Henrik Knudsen

Contramodules

Bachelor's thesis in BMAT
Supervisor: Torgeir Aambø
June 2024

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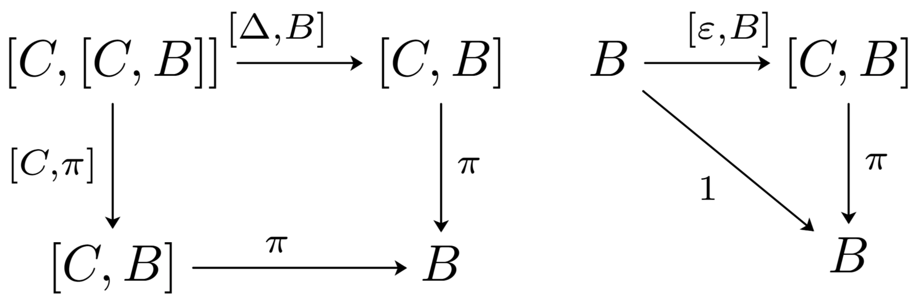
NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences



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Abstract

In this thesis we construct contramodules, both classically and in arbitrary closed symmetric monoidal categories. We then construct the categories of comodules and contramodules through categories of comodules/modules over comonads/monads. We finally relate these categories through the co-contra correspondence, based on the work of Hristova, Jones and Rumynin.

Sammendrag

I denne oppgaven konstruerer vi kontramoduler, både på klassisk vis og i arbitrære lukkede symmetrisk monoidale kategorier. Vi konstruerer deretter kategoriene av komoduler og kontramoduler via kategorier av komoduler/moduler over komonader/monader. Til slutt sammenligner vi disse kategoriene via ko-kontra korrespondansen, basert på arbedene til Hristova, Jones og Rumynin.

Contents

1 Contramodules

Contramodules were introduced by Eilenberg-Moore in 1965 [1], but mostly went under the radar until Positselski picked it back up relatively recently [2]. They are an adjoint dual to modules in closed symmetric monoidal categories, but are only in some cases equivalent to comodules.

1.1 Contramodules in vector spaces

We begin by defining contramodules over a coalgebra C over a commutative ring k the way Eilenberg-Moore first did in 1965 [1]. We will later see that this is easily generalizable to monodial categories.

Definition 1.1. A *coassociative coalgebra* C over a commutative ring k is a k-module with k-linear maps $\Delta : C \longrightarrow C \otimes_k C$ and $\varepsilon : C \longrightarrow k$ such that the following diagrams commute.

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{id}} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \varepsilon\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\varepsilon} & k \otimes C \cong C \cong C \otimes k \end{array} \tag{1}$$

In the first diagram $C \otimes (C \otimes C)$ is associated with $(C \otimes C) \otimes C$ as they are naturally isomorphic. These operations, *comultiplication* and the *counit*, are dual to the multiplication and unit maps of an associative algebra.

Dually to modules over an associative algebra one can define comodules over a coassociative coalgebra.

Definition 1.2. A right *comodule* X over a coalgebra C is a vector space over k with a structure map $\rho : X \longrightarrow X \otimes C$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & X \otimes C \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \rho\otimes\mathrm{id}} \\ X \otimes C & \xrightarrow{\mathrm{id}\otimes\Delta} & X \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\rho} & X \otimes C \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \mathrm{id}\otimes\varepsilon} \\ & & X \otimes k \cong X \end{array} \tag{2}$$

These conditions are called *coassociativity* and *counitality*, and are dual to the associativity and unitality of a module over an algebra.

For a module M over an associative algebra A, there is a structure map $n : A \otimes_k M \longrightarrow M$ with the aforementioned associativity and unitality conditions. By the Hom-tensor adjunction

$$\mathrm{Hom}_k(A \otimes_k M, M) \simeq \mathrm{Hom}_k(M, \mathrm{Hom}_k(A, M))$$

this is the same thing as having a map $p : M \longrightarrow \mathrm{Hom}_k(A, M)$ such that the following associativity and unitality diagrams commute.

$$
\begin{array}{ccc}
M & \xrightarrow{\quad p \quad} & \mathrm{Hom}_k(A, M) \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(m,\mathrm{id})} \\
\mathrm{Hom}_k(A, M) & \xrightarrow{\mathrm{Hom}(\mathrm{id},p)} & \mathrm{Hom}_k(A, \mathrm{Hom}_k(A, M))
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\ p\ } & \mathrm{Hom}_k(A, M) \\
 & {\scriptstyle \mathrm{id}}\searrow & \big\downarrow{\scriptstyle \mathrm{Hom}(e,\mathrm{id})} \\
 & & M
\end{array}
\tag{3}
$$

Here m is the multiplication from the algebra and e is the unit map.

Observe that we get two identifications via the Hom-tensor adjunction:

$$\mathrm{Hom}_k(U \otimes_k V, W) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W)) \tag{4}$$

and

$$\mathrm{Hom}_k(U \otimes_k V, W) \simeq \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \tag{5}$$

By using the first we obtain a left module structure, and by using the second we obtain a right module structure.

We now dualize this to arrive at our definition.

Definition 1.3. A (left) *contramodule* B over a coalgebra C with comultiplication μ and counit map ε, is a vector space over k with a (left) *contraaction* map $\pi : \mathrm{Hom}_k(C, B) \longrightarrow B$, such that the following diagrams commute.

$$
\begin{array}{ccc}
\mathrm{Hom}_k(C, \mathrm{Hom}_k(C, B)) & \xrightarrow{\mathrm{Hom}(\mu,B)} & \mathrm{Hom}_k(C, B) \\
{\scriptstyle \mathrm{Hom}(C,\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathrm{Hom}_k(C, B) & \xrightarrow{\quad \pi \quad} & B
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{\mathrm{Hom}(\varepsilon,B)} & \mathrm{Hom}_k(C, B) \\
 & {\scriptstyle \mathrm{id}}\searrow & \big\downarrow{\scriptstyle \pi} \\
 & & B
\end{array}
$$

where we in the first diagram identify

$$\mathrm{Hom}_k(C \otimes_k C, B) \simeq \mathrm{Hom}_k(C, \mathrm{Hom}_k(C, B))$$

by (4). Using (5) instead gives a right contramodule. These conditions are called *contraassociativity* and *contraunitality*.

1.2 Contramodules in general categories

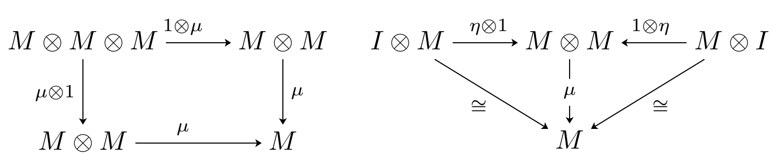
We now generalize the two types of modules—comodules and contramodules—discussed in Section 1.1 to closed symmetric monoidal categories.

Let $\mathcal{C}$ be a closed symmetric monoidal category. This means it has a symmetric, associative bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ with unit element I. It has a left adjoint, $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the *internal hom bifunctor*. In particular, there is a natural isomorphism of functors of X

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes A, B) \cong \mathrm{Hom}_{\mathcal{C}}(X, [A, B]).$$

Since the diagrams in (1) only use the tensor product, we can use them to define a *comonoid* in a symmetric monoidal category. We can then use the diagrams in (2) and (3), replacing $\mathrm{Hom}_k(-,-)$ with $[-,-]$ to define co- and contramodules over these comonoids.

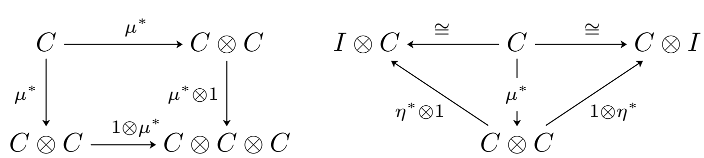
Definition 1.4. A monoid in a monoidal category $\mathcal{C}$ is an object $M \in \mathcal{C}_0$ with a multiplication map $\mu : M \otimes M \longrightarrow M$ and a unit map $\eta : I \longrightarrow M$ such that the following diagrams commute.



Here we have omitted the associator and unitors.

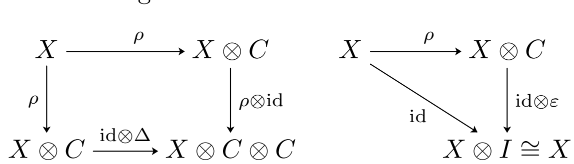
Dually we have the following.

Definition 1.5. A comonoid in a monoidal category is an object $C \in \mathcal{C}_0$ with maps $\mu^* : C \longrightarrow C \otimes C$ and $\eta^* : C \longrightarrow I$ such that the following diagrams commute:



Comodules over comonoids work very similarly to comodules over coassociative coalgebras as we saw them earlier.

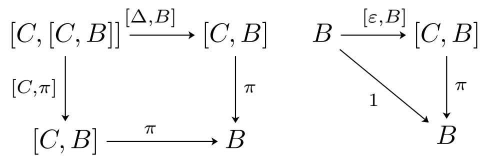
Definition 1.6. Let $\mathcal{C}$ be a monoidal category and $C \in \mathcal{C}$ a comonoid. A C-comodule is an object $X \in \mathcal{C}$ with a map $\rho : X \longrightarrow X \otimes C$ subject to the following commutative diagrams.



The category of C-comodules is denoted $\mathcal{C}_C$.

For contramodules, we replace $\mathrm{Hom}_{\mathcal{C}}(-,-)$ with $[-,-]$ everywhere.

Definition 1.7. Let $\mathcal{C}$ be a closed symmetric monoidal category and $C \in \mathcal{C}$ a comonoid. A (general) C-contramodule B is an object in $\mathcal{C}$ with a map $\pi : [C, B] \longrightarrow B$ such that the following diagrams commute.



The category of contramodules over C is denoted by $\mathcal{C}^C$.

This definition is in fact a generalization of Definition 1.3 by the following result.

Theorem 1.8. *Contramodules over vector spaces as constructed in Definition 1.3 are an example of general contramodules.*

Proof. The category of vector spaces over k is a closed symmetric monoidal category, where the with monoidal product is $- \otimes_k -$, and the internal hom is given by $\mathrm{Hom}_k(-,-)$.

A comonoid in $\boldsymbol{Vect}_k$ is precisely a coassociative coalgebra, hence the two definitions coincide by identifying the commutative diagram. ☐

2 Examples

We illustrate the concept of co- and contramodules in ***Set***, the category of sets, and ***sSet***, the category of simplicial sets, following [3].

2.1 In sets

The category ***Set*** is a symmetric monoidal closed category. The monoidal product is the Cartesian product, and internal hom is the set of functions. The unit is the set with one element, $\{*\}$ which will simply be denoted by $*$. Since internal and external hom agree (canonically, since the internal hom is already a set), we will refer to it as $[X, Y]$.

Lemma 2.1. *Any set $C \in \boldsymbol{Set}$ can uniquely be given the structure of a comonoid.*

Proof. The comultiplication is given by the diagonal map $\Delta : C \longrightarrow C \times C$ defined by $c \mapsto (c, c)$, and the counit map is the unique map $\varepsilon : C \longrightarrow *$. The comonoid structure is unique since by the counitality axiom, any comultiplication $\psi = (\psi_1, \psi_2) : C \longrightarrow C \times C$, both ψ_1 and ψ_2 will be the identity, so ψ is the diagonal map. ☐

By Definition 1.2, a (right) C-comodule is a set X with a map $\rho : X \longrightarrow X \times C$ satisfying the coassociativity and counitality conditions. We let $\boldsymbol{Set}_C$ denote the category of comodules over the comonoid C.

From Definition 1.2 we get the following diagram of coassociativity:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & X \times C \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow {\scriptstyle \mathrm{id} \times \varepsilon} \\ & & X \times k \end{array} \tag{6}$$

Since ***Set*** is Cartesian, we can describe ρ as the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times \phi} X \times C$$

for a unique map $\phi : X \longrightarrow C$. Here Δ is the diagonal map described above.

There is an alternative way of describing the category $\boldsymbol{Set}_C$, via the category of sets over C, denoted by $(\boldsymbol{Set} \downarrow C)$.

Objects of this category are pairs (X, ϕ_X), where X is a set and $\phi_X : X \longrightarrow C$ is a function. A morphism $f \in \mathrm{Hom}_{(\boldsymbol{Set} \downarrow C)}(X, Y)$ is a function $f : X \longrightarrow Y$ such that the following square commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \phi_X} \downarrow & & \downarrow {\scriptstyle \phi_Y} \\ C & \xrightarrow{1} & C \end{array} \tag{7}$$

Clearly, a set over C can be given a C-comodule structure by setting $\rho = 1 \times \phi_X : X \longrightarrow X \times C$. In fact, all C-comodules can be created this way, as the following proposition shows.

Proposition 2.2. *There is an equivalence of categories* $(\boldsymbol{Set} \downarrow C) \longrightarrow \boldsymbol{Set}_C$*, given by sending an object* $X \xrightarrow{\phi_X} C$ *in* $(\boldsymbol{Set} \downarrow C)$ *to the comodule* X*, where the comodule structure is given as above.*

Proof. $(X, \rho) = (X, 1 \otimes \phi_X) \mapsto (X, \phi)$ clearly gives an isomorphism between C-comodules and sets over C. □

We have a special type of comodules in $\boldsymbol{Set}$ which are easy to describe, and will be useful in describing the co-contra correspondence later.

Definition 2.3. Let $X = \coprod_{a \in C} X_a$ be a disjoint union of a family of sets indexed by C. Then X has a map $\phi : X \longrightarrow C$ defined by

$$\phi(x) = a \text{ for all } x \in X_a$$

which defines a C-comodule structure on X. We call this a *coproduct comodule*.

Proposition 2.4. *Every* C*-comodule* X *in* $\boldsymbol{Set}$ *is canonically isomorphic to a coproduct comodule.*

Proof. Let X be a C-comodule in $\boldsymbol{Set}$. For every $a \in C$, let X_a be the set $\phi_X^{-1}(a)$ where ϕ_X is the structure map when X is regarded as a set over C. $\coprod_{a \in C} X_a$ is then a coproduct module which is isomorphic to X. □

Definition 2.5. If the structure map ϕ of a set over C is surjective, we call the corresponding comodule *nondegenerate*.

We now turn our attention to contramodules. From Definition 1.7, a contramodule in $\boldsymbol{Set}$ over a comonoid C is a set X, together with a function $\theta : [C, X] \longrightarrow X$ satisfying the contraassociativity and contraunitality conditions. Since $[C, X] = \prod_{a \in C} X$ we can think of a function $\beta : C \longrightarrow X$ as a list of elements $(\beta(a))_{a \in C}$ in X. $\theta(\beta)$ is then the θ-product of the possibly infinite list of elements $(\beta(a))_{a \in C}$ in X.

The obvious choice for counit map is to send an element $x \in X$ to the map $f : C \longrightarrow X$, defined by $a \mapsto x$. Thus every constant map $g(a) = x, \forall a \in C$, gets sent to x by the contraaction map.

For contraassociativity, consider a map $\gamma : C \times C \longrightarrow X$, which is an element in the top left of the square in Definition 1.3 up to isomorphism coming from the hom-tensor adjunction. We can consider this as a matrix of elements of X, indexed row- and columnwise by elements in C. The row fixed by an element $a \in C$ gives rise to a function

$$r_a(\gamma) : C \longrightarrow X, \quad b \mapsto \gamma(a, b),$$

which lets us define a function

$$\rho_\gamma : C \longrightarrow X \quad a \mapsto \theta(r_a(\gamma)),$$

which we call the *row function*. We also have the *diagonal function*, defined by

$$\delta_\gamma : C \longrightarrow X, \quad a \mapsto \gamma(a, a).$$

Now contraassociativity is the condition that $\phi(\rho_\gamma) = \phi(\delta_\gamma)$ which is referred to as the *row-diagonal identity*.

Example 2.6. Let C be a set with 3 elements. We associate $\theta : [C, X] \longrightarrow X$ with a function $\theta' : X \times X \times X \longrightarrow X$, and write $\gamma : C \times C \longrightarrow X$ as a matrix with coefficients in X:

$$\gamma = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Thus, the row-diagonal identity is

$$\theta'(\theta'(x_{11}, x_{12}, x_{13}), \theta'(x_{21}, x_{22}, x_{23}), \theta'(x_{31}, x_{32}, x_{33})) = \theta'(x_{11}, x_{22}, x_{33})$$

We will now show that every contramodule in ***Set*** can be uniquely expressed as a product. We start by showing that a product actually admits a contramodule structure.

Theorem 2.7. *Every product of sets indexed over a comonoid C is a contramodule over C.*

Proof. Let $X = \prod_{a \in C} X_a$ and $Y = \coprod_{a \in C} X_a$. Define $\phi : Y \longrightarrow C$ by $y \in X_a \mapsto a$, which makes (Y, ϕ) a set over C. Now let $[C, Y]_C$ be the set of sections of ϕ. However, $[C, Y]_C = \prod_{a \in C} X_a = X$. Thus we only need to create a contramodule structure on $[C, Y]_C$.

What we need is a structure map $\theta : [C, [C, Y]_C] \longrightarrow [C, Y]_C$. A map $C \longrightarrow [C, Y]_C$ is a list $\tau = (\tau_a)_{a \in C}$ of sections of ϕ. Then we define $\theta(\tau) \in [C, Y]_C$ to be the map $\theta(\tau)(a) = \tau_a(a)$, which does satisfy the contraassociativity and contraunitality conditions. □

It is in fact true that any contramodule is a product contramodule.

Theorem 2.8. *Every contramodule is isomorphic to a product contramodule.*

Proof. The proof works by first showing it for a contramodule X over the comonoid with 2 elements. We construct maps $\pi_1, \pi_2 : X \longrightarrow X$ and show that $\pi = (\pi_1, \pi_2) : X \longrightarrow X \times X$ is an isomorphism of contramodules. This is then generalized to comonoids with arbitrary elements. See [3, Section 3.6] for a thorough proof. □

2.2 In simplicial sets

Recall that a simplicial set is a functor $\Delta^{op} \longrightarrow \boldsymbol{Set}$, where Δ is the simplex category consisting of lists of numbers $[n] = \{0, 1, 2, \ldots, n\}$ and order preserving maps. For a comprehensive treatment, see [4].

The category of simplicial sets, $\boldsymbol{sSet}$ is a Cartesian closed symmetric monoidal category, so it admits the internal hom as a right adjoint to the product. The construction is as follows:

Construction 2.9. We want an adjunction

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, [Y, Z])$$

Letting $X = \Delta^n$ be representable we get

$$\mathrm{Hom}(\Delta^n \times Y, Z) \cong \mathrm{Hom}(\Delta^n, [Y, Z]) \cong ([Y, Z])_n$$

by applying the Yoneda lemma to get the last isomorphism. Thus, the set of n-simplices is the set of maps $\Delta^n \times Y \longrightarrow Z$. We then use the Density theorem [5, Thm. III.7.1], which says that every presheaf of sets is isomorphic to a colimit of representables. This gives us the full adjunction.

A comonoid in $\boldsymbol{sSet}$ is a simplicial set $C = (C_n)$ where the comultiplication is the diagonal map. As in $\boldsymbol{Set}$, $\boldsymbol{sSet}_C$ is isomorphic to $(\boldsymbol{sSet} \downarrow C)$. As such, a C-comodule is a simplicial set with a C_n set structure at each level n compatible with order-preserving maps.

A contramodule in $\boldsymbol{sSet}$ is a simplicial set $X = (X_n)$ with a structure map

$$\theta : [C, X] \longrightarrow X, \quad \theta_n : \mathrm{Hom}_{\boldsymbol{sSet}}(C \times \Delta[n], X) \longrightarrow X_n$$

for each n.

3 Enriched Category Theory

We give an introduction to enriched category theory, adjunctions, monads and modules over them. This is useful as it gives us another way to construct the categories of co- and contramodules in closed symmetric monoidal categories.

This section uses some 2-category theory, and we refer the readers who might be unfamiliar to this concept to Appendix A.1.

3.1 Definition and examples

Definition 3.1. Given a symmetric monoidal category $\mathcal{V}$, a *$\mathcal{V}$-category* $\mathcal{A}$, also called a *category enriched over* $\mathcal{V}$, consists of

- A class of objects $\mathcal{A}_0$
- For any two $X, Y \in \mathcal{A}_0$, a *hom-object* $\mathrm{Hom}(X,Y) \in \mathcal{V}_0$
- For each $X \in \mathcal{A}_0$, a map $\mathrm{id}_x : * \longrightarrow \mathrm{Hom}(X,X)$ in $\mathcal{V}$
- For each triple $X, Y, Z \in \mathcal{A}_0$, a map $\circ : \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(X,Z)$ in $\mathcal{V}$

such that the following associativity and unity diagrams commute:

$$\begin{array}{ccc} \mathrm{Hom}(Z,W)\times\mathrm{Hom}(Y,Z)\times\mathrm{Hom}(X,Y) & \xrightarrow{1\times\circ} & \mathrm{Hom}(Z,W)\times\mathrm{Hom}(X,Z) \\ \downarrow{\scriptstyle \circ\times 1} & & \downarrow{\scriptstyle \circ} \\ \mathrm{Hom}(Y,W)\times\mathrm{Hom}(X,Y) & \xrightarrow{\circ} & \mathrm{Hom}(X,W) \end{array}$$

$$\begin{array}{ccccc} * \times \mathrm{Hom}(X,Y) & & & & \mathrm{Hom}(X,Y)\times * \\ \downarrow{\scriptstyle \mathrm{id}_Y\times 1} & \searrow{\scriptstyle \cong} & & \swarrow{\scriptstyle \cong} & \downarrow{\scriptstyle 1\times \mathrm{id}_X} \\ \mathrm{Hom}(Y,Y)\times\mathrm{Hom}(X,Y) & \xrightarrow[\circ]{} & \mathrm{Hom}(X,Y) & \xleftarrow[\circ]{} & \mathrm{Hom}(X,Y)\times\mathrm{Hom}(X,X) \end{array}$$

We have omitted the associativity isomorphism from the first diagram.

Note that in the classical source [6], Kelly only presumes $\mathcal{A}_0$ to be a set, and appears to use tensor instead of product in his commutative diagrams. Both of these points are remedied in more recent literature, such as [7], which is also the suggested source for further reading on the concept.

Definition 3.2. For two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a *$\mathcal{V}$-enriched functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of the following:

- A map $F : \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ from the objects of $\mathcal{A}$ to the objects of $\mathcal{B}$.

- Maps on the hom objects

$$H_{X,Y} : [X, Y]_{\mathcal{A}} \longrightarrow [HX, HY]_{\mathcal{B}}$$

that respect enriched composition and unitality.

Example 3.3. The category ***sSet*** is enriched over itself, via the internal hom as constructed in Construction 2.9.

Example 3.4. Given the categories $\mathcal{C}_C$ and $\mathcal{C}^C$, the categories of comodules and contramodules over a comonoid C in some category $\mathcal{C}$, we can construct hom-objects so that they become enriched over $\mathcal{C}$ under the right assumptions. We explore this and more in the next section.

3.2 Adjunctions and monads

We start by defining the three different notions of adjointness for functors on enriched categories.

Definition 3.5. Let $F, G : \mathcal{C} \longrightarrow \mathcal{C}$, not necessarily enriched, for some fixed symmetric monoidal enriched category $\mathcal{C}$.

- If there is a natural isomorphism of functors $\mathrm{Hom}_{\mathcal{C}}(F-, -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G-)$, then F and G are *externally adjoint*.
- If there is a natural isomorphism of functors $[F-, -] \cong [-, G-]$, F and G are *internally adjoint*.
- If F and G are internally adjoint and the natural isomorphism of functors is enriched, F and G are *enriched adjoint*.

In all cases, we denote this $(F \dashv G)$.

Clearly enriched adjoint implies internally adjoint, and internally adjoint implies externally adjoint by using the global sections functor which is defined as follows.

Definition 3.6. The *global sections functor* is defined as

$$\Gamma : \mathcal{C} \longrightarrow \boldsymbol{Set}, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(I, X)$$

It gives us a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \Gamma([X, Y])$$

since the internal hom determines the hom set as follows:

$$\mathrm{Hom}_{\mathcal{C}}(I, [X, Y]) = \mathrm{Hom}_{\mathcal{C}}(I \otimes X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

We can also sometimes move the other way.

Theorem 3.7. *If F, G are enriched functors that are internally adjoint, they are enriched adjoint.*

Proof. We want to show that we have natural isomorphisms such that

$$G \cong [FI, -], \quad F \cong - \otimes FI$$

Using the fact that $X \cong [I, X]$ for all X, we get

$$[FI, -] \xrightarrow{\cong} [I, G(-)] \xrightarrow{\cong} G$$

which when composed we denote $\cong^*$. We assemble this into a diagram:

$$\begin{array}{ccc} [F(X), Y] & \xrightarrow{\cong} & [X, G(Y)] \\ & & \big\downarrow \cong^* \\ [X \otimes FI, Y] & \xrightarrow{\cong} & [X, [FI, Y]] \end{array}$$

where the top and bottom isomorphisms are given by the adjunction $(F \dashv G)$ and the internal hom-tensor adjunction respectively. Thus we only need the left isomorphism which is constructed as the composite

$$[X \otimes FI, Y] \xrightarrow{\cong} [X, [FI, Y]] \xrightarrow{\cong} [X, G(Y)] \xrightarrow{\cong} [F(X), Y]$$

To show $F(-) \cong - \otimes FI$, we use that we have natural isomorphisms of representable functors

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}}(X \otimes FI, -) & & & & \\ \| & & & & \\ \Gamma([X \otimes FI, -]) & \xrightarrow{\cong} & \Gamma([X, G(-)]) & \xrightarrow{\cong} & \Gamma([FX, -]) \\ & & & & \| \\ & & & & \mathrm{Hom}_{\mathcal{C}}(FX, -) \end{array}$$

which by the Yoneda Lemma gives us the natural isomorphism of representing objects natural in X

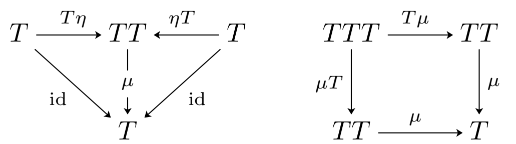
$$X \otimes FI \cong FX$$

Thus, since internal hom and tensor are enriched adjoint, we can change the internal adjunction between F and G to an enriched adjunction. □

3.3 Modules over monads

We now introduce another way to construct $\mathcal{C}_C$ and $\mathcal{C}^C$, via monads and comonads.

Definition 3.8. A *monad* in a category $\mathcal{C}$ is a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$ with natural transformations $\eta : \mathrm{id} \longrightarrow T$ and $\mu : TT \longrightarrow T$ such that the following diagrams commute:



In other words, it is a monoid in the monoidal category of endofunctors $\mathcal{C} \longrightarrow \mathcal{C}$. A *comonad* is then a comonoid in the same category of endofunctors, i.e. a functor with natural transformations $\mu^* : T \longrightarrow TT$ and $\eta^* : T \longrightarrow \mathrm{id}$ subject to commutative diagrams dual to those above.

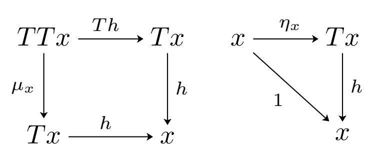
Stated 2-categorically, a monad T in a bicategory K (see Definition A.1) is given by:

- An object $a \in K$
- An endomorphism $T : a \longrightarrow a$
- A 2-morphism $\eta : 1_a \longrightarrow T$
- A 2-morphism $\mu : TT \longrightarrow T$

subject to the same diagrams as above.

The following definition of a module is by Maclane [5].

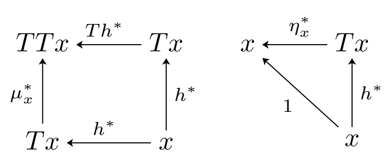
Definition 3.9. A *module over a monad* T in K, also called a T-algebra, is a pair (x, h) consisting of an object $x \in K$ and a map $h : Tx \longrightarrow x$ making the following diagrams commute:



The category of all modules over a monad $T : \mathcal{C} \longrightarrow \mathcal{C}$ is called the *Eilenberg-Moore* category of T and will be denoted $\boldsymbol{Mod}(T)$.

We can also do this dually:

Definition 3.10. A *comodule over a comonad* F in K is a pair (x, h^*) where $x \in K$, $h^* : x \longrightarrow Tx$ such that the following diagrams commute:



The Eilenberg-Moore category of a comonad $F : \mathcal{C} \longrightarrow \mathcal{C}$ is the category of all comodules over it and is denoted $\boldsymbol{Comod}(F)$. Clearly comodules over a monad do not inherently work, nor do modules over a comonad, so whether F is a monad or a comonad will be clear from the context.

Example 3.11. An example of modules (resp. comodules) over a monad (resp. comonad) are the *free modules* (resp. *cofree comodules*). They are on the form $T(X)$ (resp. $F(X)$) for $X \in \mathcal{C}$.

Definition 3.12. The Kleisli category of a monad (resp. comonad) is the full subcategory of the Eilenberg-Moore category consisting of free modules (resp. cofree comodules). They are denoted $\widetilde{C_T}$ and $\widetilde{C^F}$, and are spanned by $T(X)$ and $F(X)$ correspondingly.

Theorem 3.13. *If $(F \dashv G)$ is an adjoint comonad-monad pair, their Kleisli categories are equivalent.*

Proof. This comes from the following chain of isomorphisms for any $X, Y \in \mathcal{C}$:

$$\begin{array}{ccc} \mathrm{Hom}_{\boldsymbol{Comod}(F)}(F_\# X, F_\# Y) \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{C}}(FX, Y) & \\ & \downarrow \cong & \\ & \mathrm{Hom}_{\mathcal{C}}(X, GY) & \xrightarrow{\cong} \mathrm{Hom}_{\boldsymbol{Mod}(G)}(G^\# X, G^\# Y) \end{array}$$

Where $F_\#$ is the cofree comodule functor and $G^\#$ is the free contramodule functor

$$F_\# : \mathcal{C} \longrightarrow \boldsymbol{Comod}(F), \quad F_\#(X) = F(X),$$
$$G^\# : \mathcal{C} \longrightarrow \boldsymbol{Mod}(G), \quad G^\#(X) = G(X).$$

such that the structure map is given by the comonad (resp. monad) map $F(X) \longrightarrow FF(X)$ (resp. $GG(X) \longrightarrow G(X)$). $\square$

3.4 Another view on co- and contramodule categories

We construct the comonad $C \otimes_{\mathcal{C}} (-)$ for C a comonoid in a symmetric monoidal category $\mathcal{C}$. We have a composite map

$$C \otimes X \xrightarrow{\Delta \otimes \mathrm{id}} (C \otimes C) \otimes X \longrightarrow C \otimes (C \otimes X)$$

where Δ is the comultiplication in the comonoid, which is the natural transformation $\mu^* : C \otimes (-) \longrightarrow C \otimes (C \otimes (-))$. The comonoid map $\varepsilon : C \longrightarrow I$ gives a composite map

$$C \otimes X \xrightarrow{\varepsilon \otimes \mathrm{id}} I \otimes X \xrightarrow{\cong} X$$

which becomes our natural transformation $\eta^* : C \otimes (-) \longrightarrow 1$.

We now check that the required diagrams commute. Unravelling the definitions, we see that in the following diagram

$$C \otimes X \xrightarrow{\Delta\otimes 1} C \otimes (C \otimes X) \underset{\epsilon\otimes 1}{\overset{C\otimes(\varepsilon\otimes 1)}{\rightrightarrows}} C \otimes (I \otimes X) \cong I \otimes (C \otimes X) \cong C \otimes X$$

both compose to 1 by the monoid structure on C, so the coassociativity diagram is fulfilled.

Clearly, $C \otimes (C \otimes C \otimes X) \cong C \otimes C \otimes (C \otimes X)$, so the counitality diagram commutes, and $C \otimes (-)$ is a comonad over $\mathcal{C}$.

Next we look at comodules over $C \otimes (-)$. From our definition, it's an object $X \in \mathcal{C}$ with a map $h : X \longrightarrow C \otimes X$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{h} & C \otimes X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Delta\otimes 1} \\ X \otimes C & \xrightarrow{1\otimes h} & C \otimes C \otimes X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{h} & C \otimes X \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \varepsilon\otimes 1} \\ & & X \cong I \otimes X \end{array}$$

This is however simply a relabelling of Definition 1.6! We state this as a theorem.

Theorem 3.14. *Let $\mathcal{C}$ be a closed symmetric monoidal category and $C \in \mathcal{C}$ a comonoid. There is an equivalence between the Eilenberg-Moore category* ***Comod***(C) *and the category $\mathcal{C}_C$ of comodules over C.*

We want to show a similar theorem for contramodules and will need the following theorem.

Theorem 3.15. *Let $(F \dashv G)$ be an enriched adjoint pair of endofunctors on a closed symmetric monoidal category $\mathcal{C}$. Then F is a monad if and only if G is a comonad.*

Proof. See [1, Prop. 3.1]. □

This, combined with the fact that internal hom and tensor are enriched adjoint, and already having shown that $C \otimes (-)$ is a comonad gives us the following lemma:

Lemma 3.16. *Let $\mathcal{C}$ be a closed symmetric monoidal category. If $C \in \mathcal{C}$ is a comonoid, then $[C, -]$ is a monad.*

Naturally, we want to look at modules over this monad. Filling into our definition, we get that a module over $[C, -]$ is an object $M \in \mathcal{C}$ with a map $h : [C, X] \longrightarrow X$ such that the following diagrams commute.

$$
\begin{array}{ccc}
[C,[C,X]] & \xrightarrow{[1,h]} & [C,X] \\
\Big\downarrow{\scriptstyle [\mu,1]} & & \Big\downarrow{\scriptstyle h} \\
[C,X] & \xrightarrow{h} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{[\eta,1]} & [C,X] \\
& {\scriptstyle 1}\searrow & \Big\downarrow{\scriptstyle h} \\
& & X
\end{array}
$$

But this is again a relabelling of Definition 1.7. We again state this as a theorem.

Theorem 3.17. *The Eilenberg-Moore category* $\boldsymbol{Mod}([C,-])$ *of modules over the monad* $[C,-]$ *for a comonoid* C *in a closed symmetric monoidal category* $\mathcal{C}$ *is equivalent to* $\mathcal{C}^C$*, the category of contramodules over* C *in* $\mathcal{C}$*.*

Having seen all this, we are equipped to show the co-contra correspondence in the next section.

4 The co-contra Correspondence

In this section we show the co-contra correspondence, delegating several proofs to [3]. We have two different concepts for a dual to modules over a comonoid C in a closed symmetric monoidal category and want to compare them. We show requirements for functors between their categories to exist and illustrate in ***Set***.

4.1 General correspondence

We start by stating the theorem.

Theorem 4.1. *[3, Theorem 1] Suppose we have a closed symmetric monoidal category $\mathcal{C}$ such that each pair of morphisms $X \rightrightarrows Y$ with a common left inverse admits an equalizer and each pair of morphisms $X \rightrightarrows Y$ with a common right inverse admits a coequalizer. Then there is an enriched adjoint pair of enriched functors*

$$(L \dashv R), \quad L : \mathcal{C}^C \rightleftarrows \mathcal{C}_C : R$$

The proof is broken down in several stages, and we do some preparatory work before actually proving it.

Proposition 4.2. *The categories $\mathcal{C}_C$ and $\mathcal{C}^C$ of comodules and contramodules over a comonoid C in a closed symmetric monoidal category $\mathcal{C}$ admit enrichments in $\mathcal{C}$.*

Proof. See [3, Section 2.7]. □

We now explain the reasoning behind requiring equalizers and coequalizers of maps with common left and right inverse, respectively.

Given an adjoint comonad-monad pair $(F \dashv G)$, the comodule maps object $[X, Y]_{\mathcal{C}_C}$ is the equalizer of the two maps

$$[X, Y] \xrightarrow{[1, \rho_Y]} [X, GY]$$

$$[X, Y] \longrightarrow [GX, GY] \xrightarrow{[\rho_X, 1]} [X, GY]$$

Where the first map on the second row comes from the enrichment of T. Similarly, the contramodule maps object $[X, Y]_{\mathcal{C}^C}$ is the equalizer of the two maps

$$[X,Y] \xrightarrow{[\theta_X,1]} [FX,Y]$$

$$[X,Y] \longrightarrow [FX,FY] \xrightarrow{[1,\theta_Y]} [FX,Y]$$

Both pairs of maps admit a common left inverse, the counit of C.

For the coequalizer requirement, we start by looking at the functor $R : \mathcal{C}_C \longrightarrow \mathcal{C}^C$.

Theorem 4.3. *Consider the enriched adjunction between $G = - \otimes C$ and $F = [C, -]$ in a closed symmetric monoidal category $\mathcal{C}$ where each pair of morphisms $X \rightrightarrows Y$ with a common left inverse admits an equalizer. The assignment $X \mapsto [C, X]_G$ determines an enriched functor $R : \mathcal{C}_G \longrightarrow \mathcal{C}^F$.*

Proof. See [3, Section 4.2] □

This is precisely the functor R in the co-contra correspondance of Theorem 4.1. The other functor is a little more intricate. We look at the morphisms

$$GFY \xrightarrow{G\theta_Y} GY$$

$$GFY \xrightarrow{\Delta FY} GGFY \xrightarrow{G\varepsilon 1} GY$$

The map assigning a contramodule Y to the coequalizer of these maps is our functor L.

Theorem 4.4. *Under the assumptions of Theorem 4.1, $(L \dashv R)$ is a $\mathcal{C}$-enriched adjoint pair.*

Proof. See [3, Section 4.3, 4.4] □

From this and Theorem 3.13, which says that the Kleisli categories of an adjoint comonad-monad pair are equivalent we get the following corollary.

Corollary 4.5. *Let C be a comonoid in a closed symmetric monoidal category $\mathcal{C}$. There is an equivalence $\widetilde{\mathcal{C}_C} \simeq \widetilde{\mathcal{C}^C}$ between cofree comodules and free contramodules over C.*

4.2 In sets

We finally describe the co-contra correspondence in $\boldsymbol{Set}$.

Our functor $R : \boldsymbol{Set}_C \longrightarrow \boldsymbol{Set}^C$ takes a comodule Y and sends it to the set of sections of ϕ, the structure map when Y is regarded as a set over C. So

$$R(Y) = [C, Y]_C = \prod_{a \in C} Y_a, \quad Y_a = \phi^{-1}(a)$$

Note that if Y is degenerate (ϕ is not surjective), then $R(Y)$ is empty. However, notice that from Proposition 2.4 and Theorem 2.8 we have that every comodule is isomorphic to a coproduct and every contramodule is isomorphic to a product, i.e.

$$Y = \coprod_{a \in C} Y_a, \quad Y_a = \phi^{-1}(a)$$

This means that for a non-degenerate comodule Y, $R(Y)$ is simply flipping the coproduct sign!

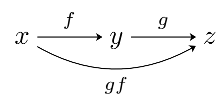
A Appendix

A.1 n-categories

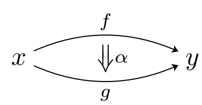
We describe n-categories, loosely following (but not adhering to the notation of) [8].

Crudely, a n-category is a category with objects, morphisms between objects, 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on all the way up to n, such that there are sufficiently nice composition rules for k-morphisms for $k \leq n$.

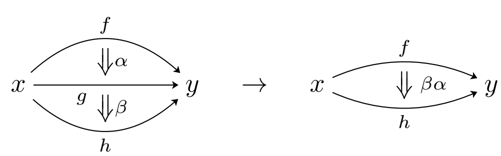
For the (1-)morphisms f and g, there's only one way to compose - the standard way of gluing them together like such:



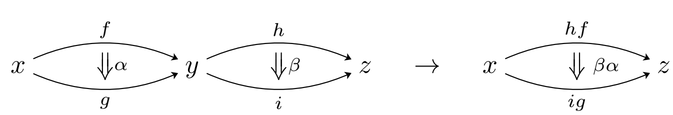
For 2-morphisms it's slightly more complicated. We again have two morphisms $f, g : x \longrightarrow y$ and a 2-morphism $\alpha : f \longrightarrow g$ and picture it as follows:



There are two ways to compose 2-morphisms. The first is vertical and goes as follows. We have three morphisms $f, g, h : x \longrightarrow y$ and two 2-morphisms $\alpha : f \longrightarrow g$, $\beta : g \longrightarrow h$, then we cam compose α and β as $\beta\alpha : f \longrightarrow h$



We can also compose 2-morphisms *horizontally*. If we have four morphisms $f, g : x \longrightarrow y$ and $h, i : y \longrightarrow z$ and two 2-morphisms $\alpha : f \longrightarrow g$ and $\beta : h \longrightarrow i$ then we can compose these as $\beta\alpha : hf \longrightarrow ig$.



There are three ways to compose 3-morphisms, four ways to compose 4-morphisms etc.

A 0-category is simply a set. The objects are the elements and there is no notion of (1-)morphisms. A 1-category is what is usually called a category (a category with small hom sets), since the collections of (1-)morphisms are sets. Another way to think about this idea is looking at equivalences. In a set, elements are either the same or different, there is no notion of equivalence. In a category, objects can be different but still isomorphic. However, comparing isomorphisms in a category brings us back to the set case: the isomorphisms are either the same or different. Comparing them beyond this only makes sense if there are 2-morphisms between them, comparing 2-isomorphisms only makes sense if there are 3-morphisms between them et cetera.

A straightforward way to create an n-category is by taking as objects all $(n-1)$-categories and as n-morphisms the functors between them. This way, we get ***Set*** for $n = 1$ and the 2-category of categories, where the 2-morphisms are natural transformations for $n = 2$.

Definition A.1. A *bicategory* K is a category weakly enriched over ***Cat***, so the hom-objects are still hom-categories but the associativity and unity laws only hold up to coherent isomorphism.

For a more precise treatment of bicategories, see the classical source [9].

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