{i,j} = A_{|i-j|}$) [40]. The following section is fully developed by the author, and all code including examples are found on the author's GitHub[1]. Note that the code is written in the high-performance language Julia, but its vectorized nature makes it fast also in languages like Python. Tables with values of different parameters used in the different experiments are given in Appendix B for reproducibility purposes.

5.1 Implementation and algorithms

5.1.1 PDL matrix

First, we are going to create a fast method for computing the PDL of a vector $\mathbf{v}$ defined on the torus. Since the PDL is a linear operator, we are going to design a PDL matrix L_α for efficiently calculating the discrete fractional Laplacian (PDL) given a vector $\mathbf{v}$, such that

$$L_\alpha \mathbf{v} = (-\Delta_h)^{\frac{\alpha}{2}} \mathbf{v}.$$

Recall our approximation formula (3.27),

$$-(-\Delta_h)^{\frac{\alpha}{2}} u_\gamma \approx F_{1,\gamma} + F_{2,\gamma},$$

[1] https://github.com/tullebulle/FractionalMFGs.jl

where

$$F_{1,\gamma} = \frac{c_\alpha}{h^\alpha} \sum_{\beta \in \mathcal{I}_h} \sum_{|\nu| \le R} (u_\beta - u_\gamma) \widetilde{K}_\alpha(\beta - \gamma - N_h \nu),$$
$$F_{2,\gamma} = -\frac{c_\alpha}{h^\alpha} u_\gamma \bigg(2\zeta(1+\alpha) - \sum_{\beta \in \mathcal{I}_h} \sum_{|\nu| \le R} \widetilde{\widetilde{K}}_\alpha(\beta - \gamma - N_h \nu) \bigg),$$

and $c_\alpha, \widetilde{K}_\alpha, \widetilde{\widetilde{K}}_\alpha$ are defined in (3.20) - (3.22). We begin by splitting up the sum in $F_{1,\gamma}$,

$$\begin{aligned} F_{1,\gamma} &= \frac{c_\alpha}{h^\alpha} \sum_{\beta \in \mathcal{I}_h} \sum_{|\nu| \le R} (u_\beta - u_\gamma) \widetilde{K}_\alpha(\beta - \gamma - N_h \nu) \\ &= \frac{c_\alpha}{h^\alpha} \bigg(\sum_{\beta \in \mathcal{I}_h} (u_\beta - u_\gamma) \sum_{|\nu| \le R} \widetilde{K}_\alpha(\beta - \gamma - N_h \nu) \bigg) \\ &= \frac{c_\alpha}{h^\alpha} \bigg(\sum_{\beta \in \mathcal{I}_h} u_\beta \mathcal{K}(\beta - \gamma) - u_\gamma \sum_{\beta \in \mathcal{I}_h} \mathcal{K}(\beta - \gamma) \bigg), \end{aligned}$$

where we defined

$$\mathcal{K}(m) := \begin{cases} \sum_{|\nu| \le R} \widetilde{K}_\alpha(m - N_h \nu) & m \neq 0 \\ 0 & m = 0. \end{cases} \tag{5.1}$$

Writing out the last sum in $F_{2,\gamma}$ gives

$$\begin{aligned} F_{2,\gamma} &= -\frac{c_\alpha}{h^\alpha} u_\gamma \bigg(2\zeta(1+\alpha) - \sum_{\beta \in \mathcal{I}_h} \sum_{|\nu| \le R} \widetilde{\widetilde{K}}_\alpha(\beta - \gamma - N_h \nu) \bigg) \\ &= -\frac{c_\alpha}{h^\alpha} u_\gamma \bigg(2\zeta(1+\alpha) - \sum_{k=1}^{(R+1)N_h - 1 - \gamma} \frac{1}{k^{1+\alpha}} - \sum_{k=1}^{RN_h + \gamma} \frac{1}{k^{1+\alpha}} \bigg). \end{aligned}$$

Hence, the total approximation at any point indexed by $\gamma \in \mathcal{I}_h$ is

$$F_{1,\gamma} + F_{2,\gamma} =$$
$$\frac{c_\alpha}{h^\alpha} \bigg[\sum_{\beta \in \mathcal{I}_h} u_\beta \mathcal{K}(\beta - \gamma) - u_\gamma \bigg(\sum_{\beta \in \mathcal{I}_h} \mathcal{K}(\beta - \gamma) + 2\zeta(1+\alpha) - \sum_{k=1}^{(R+1)N_h - 1 - \gamma} \frac{1}{k^{1+\alpha}} - \sum_{k=1}^{RN_h + \gamma} \frac{1}{k^{1+\alpha}} \bigg) \bigg].$$

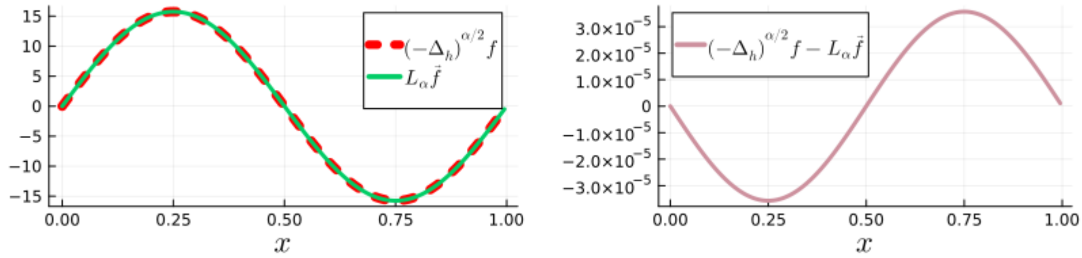
We observe the symmetric Toeplitz structure of the operator, since $\mathcal{K}(m) = \mathcal{K}(-m)$, and hence the coefficient working on u_β is only dependent on $|\beta - \gamma|$. Defining

$$D_\gamma := -\bigg(\sum_{\beta \in \mathcal{I}_h} \mathcal{K}(\beta - \gamma) + 2\zeta(1+\alpha) - \sum_{k=1}^{(R+1)N_h - \gamma} \frac{1}{k^{1+\alpha}} - \sum_{k=1}^{RN_h + \gamma - 1} \frac{1}{k^{1+\alpha}} \bigg),$$

we can define the PDL matrix as

$$L_\alpha := \frac{c_\alpha}{h^\alpha} \begin{bmatrix} D_0 & \mathcal{K}(1) & \mathcal{K}(2) & \mathcal{K}(3) & \dots & \dots & \mathcal{K}(N_h-2) & \mathcal{K}(N_h-1) \\ \mathcal{K}(1) & D_1 & \mathcal{K}(1) & \mathcal{K}(2) & \dots & \dots & \mathcal{K}(N_h-3) & \mathcal{K}(N_h-2) \\ \mathcal{K}(2) & \mathcal{K}(1) & D_2 & \mathcal{K}(1) & \dots & \dots & \mathcal{K}(N_h-4) & \mathcal{K}(N_h-3) \\ \mathcal{K}(3) & \mathcal{K}(2) & \mathcal{K}(1) & D_4 & \dots & \dots & \mathcal{K}(N_h-5) & \mathcal{K}(N_h-4) \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & \vdots \\ \mathcal{K}(N_h-2) & & & \dots & & & D_{N_h-2} & \mathcal{K}(1) \\ \mathcal{K}(N_h-1) & & & \dots & & & \mathcal{K}(1) & D_{N_h-1} \end{bmatrix}. \tag{5.2}$$

A verification plot is given in Figure 5.1.1, demonstrating that it corresponds to the approximation given in (3.27), which we in turn verified against the analytical fractional Laplacian earlier.



(a) Comparison plot. **(b)** Plot of the error.

Figure 5.1.1: Validation plot of PDL matrix L_α, for $f(x) = \sin(2\pi x)$, $\alpha = 1.5$.

5.1.2 Numerical Hamiltonian

We are going to use a quadratic Hamiltonian,

$$H(x, \mathrm{D}u) = |\mathrm{D}u|^2,$$

with numerical Hamiltonian given by

$$g(x, q_1, q_2) = (q_1^-)^2 + (q_2^+)^2.$$

Now, to solve the HJB equation, we are interested in computing the Jacobian matrix

$$\mathcal{J}_g(U) := \nabla_U g(x, [D_h U]) = \left(\frac{\partial}{\partial U_j} g(x_i, [D_h U]_i) \right)_{(i,j) \in \mathcal{I}_h^2}.$$

The matrix is periodic tridiagonal, meaning it is tridiagonal with additional two entries at the top right and bottom left corner, as the difference vector $D_h U$ is a vector of forward and backward difference operators, only dependent on adjacent points. Since $(a)^- = (-a)^+, \forall a \in \mathbb{R}$,

$$\begin{aligned} g(x_i, [D_h U]_i) &= [(D^+ U_i)^-]^2 + [(D^- U_i)^+]^2 \\ &= \frac{1}{h^2} \Big([(U_{i+1} - U_i)^-]^2 + [(U_i - U_{i-1})^+]^2 \Big) \\ &= \frac{1}{h^2} \Big([(U_i - U_{i+1})^+]^2 + [(U_i - U_{i-1})^+]^2 \Big). \end{aligned}$$

Applying that $\partial_x (f(x))^+ = f'(x)\mathbb{1}_{f(x)\geq 0}$ and letting the indices be modulo N_h, we have

$$
\begin{aligned}
(\mathcal{J}_g(U))_{i,i} := \frac{\partial}{\partial U_i} g(x_i, [D_h U]_i) &= \frac{1}{h^2}\frac{\partial}{\partial U_i}\bigg([(U_i - U_{i+1})^+]^2 + [(U_i - U_{i-1})^+]^2\bigg) \\
&= \frac{2}{h^2}\big((U_i - U_{i+1})^+ + (U_i - U_{i-1})^+\big) \\
&= \frac{2}{h^2}\big((U_{i+1} - U_i)^- + (U_i - U_{i-1})^+\big) \\
&= \frac{2}{h}\big((D^+U_i)^- + (D^-U_i)^+\big), \ \forall i \in \mathcal{I}_h.
\end{aligned}
$$

Similar calculations on the sub- and superdiagonal gives

$$
\begin{aligned}
(\mathcal{J}_g(U))_{i,i-1} := \frac{\partial}{\partial U_{i-1}} g(x_i, [D_h U]_i) &= \frac{-2}{h}(D^-U_i)^+, \ \forall i \in \mathcal{I}_h, \\
(\mathcal{J}_g(U))_{i,i+1} := \frac{\partial}{\partial U_{i+1}} g(x_i, [D_h U]_i) &= \frac{-2}{h}(D^+U_i)^-, \ \forall i \in \mathcal{I}_h.
\end{aligned}
$$

All other elements are zero. Defining the general periodic tridiagonal matrix as

$$
\text{Tridiag}_P(\mathbf{l}, \mathbf{d}, \mathbf{u}) = \begin{bmatrix}
d_1 & u_1 & 0 & 0 & \dots & l_1 \\
l_2 & d_2 & u_2 & 0 & \dots & 0 \\
0 & l_3 & d_2 & u_3 & \dots & 0 \\
\vdots & & & \ddots & & \vdots \\
 & & & & & u_{N_h-1} \\
u_{N_h} & \dots & & \dots & l_{N_h} & d_{N_h},
\end{bmatrix} \tag{5.3}
$$

we have

$$
\mathcal{J}_g(U) = \frac{2}{h}\text{Tridiag}_P(-(D^-U)^+, (D^+U)^- + (D^-U)^+, -(D^+U)^-). \tag{5.4}
$$

5.1.3 HJB equation

In this section, we are going to derive the numerical solver for the HJB equation. We have the implicit equation

$$
\frac{U^{n+1} - U^n}{\Delta t} + \nu(-\Delta_h)^{\frac{\alpha}{2}} U^{n+1} + g(x, [D_h U^{n+1}]) = F_h[M^n]
$$

$$
U^{n+1} - U^n + \Delta t \nu(-\Delta_h)^{\frac{\alpha}{2}} U^{n+1} + \Delta t g(x, [D_h U^{n+1}]) - \Delta t F_h[M^n] = 0,
$$

where the unknown is U^{n+1}, we state the problem as a root-finding problem. We want to find $x \in \mathbb{R}^{N_h}$ such that

$$
\mathcal{F}(x) = x - U^n + \Delta t\bigg(\nu L_\alpha x + g(x, [D_h x]) - F_h[M^n]\bigg) = 0.
$$

For this, we can use Newton's method, which is given by the iterative method

$$
x^{(k+1)} = x^{(k)} - J_{\mathcal{F}}^{-1}(x^{(k)})\mathcal{F}(x^{(k)}),
$$

or rather solve

$$J_{\mathcal{F}}(x^{(k)})\delta^{(k)} = \mathcal{F}(x^{(k)}),$$

and

$$x^{(k+1)} = x^{(k)} - \delta^{(k)}.$$

We thus need an expression for $J_{\mathcal{F}}(x)$. Since we have already computed $\mathcal{J}_g$, a linearity of the Jacobian gives

$$J_{\mathcal{F}}(x) = I_{N_h} + \Delta t\Big(\nu L_\alpha + \mathcal{J}_g(x)\Big).$$

The pseudocode for solving the HJB equation is given below. To minimize clutter, we let $\mathbf{C}$ denote a vector of all relevant constants. We further omit writing out the methods Create-PDL-Matrix, and Create-Jacobian-$\mathcal{F}$, as they are trivial to implement, and well-described in the preceding sections.

Algorithm 1 HJB-Step

procedure HJB-Step($U^n, M^n, L_\alpha, \mathbf{C}$)
  $k \leftarrow 0$, $\varepsilon \leftarrow +\infty$
  $x^{(k)} \leftarrow U^n$ ▷ Initial guess for U^{n+1} is simply U^n.
  while $k \leq N_{\text{HJB}}$ **and** $\varepsilon > \text{tol}$ **do** ▷ The Newton iteration.
    $\mathcal{J}_F \leftarrow$ Create-Jacobian-$\mathcal{F}(x^{(k)}, U^n, M^n, L_\alpha, \mathbf{C})$
    Solve $\mathcal{J}_F(x^{(k)})\delta = \mathcal{F}(x^{(k)})$
    $x^{(k+1)} = x^{(k)} - \delta$
    $\varepsilon = \|\delta\|_\infty$; $k = k + 1$
  end while
return $x^{(k)}$
end procedure

Algorithm 2 HJB-Solve

procedure HJB-Solve($(M^n)_{n=0}^{N_T}, \mathbf{C}$)
  Initialize $(N_{\text{HJB}}, h, \Delta t, \nu, \alpha, R) \leftarrow \mathbf{C}$
  Let $(U^n)_{n=0}^{N_T}$ be a new empty array.
  $U^0 \leftarrow G_h[M^0]$
  $L_\alpha \leftarrow$ Create-PDL-Matrix($\alpha, \mathbf{C}$)
  for $n = 0$ **to** $N_T - 1$ **do**
    $U^{n+1} \leftarrow$ HJB-Step($U^n, M^n, L_\alpha, \mathbf{C}$)
  end for
return $(U^n)_{n=0}^{N_T}$
end procedure

5.1.4 FPK equation

Unlike the HJB equation, the FPK equation is linear, and we hence only need to solve a linear system. We recall the discrete equation

$$\begin{aligned}\frac{M_i^{n+1}-M_i^n}{\Delta t}-\nu(-\Delta_h)^{\frac{\alpha}{2}}M_i^n+\mathcal{T}_i(U^{n+1},M^n)&=0,\quad \forall n\in[0,N_T-1]_{\mathbb{Z}}\\ M_i^n&\geq 0,\\ M^{N_T}&=m_T\in P_h\\ h(M^n,1)_2&=1,\ \forall n\in[0,N_T-1]\end{aligned}$$

which in vector form can be written

$$M^n+\Delta t A^n M^n=M^{n+1},\quad \forall n\in[0,N_h-1]_{\mathbb{Z}},$$

where

$$(A^nM)_i=\nu(-\Delta_h)^{\frac{\alpha}{2}}M_i-\mathcal{T}_i(U^{n+1},M),\ \forall i\in\mathcal{I}_h.$$

Furthermore, the transport operator (4.14) is in one dimension given by

$$\begin{aligned}\mathcal{T}_i(U,M):=\frac{1}{h}\Bigg(&M_i\frac{\partial g}{\partial q_1}(x_i,[D_hU]_i)-M_{i-1}\frac{\partial g}{\partial q_1}(x_{i-1},[D_hU]_{i-1})\\ &+M_{i+1}\frac{\partial g}{\partial q_2}(x_{i+1},[D_hU]_{i+1})-M_i\frac{\partial g}{\partial q_2}(x_i,[D_hU]_i)\Bigg).\end{aligned}$$

With the numerical Hamiltonian given by

$$g(x,q_1,q_2)=(q_1^-)^2+(q_2^+)^2,$$

we find that

$$\partial_{q_1}g(x,q_1,q_2)=-2(q_1^-)\quad \partial_{q_2}g(q_1,q_2)=2(q_2^+).$$

Hence,

$$\frac{\partial g}{\partial q_1}(x_i,[D_hU]_i)=-2((D^+U_i)^-),\quad \frac{\partial g}{\partial q_2}(x_i,[D_hU]_i)=2((D^-U_i)^+)$$

and thus

$$\mathcal{T}_i(U,M)=\frac{2}{h^2}\left(\begin{array}{c}-M_i(U_{i+1}-U_i)^-+M_{i-1}(U_i-U_{i-1})^-\\ +M_{i+1}(U_{i+1}-U_i)^+-M_i(U_i-U_{i-1})^+\end{array}\right).$$

For efficient computation, we create a transport operator matrix $\mathcal{T}(U)$ such that $\mathcal{T}(U)M=\mathcal{T}(U,M)$. This matrix is also periodic tridiagonal, and given by

$$\mathcal{T}(U)=\frac{2}{h^2}\text{Tridiag}_P((D^-U)^-,-((D^+U)^-+(D^-U)^+),(D^+U)^+).$$

In summary, the linear system for the FPK equation is given by

$$\Big(I_{N_h}+\Delta t\big(\nu(-\Delta_h)^{\frac{\alpha}{2}}-\mathcal{T}(U^{n+1})\big)\Big)M^n=M^{n+1},\quad \forall n\in[0,N_h-1]_{\mathbb{Z}}.\tag{5.5}$$

The algorithm for solving the FPK equation is given below.

5.1.5 MFG system

To solve the full MFG system, it's only a matter of building on the algorithms we already constructed. We define it as follows. Note that if one seeks to solve a problem which is forward in m and backwards in u, one can simply treat m_T as an initial condition, and reverse the output arrays over the time-axis.

Algorithm 3 FPK-Step

procedure FPK-Step($U^{n+1}, M^{n+1}, L_\alpha, \mathbf{C}$)
    $\mathcal{T}(U^{n+1}) \leftarrow$ Create-$\mathcal{T}((U^{n+1}, \mathbf{C}))$
    Solve $\big(I_{N_h} + \Delta t(\nu L_\alpha - \mathcal{T}(U^{n+1}))\big)x = M^{n+1}$
return x
end procedure

Algorithm 4 FPK-Solve

procedure FPK-Solve($(U^n)_{n=0}^{N_T}, m_T, \mathbf{C}$)
    Initialize $(h, \Delta t, \nu, \alpha, R) \leftarrow \mathbf{C}$
    Let $(M^n)_{n=0}^{N_T}$ be a new empty array.
    $(M^{N_T})_i \leftarrow m_T(x_i), \ \forall i \in \mathcal{I}_h$.
    $L_\alpha \leftarrow$ Create-PDL-Matrix(α)
    for $n = N_T - 1$ **downto** 0 **do**
        $M^n \leftarrow$ FPK-Step($U^{n+1}, M^{n+1}, L_\alpha, \mathbf{C}$)
    end for
return $(M^n)_{n=0}^{N_T}$
end procedure

Algorithm 5 MFG-Solve

procedure MFG-Solve($m_T, \mathbf{C}$)
    Initialize $(N_{\text{MFG}}, h, \Delta t, \nu, \alpha, R) \leftarrow \mathbf{C}$
    Let $(U^n)_{n=0}^{N_T}, (M^n)_{n=0}^{N_T}$ be new empty arrays.
    $(M^{N_T})_i \leftarrow m_T(x_i), \ \forall i \in \mathcal{I}_h$.
    Let $M^n = M^{N_T}$ $\forall n \in [0, N_T]_{\mathbb{Z}}$ ▷ Choosing the constant-in-time m_T as initial guess for $(M^n)_{n=0}^{N_T}$.
    for $\ell = 1$ **to** N_{MFG} **do**
        $(U^n)_{n=0}^{N_T} \leftarrow$ HJB-Solve($(M^n)_{n=0}^{N_T}, \mathbf{C}$)
        $(M^n)_{n=0}^{N_T} \leftarrow$ FPK-Solve($(U^n)_{n=0}^{N_T}, m_T, \mathbf{C}$)
    end for
return $(U^n)_{n=0}^{N_T}, (M^n)_{n=0}^{N_T}$
end procedure

5.2 Verification

In this section, we are going to validate the algorithms on test problems where an analytical solution is known, to ensure correctness. We begin by testing the uncoupled HJB and FPK separately, before we test the coupled MFG system.

5.2.1 Verifying HJB-Solve

We create the test problem

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\frac{\alpha}{2}} u + (\partial_x u)^2 = f(x,t) & \text{in } \mathbb{T} \times (0,2), \\ u(0,x) = u_0(x), & \text{in } \mathbb{T}, \end{cases}$$

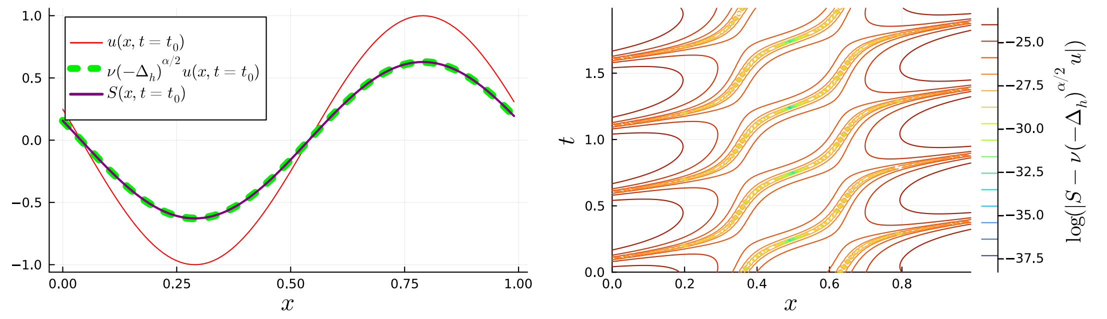
for a right-hand side to be determined. Let $\alpha = 1$, and $\nu = 0.1$. For simplicity, we let the solution be a simple traveling cosine,

$$u(x,t) = \cos(2\pi(x-t)).$$

While we can't calculate $\nu(-\Delta)^{\frac{\alpha}{2}} u(x,t)$ analytically, we compute it with the PDL matrix for a large R, ($R = 500$). By observing that the result appears to be a sinusoidal, in the same phase as u, we create a surrogate function

$$\nu(-\Delta)^{\frac{\alpha}{2}} u(x,t) \simeq S(x,t) := c_0 \cos(2\pi(x-t)),$$

where we match the amplitude c_0 with the computed result. We compare the surrogate with the PDL in Figure 5.2.1. As we can see, the surrogate is very accurate. With this surrogate, we can compute the right-hand side to be

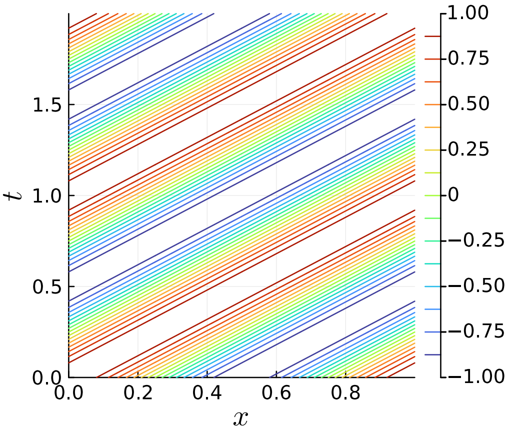


(a) Surrogate of fractional Laplacian. **(b)** Log of the error $|S(x,t) - \nu(-\Delta_h)^{\frac{\alpha}{2}} u(x,t)|$

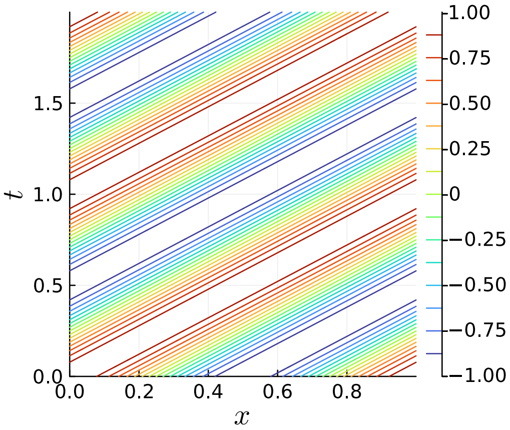
Figure 5.2.1: Validation of surrogate, HJB.

$$f(x,t) = 2\pi \sin(2\pi(x-t)) + c_0 cos(2\pi(x-t)) + 4\pi^2 \sin^2(2\pi(x-t)).$$

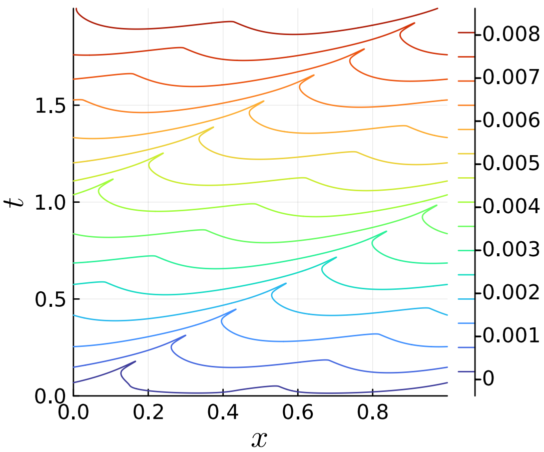
Together with the initial condition $u(x,0) = \cos(2\pi x)$, we have an initial value problem. Modeling the coupling F_h to be equal to f, we can use HJB-Solve to solve the problem. Note that F_h in this instance is independent of $(M^n)_{n=0}^{N_T}$, and just a function of space and time. This induces a small modification on HJB-Solve, in order to solve the test problem, but as the modification is minor, it serves as a validation of the algorithm. The result is given in Figure 5.2.2. While we notice some apparent energy gain in the numerical solution increasing with time, the solver seems accurate.



(a) Analytical solution, u.



(b) Numerical solution, U.



(c) Error, $U - u$.

Figure 5.2.2: Analytical and numerical solution of the test problem, together with the difference.

5.2.2 Verification of FPK-Solve

To verify FPK-Solve, we follow a similar procedure. We create a test-problem, choose a solution, and modify the function of freedom to satisfy the PDE. Consider the fractional uncoupled FPK equation,

$$\begin{cases} \partial_t m(x,t) - \nu(-\Delta)^{\frac{\alpha}{2}} m(x,t) - \partial_x(mv)(x,t) = 0 & \text{in} \quad \mathbb{T} \times (0,T) \\ \int_{\mathbb{T}^2} m(x,t)\, dx = 1, \quad m \geq 0, & \text{in} \quad (0,T) \\ m(x,T) = m_T(x) \in \mathcal{P}(\mathbb{T}) & \text{in} \quad \mathbb{T}. \end{cases} \tag{5.6}$$

Let $\alpha = 1.5, \nu = 0.1$ and $T = 2$. Compared to the HJB equation, we have some further requirements for the solution. We need the solution to be non-negative and integrate to one. Also, we here don't have an explicit right-hand side, but rather the control v is given implicitly in the divergence term (derivative in 1D). We will solve this by integration. To apply the same trick of using a surrogate for the fractional Laplacian, we aim to have a solution which is sinusoidal. We choose the solution to be

$$m(x,t) = Z\cos^2(2\pi x - c_1 t^2) + \epsilon,$$

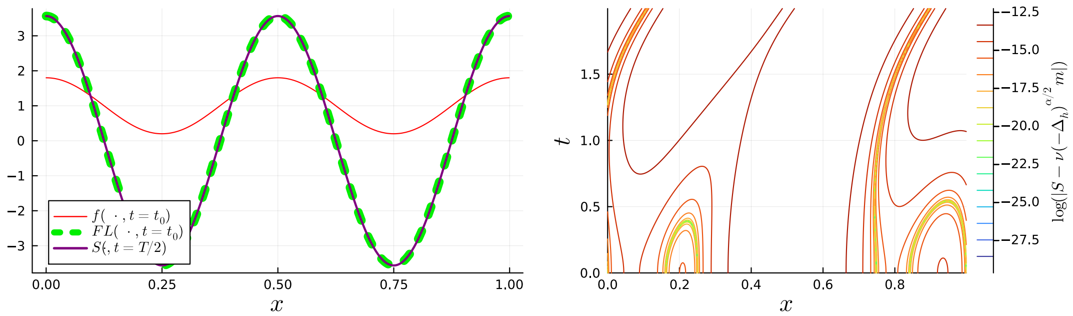
where $\epsilon > 0$ is some small number which ensures that m has support on the whole torus, and Z is a normalizing constant. Finding v now reduces to a simple integral,

$$\frac{1}{m(x,t)} \int_0^x \Big(\partial_t m(r,t) - \nu(-\Delta)^{\frac{\alpha}{2}} m(r,t) \Big) \mathrm{d}r = v(x,t).$$

We compute the PDL with a large R, and we again find an appropriate surrogate S of the fractional Laplacian, this time to be of the form

$$\nu(-\Delta)^{\frac{\alpha}{2}} m(x,t) \simeq S(x,t) := c_0 \cos(4\pi x - c_1 t^2), \tag{5.7}$$

for a constant c_0 we fit. The comparison plot of the surrogate with the PDL is given in Figure 5.2.3. We again conclude that the approximation is satisfactory.



(a) Surrogate of fractional Laplacian. **(b)** Log of the error $|S(x,t) - \nu(-\Delta_h)^{\frac{\alpha}{2}} m(x,t)|$

Figure 5.2.3: Validation of surrogate, FPK.

We compute

$$\partial_t m(x,t) = Z c_1 4t \cos(2\pi x - c_1 t^2)\sin(2\pi x - c_1 t^2),$$

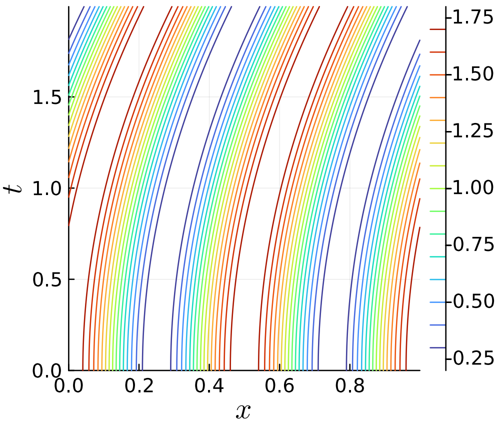
and we notice that

$$\partial_t m(x,t) = -\frac{c_1 t}{\pi} \partial_x m(x,t).$$

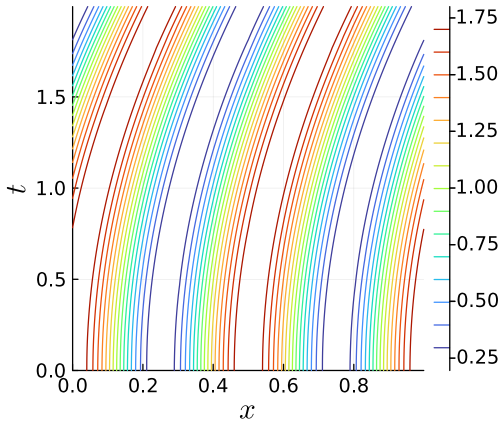
From this, we find an analytical expression for v,

$$\begin{aligned} v(x,t) &= \frac{1}{m(x,t)} \int_0^x \big(-\frac{c_1 t}{\pi} \partial_x m(r,t) + c_0 \cos(4\pi r - c_1 t^2) \big) \mathrm{d}r \\ &= \frac{1}{m(x,t)} \bigg(\frac{Zt}{\pi} [\cos^2(c_1 t^2) - \cos^2(2\pi x - c_1 t^2)] - \frac{c_0}{4\pi} [\sin(4\pi x - 2c_1 t^2) + \sin(2c_1 t^2)] \bigg). \end{aligned}$$

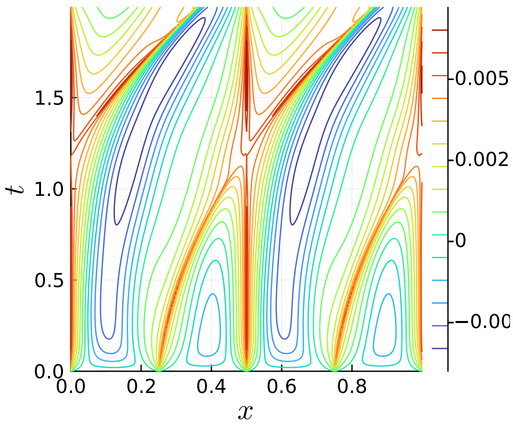
The terminal condition is given by $m(x,T) = Z\cos(2\pi x - c_1 4) + \epsilon$. Inserting $U_i^n = -\frac{1}{2}\int_0^{x_i} v(x,t_n)\mathrm{d}x,\ \forall i \in \mathcal{I}_h, n \in [0, N_T]_{\mathbb{Z}}$, which we compute numerically, we have a value for U, required to compute $\mathcal{T}(U)$ in FPK-Solve. The results are given in Figure 5.2.4. Based on the plot, FPK-Solve also seems correct.



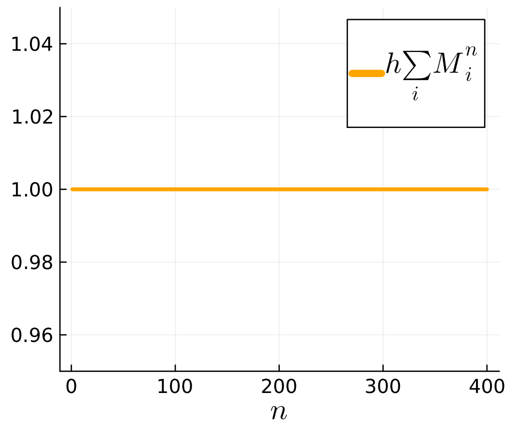
(a) Analytical solution, m.



(b) Numerical solution, M.



(c) Error, $M - m$.



(d) Plot of mass conservation for each time-step n.

Figure 5.2.4: Analytical and numerical solution of the FPK test problem, with errors and mass conservation through time.

5.2.3 Verification of MFG-Solve

While we seem to have verified the individual uncoupled solvers, we seek some verification of MFG-Solve for the coupled system. To find a test problem to verify against was somewhat challenging, as designing a test problem with a known solution was less trivial. Instead, we test the algorithm on an adjacent problem to one which has been computed before. Ersland et al. [41] have, as mentioned, developed a semi-Lagrangian method for fractional Mean Field Games. By solving a similar problem with our solver, we should get similar results. One key difference is that they study MFGs one the whole space, and not on the torus. In their numerical simulations, they create artificial Dirichlet boundary conditions for both u and m to avoid mass flowing out of some bounded domain. These sorts of boundary conditions, together with their coupling F, which is discontinuous on the torus boundary, created some difficulties. To cope with this issue, we extend the domain in both directions, and hope that the boundary effects won't influence the global solution too greatly. As the problems fundamentally differs, we can't expect identical solutions, but we expect similar trends.

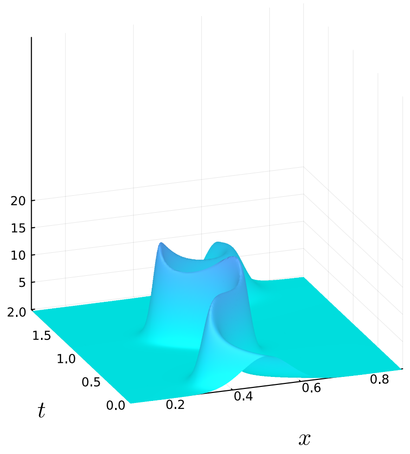
As described in Example 1 in [41], we consider the following fractional MFG system:

$$\begin{cases} -\partial_t u + \nu(-\Delta)^{\frac{\alpha}{2}} u + \frac{1}{2}(\partial_x u)^2 = F[m] & \text{in} \quad Q_T \\ \partial_t m + \nu(-\Delta)^{\frac{\alpha}{2}} m - \operatorname{div}(m\partial_x u) = 0 & \text{in} \quad Q_T \\ m(\cdot, 0) = C\exp(-\frac{(x-0.5)^2}{0.1^2}), \qquad u(\cdot, T) = 0 & \text{in} \quad \mathbb{T}^d \\ \int_{\mathbb{T}^d} m \mathrm{d}x = 1, \qquad m \geq 0 & \text{in} \quad Q_T. \end{cases}$$

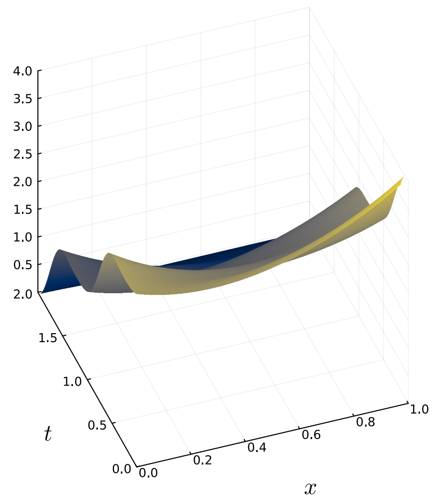
The problem is forward in m and backwards in u, and defined on $Q_T = \mathbb{T} \times (0, 2)$, $G = 0$, and the coupling is non-local with spatial-temporal dependence,

$$F[m](x, t) = 5(x - 0.5(1 - \sin(2\pi t)))^2 + \phi_\delta * m(x),$$

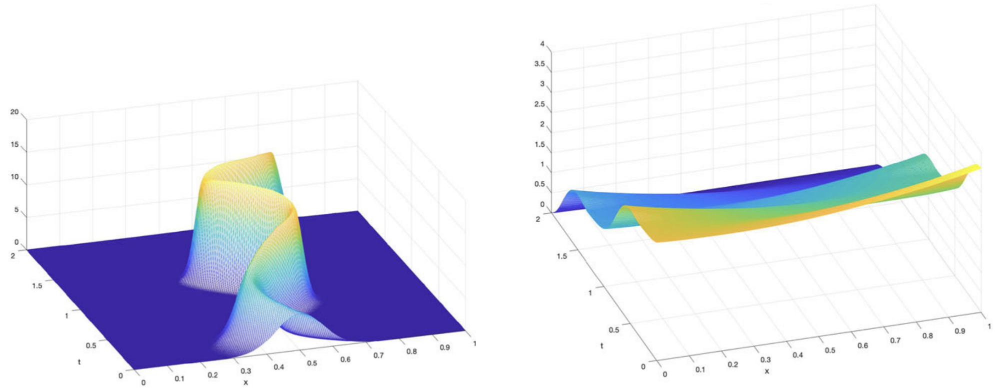
where we use Gaussian mollifier $\phi_\delta = \frac{1}{\delta\sqrt{2\pi}}\exp(-\frac{x^2}{2\delta^2})$. The initial condition of m is $m_0(x) = C\exp(-\frac{(x-0.5)^2}{0.1^2})$, with C being the normalization constant. The result is given in Figure 5.2.5. While we notice some small differences in the plot of m, in particular at its mode height, the plots generally seem to capture the same behavior. As mentioned, Ersland et al.'s problem differ to ours in that we model on a periodic domain, and they model on the whole space with artificial boundary conditions at $x = 0$ and $x = 1$.



(a) Simulation of m with $\alpha = 1.5$.



(b) Simulation of u with $\alpha = 1.5$.



(c) Ersland's simulation of a similar MFG system.

Figure 5.2.5: Comparison with Ersland et al.'s simulation of similar setup.

5.3 Application - Astroworld Crowd Crush

One of MFGs most prominent applications is crowd motion. Typical crowd motion problems include evacuation and optimal transport. We will here study another related problem, namely crowd dynamics in highly dense festivals. In particular, we will create a simple 1D model which aims to simulate the dynamics happening during the Astroworld Festival crowd crush in 2021. The happening has been modeled through crowd physics [42] [43], and we will base our assumptions on these resources.

5.3.1 Modeling and Assumptions

The area where the accident happened was in the square-shaped region marked in Figure 5.3.1 (a). We model the festival area as the unit square in the xy-plane, with the stage area between $x \simeq 0.85$ and $x = 1$. Furthermore, we assume the crowd is uniform in the y-direction, and hence we will conveniently only work in one spatial dimension. We will work on the torus, between $x = 0$ inclusive and $x = 1$ exclusive. While working on the torus might seem unfit for this application,

(a) Overview of the Astroworld Festival area. Image taken from Reuters.

(b) The artist Travis Scott is known for his intense festivals, with aggressive crowds and "mosh pits" [44]. Image taken from [45].

Figure 5.3.1

we will create a spatial potential to minimize mass flow through the boundaries. We will also let $T = 2$ throughout this section. An illustration of the festival area is given in Figure 5.3.2. We recall that each agent moves according to

Figure 5.3.2: Model of the festival area.

$$\mathrm{d}X_t = v_t \mathrm{d}t + \mathrm{d}L_{\alpha,t},$$

where $\mathrm{d}L_{\alpha,t}$ is an increment of an α-stable Lévy process. Each agent is rational and seek to minimize the functional

$$J(x,v) = \mathbb{E}\left[\int_0^t [L(X_s, s, v_s) + F[m(s)](X_s, s)\mathrm{d}s + G[m(T)](X_T) \right],$$

where L is the Fenchel conjugate

$$L(x,t,v) := \sup_{p\in\mathbb{R}} \{p \cdot v - H(x,t,p)\}.$$

Throughout this section, we will let

$$H(x,t,p) = C_H p^2,$$

which gives

$$L(x,t,v) := C_L v^2 = \frac{v^2}{4C_H},$$

where $C_H > 0$ is some parameter, possibly time-dependent. Note that we here model both L and F as time-dependent. While not including this time-dependence in our previous chapters, it is still interesting to experiment with relaxed assumptions. We will further let the terminal cost be proportional to the coupling F,

$$G[m](x) = C_G F[m](x, T).$$

We will here test a local coupling F of the form

$$F[m](x, t) = C_Q Q(x) + C_B B(m(x, t), t),$$

where Q is a spatial potential, and B is a congestion cost.

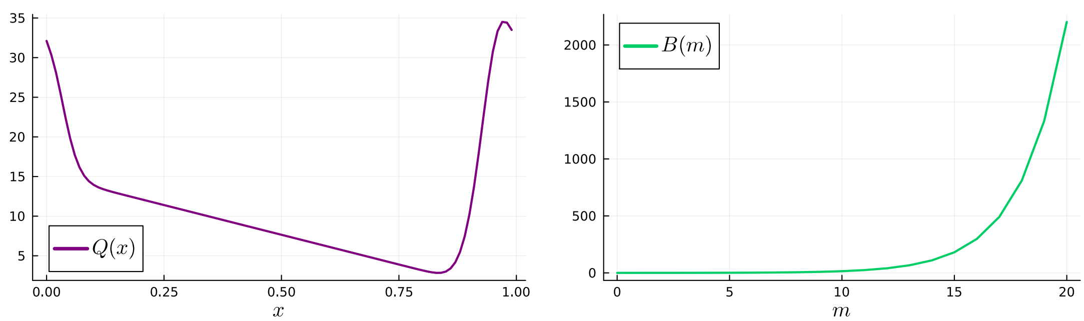
The spatial potential Q aims to model how the crowd is drawn towards the stage to get closer to the performer, but also to block the crowd to get on the stage. Specifically, we will use a spiky Gaussian to account for the high potential exactly at the stage, because of physical barriers and security. Furthermore, we let it be a linearly decreasing towards the stage, to incentivize agents to get as close to the stage as possible. As we are working on a periodic domain, we create periodic extensions outside $[0, 1]$. To smooth out the linear function at the periodic boundaries, we convolve it with a Gaussian mollifier. Explicitly,

$$\begin{aligned} Q(x) &:= (l * \phi_\delta)_{\mathbb{T}}(x) + \frac{1}{\sigma} \exp_{\mathbb{T}}\left(-\frac{(x - 0.95)^2}{4\sigma^2}\right), \\ l(x) &:= -20x + 15, \end{aligned}$$

where the subscript $\mathbb{T}$ indicate periodic extension. We will begin modelling the congestion cost B as an exponential with a constant cutoff,

$$B(m) = \begin{cases} 0.1 \exp(0.5m) & M \leq 50 \\ 0.1 \exp(25) & M > 50, \end{cases}$$

to ensure boundedness. B thus grows rapidly with m, modeling a crowd which prefers spacing. Visualizations of the spatial potential and the congestion cost is given in Figure 5.3.3. The actual cost values are controlled with parameters C_Q, C_B, so the scale of Q, B are of less importance than their shape. We will solve



(a) Spatial potential Q.

(b) Congestion term B.

Figure 5.3.3: Plot of the spatial potential and congestion terms.

the MFG forward in m and backwards in u. The initial density is a Gaussian centered near the entrance of the festival area,

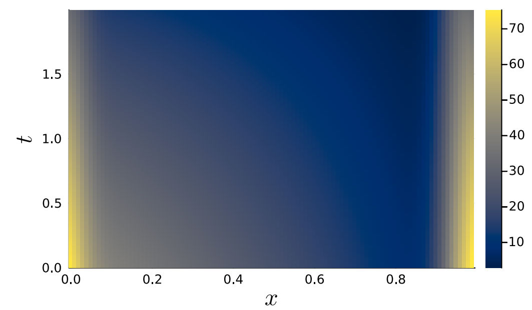
$$m(x, 0) = C \exp_{\mathbb{T}}(-50(x - 0.2)^2),$$

to simulate the crowd entering the festival area at $t = 0$.

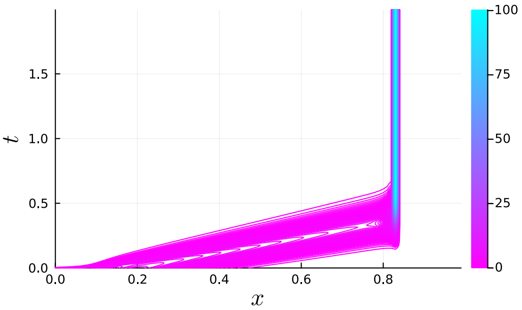
Finally, we will use the fractional parameter $\alpha = 1.5$. Each agent will hence have a non-local diffusion, and will perform discontinuous jumps in space. This is meant to model how people will move across large distances in a relative short period of time (assuming a scaled time-axis), for buying drinks, linking up with friends, or going to the restroom.

5.3.2 Example 1 - No congestion cost and no diffusion

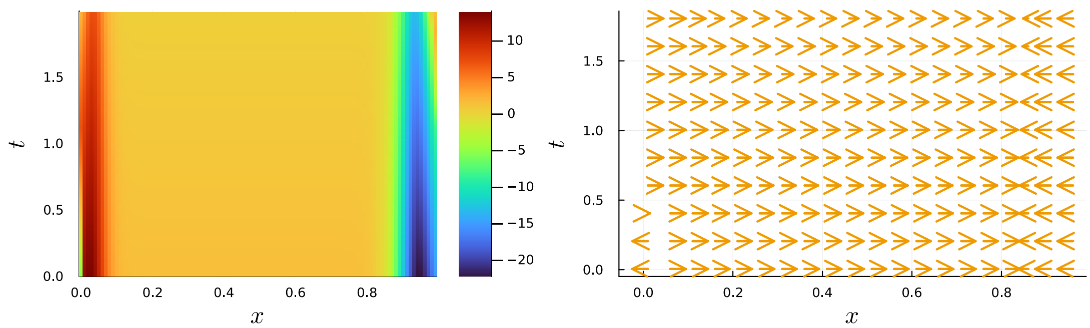
We begin with a simple instance of the problem, where there is no congestion penalty, $C_B = 0$, and no diffusion, $\nu = 0$. Hence, it reduces to a first order system. Here, we expect the agents to move close to the stage area, without worrying about crowded spaces. The pace at which agents move with will dependent on C_H. We plot the solutions of u and m, together with the control in Figure 5.3.4. Note that we plot the solutions of m as a contour-plot, which easier distinguishes small values from zero values. The density's trajectory is then easier to trace. Hence, where there is no color in Figure 5.3.4 (b) can be considered as close-to-zero values of m. We plot $u(x, t)$ as a heatmap. As we can see, the crowd moves to form a spiky Gaussian near the stage, where they stay put. Since there is no congestion cost ($C_B = 0$), we observe that u is the lowest in the dense area near the stage. We also plot the optimal feedback control, given by $\mathrm{D}_p H$. The spatial potential ensures minimum mass flow through the boundary, as modeled.



(a) Solution of u, $C_B = 0$.



(b) Solution of m, $C_B = 0$.

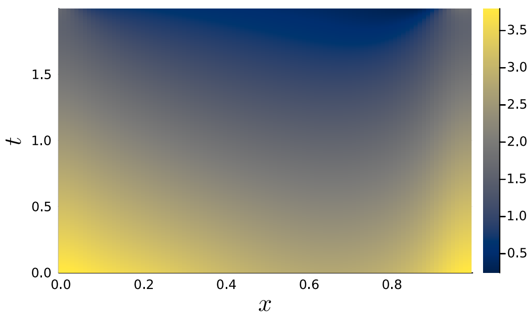


(c) Optimal feedback control (velocity field) $v(x, t) = -\mathrm{D}_p H(x, \mathrm{D}u)$.

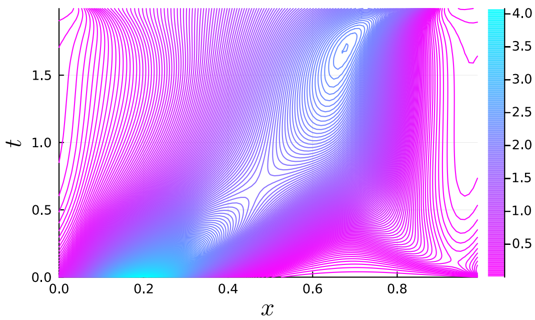
Figure 5.3.4: Plot of the solution of Example 1, together with the optimal feedback control.

5.3.3 Example 2 - Calm concert

We will now include congestion cost and diffusion. We let the terminal cost be half of the running cost $G[m(\cdot,T)] = C_G F[m(\cdot,T)]$, with $C_G = \frac{1}{2}$, to incentivize the crowd getting closer to the stage towards the end of the concert. This is not an unreasonable assumption, as the artist might save its classics until the end, and people might be more influenced. We observe that while the initial density is similar, the crowd is more evenly spread out, but seek to get closer to the stage throughout the concert.



(a) Solution of u, with exponential congestion cost.



(b) Solution of m, with exponential congestion cost.

Figure 5.3.5: Plot of the solution of Example 2.

5.3.4 Example 3 - High-intensity festival

We are now going to model the Astroworld crowd crush, and figure out if the agents in the MFG are able to avoid the dramatic turn of events in Houston 2021. According to resources on what happened [46] [47], there was supposed to be entertainment on two different stages, such that the crowd would have more room. However, the main performer Travis Scott arrived late at the main stage, which resulted in the performance on the secondary stage ending, and hence everyone from the secondary stage got impatient and pushed towards the main stage. Hence, people from the far back pushed forward, all wanting to get closer to the stage in a too small quadrant. As the density of people got higher (over 6 people per square meter), these pushes traveled as waves through the crowd, demonstrating that the crowd suddenly acted as a continuous medium [43]. At this point, people could hardly move anywhere, and the only way of getting out of the crowd was to be physically lifted above the crowd.

We are aiming to model some of the same effects in the following MFG system. To model this intense, highly dense festival, we need to do some modifications on the setup. Unlike in the previous examples, we will now model that the density increases with time, as in the Astroworld incident. Furthermore, we let the crowd have very high tolerance of standing close together (forming mosh pits, for instance) right until the density is dangerously high. We also need to simulate that mobility of the agents reduces when the concert intensifies, and the density increases.

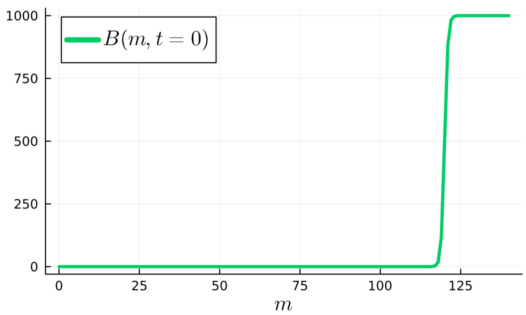
As mentioned, we will let the congestion cost be low right until the point of fatality, M_{fatal}, where the congestion cost increases rapidly. Furthermore, to simulate that the crowds gets denser as t increases, we will let this point of fatality decrease as time goes. For this, we will use a traveling sigmoid function,

$$B(m,t) = \frac{B_{\max}}{1 + \exp[-2(m - M_{\text{fatal}} + \Delta M_{\text{fatal}} t)]},$$

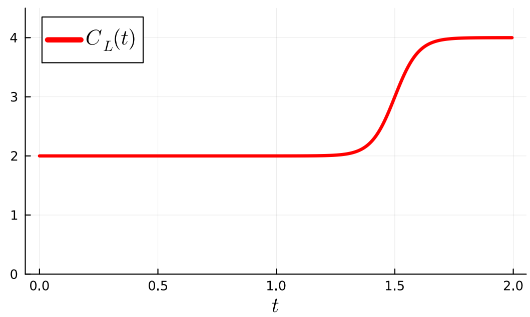
where ΔM_{fatal} is the negative change in M_{fatal} per time unit. We will let $M_{\text{fatal}} = 100$, and $\Delta M_{\text{fatal}} = 40$. To model the decrease in mobility throughout the concert, we will let the running cost suddenly increase at $t_{\text{crit}} = 1.5$. Also here, we let the coefficient increase with a sigmoid:

$$C_L(t) = C_{L,0} + \frac{\Delta C_L}{1 + \exp(-20(t - t_{\text{crit}}))}.$$

A figure of the congestion cost and the running cost is given in Figure 5.3.6.



(a) New congestion cost.



(b) New time-dependent coefficient of the running cost.

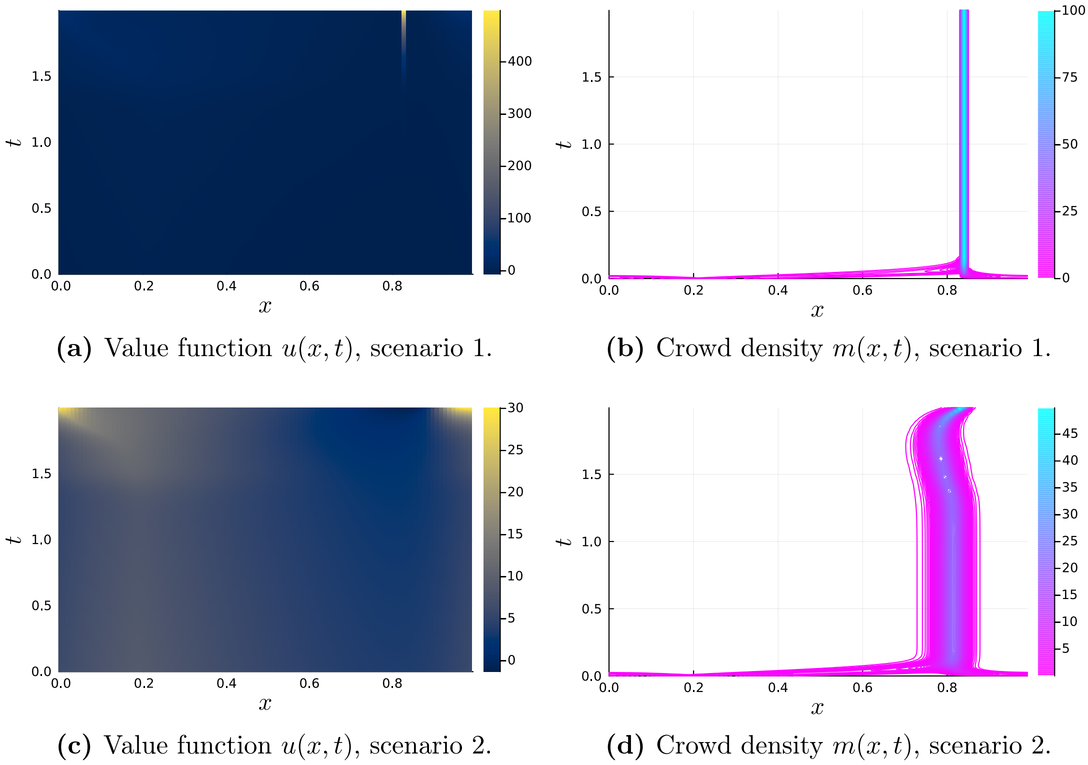
Figure 5.3.6: Congestion and coefficient of running cost meant to model a high-intensity festival.

Scenario 1 - Crowd crush

We are first going to let $\Delta C_L = 2$, which essentially triples the running cost penalty at $t \simeq t_{\text{crit}} = 1.5$. Furthermore, we will here not have any diffusion, $\nu = 0$, as the dense crowd prevents people from traveling large distances in short periods of time. As we can see in Figure 5.3.7 (a)-(b), the crowd rushes towards the stage as in Example 1, which is no problem until the sudden increase in density combined with the sudden lowered mobility. As we see in Figure 5.3.6 (a), the fatal density initially is $M_{\text{fatal}} = 120$. However, as $t \simeq 1.5$, the fatal density is around 60. With a density of over 100, the congestion cost gets extremely large. The agents seem to be stuck in front of the stage, and we see the value function u eventually gets very large right in front of the stage. This might simulate a crowd crush, where suddenly people are trapped in front of the stage, and gets crushed as the density increases.

Scenario 2 - Evacuation and increased mobility

We will here reduce the change in running cost to $\Delta C_L = 1$, in addition to allowing some fractional diffusion, $\nu = 0.05$, with $\alpha = 1.8$. This will enable discontinuous jumps of the agents with no additional cost, which is meant to model sudden random evacuation of agents outside the dense area. Ordinary second order Mean Field Games doesn't fully capture this type of behavior, demonstrating a specific real world application of fractional Mean Field Games. As we can see in Figure 5.3.7 (a)-(b), this drastically changes the dynamics. While the crowd still is tightly located near the stage, the sporadic evacuation gives a less spiky Gaussian shape, also when the festival intensifies near t_{crit}. We interestingly observe that when given some more mobility, the whole crowd "escapes" the stage by moving backwards as $t \to t_{\text{crit}}$, spreads more out and taking advantage of the space. It is further not unreasonable that they wait with this expansion until the congestion cost increases. Finally, since $C_G = \frac{1}{2}C_F$, they return to the stage as $t \to 2$.



(a) Value function $u(x, t)$, scenario 1. **(b)** Crowd density $m(x, t)$, scenario 1.

(c) Value function $u(x, t)$, scenario 2. **(d)** Crowd density $m(x, t)$, scenario 2.

Figure 5.3.7: Congestion and coefficient of running cost meant to model a high-intensity festival.

5.3.5 Interpretation of results

We have in the preceding section applied our scheme to a practical problem, namely crowd dynamics at concerts. The model is of course a major simplification of the actual dynamics in a crowd, and the goal of this example was certainly not to provide any new insight into concert logistics. The goal was rather to demonstrate how a theoretical field like fractional Mean Field Games can be applied to a real world problem many of us can relate to. In that sense, it served both as a showcase of our numerical scheme, as well as to showcasing potential real world applications of fractional Mean Field Games. In that spirit, as the results mostly agree with our intuition, it seems that at least some real world dynamics have been captured.

Perhaps the most interesting result is what happens in Figure 5.3.7 (a)-(b), which is meant to simulate the crowd crush happening in 2021. It's interesting that the agents doesn't seem to even try to escape when the cost goes drastically up, or move away prior to the high-intensity phase. This might be a consequence of including time-dependence in H and F, which we have not studied in our numerical analysis. The less dense distribution of m in the second scenario, Figure 5.3.7 (c-d), is likely a consequence of the presence of diffusion. We also observe in this scenario that the density increases somewhat towards the end, which is to be expected given that the terminal cost G is lower than the running cost ($C_G = \frac{1}{2}$).

There has of course been made several questionable assumptions along the way. First off, in MFGs, each agent are free to choose its control without physical limitations. Hence, any agent can escape a highly dense position without being physically blocked by other agents, which contrasts what actually happened in the Astroworld crowd crush. We created a surrogate of this effect by dividing into a low-intensity and high-intensity phase, and increasing L in the latter. This assumption is rather artificial, and to better capture decrease in mobility due to crowded spaces, we could rather let the running cost L also depend on m. Secondly, modeling in one spatial dimension fails to capture 2D (or, more realistically, 3D) dynamics.

5.4 Implementation discussion

We will briefly discuss our schemes strengths and weaknesses. First off, as the method is a finite difference approach, it's quite easy to understand its derivation and algorithms. This is beneficial, as it's easy to understand the code and thus modify to accommodate for any specific need. It's further highly optimized, as almost all operations are linear matrix-vector operations, making it suitable for languages like Python. The only explicit loops are in the Newton iteration in HJB-Solve and in the fixed point iteration in MFG-Solve. Translating the Julia code to Python should be quite manageable, as the languages are similar and the pseudocode for the algorithms are provided.

As there are limited alternatives to numerical methods for fractional Mean Field Games, we will compare with the one other method the author has found, which is the Semi-Lagrangian solver created by Ersland et al. [10]. One benefit with our scheme, is that we get an explicit optimal feedback control, as we get an approximation for $u(x, t)$ in all points in Q_T (see Figure 5.3.4c). Another benefit is that we don't need any CFL conditions, as our scheme is implicit.

One drawback of the scheme as it currently is, is that it solves the non-linear

HJB-equation in each time-step, rather than only once for the entire system. Even though the Newton method seems to converge quickly in each time-step, it requires $\mathcal{O}(N_h^3 N_{\text{HJB}})$ operations in the worst case for each time step. Hence, an interesting and quite simple way to possibly improve HJB-Solve would be to change it to solve for all time-steps simultaneously in one large Newton-solver, similar to what Achdou et al. does in [7]. It would be a simple matter of simply stacking the vectors for each time-step on top of each other, and creating a large Jacobian matrix using the modular algorithms provided in this paper.

CHAPTER

SIX

DISCUSSION AND FURTHER WORK

6.1 Conclusion

The primary goal of this project was to develop a new numerical method for fractional Mean Field Games with symmetric α-stable diffusion, using a finite difference approach. To achieve this goal, we first derived a monotone discretization of the system, inspired by Achdou et al.'s method for local diffusion [7]. We discretized the fractional Laplacian by the (fractional) powers of the discrete Laplacian, and derive properties required for the convergence theory to generalize to non-local diffusion. Given that we have proved existence, uniqueness and convergence of the method, the primary goal has been achieved. The secondary goal was to create a software library and provide enough implementation details to avail the method to others, as no source code for solving fractional Mean Field Games is currently publicly available. To achieve this goal, we have provided pseudocode and derivation for the necessary algorithms in the one-dimensional case. We validated our algorithms on several test problems, before we demonstrated the method on a crowd dynamics problem inspired by the Astroworld crowd crush in 2021. All source code together with all simulations used in this paper is available at the author's GitHub[1]. We have therefore also succeeded in our secondary goal, but we would if given more time explore the possibility of implementing the method in higher dimensions. As mentioned in Chapter 5, the main challenge here lies in finding an efficient and accurate computation of the fractional Laplacian.

6.2 Suggestions for further work

Continuing the discussion mentioned above, a natural extension of our current work is to generalize Chapter 5 to encapsulate higher dimensions, as most real world problems are in at least two dimensions. The biggest challenge lies in finding an efficient way of computing the discrete fractional Laplacian, as we no longer have a closed-form formula for the coefficients K_α. The authors of [24] suggests some asymptotic results which can enhance the approximations of the fractional Laplacian in two dimensions, and it would be interesting to implement and test these. Given an efficient approximation of the fractional Laplacian, extending

[1] `https://github.com/tullebulle/FractionalMFGs.jl`

the rest of the framework should be a simple matter of vectorizing the 2D grid functions, and modifying the periodic boundary conditions to accommodate for the additional dimension.

Regarding performance, there are also several measures one can make for improvement. As discussed in Chapter 5, one could modify HJB-SOLVE to rather set up one Newton solver for all time-steps, rather than performing the Newton iterations for each time-step. This could potentially speed up the algorithm, and should not require any other measures than simply concatenating the Jacobian matrices we derived for each time-step, to one large Jacobian matrix. It could particularly gain a massive speedup especially for interpreted languages like Python, where for-loops in general are very slow. We have not performed an explicit study of the order of convergence in this paper, so this would be a natural starting point when seeking to optimize the performance.

We have in this paper restricted ourselves to only work with the fractional Laplacian, but our results would probably generalize to more general fractional operators $\mathcal{L}$ satisfying the properties in Lemma 2. To use our results directly, one would only need to find a degenerate elliptic discretization $\mathcal{L}_h$, and carry out the details to prove the properties in Chapter 3. Examples of useful operators to consider are non-symmetric Lévy operators as the one arising in the CGMY model in finance [48].

Enhancement measures aside, there are several other ways one could solve fractional Mean Field Games. Other numerical methods for PDEs include but are not restricted to finite element methods, spectral methods and finite volume methods. There has also recently been developed machine learning methods for solving first order games [11], which the author has studied in his project thesis [13]. This method uses the variational formulation derived using convex duality, and models the value function u as a residual neural network. As a Lagrangian method, it benefits from parallelization, particularly powerful in the high-dimensional regime. One could possibly extend this method to fractional Mean Field Games by using a similar architecture, and apply the semi-Lagrangian implementation of the fractional operators from [41]. This could possibly be a very efficient way of dealing with the expense of computing the fractional Laplacian in higher dimension.

The field of Mean Field Games is a rapidly growing field which has gained massive traction in the mathematical and computational community. The subfield of fractional Mean Field Games is by the time of writing a less explored topic in the literature, and we hope that this paper has demonstrated some of its applicability and its beautiful nature.

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APPENDICES

APPENDIX

A

MATHEMATICS

Heat equations

We here give provide some well-known results on the continuous and semi-discrete heat equation.

Lemma 38 (Heat equation)**.** *Let $\phi \in \mathcal{C}^2 \cap \mathcal{C}_b(\mathbb{R}^d)$. The heat equation*

$$\partial_t \Psi(x,t) = \Delta \Psi(x,t), \quad \text{in } \mathbb{R}^d \times (0,\infty) \tag{A.1}$$

$$\Psi(x,0) = \psi(x), \quad \text{in } \mathbb{R}^d \tag{A.2}$$

has a solution of the form

$$\Psi(x,t) = e^{t\Delta}\psi(x) := \int_{\mathbb{R}^d} \psi(x-y) G_c(y,t)\, dy = \int_{\mathbb{R}^d} \psi(y) G_c(x-y,t)\, dy, \tag{A.3}$$

where

$$G_c(x,t) := \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t} \right) = \mathcal{N}(x; \mathbf{0}, 2tI_d).$$

Proof. The result is well-known. For more details, see [16, Lemma 2.2], and the references therein. Boundedness of Ψ for all $t > 0$ follows easily from boundedness of ψ, since the heat kernel is in $L^1(\mathbb{R}^d)$. □

Lemma 39 (Solution of semi-discrete heat equation)**.** *Let $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. Then the semi-discrete heat equation*

$$\begin{aligned} \partial_t \Phi(x,t) &= \Delta_h \Phi(x,t) && \text{in } \mathbb{R}^d \times (0,\infty) \\ \Phi(0,t) &= \phi(x) && \text{in } \mathbb{R}^d \end{aligned} \tag{A.4}$$

has a solution given by the discrete convolution

$$\Phi(x,t) := e^{t\Delta_h}\phi(x) = \sum_{\beta \in \mathbb{Z}^d} \phi(x - h\beta) G_d(\beta, \frac{t}{h^2}), \tag{A.5}$$

where

$$G_d(\beta,t) := e^{-2td} \prod_{i=1}^{d} I_{|\beta_i|}(2t) \geq 0 \quad \forall \beta \in \mathbb{Z}^d, \forall t \geq 0, \tag{A.6}$$

where I_k is the k-th order modified Bessel function of first kind (A.10)*. We have*

$$\sum_{\beta \in \mathbb{Z}^d} G_d(\beta,t) = 1. \tag{A.7}$$

Proof. The proof of showing that (A.5) solves the equation is given in Theorem 1.1 in [49]. We will instead show that the solution is well-defined. Since ϕ is bounded,

$$|\Phi(x,t)| = |\sum_{\beta\in\mathbb{Z}^d}\phi(x-h\beta)G_d(\beta,\frac{t}{h^2})(x)| \leq \|\phi\|_{\mathcal{C}_b}\left|\sum_{\beta\in\mathbb{Z}^d}G_d(\beta,\frac{t}{h^2})\right| = \|\phi\|_{\mathcal{C}_b}< \infty.$$

□

An accurate approximation of the PDL on the whole space

The following is an approximation formula for the PDL on the whole space, and gave the inspiration for the approximation formula on the torus. We provide the derivation here in case it is of interest to implement the fractional Laplacian on the whole space. Using (A.15) we can write

$$(-\Delta_h)^{\alpha/2}u_j \approx F_1(j,R,\alpha) + F_2(j,R,\alpha), \quad j\in\mathbb{Z},$$

where

$$F_1(j,R,\alpha) := \frac{c_\alpha}{h^\alpha}\sum_{\substack{|m|\leq R\\ m\neq 0}}(u_j-u_{j-m})\frac{\Gamma(|m|-\alpha/2)}{\Gamma(|m|+1+\alpha/2)}, \tag{A.8}$$

$$F_2(j,R,\alpha) := \frac{c_\alpha}{h^\alpha}u_j\sum_{|m|\geq R}\frac{1}{|m|^{1+\alpha}}, \tag{A.9}$$

We will let $(-\Delta_h)^{\alpha/2}u_j \approx F_1 + F_2$. Now, to get an almost exact implementation of F_2, we observe the following:

$$\begin{aligned}
F_2(j,R,\alpha) &= \frac{c_\alpha}{h^\alpha}u_j\sum_{|m|\geq R}\frac{1}{|m|^{1+\alpha}}\\
&= 2\frac{c_\alpha}{h^\alpha}u_j\sum_{m:\ m\geq R}\frac{1}{m^{1+\alpha}}\\
&= 2\frac{c_\alpha}{h^\alpha}u_j\left(\sum_{m=1}^{\infty}\frac{1}{m^{1+\alpha}} - \sum_{m=1}^{R-1}\frac{1}{m^{1+\alpha}}\right)\\
&= 2\frac{c_\alpha}{h^\alpha}u_j\left(\zeta(1+\alpha) - \sum_{m=1}^{R-1}\frac{1}{m^{1+\alpha}}\right).
\end{aligned}$$

Lemma 40 (Limiting cases of the fractional Laplacian)**.** *For any* $f\in\mathcal{C}_b^\infty(\mathbb{R}^d)$,

$$\lim_{\alpha\to 2}(-\Delta)^{\frac{\alpha}{2}}f = -\Delta f$$
$$\lim_{\alpha\to 0}(-\Delta)^{\frac{\alpha}{2}}f = f$$

Proof. The proof can be found on page 10 in [16]. □

Bessel functions

We define the Bessel functions I_k and give some properties that is used in the paper (see [24], section 8.2). We have that

$$I_k(t) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+k+1)} \left(\frac{t}{2}\right)^{2m+k}, \quad k \in \mathbb{Z}. \tag{A.10}$$

Bessel functions are symmetric in k,

$$I_k(t) = I_{-k}(t), \quad k \in \mathbb{Z}, \tag{A.11}$$

and

$$I_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases} \tag{A.12}$$

Furthermore, we have that as $k \to \infty$,

$$I_k \sim \frac{1}{\sqrt{2\pi k}} \left(\frac{ez}{2k}\right)^k \sim \frac{z^k}{2^k k!}, \tag{A.13}$$

where the second proportionality $\sim$ comes from Stirling's approximation formula [50]. The following formula is given in [51], p. 305. For $\Re c > 0, -\Re k < \Re\gamma < 1/2$,

$$\int_0^{\infty} e^{-ct} I_k(ct) t^{\gamma-1}\,\mathrm{d}t = \frac{(2c)^{-\gamma}\Gamma(1/2-\gamma)\Gamma(\gamma+k)}{\sqrt{\pi}\Gamma(k+1-\gamma)}. \tag{A.14}$$

Useful lemmas

Lemma 41. *Let $z \to \infty$, then*

$$\begin{aligned} \frac{\Gamma(z+a)}{\Gamma(z+b)} &= z^{a-b}\left(1 + \frac{(a-b)(a+b-1)}{2z} + \frac{1}{12}\binom{a-b}{2}(3(a+b)^2 - 7a - 5b + 2)\frac{1}{z^2}\right) + E(a,b,z) \\ &= z^{a-b} + E(a,b,z) + \mathcal{O}\left(\frac{1}{z}\right) \end{aligned} \tag{A.15}$$

Proof. See [24], (which in turn cite [52], Chapter 4, 5.05). □

Lemma 42 (Duhamel's formula)**.** *Let*

$$\partial_t E(x,t) = \Delta_h E(x,t) - \tau(x,t), \quad E(x,0) = 0. \tag{A.16}$$

Then

$$E(x,t) = -\int_0^t e^{(t-s)\Delta_h} \tau(x,s)\,ds$$

solves (A.16).

Proof. We use Leibniz' integral rule [53, Theorem 3, page 425], since $\partial_t e^{(t-s)\Delta_h}\tau(x,s) = \Delta_h e^{(t-s)\Delta_h}\tau(x,s)$ is well-defined for all $x \in \mathbb{R}^d$ and $t - s \geq 0$, and obtain

$$\begin{aligned}\partial_t E(x,t) &= -\partial_t \int_0^t e^{(t-s)\Delta_h}\tau(x,s)\mathrm{d}s \\ &= -\tau(x,t) - \int_0^t \partial_t e^{(t-s)\Delta_h}\tau(x,s)\mathrm{d}s \\ &= -\tau(x,t) - \int_0^t \Delta_h e^{(t-s)\Delta_h}\tau(x,s)\mathrm{d}s = -\tau(x,t) + \Delta_h E(x,t),\end{aligned}$$

where we used the fact that $e^{0\Delta_h}\phi(x) = \phi(x)$ by the initial condition of (39). □

Lemma 43. *Let F, G satisfy Assumption 1 and let F and G satisfy strict monotonicity conditions. That is, for all $t \in [0,T]$,*

$$\int_{\mathbb{R}^2} (F[m_1](x,t) - F[m_2](x,t))\, d(m_1 - m_2)(x) \leq 0 \implies m_1 \equiv m_2, \quad \forall m_1, m_2 \in \mathcal{P}(\mathbb{R}^2),$$

$$\int_{\mathbb{R}^2} (G[m_1](x) - G[m_2](x))\, d(m_1 - m_2)(x) \leq 0 \implies m_1 \equiv m_2, \quad \forall m_1, m_2 \in \mathcal{P}(\mathbb{R}^2).$$

Let then $F_h[M], G_h[M]$ be as defined in (4.21)*. Then Assumption 8 and 9 follows.*

Proof. To prove Assumption 8, it's enough to show that $M \to F_h[M](x)$, $M \to G_h[M](x)$ are both continuous for all $x \in \mathbb{T}^2$. This is because F_h, G_h map to the same space as F, G respectively. First note that $\mathcal{P}(\mathbb{T}^2) \ni m \to F[m](x) \in \mathbb{R}$ is a continuous mapping by Assumption (1). Hence, we only need to show that the mapping $M \to \mathcal{J}_h M$, which is a mapping between $\mathcal{P}_h$ to $\mathcal{P}(\mathbb{T}^2)$, is continuous, since a composition of two continuous functions is continuous. Let $\epsilon = \delta$. Let then $M_1, M_2 \in \mathcal{P}_h$, and let

$$\|M_1 - M_2\|_\infty < \delta.$$

Then,

$$\begin{aligned} d(\mathcal{J}_h M_1, \mathcal{J}_h M_2) &= \sup_{f \in \mathrm{Lip}_{1,1}(\mathbb{T}^2)} \left\{ \int_{\mathbb{T}^2} f(x)\mathrm{d}(\mathcal{J}_h M_1 - \mathcal{J}_h M_2)(x) \right\} \\ &< \sup_{f \in \mathrm{Lip}_{1,1}(\mathbb{T}^2)} \left\{ \int_{\mathbb{T}^2} |f(x)|\delta \mathrm{d}x \right\} \\ &\leq \sup_{f \in \mathrm{Lip}_{1,1}(\mathbb{T}^2)} \|f\|_\infty \int_{\mathbb{T}^2} \delta \mathrm{d}x = \epsilon. \end{aligned}$$

An identical proof holds for G_h.

To show Assumption 9, it's enough to notice that $\mathcal{H}_h m \in \mathcal{P}_h$. Thus, the strict monotonicity assumption implies Assumption 9. □

Lemma 44 (A discrete Gronwall lemma)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence, and let*

$$a_{n+1} \leq C_1 a_n + C_2, \tag{A.17}$$

where $C_1 > 0$. Then

$$a_n \leq C_1^n \left(a_0 + C_2 \sum_{j=1}^{n} C_1^{-j} \right) \tag{A.18}$$

Proof. A proof of a slightly more general instance of the discrete Gronwall lemma is given in Proposition 3.2 in [54]. □

Lemma 45 (Non-expansiveness of $(I_{N_h^2}+\Delta t A^n)^{-1}$ in induced 1-norm)**.** *The matrix $(I_{N_h^2}+\Delta t A^n)^{-1}$, where A^n is defined in* (4.38)*, is non-expansive in the induced* 1*-norm for all $\Delta t > 0$.*

Proof. Note that we will here use i, j to denote row and column indices of a matrix, and should not be confused with the spatial indices used to denote the first and second index used to denote each direction in $x_{i,j} \in \mathbb{T}^2$. Recall from Lemma 31 that the adjoint of A^n is a weakly diagonal dominant Z-matrix. As the matrix is real, the adjoint is equal to the transpose, and thus A^n is a weakly column diagonal dominant matrix. We found in Lemma 32 that $(I_{N_h^2}+\Delta t A^n)$ is an M-matrix. Hence, by Definition 10, we can write it as

$$I_{N_h^2}+\Delta t A^n = sI_{N_h^2} - P,$$

where $s > \rho(P)$, and $P_{i,j} \geq 0$ for all i, j. As stated in Lemma 2.5.2.1 in [37], we have that we can always let

$$s = \max_i (I_{N_h^2}+\Delta t A^n)_{i,i} = 1 + \Delta t \max_i (A^n)_{i,i}$$

for the properties to hold. P is thus given by

$$\begin{aligned} P &= sI_{N_h^2} - (I_{N_h^2}+\Delta t A^n) \\ &= (1+\Delta t \max_i (A^n)_{i,i}) I_{N_h^2} - (I_{N_h^2}+\Delta t A^n) \\ &= \Delta t \left(\max_i (A^n)_{i,i} I_{N_h^2} - A^n \right), \end{aligned}$$

which is point-wise non-negative, since A^n is a Z-matrix with non-positive off-diagonals and positive diagonal. The induced 1-norm of P is given by

$$\|P\|_1 := \max_j \sum_i |P_{i,j}|.$$

Since P is non-negative, we can commit the absolute value within the sum. Computing it, we find

$$\begin{aligned} \|P\|_1 &= \max_j \sum_i P_{i,j} \\ &= \max_j \Delta t \left(\underbrace{\max_k (A^n)_{k,k} - (A^n)_{j,j}}_{=P_{j,j}\geq 0} + \sum_{i\neq j} \underbrace{-(A^n)_{i,j}}_{=P_{i,j}\geq 0} \right) \\ &= \Delta t \left(\max_k (A^n)_{k,k} + \max_j \left(-(A^n)_{j,j} + \sum_{i\neq j} |(A^n)_{i,j}| \right) \right) \\ &\leq \Delta t \max_k (A^n)_{k,k} \\ &= s - 1 \end{aligned} \tag{A.19}$$

where we used that $-A_{i,j} = |A_{i,j}|$, $i \neq j$, and that $\max_j \Big(-(A^n)_{j,j} + \sum_{i\neq j} |(A^n)_{i,j}| \Big) \leq 0$, given column diagonal dominance of A^n. Let's invert $I_{N_h^2} + \Delta t A^n$. Since $\frac{1}{s}||P||_1 < 1$, we can use a Neumann series [55] for the inversion. We find that

$$\begin{aligned}
(\frac{1}{s}(I_{N_h^2} + \Delta t A^n))^{-1} &= (I_{N_h^2} - \frac{1}{s}P)^{-1} \\
&= \sum_{k=0}^{\infty} (\frac{1}{s}P)^k.
\end{aligned}$$

Finally, we find an upper bound for $\|(I_{N_h^2} + \Delta t A^n)^{-1}\|_1$ using geometric series,

$$\begin{aligned}
\|(I_{N_h^2} + \Delta t A^n)^{-1}\|_1 &= \left\| \frac{1}{s} \sum_{k=0}^{\infty} (\frac{1}{s}P)^k \right\|_1 \\
&\leq \frac{1}{s} \sum_{k=0}^{\infty} \|(\frac{1}{s}P)^k\|_1 \\
&\leq \frac{1}{s} \sum_{k=0}^{\infty} (\frac{1}{s}\|P\|_1)^k \\
&= \frac{1}{s} \cdot \frac{1}{1 - \frac{1}{s}\|P\|_1} \\
&= \frac{1}{s - \|P\|_1} \\
&\leq 1,
\end{aligned}$$

since $s - \|P\|_1 \geq 1$, which follows from (A.19). □

Lemma 46 (Uniform L^∞-bound of $(M)_{n=0}^{N_T}$). *Let $(M)_{n=0}^{N_T}$ be given by (4.43). Then, assuming $\Delta t = \mathcal{O}(h^s)$, for any $s > \max(\alpha, 1)$, there exists $h_0, \Delta t_0 > 0$ such that for all $h < h_0, \Delta t < \Delta t_0$,*

$$\sup_n \|M^n\|_{L^\infty(\mathbb{T}_h^2)} \leq \|M_T\|_{L^\infty(\mathbb{T}_h^2)}.$$

Proof. Writing out

$$M_{i,j}^n + \Delta t \big(\nu(-\Delta_h)^{\frac{\alpha}{2}} M_{i,j}^n - \mathcal{T}_{i,j}(U^{n+1}, M^n)\big) = M_{i,j}^{n+1}$$

$$M_{i,j}^n + \frac{\nu \Delta t}{h^\alpha} \sum_{\beta \in \mathcal{I}_h^2} (M_{i,j}^n - M_{(i,j)+\beta}^n) K_\alpha(\beta) - \Delta t \mathcal{T}_{i,j}(U^{n+1}, M^n)\big) = M_{i,j}^{n+1}.$$

Using Corollary 1 with the definition of the transport operator (4.14), we get

$$\sup_{i,j} |\mathcal{T}_{i,j}(U^{n+1}, M^n)| \leq \frac{C_2}{h} \|M^n\|_\infty,$$

where $C_2 > 0$ depends only on T, C_g, C_F, C_G. Let $C_1 = \sum_{\beta \in \mathcal{I}_h^2} K_\alpha(\beta)$ from Lemma 7. Hence,

$$\|M^n\|_\infty |1 + \frac{C_1 \Delta t}{h^\alpha} - \frac{C_2 \Delta t}{h}| \leq \|M^{n+1}\|_\infty.$$

Assuming $\Delta t = ch^s$ where $s > \max(1, \alpha)$ gives that there exists constants $h_0, \Delta t_0 > 0$ such that for all $h < h_0, \Delta t < \Delta t_0$,

$$\|M^n\|_\infty C \leq \|M^{n+1}\|_\infty$$

for a constant C. Assuming an $L^\infty(\mathbb{T}^2)$-bounded terminal condition gives a uniform $L^\infty(\mathbb{T}^2)$-bound on $(M)_{n=0}^{N_T}$. □

Lemma 47 (Monotone Convergence Theorem)**.** *Let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of non-negative integrable functions $f_n : X \to \mathbb{R}_+$ where X is a measurable set, such that for every n and every x,*

$$0 \leq f_n(x) \leq f_{n+1}(x) \leq \infty.$$

If $\lim_{n\to\infty} \int_X f_n(x)dx < +\infty$, then $f(x) := \lim_{n\to\infty} f_n(x)$ is finite almost everywhere, and is integrable with

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

Proof. The theorem is well-known and can be found in [56] and references therein. □

Lemma 48 (Dominated Convergence Theorem)**.** *Let $g : X \to \mathbb{R}_+$ satisfy $\int_X |g|dx < +\infty$, and let $\{f_n(x)\}_{n\in\mathbb{N}}$ be a sequence of integrable functions such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in X$. Then, for $f(x) := \lim_{n\to\infty} f_n(x)$, we have*

$$\int_X f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx.$$

Proof. See Theorem 1.19 in [57]. □

APPENDIX

B

IMPLEMENTATION DETAILS

All code and latex-files used in this document are included in the GitHub repository linked below. Further explanations are given in the `readme`-file.

GitHub repository link

- `https://github.com/tullebulle/FractionalMFGs.jl`

B1 - Tables of parameters

Below are tables of all parameters and coefficients used in our numerical simulation, given such that results can be reproduced.

Verification of discretization of fractional Laplacian

Figure	α	$[x_{\min}, x_{\max}]$	h	R	R_2
3.6.1 (a)	0.5	$[-10^3, 10^3]$	0.2	10^4	
3.6.1 (b)	1.5	$[-10^3, 10^3]$	0.2	10^4	
3.6.2 (a)	0.5	$[-5, 5]$	0.01	10^4	10^8
3.6.2 (b)	0.2	$[-5, 5]$	0.01	10^4	10^8
3.6.2 (c)	0.1	$[-5, 5]$	0.01	10^4	10^8
3.6.2 (d)				10^4	10^7
3.6.3 (a)	1.99	$[-5, 5]$	0.01	10^5	10^8
3.6.3 (b)	0.01	$[-5, 5]$	0.01	10^5	10^8

Table B.1.1: Table of coefficients for simulations performed when testing the PDL implementation.

Iteration	α	$[x_{\min}, x_{\max}]$	h	R
Benchmark	1.5	$[-5, 5]$	10^{-3}	10^4
1	1.5	$[-5, 5]$	5^{-1}	10^4
2	1.5	$[-5, 5]$	5^{-2}	10^4
3	1.5	$[-5, 5]$	5^{-3}	10^4
4	1.5	$[-5, 5]$	5^{-4}	10^4

Table B.1.2: Table of coefficients for convergence plot of PDL, Figure 3.6.4.

Iteration	α	$[x_{\min}, x_{\max}]$	h	R
$(-\Delta_h)^{\frac{\alpha}{2}} u$	1.5	$[0, 1]$	0.005	10^5
$L_\alpha u$	1.5	$[0, 1]$	0.005	10^2

Table B.1.3: Table of coefficients for verification plot of PDL matrix, Figure 5.1.1.

Verification of PDE solvers

From here on, we let $[x_{\min}, x_{\max}] = [0, 1]$, $N_{\text{HJB}} = 5$, and $N_{\text{MFG}} = 5$.

Verification of the HJB solver

α	h	Δt	ν	R
1.0	10^{-2}	10^{-2}	0.1	500

Table B.1.4: Table of coefficients for testing the surrogate in the HJB test case, Figure 5.2.1.

α	h	Δt	ν	R
1.0	10^{-4}	10^{-4}	0.1	10

Table B.1.5: Table of coefficients for verification of HJB-Solve, Figure 5.2.2.

Verification of the FPK solver

α	h	Δt	ν	ϵ	c_1
1.5	10^{-3}	$5 \cdot 10^{-3}$	0.1	0.2	0.4

Table B.1.6: Table of coefficients used both in the verification of surrogate, Figure 5.2.3, and for verifying FPK-Solve, Figure 5.2.4.

Verification of MFG solver

α	h	Δt	ν	δ	c_1
1.5	10^{-3}	$5 \cdot 10^{-3}$	0.09^2	0.2	0.4

Table B.1.7: Table of coefficients for verification plot of PDL matrix, Figure 5.1.1.

Astroworld Crowd Crush

Ex.	α	h	Δt	ν	C_Q	C_B	C_L	ΔC_L	$C_{L,0}$	C_G
1	1.5	10^{-2}	0.005	0	1	0	70			1
2	1.5	10^{-2}	0.005	0.1	0.1	1	10			0.5
3, S1	1.8	10^{-2}	0.005	0	2	1	$C_L(t)$	2	1	0.5
3, S2	1.8	10^{-2}	0.005	0.05	2	1	$C_L(t)$	1	1	0.5

Table B.1.8: Table of coefficients for verification plot of PDL matrix, Figure 5.1.1.

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