Triangulated structure of stable Frobenius categories and derived abelian categories

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17th of June, 2024

Abstract

Triangulated categories are one of the main areas of study in homological algebra, especially given that there are several important categorical structures appearing in related fields that can be equipped with a triangulated structure. Stable module categories are found naturally in representation theory, and to a certain degree algebraic topology. Derived categories appear most often in homological algebra itself but are also widely used in algebraic geometry.

In this thesis we will build up the notion of triangulated categories, starting with basic category theory. We will build on the notion of a category by giving different restrictions and requirements revealing different categorical structures, before defining triangulated categories. Then we'll consider two different types of categories, the stable category of a Frobenius category and the derived category of an abelian category and prove that they both carry triangulated structures.

Contents

1 Introduction

This thesis serves to function as a quick introduction to triangulated categories, not assuming any previous background in category theory, but a bit of basic abstract algebra knowledge.
It starts off with introducing the structure of a category, and then introduce several terms and structures seen in category theory whilst discussing additive, abelian and exact categories. We will then introduce the notion of a triangulated category, before moving on to Frobenius categories and Derived categories.

Sustainability

Starting this semester NTNU decided that it was a great idea to force everyone writing a bachelor thesis or a master thesis to implement and reflect on their thesis' relevance with respect to the UN sustainable development goals. The idea is all good and fine, but for a quite theoretical thesis just like this one, there just is not any real relevance. The only real relevance a thesis about triangulated categories have, is the fact that this thesis is a possible guide for people to learn some category theory and homological algebra. This again can potentially help with spread more knowledge about the topics discussed in the thesis, and then potentially play its small part in developing science. Developing science is part of goal 17: "Strengthen the means of implementation and revitalize the Global Partnership for Sustainable Development". [UN]

Acknowledgements

I would first and foremost like to give a big thanks to my supervisor Laertis Vaso. He was positive and helpful and gave me a lot of inspiration and potential road maps, even though I had in no way or form any previous experience or knowledge with category theory. I most definitely took on way too much, but he still managed to make me feel like it was doable.

Secondly I would like to thank my dear fellow Matteland inhabitant and abstract algebra addict John Aslak for helping me so much with getting into and understand category theory, and for seriously considering and answering all my ridiculous hypotheticals.

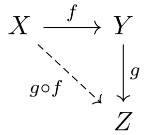
I would also like to thank my dear friend and certified LaTeX-monkey Oskar. Without him dedicating several hours to help and guidance, this thesis would be very ugly and quite unreadable indeed.

The last honours goes out to my dear friends Line and Inga. I have no idea were I would be without your guidance, support and friendship, but there would most certainly not be any thesis. Mia thanks you both.

2 Categories

Definition 2.1 (Category [Lan78, Ch. I.2])**.** A *Category* has the following:

- A class of objects
- A set of morphisms between every two objects
- Composition of morphisms are well defined, i.e. for morphism $f : X \to Y$ and $g : Y \to Z$, there exist a unique morphism $g \circ f : X \to Z$, such that the following diagram commutes:



Remark 2.2. We will use $\mathcal{C}$ to denote an arbitrary Category. For the "set" of all morphisms between two objects in $\mathcal{C}$, say X, Y, also called the hom-set, we use the notation $\text{Hom}_{\mathcal{C}}(X, Y) = \{f \mid f \text{ in } \mathcal{C} \text{ such that } \textit{dom}\, f = X \text{ and } \textit{cod}\, f = Y\}$ [Lan78, Ch. I.8].
dom and *cod* stand here for domain and codomain.

Example 2.3 ([Lan78, Ch. I.2]). The following are examples of categories:

- The category of sets, **Set**
  - Objects are sets
  - Morphisms are functions
- The category of groups, **Grp**, and similarly the category of Abelian groups, **Ab**
  - Objects are (Abelian) groups
  - Morphisms are group morphisms
- The category of vector spaces over a field K, $\textbf{Vect}_K$
  - Objects are vector spaces over K
  - Morphisms are K-linear maps

Definition 2.4 (Dual)**.** The *dual* category $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is formed by reversing all morphisms in $\mathcal{C}$. The dual can be defined for any structure in this way.

Definition 2.5 (Functor [Lan78, Ch. I.3])**.** A *functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism of categories. F maps objects, say $c \in \mathcal{C}$, to objects $F(c) \in \mathcal{D}$, and morphisms, say $f : c \longrightarrow d$ in $\mathcal{C}$, to morphisms $F(f) : F(c) \longrightarrow F(d)$.

Functors works as morphisms between categories. Thus one can easily observe that by considering objects as categories and morphisms as functors, one can define a category of categories.

Functors are often used to show how properties are translated between different categories, as when defining the quotient category.

Lemma 2.6 (Quotient category [Lan78, Ch. II.8])**.** For a given category $\mathcal{C}$, let R be a *congruence relation* that assigns to each pair of objects $X, Y \in \mathcal{C}$ a binary relation $R_{X,Y}$ on the hom-set

$\mathrm{Hom}_{\mathcal{C}}(X, Y)$, such that it preserves composition. I.e. if $fR_{X,Y}f'$ and $gR_{Y,Z}g'$, then $g \circ fR_{X,Z}g' \circ g$. Then there exists a category $\mathcal{C}/R$ and a functor $Q_R : \mathcal{C} \longrightarrow \mathcal{C}/R$ such that,

- If $fR_{X,Y}f'$ is in $\mathcal{C}$, then $Q_R(f) = Q_R(f')$;
- If $H : \mathcal{C} \longrightarrow \mathcal{D}$ is any functor from $\mathcal{C}$ for which $fR_{X,Y}f'$ implies $H(f) = H(f')$ for all f, f', then there is a unique functor $H' : \mathcal{C}/R \longrightarrow \mathcal{D}$ with $H' \circ Q_R = H$. Moreover the functor Q_R is a bijection on objects.

We call $\mathcal{C}/R$ a *quotient category.*

Proof. We prove that the quotient is indeed a category.
By construction, we can see that our quotient category $\mathcal{C}/R$ has the same objects as $\mathcal{C}$. Given construction, we can take our morphisms to be equivalence classes on the hom-set of $\mathcal{C}$. As our congruence relation preserves composition, the composite is well defined in $\mathcal{C}/R$ as it is carried over by the projection $Q_R : \mathcal{C} \to \mathcal{C}/R$. □

Definition 2.7 (Natural transformation [Lan78, ch.I.4])**.** Given two functors $S, T : \mathcal{C} \to \mathcal{B}$, a *natural transformation* $\tau : S \xrightarrow{\bullet} T$ is a function wich assigns to each object c of $\mathcal{C}$, an arrow $\tau_c = \tau c : Sc \to Tc$ of $\mathcal{B}$ in such a way that every arrow $f : c \to c'$ i $\mathcal{C}$ yields a diagram

$$\begin{array}{ccc} Sc & \xrightarrow{\tau c} & Tc \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ Sc' & \xrightarrow{\tau c'} & Tc' \end{array}$$

which is commutative.

Definition 2.8 (Equivalence [Lan78, ch.I.4])**.** An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is defined to be a pair of functors $S : \mathcal{C} \to \mathcal{D}$, $T : \mathcal{D} \to \mathcal{C}$, together with natural isomorphisms $I_{\mathcal{C}} \cong T \circ S$, $I_{\mathcal{D}} \cong S \circ T$.

3 Additive and Abelian categories

Definition 3.1 (Bilinearity [Lan78, Ch. I.8])**.** For arrows $f, f' : a \longrightarrow b$ and $g, g' : b \longrightarrow c$ from a hom-set, the hom-set is *bilinear* if

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'$$

Definition 3.2 (Preadditive category [Lan78, Ch. I.8])**.** A *preadditive category* is a category where

- All hom-sets are additive Abelian groups
- Composition of morphisms is bilinear relative to this addition

Definition 3.3 (Biproduct [Lan78, Ch. VIII.2])**.** Assume $X, Y \in \mathcal{A}$, a preadditive category. Their *biproduct*, $X \oplus Y$, is an object with morphisms p_1, p_2, i_1, i_2 as in the diagram,

$$X \underset{i_1}{\overset{p_1}{\leftrightarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y,$$

satisfying

$$p_1 \circ i_1 = id_X, \quad p_2 \circ i_2 = id_Y, \quad i_1 \circ p_1 + i_2 \circ p_2 = id_{X \oplus Y}$$

Remark 3.4. When considering morphisms between biproducts we will be using a matrix notation to describe these. For a morphism $f : X_1 \oplus X_2 \to Y_1 \oplus Y_2$, we represent it by the matrix

$$f := \begin{pmatrix} p_{Y_1} \circ f \circ i_{X_1} & p_{Y_1} \circ f \circ i_{X_2} \\ p_{Y_2} \circ f \circ i_{X_1} & p_{Y_2} \circ f \circ i_{X_2} \end{pmatrix}$$

Definition 3.5 (Additive category [Lan78, Ch. VIII.2]). We call a preadditive category $\mathcal{A}$ *additive* if it also satisfies the following:

- $\mathcal{A}$ contains a *zero object* 0, i.e. $\forall X \in \mathcal{A}$ the hom-sets $\mathrm{Hom}_{\mathcal{A}}(X, 0)$ and $\mathrm{Hom}_{\mathcal{A}}(0, X)$ have exactly one element.
- Given objects $X, Y \in \mathcal{A}$ there exists a biproduct $X \oplus Y$ in $\mathcal{A}$

Example 3.6. The category of finitely generated modules over a non-Noetherian ring is an example of an additive category that is not Abelian, something we will discuss later.

Definition 3.7 (Additive functor [Lan78, Ch. I.8]). If $\mathcal{A}$ and $\mathcal{B}$ are preaddtive categories, a functor $T : \mathcal{A} \to \mathcal{B}$ is said to be *additive* when every function $T : \mathrm{Hom}_{\mathcal{A}}(a, a') \to \mathrm{Hom}_{\mathcal{B}}(Ta, Ta')$ is a homomorphism of abelian groups.

Definition 3.8 (Epimorphism and Monomorphism [Lan78, Ch. I.5]). Consider a morphism $m : a \to b$. We say m is *monic* in a category $\mathcal{C}$ if for any two parallel arrows $f_1, f_2 : d \to a$, $m \circ f_1 = m \circ f_2$ implies $f_1 = f_2$, in other words m is left cancellable.

We say f is *epi* in $\mathcal{C}$ if for any arrows g_1, $g_2 : b \to c$, $g_1 \circ m = g_2 \circ m$ implies $g_1 = g_2$, in other words right cancellable.

We often also use the terms *monomorphisms* and *epimorphisms* for monics and epis.

Definition 3.9 (Kernel and Cokernel [Rot08]). Given a category $\mathcal{C}$ that has a zero object, and objects X, Y in $\mathcal{C}$, the *cokernel* of $f : X \longrightarrow Y$ is an arrow $u : Y \longrightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle u} \\ 0 & \longrightarrow & Z \end{array}$$

, and that for all morphisms $h : Y \to T$ such that $h \circ f$ is the zero map, there exist a unique morphism $h' : Z \to T$ such that $h' \circ u = h$.

The object Z is also sometimes called the cokernel of f, written $\mathrm{Coker}(f)$.
The *kernel* of f, written $\mathrm{Ker}(f)$ is the dual of the cokernel, with dual definition.

Remark 3.10. The literature will often use slightly different but equivalent diagrams where the use of the zero object in the diagram is omitted by instead assuming the composition of morphisms in the diagram is zero.

Definition 3.11 (Abelian category [Lan78, Ch. VIII.3]). An *Abelian category* is an additive category where

- Kernels and cokernels exist for all morphisms.
- Every monomorphism is a kernel, and every epimorphism a cokernel.

Example 3.12. [Lan78, Ch. VIII.3] The category of Abelian groups, **Ab**, is a classic example. Or to take up the thread from before, the category of finitely generated modules over a Noetherian ring is also Abelian.

4 Exact categories

Definition 4.1 (Image)**.** The *image* of a morphism $f : X \to Y$ is a monomorphism $m : I \to Y$ satisfying:

- There exist a morphism $e : X \to I$ such that $f = m \circ e$
- For any object I' with morphism $e' : X \to I'$ and a monomorphism $m' : I' \to Y$ such that $f = m' \circ e'$, there exist a unique morphism $v : I \to I'$ such that $m = m'$

Definition 4.2 (Exact sequence [Lan78, Ch. VIII.3])**.** A composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an abelian category $\mathcal{A}$, is *exact* at Y if *im* f is isomorphic to *ker* g, equivalently if *coker* f is isomorphic *coim* g

Definition 4.3 (Short exact sequence [Lan78, Ch. VIII.3])**.** A *short exact sequence* is a sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

that is exact at X, Y and Z.

Definition 4.4 (Exact category [Kra21, Part One Ch. 2.1,])**.** A *exact category* is a pair $(\mathcal{A}, \mathcal{S})$ consisting of an additive category $\mathcal{A}$ and a class $\mathcal{S}$ of short exact sequences in $\mathcal{A}$

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

(We say α is an *admissible monomorphism* and β is an *admissible epimorphism.*), closed under isomorphism satisfying:

(Ex1) The identity morphism is an admissible monomorphism and an admissible epimorphism.

(Ex2) Composites of admissible monomorphisms give admissible monomorphisms, and equivalently with epimorphisms.

(Ex3) Each pair of morphisms $X' \xleftarrow{\phi} X \xrightarrow{\alpha} Y$ with α an admissible monomorphism, can be completed to a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ X' & \xrightarrow{\alpha'} & Y' \end{array}$$

such that α' is an admissible monomorphism. And each pair of morphisms $Y \xleftarrow{\beta} Z \xrightarrow{\psi} Z'$ with β an admissible epimorphism can be completed to a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\beta'} & Z' \\ \downarrow & & \downarrow \psi \\ Y & \xrightarrow{\beta} & Z \end{array}$$

such that β' is an admissible epimorphism.

Remark 4.5. These are the axioms from Keller's definition. Compared to Quillen's original axioms, one is missing, as Keller proved in *Chain complexes and stable categories* that this missing axiom can be derived from the other axioms. [Kel90]

Additionally admissible monomorphisms are also sometimes called inflation denoted by $\rightarrowtail$, and admissible epimorphisms are also sometimes called deflation denoted by $\twoheadrightarrow$ in the literature.

Example 4.6. [Kra21, Part One Ch. 2.1] Let $\mathcal{A}$ be abelian and $\mathcal{S}$ be the class of short exact sequences. Then $(\mathcal{A}, \mathcal{S})$ is exact.

5 Triangulated Categories

Definition 5.1 (Suspended category [Kra21, Part One Ch. 3.1])**.** A *suspended category* is a pair $(\mathcal{T}, \Sigma)$ consisting of an additive category $\mathcal{T}$, and an equivalence automorphism $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ called suspension.

Remark 5.2. The suspension functor is additive, as it is an equivalence between two additive categories. We will be using this fact from now.

Equivalence of additive categories is an additive functor.

Definition 5.3 (Triangle [Kra21, Part One Ch. 3.1])**.** A *triangle* in $(\mathcal{T}, \Sigma)$ is a sequence (α, β, γ) of morphisms: $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$, and a *morphism between triangles* (α, β, γ) and $(\alpha', \beta', \gamma')$ is given by a triple (ϕ_1, ϕ_2, ϕ_3) of morphisms in $\mathcal{T}$ making

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow & & \downarrow \Sigma\phi_1 \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

commute.

Definition 5.4 (Triangulated category [Kra21, Part One Ch. 3.1])**.** A *triangulated category* is a triple $(\mathcal{T}, \Sigma, \mathcal{E})$ consisting of a suspended category $(\mathcal{T}, \Sigma)$ and a class $\mathcal{E}$ of distinguished triangles in $(\mathcal{T}, \Sigma)$ satisfying:

(TR0) The triangle $X \xrightarrow{id} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished.

(TR1) A triangle isomorphic to an distinguished triangle is distinguished.

(TR2) Given any morphism $X \xrightarrow{\alpha} Y$, there exists a distinguished triangle

$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$

(TR3) A triangle (α, β, γ) is distinguished if and only if $(\beta, \gamma, -\Sigma\alpha)$ is. In other words, if either $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ or $Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X \xrightarrow{-\Sigma\alpha} \Sigma Y$ is distinguished, both are.

(TR4) Given two distinguished triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, and morphisms $f : X \to X', g : Y \to Y'$, then there is a morphism $h : Z \to Z'$ making the following diagram commute:

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

(TR5) (The octahedral axiom) Given distinguished triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1\alpha_1$, there exist an distinguished triangle $(\delta_1, \delta_2, \delta_3)$ making

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\ \| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\ X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\ & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\ & & W & = & W & \xrightarrow{\beta_3} & \Sigma Y \\ & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\ & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U & & \end{array}$$

commute.

Remark 5.5. If $\mathcal{T}$ is a triangulated category, then so is its dual $\mathcal{T}^{op}$. We can see this by considering Σ^{-1} as the suspension automorphism and by flipping triangles.

Additionally distinguished triangles are sometimes called exact triangles, but we are avoiding this notion due to potential ambiguity in meaning.

Example 5.6. The category $\mathbf{Vect}_K$ of vector spaces over K can be seen with a triangulated structure. The shift functor will be $\Sigma X := X$ for all vector spaces X, and a distinguished triangle is a sequence $X \longrightarrow Y \longrightarrow Z \longrightarrow X$ of K-linear maps.

Definition 5.7 (Cohomological functor [Kra21, Part One Ch. 3.1])**.** Given a triangulated category $\mathcal{T}$ and an Abelian category $\mathcal{A}$, then the functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ is called a *cohomological functor* if for every distinguished triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in $\mathcal{T}$, the sequence $\mathrm{H}(X) \xrightarrow{\mathrm{H}(\alpha)} \mathrm{H}(Y) \xrightarrow{\mathrm{H}(\beta)} \mathrm{H}(Z)$ is exact in $\mathcal{A}$.

Remark 5.8. If H is a contravariant functor satisfying all the same conditions, then H is said to be a homological functor.

Proposition 5.9. [Kra21, Part One Ch. 3.1] For each object $W \in \mathcal{T}$, the functor $Hom_{\mathcal{T}}(W, -) : \mathcal{T} \longrightarrow \boldsymbol{Ab}$ is homological.

Proof. Choose a given distinguished triangle in $\mathcal{T}$, $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$. We would like to show the exactness of

$$\mathrm{Hom}_{\mathcal{T}}(W, X) \xrightarrow{\alpha\circ-} \mathrm{Hom}_{\mathcal{T}}(W, Y) \xrightarrow{\beta\circ-} \mathrm{Hom}_{\mathcal{T}}(W, Z) .$$

Fix a morphism $\phi : W \longrightarrow Y$, and consider;

$$\begin{array}{ccccccc} W & \xrightarrow{id} & W & \longrightarrow & 0 & \longrightarrow & \Sigma W \\ \downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\ X & \xrightarrow[\alpha]{} & Y & \xrightarrow[\beta]{} & Z & \xrightarrow[\gamma]{} & \Sigma X \end{array}$$

If ϕ factors through α, then (TR4) implies the existence of a morphism $0 \longrightarrow Z$ making the diagram commute. Thus $\beta \circ \phi = 0$.

Now assume $\beta \circ \phi = 0$. Applying (TR3) and (TR4), we find a morphism $W \longrightarrow X$ making the diagram commute. Thus ϕ factors through α. □

6 Frobenius Categories

Definition 6.1 (Injective and projective [Kra21, Part One Ch.3.3])**.** An object I of $\mathcal{A}$ is called *injective* if every exact triple in $\mathcal{E}$ of the form $I \longrightarrow Y \longrightarrow Z$ splits; and an object P of $\mathcal{A}$ is called *projective* if every exact triple in $\mathcal{E}$ of the form $X \longrightarrow Y \longrightarrow P$ splits.

Definition 6.2 (Having enough projectives and injectives)**.** A category has *enough projective* objects and enough *injective objects* if $\forall$ objects $X \in \mathcal{A}$ $\exists$ triples from $\mathcal{E}$ of the form $X' \longrightarrow I \longrightarrow X$ and $X \longrightarrow I' \longrightarrow X$" where I and I' are injective.

Definition 6.3 (Frobenius Category [Kra21, Part One Ch. 3.3])**.** An exact category $(\mathcal{A}, \mathcal{E})$ is called a *Frobenius category* if the following holds:

- Projective and injective objects coincide
- The category has enough projective objects and enough injective objects

Remark 6.4. [Wik24] In the next example we'll use the notion of a group ring. A group ring is a free module and at the same time a ring, constructed in a natural way from any given ring and any given group.

Example 6.5. [Kel96] The category of modules over a unital ring R is an exact category that has enough projectives and injectives. Projectives and injectives coincide as well if for example R is the group ring of a finite group over a field.

Lemma 6.6 (Stable Category [Hap88])**.** Let $(\mathcal{B}, \mathcal{E})$ be a Frobenius category. For objects X and Y in $\mathcal{B}$ let $\mathrm{Inj}(X, Y)$ denote those morphisms from X to Y which factor through some injective object. The *stable category* $\underline{\mathcal{B}}$ of the Frobenius category $(\mathcal{B}, \mathcal{E})$ has the same objects as $\mathcal{B}$; the morphisms are equivalence classes of morphisms modulo those factoring through injective objects, i.e.

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y) := \underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)/\mathrm{Inj}(X, Y)$$

Proof. We prove that the stable category is indeed a category.

In 2.5 we already proved that the quotient category is indeed a category, so we only need to show that Inj preserves composition. Assume we have $f : X \to Y$ factoring an injective $I(X)$ and $g : Y \to Z$ factoring through an injective $I(Y)$. We can then define the map from $I(X)$ to $I(Y)$ which will evidently factor through an injective. Thus Inj preserves composition. □

Definition 6.7 (Proper monomorphism)**.** A morphism $u : X \to Y$ in $\mathcal{B}$ is called a *proper monomorphism* if there exists an exact sequence $0 \longrightarrow X \xrightarrow{u} Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{E}$.

Definition 6.8 ($\mathcal{E}$-injective)**.** An object I in $\mathcal{B}$ is called $\mathcal{E}$-injective if for all proper monomorphisms $u : X \to Y$ and morphisms $f : X \to I$ in $\mathcal{B}$ there exists $g : Y \to I$ such that $f = g \circ u$.

The following lemma is given without proof.

Lemma 6.9. Say we have two short exact sequences $0 \longrightarrow X \xrightarrow{\mu(X)} I(X) \xrightarrow{\pi(X)} S(X) \longrightarrow 0$ and $0 \longrightarrow X \xrightarrow{\mu'(X)} I'(X) \xrightarrow{\pi'(X)} S'(X) \longrightarrow 0$, with morphisms $0 : 0 \to 0$, $id_X : X \to X$ and $f_x : I(x) \to I'(X)$, then there exist as morphism from $S(X)$ to $S'(X)$.

Lemma 6.10. [Hap88, chapter I.2.2] Let $(\mathcal{B}, \mathcal{E})$ be a Frobenius category, $0 \longrightarrow X \xrightarrow{\mu'} I' \xrightarrow{\pi'} X' \longrightarrow 0$ and $0 \longrightarrow X \xrightarrow{\mu''} I'' \xrightarrow{\pi''} X'' \longrightarrow 0$ be in $\mathcal{E}$ such that I', I'' are $\mathcal{E}$-injective. Then X' and X'' are isomorphic in $\underline{\mathcal{B}}$.

Proof. Since μ' and μ'' are proper monomorphisms and I' and I'' are $\mathcal{E}$-injective, we obtain the following commutative diagram such that the rows belong to $\mathcal{E}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\mu'} & I' & \xrightarrow{\pi'} & X' & \longrightarrow & 0 \\
 & & \| & & \downarrow f & & \downarrow g & & \\
0 & \longrightarrow & X & \xrightarrow{\mu'} & I'' & \xrightarrow{\pi''} & X'' & \longrightarrow & 0 \\
 & & \| & & \downarrow f' & & \downarrow g' & & \\
0 & \longrightarrow & X & \xrightarrow{\mu'} & I' & \xrightarrow{\pi'} & X' & \longrightarrow & 0
\end{array}$$

Thus $(f' \circ f - id_{I'}) \circ \mu' = 0$. So there exists $h : X' \to I'$ such that $h \circ \pi' = f \circ f' - id_{I'}$. Therefore $\pi' \circ h \circ \pi' = \pi' \circ (f' \circ f - id_{I'}) = g' \circ g \circ \pi' - \pi' = (g' \circ g - id_{X'}) \circ \pi'$. Thus we have that $g' \circ g - id_{X'} = \pi' \circ h$. In other words, $\underline{g'} \circ \underline{g} = \underline{id}_{X'}$. Similarly, we can show that $\underline{g} \circ \underline{g'} = \underline{id}_{X''}$. □

Let $X \in \mathcal{B}$. We denote by $[X]$ the isomorphism class of X in $\underline{\mathcal{B}}$. Moreover let $0 \longrightarrow X \longrightarrow I(X) \longrightarrow X' \longrightarrow 0$ be in $\mathcal{E}$ such that I(X) is $\mathcal{E}$-injective. We assume that for all $X \in \mathcal{B}$ there is a bijection $\gamma_X : [X] \to [X']$. The preceding lemma shows that this assumption does not depend on the choice of $0 \to X \to I(X) \to X' \to 0$. It is easily seen that this assumption is satisfied in all our applications.

For all $X \in \mathcal{B}$, we now choose elements $0 \longrightarrow X \xrightarrow{\mu(X)} I(X) \xrightarrow{\pi(X)} \Sigma X \longrightarrow 0$ in $\mathcal{B}$, with $I(X)$ an $\mathcal{E}$-injective such that $\Sigma X = \gamma_X(X)$. Let $f : X \to Y$ be a morphism in $\mathcal{B}$. Since $\mu(X)$ is a proper monomorphism and $I(Y)$ is $\mathcal{E}$-injective, we obtain a morphism $\Sigma(f)$ such that $\Sigma(f) \circ \pi(X) = \pi(Y) \circ I(f)$. It is easily seen that the residue class of $\Sigma(f)$ in $\underline{\mathcal{B}}$ does not depend on the choice of $I(f)$. Thus we may consider Σ as a functor from $\underline{\mathcal{B}}$ to $\underline{\mathcal{B}}$. The following commuting diagram illustrates the above:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\mu(X)} & I(X) & \xrightarrow{\pi(X)} & \Sigma X & \longrightarrow & 0 \\
 & & f \downarrow & \swarrow I(f) & \downarrow \mu(f) \circ I(f) & & \vdots \exists! \Sigma(f) & & \\
0 & \longrightarrow & Y & \xrightarrow[\mu(Y)]{} & I(Y) & \xrightarrow{\pi(X)} & \Sigma Y & \longrightarrow & 0
\end{array}$$

Lemma 6.11. [Hap88, Ch. I.2.2] The morphism Σ is a well-defined automorphism in the stable category $\underline{\mathcal{B}}$.

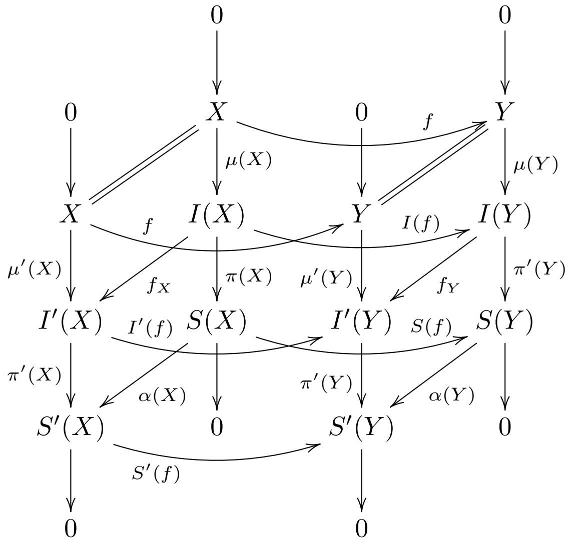
Proof. We may choose for $X \in \mathcal{B}$ elements $0 \to X \to I(X) \to S(X) \to 0$ in $\mathcal{E}$ with $I(x)$ an $\mathcal{E}$-injective. The same construction as above then yields that S may be considered a functor on $\underline{\mathcal{B}}$. This time providing us with a self-equivalence. We claim that any two such choices yield isomorphic functors. In fact, suppose that two such assignments have been made. So we have for all $X \in \mathcal{B}$ elements $0 \longrightarrow X \xrightarrow{\mu(X)} I(X) \xrightarrow{\pi(X)} S(X) \longrightarrow 0$ and $0 \longrightarrow X \xrightarrow{\mu'(X)} I'(X) \xrightarrow{\pi'(X)} S'(X) \longrightarrow 0$ in $\mathcal{E}$ such that $I(X)$ and $I'(X)$ are $\mathcal{E}$-injective. Since $\mu(X)$ is a proper monomorphism and $I'(X)$ is $\mathcal{E}$-injective, we obtain $f_X : I(x) \to I'(X)$ such that $f_X \circ \mu(X) = \mu'(X)$. This induces by lemma 6.9. a morphism $\alpha(X) : S(X) \to S'(X)$ such that $\alpha(X) \circ \pi(X) = \pi'(X) \circ f_X$. We know that $\underline{\alpha}(X)$ is an isomorphism. Thus it remains to be seen that this assignment is a natural transformation. In fact, let $f : X \to Y$. This induces the following two commutative diagrams:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
{\scriptstyle \mu(X)}\downarrow & & \downarrow{\scriptstyle \mu(X)} \\
I(X) & \xrightarrow{I(f)} & I(Y) \\
{\scriptstyle \pi(X)}\downarrow & & \downarrow{\scriptstyle \pi(Y)} \\
S(X) & \xrightarrow{S(f)} & S(Y) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\qquad
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
{\scriptstyle \mu'(X)}\downarrow & & \downarrow{\scriptstyle \mu'(Y)} \\
I'(X) & \xrightarrow{I'(X)} & I'(Y) \\
{\scriptstyle \pi'(X)}\downarrow & & \downarrow{\scriptstyle \pi'(Y)} \\
S'(X) & \xrightarrow{S'(f)} & S'(Y) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

We have morphisms $f_X : I(X) \to I'(X)$ and $f_Y : I(Y) \to I'(Y)$ such that $f_X \circ \mu(X) = \mu'(X)$ and $f_Y \circ \mu(Y) = \mu'(Y)$. Moreover we have morphisms $\alpha(X) : S(X) \to S'(X)$ and $\alpha(Y) : S(Y) \to S'(Y)$ such that $\alpha(X) \circ \pi(X) = \pi'(X) \circ f_X$ and $\alpha(Y) \circ \pi(Y) = \pi'(Y) \circ f_Y$. We claim that $\underline{\alpha(Y) \circ S(f)} = \underline{S'(f) \circ \alpha(X)}$. We show that $\alpha(Y) \circ S(f) - S'(f) \circ \alpha(X)$ factors over an $\mathcal{E}$-injective. Observe that $(f_Y \circ I(f) - I'(f) \circ f_X) \circ \mu(X) = 0$. Thus there exists $g : S(X) \to I'(Y)$ such that $g \circ \pi(X) = f_Y \circ I(f) - I'(f) \circ f_X$. Now

$$
\begin{aligned}
\pi'(Y) \circ g \circ \pi(X) &= \pi'(Y) \circ (f_Y \circ I(f) - I'(f) \circ f_X) \\
&= \pi'(Y) \circ f_Y \circ I(f) - \pi'(Y) \circ I'(f) \circ f_X \\
&= \alpha(Y) \circ \pi(Y) \circ I(f) - S'(f) \circ \pi'(X) \circ f_X \\
&= \alpha(Y) \circ S(f) \circ \pi(X) - S'(f) \circ \alpha(X) \circ \pi(X) \\
&= (\alpha(Y) \circ S(f) - S'(f) \circ \alpha(X)) \circ \pi(X).
\end{aligned}
$$

In particular $\pi'(Y) \circ g = \alpha(Y) \circ S(f) - S'(f) \circ \alpha(X)$. This can also be represented as in the following commuting diagram:



□

Theorem 6.12. [Hap88, chap.I.2.6] Let $(\mathcal{B}, \mathcal{E})$ be a Frobenius category. With the above suspension functor Σ and the collection of distinguished triangles just as defined, the stable category $\underline{\mathcal{B}}$ is a triangulated category.

Proof. [Hap88] Let $\underline{\mathcal{B}}$ be the stable of our Frobenius. We define a set $\mathcal{T}$ of sextuples in $\underline{\mathcal{B}}$. Let $X, Y \in \underline{\mathcal{B}}$ and $u \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$. Consider the following diagram in $\mathcal{B}$:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle v} \\ I(X) & \xrightarrow[\overline{u}]{} & C_u \\ {\scriptstyle \overline{x}}\downarrow & & \downarrow{\scriptstyle w} \\ \Sigma X & = & \Sigma X \end{array}$$

where $0 \longrightarrow X \xrightarrow{x} I(X) \xrightarrow{\overline{x}} \Sigma X \longrightarrow 0$ is in $\mathcal{E}$ and $I(X)$ is $\mathcal{E}$-injective. C_u is the pushout of u and x.

Since $\mathcal{B}$ is closed under extensions in some abelian category $\mathcal{A}$, the pushout C_u in $\mathcal{B}$ coincides with the pushout in $\mathcal{A}$. A *standard* is a sextuple of the form $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X$ and it's image in $\underline{\mathcal{B}}$.

A sextuple $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} \Sigma X$ of objects and morphisms in $\underline{\mathcal{B}}$ lies in $\mathcal{T}$ if it is isomorphic in $\underline{\mathcal{B}}$ to a standard sextuplet.

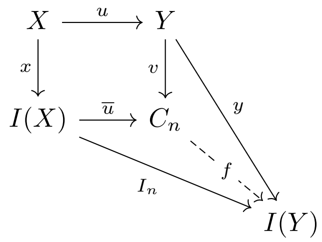
Let us now show that $\mathcal{T}$ is a triangulated structure of $\underline{\mathcal{B}}$:

(TR0) Clearly the sextuple $X \xrightarrow{1_X} X \xrightarrow{0} 0 \xrightarrow{0} \Sigma X$ lies in $\mathcal{T}$.

(TR1) $\mathcal{T}$ is closed under isomorphisms by definition, thus a triangle isomorphic to a distinguished triangle is as well.

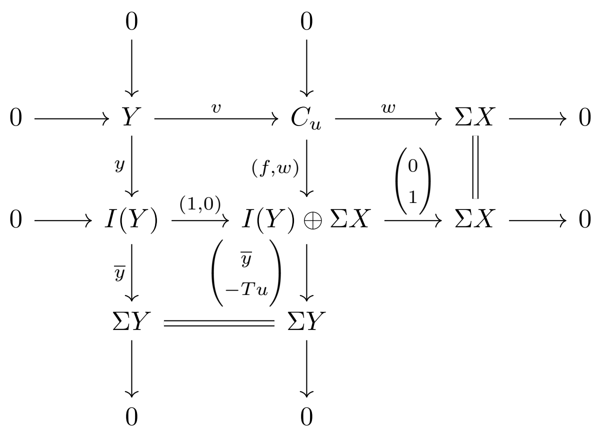
(TR2) Every morphism can also by definition be embedded into a triangle.

(TR3) It easily seen that it suffices to consider the case of a standard triangle. Suppose that $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X$ is a standard triangle. Let $0 \longrightarrow Y \xrightarrow{y} I(Y) \xrightarrow{\overline{y}} \Sigma Y \longrightarrow 0$ be in $\mathcal{E}$ with $I(Y)$ being $\mathcal{E}$-injective. There exists I_u such that $I_u \circ x = y \circ u$ and there exists Σu such that $\overline{y} \circ I_u = \Sigma u \circ \overline{x}$. Define $f : C_u \to I(Y)$ by using the pushout property:



thus $y = f \circ v$, $I_u = f \circ \overline{u}$. Since $\overline{y} \circ f \circ v = \overline{y} \circ y = 0 = \Sigma u \circ w \circ v$, and $\overline{y} \circ f \circ \overline{u} = \overline{y} \circ I_u = \Sigma u \circ \overline{x} = \Sigma u \circ w \circ \overline{u}$, we infer that $\overline{y} \circ f = \Sigma u \circ w$, for a pushout C_u.

In this way, we obtain the following commutative diagram:



The upper two rows are exact, therefore the middle one is the induced exact sequence of the upper one, in particular the left upper square is a pushout. Therefore, $Y \xrightarrow{v} C_u \xrightarrow{(f,w)} I(Y) \oplus \Sigma X \xrightarrow{\begin{pmatrix} \overline{y} \\ -\Sigma u \end{pmatrix}} \Sigma Y$ is a standard sextuplet, which is obviously isomorphic in $\underline{\mathcal{B}}$ to $Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} \Sigma X \xrightarrow{-\Sigma \underline{u}} \Sigma Y$.

(TR4) Let us first consider the case of standard triangles. Consider the following two standard sextuplets:

$$
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle v} \\
I(X) & \xrightarrow[\overline{u}]{} & C_u \\
\downarrow{\scriptstyle \overline{x}} & & \downarrow{\scriptstyle w} \\
\Sigma X & = & \Sigma X
\end{array}
\qquad\qquad
\begin{array}{ccc}
X' & \xrightarrow{u'} & Y' \\
{\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle v'} \\
I(X') & \xrightarrow[\overline{u}']{} & C_{u'} \\
{\scriptstyle \overline{x}'}\downarrow & & {\scriptstyle w'}\downarrow \\
\Sigma X' & = & \Sigma X'
\end{array}
$$

and two morphisms f, g such that $\underline{u' \circ f} = \underline{g \circ u}$ in $\underline{\mathcal{B}}$. Then there exists $\alpha : I(X) \to Y'$ such that $g \circ u = u' \circ f + \alpha \circ x$. We have morphisms $I_f : I(X) \to I(X')$ such that $x' \circ f = I_f \circ x$ and $\Sigma f : \Sigma X \to \Sigma X'$ such that $\overline{x}' \circ I_f = \Sigma f \circ \overline{x}$. Thus we obtain morphisms $v' \circ g : Y \to C_{u'}$ and $\overline{u}' \circ I_f + v' \circ \alpha : I(X) \to C_{u'}$ such that $v' \circ g \circ u = v' \circ u' \circ f + v' \circ \alpha \circ x = \overline{u}' \circ x' \circ f + v' \circ \alpha \circ x = (\overline{u}' \circ I_f + v' \circ \alpha) \circ x$. This yields a morphism $h : C_u \to C_{u'}$ such that $h \circ v = v' \circ g$ and $h \circ \overline{u} = \overline{u}' \circ I_f + v' \circ \alpha$, for a pushout C_u. We claim that $w' \circ h = \Sigma f \circ w$. For this it is enough to show that $w' \circ h \circ v = \Sigma f \circ w \circ v$ and $w' \circ h \circ \overline{u} = \Sigma f \circ w \circ \overline{u}$. For the first, observe that $\Sigma f \circ w \circ v = 0$ and $w' \circ h \circ v = w' \circ v' \circ g = 0$. For the second, we have $\Sigma f \circ w \circ \overline{u} = \Sigma f \circ \overline{x} = \overline{x}' \circ I_f = w \circ \overline{u}' \circ I_f = w' \circ \overline{u}' \circ I_f + w' \circ v' \circ \alpha = w' \circ (\overline{u}' \circ I_f + v' \circ \alpha) = w' \circ h \circ \overline{u}$. Thus $(\underline{f}, \underline{g}, \underline{h})$ is a morphism of triangles, or in other words there exists a unique morphism $\underline{h}$ as described in (TR4).

The general case is easily deduced from the previous one. In fact, let $(X, Y, Z, \underline{u}, \underline{v}, \underline{w})$ and $(X', Y', Z', \underline{u}', \underline{v}', \underline{w}')$ be arbitrary elements in $\mathcal{T}$ and f, g two morphisms such that $\underline{u' \circ f} = \underline{g \circ u}$ in $\underline{\mathcal{B}}$. Then we have isomorphisms to the corresponding standard triangles. Using the first part, we obtain the following commutative diagram, where the rows are in $\mathcal{T}$ and $\underline{h}_1, \underline{h}_2$ are isomorphisms:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\underline{u}} & Y & \xrightarrow{\underline{v}} & Z & \xrightarrow{\underline{w}} & \Sigma X \\
\| & & \| & & \downarrow{\scriptstyle \underline{h}_1} & & \| \\
X & \xrightarrow{\underline{u}} & Y & \xrightarrow{\underline{\tilde{v}}} & C_u & \xrightarrow{\underline{\tilde{w}}} & \Sigma X \\
\downarrow{\scriptstyle \underline{f}} & & \downarrow{\scriptstyle \underline{g}} & & \downarrow{\scriptstyle \underline{h}} & & \downarrow{\scriptstyle \Sigma \underline{f}} \\
X' & \xrightarrow{\underline{u}'} & Y' & \xrightarrow{\underline{\tilde{v}}'} & C_{u'} & \xrightarrow{\underline{\tilde{w}}'} & \Sigma X' \\
\| & & \| & & \uparrow{\scriptstyle \underline{h}_2} & & \| \\
X' & \xrightarrow{\underline{u}'} & Y' & \xrightarrow{\underline{v}'} & Z' & \xrightarrow{\underline{w}'} & \Sigma X'
\end{array}
$$

Then $(\underline{f}, \underline{g}, \underline{h}_2^{-1} \circ \underline{h} \circ \underline{h}_1)$ is a morphism of triangles. In fact, $\underline{h}_2^{-1} \circ \underline{h} \circ \underline{h}_1 \circ \underline{v} = \underline{h}_2^{-1} \circ \underline{h} \circ \underline{\tilde{v}} = \underline{h}_2^{-1} \circ \underline{\tilde{v}}' \circ \underline{g} = \underline{v}' \circ \underline{g}$ and $\underline{w}' \circ \underline{h}_2^{-1} \circ \underline{h} \circ \underline{h}_1 = \underline{\tilde{w}}' \circ \underline{h} \circ \underline{h}_1 = \underline{\Sigma f} \circ \underline{\tilde{w}} \circ \underline{h}_1 = \underline{\Sigma f} \circ \underline{w}$. Thus in general $\underline{h}_2^{-1} \circ \underline{h} \circ \underline{h}_1$ is our unique completion of the corresponding diagram given in (TR4).

(TR5) Again it is enough to consider the case of standard triangles. Suppose we are given three standard sextuplets:

$$
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle i} \\
I(X) & \xrightarrow{\overline{u}} & Z' \\
{\scriptstyle \overline{x}}\downarrow & & \downarrow{\scriptstyle i'} \\
\Sigma X & = & \Sigma X
\end{array}
\qquad
\begin{array}{ccc}
Y & \xrightarrow{v} & Z \\
{\scriptstyle y}\downarrow & & \downarrow{\scriptstyle j} \\
I(Y) & \xrightarrow{\overline{v}} & X' \\
{\scriptstyle \overline{y}}\downarrow & & \downarrow{\scriptstyle j'} \\
\Sigma Y & = & \Sigma Y
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{w} & Z \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle k} \\
I(X) & \xrightarrow{\overline{w}} & Y' \\
{\scriptstyle \overline{x}}\downarrow & & \downarrow{\scriptstyle k'} \\
\Sigma X & = & \Sigma X
\end{array}
$$

with $w = v \circ u$.

Let us replace $I(Y), \Sigma Y, y, \overline{y}$ as follows: Assume we have the sequence $0 \longrightarrow Y \xrightarrow{i} Z' \longrightarrow \Sigma X \longrightarrow 0$ in $\mathcal{E}$, and consider a sequence $0 \longrightarrow Z' \longrightarrow I(Z') \xrightarrow{\overline{l}} \Sigma Z' \longrightarrow 0$ in $\mathcal{E}$ with $I(Z')$ being $\mathcal{E}$-injective. Consider the following commutative diagram of exact sequences in $\mathcal{A}$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{i} & Z' & \longrightarrow & \Sigma X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle l} & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{l \circ i} & I(Z') & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Sigma Z' & = & \Sigma Z' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathcal{B}$ is closed under extensions in $\mathcal{A}$, we infer that the second row belongs to $\mathcal{E}$. Thus instead of $0 \longrightarrow Y \xrightarrow{y} I(Y) \xrightarrow{\overline{y}} \Sigma Y \longrightarrow 0$, we may take

$0 \longrightarrow Y \xrightarrow{i} I(Z') \longrightarrow M \longrightarrow 0$.

Changing the notation, we may assume that $I(Y) = I(Z')$ and $y = l \circ i$. Since $y \circ u = l \circ i \circ u = l \circ \overline{x}$, we denote $l \circ \overline{u}$ by I_u, and define by $\Sigma u \circ \overline{x} = \overline{y} \circ I_u = \overline{y} \circ l \circ \overline{u}$. The identity map $I(Y) = I(Z')$ can be denoted by I_i, for $l \circ i = id_{Z'} \circ y = I_i \circ y$, and there is the induced map $\Sigma i : \Sigma Y \to \Sigma Z'$ such that $\overline{l} = \overline{l} \circ I_i = \Sigma i \circ \overline{y}$. Since $\overline{w} \circ x = k \circ w = k \circ v \circ u$, there exists $f : Z' \to Y'$ such that $f \circ \overline{u} = \overline{w}$ and $f \circ i = k \circ v$, using the pushout property of Z'. Similarly, $j \circ w = j \circ v \circ u = \overline{v} \circ y \circ u = \overline{v} \circ l \circ i \circ u = \overline{v} \circ l \circ \overline{u} \circ x$, shows that there exists $g : Y' \to X'$ such that $g \circ \overline{w} = \overline{v} \circ l \circ \overline{u}$ and $g \circ k = j$, using this time the pushout property of Y'.

We claim that $g \circ f = \overline{v} \circ l$. For this it is enough to show that $g \circ f \circ i = \overline{v} \circ l \circ i$ and $g \circ f \circ \overline{u} = \overline{v} \circ l \circ \overline{u}$. In fact, $g \circ f \circ i = g \circ k \circ v = j \circ v = \overline{v} \circ y = \overline{v} \circ l \circ i$, and

$g \circ f \circ \overline{u} = g \circ \overline{w} = \overline{v} \circ l \circ \overline{u}$. Altogether, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
{\scriptstyle x}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle k}\downarrow \\
I(X) & \xrightarrow{\overline{u}} & Z' & \xrightarrow{f} & Y' \\
{\scriptstyle \overline{x}}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle g}\downarrow \\
\Sigma X & & I(Z') & \xrightarrow{\overline{v}} & X' \\
& \searrow^{Tu} & {\scriptstyle \overline{y}}\downarrow & & \downarrow \\
& & \Sigma Y & \longrightarrow & \Sigma Z'
\end{array}$$

with $f \circ \overline{u} = \overline{w}$, $l \circ i = y$ and $g \circ k = j$.

Let us first check the various relations to be satisfied. By construction $f \circ i = k \circ v$ and $g \circ k = j$.

Let us show next that $k' \circ f = i'$. For this it is enough to show that $k' \circ f \circ i = i' \circ i$ and $k' \circ f \circ \overline{u} = i' \circ \overline{u}$ using the pushout property of Z'. In fact, $k' \circ f \circ i = k' \circ k \circ v = 0 = i' \circ i$, and $k' \circ f \circ \overline{u} = k' \circ \overline{w} = \overline{x} = i' \circ \overline{u}$.

Finally let us show that $\Sigma u \circ k' = j' \circ g$. For this it is enough to show that $\Sigma u \circ k' \circ k = j' \circ g \circ k$, and $\Sigma u \circ k' \circ \overline{w} = j' \circ g \circ \overline{w}$ using this time the pushout property of Y'. In fact $\Sigma u \circ k' \circ k = 0 = j' \circ j = j' \circ g \circ k$, and $j' \circ g \circ \overline{w} = j' \circ g \circ f \circ u \circ = j' \circ \overline{v} \circ l \circ \overline{u} = \overline{y} \circ l \circ \overline{u} = \Sigma u \circ \overline{x} = \Sigma u \circ k' \circ \overline{w}$.

It remains to be seen that $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{\Sigma i \circ j'} \Sigma Z'$ is a standard sextuplet. Since X' is a pushout of $l \circ i$ and v, and Y' is a pushout of i and v, it follows that the following diagram is a pushout:

$$\begin{array}{ccc}
Z' & \xrightarrow{f} & Y' \\
{\scriptstyle l}\downarrow & & \downarrow{\scriptstyle g} \\
I(Z') & \xrightarrow{\overline{v}} & X'
\end{array}$$

Recall that $\bar{l} = \Sigma i \circ \overline{y}$, thus $\bar{l} = \Sigma i \circ \overline{y} = \Sigma i \circ j' \circ \overline{v}$. Therefore we obtain the following commutative diagram with columns in $\mathcal{E}$:

$$\begin{array}{ccc}
Z' & \xrightarrow{f} & Y' \\
{\scriptstyle l}\downarrow & & \downarrow{\scriptstyle g} \\
I(Z') & \xrightarrow{\overline{v}} & X' \\
{\scriptstyle \bar{l}}\downarrow & & \downarrow{\scriptstyle \Sigma i \circ j'} \\
\Sigma Z' & \longrightarrow & \Sigma Z'
\end{array}$$

Hence $(Z', Y', X', f, g, \Sigma i \circ j')$ is a standard sextuplet.

Thus we have now showed that $\mathcal{T}$ is a triangulation of $\underline{\mathcal{B}}$.

Let us now show that the elements of $\mathcal{E}$ does indeed give rise to triangles in $\underline{\mathcal{B}}$.

Consider the following two exact sequences $0 \longrightarrow X \xrightarrow{y} Y \xrightarrow{v} Z \longrightarrow 0$ and $0 \longrightarrow Y \xrightarrow{y} I(Y) \xrightarrow{\overline{y}} \Sigma Y \longrightarrow 0$ in $\mathcal{E}$, with $I(Y)$ being $\mathcal{E}$-injective. They induce the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle y} & & \big\downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & X & \xrightarrow{y \circ u} & I(Y) & \xrightarrow{p} & \Sigma X & \longrightarrow & 0
\end{array}$$

Let us now show that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} \Sigma X$ belongs to $\mathcal{T}$.

By construction the following square is a pullback:

$$\begin{array}{ccc}
Y & \xrightarrow{v} & Z \\
{\scriptstyle y}\big\downarrow & & \big\downarrow{\scriptstyle w} \\
I(Y) & \xrightarrow{p} & \Sigma X
\end{array}$$

This induces the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \big\downarrow & & \big\downarrow & & & & \\
0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle y \circ u}\big\downarrow & & {\scriptstyle (v,i)}\big\downarrow & & \Big\| & & \\
0 & \longrightarrow & I(Y) & \xrightarrow{(0,1)} & Z \oplus I(Y) & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle p}\big\downarrow & \begin{pmatrix} -w \\ p \end{pmatrix} & \big\downarrow & & & & \\
 & & \Sigma X & = & \Sigma X & & & & \\
 & & \big\downarrow & & \big\downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The assertion now follows as in the proof of (TR2).

Thus $\underline{\mathcal{B}}$ is a triangulated category. $\square$

7 Derived Categories

Definition 7.1 (Category of complexes [Kra21, Ch. 4.1])**.** A *cochain complex* (or simply a complex) in $\mathcal{A}$, an Abelian category, is a sequence of morphisms

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots$$

such that $d^n \circ d^{n-1} = 0, \forall n \in \mathbb{Z}$. We think of a complex X as a graded object with differential d, and refer to n as the degree. We denote by $\mathbf{C}(\mathcal{A})$ the category of complexes, where a morphism $\phi : X \longrightarrow Y$ between complexes consists of morphisms $\phi^n : X^n \longrightarrow Y^n$ with $d_Y^n \circ \phi^n = \phi^{n+1} \circ d_X^n, \forall n \in \mathbb{Z}$.

Definition 7.2 (Null-homotopic [Kra21, Ch. 4.1])**.** A morphism $\phi : X \longrightarrow Y$ is *null-homotopic* if there are morphisms $\rho^n : X^n \longrightarrow Y^{n-1}$ such that $\phi^n = d_Y^{n-1} \circ \rho^n + \rho^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$
The null-homotopic morphisms form an ideal $\mathfrak{I}$ in $\mathbf{C}(\mathcal{A})$.

Definition 7.3. [Kra21, chap. 4.1] The *homotopy category* $\mathbf{K}(\mathcal{A})$ is the quotient of $\mathbf{C}(\mathcal{A})$ with respect to this ideal. Thus

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y) = \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X, Y)/\mathfrak{I}(X, Y)$$

Definition 7.4 (Mapping cone [Ste98, Ch.2.3])**.** Let $X, Y \in \mathbf{C}(\mathcal{A})$ and $f : X \to Y$ in $\mathbf{C}(\mathcal{A})$. The *mapping cone* of f is the following complex in $\mathbf{C}(\mathcal{A})$: For each $n \in \mathbb{Z}$, we define

$$M(f)^n := X^{n+1} \oplus Y^n \text{ and } d^n_{M(f)} := \begin{pmatrix} (-1)^n d_X^{n+1} & 0 \\ f^{n+1} & (-1)^n d_Y^n \end{pmatrix}.$$

We can identify the relation between the mapping cone $M(f)$ and X, Y by defining $\alpha(f) : Y \to M(f)$ by $\alpha(f) := \begin{pmatrix} 0 \\ id_{Y^n} \end{pmatrix}$ and $\beta(f) : M(f) \to \Sigma X$ by $\beta(f)^n := \begin{pmatrix} id_{X^{n+1}} & 0 \end{pmatrix}$.

The following lemma is given without proof.

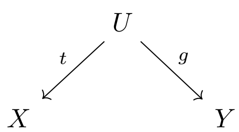
Lemma 7.5. $\mathbf{K}(\mathcal{A})$ with a class of distinguished triangles of the form
$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma X$ is a triangulated category.

Definition 7.6 (Acyclic complex [Chr24])**.** A *acyclic complex* is a complex where it's nth homology group, is trivial, i.e. the quotient group
$H^n(X) := ker(d^n)/im(d^{n-1})$ has only one element for all $n \geq 0$.

Definition 7.7 (Quasi-isomorphism [Kra21, Ch.4.1] [Ste98, Ch.2.5])**.** We denote by $\mathbf{Ac}(\mathcal{A})$ the full subcategory of complexes in $\mathbf{C}(\mathcal{A})$ that are isomorphic to an acyclic complex in $\mathbf{K}(\mathcal{A})$. A morphism of complexes is a *quasi-isomorphism* if its mapping cone is in $\mathbf{Ac}(\mathcal{A})$, and we write **Qis** for the class of all quasi-isomorphism with its mapping cone is in $\mathbf{C}(\mathcal{A})$.
Equivalently, we call a morphism $f : X \to Y$ a quasi-isomorphism if $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism in $\mathcal{A}$ for all integers n.

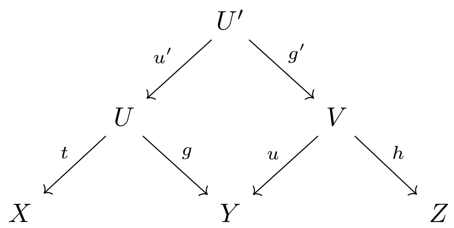
Definition 7.8 (Derived category of an Abelian category [HJ10] [Kra21, chap. 4.1])**.** The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$, an Abelian category, is obtained from $\mathbf{K}(\mathcal{A})$ by quotienting out all exact complexes.

Definition 7.9 (Roofs [Ste98])**.** The morphisms of a derived category can be seen as *roofs*. A roof for example from X to Y, is represented by a triple:



with $t : U \to X$ a quasi-isomorphism and $g : U \to Y$ any morphism. We write $g \cdot t^{-1}$ for these roofs.
Composition of roofs are given such:

Say we have $g \cdot t^{-1} : X \to Y$ and $h \cdot u^{-1} : Y \to Z$. Then by completing g and u to a pullback, we get a roof $g' \cdot u' : U \to V$. By this construction we get that $(h \cdot u^{-1}) \circ (g \cdot t^{-1}) = (h \circ g') \cdot (t \circ u')^{-1}$. This can be illustrated by the diagram:



Theorem 7.10. $\mathbf{D}(\mathcal{A})$ is triangulated.

Proof. Let's first show that $\mathbf{D}(\mathcal{A})$ is actually a category.
The objects are complexes by definition, where all exact complexes from $\mathbf{K}(\mathcal{A})$ have been mapped to the zero complex in $\mathbf{D}(\mathcal{A})$.
The morphisms are roofs as described above.
It's easily seen that composition is preserved under the collection of exact complexes in $\mathbf{K}(\mathcal{A})$, and that this collection constitutes an equivalence class. Using the fact that the quotient category is a category, we get that $\mathbf{D}(\mathcal{A})$ is indeed a category.

Now let's show the triangulated structure.
We claim that $\mathbf{D}(\mathcal{A})$ with a class of distinguished triangles consisting of all triangles isomorphic to distinguished triangles $X \xrightarrow{f} Y \longrightarrow M(f) \longrightarrow \Sigma X$, is a triangulated category.
Given that we know $\mathbf{K}(\mathcal{A})$ is triangulated, we will inherit the shift functor from our construction there. Observe that the shift of a quasi-isomorphism is then also an quasi-isomorphism.

Now we check for the axioms of distinguished triangles.

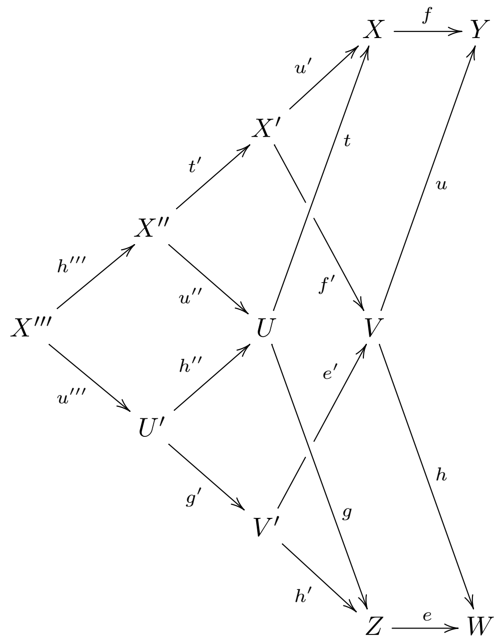
(TR0) The statement follows from the homotopy category $\mathbf{K}(\mathcal{A})$ being triangulated.

(TR1) Is evident by construction.

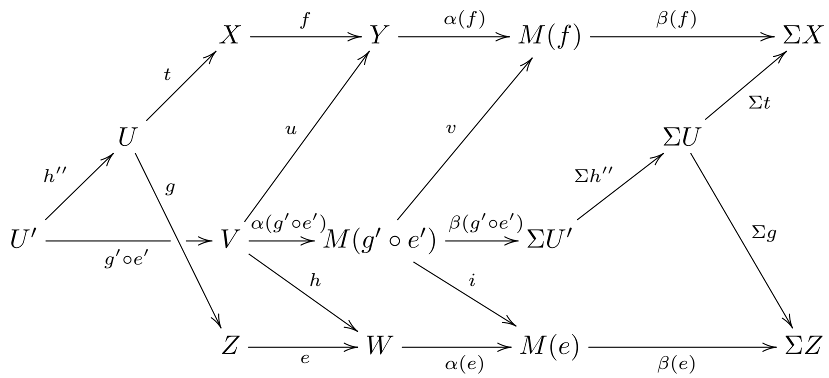
(TR2) We can complete a given g to a distinguished triangle as it lives in $\mathbf{K}(\mathcal{A})$, then we alter it by t.

(TR3) Say we have a distinguished triangle $(f, \alpha(f), \beta(f))$. We know for $\mathbf{K}(\mathcal{A})$ that $(\alpha(f), \beta(f), -\Sigma f)$ is isomorphic to it, and thus also a distinguished triangle. Therefore the same holds in $\mathbf{D}(\mathcal{A})$ as well.

(TR4) Assume we have distinguished triangles $(f, \alpha(f), \beta(f))$ and $(e, \alpha(e), \beta(e))$ with roofs $g \cdot t^{-1} : X \to Z$ and $h \cdot u^{-1} : Y \to W$. Now complete f and u to a pullback giving the roof $f' \cdot u'^{-1} : X \to V$, and then do the same for h and e giving the roof $h' \cdot e'^{-1} : V \to X'$. Then repeat, giving $u'' \cdot t'^{-1} : U \to X'$ for t and u', and $g' \cdot h''^{-1} : U \to V'$ for g and h'. Now do it a last time for u'' and h'' giving $u''' \cdot h'''^{-1} : X'' \to U'$. This gives us the following commuting diagram:



Observe that we get the roof $(g' \circ e') \cdot h''^{-1} : U \to V$. Now complete $(g' \circ e')$ to a triangle $((g' \circ e'), \alpha((g' \circ e')), \beta((g' \circ e')))$. Applying the shift functor to the morphism h'' gives a morphism $\Sigma h'' : \Sigma U' \to \Sigma U$. As a distinguished triangle in $\mathbf{D}(\mathcal{A})$ is distinguished in $\mathbf{K}(\mathcal{A})$, we can use (TR4) from the $\mathbf{K}(\mathcal{A})$ to find morphisms v and i making the following diagram commute:



It can then be showed that since u and $h'' \circ t$ are quasi-isomorphisms, then v is such as well. This gives us the roof $i \cdot v^{-1} : M(f) \to M(e)$.

(TR5) The same argument can be made as in (TR3) as all the distinguished triangles in question lie in $\mathbf{K}(\mathcal{A})$, and follows thus from the octahedral axiom in $\mathbf{K}(\mathcal{A})$.

□

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