Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Asymptotic behaviour for parabolic-hyperbolic partial differential equations

Master's thesis in Applied Physics and Mathematics
Supervisor: Jørgen Endal Letnes
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Preface

This master's thesis concludes my five years of studying Applied Physics and Mathematics at the Norwegian University of Science and Technology, specialising in Industrial Mathematics. I have really enjoyed being able to learn about many interesting topics and getting to know so many friendly people along the way.

The thesis is written under the supervision of Jørgen Endal Letnes. I would like to thank Jørgen for proposing the interesting topic of asymptotic behaviour for parabolic-hyperbolic PDEs, for enlightening discussions and for providing very thorough and good help throughout my work.

Finally, I would like to thank my friends and family for all their encouragement and support.

Sander Evensen

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Abstract

We study the asymptotic behaviour of the heat equation and a scalar convection-diffusion equation with a non-linear convection term, mainly following Zuazua [1]. The work is carried out in a framework of strong solutions on $\mathbb{R}^n \times (0, \infty)$ with initial data in $L^1(\mathbb{R}^n)$. First, using a scaling argument and parabolic L^1–L^∞-smoothing, we show that the asymptotic behaviour of the heat equation is given by the mass of its solution times the heat kernel, and that the same essentially applies when introducing a linear convection term in the equation. Then, we introduce the scalar convection-diffusion equation, and after developing some results regarding well-posedness, we investigate its asymptotic behaviour when the non-linearity is of the form $a\partial_x(u^q)$ for $q > 1$. By a similar approach as for the heat equation, we find weakly non-linear asymptotic behaviour for $q > 2$, in the sense that the non-linear term disappears, leaving us with the same behaviour as for the heat equation. The case $q = 2$ is not considered, here we refer to [2]. For $1 < q < 2$, we find that the non-linear term dominates, yielding a hyperbolic L^1–L^∞-smoothing result, where the rate is different from the parabolic result. The resulting asymptotic behaviour is called strongly non-linear, and is given by the solution of a purely convective equation with initial data given by the mass of the solution times Dirac's delta. Finally, we present some of the theory needed for this equation, specifically the uniqueness for entropy solutions with a non-negative finite Radon measure on $\mathbb{R}$ as initial data.

Sammendrag

Vi studerer den asymptotiske oppførselen til varmelikninga og en skalar konveksjon-diffusjonslikning med et ikke-lineært konveksjonsledd, hvor vi følger Zuazua [1]. Rammeverket for arbeidet er sterke løsninger i $\mathbb{R}^n \times (0, \infty)$ med initialdata i $L^1(\mathbb{R}^n)$. Først, ved bruk av et skaleringsargument og parabolsk L^1–L^∞-smoothing, viser vi at den asymptotiske oppførselen til varmelikninga er gitt ved massen til løsninga ganger varmekjernen, og at det samme essensielt også gjelder dersom vi legger til et lineært konveksjonsledd i likninga. Deretter introduserer vi den skalare konveksjon-diffusjonslikninga, og etter å ha vist noen resultater om velstiltheten til denne likninga, undersøker vi dens asymptotiske oppførsel når ikke-lineariteten er på formen $a\partial_x(u^q)$ for $q > 1$. Med en liknende fremgangsmåte som for varmelikninga, finner vi svak ikke-lineær asymptotisk oppførsel for $q > 2$ i den forstand at det ikke-lineære leddet forsvinner, slik at oppførselen blir den samme som for varmelikninga. Tilfellet $q = 2$ diskuteres ikke her, vi viser heller til [2]. For $1 < q < 2$ finner vi at det ikke-lineære leddet dominerer, noe som gir hyperbolsk L^1–L^∞-smoothing, hvor raten er endret i forhold til det parabolske tilfellet. Den resulterende asymptotiske oppførselen kalles sterk ikke-lineær, og er gitt av løsninga til en ren konveksjonslikning med initialdata gitt av massen til løsninga ganger Diracs delta. Til slutt presenterer vi noe teori for denne likninga, mer spesifikt entydighet for entropiløsninger med et ikke-negativt endelig Radonmål på $\mathbb{R}$ som initialdata.

Contents

Notation

$:=$, defined to be equal.

$\overset{*}{\rightharpoonup}$, weak* convergence.

a.e., almost everywhere, that is, everywhere except on a set of measure zero.

C, a constant.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, natural numbers.

$\mathbb{R}^n$, n-dimensional real Euclidean space. A point in $\mathbb{R}^n$ is $x = (x_1, \ldots, x_n)$.

$x \cdot y = \sum_{i=1}^{n} x_i y_i$, the scalar product between $x, y \in \mathbb{R}^n$.

$|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, the Euclidean norm for some $x \in \mathbb{R}^n$.

$\partial_x = \frac{\partial}{\partial x}$, the partial derivative with respect to the variable x. $\partial_x^2 = \frac{\partial^2}{\partial x^2}$ denotes the second derivative with respect to the variable x, while $'$ or $\frac{d}{dx}$ denotes the derivative of a function of one variable.

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, the gradient operator on $\mathbb{R}^n$. As a shorthand, we will sometimes let ∇u denote all the first order partial derivatives of u instead of the gradient vector. For example, we write $\nabla u \in L^\infty(\mathbb{R}^n)$ to say that $\partial_{x_i} u \in L^\infty(\mathbb{R}^n)$ $\forall\, i \in \{1, \ldots, n\}$.

$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, the Laplacian operator on $\mathbb{R}^n$.

U, an open set in $\mathbb{R}^n$. ∂U denotes its boundary, and $\overline{U} = U \cap \partial U$ denotes its closure.

sup, supremum, the least upper bound.

inf, infimum, the greatest lower bound.

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf_{m\geq n} x_m)$, the limit inferior of a sequence $\{x_n\}_{n=1}^{\infty}$.

$\operatorname{ess\,lim}$, the essential limit, meaning we consider the limit a.e.

$f = O(g)$ as $x \to x_0$, big-oh notation. This means that there exists a constant C such that for all x sufficiently close to x_0,

$$|f(x)| \leq C|g(x)|,$$

where $f, g : U \to \mathbb{R}$ and $x, x_0 \in U$. We will drop the limit when it is obvious.

$f(A) = \{f(x) \mid x \in A\}$, the image under the function $f : U \to \mathbb{R}$ of the subset A of U.

$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy$, the convolution of the functions $f, g : \mathbb{R}^n \to \mathbb{R}$.

$\text{supp}(f) = \overline{\{x \in U \mid f(x) \neq 0\}}$, the closed support of a function $f : U \to \mathbb{R}$. If the closed support of f is a compact subset of U, then f is said to have compact support.

$[f]^+(x) = \max\{f(x), 0\}$, $[f]^-(x) = \max\{-f(x), 0\}$, the positive and negative parts of a function $f : U \to \mathbb{R}$. $f = [f]^+ - [f]^-$, $|f| = [f]^+ + [f]^-$.

$B(x, r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$, the open ball in $\mathbb{R}^n$, centered in $x \in \mathbb{R}^n$ with radius $r > 0$.

$\text{sign} : \mathbb{R} \to \{-1, 0, 1\}$, the sign function, defined by

$$\text{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Similarly, we define the positive and negative sign functions respectively, by

$$\text{sign}^+(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0, \end{cases} \qquad \text{sign}^-(x) = \begin{cases} -1, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

$\mathbb{1}_A : U \to \{0, 1\}$, the indicator function of a subset A of the set U, defined by

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

$C(U) = \{u : U \to \mathbb{R} \mid u \text{ is continuous}\}$.

$C^k(U) = \{u : U \to \mathbb{R} \mid u \text{ is } k\text{-times continuously differentiable}\}$.

$C^\infty(U) = \{u : U \to \mathbb{R} \mid u \text{ is infinitely differentiable}\}$. Referred to as smooth functions.

$C_c^\infty(U) = \{u : U \to \mathbb{R} \mid u \text{ is infinitely differentiable with compact support}\}$. Referred to as test functions.

$L^p(U) = \{u : U \to \mathbb{R} \mid u$ is Lebesgue measurable, $\|u\|_{L^p(U)} < \infty\}$, where

$$\|u\|_{L^p(U)} = \left(\int_U |u|^p dx\right)^{\frac{1}{p}} \quad (1 \le p < \infty).$$

The norm is written as $\|\cdot\|_{L^p}$, when the domain U is clear. We shall refer to functions in $L^1(U)$ as integrable on U.

$L^\infty(U) = \{u : U \to \mathbb{R} \mid u$ is Lebesgue measurable, $\|u\|_{L^\infty(U)} < \infty\}$, where

$$\|u\|_{L^\infty(U)} = \operatorname*{ess\,sup}_U |u| = \inf\{C \ge 0 \mid \mu(\{x \in U \mid |u(x)| > C\}) = 0\}.$$

μ is the Lebesgue measure.

$L^p_{loc}(U) = \{u : U \to \mathbb{R} \mid u \in L^p(K)\ \forall$ compact $K \subset U\}$. We shall refer to functions in $L^1_{loc}(U)$ as locally integrable on U.

$L^1(U; 1+|x|) = \{u : U \to \mathbb{R} \mid u \in L^1(U)$ and $\|u\|_{L^1(U;1+|x|)} < \infty\}$, where

$$\|u\|_{L^1(U;1+|x|)} = \int_{\mathbb{R}^n} |u|(1+|x|)dx.$$

Referred to as $L^1(U)$ with weights. The weights may vary.

$W^{k,p}(U) = \{u : U \to \mathbb{R} \mid u$ is locally integrable and $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u \in L^p(U)\ \forall\ |\alpha| \le k$ in the weak sense$\}$, where $1 \le p \le \infty$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \sum_{i=1}^n \alpha_i$, and k, α_i are non-negative integers. Referred to as Sobolev spaces.

Chapter 1

Introduction

In the study of parabolic partial differential equations (PDEs), the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

plays a vital role as the prime example for this class of equations. Because it is so fundamental, this equation has been widely studied for centuries, resulting in a well-developed and rich theory. In the specialisation project [3], we studied classical results regarding well-posedness for the heat equation. By well-posedness, we mean that [4, p. 7]:

(i) A solution exists.

(ii) The solution is unique.

(iii) The solution depends continuously on the initial data.

Now, by adding a non-linear convection term on the right hand side of (1.1), we get a general scalar convection-diffusion equation

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla(F(u)), & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Details on the terms of this equation will be made clear later, but by adding a hyperbolic term we now have a parabolic-hyperbolic equation. This combination makes (1.2) suitable as a simple model for physical phenomena containing both diffusion and convection [5, p. 43]. Depending on the choice of a and F, some examples of such phenomena are fluid displacement in a porous medium, as well as other types of fluid flow with both non-linear wave propagation and heat conduction [1, p. 1].

We are interested in the mathematical properties of (1.2). Because of its relative simplicity as a model, it is suitable for rigorous mathematical investigation in order to get a better understanding of the mathematics of diffusive-convective phenomena, as stated in [5, p. 43].

The main property we wish to study in this thesis is asymptotic behaviour, that is, the behaviour of solutions for large times. We will work in a framework of strong solutions with initial data $u_0 \in L^1(\mathbb{R}^n)$. Strong solutions will be defined later on, but we essentially require the solutions to solve the equations a.e., thus requiring them to be actual functions and not just distributions.

We will mainly follow the work done by Zuazua in [1], but we will expand his arguments and pay more attention to details, to gain a better understanding of the asymptotic behaviour and how it arises. An important technique in this regard is a scaling argument, as we will see.

The asymptotic behaviour will be studied both for the heat equation (1.1) and the scalar convection-diffusion equation (1.2), not only with a non-linear convection term, but also with a linear one. In particular, we wish to compare the behaviour of these equations to the heat equation, to see how the introduction of a convection term influences the behaviour.

1.1 Thesis outline

The thesis is structured as follows:

- *Chapter 1: Introduction.* We introduce some background and present the goal and outline of this thesis, as well as its relevance to the Sustainable Development Goals of the United Nations.

- *Chapter 2: A priori estimates for the heat equation.* We repeat useful definitions and results developed for the heat equation (1.1) in the specialisation project [3].

- *Chapter 3: Asymptotic behaviour for the heat equation* The theory on the heat equation from the specialisation project [3] is continued and completed by characterising its asymptotic behaviour. Following Chapter 1 of [1], by introducing a scaling argument and parabolic L^1–L^∞-smoothing (3.1), we motivate that we expect the asymptotic behaviour to be given by the heat kernel G times the mass M of the solution. This is proved rigorously in Theorem 3.3. Finally, we introduce a linear convection term in the heat equation, and see in Theorem 3.5 that it essentially does not affect the asymptotic behaviour, except by a shift of the heat kernel.

- *Chapter 4: Scalar convection-diffusion equations.* A general scalar convection-diffusion equation of the form (1.2) is introduced within the suitable framework of strong solutions. Following Chapter 4 of [1], we develop two main results which establish the necessary well-posedness for this equation, first with initial data in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ in Theorem 4.6, then in $L^1(\mathbb{R}^n)$ in Theorem 4.11 by an approximation argument. One of several important results in this theorem is that the parabolic L^1–L^∞-smoothing still holds for this equation.

- *Chapter 5: Asymptotic behaviour for a convection-diffusion equation.* We investigate the asymptotic behaviour of the convection-diffusion equation in one dimension with a particular convection term of the form $a\partial_x(u^q)$ with $q > 1$. Through a scaling argument similar to the one for the heat equation, we find that the behaviour depends on the value of q. For $q > 2$, the convection term disappears, leading to the same asymptotic behaviour as for the heat equation. For this reason, this behaviour is said to be weakly non-linear. The result is proved in Theorem 5.2 following Chapter 5 of [1] and depends on the L^1–L^∞-smoothing from the previous chapter. We do not consider the case of $q = 2$, but rather refer to [2]. For $1 < q < 2$, on the other hand, the non-linear convection term dominates, leading to a hyperbolic L^1–L^∞-smoothing result given in Lemma 5.5, where the rate is sharper than in the parabolic result. The resulting asymptotic behaviour is given by a solution of a purely convective equation (5.8) with the mass M of the solution times Dirac's delta function δ as initial data. This behaviour is characterised as strongly non-linear, and since it differs from the parabolic equations we have worked with so far, the proof is more difficult in this case. We show this result in Theorem 5.9 by following [5, 6].

- *Chapter 6: Entropy solutions for a convection equation.* The purely convective equation resulting from the strongly non-linear behaviour in the previous chapter is discussed in more detail. Following [7, 8], we discuss uniqueness for so-called entropy solutions for a general convective equation (6.1) with a non-negative finite Radon measure on $\mathbb{R}$ as initial data.

- *Chapter 7: Further work.* We propose some possibilities for further work, such as proving some results which we only stated in the thesis, and introducing new terms in order to consider the asymptotic behaviour of other types of parabolic-hyperbolic equations.

Finally, for convenience and completeness, there are two appendices containing preliminary results and heat kernel estimates, respectively.

1.2 Relevance to the Sustainable Development Goals

The 17 Sustainable Development Goals (SDGs) of the United Nations were created with the goal of achieving a more sustainable development of the world [9]. Although this thesis is theoretical in nature, dealing with the mathematical theory on PDEs, it can still be relevant for several of the SDGs.

First, Goal 4 is concerning quality education, and the development and study of mathematical theory is important with regards to this.

Second, Goal 9 is about industry, innovation and infrastructure, and the theory discussed in this thesis may be relevant because the equations we study arise from physical phenomena, as discussed earlier. Therefore, a better mathematical understanding of the equations may lead to better modelling and understanding of these phenomena, which may have applications in many industries.

Third, the results discussed are all results of global scientific research collaborations, which is relevant for Goal 17, partnerships for the goals.

Finally, the equations studied may be used in many types of applications, so several of the other goals may also be relevant depending on the type of application one considers.

Chapter 2

A priori estimates for the heat equation

In the specialisation project [3], we considered the initial-value problem for the heat equation in $\mathbb{R}^n$ $(n \geq 1)$

$$\begin{cases} \partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,T) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $T > 0$, u_0 is some given initial value and u is the solution to be found. We showed existence and uniqueness of solutions to (2.1), both in the so-called classical and very weak sense, before we finally developed some well-known regularity estimates in L^p-norm for the class of strong solutions. This class lies between the classical and very weak solutions, in the sense that we have the inclusions classical solutions $\subset$ strong solutions $\subset$ very weak solutions. The regularity estimates are useful on their own, by providing information on how the solutions depend on their initial data, but they will also prove to be useful when we later turn our attention to other parabolic differential equations. For this reason, we will repeat these results without proof in this chapter, referring to Chapter 3 and 4 of the specialisation project [3] for details.

Definition 2.1. *We call u a very weak solution of the heat equation on $\mathbb{R}^n \times [0,T)$ with initial value u_0, if $u \in L^1_{loc}(\mathbb{R}^n \times (0,T))$, $u_0 \in L^1_{loc}(\mathbb{R}^n)$, and if*

$$\int_0^T \int_{\mathbb{R}^n} [u(x,t)\partial_t\varphi(x,t) + u(x,t)\Delta\varphi(x,t)]dxdt + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)dx = 0,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,T))$.

Theorem 2.2. *Assume* $u_0 \in L^1(\mathbb{R}^n)$*, and define*

$$u(x,t) = (G(\cdot,t) * u_0)(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \quad (x \in \mathbb{R}^n,\ t > 0). \tag{2.2}$$

Then $u \in C([0,\infty); L^1(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T))$*, and the initial-value problem* (2.1) *admits a unique*[1] *very weak solution given by* (2.2)*.*

Remark 2.3. *The function*

$$G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \quad (x \in \mathbb{R}^n,\ t > 0),$$

is called the heat kernel on $\mathbb{R}^n$*.*

Lemma 2.4. *Let* $u \in L^1(\mathbb{R}^n \times (0,T)) \cap C([0,T]; L^1(\mathbb{R}^n))$ *be a very weak solution of the heat equation with initial data* $u_0 \in L^1(\mathbb{R}^n)$*. Then, the mass of* u *is conserved, i.e.*

$$\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx \qquad \forall\ t \in [0,T).$$

Remark 2.5. *This result still holds when we only assume* $u \in C([0,T]; L^1(\mathbb{R}^n))$*, that is, when we remove the assumption of* u *to be integrable in time. The reason for this is that we in the proof [3, pp. 27–30] multiply* u *with a test function that has compact support in time, and thus we only need to consider the integral of* u *on a compact time interval. Since continuous functions on a compact set are bounded, it is sufficient with* $u \in C([0,T]; L^1(\mathbb{R}^n))$ *for the proof to hold.*

Definition 2.6. *We call* u *a strong solution of the heat equation with initial value* u_0 *if:*

(i) $\partial_t u, \Delta u \in L^1_{loc}((0,T); L^1(\mathbb{R}^n))$*.*

(ii) $\partial_t u - \Delta u = 0$ *a.e. in* $\mathbb{R}^n \times (0,T)$*.*

(iii) $u(\cdot,0) = u_0$ *a.e. in* $\mathbb{R}^n$*.*

Lemma 2.7. *Let* $u, v \in C([0,T]; L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,T))$ *be two strong solutions of the heat equation with initial values* $u_0, v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Then we have the following:*

(i) If $u_0(x) \leq v_0(x)$ *for a.e.* $x \in \mathbb{R}^n$*, then* $u(x,t) \leq v(x,t)$ *for a.e.* $(x,t) \in \mathbb{R}^n \times [0,T)$*.*

(ii) If $u_0(x) \geq 0$ *for a.e.* $x \in \mathbb{R}^n$*, then* $u(x,t) \geq 0$ *for a.e.* $(x,t) \in \mathbb{R}^n \times [0,T)$*.*

(iii) $\|(u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^n)}$ $\forall\ t \in [0,T)$*.*

[1] Note that there is an error in Theorem 3.5 in the specialisation project [3, pp. 24–26], which deals with uniqueness for very weak solutions of the heat equation. The theorem falsely states that we only require initial data $u_0 \in L^1_{loc}(\mathbb{R}^n)$ to have a unique solution. The correct statement is that we require $u_0 \in L^1(\mathbb{R}^n)$, which guarantees that the solution exists and is given by the convolution solution. The error in the statement of the theorem does not affect its proof, which correctly assumes $u_0 \in L^1(\mathbb{R}^n)$.

(iv) $\|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)}\ \forall\ t \in [0,T)$.

Lemma 2.8. *Let $u \in C([0,T);L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,T))$ be a strong solution of the heat equation with initial value $0 \leq u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $p \in (1,\infty)$. Then, for $t \in [0,T)$, we have the following:*

(i) $\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leq \|u_0\|_{L^p(\mathbb{R}^n)}$.

(ii) $\left\|\nabla u^{\frac{p}{2}}\right\|^2_{L^2(\mathbb{R}^n\times[0,t))} \leq \frac{p}{4(p-1)}\|u_0\|^p_{L^p(\mathbb{R}^n)}$.

Lemma 2.9. *Let $u \in C([0,T);L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,T))$ be a strong solution of the heat equation with initial value $0 \leq u_0 \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Let $p \in (1,\infty)$ and $q \in [1,p)$. Then, for $t \in (0,T)$, we have the following:*

$$\|u(\cdot,t)\|_{L^p} \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q},$$

where

$$C = \left(\frac{np^2}{8(p-1)}(\frac{1}{q}-\frac{1}{p})\tilde{C}^2\right)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}.$$

$\tilde{C} = \tilde{C}(n)$ is the constant from the Sobolev inequality (Theorem A.1).

Remark 2.10. *We did not cover the case $p = \infty$ in the spesialisation project. It is tempting to simply let $p \to \infty$ in Lemma 2.9, but this leads to $C \to \infty$, thus not giving us any useful information. We will return to this problem in Chapter 4.*

Chapter 3

Asymptotic behaviour for the heat equation

We now turn our attention to asymptotic behaviour, asking what happens to solutions of the heat equation (2.1) as $t \to \infty$. Our previous results have been developed for the time interval $[0, T)$ for some $T > 0$, but we will now extend this to $[0, \infty)$, since the validity of these results does not depend on the choice of T.

In this chapter we will follow the approach made in Chapter 1 of Zuazua [1], replicating his results while trying to fill out some of his arguments in more detail. In the first section, we will motivate what type of asymptotic behaviour we should expect from the heat equation by the use of a scaling argument. This technique will play a crucial role throughout the thesis when we are discussing asymptotic behaviour.

3.1 Motivation

Throughout this chapter, we will work in the framework of initial data $u_0 \in L^1(\mathbb{R}^n)$. As we saw in Theorem 2.2 in Chapter 2, the heat equation is well-posed in this case. Furthermore, with initial data in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, it fulfills an L^p–L^q estimate given in Lemma 2.9 for strong solutions. Through an approximation argument, this estimate may be extended to still hold for $q = 1$ in the case of initial data in $L^1(\mathbb{R}^n)$, analogously to what we will see in Chapter 4. Another way to obtain the estimate, is as follows:

$$\|u(\cdot,t)\|_{L^p} = \|G(\cdot,t) * u_0\|_{L^p} \le \|G(\cdot,t)\|_{L^p}\|u_0\|_{L^1} \le C_p t^{-\frac{n}{2}(1-\frac{1}{p})}\|u_0\|_{L^1} \ \forall\ t > 0, \quad (3.1)$$

where $p \in [1, \infty]$. We have used the solution formula from Theorem 2.2, Young's inequality (Theorem A.11) and an estimate for the L^p-norm of the heat kernel (Lemma B.2). The estimate shows that the solution decays in L^p-norm over time, and also implies that the expression $t^{\frac{n}{2}(1-\frac{1}{p})}\|u(\cdot,t)\|_{L^p}$ is bounded for all $t > 0$, with a bound independent of time. This suggests that it might be interesting to investigate the behaviour of $t^{\frac{n}{2}(1-\frac{1}{p})}u(\cdot,t)$ in L^p-norm as $t \to \infty$.

Furthermore, we define

$$M := \int_{\mathbb{R}^n} u_0(x)dx, \tag{3.2}$$

i.e. the initial mass of the solution. From Lemma 2.4 and Remark 2.5, we have that this mass is conserved. Thus this property must be fulfilled for $u(\cdot, t)$ when $t \to \infty$ as well, suggesting that M plays a role in the asymptotic behaviour.

To gain some more insight into what kind of asymptotic behaviour we should expect for solutions of the heat equation, we make use of the well-known fact that they are invariant under a specific scaling. Indeed, with u as defined above, the rescaled function

$$u_\lambda(x,t) := \lambda^n u(\lambda x, \lambda^2 t),$$

where $\lambda > 0$, is a solution to (2.1) with initial data $\lambda^n u_0(\lambda x)$, i.e.

$$\begin{cases} \partial_t u_\lambda - \Delta u_\lambda = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u_\lambda(x,0) = \lambda^n u_0(\lambda x), & x \in \mathbb{R}^n. \end{cases}$$

The term λ^n which we multiply u with is chosen so that u_λ conserves its mass like u. We now claim that the initial data fulfills

$$\int_{\mathbb{R}^n} u_\lambda(x,0)\varphi(x)dx \to M\varphi(0) \text{ as } \lambda \to \infty,$$

for all $\varphi \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. This in turn implies

$$u_\lambda(\cdot, 0) \to M\delta \text{ as } \lambda \to \infty,$$

where δ denotes Dirac's delta function. To justify this claim, we use (3.2) and a change of variables $z := \lambda x$ to get:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u_\lambda(x,0)\varphi(x)dx - M\varphi(0) \right| &= \left| \int_{\mathbb{R}^n} \lambda^n u_0(\lambda x)\varphi(x)dx - \int_{\mathbb{R}^n} u_0(z)\varphi(0)dz \right| \\ &\overset{z:=\lambda x}{=} \left| \int_{\mathbb{R}^n} u_0(z)\varphi(\frac{z}{\lambda})dz - \int_{\mathbb{R}^n} u_0(z)\varphi(0)dz \right| \\ &\leq \int_{\mathbb{R}^n} |u_0(z)| \left| \varphi(\frac{z}{\lambda}) - \varphi(0) \right| dz \\ &\to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

Passing the limit inside the integral is justified by the Dominated convergence theorem (Theorem A.8), since the integrand is bounded by the integrable function $2\|\varphi\|_{L^\infty} u_0$.

Thus, we know that the initial data $u_\lambda(\cdot, 0)$ converges to $M\delta$ as $\lambda \to \infty$. It is well-known that the heat kernel G is a solution to the heat equation with δ as initial data, and thus we should expect $u_\lambda(\cdot, t)$ to converge to $MG(\cdot, t)$ for all $t > 0$ as $\lambda \to \infty$.

To connect this to our question of what happens when $t \to \infty$, consider convergence in $L^1(\mathbb{R}^n)$ with $t = 1$. We can write this as

$$\|u_\lambda(\cdot, 1) - MG(\cdot, 1)\|_{L^1} \to 0 \text{ as } \lambda \to \infty.$$

This statement is in fact equivalent to

$$\|u(\cdot, t) - MG(\cdot, t)\|_{L^1} \to 0 \text{ as } t \to \infty,$$

which can be seen through a change of variables $z := \lambda x$ together with the relation $G(x,t) = t^{-\frac{n}{2}} G(\frac{x}{\sqrt{t}}, 1)$:

$$\begin{aligned}
&\|u_\lambda(\cdot, 1) - MG(\cdot, 1)\|_{L^1} \\
&= \int_{\mathbb{R}^n} \left|\lambda^n u(\lambda x, \lambda^2) - MG(x, 1)\right| dx \overset{z:=\lambda x}{=} \int_{\mathbb{R}^n} \left| u(z, \lambda^2) - M \frac{G(\frac{z}{\lambda}, 1)}{\lambda^n} \right| dz \\
&= \int_{\mathbb{R}^n} \left| u(z, \lambda^2) - MG(z, \lambda^2)\right| dz = \left\| u(\cdot, \lambda^2) - MG(\cdot, \lambda^2) \right\|_{L^1}.
\end{aligned}$$

Thus, we obtain information about the expected asymptotic behaviour of solutions u through the rescaled solutions u_λ. In the next section we will prove that our expectations are indeed correct, so that the solutions do converge to $MG(\cdot, t)$ as $t \to \infty$, not just in $L^1(\mathbb{R}^n)$, but also in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$.

3.2 Asymptotic behaviour

Before we can show our main theorem, we need a growth bound under the strengthened assumption of initial data in the space of L^1-functions with weights, i.e. $u_0 \in L^1(\mathbb{R}^n; 1 + |x|)$ (see Notation).

Lemma 3.1. *Let $p \in [1, \infty]$, then there exists a constant $C_p > 0$ such that*

$$\|G(\cdot, t) * u_0 - MG(\cdot, t)\|_{L^p} \le C_p t^{-\frac{n}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}^n; |x|)} \ \forall\ t > 0,$$

for all $u_0 \in L^1(\mathbb{R}^n; 1 + |x|)$ with $\int_{\mathbb{R}^n} u_0(x)dx = M \neq 0$.

Proof. We follow the proof of Lemma 1.2 in Zuazua [1, pp. 9–10].

Using that $M = \int_{\mathbb{R}^n} u_0(x)dx$, we get

$$\begin{aligned}
(G(\cdot, t) * u_0)(x) - MG(x, t) &= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy - \int_{\mathbb{R}^n} u_0(y)dy (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \\
&= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left[e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}} \right] u_0(y)dy.
\end{aligned}$$

To rewrite the integrand, we use the following generalised mean value theorem, which is valid for $f \in C^1$:

$$f(x - y) - f(x) = \left(\int_0^1 \nabla f(x - \theta y) d\theta \right) \cdot (-y). \tag{3.3}$$

In our case this yields

$$e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x|^2}{4t}} = \frac{1}{2t}\int_0^1 y\cdot(x-\theta y)e^{-\frac{|x-\theta y|^2}{4t}}d\theta,$$

and so

$$(G(\cdot,t)*u_0)(x) - MG(x,t) = \frac{(4\pi t)^{-\frac{n}{2}}}{\sqrt{t}}\int_0^1\int_{\mathbb{R}^n} y\cdot\frac{(x-\theta y)}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}u_0(y)dyd\theta,$$

where we have used Fubini's theorem (Theorem A.10) to interchange the integrals.

Next, we further apply Fubini's theorem and a change of variables $z := \frac{x-\theta y}{2\sqrt{t}}$ to calculate the norm for $p=1$:

$$\begin{aligned}
&\|(G(\cdot,t)*u_0)(\cdot) - MG(\cdot,t)\|_{L^1} \\
&\leq \frac{(4\pi t)^{-\frac{n}{2}}}{\sqrt{t}}\int_{\mathbb{R}^n}\int_0^1\int_{\mathbb{R}^n}|y|\frac{|x-\theta y|}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}|u_0(y)|dyd\theta dx \\
&= (4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-\frac{1}{2}}\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|x-\theta y|}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}|y||u_0(y)|dxdyd\theta \\
&\overset{z:=\frac{x-\theta y}{2\sqrt{t}}}{=} (4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-\frac{1}{2}}\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|z|e^{-|z|^2}|y||u_0(y)|2^n t^{\frac{n}{2}}dzdyd\theta \\
&= \pi^{-\frac{n}{2}}t^{-\frac{1}{2}}\left\||z|e^{-|z|^2}\right\|_{L^1}\int_0^1\int_{\mathbb{R}^n}|y||u_0(y)|dyd\theta \\
&= \pi^{-\frac{n}{2}}\left\||z|e^{-|z|^2}\right\|_{L^1}t^{-\frac{1}{2}}\|u_0\|_{L^1(\mathbb{R}^n;|x|)}\ \forall\ t>0.
\end{aligned}$$

Thus we have our result for $p=1$ with $C_1 = \pi^{-\frac{n}{2}}\left\||z|e^{-|z|^2}\right\|_{L^1}$.

For $p=\infty$, we get

$$\begin{aligned}
&\|(G(\cdot,t)*u_0)(\cdot) - MG(\cdot,t)\|_{L^\infty} \\
&= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left|\frac{(4\pi t)^{-\frac{n}{2}}}{\sqrt{t}}\int_0^1\int_{\mathbb{R}^n} y\cdot\frac{(x-\theta y)}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}u_0(y)dyd\theta\right| \\
&\leq (4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-\frac{1}{2}}\int_{\mathbb{R}^n}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\int_0^1\frac{|x-\theta y|}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}|y||u_0(y)|d\theta dy \\
&\leq (4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-\frac{1}{2}}\int_{\mathbb{R}^n}\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n}\left\{\int_0^1\frac{|x-\theta y|}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}d\theta\right\}|y||u_0(y)|dy \\
&= (4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-\frac{1}{2}}\|u_0\|_{L^1(\mathbb{R}^n;|x|)}\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n}\left\{\int_0^1\frac{|x-\theta y|}{2\sqrt{t}}e^{-\frac{|x-\theta y|^2}{4t}}d\theta\right\} \\
&\leq (4\pi)^{-\frac{n}{2}}t^{-\frac{n}{2}-\frac{1}{2}}\|u_0\|_{L^1(\mathbb{R}^n;|x|)}\int_0^1\operatorname*{ess\,sup}_{z\in\mathbb{R}^n}\left\{|z|e^{-|z|^2}\right\}d\theta \\
&= (4\pi)^{-\frac{n}{2}}\left\||z|e^{-|z|^2}\right\|_{L^\infty}t^{-\frac{n}{2}-\frac{1}{2}}\|u_0\|_{L^1(\mathbb{R}^n;|x|)}\ \forall\ t>0,
\end{aligned}$$

and so $C_\infty = (4\pi)^{-\frac{n}{2}} \left\| |z| e^{-|z|^2} \right\|_{L^\infty}$.

For $p \in (1,\infty)$, we use the interpolation inequality for L^p-norms (Theorem A.4) to interpolate between our results for $p = 1$ and $p = \infty$. This yields

$$\begin{aligned}
&\|(G(\cdot,t) * u_0)(\cdot) - MG(\cdot,t)\|_{L^p} \\
&\leq \|(G(\cdot,t) * u_0)(\cdot) - MG(\cdot,t)\|_{L^1}^{\frac{1}{p}} \|(G(\cdot,t) * u_0)(\cdot) - MG(\cdot,t)\|_{L^\infty}^{\frac{p-1}{p}} \\
&\leq \pi^{-\frac{n}{2}\frac{1}{p}} \left\| |z| e^{-|z|^2} \right\|_{L^1}^{\frac{1}{p}} t^{-\frac{1}{2}\frac{1}{p}} \|u_0\|_{L^1(\mathbb{R}^n;|x|)}^{\frac{1}{p}} (4\pi)^{-\frac{n}{2}\frac{p-1}{p}} \left\| |z| e^{-|z|^2} \right\|_{L^\infty}^{\frac{p-1}{p}} t^{-(\frac{n}{2}+\frac{1}{2})\frac{p-1}{p}} \|u_0\|_{L^1(\mathbb{R}^n;|x|)}^{\frac{p-1}{p}} \\
&= 2^{-\frac{n}{2}(1-\frac{1}{p})} \pi^{-\frac{n}{2}} \left\| |z| e^{-|z|^2} \right\|_{L^1}^{\frac{1}{p}} \left\| |z| e^{-|z|^2} \right\|_{L^\infty}^{(1-\frac{1}{p})} t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}^n;|x|)} \ \forall t > 0.
\end{aligned}$$

Thus we have our result for $p \in (1,\infty)$ with

$$C_p = 2^{-\frac{n}{2}(1-\frac{1}{p})} \pi^{-\frac{n}{2}} \left\| |z| e^{-|z|^2} \right\|_{L^1}^{\frac{1}{p}} \left\| |z| e^{-|z|^2} \right\|_{L^\infty}^{(1-\frac{1}{p})},$$

which concludes the proof. □

With this lemma at hand, we are now able to show rigorously the asymptotic behaviour which we motivated in Section 3.1, starting with the case $u_0 \in L^1(\mathbb{R}^n; 1 + |x|)$:

Theorem 3.2. *Let $u_0 \in L^1(\mathbb{R}^n; 1 + |x|)$ with $\int_{\mathbb{R}^n} u_0(x)dx = M \neq 0$. Then, the solution u given by* (2.2) *of the heat equation* (2.1) *satisfies*

$$t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_{L^p} \to 0 \text{ as } t \to \infty, \tag{3.4}$$

for all $p \in [1,\infty]$.

Proof. The result follows directly from Lemma 3.1. Indeed, using this result gives that for $p \in [1,\infty]$ and $\forall\, t > 0$,

$$0 \leq t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_{L^p} \leq C_p t^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}^n;|x|)} \to 0 \text{ as } t \to \infty. \tag{3.5}$$

□

Finally, we obtain the result for $u_0 \in L^1(\mathbb{R}^n)$ through an approximation:

Theorem 3.3. *Let $u_0 \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} u_0(x)dx = M \neq 0$. Then, the solution u given by* (2.2) *of the heat equation* (2.1) *satisfies*

$$t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,t) - MG(\cdot,t)\|_{L^p} \to 0 \text{ as } t \to \infty, \tag{3.6}$$

for all $p \in [1,\infty]$.

Proof. We follow the proof of Theorem 1.1 in Zuazua [1, pp. 10–11].

To extend Theorem 3.2 to $u_0 \in L^1(\mathbb{R}^n)$, we will argue that $L^1(\mathbb{R}^n; 1 + |x|)$ is dense in $L^1(\mathbb{R}^n)$. Indeed, we know that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ (see for example Bresiz [10, pp. 97–98]), and that $C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n; 1 + |x|)$. Therefore, given any $u_0 \in L^1(\mathbb{R}^n)$

with $\int_{\mathbb{R}^n} u_0(x)dx = M$, there exists a sequence $\{\varphi_k\} \subset C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n; 1+|x|)$ such that for each $k \in \mathbb{N}$, $\varphi_k \in L^1(\mathbb{R}^n; 1+|x|)$ with $\int_{\mathbb{R}^n} \varphi_k(x)dx = M$. Furthermore, $\varphi_k \to u_0$ in $L^1(\mathbb{R}^n)$ as $k \to \infty$.

Thus, for $p \in [1, \infty]$ and $\forall\, t > 0$, we get

$$\begin{aligned} &t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * u_0 - MG(\cdot,t)\|_{L^p} \\ &= t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * u_0 - MG(\cdot,t) \pm G(\cdot,t) * \varphi_k\|_{L^p} \\ &\leq t^{\frac{n}{2}(1-\frac{1}{p})}\left(\|G(\cdot,t) * \varphi_k - MG(\cdot,t)\|_{L^p} + \|G(\cdot,t) * (\varphi_k - u_0)\|_{L^p}\right) \end{aligned}$$

Let $\varepsilon > 0$. Since $\varphi_k \in L^1(\mathbb{R}^n; 1+|x|)$, by Theorem 3.2 we have

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * \varphi_k - MG(\cdot,t)\|_{L^p} \to 0 \text{ as } t \to \infty.$$

So there exists a $t_0 > 0$ sufficiently large so that for each fixed $k \in \mathbb{N}$,

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * \varphi_k - MG(\cdot,t)\|_{L^p} < \frac{\varepsilon}{2} \;\forall\, t \geq t_0. \tag{3.7}$$

For the second part, we use (3.1) to get

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * (\varphi_k - u_0)\|_{L^p} \leq C_p\|\varphi_k - u_0\|_{L^1}.$$

Since $\varphi_k \to u_0$ in $L^1(\mathbb{R}^n)$ as $k \to \infty$, we know there exists a $K \in \mathbb{N}$ large enough so that the expression above is arbitrarily small, i.e.

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * (\varphi_K - u_0)\|_{L^p} < \frac{\varepsilon}{2} \;\forall\, t > 0. \tag{3.8}$$

Finally, combining (3.7) and (3.8) we get

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|G(\cdot,t) * u_0 - MG(\cdot,t)\|_{L^p} < \varepsilon \;\forall\, t \geq t_0.$$

Since $\varepsilon > 0$ was arbitrary, this implies (3.6), and we are done. □

In Theorem 3.2, we saw that when we assumed $u_0 \in L^1(\mathbb{R}^n; 1+|x|)$, we obtained (3.5). This result contains more information than (3.6), in the sense that it tells us not only that the solution converges as $t \to \infty$, but also that the convergence follows the rate $O(t^{-\frac{1}{2}})$. Thus, we have obtained a better result by putting higher constraints on the solution. We will now briefly investigate how the assumption $u_0 \in L^1(\mathbb{R}^n; 1+|x|)$ affects the conservation properties of the solution. In Section 3.1, we saw that the mass of u is conserved when $u_0 \in L^1(\mathbb{R}^n)$. A natural question to ask is therefore if the expression

$$\int_{\mathbb{R}^n} xu(x,t)dx$$

is conserved when $u_0 \in L^1(\mathbb{R}^n; 1+|x|)$. This expression is called the first (signed) moment, and is to be understood as a vector in $\mathbb{R}^n$ with components $\int_{\mathbb{R}^n} x_i u(x,t)dx$, $i = 1, \ldots, n$. The next result shows that the first moment indeed is conserved:

Proposition 3.4. *Let u be a solution to the heat equation with initial data $u_0 \in L^1(\mathbb{R}^n; 1+|x|)$, given by (2.2). Then, the first moment of u is conserved, i.e.*

$$\int_{\mathbb{R}^n} xu(x,t)dx = \int_{\mathbb{R}^n} xu_0(x)dx \quad \forall\, t \geq 0. \tag{3.9}$$

Proof. The case $t = 0$ is obvious. For $t > 0$, we will use estimates for the heat kernel given by Lemma B.1 and Lemma B.5. The latter gives that the heat kernel conserves the first moment, i.e.

$$\int_{\mathbb{R}^n} xG(x,t)dx = 0 \;\forall\, t > 0.$$

We will only show the result for the first component of (3.9), as all components are similar. The result follows directly by applying the lemmas mentioned above:

$$\begin{aligned}
\int_{\mathbb{R}^n} x_1 u(x,t)dx &= \int_{\mathbb{R}^n} x_1 \int_{\mathbb{R}^n} G(x-y)u_0(y)dydx \\
&= \int_{\mathbb{R}^n} (y_1 + (x_1 - y_1)) \int_{\mathbb{R}^n} G(x-y)u_0(y)dydx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G(x-y)y_1u_0(y)dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (x_1-y_1)G(x-y)u_0(y)dydx \\
&\overset{z:=x-y}{=} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G(z)y_1u_0(y)dydz + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} z_1G(z)u_0(y)dydz \\
&= \underbrace{\int_{\mathbb{R}^n} G(z)dz}_{=1} \int_{\mathbb{R}^n} y_1u_0(y)dy + \underbrace{\int_{\mathbb{R}^n} z_1G(z)dz}_{=0} \int_{\mathbb{R}^n} u_0(y)dy \\
&= \int_{\mathbb{R}^n} y_1u_0(y)dy \\
&\overset{x:=y}{=} \int_{\mathbb{R}^n} x_1u_0(x)dx \;\forall\, t > 0.
\end{aligned}$$

□

Once again, we see how the solution inherits many of its properties from those of the heat kernel. It is possible to further assume $u_0 \in L^1(\mathbb{R}^n; 1+|x|^2)$ and ask if the second moment $\int_{\mathbb{R}^n} |x|^2 u(x,t)dx$ is conserved. We will not pursue this question any further here, but one would then find that neither the heat kernel nor the solution u conserves the second moment. However, the strengthened assumption on u_0 does lead to a better asymptotic convergence result, as can be seen in [11, pp. 16–19].

3.3 Asymptotic behaviour with linear convection

We now turn to the asymptotic behaviour of a slightly moderated equation, where we have added a linear convection term:

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla u, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.10}$$

Here, $a \in \mathbb{R}^n$ is a constant and the initial data u_0 is still in $L^1(\mathbb{R}^n)$.

This problem is very closely related to the original heat equation, and thus we should expect the asymptotic behaviour to be very similar to the one we uncovered in the previous section.

Indeed, if we let u be a solution to (3.10), then the chain rule yields that the translated function

$$v(x,t) = u(x - at, t)$$

is a solution to the heat equation, i.e.

$$\begin{cases} \partial_t v - \Delta v = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ v(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.11}$$

By Section 3.1, this problem admits a unique solution in $C([0,\infty); L^1(\mathbb{R}^n))$ given by

$$v(x,t) = (G(\cdot,t) * u_0)(x).$$

Rewriting back in terms of u, we get a unique solution of (3.10) in $C([0,\infty); L^1(\mathbb{R}^n))$ given by

$$u(x,t) = (G(\cdot,t) * u_0)(x + at). \tag{3.12}$$

This solution corresponds to a translated solution to the heat equation, and thus fulfills the same L^p-decay estimate as the one given in (3.1). For the asymptotic behaviour, we simply get a translated version of Theorem 3.3:

Theorem 3.5. *Let $u_0 \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} u_0(x)dx = M \neq 0$. Then, the solution u given by* (3.12) *of the heat equation with linear convection* (3.10) *satisfies*

$$t^{\frac{n}{2}(1-\frac{1}{p})} \| u(\cdot,t) - MG(\cdot + at, t) \|_{L^p} \to 0 \text{ as } t \to \infty,$$

for all $p \in [1,\infty]$.

Chapter 4

Scalar convection-diffusion equations

The previous chapter ended with a discussion around the heat equation with a linear convection term. We will now generalise this term into a non-linear one, which leads to the following scalar convection-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u = a \cdot \nabla(F(u)), & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.1}$$

Here, the initial data u_0 is assumed to be either in $L^1(\mathbb{R}^n)$ or in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, while $F \in C^1(\mathbb{R})$ with $F(0) = 0$. $a \in \mathbb{R}^n$ is a constant.

We want to investigate the asymptotic behaviour of solutions to (4.1) similarly as we did in the previous chapter, and also compare the behaviour of the different equations. However, we first need to establish fundamental results regarding existence, uniqueness and regularity of the solutions. The results in Chapter 2 were developed in the framework of strong solutions to the heat equation, meaning solutions fulfilling the equation almost everywhere. As we discussed in the specialisation project [3, p. 30], this class of solutions imposes higher regularity demands than very weak solutions, as we require the derivatives to be functions, not just distributions.

The scalar convection-diffusion equation (4.1) differs from the heat equation with the introduction of a non-linear convection term, but the diffusion term Δu is still the leading order of the equation. The leading order controls much of the behaviour of the equation, and we therefore hope that (4.1) is so closely related to the heat equation that we will be able to make use of our results from Chapter 2. It is therefore natural to continue working in a strong solution framework, which leads us to the following definition:

Definition 4.1. *Let $F \in C^1(\mathbb{R})$ with $F(0) = 0$. We call u a strong solution of the scalar convection-diffusion equation* (4.1) *with initial value $u_0 \in L^1(\mathbb{R}^n)$ if $u \in C([0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$, and if:*

(i) $\partial_t u, \Delta u, \nabla(F(u)) \in L^p_{loc}((0,\infty); L^p(\mathbb{R}^n))$ *for* $p \in (1,\infty)$.

(ii) $\partial_t u - \Delta u = a \cdot \nabla(F(u))$ *a.e. in* $\mathbb{R}^n \times (0, \infty)$.

(iii) $u(\cdot, 0) = u_0$ *a.e. in* $\mathbb{R}^n$.

Throughout this chapter, similarly as in the previous one, we have replicated and adapted the results in Chapter 4 of Zuazua [1], expanding or rewriting some of his more brief arguments.

4.1 Well-posedness

Before we present the main theorem which establishes the well-posedness of solutions to (4.1), we start with four lemmas which we will need in its proof. This section follows the proof of Theorem 4.1 in Chapter 4 in [1, pp. 27–32], but we have restructured it into several parts to suit our needs. The first lemma deals with regularity and is stated without proof (see Zuazua [1, p. 29]).

Lemma 4.2. *Let* $u_0 \in L^1(\mathbb{R}^n)$*. Suppose* $u \in C([0,T]; L^1(\mathbb{R}^n)) \cap L^\infty_{loc}((0,T); L^\infty(\mathbb{R}^n))$ *solves the integral equation*

$$u(x,t) = (G(\cdot,t) * u_0)(x) + \int_0^t \left[a \cdot \nabla G(\cdot, t-s) * F(u(\cdot,s))\right](x) ds,$$

for $(x,t) \in \mathbb{R}^n \times [0,T)$*, where* $T > 0$*. Then, we have* $u \in C((0,T); W^{2,p}(\mathbb{R}^n)) \cap C^1((0,T); L^p(\mathbb{R}^n))$ *for all* $p \in (1, \infty)$*.*

The second lemma is a Kato inequality, resulting from the use of the inequality in Theorem A.17 introduced by Kato [12].

Lemma 4.3. *Let* u, v *be two strong solutions of* (4.1) *up to some time* $T > 0$*, with initial values* $u_0, v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Then*

$$\int_0^T \int_{\mathbb{R}^n} \big([u(x,t) - v(x,t)]^+ \partial_t \varphi(x,t) + [u(x,t) - v(x,t)]^+ \Delta\varphi(x,t) \\ - \operatorname{sign}^+(u(x,t) - v(x,t))(F(u(x,t)) - F(v(x,t))) a \cdot \nabla\varphi(x,t)\big) dxdt \geq 0,$$

for all $0 \leq \varphi \in C_c^\infty(\mathbb{R}^n \times (0,T))$*.*

Proof. Since u and v both satisfy (4.1) a.e., we get that $u - v$ satisfies

$$\partial_t(u-v) - \Delta(u-v) = a \cdot \nabla(F(u)) - a \cdot \nabla(F(v)) = a \cdot \nabla(F(u) - F(v)) \text{ a.e.}$$

We will multiply this equation with $p(u-v)\varphi$ and integrate over $\mathbb{R}^n \times (0,T)$, where $0 \leq \varphi \in C_c^\infty(\mathbb{R}^n \times (0,T))$. We define p to approximate the sign^+ function (see Notation) in such a way that $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ fulfills $0 \leq p \leq 1$, $p(y) = 0$ for $y \leq 0$ and $p'(y) > 0$ for $y > 0$. Thus, we get

$$\begin{aligned} 0 = &\int_0^T \int_{\mathbb{R}^n} \partial_t(u-v) p(u-v)\varphi dxdt - \int_0^T \int_{\mathbb{R}^n} \Delta(u-v) p(u-v)\varphi dxdt \\ &- \int_0^T \int_{\mathbb{R}^n} a \cdot \nabla(F(u) - F(v)) p(u-v)\varphi dxdt. \end{aligned} \tag{4.2}$$

Now, consider the first term on the right hand side of (4.2). We wish to let p tend to the sign^+ function to get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}\partial_t(u-v)\text{sign}^+(u-v)\varphi dxdt=\int_0^T\int_{\mathbb{R}^n}\partial_t[u-v]^+\varphi dxdt\\&=\int_{\mathbb{R}^n}\int_0^T\partial_t[u-v]^+\varphi dtdx=-\int_0^T\int_{\mathbb{R}^n}[u-v]^+\partial_t\varphi dtdx=-\int_0^T\int_{\mathbb{R}^n}[u-v]^+\partial_t\varphi dxdt.\end{aligned}\tag{4.3}$$

To justify taking the limit for p, we make use of the Dominated convergence theorem (Theorem A.8). Using the theorem is justified since

$$\left|\partial_t(u-v)\underbrace{p(u-v)}_{\leq 1}\varphi\right|\leq|\partial_t(u-v)\varphi|,$$

where the right hand side is integrable in $\mathbb{R}^n\times(0,T)$ by Hölder's inequality (Theorem A.3). Indeed, $\varphi\in C_c^\infty(\mathbb{R}^n\times(0,T))$, while $\partial_t(u-v)\in C((0,T);L^p(\mathbb{R}^n))$ for all $p\in(1,\infty)$ by Lemma 4.2. It is not directly clear that this lemma is applicable here, but as we will see in Theorem 4.6, the solutions are constructed to fulfill the integral equality given in Lemma 4.2. Thus, we get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}|\partial_t(u(x,t)-v(x,t))\varphi(x,t)|dxdt\leq\int_0^T\|\partial_t(u(\cdot,t)-v(\cdot,t))\|_{L^p}\|\varphi(\cdot,t)\|_{L^q}dt\\&\leq\operatorname*{ess\,sup}_{t\in(0,T)}\{\|\partial_t(u(\cdot,t)-v(\cdot,t))\|_{L^p}\|\varphi(\cdot,t)\|_{L^q}\}T<\infty,\end{aligned}$$

where q is the conjugate exponent of p, i.e. $1/p+1/q=1$. Similarly, we may use this approach to justify the use of Fubini's theorem (Theorem A.10) to interchange the integrals in (4.3).

Furthermore, we may also let p tend to sign^+ to deal with the diffusion term, since $\Delta(u-v)\in C((0,T);L^p(\mathbb{R}^n))$. Then, we make use of Kato's inequality (Theorem A.17) and Green's first identity (Theorem A.6), ending up with

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\Delta(u-v)\text{sign}^+(u-v)\varphi dxdt&\leq\int_0^T\int_{\mathbb{R}^n}\Delta[u-v]^+\varphi dxdt\\&=\int_0^T\int_{\mathbb{R}^n}[u-v]^+\Delta\varphi dxdt.\end{aligned}$$

For the final term, we use integration by parts to get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}a\cdot\nabla(F(u)-F(v))p(u-v)\varphi dxdt\\&=-\int_0^T\int_{\mathbb{R}^n}a\cdot(p(u-v)\nabla\varphi+\varphi\nabla p(u-v))(F(u)-F(v))dxdt\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^T \int_{\mathbb{R}^n} p(u-v)(F(u)-F(v))a \cdot \nabla\varphi dxdt \\
&\quad - \int_0^T \int_{\mathbb{R}^n} a \cdot \nabla(u-v)p'(u-v)(F(u)-F(v))\varphi dxdt.
\end{aligned}
$$

The second term in the last equality above vanishes in the limit because $p'(u-v)$ approaches δ_{u-v}, thus yielding a contribution only when $u = v$. The first term becomes, after taking the limit,

$$
-\int_0^T \int_{\mathbb{R}^n} \operatorname{sign}^+(u-v)(F(u)-F(v))a \cdot \nabla\varphi dxdt.
$$

The use of dominated convergence is justified here since $F \in C^1(\mathbb{R})$, while $u, v \in L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$, while $\nabla\varphi$ has compact support. Thus, u and v are bounded, which means $F(u)$ and $F(v)$ attains their maximas, and consequently, the integrand is bounded by the integrable function $(|F(u)| + |F(v)|)|a||\nabla\varphi|$.

Returning to (4.2), we finally get

$$
\begin{aligned}
0 \geq &-\int_0^T \int_{\mathbb{R}^n} [u-v]^+ \partial_t\varphi dxdt - \int_0^T \int_{\mathbb{R}^n} [u-v]^+ \Delta\varphi dxdt \\
&+ \int_0^T \int_{\mathbb{R}^n} \operatorname{sign}^+(u-v)(F(u)-F(v))a \cdot \nabla\varphi dxdt.
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
0 \leq &\int_0^T \int_{\mathbb{R}^n} [u-v]^+ \partial_t\varphi dxdt + \int_0^T \int_{\mathbb{R}^n} [u-v]^+ \Delta\varphi dxdt \\
&- \int_0^T \int_{\mathbb{R}^n} \operatorname{sign}^+(u-v)(F(u)-F(v))a \cdot \nabla\varphi dxdt,
\end{aligned}
$$

as we wanted. □

By making use of this result, we obtain the third lemma which we will see yields the important property of L^1-contraction.

Lemma 4.4. *Let u, v be two strong solutions of* (4.1) *up to some time $T > 0$, with initial values $u_0, v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$
\int_{\mathbb{R}^n} [u(x,t) - v(x,t)]^+ dx \leq \int_{\mathbb{R}^n} [u_0(x) - v_0(x)]^+ dx \;\forall\; t \in [0,T).
$$

Proof. We will show that this result follows from the result in Lemma 4.3 by choosing a specific test function $0 \leq \varphi \in C_c^\infty(\mathbb{R}^n \times (0,T))$. The approach we make is very similar to the one we did in the proof of Lemma 4.1 in the specialisation project [3, pp. 27–30].

First, let us choose the test function

$$
\varphi(x,t) := \theta_\varepsilon(t)\xi_R(x),
$$

where $0 \leq \xi_R \in C_c^\infty(\mathbb{R}^n)$ is a bump function approximating 1 as $R \to \infty$. The function fulfills $\xi_R(x) = \xi(\frac{x}{R})$, $\xi_R(x) = 1$ for $|x| \leq R$ and $\xi_R(x) = 0$ for $|x| > 2R$. We also assume $\nabla\xi, \Delta\xi \in L^\infty(\mathbb{R}^n)$, so that we obtain the following bounds:

$$\begin{aligned} |\nabla\xi_R(x)| &= \left|\nabla\xi(\frac{x}{R})\right| = \frac{1}{R}\left|\nabla\xi(\frac{x}{R})\right| \leq \frac{1}{R}\|\nabla\xi\|_{L^\infty}, \\ |\Delta\xi_R(x)| &= \left|\Delta\xi(\frac{x}{R})\right| = \frac{1}{R^2}\left|\Delta\xi(\frac{x}{R})\right| \leq \frac{1}{R^2}\|\Delta\xi\|_{L^\infty}. \end{aligned} \tag{4.4}$$

Moreover, $0 \leq \theta_\varepsilon \in C_c^\infty((0,T))$ approximates a jump from 0 to 1 at some time t_1 and a jump from 1 to 0 at some time $t_2 > t_1$, as $\varepsilon \to 0$. This function fulfills $\theta_\varepsilon(t) = 1$ for $t \in (t_1 + \varepsilon, t_2 - \varepsilon)$ and $\theta_\varepsilon(t) = 0$ for $t \in (0, t_1 - \varepsilon) \cup (t_2 + \varepsilon, T)$, where $t_1, t_2 \in (0,T)$ such that $t_2 > t_1$. Furthermore, $|\theta'_\varepsilon(t)| \leq C/\varepsilon$ for $t \in (0,T)$, where $C > 0$ is some constant.

Applying Lemma 4.3 with this test function, we get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \big([u-v]^+\theta'_\varepsilon\xi_R + [u-v]^+\theta_\varepsilon\Delta\xi_R \\ + \operatorname{sign}^+(u-v)(F(u)-F(v))a \cdot \nabla\xi_R\theta_\varepsilon\big)dxdt \geq 0. \end{aligned}$$

Observe that the diffusion term vanishes as $R \to \infty$:

$$\begin{aligned} \left|\int_0^T \int_{\mathbb{R}^n} [u-v]^+|\theta_\varepsilon|\Delta\xi_R dxdt\right| &\leq \int_0^T \int_{\mathbb{R}^n} [u-v]^+ \underbrace{|\theta_\varepsilon|}_{\leq 1} |\Delta\xi_R| dxdt \\ &\leq \frac{1}{R^2}\|\Delta\xi\|_{L^\infty} \int_0^T \int_{\mathbb{R}^n} [u-v]^+ dxdt \\ &\to 0 \text{ as } R \to \infty. \end{aligned}$$

The last integral is finite since $u, v \in C([0,\infty); L^1(\mathbb{R}^n))$, and the identity $|u-v| = [u-v]^+ + [u-v]^-$ yields $[u-v]^+ \leq |u-v|$.

We will also show that the convection term vanishes. To deal with $F \in C^1(\mathbb{R})$, we make use of Lipschitz continuity, since a function in $C^1(\mathbb{R})$ is uniformly Lipschitz continuous on any compact set in $\mathbb{R}$. Definition 4.1 only gives $u, v \in L^\infty_{loc}((0,T); L^\infty(\mathbb{R}^n))$, but since we also assume $u_0, v_0 \in L^\infty(\mathbb{R}^n)$, this in fact leads to $u, v \in L^\infty(\mathbb{R}^n \times (0,T))$. This result will be shown in Theorem 4.6[1].

Now, with $u, v \in L^\infty(\mathbb{R}^n \times (0,T))$, we know that $M := \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\} < \infty$. Therefore, for all $(x,t) \in \mathbb{R}^n \times [0,T)$, u, v only take values on the compact set $[-M, M]$, which implies there exists a constant $L > 0$ such that

$$|F(u(x,t)) - F(v(x,t))| \leq L|u(x,t) - v(x,t)| \;\forall\; (x,t) \in \mathbb{R}^n \times [0,T).$$

[1]Note that the proof of Theorem 4.6 depends on Lemma 4.4 which we are currently proving, but the regularity result we are using here is proved independently of the lemma, so we do not have a circular reasoning.

Applying this yields as wanted

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^n} \operatorname{sign}^+(u-v)(F(u)-F(v))a\cdot\nabla\xi_R\theta_\varepsilon dxdt\right| \\
&\leq \int_0^T \int_{\mathbb{R}^n} \underbrace{\left|\operatorname{sign}^+(u-v)\right|}_{\leq 1}|F(u)-F(v)||a||\nabla\xi_R|\underbrace{|\theta_\varepsilon|}_{\leq 1}dxdt \\
&\leq \int_0^T \int_{\mathbb{R}^n} |a||\nabla\xi_R|L|u-v|dxdt \\
&\leq \frac{1}{R}\|\nabla\xi\|_{L^\infty}|a|L\int_0^T \int_{\mathbb{R}^n} |u-v|dxdt \\
&\to 0 \text{ as } R\to\infty,
\end{aligned}$$

where the last integral is finite since $u, v \in C([0,\infty); L^1(\mathbb{R}^n))$.

Thus, we are only left with the term involving the time derivative. Observing that θ'_ε only is supported on $(t_1-\varepsilon, t_1+\varepsilon)$ and $(t_2-\varepsilon, t_2+\varepsilon)$, we can change the area of integration and get

$$\begin{aligned}
0 &\leq \int_0^T \int_{\mathbb{R}^n} [u-v]^+\theta'_\varepsilon\xi_R dxdt \\
&= \int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} [u-v]^+\theta'_\varepsilon\xi_R dxdt + \int_{t_2-\varepsilon}^{t_2+\varepsilon} \int_{\mathbb{R}^n} [u-v]^+\theta'_\varepsilon\xi_R dxdt.
\end{aligned} \tag{4.5}$$

As both terms are very similar, we only focus on the first term. By the Dominated convergence theorem (Theorem A.8), we may take the limit $R\to\infty$, in which $\xi_R\to 1$. This is justified since the integrand is bounded by the integrable function $[u-v]^+|\theta'_\varepsilon|$. We get

$$\int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} [u-v]^+\theta'_\varepsilon\xi_R dxdt \to \int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} [u-v]^+\theta'_\varepsilon dxdt \text{ as } R\to\infty.$$

To continue, we add and subtract a new integral to get

$$\begin{aligned}
&\int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} [u(x,t)-v(x,t)]^+\theta'_\varepsilon(t)dxdt \pm \int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} [u(x,t_1)-v(x,t_1)]^+\theta'_\varepsilon(t)dxdt \\
&= \int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} \theta'_\varepsilon(t)\left([u(x,t)-v(x,t)]^+ - [u(x,t_1)-v(x,t_1)]^+\right)dxdt \\
&\quad + \int_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\mathbb{R}^n} [u(x,t_1)-v(x,t_1)]^+\theta'_\varepsilon(t)dxdt.
\end{aligned}$$

The first integral vanishes as $\varepsilon\to 0$. To see this, we use that $|\theta'_\varepsilon(t)| \leq C/\varepsilon$ for $t\in(0,T)$

and Lebesgue's differentiation theorem (Theorem A.9):

$$\begin{aligned}
&\left|\int_{t_1-\varepsilon}^{t_1+\varepsilon}\int_{\mathbb{R}^n}\theta_\varepsilon'(t)\left([u(x,t)-v(x,t)]^+-[u(x,t_1)-v(x,t_1)]^+\right)dxdt\right|\\
&\le \int_{t_1-\varepsilon}^{t_1+\varepsilon}\left|\theta_\varepsilon'(t)\right|\int_{\mathbb{R}^n}\left|[u(x,t)-v(x,t)]^+-[u(x,t_1)-v(x,t_1)]^+\right|dxdt\\
&\le \int_{t_1-\varepsilon}^{t_1+\varepsilon}\frac{C}{\varepsilon}\int_{\mathbb{R}^n}\left|[u(x,t)-v(x,t)]^+-[u(x,t_1)-v(x,t_1)]^+\right|dxdt\\
&\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}$$

The second integral may be split up in two independent integrals:

$$\begin{aligned}
&\int_{t_1-\varepsilon}^{t_1+\varepsilon}\int_{\mathbb{R}^n}[u(x,t_1)-v(x,t_1)]^+\theta_\varepsilon'(t)dxdt\\
&=\int_{t_1-\varepsilon}^{t_1+\varepsilon}\theta_\varepsilon'(t)dt\int_{\mathbb{R}^n}[u(x,t_1)-v(x,t_1)]^+dx\\
&=(\theta_\varepsilon(t_1+\varepsilon)-\theta_\varepsilon(t_1-\varepsilon))\int_{\mathbb{R}^n}[u(x,t_1)-v(x,t_1)]^+dx.
\end{aligned}$$

Returning to (4.5), we thus get

$$\begin{aligned}
0 &\le (\theta_\varepsilon(t_1+\varepsilon)-\theta_\varepsilon(t_1-\varepsilon))\int_{\mathbb{R}^n}[u(x,t_1)-v(x,t_1)]^+dx\\
&\quad+(\theta_\varepsilon(t_2+\varepsilon)-\theta_\varepsilon(t_2-\varepsilon))\int_{\mathbb{R}^n}[u(x,t_2)-v(x,t_2)]^+dx\\
&=(1-0)\int_{\mathbb{R}^n}[u(x,t_1)-v(x,t_1)]^+dx+(0-1)\int_{\mathbb{R}^n}[u(x,t_2)-v(x,t_2)]^+dx\\
&=\int_{\mathbb{R}^n}\left([u(x,t_1)-v(x,t_1)]^+-[u_0(x)-v_0(x)]^+\right)dx+\int_{\mathbb{R}^n}[u_0(x)-v_0(x)]^+dx\\
&\quad-\int_{\mathbb{R}^n}[u(x,t_2)-v(x,t_2)]^+dx.
\end{aligned}$$

Next, we use that $[u(\cdot,t_1)-v(\cdot,t_1)]^+\to[u_0-v_0]^+$ in $L^1(\mathbb{R}^n)$ as $t_1\to 0$, because $u,v\in C([0,T);L^1(\mathbb{R}^n))$, to get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left([u(x,t_1)-v(x,t_1)]^+-[u_0(x)-v_0(x)]^+\right)dx\right|\\
&\le\int_{\mathbb{R}^n}\left|[u(x,t_1)-v(x,t_1)]^+-[u_0(x)-v_0(x)]^+\right|dx\\
&\to 0 \text{ as } t_1\to 0.
\end{aligned}$$

Thus, we are left with

$$\int_{\mathbb{R}^n}[u(x,t_2)-v(x,t_2)]^+dx\le\int_{\mathbb{R}^n}[u_0(x)-v_0(x)]^+dx.$$

Since $t_2\in(0,T)$ such that $t_2>t_1$ was arbitrary, we have our result. □

We will also need this fourth lemma to bound our solutions in L^∞ by their initial data:

Lemma 4.5. *Let u be a strong solution of (4.1) up to some time $T > 0$, with initial value $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\|u(\cdot,t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} \ \forall\ t \in [0,T).$$

Proof. Let $\mu := \|u_0\|_{L^\infty}$. We wish to show that

$$\int_{\mathbb{R}^n} [u(x,t) - \mu]^+ dx \leq \int_{\mathbb{R}^n} [u_0(x) - \mu]^+ dx \ \forall\ t \in [0,T), \tag{4.6}$$

because this yields

$$[u_0 - \mu]^+ = 0 \implies [u(x,t) - \mu]^+ \leq 0 \implies u(x,t) \leq \mu \ \forall\ t \in [0,T). \tag{4.7}$$

It is tempting to apply Lemma 4.4 with $v = \mu$, but $\mu \notin L^1(\mathbb{R}^n)$, and so it does not fulfill the assumptions needed to use this result directly. Instead we make a similar argument, also using techniques from Lemma 4.3. First, observe that since u satisfies (4.1) a.e., it also satisfies

$$\partial_t(u - \mu) - \Delta(u - \mu) = a \cdot \nabla(F(u)) \text{ a.e.}$$

Next, we multiply this equation with $p(u - \mu)\xi_R$, where p is as defined in Lemma 4.3 and ξ_R is as defined in Lemma 4.4, and integrate over $\mathbb{R}^n$ to get

$$\begin{aligned} &\int_{\mathbb{R}^n} \partial_t(u - \mu)p(u - \mu)\xi_R dx - \int_{\mathbb{R}^n} \Delta(u - \mu)p(u - \mu)\xi_R dx \\ &= \int_{\mathbb{R}^n} a \cdot \nabla(F(u))p(u - \mu)\xi_R dx. \end{aligned} \tag{4.8}$$

Now, defining

$$H(z) := \int_0^z F'(s)p(s - \mu)ds,$$

we move the gradient over to the bump function ξ_R, before making use of Lipschitz continuity and $F(0) = 0$ to get

$$\begin{aligned} \left| \int_{\mathbb{R}^n} a \cdot \nabla(F(u))p(u - \mu)\xi_R dx \right| &= \left| \int_{\mathbb{R}^n} a \cdot \nabla H(u)\xi_R dx \right| \\ &= \left| \int_{\mathbb{R}^n} a \cdot \nabla \xi_R H(u) dx \right| \\ &\leq \frac{1}{R}|a| \|\nabla \xi\|_{L^\infty} \int_{\mathbb{R}^n} |H(u)| dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{R}|a|\|\nabla\xi\|_{L^\infty} \int_{\mathbb{R}^n} \left| \int_0^u F'(s) \underbrace{p(s-\mu)}_{\leq 1} ds \right| dx \\
&\leq \frac{1}{R}|a|\|\nabla\xi\|_{L^\infty} \int_{\mathbb{R}^n} |F(u) - F(0)| dx \\
&\leq \frac{1}{R} L|a|\|\nabla\xi\|_{L^\infty} \|u(\cdot,t)\|_{L^1} \\
&\overset{R\to\infty}{\to} 0 \ \forall\ t \in [0,T).
\end{aligned}
$$

In Theorem 4.5 of [3, pp. 32–35], we proved a similar result for the heat equation. Since we now have removed the convection term from (4.8) in the limit $R \to \infty$, this means we are left with the same expression as in the case of the heat equation. The rest of this proof is therefore almost identical to the one in [3, pp. 32–35]. We will thus refer to the proof of this theorem for details, but the main idea is to obtain

$$
\int_{\mathbb{R}^n} \Delta(u-\mu) p(u-\mu) \xi_R dx \leq 0 \text{ as } R \to \infty,
$$

so that (4.8) becomes

$$
\int_{\mathbb{R}^n} \partial_t (u-\mu) p(u-\mu) \xi_R dx \leq 0.
$$

Next, we integrate over $[0, t']$ in time, where $0 \leq t' < T$, and let p tend to the sign$^+$ function to get

$$
\int_0^{t'} \int_{\mathbb{R}^n} \partial_t (u-\mu) \text{sign}^+(u-\mu) \xi_R dx dt = \int_0^{t'} \int_{\mathbb{R}^n} \partial_t [u-\mu]^+ \xi_R dx dt \leq 0.
$$

Taking this limit using dominated convergence (Theorem A.8) is justified just as in Lemma 4.3.

Finally, the idea is to first move the derivative out of the spatial integral, then applying the fundamental theorem of calculus, before finally letting $R \to \infty$ to obtain

$$
\int_{\mathbb{R}^n} [u(x,t') - \mu]^+ dx - \int_{\mathbb{R}^n} [u_0(x) - \mu]^+ dx \leq 0 \ \forall\ t' \in [0,T),
$$

which is the claim in (4.6).

To obtain a lower bound for the L^∞-norm of u, we claim that

$$
\int_{\mathbb{R}^n} [u(x,t) + \mu]^- dx \leq \int_{\mathbb{R}^n} [u_0(x) + \mu]^- dx \ \forall\ t \in [0,T), \tag{4.9}
$$

because this yields

$$
[u_0 + \mu]^- = 0 \implies [u(x,t) + \mu]^- \leq 0 \implies u(x,t) \geq -\mu \ \forall\ t \in [0,T).
$$

This together with (4.7) implies Lemma 4.5. To show this claim, we observe that u satisfies

$$\partial_t(u+\mu) - \Delta(u+\mu) = a \cdot \nabla(F(u)) \text{ a.e.}$$

To adjust for the fact that we are dealing with the negative part of a function, $[\cdot]^-$, we multiply this equation by $q(u+\mu)\xi_R$, where $q \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ fulfills $-1 \leq q \leq 0$, $q(y) = 0$ for $y \geq 0$ and $q'(y) > 0$ for $y < 0$. Thus, q approximates the function sign$^-$ function (see Notation), and we get

$$\begin{aligned}&\int_{\mathbb{R}^n} \partial_t(u+\mu)q(u+\mu)\xi_R dx - \int_{\mathbb{R}^n} \Delta(u+\mu)q(u+\mu)\xi_R dx\\ &= \int_{\mathbb{R}^n} a \cdot \nabla(F(u))q(u+\mu)\xi_R dx.\end{aligned}$$

From this we proceed in the same way as we did in the first part, with some obvious modifications, to obtain (4.9). □

Finally, we are ready to show the following well-posedness theorem:

Theorem 4.6. *Suppose $F \in C^1(\mathbb{R})$ with $F(0) = 0$. If $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then there exists a unique strong solution u to* (4.1) *with $u \in L^\infty(\mathbb{R}^n \times (0,\infty))$. For all $p \in (1,\infty)$, we also have $u \in C((0,\infty); W^{2,p}(\mathbb{R}^n)) \cap C^1((0,\infty); L^p(\mathbb{R}^n))$.*

Furthermore, if u and v are strong solutions to (4.1) *with initial data $u_0, v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ respectively, then*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \ \forall\ t \geq 0.$$

Proof. We follow and expand the proof of Theorem 4.1 in Zuazua [1, pp. 27–32].

First, we consider existence of a solution. To find such a solution, we will take inspiration from the inhomogeneous heat equation

$$\begin{cases}\partial_t u - \Delta u = f, & (x,t) \in \mathbb{R}^n \times (0,\infty)\\ u(x,0) = u_0(x), & x \in \mathbb{R}^n.\end{cases}$$

By using Duhamel's principle, one can obtain the following classical solution to this problem (see Evans [4, pp. 49–51]):

$$u(x,t) = (G(\cdot,t) * u_0)(x) + \int_0^t [G(\cdot,t-s) * f(\cdot,s)](x)ds.$$

We wish to make use of this solution formula to find a solution to (4.1), by using the same idea as we did when showing existence of very weak solutions to the heat equation (Theorem 3.4 of [3, pp. 22–24]): we plug in less regular functions into a classical solution formula in order to obtain a weaker solution. In our case this means weakened assumptions on u_0 and setting $f = a \cdot \nabla(F(u))$, which leads us to the following integral equation:

$$\begin{aligned} u(x,t) &= (G(\cdot,t) * u_0)(x) + \int_0^t [G(\cdot,t-s) * a \cdot \nabla(F(u(\cdot,s)))]\,(x)ds \\ &= (G(\cdot,t) * u_0)(x) + \int_0^t [a \cdot \nabla G(\cdot,t-s) * F(u(\cdot,s))]\,(x)ds. \end{aligned} \tag{4.10}$$

Thus, if u solves the last equality in (4.10) and fulfills the regularity demands in Definition 4.1, then it is a strong solution to (4.1).

To find a solution to (4.10), we first restrict ourselves to local time, $t \in [0,T]$, where $T > 0$ is some sufficiently small time which we will choose later. Next, we define the operator $\phi : X_T \to X_T$ by

$$\phi[u](x,t) := (G(\cdot,t) * u_0)(x) + \int_0^t [a \cdot \nabla G(\cdot,t-s) * F(u(\cdot,s))]\,(x)ds,$$

where X_T is the Banach space

$$X_T := C([0,T]; L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,T)),$$

with corresponding norm

$$\|u\|_{X_T} := \operatorname*{ess\,sup}_{t\in[0,T]} \{\|u(\cdot,t)\|_{L^1} + \|u(\cdot,t)\|_{L^\infty}\}.$$

By the Banach fixed-point theorem (Theorem A.14), we have that if ϕ is a contraction in X_T, then it admits a unique fixed point $u \in X_T$, i.e. $\phi[u](x,t) = u(x,t)\ \forall\ (x,t) \in X_T$. In other words, we have local existence of a unique solution $u \in C([0,T]; L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,T))$ to (4.10).

To check this, we will show the sufficient condition that ϕ is a contraction on B_R and that $\phi[B_R] \subset B_R$, where

$$B_R := \{u \in X_T \mid \|u\|_{X_T} < R\},$$

i.e. on a ball in X_T with radius $R > 0$ yet to be determined.

Now, let $u \in B_R$ and $r = 1, \infty$. We will first estimate the X_T-norm of $\phi[u]$ to show that $\phi[B_R] \subset B_R$:

$$\begin{aligned} \|\phi[u](\cdot,t)\|_{L^r} &\le \|G(\cdot,t) * u_0\|_{L^r} + \left\| \int_0^t [a \cdot \nabla G(\cdot,t-s) * F(u(\cdot,s))]\,ds \right\|_{L^r} \\ &\le \|G(\cdot,t) * u_0\|_{L^r} + \int_0^t \|a \cdot \nabla G(\cdot,t-s) * F(u(\cdot,s))\|_{L^r} ds \\ &\le \|G(\cdot,t)\|_{L^1} \|u_0\|_{L^r} + \int_0^t \|a \cdot \nabla G(\cdot,t-s)\|_{L^1} \|F(u(\cdot,s))\|_{L^r} ds. \end{aligned} \tag{4.11}$$

On the second line, we have used Minkowski's integral inequality (Theorem A.12), while on the third we have used Young's inequality (Theorem A.11).

Turning to $F(u)$, we use that $u \in B_R$, which implies that u only takes values on the compact set $[-R, R]$. $F \in C^1(\mathbb{R})$ is uniformly Lipschitz continuous on any compact set in $\mathbb{R}$, and so there exists a constant $L(R) > 0$ depending on R, such that

$$|F(x) - F(y)| \leq L(R)|x - y| \ \forall\ x, y \in [-R, R]. \tag{4.12}$$

Using that $F(0) = 0$, we get

$$|F(x)| \leq L(R)|x| \ \forall\ x \in [-R, R].$$

Thus we have

$$|F(u(x,t))| \leq L(R)|u(x,t)| \ \forall\ (x,t) \in \mathbb{R}^n \times [0,T],$$

which gives

$$\|F(u(\cdot,t))\|_{L^r} \leq L(R)\|u(\cdot,t)\|_{L^r} < L(R)R \ \forall\ (x,t) \in \mathbb{R}^n \times [0,T]. \tag{4.13}$$

We will also use the following heat kernel estimates: $\|G(\cdot,t)\|_{L^1} = 1 \ \forall\ t > 0$ (Lemma B.1), and $\|\nabla G(\cdot,t)\|_{L^1} \leq Ct^{-\frac{1}{2}} \ \forall\ t > 0$ (Lemma B.3). Furthermore, let $M := \max\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\}$. Returning to (4.11) with these estimates we get:

$$\begin{aligned}
\|\phi[u](\cdot,t)\|_{L^r} &\leq M + \int_0^t |a|C(t-s)^{-\frac{1}{2}}L(R)R ds \\
&\leq M + 2C|a|L(R)Rt^{\frac{1}{2}} \\
&\leq M + 2C|a|L(R)RT^{\frac{1}{2}} \ \forall\ t \in [0,T].
\end{aligned}$$

To guarantee that $\phi[B_R] \subset B_R$, we need $\|\phi[u]\|_{X_T} < R$, and this holds if

$$M + 2C|a|L(R)RT^{\frac{1}{2}} < \frac{R}{2}. \tag{4.14}$$

Choosing $R \geq 4M$, (4.14) then becomes

$$2C|a|L(R)RT^{\frac{1}{2}} < \frac{R}{4} \implies T < \frac{1}{(8C|a|L(R))^2}. \tag{4.15}$$

With these choices for R and T, $\phi[u]$ remains in B_R as long as u is in B_R.

Next, we let $u, v \in B_R$ and estimate the difference $\phi[u] - \phi[v]$ in X_T-norm in a similar way:

$$\begin{aligned}
\|\phi[u](\cdot,t) - \phi[v](\cdot,t)\|_{L^r} &\leq \|G(\cdot,t) * u_0 - G(\cdot,t) * u_0\|_{L^r} \\
&\quad + \left\| \int_0^t [a \cdot \nabla G(\cdot, t-s) * (F(u(\cdot,s)) - F(v(\cdot,s)))]\, ds \right\|_{L^r} \\
&\leq \int_0^t \|a \cdot \nabla G(\cdot,t-s)\|_{L^1} \|F(u(\cdot,s)) - F(v(\cdot,s))\|_{L^r} ds.
\end{aligned}$$

With $u, v \in B_R$, we make use of Lipschitz continuity (4.12) once again to obtain

$$\|F(u(\cdot,t)) - F(v(\cdot,t))\|_{L^r} \leq L(R)\|u(\cdot,t) - v(\cdot,t)\|_{L^r} \ \forall\ t \in [0,T].$$

Thus, for all $t \in [0,T]$

$$\|\phi[u](\cdot,t) - \phi[v](\cdot,t)\|_{L^r} \leq \int_0^t |a|C(t-s)^{-\frac{1}{2}} L(R)\|u(\cdot,t) - v(\cdot,t)\|_{L^r} ds,$$

and so

$$\begin{aligned}
&\|\phi[u](\cdot,t) - \phi[v](\cdot,t)\|_{L^1} + \|\phi[u](\cdot,t) - \phi[v](\cdot,t)\|_{L^\infty} \\
&\leq \int_0^t |a|C(t-s)^{-\frac{1}{2}} L(R) \left(\|u(\cdot,t) - v(\cdot,t)\|_{L^1} + \|u(\cdot,t) - v(\cdot,t)\|_{L^\infty}\right) ds \\
&\leq \int_0^t |a|C(t-s)^{-\frac{1}{2}} L(R)\|u - v\|_{X_T} ds \\
&\leq 2C|a|L(R)T^{\frac{1}{2}}\|u - v\|_{X_T}.
\end{aligned}$$

Taking supremum over time, we get

$$\|\phi[u] - \phi[v]\|_{X_T} \leq 2C|a|L(R)T^{\frac{1}{2}}\|u - v\|_{X_T} \ \forall\ u, v \in B_R.$$

Therefore, ϕ is a contraction on B_R with $\phi[B_R] \subset B_R$ if T is chosen so that

$$2C|a|L(R)T^{\frac{1}{2}} < 1 \implies T < \frac{1}{(2C|a|L(R))^2} \tag{4.16}$$

holds, in addition to (4.15), where $R \geq 4M$.

From this we have existence of a unique fixed point $u \in B_R$, in other words, a local solution $u \in C([0,T]; L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,T))$ to (4.10) for $t \in [0,T]$. By Lemma 4.2, we further have $u \in C((0,T); W^{2,p}(\mathbb{R}^n)) \cap C^1((0,T); L^p(\mathbb{R}^n))$ for $p \in (1,\infty)$, and thus u is a strong local solution to (4.1).

We wish to extend the solution in time to some maximal time of existence $T_m > 0$. To do this, we use a classical extension method, where the idea is to cover the timeline with overlapping local solutions, and then using the fact that each of these solutions are unique to uncover a global solution. Indeed, we have just found a unique solution u to (4.1) in the time interval $[0,T]$, with regularity as stated in the paragraph above. As we can see from (4.15) and (4.16), T is not dependent on the initial time. Define $u_1(x) := u(x, T/2)$, then $u_1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. By using u_1 as initial data in our proof above, we obtain a unique local solution v to (4.1) in the time interval $[T/2, 3T/2]$, with regularity as above. Both solutions are overlapping on the interval $[T/2, T]$, but they should be unique. Thus, they must coincide on this interval, giving us existence of a unique solution on $[0, 3T/2]$.

Continuing in the same way, we can also cover the intervals $[T, 2T], [3T/2, 5T/2], \ldots$ with unique local solutions until we reach T_m. This process yields existence of a unique maximal solution u to (4.1), with regularity $C([0,T_m);L^1(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n\times(0,T_m))$ and $C((0,T_m);W^{2,p}(\mathbb{R}^n))\cap C^1((0,T_m);L^p(\mathbb{R}^n))$ for $p\in(1,\infty)$.

What remains is to decide what the maximal time T_m must be, and here we have two possibilities: either $T_m=\infty$, or $T_m<\infty$. In the latter case, the solution must leave the space $C([0,T_m);L^1(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n\times(0,T_m))$ as the time approaches T_m, i.e.

$$\limsup_{t\to T_m^-}\{\|u(\cdot,t)\|_{L^1}+\|u(\cdot,t)\|_{L^\infty}\}=\infty.$$

So if we can show that

$$\operatorname*{ess\,sup}_{t\in[0,T_m)}\{\|u(\cdot,t)\|_{L^1}+\|u(\cdot,t)\|_{L^\infty}\}<\infty, \tag{4.17}$$

then we have $T_m=\infty$, and thus a global solution.

To show (4.17), we will make use of the lemmas above. Let v be a solution to (4.1) similarly as u, but with $v_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ as initial data instead of u_0. By Lemma 4.4, we then have

$$\int_{\mathbb{R}^n}[u(x,t)-v(x,t)]^+dx\le\int_{\mathbb{R}^n}[u_0(x)-v_0(x)]^+dx\ \forall\ t\in[0,T_m).$$

By using that $[v(x,t)-u(x,t)]^+=[-(u(x,t)-v(x,t))]^+=[u(x,t)-v(x,t)]^-$, we can apply Lemma 4.4 once again with u and v switched to get

$$\int_{\mathbb{R}^n}[u(x,t)-v(x,t)]^-dx\le\int_{\mathbb{R}^n}[u_0(x)-v_0(x)]^-dx\ \forall\ t\in[0,T_m).$$

Finally, since $|u(x,t)-v(x,t)|=[u(x,t)-v(x,t)]^++[u(x,t)-v(x,t)]^-$, we add the results above to get

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1}\le\|u_0-v_0\|_{L^1}\ \forall\ t\in[0,T_m). \tag{4.18}$$

By letting $v=0$, we get that the L^1-norm of u is bounded by its initial data u_0:

$$\|u(\cdot,t)\|_{L^1}\le\|u_0\|_{L^1}\ \forall\ t\in[0,T_m). \tag{4.19}$$

Lemma 4.5 directly yields a similar bound for the L^∞-norm of u:

$$\|u(\cdot,t)\|_{L^\infty}\le\|u_0\|_{L^\infty}\ \forall\ t\in[0,T_m).$$

This result together with (4.19) implies (4.17), and so we have $T_m=\infty$ and a global solution to (4.1). What remains is to show uniqueness and the L^1-contraction property, but observe that we already showed the latter in (4.18). Furthermore, uniqueness also follows directly from this, as (4.18) implies that any two solutions u,v with the same initial data $u_0=v_0$, must be equal themselves. This concludes the proof. □

4.2 Regularity estimates

As we saw in the previous section, we are often able to remove the influence of the convection term and thus reduce the properties of the convection-diffusion equation back to those of the heat equation. In this section, we will see that this is also the case when dealing with estimates in $L^p(\mathbb{R}^n)$-norm of the solutions we found in Theorem 4.6. The case $p = 1$ is already covered with the L^1-contraction result from the mentioned theorem, and we will consider the rest here, treating the cases $p \in (1, \infty)$ and $p = \infty$ separately. We will assume, as we did with the heat equation, that $u_0 \geq 0$, which implies that $u \geq 0$ by Lemma 4.4.

Theorem 4.7. *Let u be a strong solution of* (4.1) *with initial data $0 \leq u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $p \in (1, \infty)$ and $q \in [1, p)$. Then, we have*

$$\|u(\cdot, t)\|_{L^p} \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^q} \ \forall \ t > 0, \tag{4.20}$$

where

$$C = \left(\frac{np^2}{8(p-1)}(\frac{1}{q} - \frac{1}{p})\tilde{C}^2\right)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}.$$

$\tilde{C} = \tilde{C}(n)$ is the constant from the Sobolev inequality (Theorem A.1).

Proof. This proof is almost identical to the one we performed in [3, pp. 36–41] to prove Lemma 2.9, so we will only focus on the differences between the proofs.

We start by multiplying (4.1) with $u^{p-1}\xi_R$ and integrate over $\mathbb{R}^n$, where ξ_R is a bump function as defined in the proof of Lemma 4.4. We get

$$\int_{\mathbb{R}^n} \partial_t u u^{p-1}\xi_R dx - \int_{\mathbb{R}^n} \Delta u u^{p-1}\xi_R dx = \int_{\mathbb{R}^n} a \cdot \nabla(F(u))u^{p-1}\xi_R dx.$$

Once again, our goal is to get rid of the convection term on the right hand side to reduce back into the heat equation. Let

$$H(z) := \int_0^z F'(s)s^{p-1}ds,$$

then, analogously as in the previous section, we use Lipschitz continuity to get

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} a \cdot \nabla(F(u))u^{p-1}\xi_R dx\right| &= \left|\int_{\mathbb{R}^n} a \cdot \nabla\xi_R H(u)dx\right| \\
&\leq \frac{1}{R}|a|\|\nabla\xi\|_{L^\infty}\int_{\mathbb{R}^n} |H(u)|dx \\
&= \frac{1}{R}|a|\|\nabla\xi\|_{L^\infty}\int_{\mathbb{R}^n} \left|\int_0^u F'(s)s^{p-1}ds\right|dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{R}|a|\|\nabla \xi\|_{L^\infty} \int_{\mathbb{R}^n} \left| u^{p-1} \int_0^u F'(s)ds \right| dx \\
&= \frac{1}{R}|a|\|\nabla \xi\|_{L^\infty} \int_{\mathbb{R}^n} u^{p-1}|F(u) - F(0)|dx \\
&\leq \frac{1}{R}L|a|\|\nabla \xi\|_{L^\infty} \int_{\mathbb{R}^n} u^p dx \\
&= \frac{1}{R}L|a|\|\nabla \xi\|_{L^\infty} \|u(\cdot,t)\|_{L^p}^p \\
&\overset{R\to\infty}{\to} 0 \ \forall \ t > 0.
\end{aligned}
$$

We have also used $u \in C((0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,\infty))$ to ensure that $\|u(\cdot,t)\|_{L^p}$ is finite.

Now, we are left with exactly the same expression as in the case of the heat equation, namely

$$\int_{\mathbb{R}^n} \partial_t u u^{p-1} \xi_R dx = \int_{\mathbb{R}^n} \Delta u u^{p-1} \xi_R dx. \tag{4.21}$$

From this point we can proceed just as in [3, pp. 36–41] to obtain the result. □

Remark 4.8. *To prove the next result, we will need an estimate of the form as in point* (i) *of Lemma 2.8 for the convection-diffusion equation. In the case of the heat equation, we saw in [3, pp. 38–39] that this estimate followed from an estimate of the exact form as in* (4.21)*. Since we now have shown this estimate for the convection-diffusion equation* (4.1)*, we may extend the result in Lemma 2.8 to solutions of* (4.1) *as well, i.e.*

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p} \ \forall \ t \geq 0,$$

where $p \in (1,\infty)$*.*

As we mentioned in Remark 2.10, in order to obtain an estimate when $p = \infty$, we cannot simply let $p \to \infty$ in Theorem 4.7, as $C \to \infty$ in this case as well. The next result shows that we can overcome this by gaining control over how C grows as $p \to \infty$, leading to the following:

Proposition 4.9. *Let* u *be a strong solution of* (4.1) *with initial data* $0 \leq u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Let* $q \in [1,\infty)$*. Then, we have*

$$\|u(\cdot,t)\|_{L^\infty} \leq C t^{-\frac{n}{2q}} \|u_0\|_{L^q} \ \forall \ t > 0, \tag{4.22}$$

where

$$C = \left(2n\tilde{C}^2\right)^{\frac{n}{2q}}.$$

$\tilde{C} = \tilde{C}(n)$ *is the constant from the Sobolev inequality (Theorem A.1).*

Remark 4.10. *Proposition 4.9 holds for the heat equation as well, which can be seen by setting $F = 0$. Thus we have solved the problem which we commented in Remark 2.10, and we have finally covered all $p \in [1, \infty]$ for both the heat equation and the convection-diffusion equation.*

Proof. We are following the approach by Zuazua [1, pp. 34–35].

Observe that solutions of (4.1) are translation invariant in time, and so if $u(x,t)$ solves $\partial_t u(x,t) - \Delta u(x,t) = a \cdot \nabla(F(u(x,t)))$ with initial data $u(x,0)$, then $u(x,t+s)$ solves $\partial_t u(x,t+s) - \Delta u(x,t+s) = a \cdot \nabla(F(u(x,t+s)))$ with initial data $u(x,s)$ for any $s > 0$. Applying Proposition 4.9, we get for $s < t$,

$$\|u(\cdot,t)\|_{L^p} \leq C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u(\cdot,s)\|_{L^q}, \tag{4.23}$$

where $p \in (1,\infty)$, $q \in [1,p)$ and

$$C = \left(\frac{np^2}{8(p-1)}(\frac{1}{q}-\frac{1}{p})\tilde{C}^2\right)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}.$$

As mentioned in our discussion above, we must gain control over C as $p \to \infty$, and in order to do this, we use Moser iteration, introduced by Jürgen Moser in [13, 14]. The idea behind this method is to iterate over a sequence of values for p and t, repeatedly making use of (4.23), so that we obtain a constant which does not blow up as $p \to \infty$.

We start by letting $p_0 = q$ and $p_k = 2^k q$ for $k = 1, 2, \ldots$, thus creating an increasing sequence of values such that $p_k \to \infty$ as $k \to \infty$. For the time, we let t_0 be some starting point with $0 < t_0 < t$, and $t_k - t_{k-1} = \frac{t-t_0}{2^k}$ for $k = 1, 2, \ldots$. This creates a sequence moving from t_0 to t with decreasing step size as $k \to \infty$.

Next, we insert our new values into eq. (4.23) with $p = p_k$, $q = p_{k-1}$, $t = t_k$ and $s = t_{k-1}$, leading to

$$\|u(\cdot,t_k)\|_{L^{p_k}} \leq I_k^{\frac{n}{2}\left(\frac{1}{p_{k-1}}-\frac{1}{p_k}\right)}\|u(\cdot,t_{k-1})\|_{L^{p_{k-1}}}, \tag{4.24}$$

where

$$\begin{aligned} I_k &= \frac{\frac{np_k^2}{8(p_k-1)}(\frac{1}{p_{k-1}}-\frac{1}{p_k})\tilde{C}^2}{t_k - t_{k-1}} = \frac{\frac{n(2^k q)^2}{8(2^k q-1)}(\frac{1}{2^{k-1}q}-\frac{1}{2^k q})\tilde{C}^2}{\frac{t-t_0}{2^k}} \\ &= \frac{4^k n\tilde{C}^2 q}{8(t-t_0)(2^k q-1)} \leq \frac{4^k n\tilde{C}^2}{8(t-t_0)} =: 4^k\frac{\hat{C}}{t-t_0}. \end{aligned}$$

The last inequality makes use of the fact that $q \geq 1 \iff \frac{q}{2^k q-1} \leq 1$, thus giving an upper bound independent of q. Now, we make repeated use of (4.24) to write out the right hand side, also using the bounds for I_k:

$$\begin{aligned}
\|u(\cdot,t_k)\|_{L^{p_k}} &\leq I_k^{\frac{n}{2}\left(\frac{1}{p_{k-1}}-\frac{1}{p_k}\right)}\|u(\cdot,t_{k-1})\|_{L^{p_{k-1}}} \\
&\leq I_k^{\frac{n}{2}\left(\frac{1}{p_{k-1}}-\frac{1}{p_k}\right)} I_{k-1}^{\frac{n}{2}\left(\frac{1}{p_{k-2}}-\frac{1}{p_{k-1}}\right)}\|u(\cdot,t_{k-2})\|_{L^{p_{k-2}}} \\
&\leq I_k^{\frac{n}{2}\left(\frac{1}{p_{k-1}}-\frac{1}{p_k}\right)} I_{k-1}^{\frac{n}{2}\left(\frac{1}{p_{k-2}}-\frac{1}{p_{k-1}}\right)}\cdots I_1^{\frac{n}{2}\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}\|u(\cdot,t_0)\|_{L^{p_0}} \\
&\leq \left(\frac{4^k\hat{C}}{t-t_0}\right)^{\frac{n}{2}\frac{1}{2^k q}}\left(\frac{4^{k-1}\hat{C}}{t-t_0}\right)^{\frac{n}{2}\frac{1}{2^{k-1}q}}\cdots\left(\frac{4\hat{C}}{t-t_0}\right)^{\frac{n}{2}\frac{1}{2q}}\|u(\cdot,t_0)\|_{L^q} \\
&= 4^{\frac{n}{2q}\sum_{i=1}^{k}\frac{i}{2^i}}\left(\frac{\hat{C}}{t-t_0}\right)^{\frac{n}{2q}\sum_{i=1}^{k}\frac{1}{2^i}}\|u(\cdot,t_0)\|_{L^q} \qquad (4.25)\\
&\leq 4^{\frac{n}{2q}\sum_{i=1}^{\infty}\frac{i}{2^i}}\left(\frac{\hat{C}}{t-t_0}\right)^{\frac{n}{2q}\sum_{i=1}^{\infty}\frac{1}{2^i}}\|u(\cdot,t_0)\|_{L^q} \\
&= 4^{\frac{n}{2q}2}\left(\frac{\hat{C}}{t-t_0}\right)^{\frac{n}{2q}}\|u(\cdot,t_0)\|_{L^q} \\
&= \left(\frac{2n\tilde{C}^2}{t-t_0}\right)^{\frac{n}{2q}}\|u(\cdot,t_0)\|_{L^q}.
\end{aligned}$$

To calculate the infinite series, we have made use of the geometric series $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, for $|x| < 1$. Inserting $x = \frac{1}{2}$ and removing the first term, this yields $\sum_{i=1}^{\infty}\frac{1}{2^i} = 1$. Furthermore, differentiating this geometric series and multiplying by x, we get $\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$, for $|x| < 1$. Thus, with $x = \frac{1}{2}$ we get $\sum_{i=1}^{\infty}\frac{i}{2^i} = 2$.

Observe that the constant on the last line of (4.25) now is independent of k, meaning that we have gained control over the right hand side of the inequality.

Furthermore, since $t_k \leq t$ for all k, by Remark 4.8 we can use the result in point (i) of Lemma 2.8 to remove k from inside the norm on the left hand side, viewing t_k as initial time:

$$\|u(\cdot,t)\|_{L^{p_k}} \leq \|u(\cdot,t_k)\|_{L^{p_k}} \leq \left(\frac{2n\tilde{C}^2}{t-t_0}\right)^{\frac{n}{2q}}\|u(\cdot,t_0)\|_{L^q}.$$

Next, we want to let $t_0 \to 0$ so that we have the initial data on the right hand side. Making use of the reverse triangle inequality, the interpolation inequality for L^p-norms (Theorem A.4) and the fact that $u \in C([0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0,\infty))$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we get

$$\begin{aligned}
|\|u(\cdot,t_0)\|_{L^q} - \|u_0\|_{L^q}| &\leq \|u(\cdot,t_0) - u_0\|_{L^q} \leq \|u(\cdot,t_0) - u_0\|_{L^\infty}^{\frac{q-1}{q}}\|u(\cdot,t_0) - u_0\|_{L^1}^{\frac{1}{q}} \\
&\leq (\|u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} + \|u_0\|_{L^\infty(\mathbb{R}^n)})^{\frac{q-1}{q}}\|u(\cdot,t_0) - u_0\|_{L^1}^{\frac{1}{q}} \to 0 \text{ as } t_0 \to 0.
\end{aligned}$$

Thus, we end up with

$$\|u(\cdot,t)\|_{L^{p_k}} \leq \left(2n\tilde{C}^2\right)^{\frac{n}{2q}} t^{-\frac{n}{2q}} \|u_0\|_{L^q} \ \forall\ t > 0.$$

Finally, with $u(\cdot,t) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we can let $p_k \to \infty$ to conclude that (4.22) holds. □

4.3 Initial data in $L^1(\mathbb{R}^n)$

In this section, we will remove the assumption of $u_0 \in L^\infty(\mathbb{R}^n)$, and see that the results we developed in the previous section may be extended to this more general setting through an approximation.

Theorem 4.11. *Suppose $F \in C^1(\mathbb{R})$ with $F(0) = 0$. If $u_0 \in L^1(\mathbb{R}^n)$, then there exists a unique strong solution u to* (4.1)*. Furthermore:*

(i) $u \in C((0,\infty); W^{2,p}(\mathbb{R}^n)) \cap C^1((0,\infty); L^p(\mathbb{R}^n))$ *for all* $p \in (1,\infty)$.

(ii) *u satisfies the regularity estimates* (4.20) *and* (4.22) *with* $q = 1$.

(iii) *If u and v are strong solutions to* (4.1) *with initial data $u_0, v_0 \in L^1(\mathbb{R}^n)$ respectively, then*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \ \forall\ t \geq 0$$

If $u_0 \leq v_0$, then

$$u(x,t) \leq v(x,t) \ \forall\ t > 0, \ \text{a.e. } x \in \mathbb{R}^n.$$

Proof. We follow the proof of Theorem 4.1 in Zuazua [1, pp. 35–36].

Let $u_0 \in L^1(\mathbb{R}^n)$. By Bresiz [10, pp. 97–98], we know that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Thus, there exists a sequence $\{u_{0,k}\} \subset C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$u_{0,k} \to u_0 \text{ in } L^1(\mathbb{R}^n) \text{ as } k \to \infty, \tag{4.26}$$

and

$$\|u_{0,k}\|_{L^1} \leq \|u_0\|_{L^1}. \tag{4.27}$$

Since each $u_{0,k}$ lies in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, Theorem 4.6 gives that there exists a strong solution u_k to (4.1) with initial data $u_{0,k}$ for each $k \in \mathbb{N}$. To show that this sequence of strong solutions $\{u_k\}$ converges to some $u \in C([0,\infty); L^1(\mathbb{R}^n))$, we will show that it is a Cauchy sequence in the Banach space $C([0,\infty); L^1(\mathbb{R}^n))$. From Theorem 4.6, we have the L^1-contraction property, so that

$$\|u_k(\cdot,t) - u_m(\cdot,t)\|_{L^1} \leq \|u_{0,k} - u_{0,m}\|_{L^1} \ \forall\ t \geq 0, \ \forall\ k,m \in \mathbb{N}.$$

Now, since $\{u_{0,k}\}$ converges in the Banach space $L^1(\mathbb{R}^n)$, it is also Cauchy in $L^1(\mathbb{R}^n)$. Thus, for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that for $k, m \geq N$,

$$\|u_k(\cdot,t) - u_m(\cdot,t)\|_{L^1} \leq \|u_{0,k} - u_{0,m}\|_{L^1} < \varepsilon \;\forall\; t \geq 0.$$

Taking supremum in time, we get

$$\|u_k(\cdot,t) - u_m(\cdot,t)\|_{C([0,\infty);L^1(\mathbb{R}^n))} < \varepsilon.$$

Thus $\{u_k\}$ is Cauchy in $C([0,\infty); L^1(\mathbb{R}^n))$, which means it converges to some u in $C([0,\infty); L^1(\mathbb{R}^n))$, i.e.

$$u_k \to u \text{ in } C([0,\infty); L^1(\mathbb{R}^n)) \text{ as } k \to \infty. \tag{4.28}$$

We are yet to show that $u \in L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$ as well. The reason we work with this space and not $L^\infty(\mathbb{R}^n \times (0,\infty))$ as in Theorem 4.6, is that we no longer have $u_0 \in L^\infty(\mathbb{R}^n)$. Thus, we can only expect u to be locally bounded in time.

To show $u \in L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$, it is tempting to try to extend the argument above and show that $\{u_k\}$ is Cauchy in the space $C([0,\infty); L^1(\mathbb{R}^n)) \cap L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$. However, we do not have a $L^\infty(\mathbb{R}^n)$-contraction result needed to show that $\{u_k\}$ is Cauchy in this space, and therefore, we must try another approach. Observe that by Proposition 4.9 and (4.27), each u_k fulfills

$$\|u_k(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{n}{2}}\|u_{0,k}\|_{L^1} \leq Ct^{-\frac{n}{2}}\|u_0\|_{L^1} \;\forall\; t > 0, \tag{4.29}$$

where C only depends on n. Thus, $u_k \in L^\infty(\mathbb{R}^n \times (\tau,T))$ for $0 < \tau \leq T \leq \infty$. By the Banach-Alaoglu theorem (Theorem A.15), this implies that $u_k \overset{*}{\rightharpoonup} w$ in $L^\infty(\mathbb{R}^n \times (\tau,T))$, for some w. If we can show that $u_k \overset{*}{\rightharpoonup} u$, then we have $w = u$. This means that $u \in L^\infty(\mathbb{R}^n \times (\tau,T))$, in other words $u \in L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$ as wanted. By definition of weak* convergence, we therefore need to show

$$\int_\tau^T \int_{\mathbb{R}^n} u_k \varphi dx dt \to \int_\tau^T \int_{\mathbb{R}^n} u\varphi dx dt \text{ as } k \to \infty \;\forall\; \varphi \in L^1(\mathbb{R}^n \times (\tau,T)).$$

Since $u_k \to u$ in $C([0,\infty); L^1(\mathbb{R}^n))$, we have that there exists a subsequence of $\{u_k\}$ which converges to u a.e. Thus, for k large enough, we get

$$|u| \leq |u_k - u| + |u_k| \leq 1 + Ct^{-\frac{n}{2}}\|u_0\|_{L^1},$$

so that u is bounded. By dominated convergence (Theorem A.8), we thus get

$$\left|\int_\tau^T \int_{\mathbb{R}^n} u_k\varphi dx dt - \int_\tau^T \int_{\mathbb{R}^n} u\varphi dx dt\right| \leq \int_\tau^T \int_{\mathbb{R}^n} |u_k - u||\varphi| dx dt \to 0 \text{ as } k \to \infty.$$

It is clear by construction that $u(\cdot,0) = u_0$ a.e. in $\mathbb{R}^n$, but we also need to show that u still solves (4.1) a.e. In Theorem 4.6, we argued that this follows if u satisfies the integral equation (4.10). We know from this theorem that u_k satisfies (4.10) for each $k \in \mathbb{N}$, i.e.

$$u_k(x,t) = (G(\cdot,t) * u_{0,k})(x) + \int_0^t \left[a \cdot \nabla G(\cdot, t-s) * F(u_k(\cdot,s))\right](x)ds.$$

If we can show that each term above converges to u in $L^1(\mathbb{R}^n)$ for all $t > 0$, then we have that u solves (4.10), and thus also (4.1).

First, the left hand side simply converges by (4.28). Second, we apply Young's inequality (Theorem A.11), Lemma B.2 and (4.26) to get

$$\begin{aligned}\|G(\cdot,t) * u_{0,k} - G(\cdot,t) * u_0\|_{L^1} &= \|G(\cdot,t) * (u_{0,k} - u_0)\|_{L^1} \le \|G(\cdot,t)\|_{L^1}\|u_{0,k} - u_0\|_{L^1} \\ &\le Ct^{-\frac{n}{2}}\|u_{0,k} - u_0\|_{L^1} \to 0 \text{ as } k \to \infty \ \forall\, t > 0.\end{aligned}$$

Finally, we proceed with the last term similarly as in the proof of Theorem 4.6:

$$\begin{aligned}&\left\|\int_0^t a \cdot \nabla G(\cdot, t-s) * F(u_k(\cdot,s))ds - \int_0^t a \cdot \nabla G(\cdot, t-s) * F(u(\cdot,s))ds\right\|_{L^1} \\ &= \left\|\int_0^t a \cdot \nabla G(\cdot, t-s) * \left[F(u_k(\cdot,s)) - F(u(\cdot,s))\right]ds\right\|_{L^1} \\ &\le \int_0^t \|a \cdot \nabla G(\cdot, t-s) * [F(u_k(\cdot,s)) - F(u(\cdot,s))]\|_{L^1} ds \\ &\le \int_0^t \|a \cdot \nabla G(\cdot, t-s)\|_{L^1}\|F(u_k(\cdot,s)) - F(u(\cdot,s))\|_{L^1} ds \\ &\le |a|C\int_0^t (t-s)^{-\frac{1}{2}}\|F(u_k(\cdot,s)) - F(u(\cdot,s))\|_{L^1} ds \\ &= |a|C\int_0^\varepsilon (t-s)^{-\frac{1}{2}}\|F(u_k(\cdot,s)) - F(u(\cdot,s))\|_{L^1} ds \\ &\quad + |a|C\int_\varepsilon^t (t-s)^{-\frac{1}{2}}\|F(u_k(\cdot,s)) - F(u(\cdot,s))\|_{L^1} ds.\end{aligned}$$

On the last line, we have split the integral in two parts by some $\varepsilon > 0$. The reason for this is to overcome the non-linearity introduced by F. We wish to use Lipschitz continuity once again, but with $u \in L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$, i.e. only locally bounded in time, we must be careful. As we discussed above, the problem with boundedness for u arises at time 0, so if we restrict our time interval to $s \ge \varepsilon$, we get local Lipschitz continuity, i.e. for all $s \ge \varepsilon > 0$ and $t > 0$,

$$\|F(u_k(\cdot,s)) - F(u(\cdot,s))\|_{L^1} \le L\|u_k(\cdot,s) - u(\cdot,s)\|_{L^1} \to 0 \text{ as } k \to \infty.$$

Thus, using this, (4.28) and dominated convergence (Theorem A.8), the second integral becomes

$$\begin{aligned}&|a|C\int_\varepsilon^t (t-s)^{-\frac{1}{2}}\|F(u_k(\cdot,s)) - F(u(\cdot,s))\|_{L^1} ds \\ &\le |a|CL\int_\varepsilon^t (t-s)^{-\frac{1}{2}}\|u_k(\cdot,s) - u(\cdot,s)\|_{L^1} ds \to 0 \text{ as } k \to \infty \ \forall\, t > 0.\end{aligned}$$

The first integral yields,

$$|a|C\int_0^{\varepsilon}(t-s)^{-\frac{1}{2}}\|F(u_k(\cdot,s))-F(u(\cdot,s))\|_{L^1}ds\to 0 \text{ as } \varepsilon\to 0\ \forall\ t>0,$$

where the integral is bounded for k large enough since $F\in C^1(\mathbb{R}^n)$ and u, u_k are bounded.

So u satisfies (4.10), and by Lemma 4.2 we also obtain (i). With this, we have that u is a strong solution to (4.1).

To show uniqueness of the solution, we can proceed similarly as we did in Lemma 4.4 to show that this result still holds for strong solutions u, v given by initial values $u_0, v_0\in L^1(\mathbb{R}^n)$. Thus, we have

$$\int_{\mathbb{R}^n}[u(x,t)-v(x,t)]^+dx\le\int_{\mathbb{R}^n}[u_0(x)-v_0(x)]^+dx\ \forall\ t\ge 0,$$

which we saw implies both $L^1(\mathbb{R}^n)$-contraction and uniqueness in Theorem 4.6. The comparison principle also follows from this result, as

$$u_0\le v_0\implies[u_0-v_0]^+=0\implies[u-v]^+\le 0\implies u\le v.$$

Thus, we have (iii).

Finally, it remains to show the regularity estimates in (ii). We must restrict ourselves to $q=1$, since u_0 only is assumed to be in $L^1(\mathbb{R}^n)$. Theorem 4.7 gives that each u_k satisfies (4.20), and together with (4.27), this yields for $p\in(1,\infty)$,

$$\|u_k(\cdot,t)\|_{L^p}\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_{0,k}\|_{L^1}\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^1}\ \forall\ t>0,$$

where C only depends on n and p. The right hand side is uniformly bounded in k, while u_k on the left hand side converges to u in $C([0,\infty);L^1(\mathbb{R}^n))$, thus also a.e. We may therefore apply Fatou's lemma (Theorem A.7) to take the limit and obtain the estimate for $p\in(1,\infty)$ and $t>0$:

$$\|u(\cdot,t)\|_{L^p}\le\liminf_{k\to\infty}\|u_k(\cdot,t)\|_{L^p}\le\liminf_{k\to\infty}Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^1}=Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_{L^1}.$$

For $p=\infty$, we have from (4.29) that

$$\|u_k(\cdot,t)\|_{L^\infty}\le Ct^{-\frac{n}{2}}\|u_0\|_{L^1}\ \forall\ t>0,$$

where C only depends on n. By Theorem A.16, $u_k(\cdot,t)\overset{*}{\rightharpoonup}u(\cdot,t)$ in $L^\infty(\mathbb{R}^n)$ implies lower semicontinuity in $L^\infty(\mathbb{R}^n)$, thus

$$\|u(\cdot,t)\|_{L^\infty}\le\liminf_{k\to\infty}\|u_k(\cdot,t)\|_{L^\infty}\le\liminf_{k\to\infty}Ct^{-\frac{n}{2}}\|u_0\|_{L^1}=Ct^{-\frac{n}{2}}\|u_0\|_{L^1}\ \forall\ t>0.$$

This concludes the proof. □

Chapter 5

Asymptotic behaviour for a convection-diffusion equation

Having established in the previous chapter that the convection-diffusion equation (4.1) is well-posed with initial data $u_0 \in L^1(\mathbb{R}^n)$, we are now ready to investigate the asymptotic behaviour of the solutions. We will start by motivating what type of behaviour we should expect similarly as in Section 3.1.

5.1 Motivation

We have seen that mass conservation is an important property to understand the asymptotic behaviour, and so we start by showing mass conservation for solutions of (4.1):

Lemma 5.1. *Let u be a strong solution of* (4.1) *with initial value $u_0 \in L^1(\mathbb{R}^n)$. Then, the mass of u is conserved, i.e.*

$$\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx \qquad \forall\ t > 0.$$

Proof. We multiply (4.1) by ξ_R as defined in Lemma 4.4 and integrate over $\mathbb{R}^n$ to get

$$\int_{\mathbb{R}^n} \partial_t u \xi_R dx = \int_{\mathbb{R}^n} \Delta u \xi_R dx + \int_{\mathbb{R}^n} a \cdot \nabla(F(u))\xi_R dx.$$

Proceeding as we have done several times by now, we wish to show that both terms on the right hand side tend to 0 as $R \to \infty$. For the first integral, we apply Green's first identity (Theorem A.6) to move the Laplacian over to ξ_R, and since $u \in C([0,\infty); L^1(\mathbb{R}^n))$, we can bound this integral as follows:

$$\left| \int_{\mathbb{R}^n} \Delta u \xi_R dx \right| \leq \int_{\mathbb{R}^n} |u||\Delta \xi_R| dx \leq \frac{1}{R^2}\|\xi\|_{L^\infty}\|u(\cdot,t)\|_{L^1} \to 0 \text{ as } R \to \infty\ \forall\ t > 0.$$

For the second term, we similarly move the gradient over to ξ_R, and since Theorem 4.11 gives that $u \in L^\infty_{loc}((0,\infty); L^\infty(\mathbb{R}^n))$, we can apply Lipschitz continuity, with some care as we did in the proof of Theorem 4.11, to move from $F(u)$ to u:

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a\cdot\nabla(F(u))\xi_R dx\right| &\le \int_{\mathbb{R}^n} |a\cdot\nabla\xi_R F(u)|dx \le \frac{1}{R}|a|\|\nabla\xi\|_{L^\infty}\int_{\mathbb{R}^n}|F(u)|dx\\ &\le \frac{1}{R}L|a|\|\nabla\xi\|_{L^\infty}\|u(\cdot,t)\|_{L^1} \to 0 \text{ as } R\to\infty\ \forall\ t\ >0.\end{aligned}$$

Thus we are left with

$$\int_{\mathbb{R}^n}\partial_t u(x,t)\xi_R(x)dx = O\left(1/R\right).$$

Theorem 4.11 gives $u \in C^1((0,\infty); L^p(\mathbb{R}^n))$ for $p \in (1,\infty)$, and together with $\xi_R \in C_c^\infty(\mathbb{R}^n)$, this implies that we can move the derivative out of the integral. Finally, by dominated convergence (Theorem A.8), we can let $\xi_R \to 1$ as $R \to \infty$, since $u \in C([0,\infty); L^1(\mathbb{R}^n))$. This gives

$$\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)dx = 0\ \forall\ t>0.$$

which implies our result. □

The equation (4.1) contains an unspecified function $F \in C^1(\mathbb{R})$ with $F(0)=0$. The asymptotic behaviour of this equation may vary a lot depending on F, and so we need to specify this function somewhat. For the rest of this chapter, we will therefore work in one spatial dimension ($n=1$) with $F(u) := u^q$ for some $q>1$, assuming $0 \le u_0 \in L^1(\mathbb{R})$, which implies $u \ge 0$ by Theorem 4.11. Thus, (4.1) becomes

$$\begin{cases}\partial_t u - \partial_x^2 u = a\partial_x(u^q), & (x,t)\in\mathbb{R}\times(0,\infty)\\ u(x,0)=u_0(x), & x\in\mathbb{R}.\end{cases} \tag{5.1}$$

Although we assume $q>1$, observe that $q=1$ yields a linear convection term on the right hand side. In fact, we covered the asymptotic behaviour in this case in Section 3.3, and found that it was given by a translation of the asymptotic behaviour for the heat equation.

To motivate what type of asymptotic behaviour we should expect from the equation (5.1), we will make a scaling argument similarly as is Section 3.1. The idea is to investigate what equation the scaled solution u_λ fulfills for a fixed time as $\lambda\to\infty$. Through a change of variables, this provides us with the asymptotic behaviour of u as $t\to\infty$.

Let $\lambda>0$, assume u solves (5.1) and consider the rescaled function

$$u_\lambda(x,t) := \lambda^\alpha u(\lambda^\beta x, \lambda^\gamma t),$$

with initial value $u_\lambda(x,0) = \lambda^\alpha u_0(\lambda^\beta x) =: u_{0,\lambda}(x)$. We will now determine appropriate choices for α, β, and γ. Mass conservation was an important property in the asymptotic behaviour of the heat equation, and since u conserves its mass by Lemma 5.1, we also want u_λ to conserve its mass. To ensure this holds, we require that $\alpha = \beta$, which can be seen from the following:

$$\begin{aligned}\int_{\mathbb{R}} u_\lambda(x,t)dx &= \int_{\mathbb{R}} \lambda^\alpha u(\lambda^\beta x, \lambda^\gamma t)dx \overset{z:=\lambda^\beta x}{=} \int_{\mathbb{R}} \lambda^\alpha u(z,\lambda^\gamma t)\lambda^{-\beta}dz \overset{\alpha=\beta}{=} \int_{\mathbb{R}} u(z,\lambda^\gamma t)dz \\ &= \int_{\mathbb{R}} u_0(z)dz \overset{x:=\lambda^{-\alpha}z}{=} \int_{\mathbb{R}} \lambda^\alpha u_0(\lambda^\alpha x)dx = \int_{\mathbb{R}} u_{0,\lambda}(x)dx.\end{aligned}$$

With this condition, we also get

$$u_{0,\lambda} \to M\delta \text{ as } \lambda \to \infty. \tag{5.2}$$

This can be seen by proceeding exactly as in Section 3.1 to show that

$$\int_{\mathbb{R}} u_\lambda(x,0)\varphi(x)dx \to M\varphi(0) \text{ as } \lambda \to \infty, \tag{5.3}$$

for all $\varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Furthermore, differentiating u_λ yields

$$\begin{aligned}\partial_t u_\lambda(x,t) &= \lambda^{\alpha+\gamma}\partial_t u(\lambda^\alpha x, \lambda^\gamma t) \\ \partial_x^2 u_\lambda(x,t) &= \lambda^{3\alpha}\partial_x^2 u(\lambda^\alpha x, \lambda^\gamma t) \\ \partial_x((u_\lambda(x,t))^q) &= \lambda^{\alpha(q+1)}\partial_x(u^q)(\lambda^\alpha x, \lambda^\gamma t).\end{aligned} \tag{5.4}$$

Observe that if $\gamma = 2\alpha$, then the λ's in front of the first two terms above have the same power. We choose $\alpha = 1$ and $\gamma = 2$ and thus get the same scaling we had for the heat equation in Section 3.1. This u_λ solves the equation

$$\partial_t u_\lambda - \partial_x^2 u_\lambda = \lambda^{2-q} a \partial_x(u_\lambda^q).$$

To see what happens to the equation as $\lambda \to \infty$, we must consider different choices for q separately.

First, let us mention that if $q = 2$, then (5.1) is invariant with respect to this scaling. We will not study this case here, but rather refer to the work by Escobedo and Zuazua [2], where they find that the asymptotic behaviour in this case is given by explicit self-similar solutions.

Second, if $q > 2$, then $\lambda^{2-q} \to 0$ as $\lambda \to \infty$, in which case the convective term disappears and the equation approaches the heat equation. For this reason, together with (5.2), we expect the same asymptotic behaviour as we saw for the heat equation, i.e. that the solution converges to M times the heat kernel G. From the estimates given in point (ii) of Theorem 4.11, we also see that $t^{\frac{1}{2}(1-\frac{1}{p})}\|u(\cdot,t)\|_{L^p}$ is bounded independent of the time t for all $t > 0$ and $p \in [1,\infty]$. This suggests that the convergence takes place in this setting, which we also saw in the case of the heat equation.

This type of asymptotic behaviour is referred to as weakly non-linear, as the non-linear convection term disappears from the equation in the long run, thus not influencing the asymptotic behaviour of the solutions. For the time being, we have only motivated that we expect this type of behaviour in the case $q > 2$, but in the next section we will show rigorously that this indeed holds.

Finally, what remains is the case $1 < q < 2$, but here we see that $\lambda^{2-q} \to \infty$ as $\lambda \to \infty$, and thus we can no longer expect the convective term to disappear in the long run. To overcome this, we search for a different scaling, where we try to remove the diffusion term instead as $\lambda \to \infty$. Observe that if $\gamma = q\alpha$, then the λ's in front of the first and last term of (5.4) have the same power. Choosing $\alpha = 1$ and $\gamma = q$, we get that u_λ solves the equation

$$\partial_t u_\lambda - \lambda^{q-2}\partial_x^2 u_\lambda = a\partial_x(u_\lambda^q).$$

With $1 < q < 2$, we get that $\lambda^{q-2} \to 0$ as $\lambda \to \infty$, in which case the diffusion term disappears and we end up with a convective partial differential equation in the limit:

$$\partial_t u - a\partial_x(u^q) = 0. \tag{5.5}$$

Thus, we expect the asymptotic behaviour to be given by the solution of this equation with $M\delta$ as initial data, by (5.2). However, this hyperbolic equation is fundamentally different from the parabolic heat equation which we have based our studies on previously, and it is therefore not as obvious what kind of approach to use in order to show this rigorously. We will return to this problem in Section 5.3.

Since the non-linear term does not disappear in the case of $1 < q < 2$, this type of asymptotic behaviour is referred to as strongly non-linear.

5.2 Weakly non-linear asymptotic behaviour

In this section we consider the case $q > 2$. We will show rigorously what we motivated in the previous section, namely that the equation (5.1) admits a weakly non-linear asymptotic behaviour in this case, resulting in the same asymptotic behaviour as for the heat equation.

Theorem 5.2. *Let $0 \leq u_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)dx = M > 0$. Then, the strong solution u to* (5.1) *with $q > 2$ satisfies*

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|u(\cdot,t) - MG(\cdot,t)\|_{L^p} \to 0 \text{ as } t \to \infty, \tag{5.6}$$

for all $p \in [1, \infty]$.

Remark 5.3. *This result may be extended to hold in dimensions $n \geq 1$ and for more general functions $F \in C^1(\mathbb{R})$ than $F(u) = u^q$, under some conditions on their behaviour near zero. This is a quite natural extension, since the solution u approaches zero for large time values, thus making the behaviour of F around zero the most influential for the asymptotic behaviour, as commented in [5, p. 63].*

The proof presented below is in fact a simplification of Theorem 5.1 *in Zuazua [1, pp. 42–43], which covers this more general setting. The reason for our simplification here* *is that we want to work in the same framework as the one we use in the next section, when we cover the more difficult case of* $1 < q < 2$.

Proof. We follow the proof of Theorem 5.1 in Zuazua [1, pp. 42–43].

We saw in the proof of Theorem 4.11 that u satisfies the integral equation

$$u(x,t) = (G(\cdot,t) * u_0)(x) + \int_0^t a[\partial_x G(\cdot,t-s) * (u(\cdot,s))^q](x)ds.$$

Inserting this directly into (5.6) yields

$$\begin{aligned}
&t^{\frac{1}{2}(1-\frac{1}{p})}\|u(\cdot,t) - MG(\cdot,t)\|_{L^p} \\
&= t^{\frac{1}{2}(1-\frac{1}{p})}\left\|G(\cdot,t) * u_0 + \int_0^t a\partial_x G(\cdot,t-s) * (u(\cdot,s))^q ds - MG(\cdot,t)\right\|_{L^p} \\
&\le t^{\frac{1}{2}(1-\frac{1}{p})}\|G(\cdot,t) * u_0 - MG(\cdot,t)\|_{L^p} + t^{\frac{1}{2}(1-\frac{1}{p})}\left\|\int_0^t a\partial_x G(\cdot,t-s) * (u(\cdot,s))^q ds\right\|_{L^p}.
\end{aligned}$$

We wish to show that both of these two terms go to zero as $t \to \infty$. The first part simply corresponds to the asymptotic behaviour of the heat equation, which by Theorem 3.3 gives us

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|G(\cdot,t) * u_0 - MG(\cdot,t)\|_{L^p} \to 0 \text{ as } t \to \infty.$$

For the second part, we proceed similarly as done before by using Minkowski's integral inequality (Theorem A.12) and Young's inequality Theorem A.11:

$$\begin{aligned}
&\left\|\int_0^t a\partial_x G(\cdot,t-s) * (u(\cdot,s+1))^q ds\right\|_{L^p} \\
&\le \int_0^t \|a\partial_x G(\cdot,t-s) * (u(\cdot,s+1))^q\|_{L^p} ds \\
&= \int_0^{\frac{t}{2}} \|a\partial_x G(\cdot,t-s) * (u(\cdot,s+1))^q\|_{L^p} ds \\
&\quad + \int_{\frac{t}{2}}^t \|a\partial_x G(\cdot,t-s) * (u(\cdot,s+1))^q\|_{L^p} ds \\
&\le \int_0^{\frac{t}{2}} \|a\partial_x G(\cdot,t-s)\|_{L^p} \|(u(\cdot,s+1))^q\|_{L^1} ds \\
&\quad + \int_{\frac{t}{2}}^t \|a\partial_x G(\cdot,t-s)\|_{L^1} \|(u(\cdot,s+1))^q\|_{L^p} ds.
\end{aligned}$$

We have split this integral in two parts in order to control the singularity which arises at time 0, so that we still obtain a decay in t which is stronger than the growth $t^{\frac{1}{2}(1-\frac{1}{p})}$. To bound the second integral, we use the L^1-norm estimate for the gradient of the heat kernel given in Lemma B.3 and the L^p-norm estimates for u given in Theorem 4.11, which yields

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \|a\partial_x G(\cdot,t-s)\|_{L^1}\|(u(\cdot,s))^q\|_{L^p} ds \\
&\le \int_{\frac{t}{2}}^{t} |a|\tilde{C}(t-s)^{-\frac{1}{2}}\|u(\cdot,s)\|_{L^{pq}}^q ds \\
&\le |a|\tilde{C}\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} C_{pq} s^{-\frac{1}{2}(1-\frac{1}{pq})q}\|u_0\|_{L^1}^q ds \\
&\le |a|C\|u_0\|_{L^1}^q \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}(q-\frac{1}{p})} ds \\
&\le |a|C\|u_0\|_{L^1}^q \left(\frac{t}{2}\right)^{-\frac{1}{2}(q-\frac{1}{p})} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}} ds \\
&= 2|a|C\|u_0\|_{L^1}^q \left(\frac{t}{2}\right)^{-\frac{1}{2}(q-\frac{1}{p})} \left(\frac{t}{2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&t^{\frac{1}{2}(1-\frac{1}{p})}\int_{\frac{t}{2}}^{t} \|a\partial_x G(\cdot,t-s)\|_{L^1}\|(u(\cdot,s+1))^q\|_{L^p} ds \\
&\le 2|a|C\|u(\cdot,1)\|_{L^1}^q \left(\frac{1}{2}\right)^{-\frac{1}{2}(q-\frac{1}{p})+\frac{1}{2}} t^{\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}(q-\frac{1}{p})+\frac{1}{2}} \\
&= 2|a|C\|u(\cdot,1)\|_{L^1}^q \left(\frac{1}{2}\right)^{-\frac{1}{2}(q-\frac{1}{p})+\frac{1}{2}} t^{\frac{2-q}{2}} \to 0 \text{ as } t\to\infty.
\end{aligned}$$

For the first integral, we use a stronger estimate for the gradient of the heat kernel given in Lemma B.4. Together with the usual L^1-norm estimate for u, this yields

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \|a\partial_x G(\cdot,t-s)\|_{L^p}\|(u(\cdot,s))^q\|_{L^1} ds \\
&\le \int_0^{\frac{t}{2}} |a|\tilde{C}(t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u(\cdot,s)\|_{L^q}^q ds \\
&\le |a|\tilde{C}\left(\frac{t}{2}\right)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_0^{\frac{t}{2}} \|u(\cdot,s)\|_{L^q}^q ds \\
&\le |a|\tilde{C}\left(\frac{t}{2}\right)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_0^{\frac{t}{2}} C_q s^{-\frac{1}{2}(1-\frac{1}{q})q}\|u_0\|_{L^1}^q ds \\
&= |a|C\|u_0\|_{L^1}^q \left(\frac{t}{2}\right)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_0^{\frac{t}{2}} s^{-\frac{1}{2}(q-1)} ds \\
&= |a|C\|u_0\|_{L^1}^q \left(\frac{t}{2}\right)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \frac{2}{3-q}\left(\frac{t}{2}\right)^{\frac{3-q}{2}}.
\end{aligned}$$

Finally, we get

$$
\begin{aligned}
&t^{\frac{1}{2}(1-\frac{1}{p})}\int_0^{\frac{t}{2}} \|a\partial_x G(\cdot,t-s)\|_{L^p}\|(u(\cdot,s))^q\|_{L^1}ds\\
&\le \frac{2}{3-q}|a|C\|u_0\|_{L^1}^q\left(\frac{1}{2}\right)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\frac{3-q}{2}} t^{\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\frac{3-q}{2}}\\
&= \frac{2}{3-q}|a|C\|u_0\|_{L^1}^q\left(\frac{1}{2}\right)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\frac{3-q}{2}} t^{\frac{2-q}{2}} \to 0 \text{ as } t\to\infty.
\end{aligned}
$$

This concludes the proof. □

5.3 Strongly non-linear asymptotic behaviour

In this section we consider the case of $1 < q < 2$. To show the asymptotic behaviour of (5.1) in this case, we will follow the approach made by Escobedo, Vázquez and Zuazua in [5] and [6]. Consequently, we also assume $a = -1/q$ in (5.1), yielding the following problem:

$$
\begin{cases}\partial_t u = \partial_x^2 u - \frac{1}{q}\partial_x\left(u^q\right), & (x,t)\in\mathbb{R}\times(0,\infty)\\ u(x,0) = u_0(x), & x\in\mathbb{R}.\end{cases} \tag{5.7}
$$

As we saw in Section 5.1, for $\lambda > 0$, the rescaled function

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^q t)$$

conserves its mass and solves the equation

$$\partial_t u_\lambda = \lambda^{q-2}\partial_x^2 u_\lambda - \frac{1}{q}\partial_x\left(u_\lambda^q\right),$$

with initial data

$$u_\lambda(x,0) = \lambda u_0(\lambda x).$$

In the limit $\lambda\to\infty$, this turns into the convective equation

$$
\begin{cases}\partial_t u + \frac{1}{q}\partial_x\left(u^q\right) = 0, & (x,t)\in\mathbb{R}\times(0,\infty)\\ u(x,0) = M\delta, & x\in\mathbb{R}.\end{cases} \tag{5.8}
$$

We expect the solution of (5.8) to give the asymptotic behaviour of (5.7). The equation (5.8) is hyperbolic and therefore of a different nature than the parabolic heat equation for which we have based most of our theory upon thus far. Liu and Pierre show in [7, p. 432] that (5.8) admits a so-called entropy solution given by the function

$$
u_M(x,t) := \begin{cases}\left(\frac{x}{t}\right)^{\frac{1}{q-1}} & \text{if } 0 < x < r(t)\\ 0 & \text{otherwise,}\end{cases} \tag{5.9}
$$

where

$$r(t) := \frac{q}{q-1}^{\frac{q-1}{q}} M^{\frac{q-1}{q}} t^{\frac{1}{q}}.$$

This solution also fulfills the following decay in time:

$$\|u_M(\cdot,t)\|_{L^\infty} \le Ct^{-\frac{1}{q}} \ \forall\ t > 0. \tag{5.10}$$

Finally, we claim that this solution is unique. The question of uniqueness of solutions to (5.8) and its relation to entropy solutions is the main focus of Chapter 6, and we will therefore return to this claim there. Right now, however, our focus is to show that the solution u of (5.7) indeed converges to u_M given by (5.9), and find out in what sense this happens.

A natural starting point may be to investigate if the method applied in the weakly non-linear case, where $q > 2$, still is useful here. The answer is unfortunately negative, even though the L^p-estimates and the integral equality from Theorem 4.11 are still valid. As we will see below, the dominating convection term in (5.7) yields a norm estimate of the form (5.10) for solutions of (5.7). With $1 < q < 2$, this estimate is sharper than the corresponding estimate from Theorem 4.11, which has a decay in time of the form $t^{-\frac{1}{2}}$. With a sharper expected convergence rate, our previous results are in a sense "too slow" to help us out.

To cope with this, we instead make use of what we have, namely the scaled solutions u_λ. We wish to show that these solutions do in fact converge to u_M as $\lambda \to \infty$ with a sharper rate, and then connect this to $t \to \infty$ in the same way as we did in Section 3.1.

Before we can show this result rigorously, however, we first need some estimates on the solutions of (5.7), starting with the so-called entropy inequality.

Lemma 5.4. *Let $u_0 \in C^\infty(\mathbb{R})$ be strictly positive and bounded. Then, the classical solution u to (5.7) with $1 < q < 2$ satisfies*

$$\partial_x \left[(u(x,t))^{q-1}\right] \le \frac{1}{t} \ \forall\ (x,t) \in \mathbb{R} \times (0,\infty). \tag{5.11}$$

Proof. We will only make a sketch of the proof here, in the same way as it is done in the proof of Lemma 1.1 in [5, p. 48]. With $u_0 \in C^\infty(\mathbb{R})$ such that u_0 is strictly positive and bounded, the resulting classical solution u is also strictly positive, bounded and lies in $C^\infty(\mathbb{R} \times [0,\infty))$, since we have a parabolic equation.

Let $z := u^{q-1}$, then $u = z^{\frac{1}{q-1}}$, so that (5.7) becomes

$$\partial_t z + z\partial_x z - \beta\frac{(\partial_x z)^2}{z} = \partial_x^2 z,$$

where $\beta := (2-q)/(q-1)$. By differentiating this equation with respect to x and defining $w := \partial_x z$, we obtain

$$\partial_t w - \partial_x^2 w + \left(z - 2\beta\frac{w}{z}\right)\partial_x w + w^2 + \beta\frac{w^3}{z^2} = 0. \tag{5.12}$$

Next, observe that the function

$$W(t) := \frac{1}{t},$$

solves the ordinary differential equation

$$\partial_t w + w^2 = 0. \tag{5.13}$$

Furthermore, it is also a supersolution for (5.12), i.e. it solves

$$\partial_t w - \partial_x^2 w + \left(z - 2\beta\frac{w}{z}\right)\partial_x w + w^2 + \beta\frac{w^3}{z^2} \geq 0.$$

To see this, we observe that the first and fifth term on the left hand side yield 0 by (5.13), the second, third and fourth term involve spacial derivatives and thus disappear, while the sixth term is non-negative since $\beta > 0$ when $1 < q < 2$.

To obtain the result (5.11), we apply the results above together with the fact that $W(0) = \infty$, so that the maximum principle yields

$$w(x,t) \leq W(t) \; \forall \; (x,t) \in \mathbb{R} \times (0,\infty).$$ □

From this result, we may now show the hyperbolic L^1–L^∞-smoothing result which we claimed earlier:

Lemma 5.5. *Let $0 \leq u_0 \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that u_0 is bounded and $\int_{\mathbb{R}} u_0(x)dx = M > 0$. Then, the classical solution u to (5.7) with $1 < q < 2$ satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{1}{q}}\|u_0\|_{L^1}^{\frac{1}{q}} \; \forall \; t > 0, \tag{5.14}$$

where $C > 0$ is a constant depending on q.

Remark 5.6.

(i) For simplicity, we will sometimes write this result as

$$\|u(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{1}{q}} \; \forall \; t > 0,$$

where $\|u_0\|_{L^1}^{\frac{1}{q}}$ is included in the constant C.

(ii) We will need this result in the weaker setting of strong solutions to (5.7) with initial data $0 \leq u_0 \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} u_0(x)dx = M > 0$. This extension may be achieved by means of an approximation, and we will therefore use this result with strong solutions below. The same also applies for the result in Lemma 5.4, as discussed in [5, p. 48].

Proof. We follow the proof of Lemma 1.2 in [5, pp. 48–49]. This result builds upon the result from Lemma 5.4, which assumes u_0 is strictly positive. However, by an approximation, we may still use this result here, where u_0 only is assumed to be non-negative.

First, we get the lower bound from $u_0 \geq 0$, since this implies $u \geq 0$.

To obtain the upper bound, we start by integrating the result from Lemma 5.4 to get

$$(u(x,t))^{q-1} \leq \frac{x}{t} \ \forall \ (x,t) \in \mathbb{R} \times (0,\infty).$$

Fix a time $t > 0$ and a point $x_0 \in \mathbb{R}$. Then, this result yields

$$(u(x_0,t))^{q-1} \leq (u(x,t))^{q-1} + \frac{x_0 - x}{t} \text{ for } x \leq x_0,$$

which can be rewritten as

$$(u(x,t))^{q-1} \geq (u(x_0,t))^{q-1} - \frac{x_0 - x}{t} \text{ for } x \leq x_0. \tag{5.15}$$

Let $x_1 := x_0 - (u(x_0,t))^{q-1}t$, then $x_1 \leq x_0$ since $u(x_0,t) \geq 0$, and we can use (5.15) to get

$$(u(x,t))^{q-1} \geq (u(x_0,t))^{q-1} - \frac{x_1 + (u(x_0,t))^{q-1}t - x}{t} = \frac{x - x_1}{t} \text{ for } x_1 \leq x \leq x_0.$$

Taking the root and integrating this inequality over $[x_1, x_0]$ gives

$$\int_{x_1}^{x_0} u(x,t)dx \geq \int_{x_1}^{x_0} \left(\frac{x - x_1}{t}\right)^{\frac{1}{q-1}} dx.$$

The left hand side yields

$$\int_{x_1}^{x_0} u(x,t)dx \leq \int_{\mathbb{R}} u(x,t)dx = M,$$

while the right hand side yields

$$\int_{x_1}^{x_1+(u(x_0,t))^{q-1}t} \left(\frac{x - x_1}{t}\right)^{\frac{1}{q-1}} dx \overset{z:=(x-x_1)/t}{=} \int_0^{(u(x_0,t))^{q-1}} z^{\frac{1}{q-1}} t dz = \frac{q-1}{q}(u(x_0,t))^q t.$$

Thus, we end up with

$$M \geq \frac{q-1}{q}(u(x_0,t))^q t,$$

which implies the upper bound

$$u(x_0,t) \leq \left(\frac{Mq}{q-1}\right)^{\frac{1}{q}} t^{-\frac{1}{q}}.$$

The point $x_0 \in \mathbb{R}$ was arbitrary, and so we have (5.14). $\square$

We will also need the following lemma which bounds the derivatives of u^q:

Lemma 5.7. *Let $0 \leq u_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)dx = M > 0$. Then, the strong solution u to (5.7) with $1 < q < 2$ satisfies*

$$\partial_x[(u(x,t))^q] \leq \frac{qu(x,t)}{(q-1)t} \ \forall\ (x,t) \in \mathbb{R} \times (0,\infty).$$

Proof. The result follows directly from Lemma 5.4, as stated in [5, p. 49]:

$$\begin{aligned}\partial_x[(u(x,t))^q] &= q(u(x,t))^{q-1}\partial_x u(x,t) = q\frac{q-1}{q-1}u(x,t)(u(x,t))^{q-2}\partial_x u(x,t) \\ &= \frac{qu(x,t)}{q-1}\partial_x[(u(x,t))^{q-1}] \leq \frac{qu(x,t)}{(q-1)t}.\end{aligned}$$

□

The last lemma is an energy estimate for u:

Lemma 5.8. *Let $0 \leq u_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)dx = M > 0$. Then, the strong solution u to (5.7) with $1 < q < 2$ satisfies*

$$\int_\tau^T \int_{\mathbb{R}} |\partial_x u(x,t)|^2 dxdt \leq \frac{1}{2}\int_{\mathbb{R}} (u(x,\tau))^2 dx \leq C\tau^{-\frac{1}{q}} \ \forall\ 0 < \tau < T,$$

where $C = C(q, M)$ is a constant.

Proof. The result is stated in Lemma 1.5 in [5, p. 50] without a detailed proof, but we will write it out here.

With u being a strong solution of (5.7), we have that it fulfills

$$\partial_t u(x,t) = \partial_x^2 u(x,t) - (u(x,t))^{q-1}\partial_x u(x,t),$$

a.e. for $(x,t) \in \mathbb{R} \times (0,\infty)$. We begin by multiplying this equation by $u(x,t)\xi_R(x)$ and integrating over $\mathbb{R}$, where ξ_R is a bump function as defined in (4.4):

$$\int_{\mathbb{R}} \partial_t u u \xi_R dx = \int_{\mathbb{R}} \partial_x^2 u u \xi_R dx - \int_{\mathbb{R}} u^q \partial_x u \xi_R dx. \tag{5.16}$$

Integrating by parts the first term on the right hand side yields

$$\begin{aligned}\int_{\mathbb{R}} \partial_x^2 u u \xi_R dx &= [\partial_x u u \xi_R]_{x=-\infty}^{\infty} - \int_{\mathbb{R}} \partial_x u \partial_x(u\xi_R)dx = -\int_{\mathbb{R}} \partial_x u(\partial_x u \xi_R + u\partial_x \xi_R)dx \\ &= -\int_{\mathbb{R}} |\partial_x u|^2 \xi_R dx - \int_{\mathbb{R}} u\partial_x u \partial_x \xi_R dx.\end{aligned}$$

We wish to show that the last term above vanishes as $R \to \infty$ by using the estimate on $\partial_x^2 \xi_R$ given in (4.4). To do this, we move the derivative over to $\partial_x \xi_R$ through another round of integration by parts, which gives

$$\begin{aligned}\left|-\int_{\mathbb{R}} u\partial_x u \partial_x \xi_R dx\right| &= \left|\frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2)\partial_x \xi_R dx\right| = \left|\left[\frac{1}{2}u^2 \partial_x \xi_R\right]_{x=-\infty}^{\infty} - \frac{1}{2}\int_{\mathbb{R}} u^2 \partial_x^2 \xi_R dx\right| \\ &\leq \frac{1}{2}\int_{\mathbb{R}} |u|^2 |\partial_x^2 \xi_R| dx \leq \frac{1}{2}\frac{1}{R^2}\left\|\partial_x^2 \xi\right\|_{L^\infty} \int_{\mathbb{R}} |u|^2 dx \to 0 \text{ as } R \to \infty.\end{aligned}$$

The last integral is finite by point (i) of Theorem 4.11, which states that for $p \in (1,\infty)$, $u \in C^1((0,\infty);L^p(\mathbb{R}))$. Additionally, the theorem states that $u \in C((0,\infty);W^{2,p}(\mathbb{R}))$. Therefore, we have $\partial_x^2 u(\cdot,t) \in L^2(\mathbb{R})$, and so we can apply the Dominated convergence theorem (Theorem A.8) to let $\xi_R \to 1$ as $R \to \infty$ in the first term above:

$$-\int_{\mathbb{R}} |\partial_x u|^2 \xi_R dx \to -\int_{\mathbb{R}} |\partial_x u|^2 dx \text{ as } R \to \infty.$$

We get rid of the second term on the right hand side of (5.16) in a similar manner. In particular, observe that $u(\cdot,t) \in L^{q+1}(\mathbb{R})$ since $2 < q+1 < 3$:

$$\begin{aligned}
\left| -\int_{\mathbb{R}} u^q \partial_x u \xi_R dx \right| &= \left| \frac{1}{q+1} \int_{\mathbb{R}} \partial_x (u^{q+1}) \xi_R dx \right| \\
&= \left| \left[\frac{1}{q+1} u^{q+1} \xi_R \right]_{x=-\infty}^{\infty} - \frac{1}{q+1} \int_R u^{q+1} \partial_x \xi_R dx \right| \\
&\leq \frac{1}{q+1} \int_{\mathbb{R}} |u|^{q+1} |\partial_x \xi_R| dx \\
&\leq \frac{1}{q+1} \frac{1}{R} \|\partial_x \xi\|_{L^\infty} \int_{\mathbb{R}} |u|^{q+1} dx \\
&\to 0 \text{ as } R \to \infty.
\end{aligned}$$

Next, we move out the time derivative on the left hand side of (5.16) and take the limit $R \to \infty$ inside the integral with dominated convergence (Theorem A.8), to get

$$\int_{\mathbb{R}} \partial_t u u \xi_R dx = \frac{1}{2} \int_{\mathbb{R}} \partial_t (u^2) \xi_R dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 \xi_R dx \to \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 dx \text{ as } R \to \infty.$$

Returning to (5.16), this equation simplifies to

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 dx = -\int_{\mathbb{R}} |\partial_x u|^2 dx,$$

and by integrating this in time over $[\tau, T]$ with $0 < \tau < T$, we get

$$\begin{aligned}
\int_\tau^T \int_{\mathbb{R}} |\partial_x u(x,t)|^2 dx dt &= -\frac{1}{2} \int_\tau^T \frac{d}{dt} \int_{\mathbb{R}} (u(x,t))^2 dx dt \\
&= -\frac{1}{2} \int_{\mathbb{R}} (u(x,T))^2 dx + \frac{1}{2} \int_{\mathbb{R}} (u(x,\tau))^2 dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}} (u(x,\tau))^2 dx.
\end{aligned}$$

Finally, we obtain the wanted estimate for the right hand side by using Lemma 5.5 and that $u(\cdot,\tau)$ has mass M by the mass conservation result in Lemma 5.1:

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}} (u(x,\tau))^2 dx &\leq \frac{1}{2} \int_{\mathbb{R}} u(x,\tau) \|u(\cdot,\tau)\|_{L^\infty} dx \leq \frac{1}{2} \left(\frac{qM}{q-1} \right)^{\frac{1}{q}} \tau^{-\frac{1}{q}} \int_{\mathbb{R}} u(x,\tau) dx \\
&= \frac{1}{2} \left(\frac{q}{q-1} \right)^{\frac{1}{q}} M^{\frac{q+1}{q}} \tau^{-\frac{1}{q}} =: C \tau^{-\frac{1}{q}}.
\end{aligned}$$

□

Finally, we are ready to show the main result of this section, confirming our expectations from above and thus completing our studies of strongly non-linear asymptotic behaviour.

Theorem 5.9. *Let $0 \leq u_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)dx = M > 0$. Then, for $p \in [1, \infty)$ and $1 < q < 2$, the strong solution u to (5.7) satisfies*

$$t^{\frac{1}{q}(1-\frac{1}{p})} \|u(\cdot, t) - u_M(\cdot, t)\|_{L^p} \to 0 \text{ as } t \to \infty, \tag{5.17}$$

where u_M is the entropy solution to (5.8) given by (5.9).

Remark 5.10. *As we have mentioned, this strongly non-linear behaviour is vastly different from the weakly non-linear behaviour found in the previous section. Not only does the convergence rates differ, but by comparing the asymptotic solutions, we see that the entropy solution (5.9) is both asymmetrical and discontinuous. The asymptotic solution for the weakly non-linear behaviour, on the other hand, is given by the mass M times the heat kernel G, which follows a classical Gaussian profile.*

Remark 5.11. *This result may be generalised in many different ways. Similarly as we commented in Remark 5.3, we may consider higher dimensions as done in [15]. Furthermore, in Chapter 3 of [5] the assumption of non-negative solutions is removed, while Chapter 7 generalises the result to more general non-linearities $F(u)$ behaving like u^q near 0.*

Remark 5.12. *As commented in [1, p. 5], Theorem 5.9 still holds for a general constant a, not just our convenient choice $a = -1/q$, in the sense that we just replace u_M by the unspecified entropy solution of the general problem (5.5) with initial data $M\delta$.*

Remark 5.13. *Observe that we no longer have a result in the case when $p = \infty$. The reason for this is that we converge towards a discontinuous solution u_M, while convergence in L^∞ happens pointwise a.e.*

Proof. We follow a combination of the proofs presented in [5, pp. 50–54] and [6, pp. 7–11], where the approach is to prove the convergence by scaling arguments. The proof is split into steps to cover the different ideas of the proof. This proof technique is also described in [11, p. 10] and used in [16]. Even though we follow the general approach as described above, some of the arguments are considerably expanded where the original sources are brief or unspecific. This is especially the case in Step 3 where we are justifying the passage to the limit.

Step 1. Rescaling. Let $\lambda > 0$. We have already used the rescaling

$$u_\lambda(x, t) := \lambda u(\lambda x, \lambda^q t)$$

to determine that we expect the asymptotic behaviour of u to be given by u_M, but now we will use it to show rigorously that this is the case. The idea is based on the following:

$$\begin{aligned}
&\|u_\lambda(\cdot,1)-u_M(\cdot,1)\|_{L^1}\\
&=\int_{-\infty}^{0}|\lambda u(\lambda x,\lambda^q)-0|dx+\int_0^{r(1)}\left|\lambda u(\lambda x,\lambda^q)-x^{\frac{1}{q-1}}\right|dx+\int_{r(1)}^{\infty}|\lambda u(\lambda x,\lambda^q)-0|dx\\
&\overset{y:=\lambda x}{=}\frac{1}{\lambda}\int_{-\infty}^{0}|\lambda u(y,\lambda^q)|dy+\frac{1}{\lambda}\int_0^{\lambda r(1)}\left|\lambda u(y,\lambda^q)-\left(\frac{y}{\lambda}\right)^{\frac{1}{q-1}}\right|dy+\frac{1}{\lambda}\int_{\lambda r(1)}^{\infty}|\lambda u(y,\lambda^q)|dy\\
&=\frac{1}{\lambda}\int_{-\infty}^{0}|\lambda u(y,\lambda^q)|dy+\frac{1}{\lambda}\int_0^{\lambda r(1)}\left|\lambda u(y,\lambda^q)-\lambda u_M\left(\frac{y}{\lambda^q},1\right)\right|dy+\frac{1}{\lambda}\int_{\lambda r(1)}^{\infty}|\lambda u(y,\lambda^q)|dy\\
&\overset{t:=\lambda^q}{=}\int_{-\infty}^{0}|u(y,t)-0|dy+\int_0^{r(t)}|u(y,t)-u_M(y,t)|dy+\int_{r(t)}^{\infty}|u(y,t)-0|dy\\
&=\|u(\cdot,t)-u_M(\cdot,t)\|_{L^1}.
\end{aligned}$$

Thus, if we can show that $u_\lambda(\cdot,1)\to u_M(\cdot,1)$ in L^1 as $\lambda\to\infty$, this is equivalent to showing that $u(\cdot,t)\to u_M(\cdot,t)$ in L^1 as $t\to\infty$.

Furthermore, the result (5.17) is implied by the convergence of u to u_M in L^1 combined with the L^∞-estimates (5.10) and (5.14) for u_M and u:

$$\begin{aligned}
&t^{\frac{1}{q}(1-\frac{1}{p})}\|u(\cdot,t)-u_M(\cdot,t)\|_{L^p}\\
&=t^{\frac{1}{q}(1-\frac{1}{p})}\left(\int_{\mathbb{R}}|u(x,t)-u_M(x,t)|^p dx\right)^{\frac{1}{p}}\\
&\le t^{\frac{1}{q}(1-\frac{1}{p})}\left(\int_{\mathbb{R}}|u(x,t)-u_M(x,t)|\|u(\cdot,t)-u_M(\cdot,t)\|_{L^\infty}^{p-1}dx\right)^{\frac{1}{p}}\\
&\le t^{\frac{1}{q}(1-\frac{1}{p})}\left(\|u(\cdot,t)\|_{L^\infty}+\|u_M(\cdot,t)\|_{L^\infty}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}}|u(x,t)-u_M(x,t)|dx\right)^{\frac{1}{p}}\\
&\le t^{\frac{1}{q}(1-\frac{1}{p})}\left(C_1t^{-\frac{1}{q}}+C_2t^{-\frac{1}{q}}\right)^{\frac{p-1}{p}}\|u(\cdot,t)-u_M(\cdot,t)\|_{L^1}^{\frac{1}{p}}\\
&=t^{\frac{1}{q}(1-\frac{1}{p})}\left(C_1+C_2\right)^{\frac{p-1}{p}}t^{-\frac{1}{q}(1-\frac{1}{p})}\|u(\cdot,t)-u_M(\cdot,t)\|_{L^1}^{\frac{1}{p}}\\
&=\left(C_1+C_2\right)^{\frac{p-1}{p}}\|u(\cdot,t)-u_M(\cdot,t)\|_{L^1}^{\frac{1}{p}}.
\end{aligned}$$

Therefore, we are done if we can show that $u_\lambda(\cdot,1)\to u_M(\cdot,1)$ in L^1 as $\lambda\to\infty$.

Step 2. Uniform estimates.

In order to pass to the limit $\lambda\to\infty$ for u_λ, we first need some estimates on these functions. First, observe that

$$u(x,t)\ge 0 \implies u_\lambda(x,t)\ge 0\ \forall\ (x,t)\in\mathbb{R}\times(0,\infty). \tag{5.18}$$

Through direct computation, it can be seen that the estimates on u given in Lemma 5.4, Lemma 5.5 and Lemma 5.7 still hold for u_λ, so that we have

$$\partial_x \left[(u_\lambda(x,t))^{q-1}\right] \le \frac{1}{t} \ \forall\ (x,t) \in \mathbb{R} \times (0,\infty), \tag{5.19}$$

$$\|u_\lambda(\cdot,t)\|_{L^\infty} \le Ct^{-\frac{1}{q}} \ \forall\ t > 0, \tag{5.20}$$

$$\partial_x[(u_\lambda(x,t))^q] \le \frac{qu_\lambda(x,t)}{(q-1)t} \ \forall\ (x,t) \in \mathbb{R} \times (0,\infty). \tag{5.21}$$

Furthermore, we ensured mass conservation for u_λ in Section 5.1, giving

$$\int_{\mathbb{R}} u_\lambda(x,t)dx = M \ \forall\ t > 0. \tag{5.22}$$

However, the estimate given in Lemma 5.8 is changed in the sense that it behaves like $\lambda^{2-q} \to \infty$ as $\lambda \to \infty$:

$$\int_\tau^T \int_{\mathbb{R}} |\partial_x u_\lambda(x,t)|^2 dxdt \le \frac{1}{2}\lambda^{2-q} \int_{\mathbb{R}} (u_\lambda(x,\tau))^2 dx = O(\lambda^{2-q}) \ \forall\ 0 < \tau < T. \tag{5.23}$$

This can be seen by the following:

$$\begin{aligned}
\int_\tau^T \int_{\mathbb{R}} |\partial_x u_\lambda(x,t)|^2 dxdt &= \int_\tau^T \int_{\mathbb{R}} \left|\lambda^2 \partial_x u(\lambda x, \lambda^q t)\right|^2 dxdt \\
&\overset{\substack{z:=\lambda x\\ y:=\lambda^q t}}{=} \lambda^4 \int_{\lambda^q\tau}^{\lambda^q T} \int_{\mathbb{R}} |\partial_x u(z,y)|^2 \frac{1}{\lambda^{q+1}} dzdy \le \frac{1}{2}\lambda^{3-q} \int_{\mathbb{R}} (u(z,\lambda^q\tau))^2 dz \\
&\overset{x:=z/\lambda}{=} \frac{1}{2}\lambda^{3-q} \int_{\mathbb{R}} (u(\lambda x, \lambda^q \tau))^2 \lambda dx = \frac{1}{2}\lambda^{2-q} \int_{\mathbb{R}} (u_\lambda(x,\tau))^2 dx = O(\lambda^{2-q}),
\end{aligned}$$

where the final integral is bounded using (5.20) and (5.22). We will also need an estimate in $L^1(\mathbb{R})$ on u_λ^q, and this is obtained similarly:

$$\int_{\mathbb{R}} |(u_\lambda(x,t))^q| dx \le \int_{\mathbb{R}} |u_\lambda(x,t)| \|u_\lambda(\cdot,t)\|_{L^\infty}^{q-1} dx \le C^{q-1} t^{-\frac{q-1}{q}} M \ \forall\ t > 0. \tag{5.24}$$

Next, we introduce the functions

$$v_\lambda(x,t) := \int_{-\infty}^x u_\lambda(y,t)dy.$$

The reason for this is that they will play a role in taking the limit for λ. Observe that by (5.18) and (5.22), these functions are bounded:

$$0 \le v_\lambda(x,t) \le M \ \forall\ (x,t) \in \mathbb{R} \times (0,\infty). \tag{5.25}$$

By the fundamental theorem of calculus, we have

$$\partial_x v_\lambda = u_\lambda,$$

which combined with (5.20) gives us a bound on $\partial_x v_\lambda$ as well:

$$\|\partial_x v_\lambda(\cdot,t)\|_{L^\infty} = \|u_\lambda(\cdot,t)\|_{L^\infty} \le Ct^{-\frac{1}{q}} \ \forall\ t>0. \tag{5.26}$$

We finally wish to bound $\partial_t v_\lambda$. Recall that the rescaled functions u_λ satisfy the following equation:

$$\partial_t u_\lambda = \lambda^{q-2}\partial_x^2 u_\lambda - u_\lambda^{q-1}\partial_x u_\lambda \ \forall\ (x,t) \in \mathbb{R}\times(0,\infty). \tag{5.27}$$

By integrating this equation, we get

$$\int_{-\infty}^{x} \partial_t u_\lambda dy = \int_{-\infty}^{x} \lambda^{q-2}\partial_x^2 u_\lambda dy - \int_{-\infty}^{x} u_\lambda^{q-1}\partial_x u_\lambda dy.$$

Since $u \in C([0,\infty); L^1(\mathbb{R}))$, this implies that $u_\lambda \in C([0,\infty); L^1(\mathbb{R}))$ as well, and thus we can move the derivatives out of the integrals and apply the definition of v_λ to get

$$\partial_t v_\lambda = \lambda^{q-2}\partial_x^2 v_\lambda - \frac{1}{q}(u_\lambda)^q = \lambda^{q-2}\partial_x u_\lambda - \frac{1}{q}(u_\lambda)^q \ \forall\ (x,t)\in\mathbb{R}\times(0,\infty).$$

For the last term we used the fundamental theorem of calculus to get rid of the integral.

Moreover, since u_λ is non-negative by (5.18), we get

$$\partial_t v_\lambda \le \lambda^{q-2}\partial_x u_\lambda.$$

Now, we can apply this to (5.23) to get

$$\begin{aligned}
&\int_\tau^T \int_K |\partial_t v_\lambda(x,t)|^2 dxdt \le \int_\tau^T \int_K \left|\lambda^{q-2}\partial_x u_\lambda(x,t)\right|^2 dxdt \\
&\le \lambda^{2(q-2)} \int_\tau^T \int_{\mathbb{R}} |\partial_x u_\lambda(x,t)|^2 dxdt \le \lambda^{2(q-2)}\frac{1}{2}\lambda^{2-q}\int_{\mathbb{R}} (u_\lambda(x,\tau))^2 dx \\
&\le \frac{1}{2}\lambda^{q-2}\int_{\mathbb{R}} u_\lambda(x,\tau)\|u_\lambda(\cdot,\tau)\|_{L^\infty} dx \le \frac{1}{2}\lambda^{q-2} MC\tau^{-\frac{1}{q}} \le \frac{1}{2}MC\tau^{-\frac{1}{q}}.
\end{aligned} \tag{5.28}$$

We have used (5.20), (5.22) and the fact that $\lambda^{q-2} \le 1$ for $\lambda \ge 1$ since $1<q<2 \implies -1<q-2<0$. Thus, we have a uniform bound for $\partial_t v_\lambda$ in $L^2_{loc}((0,\infty); L^2(\mathbb{R}))$.

Step 3. Passage to the limit.

Let

$$\mathcal{S} := \{\tilde{v}_\lambda \mid \lambda > 0\},$$

where

$$\tilde{v}_\lambda := v_\lambda \cdot \mathbb{1}_{K_x\times K_t},$$

for some compact $K_x \times K_t \subseteq \mathbb{R} \times (0,\infty)$. Thus, $\mathcal{S}$ is a subset of $L^p(\mathbb{R}^2)$ for $p \in [1,\infty)$, which can be seen as follows:

$$\begin{aligned}\|\tilde{v}_\lambda\|_{L^p} &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |v_\lambda(x,t)\mathbb{1}_{K_x\times K_t}(x,t)|^p dxdt\right)^{\frac{1}{p}} = \left(\int_{K_t}\int_{K_x} |v_\lambda(x,t)|^p dxdt\right)^{\frac{1}{p}}\\ &\leq \left(\int_{K_t}\int_{K_x} M^p dxdt\right)^{\frac{1}{p}} = M(\mu(K_x\times K_t))^{\frac{1}{p}} < \infty.\end{aligned}$$

Here, μ denotes the Lebesgue measure, and we have used (5.25) to bound v_λ. We wish to make use of Kolmogorov's compactness theorem (Theorem A.18) to take the limit $\lambda \to \infty$ for $\tilde{v}_\lambda$. The theorem states that $\mathcal{S}$ is relatively compact if and only if three specific conditions are fulfilled, and we will now show this indeed is the case.

For the first condition, observe that the bound on $\tilde{v}_\lambda$ in $L^p(\mathbb{R}^2)$ above is independent of λ, thus we have

$$\sup_{\lambda>0} \|\tilde{v}_\lambda\|_{L^p} < \infty,$$

and condition (i) is fulfilled.

For condition (ii), we need to show that

$$\|\tilde{v}_\lambda(\cdot+h,\cdot+k) - \tilde{v}_\lambda(\cdot,\cdot)\|_{L^p} \leq \varepsilon(|(h,k)|),$$

for a modulus of continuity ε that is independent of λ. In order to control this translation, we will make use of the bounds on the first derivatives of v_λ established in the last step.

We start by expanding the expression with the triangle inequality:

$$\begin{aligned}&\|\tilde{v}_\lambda(\cdot+h,\cdot+k) - \tilde{v}_\lambda(\cdot,\cdot)\|_{L^p} = \|\tilde{v}_\lambda(\cdot+h,\cdot+k) - \tilde{v}_\lambda(\cdot,\cdot) \pm \tilde{v}_\lambda(\cdot+h,\cdot)\|_{L^p}\\ &\leq \|\tilde{v}_\lambda(\cdot+h,\cdot+k) - \tilde{v}_\lambda(\cdot+h,\cdot)\|_{L^p} + \|\tilde{v}_\lambda(\cdot+h,\cdot) - \tilde{v}_\lambda(\cdot,\cdot)\|_{L^p} =: A + B.\end{aligned}$$

Next, we write out $\tilde{v}_\lambda$ and expand part A once more:

$$\begin{aligned}A &= \big\|v_\lambda(\cdot+h,\cdot+k)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot+k) - v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot)\\ &\quad \pm v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot+k)\big\|_{L^p}\\ &\leq \|v_\lambda(\cdot+h,\cdot+k)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot+k) - v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot+k)\|_{L^p}\\ &\quad + \|v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot+k) - v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot)\|_{L^p} =: A_1 + A_2.\end{aligned}$$

For A_1, we use the generalised mean value theorem (3.3) introduced in Chapter 3 and the bound (5.25) for v_λ to get

$$\begin{aligned}
A_1 &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |v_\lambda(x+h,t+k)-v_\lambda(x+h,t)|^p|\mathbb{1}_{K_x\times K_t}(x+h,t+k)|^p dxdt\right)^{\frac{1}{p}}\\
&\le \left(\int_{K_t-k}\int_{K_x-h} |v_\lambda(x+h,t+k)-v_\lambda(x+h,t)|^p dxdt\right)^{\frac{1}{p}}\\
&= \left(\int_{K_t-k}\int_{K_x-h} |v_\lambda(x+h,t+k)-v_\lambda(x+h,t)|^2\right.\\
&\quad \left.\cdot |v_\lambda(x+h,t+k)-v_\lambda(x+h,t)|^{p-2} dxdt\right)^{\frac{1}{p}}\\
&\le \left(\int_{K_t-k}\int_{K_x-h} \left|\left(\int_0^1 \partial_t v_\lambda(x+h,t+\theta k)d\theta\right)\cdot k\right|^2 (2M)^{p-2} dxdt\right)^{\frac{1}{p}}.
\end{aligned}$$

The notation $K_t - k$ and $K_x - h$ expresses that the sets K_t and K_x are translated by k and h, respectively, corresponding to the indicator function. To continue, we wish to make use of the fact that $\partial_t v_\lambda$ is in $L^2_{loc}((0,\infty);L^2(\mathbb{R}))$ by (5.28). Observing that the function $x \to |x|^2$ for $x \in \mathbb{R}$ is convex, we use Jensen's inequality (Theorem A.5) to move the squared absolute value inside the inner integral, while Fubini's theorem (Theorem A.10) ensures we can interchange the integral order to get

$$\begin{aligned}
&\left(\int_{K_t-k}\int_{K_x-h} \left|\left(\int_0^1 \partial_t v_\lambda(x+h,t+\theta k)d\theta\right)\cdot k\right|^2 (2M)^{p-2} dxdt\right)^{\frac{1}{p}}\\
&\le (2M)^{\frac{p-2}{p}}|k|^{\frac{2}{p}}\left(\int_{K_t-k}\int_{K_x-h}\int_0^1 |\partial_t v_\lambda(x+h,t+\theta k)|^2 d\theta dxdt\right)^{\frac{1}{p}}\\
&= (2M)^{\frac{p-2}{p}}|k|^{\frac{2}{p}}\left(\int_0^1\int_{K_t-k}\int_{K_x-h} |\partial_t v_\lambda(x+h,t+\theta k)|^2 dxdtd\theta\right)^{\frac{1}{p}}\\
&\to 0 \text{ as } |(h,k)| \to 0.
\end{aligned}$$

For A_2, we once again bound v_λ using (5.25), while continuity of translation for L^p-functions (Theorem A.13) allows us to take the limit for $\mathbb{1}_{K_x\times K_t} \in L^p(\mathbb{R}^2)$:

$$\begin{aligned}
A_2 &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |v_\lambda(x+h,t)|^p|\mathbb{1}_{K_x\times K_t}(x+h,t+k)-\mathbb{1}_{K_x\times K_t}(x+h,t)|^p dxdt\right)^{\frac{1}{p}}\\
&\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} M^p|\mathbb{1}_{K_x\times K_t}(x+h,t+k)-\mathbb{1}_{K_x\times K_t}(x+h,t)|^p dxdt\right)^{\frac{1}{p}}\\
&= M\|\mathbb{1}_{K_x\times K_t}(\cdot,\cdot+k)-\mathbb{1}_{K_x\times K_t}(\cdot,\cdot)\|_{L^p(\mathbb{R}^2)}\\
&\to 0 \text{ as } |(h,k)| \to 0.
\end{aligned}$$

Next, we turn to part B which we handle similarly as part A, first expanding and using the triangle inequality:

$$\begin{aligned}B &= \|v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot) - v_\lambda(\cdot,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot,\cdot) \pm v_\lambda(\cdot,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot)\|_{L^p}\\ &\leq \|v_\lambda(\cdot+h,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot) - v_\lambda(\cdot,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot)\|_{L^p}\\ &\quad + \|v_\lambda(\cdot,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot) - v_\lambda(\cdot,\cdot)\mathbb{1}_{K_x\times K_t}(\cdot,\cdot)\|_{L^p} =: B_1 + B_2\end{aligned}$$

For B_1 we proceed as with A_1, but this time we make use of the bound on $\partial_x v_\lambda$ in (5.26), since we are dealing with a translation in space instead of in time. This yields

$$\begin{aligned}B_1 &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |v_\lambda(x+h,t) - v_\lambda(x,t)|^p |\mathbb{1}_{K_x\times K_t}(x+h,t)|^p dxdt\right)^{\frac{1}{p}}\\ &\leq \left(\int_{K_t}\int_{K_x-h} |v_\lambda(x+h,t) - v_\lambda(x,t)|^p dxdt\right)^{\frac{1}{p}}\\ &= \left(\int_{K_t}\int_{K_x-h} \left|\left(\int_0^1 \partial_x v_\lambda(x+\theta h,t)d\theta\right)\cdot h\right|^p dxdt\right)^{\frac{1}{p}}\\ &\leq \left(\int_{K_t}\int_{K_x-h} \left|\int_0^1 Ct^{-\frac{1}{q}}d\theta\right|^p |h|^p dxdt\right)^{\frac{1}{p}}\\ &= \left(\int_{K_t}\int_{K_x-h} C^p t^{-\frac{p}{q}}|h|^p dxdt\right)^{\frac{1}{p}}\\ &= C|h|(\mu(K_x-h))^{\frac{1}{p}}\left[t^{\frac{q-p}{pq}}\right]_{\partial K_t}\\ &\to 0 \text{ as } |(h,k)|\to 0.\end{aligned}$$

Turning to B_2, we treat this as as we did with A_2, to get:

$$\begin{aligned}B_2 &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |v_\lambda(x,t)|^p |\mathbb{1}_{K_x\times K_t}(x+h,t) - \mathbb{1}_{K_x\times K_t}(x,t)|^p dxdt\right)^{\frac{1}{p}}\\ &\leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} M^p |\mathbb{1}_{K_x\times K_t}(x+h,t) - \mathbb{1}_{K_x\times K_t}(x,t)|^p dxdt\right)^{\frac{1}{p}}\\ &= M\|\mathbb{1}_{K_x\times K_t}(\cdot+h,\cdot) - \mathbb{1}_{K_x\times K_t}(\cdot,\cdot)\|_{L^p(\mathbb{R}^2)}\\ &\to 0 \text{ as } |(h,k)|\to 0.\end{aligned}$$

Thus, we have bounded all four parts of the expression independently of λ in such a way that they all approach 0 as $|(h,k)|\to 0$, meaning condition (ii) is fulfilled.

Finally, condition (iii) requires that

$$\lim_{\alpha\to\infty}\int_{\{(x,t)\in\mathbb{R}^2 | |(x,t)|\geq\alpha\}} |\tilde{v}_\lambda(x,t)|^p dxdt = 0 \text{ uniformly for } \tilde{v}_\lambda\in\mathcal{S}.$$

This condition is immediately fulfilled by construction of $\tilde{v}_\lambda = v_\lambda\cdot\mathbb{1}_{K_x\times K_t}$.

Thus, we have that $\mathcal{S}$ is relatively compact in $L^p(\mathbb{R}^2)$, meaning that the closure of $\mathcal{S}$, $\overline{\mathcal{S}}$, is compact, i.e. every infinite sequence in $\overline{\mathcal{S}}$ contains a subsequence which is strongly convergent in $L^p(\mathbb{R}^2)$. We may therefore take a subsequence λ_j of λ which goes to infinity, and conclude that $\tilde{v}_{\lambda_j} \to \tilde{v}$ in $L^p(\mathbb{R}^2)$, for some $\tilde{v}$.

From this we can recover the original function v_{λ_j} by a covering and a diagonal argument, so that v_{λ_j} converges to some $\bar{v}$ in $L^p_{loc}(\mathbb{R} \times (0,\infty))$ for $p \in [1,\infty)$ and thus also a.e. in $\mathbb{R} \times (0,\infty)$.

Next, we turn to the convergence of $(u_\lambda(\cdot,t_0))^q$, for a fixed $t_0 > 0$. This time, we let

$$\mathcal{S}' := \{\tilde{u}_\lambda \mid \lambda > 0\},$$

where

$$\tilde{u}_\lambda(\cdot) := (u_\lambda(\cdot,t_0))^q \cdot \mathbb{1}_{K_x}(\cdot),$$

for some compact $K_x \subseteq \mathbb{R}$. Proceeding analogously as we did with $\tilde{v}_\lambda$, we get that $\mathcal{S}'$ is a subset of $L^p(\mathbb{R})$ for $p \in [1,\infty)$. We can now repeat the steps we took for $\tilde{v}_\lambda$, with some simplifications (due to the lack of a time dimension) and modifications, to show that the three conditions in Kolmogorov's compactness theorem (Theorem A.18) are also fulfilled in this framework. For instance, we make use of the estimates (5.24) and (5.21) to bound u_λ^q and $\partial_x u_\lambda^q$.

Thus, we get that $\mathcal{S}'$ is relatively compact in $L^p(\mathbb{R})$, which gives that $(u_{\lambda_{j'}}(\cdot,t_0))^q$ converges along some subsequence $\lambda_{j'}$ to some $\bar{w}(\cdot,t_0)$ in $L^p_{loc}(\mathbb{R})$ for $p \in [1,\infty)$ and a.e. in $\mathbb{R}$.

To recover the convergence of u_λ from this, we will make use of Theorem A.19 to write

$$\begin{aligned} \left|u_\lambda(x,t_0) - (\bar{w}(x,t_0))^{\frac{1}{q}}\right| &= \left|((u_\lambda(x,t_0))^q)^{\frac{1}{q}} - (\bar{w}(x,t_0))^{\frac{1}{q}}\right| \\ &\leq |(u_\lambda(x,t_0))^q - \bar{w}(x,t_0)|^{\frac{1}{q}} \text{ for a.e. } x \in \mathbb{R}. \end{aligned} \tag{5.29}$$

To justify the use of this theorem, we need to ensure that u_λ^q and $\bar{w}$ are finite. For u_λ^q, observe that (5.20) yields the following bound:

$$0 \leq (u_\lambda(x,t_0))^q \leq \frac{C^q}{t_0} \text{ for a.e. } x \in \mathbb{R}. \tag{5.30}$$

For the limit $\bar{w}$, we use this bound together with the convergence of $u^q_{\lambda_{j'}}$ a.e. to say that for $\lambda_{j'}$ large enough,

$$|\bar{w}(x,t_0)| \leq \left|\bar{w}(x,t_0) - (u_{\lambda_{j'}}(x,t_0))^q\right| + \left|(u_{\lambda_{j'}}(x,t_0))^q\right| \leq 1 + \frac{C^q}{t_0} \text{ for a.e. } x \in \mathbb{R}.$$

Thus, by (5.29) we get that $u_{\lambda_{j'}}(\cdot,t_0)$ converges along the subsequence $\lambda_{j'}$ to $(\bar{w}(\cdot,t_0))^{\frac{1}{q}}$ a.e. in $\mathbb{R}$.

For the convergence locally in L^p, we similarly get

$$\begin{aligned}\left\|u_\lambda(\cdot,t_0)-(\bar{w}(\cdot,t_0))^{\frac{1}{q}}\right\|_{L^p_{loc}} &= \left(\int_K \left|((u_\lambda(x,t_0))^q)^{\frac{1}{q}}-(\bar{w}(x,t_0))^{\frac{1}{q}}\right|^p dx\right)^{\frac{1}{p}}\\ &\le \left(\int_K \left||(u_\lambda(x,t_0))^q-\bar{w}(x,t_0)|^{\frac{1}{q}}\right|^p dx\right)^{\frac{1}{p}}\\ &= \left(\int_K |(u_\lambda(x,t_0))^q-\bar{w}(x,t_0)|^{\frac{p}{q}}dx\right)^{\frac{1}{p}\cdot\frac{q}{q}}\\ &= \|(u_\lambda(\cdot,t_0))^q-\bar{w}(\cdot,t_0)\|^{\frac{1}{q}}_{L^{\frac{p}{q}}_{loc}},\end{aligned}$$

where $K\subseteq\mathbb{R}$ is some compact subset. In order to use the convergence of $(u_{\lambda_{j'}}(\cdot,t_0))^q$ to $\bar{w}(\cdot,t_0)$ in $L^p_{loc}(\mathbb{R})$ for $p\in[1,\infty)$ here, we need to ensure that $p/q\ge 1$. With $1<q<2$, we therefore require $p\in[2,\infty)$, meaning we must show the result for $p\in[1,2)$ in some other way.

To cover $p=1$, we apply Hölder's inequality (Theorem A.3) and the newly established convergence in $L^2_{loc}(\mathbb{R})$:

$$\begin{aligned}\left\|u_\lambda(\cdot,t_0)-(\bar{w}(\cdot,t_0))^{\frac{1}{q}}\right\|_{L^1_{loc}} &= \int_K \left|\left(u_\lambda(x,t_0)-(\bar{w}(x,t_0))^{\frac{1}{q}}\right)\cdot 1\right|dx\\ &\le \left(\int_K \left|u_\lambda(x,t_0)-(\bar{w}(x,t_0))^{\frac{1}{q}}\right|^2 dx\right)^{\frac{1}{2}}\left(\int_K |1|^2 dx\right)^{\frac{1}{2}}\\ &= (\mu(K))^{\frac{1}{2}}\left\|u_\lambda(\cdot,t_0)-(\bar{w}(\cdot,t_0))^{\frac{1}{q}}\right\|_{L^2_{loc}}.\end{aligned}$$

For $p\in(1,2)$, we interpolate between the convergence in L^1_{loc} and L^2_{loc} with (Theorem A.4):

$$\left\|u_\lambda(\cdot,t_0)-(\bar{w}(\cdot,t_0))^{\frac{1}{q}}\right\|_{L^p_{loc}} \le \left\|u_\lambda(\cdot,t_0)-(\bar{w}(\cdot,t_0))^{\frac{1}{q}}\right\|^{\frac{2-p}{p}}_{L^1_{loc}}\left\|u_\lambda(\cdot,t_0)-(\bar{w}(\cdot,t_0))^{\frac{1}{q}}\right\|^{\frac{2(p-1)}{p}}_{L^2_{loc}}.$$

Thus, we have shown that $u_{\lambda_{j'}}(\cdot,t_0)$ converges to $\bar{u}(\cdot,t_0):=(\bar{w}(\cdot,t_0))^{\frac{1}{q}}$ in $L^p_{loc}(\mathbb{R})$ for $p\in[1,\infty)$.

Next, recall the relation between u_λ and v_λ:

$$v_\lambda(x,t)=\int_{-\infty}^x u_\lambda(y,t)dy.$$

By taking the limit in this relation, we conclude that

$$\partial_x\bar{v}(x,t_0)=\bar{u}(x,t_0) \text{ for a.e. } x\in\mathbb{R}.$$

Thus, by this relation, we have now identified the limit of $u^q_{\lambda_{j'}}$ independently of the subsequence, meaning the whole sequences u_{λ_j} and $u^q_{\lambda_j}$ converge to $\bar{u}$ and $\bar{w}$, respectively, for the fixed time t_0. Moreover, the bounds given in (5.20) and (5.30) still hold for the limits $\bar{u}$ and $\bar{w}$, respectively. For $\bar{u}$, this can be seen from

$$\begin{aligned}|\bar{u}(x,t)| &\leq \left|\bar{u}(x,t) - u_{\lambda_j}(x,t)\right| + \left|u_{\lambda_j}(x,t)\right| \\ &\leq \left|\bar{u}(x,t) - u_{\lambda_j}(x,t)\right| + Ct^{-\frac{1}{q}} \\ &\to Ct^{-\frac{1}{q}},\end{aligned}$$

along the sequence λ_j, for $t > 0$ and a.e. $x \in \mathbb{R}$. $\bar{w}$ can be treated similarly.

What remains is to take the very weak limit in the equation (5.27) for u_λ. We multiply this equation by a test function $\varphi \in C_c^\infty(\mathbb{R})$ and integrate over $\mathbb{R} \times (\tau, t)$, for $0 < \tau < t$ to get

$$\begin{aligned}\int_\tau^t \int_{\mathbb{R}} \partial_t u_\lambda(x,s)\varphi(x)dxds = \lambda^{q-2} &\int_\tau^t \int_{\mathbb{R}} \partial_x^2 u_\lambda(x,s)\varphi(x)dxds \\ - &\int_\tau^t \int_{\mathbb{R}} (u_\lambda(x,s))^{q-1}\partial_x u_\lambda(x,s)\varphi(x)dxds.\end{aligned}$$

Through integration by parts and by moving the time derivative out of the inner integral on the left hand side, this turns into

$$\begin{aligned}\int_{\mathbb{R}} u_\lambda(x,t)\varphi(x)dx - \int_{\mathbb{R}} u_\lambda(x,\tau)\varphi(x)dx = \lambda^{q-2} &\int_\tau^t \int_{\mathbb{R}} u_\lambda(x,s)\partial_x^2\varphi(x)dxds \\ + \frac{1}{q} &\int_\tau^t \int_{\mathbb{R}} (u_\lambda(x,s))^q \partial_x\varphi(x)dxds.\end{aligned} \tag{5.31}$$

We claim that by taking the limit through the sequence $\lambda_j \to \infty$, we end up with

$$\int_{\mathbb{R}} \bar{u}(x,t)\varphi(x)dx - \int_{\mathbb{R}} \bar{u}(x,\tau)\varphi(x)dx = \frac{1}{q}\int_\tau^t \int_{\mathbb{R}} (\bar{u}(x,s))^q \partial_x\varphi(x)dxds,$$

which states that $\bar{u}$ fulfills the convective equation (5.8) in the sense of distributions for time greater than 0. To show this claim, we consider each term separately. Let $s = \tau, t$, then the terms on the left hand side converge by the Dominated convergence theorem (Theorem A.8):

$$\left|\int_{\mathbb{R}} u_{\lambda_j}(x,s)\varphi(x)dx - \int_{\mathbb{R}} \bar{u}(x,s)\varphi(x)dx\right| \leq \int_{\mathbb{R}} \left|u_{\lambda_j}(x,s) - \bar{u}(x,s)\right||\varphi(x)|dx \to 0, \tag{5.32}$$

as $\lambda_j \to \infty$.

The diffusion term disappears as expected, since $\lambda_j^{q-2} \to 0$ as $\lambda_j \to \infty$, and the integral is bounded similarly as we have done several times by now:

$$\begin{aligned}&\left|\lambda_j^{q-2}\int_\tau^t \int_{\mathbb{R}} u_{\lambda_j}(x,s)\partial_x^2\varphi(x)dxds\right| \leq \lambda_j^{q-2}\left\|\partial_x^2\varphi\right\|_{L^\infty}\int_\tau^t \int_{\mathbb{R}} u_{\lambda_j}(x,s)dxds \\ &\leq \lambda_j^{q-2}\left\|\partial_x^2\varphi\right\|_{L^\infty} M(t-\tau) \to 0 \text{ as } \lambda_j \to \infty.\end{aligned}$$

Finally, for the last term, we apply the Dominated convergence theorem (Theorem A.8) together with the established convergence of $u_{\lambda_j}^q$ to get

$$\begin{aligned}
&\left|\frac{1}{q}\int_\tau^t\int_{\mathbb{R}}(u_{\lambda_j}(x,s))^q\partial_x\varphi(x)dxds - \frac{1}{q}\int_\tau^t\int_{\mathbb{R}}(\bar{u}(x,s))^q\partial_x\varphi(x)dxds\right| \\
&\leq \frac{1}{q}\int_\tau^t\int_{\mathbb{R}}\left|(u_{\lambda_j}(x,s))^q - (\bar{u}(x,s))^q\right||\partial_x\varphi(x)|dxds \\
&\to 0 \text{ as } \lambda_j \to \infty.
\end{aligned}$$

Step 4. Initial condition.

Having shown that $\bar{u}$ fulfills the convective equation (5.8) in the limit for time greater than zero, what remains is to check the initial condition. We wish to show that $\operatorname{ess\,lim}_{t\to 0^+}\bar{u}(x,t) = M\delta(x)$ in the weak sense of finite measures on $\mathbb{R}$, i.e. for all $\varepsilon > 0$,

$$\left|\int_{\mathbb{R}}\bar{u}(x,t)\varphi(x)dx - M\varphi(0)\right| \leq \varepsilon, \tag{5.33}$$

as long as $0 < t < \tau$ for some $\tau > 0$. This must hold for all $\varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

To show this, we start by assuming $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$. Expanding the expression and using the triangle inequality yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\bar{u}(x,t)\varphi(x)dx - M\varphi(0)\right| \\
&= \left|\int_{\mathbb{R}}\bar{u}(x,t)\varphi(x)dx - M\varphi(0) \pm \int_{\mathbb{R}}u_\lambda(x,t)\varphi(x)dx \pm \int_{\mathbb{R}}u_\lambda(x,0)\varphi(x)dx\right| \\
&\leq \left|\int_{\mathbb{R}}\bar{u}(x,t)\varphi(x)dx - \int_{\mathbb{R}}u_\lambda(x,t)\varphi(x)dx\right| + \left|\int_{\mathbb{R}}u_\lambda(x,t)\varphi(x)dx - \int_{\mathbb{R}}u_\lambda(x,0)\varphi(x)dx\right| \\
&+ \left|\int_{\mathbb{R}}u_\lambda(x,0)\varphi(x)dx - M\varphi(0)\right|.
\end{aligned}$$

Now, we showed in (5.32) that the first term converges to zero:

$$\left|\int_{\mathbb{R}}\bar{u}(x,t)\varphi(x)dx - \int_{\mathbb{R}}u_{\lambda_j}(x,t)\varphi(x)dx\right| \to 0 \text{ as } \lambda_j \to \infty.$$

For the third term, recall from (5.3) in Section 5.1, that for all $\varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\left|\int_{\mathbb{R}}u_\lambda(x,0)\varphi(x)dx - M\varphi(0)\right| \to 0 \text{ as } \lambda \to \infty.$$

For the second term, we use (5.31) with $\tau = 0$ to bound it when $0 < t < t_0$ and

$\lambda > \lambda_0$ for a sufficiently small $t_0 > 0$ and sufficiently large $\lambda > 0$:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} u_\lambda(x,t)\varphi(x)dx - \int_{\mathbb{R}} u_\lambda(x,0)\varphi(x)dx\right| \\
&= \left|\lambda^{q-2}\int_0^t\int_{\mathbb{R}} u_\lambda(x,s)\partial_x^2\varphi(x)dxds + \frac{1}{q}\int_0^t\int_{\mathbb{R}}(u_\lambda(x,s))^q\partial_x\varphi(x)dxds\right| \\
&\le \lambda^{q-2}\left\|\partial_x^2\varphi\right\|_{L^\infty}\int_0^t\int_{\mathbb{R}}|u_\lambda(x,s)|dxds + \frac{1}{q}\|\partial_x\varphi\|_{L^\infty}\int_0^t\int_{\mathbb{R}}|u_\lambda(x,s)|\|u_\lambda(\cdot,s)\|_{L^\infty}^{q-1}dxds \\
&\le \lambda^{q-2}\left\|\partial_x^2\varphi\right\|_{L^\infty}Mt + \frac{1}{q}\|\partial_x\varphi\|_{L^\infty}M\int_0^t C^{q-1}s^{-\frac{q-1}{q}}ds \\
&= \lambda^{q-2}\left\|\partial_x^2\varphi\right\|_{L^\infty}Mt + \|\partial_x\varphi\|_{L^\infty}MC^{q-1}t^{\frac{1}{q}} \\
&\le \varepsilon.
\end{aligned}$$

We thus have control over all parts, meaning (5.33) holds with $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$. To extend this to $\varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, one may perform an argument analogous to the one by Ignat and Stan in [17, pp. 277–278], where they control the tails of u_λ and thereby the tails of $\bar{u}$.

Step 5. Identification of the limit.

Now that we have found a limit $\bar{u}$, we wish to identify this as the solution u_M of the convective equation (5.8). We showed in the last step that $\bar{u}$ solves the convective equation (5.8) in the sense of distributions, i.e. in a very weak sense, with initial data $M\delta$ in the weak sense of finite measures. Regarding the regularity of the solution, we have seen that $\bar{u}$ is bounded away from zero, giving $\bar{u} \in L^\infty(\mathbb{R} \times (\tau,\infty))$ for some $\tau > 0$. Furthermore, we claim that $\bar{u} \in L^\infty((0,\infty); L^1(\mathbb{R}))$. Indeed, for λ_j great enough, we have

$$\begin{aligned}
\int_{\mathbb{R}} |\bar{u}(x,t)|\mathbb{1}_{B(0,r)}(x)dx &\le \int_{\mathbb{R}} \left|\bar{u}(x,t) - u_{\lambda_j}(x,t)\right|\mathbb{1}_{B(0,r)}(x)dx + \int_{\mathbb{R}} \left|u_{\lambda_j}(x,t)\right|\mathbb{1}_{B(0,r)}(x)dx \\
&\le \int_{B(0,r)} \left|\bar{u}(x,t) - u_{\lambda_j}(x,t)\right|dx + \int_{\mathbb{R}} u_{\lambda_j}(x,t)dx \\
&\le 1 + M,
\end{aligned}$$

for some $r > 0$. Applying Fatou's lemma (Theorem A.7) in the limit for r yields

$$\int_{\mathbb{R}} |\bar{u}(x,t)|dx \le \liminf_{r\to\infty}\int_{\mathbb{R}} |u_\lambda(x,t)|\mathbb{1}_{B(0,r)}(x)dx \le 1 + M.$$

Since this bound does not depend on the time, we may take essential supremum in time and conclude that $\bar{u} \in L^\infty((0,\infty); L^1(\mathbb{R}))$.

In order to identify our solution $\bar{u}$ equal to the unique entropy solution u_M, we claim that it suffices to ensure that our solution $\bar{u}$ satisfies the entropy inequality we introduced in Lemma 5.4,

$$\partial_x[(\bar{u}(x,t))^q] \le \frac{C\bar{u}(x,t)}{t}, \tag{5.34}$$

in a sense of distributions. We will justify this claim in Chapter 6.

To show the inequality (5.34) still holds in the limit, we multiply the estimate given in (5.21) by a test function $0 \leq \varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$ and integrate over $\mathbb{R} \times (0,\infty))$ to get:

$$\int_0^\infty \int_{\mathbb{R}} \partial_x[(u_\lambda(x,t))^q]\varphi(x,t)dxdt \leq \int_0^\infty \int_{\mathbb{R}} \frac{Cu_\lambda(x,t)}{t}\varphi(x,t)dxdt,$$

which through integration by parts on the left hand side becomes

$$-\int_0^\infty \int_{\mathbb{R}} (u_\lambda(x,t))^q \partial_x\varphi(x,t)dxdt \leq \int_0^\infty \int_{\mathbb{R}} \frac{Cu_\lambda(x,t)}{t}\varphi(x,t)dxdt.$$

Finally, by taking the limit along the sequence λ_j for $u_{\lambda_j}^q$ and u_{λ_j} using dominated convergence (Theorem A.8), we get

$$-\int_0^\infty \int_{\mathbb{R}} (\bar{u}(x,t))^q \partial_x\varphi(x,t)dxdt \leq \int_0^\infty \int_{\mathbb{R}} \frac{C\bar{u}(x,t)}{t}\varphi(x,t)dxdt,$$

as wanted.

Since the limit $\bar{u}$ now has been uniquely identified as u_M, and therefore the same applies to the limit $\bar{v}$, we are able to say that the convergence holds for the whole family $\{u_\lambda\}$ and not just the sequence $\{u_{\lambda_j}\}$. Therefore, we conclude that $u_\lambda \to u_M$ as $\lambda \to \infty$.

Step 6. Convergence in L^1.

Finally, we are ready to show that $u_\lambda(\cdot,1) \to u_M(\cdot,1)$ in $L^1(\mathbb{R})$ as $\lambda \to \infty$, which is the last step of the proof. Observe that u_M as defined in (5.9), only has support on a bounded subset of $\mathbb{R}$, say $[-r,r]$ for some $r > 0$. We have shown that $u_\lambda(\cdot,1)$ converges to $u_M(\cdot,1)$ in $L^1_{loc}(\mathbb{R})$, and so for every $\varepsilon > 0$ there exists a λ_0 such that for $\lambda > \lambda_0$,

$$\int_{-r}^r |u_\lambda(x,1) - u_M(x,1)|dx \leq \varepsilon. \tag{5.35}$$

Next, we must gain control outside of $[-r,r]$. Since u_M is a fundamental solution of (5.8) with initial data $M\delta$, and mass conservation holds for this equation (this can be seen by removing the diffusion term in the proof of Lemma 5.1), we deduce that the mass of u_M must be equal to M:

$$\int_{\mathbb{R}} u_M(x,1)dx = \int_{-r}^r u_M(x,1)dx = M. \tag{5.36}$$

We use (5.35) to write

$$\begin{aligned}\int_{-r}^r u_M(x,1)dx &= \int_{-r}^r u_\lambda(x,1)dx + \int_{-r}^r (u_M(x,1) - u_\lambda(x,1))dx \\ &\leq \int_{-r}^r u_\lambda(x,1)dx + \int_{-r}^r |u_M(x,1) - u_\lambda(x,1)|dx \\ &\leq \int_{-r}^r u_\lambda(x,1)dx + \varepsilon,\end{aligned}$$

and together with the mass estimate in (5.36) this yields

$$\int_{-r}^{r} u_\lambda(x,1)dx \geq \int_{-r}^{r} u_M(x,1)dx - \varepsilon = M - \varepsilon.$$

Thus, we control the outer part as well:

$$\begin{aligned}\int_{\{|x|>r\}} |u_\lambda(x,1) - u_M(x,1)|dx = \int_{\{|x|>r\}} u_\lambda(x,1)dx = M - \int_{-r}^{r} u_\lambda(x,1)dx \\ \leq M - (M-\varepsilon) = \varepsilon.\end{aligned} \tag{5.37}$$

At last, combining (5.35) and (5.37), we get

$$\begin{aligned}&\int_{\mathbb{R}} |u_\lambda(x,1) - u_M(x,1)|dx \\ &= \int_{-r}^{r} |u_\lambda(x,1) - u_M(x,1)|dx + \int_{\{|x|>r\}} |u_\lambda(x,1) - u_M(x,1)|dx \\ &\leq 2\varepsilon,\end{aligned}$$

and so we are done. □

Chapter 6

Entropy solutions for a convection equation

We saw in the end of the last chapter that the strongly non-linear asymptotic behaviour of the convection-diffusion equation in (5.1), was given by a solution $\bar{u}$ of (5.8), a strictly convective equation with $M\delta$ as initial data, where M was the mass of the solution. Furthermore, we found a unique so-called entropy solution u_M to this problem given by (5.9), and claimed that the two actually were equal since $\bar{u}$ satisfied the entropy inequality (5.34). In this chapter we will see why this was the case, by studying entropy solutions and their uniqueness, for the following more general convection equation

$$\partial_t u + \partial_x (F(u)) = 0, \tag{6.1}$$

where $(x,t) \in \mathbb{R} \times (0, \infty)$, while $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with $F(0) = 0$.

We will mainly follow the work by Liu and Pierre [7] on this matter, as well as the fundamental work by Kružkov [8], which Liu and Pierre build their work upon. These articles are working for time up to some $T > 0$, but we will extend this to $T = \infty$ similarly as we did in Chapter 3.

The content in this chapter is not treated as rigorously as the previous chapters, we will skip many proofs and rather focus on illustrating and discussing the main ideas behind them.

6.1 Defining entropy solutions

As stated in [5, p. 44], solutions to (6.1) are generally not unique, and so it is necessary to define a special class of solutions called entropy solutions. The idea is that we want to define a class of solutions to (6.1) for which L^1-contraction holds. The reason for this, is that under the assumption of sufficient regularity, L^1-contraction implies uniqueness for solutions with the same initial data, as we saw in the proof of Theorem 4.6 in Chapter 4. This leads us to a definition very similar to the result of Lemma 4.3, a result which implied L^1-contraction for the convection-diffusion equation in Chapter 4.

We present a definition of entropy solutions based on the one given by Liu and Pierre [7, p. 422], that is, in the sense as introduced by Kružkov [8, p. 220]:

Definition 6.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous with $F(0) = 0$, and μ be a non-negative finite Radon measure on $\mathbb{R}$. We say that u is an entropy solution to (6.1) with initial value μ if $u \in L^\infty((\tau,\infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (\tau,\infty))$ for all $\tau > 0$, and if:*

(i) For all $k \in \mathbb{R}$ and $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$ with $\varphi \geq 0$,

$$\int_0^\infty \int_{\mathbb{R}} (|u-k|\partial_t\varphi + \operatorname{sign}(u-k)(F(u)-F(k))\partial_x\varphi)\, dxdt \geq 0. \tag{6.2}$$

(ii) For all $\varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}} \varphi(x)u(x,t)dx = \int_{\mathbb{R}} \varphi(x)d\mu(x).$$

Note that the sign in front of the convection term in point (i) is different when comparing to the Kato inequality in Lemma 4.3. This difference comes from the fact that the convection term is on the right hand side of the convection-diffusion equation (4.1), which the Kato inequality is based on. The inequality above, on the other hand, comes from (6.1), where the convection term is on the left hand side.

The initial data is defined in this broad sense of non-negative finite Radon measures (see e.g. [18, p. 63] for a definition) to eventually allow us to choose Dirac's delta function δ as initial data. We will, however, first consider more regular initial data in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ before we get to this case.

Entropy solutions of (6.1) are contained in the class of very weak solutions, i.e. solutions taken in the distributional sense. To see this, we choose $k \leq -\|u\|_{L^\infty}$ in the definition above, which yields $u - k \geq u + \|u\|_{L^\infty} \geq 0$, and thus

$$\begin{aligned}
0 &\leq \int_0^\infty \int_{\mathbb{R}} (|u-k|\partial_t\varphi + \operatorname{sign}(u-k)(F(u)-F(k))\partial_x\varphi)\, dxdt \\
&= \int_0^\infty \int_{\mathbb{R}} ((u-k)\partial_t\varphi + (F(u)-F(k))\partial_x\varphi)\, dxdt \\
&= \int_0^\infty \int_{\mathbb{R}} (u\partial_t\varphi + F(u)\partial_x\varphi)\, dxdt - \int_0^\infty \int_{\mathbb{R}} (k\partial_t\varphi + F(k)\partial_x\varphi)\, dxdt \\
&= \int_0^\infty \int_{\mathbb{R}} (u\partial_t\varphi + F(u)\partial_x\varphi)\, dxdt.
\end{aligned}$$

We have used that the second integral on the third line is zero, since k and $F(k)$ are constants and φ has compact support in both time and space, thus integrating to zero on the boundaries of the integrals.

Similarly, choosing $k \geq \|u\|_{L^\infty}$ yields $u - k \leq u - \|u\|_{L^\infty} \leq 0$, and thus

$$\begin{aligned} 0 &\leq \int_0^\infty \int_{\mathbb{R}} (|u-k|\partial_t\varphi + \operatorname{sign}(u-k)(F(u)-F(k))\partial_x\varphi)\,dxdt \\ &= \int_0^\infty \int_{\mathbb{R}} (-(u-k)\partial_t\varphi - (F(u)-F(k))\partial_x\varphi)\,dxdt \\ &= -\int_0^\infty \int_{\mathbb{R}} (u\partial_t\varphi + F(u)\partial_x\varphi)\,dxdt + \int_0^\infty \int_{\mathbb{R}} (k\partial_t\varphi + F(k)\partial_x\varphi)\,dxdt \\ &= -\int_0^\infty \int_{\mathbb{R}} (u\partial_t\varphi + F(u)\partial_x\varphi)\,dxdt. \end{aligned}$$

Combining these two results by choosing $k = \|u\|_{L^\infty}$, we get

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t\varphi + F(u)\partial_x\varphi)\,dxdt = 0 \;\forall\; \varphi \in C_c^\infty(\mathbb{R} \times (0,\infty)), \varphi \geq 0. \tag{6.3}$$

To remove the requirement of non-negative test functions, we define $\varphi := \varphi^+ - \varphi^-$, where $\varphi^\pm \in C_c^\infty(\mathbb{R} \times (0,\infty))$ such that $\varphi^\pm \geq 0$. Thus, $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$, and since $\varphi^\pm$ each satisfy (6.3), we combine the two and get

$$\int_0^\infty \int_{\mathbb{R}} \big(u\partial_t(\varphi^+ - \varphi^-) + F(u)\partial_x(\varphi^+ - \varphi^-)\big)\,dxdt = 0,$$

which is the same as

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t\varphi + F(u)\partial_x\varphi)\,dxdt = 0 \;\forall\; \varphi \in C_c^\infty(\mathbb{R} \times (0,\infty)).$$

As we mentioned above, we have already found an entropy solution u_M to (6.1) in the case when $F(u) = \frac{1}{q}u^q$ with $1 < q < 2$, given by (5.9). The solution $\bar{u}$ found in the proof of Theorem 5.9 fulfilled this equation in the very weak sense, with initial value $M\delta$ taken in the sense of finite measures. In addition, we saw that $\bar{u}$ fulfills the entropy inequality (5.34), and this is, as discussed in [5, p. 54], an equivalent way of stating that $\bar{u}$ therefore is an entropy solution as well. We therefore have two entropy solutions of the same problem, and will see in the next section that they necessarily must be equal by uniqueness.

We will not pursue the question of existence any further in the general case, but rather refer to Chapter 3 in Kružkov [8] and Chapter 2 in Liu and Pierre [7].

6.2 Uniqueness of entropy solutions

We are now ready to look at the question of uniqueness. The idea is, as mentioned, to obtain uniqueness through L^1-contraction. Now, we will see that we can obtain this from the inequality (6.2) in the definition of entropy solutions. Indeed, Kružkov showed that we may move from this inequality to the next result from which we are able to compare two solutions:

Lemma 6.2. *Let u, v be entropy solutions on $\mathbb{R} \times (0, \infty)$ to (6.1). Then, for all $\varphi \in C_c^\infty(\mathbb{R} \times (0, \infty))$ with $\varphi \geq 0$, we have*

$$\int_0^\infty \int_{\mathbb{R}} \left[|u - v|\partial_t \varphi + \operatorname{sign}(u - v)(F(u) - F(v))\partial_x \varphi\right] dxdt \geq 0.$$

Proof. We refer to Kružkov [8, pp. 223–224] for the full proof, but the main idea is to set $k = v(y, \tau)$ for some fixed point (y, τ) in the inequality (6.2) for $u(x, t)$, before integrating over y and τ. Similarly, in the inequality (6.2) for $v(y, \tau)$, we set $k = u(x, t)$ for some fixed point (x, t) and integrate over x and t. Then, by combining the two results and through a particular choice of test function φ, the result is obtained. □

Let us first assume initial data $0 \leq u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By Kružkov [8] there exists entropy solutions to (6.1) fulfilling $u, v \in C([0, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (0, \infty))$. Now, analogously to what we did in Lemma 4.4 leading to L^1-contraction for the convection-diffusion equation, we may choose a particular test function $\varphi := \theta_\varepsilon \xi_R$ in Lemma 6.2 and get L^1-contraction here as well:

$$\int_{\mathbb{R}} |u(x, t) - v(x, t)| dx \leq \int_{\mathbb{R}} |u_0(x) - v_0(x)| dx \ \forall\ t \geq 0.$$

Thus, uniqueness follows in this case when $u_0 = v_0$.

Next, we wish to consider the case of a non-negative finite Radon measure μ as initial data, such as δ. It is often the case that when we relax our assumptions on the initial data to allow for solutions in a broader sense, showing uniqueness becomes more difficult, and this is also the case here. Liu and Pierre devote Chapter 1 in [7] to show the next result, which guarantees uniqueness for such solutions.

Theorem 6.3. *Assume $F([0, \infty)) \subset [0, \infty)$. Then there exists at most one entropy solution u to (6.1) with a non-negative finite measure μ on $\mathbb{R}$ as initial data.*

For a proof of this result, see [7, pp. 427–429]. This result finally allows us to conclude that entropy solutions to (6.1) are unique. It also yields the claim from Chapter 5, meaning that our solutions u_M and $\bar{u}$ coincide.

We will end this chapter with some discussion around why the proof works despite only assuming initial data as a non-negative finite Radon measure. The reason for this is that the solutions achieve the regularity $L^\infty((\tau, \infty); L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (\tau, \infty))$ for all $\tau > 0$, meaning we have a sufficient regularity away from zero. The property of bounded solutions away from zero is not an obvious result, and we will see an example of where it originates from in the familiar case of $F(u) = \frac{1}{q}u^q$ with $1 < q < 2$ in (6.1), i.e.

$$\partial_t u + \frac{1}{q}\partial_x (u^q) = 0. \tag{6.4}$$

For strong solutions u of the convection-diffusion equation (5.7) with initial data $u_0 \in L^1(\mathbb{R})$, we found the following L^1–L^∞-smoothing estimate in Lemma 5.5:

$$\|u(\cdot, t)\|_{L^\infty} \leq Ct^{-\frac{1}{q}} \|u_0\|_{L^1}^{\frac{1}{q}} \ \forall\ t > 0. \tag{6.5}$$

This estimate followed from the fact that the convection term was dominating over the diffusion term in this equation when $1 < q < 2$, thus it originated from the convective part of the equation. For this reason, we expect our purely convective equation (6.4) to also inherit this estimate. Indeed, we saw in (5.10) that this estimate holds for the entropy solution u_M. We will not prove this result here, but it can be obtained by considering the convection-diffusion equation with a diffusion term of the form $\varepsilon\partial_x^2 u$ for some $\varepsilon > 0$, and then make an approximation argument to let $\varepsilon \to 0$.

Thus, by claiming that entropy solutions of (6.4) with initial data $u_0 \in L^1(\mathbb{R})$ satisfy the estimate (6.5), we wish to show this still holds when we generalise to initial data μ being a non-negative finite Radon measure by means of an approximation. We do this in our final result, under some assumptions on the convergence of the solutions and their initial data:

Proposition 6.4. *Let $\{\mu_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\mu_n \geq 0$ and $\mu_n \to \mu$ in the sense that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f d\mu_n = \int_{\mathbb{R}} f d\mu \ \forall \ f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

where μ is a non-negative finite Radon measure on $\mathbb{R}$.

Let further u_n be entropy solutions to (6.1) with corresponding initial values μ_n, and assume they fulfill estimate (6.5).

If, moreover, $u_n(\cdot,t)$ converges to $u(\cdot,t)$ in some appropriate sense in $\mathbb{R}$ for all $t > 0$, then:

(i) Estimate (6.5) carries over to u as well, i.e.

$$\|u(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{1}{q}} \|\mu\|^{\frac{1}{q}} \ \forall \ t > 0.$$

(ii) u is an entropy solution to (6.1) with μ as initial data.

Proof. To show (i), we use that for each $n \in \mathbb{N}$, we have

$$\|u_n(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{1}{q}} \|\mu_n\|_{L^1}^{\frac{1}{q}} \ \forall \ t > 0.$$

By the convergence of μ_n, we get

$$\|\mu_n\|_{L^1} = \|\mu\| \ \forall \ n \in \mathbb{N},$$

which means that $u_n(\cdot,t)$ is uniformly bounded in $L^\infty(\mathbb{R})$ by the estimate above. This implies that $u_n(\cdot,t) \overset{*}{\rightharpoonup} u(\cdot,t)$ in $L^\infty(\mathbb{R})$, by the Banach-Alaoglu theorem (Theorem A.15), which further implies lower semi-continuity, by Theorem A.16. Thus we may take the limit to get our result

$$\|u(\cdot,t)\|_{L^\infty} \leq \liminf_{n\to\infty} \|u_n(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{1}{q}} \|\mu\|^{\frac{1}{q}} \ \forall \ t > 0.$$

Finally, for (ii), this is a more standard result, provided the sense of convergence for u_n is appropriate to carry the inequality in Definition 6.1 over to u in the limit. □

Chapter 7

Further work

We have now completed our investigation of the asymptotic behaviour of different types of parabolic-hyperbolic PDEs, having found that the behaviour depends on which of the terms dominate in the equation for large times.

There are several possibilities regarding further work in this field. Some natural extensions are to extend our results on asymptotic behaviour for the convection-diffusion equation in (5.1) into more general settings, such as higher dimensions $n > 1$ or more general non-linearities $F \in C^1(\mathbb{R}^n)$ instead of $F(u) = u^q$. We have already commented on this in Remark 5.3 and Remark 5.11.

Furthermore, there are several useful results which we have stated and used without proof, such as Lemma 4.2 regarding the regularity of solutions to the scalar convection-diffusion equation (4.1), and Theorem 6.3, the uniqueness result for entropy solutions of (6.1) with a non-negative finite Radon measure on $\mathbb{R}$ as initial data. For completeness, it could be useful to prove these results as well.

Finally, it could be interesting to investigate the asymptotic behaviour of other slightly more different equations, for example porous medium equations with convection as in [19, 20], where they replace the diffusion term Δu with the non-linear term $\Delta(u^m)$, for $m > 1$. The extension in [21] is also interesting, where they consider the fractional heat equation resulting from replacing the Laplace operator Δ by the fractional Laplace operator $-(-\Delta)^s$. One could also consider a fractional diffusion-convection equation as in [17].

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Appendix A

Preliminary results

For completeness, we here present some preliminary results referred to in the previous chapters. If not mentioned otherwise, the results are stated here similarly as in Evans [4, pp. 277–279, 705–708, 711–712, 732–733].

Theorem A.1 (Sobolev inequality)**.** *Assume* $1 \leq p < n$*. Define*

$$p^* := \frac{np}{n-p}.$$

Then there exists a constant $C = C(p, n)$*, depending only on* p *and* n*, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$*.*

Remark A.2. *By density, this result holds for other* u *as well, as long as the right hand side of the inequality is finite.*

Theorem A.3 (Hölder's inequality)**.** *Assume* $1 \leq p, q \leq \infty$*, with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then if* $u \in L^p(U), v \in L^q(U)$*, we have*

$$\int_U |uv| dx \leq \|u\|_{L^p} \|v\|_{L^q}.$$

Theorem A.4 (Interpolation inequality for L^p-(quasi)norms)**.** *Assume* $0 < b \leq a \leq c \leq \infty$ *and* $\nu \in [0, 1]$ *such that*

$$\frac{1}{a} = \frac{\nu}{b} + \frac{1-\nu}{c}.$$

Suppose also $u \in L^b(U) \cap L^c(U)$*, where* U *is a bounded, open subset of* $\mathbb{R}^n$*. Then* $u \in L^a(U)$*, and*

$$\|u\|_{L^a} \leq \|u\|_{L^b}^{\nu} \|u\|_{L^c}^{1-\nu}.$$

Theorem A.5 (Jensen's inequality)**.** *Assume* $f : \mathbb{R}^m \to \mathbb{R}$ *is convex and* $U \subset \mathbb{R}^n$ *is open, bounded. Let* $u : U \to \mathbb{R}^m$ *be finitely integrable. Then*

$$f\left(\frac{1}{|U|}\int_U u dx\right) \leq \frac{1}{|U|}\int_U f(u) dx.$$

Theorem A.6 (Green's first identity)**.** *Assume U is a bounded, open subset of $\mathbb{R}^n$ and ∂U is C^1. Let $u, v \in C^1(\bar{U})$. Then*

$$\int_U (\nabla v \cdot \nabla u + u\Delta v)dx = \int_{\partial U} u\frac{\partial v}{\partial \nu}dS,$$

where $\frac{\partial u}{\partial \nu} = \boldsymbol{\nu} \cdot \nabla u$ is the directional derivative of u with respect to $\boldsymbol{\nu}$, the outward unit normal on ∂U.

Theorem A.7 (Fatou's lemma)**.** *Assume the functions $\{f_k\}_{k=1}^{\infty}$ are non-negative and measurable. Then*

$$\int_{\mathbb{R}^n} \liminf_{k\to\infty} f_k dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} f_k dx.$$

Theorem A.8 (Dominated convergence theorem)**.** *Assume the functions $\{f_k\}_{k=1}^{\infty}$ are integrable and $f_k \to f$ a.e. Suppose also $|f_k| \leq g$ a.e. for some finitely integrable function g. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)dx = \int_{\mathbb{R}^n} f(x)dx.$$

Theorem A.9 (Lebesgue's differentiation theorem)**.** *Let $f \in L^p_{loc}(\mathbb{R}^n)$ $(1 \leq p < \infty)$, then for a.e. point $x_0 \in \mathbb{R}^n$ we have*

$$\frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} |f(x) - f(x_0)|^p dx \to 0 \text{ as } r \to 0,$$

where $|B(x_0, r)|$ denotes the n-dimensional volume of the ball $B(x_0, r)$.

Stated as in [22, p. 218]:

Theorem A.10 (Fubini's theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. If $f \in L^1(\mu \times \nu)$, then*

$$\iint_{X\times Y} fd(\mu \times \nu) = \int_X \int_Y f(x, y)d\nu(y)d\mu(x) = \int_Y \int_X f(x, y)d\mu(x)d\nu(y).$$

Stated as in [10, pp. 104–105]:

Theorem A.11 (Young)**.** *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then, $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}.$$

Stated as in [23, p. 271]:

Theorem A.12 (Minkowski's integral inequality)**.** *Let $F(x, y)$ be a measurable function on the σ-finite product measure space $X \times Y$. Let $1 \leq p < \infty$. Then*

$$\left(\int_Y \left(\int_X |F(x, y)|dx\right)^p dy\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |F(x, y)|^p dy\right)^{\frac{1}{p}} dx.$$

Stated as in [3, p. 52]:

Theorem A.13 (Continuity of translation in L^p). *Let $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ and $y \in \mathbb{R}^n$, then*

$$\lim_{y \to 0} \|f(\cdot - y) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0.$$

Stated as in [10, p. 138]:

Theorem A.14 (Banach fixed-point theorem). *Let (X, d) be a non-empty complete metric space and let $T : X \to X$ be a contraction, i.e. $\exists\ 0 < K < 1$ such that*

$$d(T(x), T(y)) \leq K d(x, y)\ \forall\ x, y \in X.$$

Then T has a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.

Stated as in [10, p. 66]:

Theorem A.15 (Banach-Alaoglu theorem). *The closed unit ball*

$$B_{E^*} = \{f \in E^* \mid \|f\| \leq 1\}$$

is compact in the weak topology $\sigma(E^*, E)$.*

Stated as in [10, p. 63]:

Theorem A.16 (Weak* lower semi-continuity). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in E^*. If $f_n \overset{*}{\rightharpoonup} f$ in the weak* topology $\sigma(E^*, E)$, then $\{\|f_n\|\}_{n=1}^{\infty}$ is bounded, and*

$$\|f\| \leq \liminf_{n \to \infty} \|f_n\|.$$

Stated as in [24, p. 600]:

Theorem A.17 (Kato's inequality). *Let $u \in L^1_{loc}(U)$ such that $\Delta u \in L^1_{loc}(U)$, where U is a bounded, open subset of $\mathbb{R}^n$. Then $\Delta[u]^+$ is a Radon measure, and*

$$\int_U \text{sign}^+(u(x)) \Delta u(x) \varphi(x) dx \leq \int_U \Delta[u(x)]^+ \varphi(x) dx\ \forall\ \varphi \in C_c^{\infty}(U).$$

Stated as in [25, p. 435]:

Theorem A.18 (Kolmogorov's compactness theorem). *Let M be a subset of $L^p(\Omega)$, $p \in [1, \infty)$, for some open set $\Omega \subseteq \mathbb{R}^n$. Then M is relatively compact if and only if the following three conditions are fulfilled:*

(i) M is bounded in $L^p(\Omega)$, i.e.

$$\sup_{u \in M} \|u\|_{L^p} < \infty.$$

(ii) *We have*

$$\|u(\cdot+\varepsilon)-u(\cdot)\|_{L^p} \leq \lambda(|\varepsilon|),$$

for a modulus of continuity λ *that is independent of* $u \in M$ *(we let* u *equal zero outside* Ω*).*

(iii)

$$\lim_{\alpha\to\infty}\int_{\{x\in\Omega||x|\geq\alpha\}} |u(x)|^p dx = 0 \text{ uniformly for } u \in M.$$

Theorem A.19. *Let* $0 \leq x, y < \infty$*. Assume* $1 < q < 2$*, then*

$$\left|x^{\frac{1}{q}} - y^{\frac{1}{q}}\right| \leq |x-y|^{\frac{1}{q}}.$$

Proof. In the cases $x = y$, $0 = x < y$ and $x > y = 0$, both sides are trivially equal. Assume therefore that $x > y > 0$, and define

$$f(x) := x^{\frac{1}{q}} \ (x \geq 0).$$

Differentiation yields

$$f'(x) = \frac{1}{q}x^{\frac{1}{q}-1} \ (x \geq 0).$$

By the fundamental theorem of calculus, we get

$$\begin{aligned} f(x)-f(y) &= \int_y^x f'(s)ds = \int_y^x \frac{1}{q}\left(\frac{1}{s}\right)^{1-\frac{1}{q}} ds \overset{u:=s-y}{=} \int_0^{x-y} \frac{1}{q}\left(\frac{1}{u+y}\right)^{1-\frac{1}{q}} du \\ &< \int_0^{x-y} \frac{1}{q}\left(\frac{1}{u}\right)^{1-\frac{1}{q}} du = \int_0^{x-y} f'(u)du = f(x-y)-f(0), \end{aligned}$$

where the inequality comes from the fact that $0 < 1 - 1/q < 1/2$, in which case the function $\alpha^{1-\frac{1}{q}}$ is increasing as $\alpha \geq 0$ increases. Using that $f(0) = 0$, we obtain the upper bound in this case:

$$x^{\frac{1}{q}} - y^{\frac{1}{q}} = f(x)-f(y) < f(x-y) = (x-y)^{\frac{1}{q}} = |x-y|^{\frac{1}{q}}.$$

Next, we assume $y > x > 0$. To avoid evaluating f at the negative value $x-y$, we observe that f is an increasing function, and thus $y > x$ implies $f(y)-f(x) > 0$ and $f(x)-f(y) < 0$. Using this, we rewrite so that we can apply the result above, and get

$$x^{\frac{1}{q}} - y^{\frac{1}{q}} < y^{\frac{1}{q}} - x^{\frac{1}{q}} < |y-x|^{\frac{1}{q}} = |x-y|^{\frac{1}{q}}.$$

Thus, we have the upper bound

$$x^{\frac{1}{q}} - y^{\frac{1}{q}} \leq |x-y|^{\frac{1}{q}},$$

for all $0 \leq x, y < \infty$. To obtain the lower estimate, we simply switch the roles of x and y to get

$$y^{\frac{1}{q}} - x^{\frac{1}{q}} \leq |y-x|^{\frac{1}{q}} \iff -(x^{\frac{1}{q}} - y^{\frac{1}{q}}) \leq |x-y|^{\frac{1}{q}} \iff x^{\frac{1}{q}} - y^{\frac{1}{q}} \geq -|x-y|^{\frac{1}{q}}.$$

This concludes the proof. $\square$

Appendix B

Heat kernel estimates

Recall that the heat kernel is given by

$$G(x,t) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \ (x \in \mathbb{R}^n,\ t > 0).$$

Lemma B.1.

$$\int_{\mathbb{R}^n} G(x,t)dx = 1 \ \forall\ t > 0.$$

For a proof, see Lemma 2.3 in [3, p. 7].

Lemma B.2. *Let $p \in [1,\infty]$, then there exists a constant $C_p > 0$ such that*

$$\|G(\cdot,t)\|_{L^p} \leq C_p t^{-\frac{n}{2}(1-\frac{1}{p})} \ \forall\ t > 0.$$

Proof. The case $p = 1$ follows from Lemma B.1, so we can choose $C_1 = 1$.

Next, consider $1 < p < \infty$. Through some manipulations, and using the well-known result $\int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$, we get

$$\begin{aligned}
\|G(\cdot,t)\|_{L^p}^p &= \int_{\mathbb{R}^n} \left| (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \right|^p dx \\
&= (4\pi t)^{-\frac{np}{2}} \int_{\mathbb{R}^n} e^{-\frac{p|x|^2}{4t}} dx \\
&\overset{z:=\sqrt{p/t}x}{=} (4\pi t)^{-\frac{np}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} \left(\frac{t}{p}\right)^{\frac{n}{2}} dz \\
&= (4\pi)^{-\frac{np}{2}} t^{\frac{n}{2}(1-p)} p^{-\frac{n}{2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-\frac{z_1^2}{4}} \cdots e^{-\frac{z_n^2}{4}} dz_1 \cdots dz_n \\
&= (4\pi)^{-\frac{np}{2}} t^{\frac{n}{2}(1-p)} p^{-\frac{n}{2}} \left(\int_{\mathbb{R}} e^{-\frac{z_1^2}{4}} dz_1 \right)^n \\
&\overset{y:=z_1/2}{=} (4\pi)^{-\frac{np}{2}} t^{\frac{n}{2}(1-p)} p^{-\frac{n}{2}} \left(2 \int_{\mathbb{R}} e^{-y^2} dy \right)^n
\end{aligned}$$

$$
\begin{aligned}
&= (4\pi)^{-\frac{np}{2}} t^{\frac{n}{2}(1-p)} p^{-\frac{n}{2}} \left(2\sqrt{\pi}\right)^n \\
&= (4\pi)^{\frac{n}{2}(1-p)} p^{-\frac{n}{2}} t^{\frac{n}{2}(1-p)},
\end{aligned}
$$

Taking the p-th root, we obtain as wanted

$$
\|G(\cdot,t)\|_{L^p} = (4\pi)^{-\frac{n}{2}(1-\frac{1}{p})} p^{-\frac{n}{2p}} t^{-\frac{n}{2}(1-\frac{1}{p})} =: C_p t^{-\frac{n}{2}(1-\frac{1}{p})} \ \forall\ t > 0.
$$

Finally, we consider the case $p = \infty$, which yields

$$
\begin{aligned}
\|G(\cdot,t)\|_{L^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|(4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}\right| = (4\pi)^{-\frac{n}{2}} t^{-\frac{n}{2}} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|e^{-\frac{|x|^2}{4t}}\right| \\
&= (4\pi)^{-\frac{n}{2}} t^{-\frac{n}{2}} \cdot 1 =: C_\infty t^{-\frac{n}{2}} \ \forall\ t > 0.
\end{aligned}
$$

□

Lemma B.3. *There exists a constant $C > 0$ such that*

$$
\|\nabla G(\cdot,t)\|_{L^1} \leq C t^{-\frac{1}{2}} \ \forall\ t > 0.
$$

Proof. By differentiating the heat kernel, we get

$$
\frac{\partial G}{\partial x_i}(x,t) = -\frac{x_i}{2t} G(x,t).
$$

Assume without loss of generality that $i = 1$. Calculating the norm using Lemma B.1 finally yields

$$
\begin{aligned}
\left\|\frac{\partial G}{\partial x_1}(\cdot,t)\right\|_{L^1} &= \int_{\mathbb{R}^n} \left|-\frac{x_1}{2t} G(x,t)\right| dx \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{|x_1|}{2t} (4\pi t)^{-\frac{n}{2}} e^{-\frac{x_1^2}{4t}} \cdots e^{-\frac{x_n^2}{4t}} dx_1 \cdots dx_n \\
&= \int_{\mathbb{R}} \frac{|x_1|}{2t} (4\pi t)^{-\frac{1}{2}} e^{-\frac{x_1^2}{4t}} dx_1 \left(\int_{\mathbb{R}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{z^2}{4t}} dz\right)^{n-1} \\
&= (4\pi t)^{-\frac{1}{2}} 2 \int_0^\infty \frac{x_1}{2t} e^{-\frac{x_1^2}{4t}} dx_1 \cdot 1^{n-1} \\
&\overset{u:=x_1^2/4t}{=} (\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-u} du \\
&= (\pi t)^{-\frac{1}{2}} [-e^{-u}]_0^\infty \\
&= \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} \\
&=: C t^{-\frac{1}{2}} \ \forall\ t > 0.
\end{aligned}
$$

□

Lemma B.4. *Let $p \in [1,\infty]$, then there exists a constant $C_p > 0$ such that*

$$
\|\nabla G(\cdot,t)\|_{L^p} \leq C_p t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}} \ \forall\ t > 0.
$$

Proof. Let first $p \in [1,\infty)$. We argue as in the proof of Lemma B.3, and thus get

$$\begin{aligned}
\left\|\frac{\partial G}{\partial x_1}(\cdot,t)\right\|_{L^p}^p &= \int_{\mathbb{R}^n} \left|-\frac{x_1}{2t}G(x,t)\right|^p dx \\
&= (2t)^{-p}(4\pi t)^{-\frac{np}{2}} \int_{\mathbb{R}^n} |x_1|^p e^{-\frac{p|x|^2}{4t}} dx \\
&\overset{y:=\sqrt{p/t}x}{=} (2t)^{-p}(4\pi t)^{-\frac{np}{2}} \int_{\mathbb{R}^n} \left|\sqrt{\frac{t}{p}}y_1\right|^p e^{-\frac{|y|^2}{4}} \left(\frac{t}{p}\right)^{\frac{n}{2}} dy \\
&= 2^{-p}(4\pi)^{-\frac{np}{2}} p^{-\frac{n+p}{2}} t^{-\frac{n}{2}(p-1)-\frac{p}{2}} \int_{\mathbb{R}^n} |y_1|^p e^{-\frac{|y|^2}{4}} dy \;\forall\; t>0.
\end{aligned}$$

Taking the p-th root, we get the term $t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}$, so if we can ensure the integral is finite, we are done. Calculating the integral, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |y_1|^p e^{-\frac{|y|^2}{4}} dy &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |y_1|^p e^{-\frac{y_1^2}{4}} \cdots e^{-\frac{y_n^2}{4}} dy_1 \cdots dy_n \\
&= \int_{\mathbb{R}} |y_1|^p e^{-\frac{y_1^2}{4}} dy_1 \left(\int_{\mathbb{R}} e^{-\frac{z^2}{4}} dz\right)^{n-1} \\
&\overset{u:=z/2}{=} \int_{\mathbb{R}} |y_1|^p e^{-\frac{y_1^2}{4}} dy_1 \left(2\int_{\mathbb{R}} e^{-u^2} du\right)^{n-1} \\
&= 2\left(2\pi^{\frac{1}{2}}\right)^{n-1} \int_0^\infty y_1^p e^{-\frac{y_1^2}{4}} dy_1 \\
&= \left(2\pi^{\frac{1}{2}}\right)^{n-1} \left(\frac{1}{4}\right)^{-\frac{p+1}{2}} \Gamma\left(\frac{p+1}{2}\right) < \infty,
\end{aligned}$$

as wanted. We have made use of the result $\int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$, that the integrand is an even function and the following integral result from [26, p. 155]:

$$\int_0^\infty x^k e^{-\lambda x^2} dx = \frac{1}{2}\lambda^{-\frac{k+1}{2}}\Gamma\left(\frac{k+1}{2}\right),\; k>-1,\; \lambda>0.$$

Finally, for the case $p=\infty$ we get

$$\begin{aligned}
\|\nabla G(\cdot,t)\|_{L^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|-\frac{x}{2t}G(x,t)\right| \\
&= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|-\frac{x}{2t}(4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}\right| \\
&= (4\pi)^{-\frac{n}{2}} t^{-\frac{n}{2}-\frac{1}{2}} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left\{\frac{|x|}{2t^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}}\right\} \\
&\overset{z:=|x|/2t^{\frac{1}{2}}}{=} (4\pi)^{-\frac{n}{2}} t^{-\frac{n}{2}-\frac{1}{2}} \operatorname*{ess\,sup}_{z\geq 0} \left\{z e^{-z^2}\right\} \\
&= (4\pi)^{-\frac{n}{2}} \frac{\sqrt{2}}{2} e^{-\frac{1}{2}} t^{-\frac{n}{2}-\frac{1}{2}} \;\forall\; t>0.
\end{aligned}$$

□

Lemma B.5.

$$\int_{\mathbb{R}^n} xG(x,t)dx = 0 \ \forall \ t > 0.$$

Proof. We only show the result for the first component, since all of them yield the same result. We get

$$\begin{aligned}
\int_{\mathbb{R}^n} x_1 G(x,t)dx &= \int_{\mathbb{R}^n} x_1 (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} dx \\
&= \int_{\mathbb{R}} x_1 (4\pi t)^{-\frac{1}{2}} e^{-\frac{x_1^2}{4t}} dx_1 \left(\int_{\mathbb{R}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{z^2}{4t}} dz \right)^{n-1} \\
&= (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} x_1 e^{-\frac{x_1^2}{4t}} dx_1 \cdot 1^{n-1} \\
&= 0 \ \forall \ t > 0,
\end{aligned}$$

where we have used Lemma B.1 and the fact that $x_1 e^{-\frac{x_1^2}{4t}}$ is an odd function, and thus integrates to zero over $\mathbb{R}^n$. $\square$

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