Bachelor's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Denis Bergmann

An Introduction to Grothendieck Fibrations

Bachelor's thesis in Mathematical Sciences
Supervisor: Fernando Abellan Garcia
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Abstract

We introduce the important concept of a Grothendieck fibration using the notion of a cartesian morphism with respect to a given functor. Then, we define the Grothendieck construction associated to a presheaf valued in categories, and use this to sketch a proof of the equivalence between Grothendieck fibrations and such presheaves. Finally, we define the free Grothendieck fibration on a functor, which provides a universal method of turning an arbitrary functor into a fibration.

Sammendrag

Vi introduserer det viktige konseptet grothendieckfibrasjon ved hjelp av kartesiske morfier med hensyn på en gitt funktor. Så definerer vi grothendieckkonstruksjonen assosiert til en preknippe med verdier i kategorier, og bruker dette til å skissere et bevis av ekvivalensen mellom grothendieckfibrasjoner og slike preknipper. Til slutt definer vi den frie grothendieckfibrasjonen på en funktor, som gir en universell metode for å gjøre om en vilkårlig funktor til en fibrasjon.

A reflection over the sustainability relevance of this thesis

It has been decided that all bachelor and master theses written in certain study programs at NTNU must include a reflection over their sustainability relevance based on the United Nations' sustainable development goals, starting from spring 2024. Readers wishing to get started on the mathematical content of this work may safely skip over the remaining text on this page; it is there merely to fulfill a formal requirement.

As this is a thesis in abstract mathematics, and particularly in category theory, it would be accurate to say that the subject matter presented here has *no immediate sustainability relevance* whatsoever. This is not something specific to this particular thesis, but is common to all sufficiently abstract works in mathematics. Does this mean that we have committed a sin by not taking sustainability into consideration when choosing and writing about this topic? We certainly do not think so, and hope the reader will agree with us. We believe that the value of a work of science, literature or art should not be reduced to its practical applications.

That being said, it is not *impossible* that this thesis will one day be relevant for the United Nations' sustainability goals. But it is highly doubtful that this will happen anytime soon. Therefore, we wish future generations the best of luck in finding practical applications of Grothendieck fibrations for the purposes of sustainable development.

Contents

Preface

Grothendieck fibrations and related concepts have become a foundational, indispensable tool in the category theorist's toolbox. Numerous generalizations and variations of the notion of a fibration have been put forth, and especially in higher category theory, they have proved to be extremely helpful when attempting to generalize classical notions in category theory to the higher setting. In this thesis, we will stick to the well-trodden world of ordinary category theory, discussing Grothendieck fibrations and related concepts in the context of 1-categories. The structure of this thesis is as follows:

First, we give a quick overview of some basic concepts in 2-category theory in Section 1. Even though we stick to the classical 1-categorical theory of Grothendieck fibrations, some 2-categorical concepts are nearly impossible to avoid. This is by no means a suitable introduction to the vast theory of 2-categories, it is instead intended to be just enough to get us by.

In Section 2 we introduce the main concept of this thesis: Grothendieck fibrations. These are to be thought of as functors satisfying a certain "lifting property", or simply functors that are well-behaved "projections". The lifting property will be made precise using the concept of a *cartesian morphism*, which we will introduce beforehand. After the main definitions, we move on to basic results and consequences of the definitions. At the end, we give a few examples of Grothendieck fibrations.

Afterwards, we continue by stating what is arguably the most important result of this thesis: the equivalence between **Cat**-valued presheaves and Grothendieck fibrations. Section 3 begins by considering a simpler, decategorified version of the equivalence for the case of sets and functions. We then discuss how we can categorify this to the general case of categories and functors. A key part of this generalization is the *Grothendieck construction*, which can be seen as categorification of taking the disjoint union of an indexed family of sets. Finally, we sketch a proof of the aforementioned equivalence.

The last section of this thesis is dedicated to the question of how to turn an arbitrary functor into a Grothendieck fibration. We answer this by constructing the *free fibration* on a given functor, which will be the value of a 2-categorical left adjoint to a forgetful functor. In order to motivate our construction of the free fibration, we begin Section 4 by using an analogy with topological fibrations in homotopy theory.

The reader is expected to have a working knowledge of basic category theory, including categories, functors, natural transformations, adjunctions, and equivalences of categories. We also use some examples and analogies from basic topology and homotopy theory, but these are not integral to the understanding of the main material and can be skipped if needed.

1 A quick primer on 2-categories

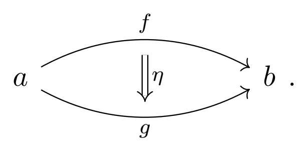
We will not make heavy use of 2-categories in this thesis, but we will need some basic facts, particularly in Section 4. Therefore, we will give a quick rundown of basic definitions. A much more comprehensive account can be found in [JY20, Ch. 2].

Definition 1.0.1. A (strict) *2-category* C consists of:

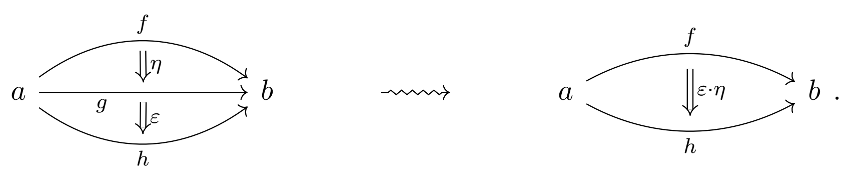
- a collection of objects $\mathrm{ob}(C)$
- for all objects $a, b \in \mathrm{ob}(C)$, a category $\mathrm{Hom}_C(a, b)$, called a *hom-category*, whose objects will be called *1-morphisms* denoted as

  $$a \xrightarrow{\quad f \quad} b\ ,$$

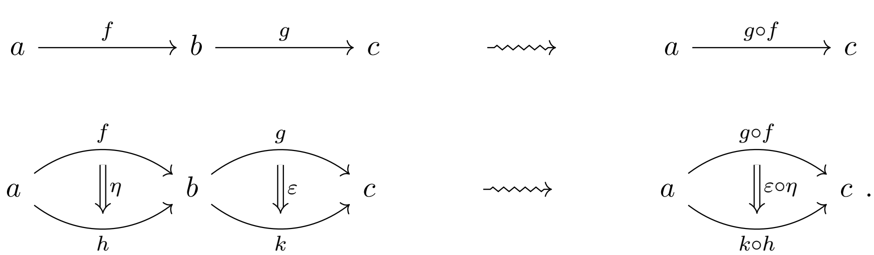
  and whose morphisms will be called *2-morphisms* and denoted as

  Composition in $\mathrm{Hom}_C(a, b)$ of 2-morphisms will be called *vertical composition* and will be denoted as

- for all objects $a, b, c \in \mathrm{ob}(C)$, a functor $\circ\colon \mathrm{Hom}_C(a, b) \times \mathrm{Hom}_C(b, c) \to \mathrm{Hom}_C(a, c)$. Its action on objects 1-morphisms will simply called composition, and its action 2-morphisms will be called *horizontal composition*. They are denoted as

We require the functors $\circ$ to be strictly associative in the sense that for all objects $a, b, c, d \in \mathrm{ob}(C)$, the two composite functors in the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_C(a,b) \times \mathrm{Hom}_C(b,c) \times \mathrm{Hom}_C(c,d) & \xrightarrow{\ \circ \times \mathrm{id}\ } & \mathrm{Hom}_C(a,c) \times \mathrm{Hom}_C(c,d) \\
{\scriptstyle \mathrm{id}\times\circ}\big\downarrow & & \big\downarrow{\scriptstyle \circ} \\
\mathrm{Hom}_C(a,b) \times \mathrm{Hom}_C(b,d) & \xrightarrow[\ \circ\ ]{} & \mathrm{Hom}_C(a,d)
\end{array}$$

are *equal*. We also require $\circ$ to be strictly unital in the sense that for all objects $b \in \mathrm{ob}(C)$ we are given a specified 1-morphism $\mathrm{id}_b \in \mathrm{Hom}_C(b,b)$ such that for all objects $a, c \in \mathrm{ob}(C)$ the two diagrams below commute.

$$\begin{array}{ccc} \mathrm{Hom}_C(a,b) & \xrightarrow{f \mapsto (f, \mathrm{id}_b)} & \mathrm{Hom}_C(a,b) \times \mathrm{Hom}_C(b,b) \\ & \overset{\mathrm{id}}{\searrow\!\!\!\!=} & \downarrow \circ \\ & & \mathrm{Hom}_C(a,b) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_C(b,b) \times \mathrm{Hom}_C(b,c) & \xleftarrow{(\mathrm{id}_b, g) \leftarrow\!\shortmid g} & \mathrm{Hom}_C(b,c) \\ \circ \downarrow & \overset{\mathrm{id}}{=\!\!\!\!\swarrow} & \\ \mathrm{Hom}_C(b,c) & & \end{array}$$

Notice that the commutativity of the diagrams imply in particular that both composition of 1-morphisms and horizontal composition of 2-morphisms are strictly associative. We do not need to require that vertical composition of 2-morphisms be associative, as that composition takes place in a category, where associativity is already required.

We list three examples of 2-categories which will be relevant for us.

Example 1.0.2. The category of all categories, **Cat**, is the prototypical example of a 2-category. Objects are categories, 1-morphisms are functors, and 2-morphisms are natural transformations.

Remark 1.0.3. Of course, in order to avoid foundational issues, objects of **Cat** should really be *small* categories, relative to some universe. We will (fortunately) keep ignoring foundational issues in this thesis.

Example 1.0.4. Similarly to how a set can be considered as a discrete category, any category C can be given the structure of a 2-category by letting its morphisms be 1-morphisms, and letting the 2-morphisms be trivial. In other words, $\mathrm{Hom}_C(a,b)$ is discrete category for all objects $a, b \in C$.

Example 1.0.5. Let B be a fixed category, and consider the *slice category* $\mathbf{Cat}_{/B}$, whose objects are tuples (A, F) where A is a category and $F\colon A \to B$ is a functor, and whose morphisms from (A, F) to (C, G) are functors $H\colon A \to C$ such that the triangle of functors

$$\begin{array}{ccc} A & \xrightarrow{H} & C \\ & \circ & \\ {\scriptstyle F}\searrow & & \swarrow{\scriptstyle G} \\ & B & \end{array}$$

strictly commutes. We make this into a 2-category by letting the 2-morphisms from functors H to K be natural transformations $\eta\colon H \Rightarrow K$ such that $G\eta = \mathrm{id}_F$, in other words, for all objects $a \in A$, $G(\eta_a) = \mathrm{id}_{F(a)}$. Such natural transformations are called *vertical*.

Next up, we need an appropriate notion of mapping between 2-categories. The naive categorification of the concept of a functor between categories is that of a *2-functor*:

Definition 1.0.6. Let C and D be 2-categories. A *2-functor* $F\colon C \to D$ consists of:

- a function $F\colon \mathrm{ob}(C) \to \mathrm{ob}(D)$,
- for all objects $a, b \in C$, a functor $F\colon \mathrm{Hom}_C(a,b) \to \mathrm{Hom}_D(F(a), F(b))$,

such that F strictly preserves composition of 1-morphisms, horizontal composition of 2-morphisms, and identity 1-morphisms.

Remark 1.0.7. Once again, we did not have to specify that F preserves vertical composition of 2-morphisms, as this is already encoded in the fact that F is a functor between hom-categories.

This is sufficient for some purposes, including Section 4 of this thesis, but for other purposes, such as Section 3, a more refined notion is preferred. When we are given a category C and two morphisms $f, g \in \mathrm{Hom}_C(a, b)$, all we can say is whether or not f is *equal* to g. Since $\mathrm{Hom}_C(a, b)$ is merely a *set*, equality is the only form of comparison we have. However, if C is a 2-category, we have a more refined notion available: we can say whether f and g are *isomorphic* in the *category* $\mathrm{Hom}_C(a, b)$. Translating this into our language, f and g are isomorphic 1-morphisms if there exists an invertible 2-morphism $\eta \colon f \to g$. Taking this into account, one can come to the conclusion that the preceding notion of 2-functor is too strong: why compare $F(g \circ f)$ and $F(g) \circ F(f)$ for equality when we have isomorphism available? This motivates the following definition.

Definition 1.0.8. Let C and D be 2-categories. A *pseudofunctor* $F \colon C \to D$ consists of:

- a function $F \colon \mathrm{ob}(C) \to \mathrm{ob}(D)$,
- for all objects $a, b \in C$, a functor $F \colon \mathrm{Hom}_C(a, b) \to \mathrm{Hom}_D(F(a), F(b))$,
- for all objects $a, b, c \in C$, for all 1-morphisms $f \colon a \to b$, $g \colon b \to c$, a specified invertible 2-morphism $F(g) \circ F(f) \Rightarrow F(g \circ f)$,
- for all objects $a \in C$, a specified invertible 2-morphism $\mathrm{id}_{F(a)} \Rightarrow F(\mathrm{id}_a)$,

such that the specified 2-morphisms satisfy certain *coherence conditions*, which we will not specify here.

2-functors are seen as a special case of pseudofunctors by setting the specified 2-morphisms equal to identities.

Next, we categorify natural transformations between functors. As with the case of functors, there should be a strict notion and a more relaxed notion. We define both below.

Definition 1.0.9. Let C and D be 2-categories, and $F, G \colon C \to D$ be pseudofunctors. A *2-natural transformation* $\eta \colon F \Rightarrow G$ consists of a collection of 1-morphisms $\eta_c \colon F(c) \to G(c)$ indexed by objects $c \in C$, such that for all 1-morphisms $f \colon c \to d$ in C the square

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(d) \\ {\scriptstyle \eta_c}\downarrow & \circ & \downarrow{\scriptstyle \eta_d} \\ G(c) & \xrightarrow[G(f)]{} & G(d) \end{array}$$

commutes. There is also an addition condition concerning compatibility with respect to 2-morphisms which we do not specify here.

Definition 1.0.10. Let C and D be 2-categories, and $F, G \colon C \to D$ be pseudofunctors. A *pseudo-natural* transformation $\eta \colon F \to G$ consists of:

- for each object $c \in C$, a 1-morphism $\eta_c \colon F(c) \to G(c)$,

- for each 1-morphism $f\colon c \to d$ in C, a specified invertible 2-morphism between the two composites of the naturality square as indicated:

$$
\begin{array}{ccc}
F(c) & \xrightarrow{F(f)} & F(d) \\
{\scriptstyle \eta_c}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \eta_d} \\
G(c) & \xrightarrow[G(f)]{} & G(d)
\end{array}
$$

This data is required to satisfy certain coherence conditions, which we once again omit.

Next up, we would like a notion of adjunction of 2-functors/pseudofunctors. For our purposes, we will use the following somewhat non-standard definition, but it will be sufficient for us, as it is exactly the notion we need in Section 4.

Definition 1.0.11. Let C and D be 2-categories and $F\colon C \to D$, $G\colon D \to C$ be pseudofunctors. A *pseudoadjunction* from F to G is a *natural equivalence* of hom-categories

$$\mathrm{Hom}_C(F(c), d) \simeq \mathrm{Hom}_D(c, G(d)).$$

That is, for all objects $c \in C$, $d \in D$ we have a specified equivalence of hom-categories as shown, and the collection of these equivalences should be (strictly) 2-natural, in the sense of Definition 1.0.9.

Remark 1.0.12. The term pseudoadjunction may already have a slightly different meaning, so we repeat that this is a non-standard definition. In particular, it seems particularly strong to require 2-naturality instead of pseudonaturality.

Finally, we would like a notion of equivalence between 2-categories. What follows will not be a proper *definition*, but more of a *characterization*, in the same vein as "a functor is an equivalence of categories if it is fully faithful and essentially surjective on objects". This time we will only focus on the strict version, as the relaxed version will not be relevant for us. In the following, two objects a, b in a 2-category C are said to be *equivalent* if there exist 1-morphisms $f\colon a \to b$ and $g\colon b \to a$ such that the composites $g \circ f$ and $f \circ g$ are isomorphic (via an invertible 2-morphism) to their respective identity 1-morphisms.

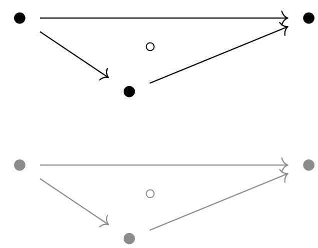
Definition/Proposition 1.0.13. Let C and D be 2-categories, and $F\colon C \to D$ a 2-functor. We say that F is a *2-equivalence* if

- for all objects $a, b \in C$, the functor $F\colon \mathrm{Hom}_C(a, b) \to \mathrm{Hom}_D(F(a), F(b))$ is an equivalence of categories,
- F is surjective on equivalence classes of objects of D, in the sense that for each object $d \in D$, there exists an object $c \in C$ such that $F(c)$ is equivalent to d.

2 Cartesian morphisms and Grothendieck fibrations

2.1 Basic definitions

In this thesis, we will often be interpreting a functor of categories as exhibiting the domain category as being *indexed*, or *fibered* over the codomain category, and in some sense lying over it. We introduce some notation for this scenario as follows. Given a functor $p\colon E \to B$, we will use diagrams having two layers: a top black layer consisting of a diagram in E, and a bottom grey layer consisting of a diagram in B.



A two-layered diagram will be called *satisfied* if:

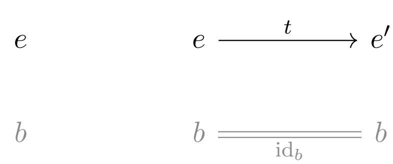
1. p takes the drawn objects in E (black vertices) to the objects in B (grey vertices) directly below them,
2. p takes the drawn morphisms in E (black arrows) to the corresponding morphisms in B (grey arrows) directly below them,
3. all things which should commute, such as triangles with circles in them, actually commute.

All drawn two-layered diagrams (which will simply be called diagrams) will be assumed to be satisfied.

The first basic concept we will need is a categorical notion of *fiber*. Recall, that for a function $f\colon A \to B$ between sets and an element $b \in B$, the (set-theoretic) fiber of f over b is the subset $f^{-1}(b) \subseteq A$ defined as $f^{-1}(b) = \{a \in A\colon f(a) = b\}$. We generalize this to functors as follows:

Definition 2.1.1. Let E, B be categories, and $p\colon E \to B$ a functor. Given an object $b \in B$, we define a subcategory $p^{-1}(b) \subseteq E$, called the (category-theoretic) *fiber* of p over b, as follows:

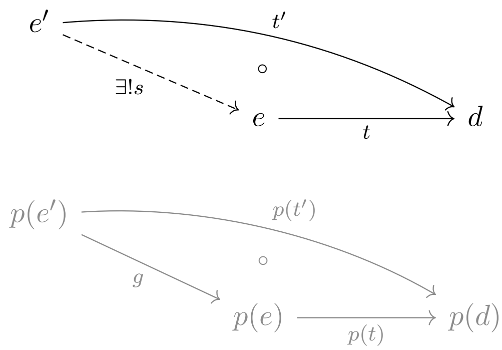
- objects in $p^{-1}(b)$ are objects $e \in E$ such that $p(e) = b$,
- morphisms in $p^{-1}(b)$ are morphisms $t\colon e \to e'$ such that $p(t) = \mathrm{id}_b$.



Given a morphism t in E, we say that t is *vertical* if $p(t)$ is an identity morphism. That is, if it belongs to the fiber $p^{-1}(b)$ over some object $b \in B$.

Next, we introduce the crucial notion of a *cartesian morphism*, which will be the building block in the definition of a Grothendieck fibration, and consequently of almost everything in this thesis.

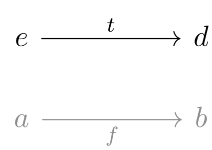
Definition 2.1.2. Let $p\colon E \to B$ be a functor between categories. A morphism $t\colon e \to d$ in E is called *cartesian* (with respect to p) if for all partial diagrams as indicated below, there exists a *unique* dashed morphism completing and satisfying the diagram.



More explicitly, for any morphism $t' \colon e' \to d$ in E and any morphism $g \colon p(e') \to p(e)$ in B such that $p(t) \circ g = p(t')$, there should exist a unique morphism $s \colon e' \to e$ in E such that $p(s) = g$ and $t \circ s = t'$.

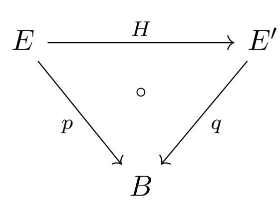
We can intuitively think of cartesian morphisms as universal *base change* morphisms. We will see later that we can use cartesian morphisms lying over $f \colon a \to b$ to transfer information between fibers; from $p^{-1}(b)$ to $p^{-1}(a)$. In order for cartesian morphisms to be as useful as possible, we need a sufficiently large supply of them. A functor having "enough" cartesian morphisms turns out to be one of the most fundamental concepts in category theory, which we define below.

Definition 2.1.3. Let $p \colon E \to B$ be a functor. We say that p is a *Grothendieck fibration* (alternatively, *cartesian fibration*, or simply *fibration*), if for every morphism $f \colon a \to b$ in B, and any object $d \in p^{-1}(b)$, there exists a cartesian morphism $t \colon e \to d$ in E with $p(t) = f$.



Intuitively, we can think of a Grothendieck fibration as a well-behaved "projection" functor, or a functor with a specific lifting property concerning cartesian morphisms.

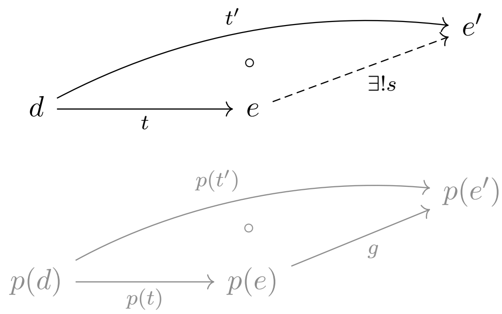
We can form a category $\mathrm{Fib}^{\mathrm{cart}}(B)$ of Grothendieck fibrations over a given base category B as a (2)-subcategory of $\mathbf{Cat}_{/B}$: a morphism from $p \colon E \to B$ to $q \colon E' \to B$ is a functor H making the triangle



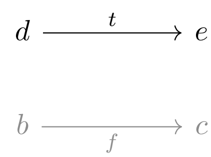
commute, *and* sending cartesian morphisms in E to cartesian morphisms in E'. We call such functors *cartesian*. Furthermore, $\mathrm{Fib}^{\mathrm{cart}}(B)$ is naturally a 2-category, whose 2-morphisms are vertical natural transformations as defined in Example 1.0.5.

Of course, these definitions can be dualized.

Definition 2.1.4. Let $f\colon E \to B$ be a functor. A morphism $t\colon d \to e$ in E is called *cocartesian* if the corresponding morphism $t^{\mathrm{op}}\colon e \to d$ in E^{op} is cartesian for the functor $p^{\mathrm{op}}\colon E^{\mathrm{op}} \to B^{\mathrm{op}}$. Translating this into a statement about the original functor $p\colon E \to B$, we get that for all partial diagrams as below, there should exist a *unique* dashed morphism s completing and satisfying the diagram.



Definition 2.1.5. Let $p\colon E \to B$ be a functor. We say that p is a *Grothendieck opfibration* (alternatively, *cocartesian fibration*) if the opposite functor $p^{\mathrm{op}}\colon E^{\mathrm{op}} \to B^{\mathrm{op}}$ is a Grothendieck fibration. In other words, for every morphism $f\colon b \to c$ in B and every object $d \in p^{-1}(b)$, there should exist a cocartesian morphism $t\colon d \to e$ such that $p(t) = f$:



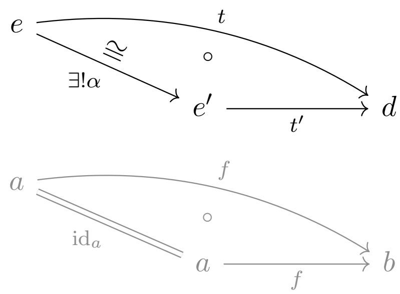
Definition 2.1.6. Let $p\colon E \to B$ be a functor. We say that p is a *Grothendieck bifibration* if it is both a Grothendieck fibration and a Grothendieck opfibration.

2.2 Basic results concerning Grothendieck fibrations

Here, we record several useful results about (co)cartesian morphisms and Grothendieck (op)fibrations. We will, for the most part, only state and prove these results for cartesian morphisms and Grothendieck fibrations, but all of them have dual analogues for cocartesian morphisms and opfibrations, with equally dual proofs.

First, we notice that cartesian morphisms with a specified codomain and projecting to a specified morphism in the base category satisfy a universal property. As with all universal properties, we expect entities exhibiting them to be unique up to a *unique, canonical* isomorphism. We record this in the following lemma.

Lemma 2.2.1. *Let $p\colon E \to B$ be a functor, and let $f\colon a \to b$ be a morphism in B. Given two cartesian morphisms $t\colon e \to d$ and $t'\colon e' \to d$ such that $p(t) = p(t') = f$, there exists a unique vertical isomorphism $\alpha\colon t \to t'$ such that the following diagram is satisfied:*



Proof. It's the usual argument. As t' is cartesian, there exists a morphism α as shown, except that it's not necessarily an isomorphism (yet). Similarly, as t is cartesian, we obtain a unique morphism $\beta \colon e' \to e$ satisfying a corresponding diagram. To see that α and β are inverses, we notice that id_e is a vertical morphism factorization of t through itself, satisfying an evident diagram. Because $\beta \circ \alpha$ satisfies the same diagram (as one readily checks), we get that $\beta \circ \alpha = \mathrm{id}_e$ by uniqueness. Similarly, we get that $\alpha \circ \beta = \mathrm{id}_{e'}$, showing that α is an isomorphism. □

Next, we derive an alternative characterization of a morphism being cartesian, and consequently, of a functor being a Grothendieck fibration, using pullback squares.

Proposition 2.2.2. *Let $p \colon E \to B$ be a functor, and $t \colon e \to d$ a morphism in E. Then, t is cartesian if and only if for all objects $e' \in E$, the following commutative square is a pullback:*

$$\begin{array}{ccc} Hom_E(e', e) & \xrightarrow{\;t_*\;} & Hom_E(e', d) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Hom_B(p(e'), p(e)) & \xrightarrow[\;p(t)_*\;]{} & Hom_B(p(e'), p(d)) \end{array}$$

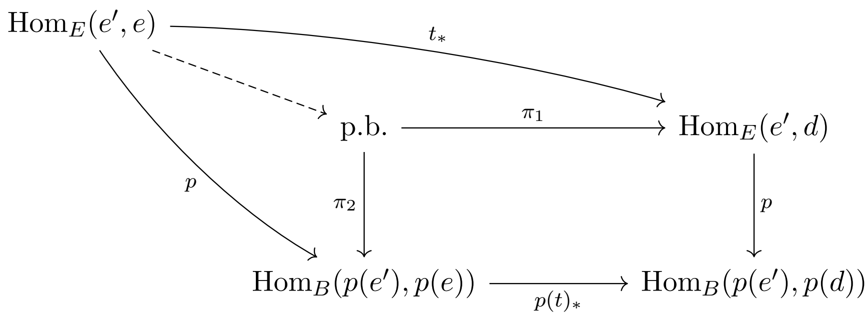
Proof. In the above diagram, the horizontal maps are postcompositions, and the vertical maps are actions of p on morphisms. We will need the following description of pullbacks in **Set**. Given diagram of sets

$$\begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\;f\;]{} & B \end{array}$$

its pullback is given by

$$A \times_B C = \{(a, c) \in A \times C : f(a) = g(c)\}.$$

Next, we use the fact that a commutative square is a pullback if and only if the induced map to any given pullback is an isomorphism. That is, our square is a pullback if and only if the canonical dashed map is a bijection of sets:



We calculate:

The dashed map is a bijection

$\iff$ for all $(t', g) \in$ p.b. such that $p(t') = p(t)_*(g)$, there exists a unique $s \in \mathrm{Hom}_E(e', e)$ such that $(t_*(s), p(s)) = (t', g)$

$\iff$ for all $t' \colon e' \to d$ and $g \colon p(e') \to p(e)$ such that $p(t') = p(t) \circ g$, there exists a unique $s \colon e' \to e$ such that $t \circ s = t'$ and $p(s) = g$

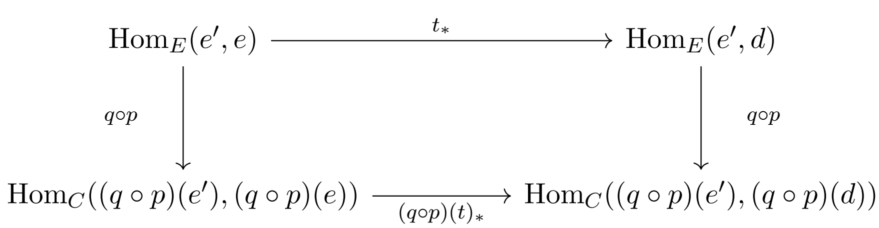
Quantifying this over all $e' \in E$, we see that this is exactly the condition for t to be cartesian. □

Corollary 2.2.3. *The composition of two Grothendieck fibrations is a Grothendieck fibration.*

Proof. Let $p \colon E \to B$ and $q \colon B \to C$ be Grothendieck fibrations. Pick any morphism $f : x \to c$ in C, and any $d \in E$ with $(q \circ p)(d) = c$. We will construct a cartesian morphism in E with codomain d projecting to f via $q \circ p$.

As $q(p(d)) = c$ and q is a Grothendieck fibration, there exists a cartesian morphism $g \colon b \to p(d)$ in B with $q(g) = f$. Once again, as p is a fibration, there exists a cartesian morphism $t \colon e \to d$ in E with $p(t) = g$.

We claim that t does the job. Clearly $(q \circ p)(t) = f$. By the preceding proposition, showing that t is cartesian amounts to showing that the following square is a pullback for all $e' \in E$:



We can factor this square as follows:

$$\begin{array}{ccc}
\mathrm{Hom}_E(e', e) & \xrightarrow{t_*} & \mathrm{Hom}_E(e', d) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathrm{Hom}_B(p(e'), p(e)) & \xrightarrow{p(t)_* = g_*} & \mathrm{Hom}_B(p(e'), p(d)) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
\mathrm{Hom}_C(q(p(e')), q(p(e))) & \xrightarrow[q(g)_*]{} & \mathrm{Hom}_C(q(p(e')), q(p(d)))
\end{array}$$

Now, again by the proposition, the upper and lower squares are pullbacks because p and q are Grothendieck fibrations, respectively. By a basic categorical result, the composite of two pullback squares is itself a pullback square, so we are done. □

Corollary 2.2.4. *Let $p\colon E \to B$ be a functor, and let $f\colon x \to y$, $g\colon y \to z$ be composable morphisms in E.*

1. *If f and g are cartesian morphisms, then so is $g \circ f$.*
2. *If g and $g \circ f$ are cartesian morphisms, then so is f.*

Proof. Pick any object $e \in E$. We can factor the square

$$\begin{array}{ccc}
\mathrm{Hom}_E(e, x) & \xrightarrow{(g\circ f)_*} & \mathrm{Hom}_E(e, z) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathrm{Hom}_B(p(e), p(x)) & \xrightarrow[p(g\circ f)_*]{} & \mathrm{Hom}_B(p(e), p(z))
\end{array}$$

as follows:

$$\begin{array}{ccccc}
\mathrm{Hom}_E(e, x) & \xrightarrow{f_*} & \mathrm{Hom}_E(e, y) & \xrightarrow{g_*} & \mathrm{Hom}_E(e, z) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathrm{Hom}_B(p(e), p(x)) & \xrightarrow[p(f)_*]{} & \mathrm{Hom}_B(p(e), p(y)) & \xrightarrow[p(g)_*]{} & \mathrm{Hom}_B(p(e), p(z))
\end{array}$$

Now, a basic categorical result says that given a composite of two squares where the right square is a pullback, the left square is a pullback if and only if the composite rectangle is a pullback. Using this and quantifying over all $e \in E$, we are done by the previous proposition. □

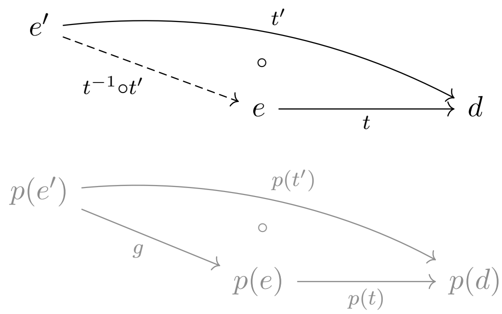
Next, we characterize cartesian morphisms lying over isomorphisms. They turn out to have a particularly simple form:

Lemma 2.2.5. *Let $p\colon E \to B$ be a functor, and let $t\colon e \to d$ be a morphism in E such that $p(t)$ is an isomorphism in B. Then, the following are equivalent:*

1. *t is cartesian,*

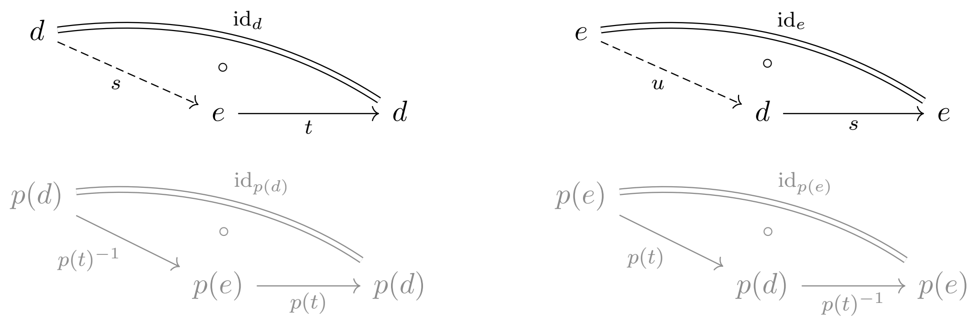
2. *t is an isomorphism.*

Proof. First, assume that t is an isomorphism. We can complete any partial diagram in Definition 2.1.2 as follows:



The diagram is satisfied because $p(t) \circ g = p(t')$ implies that $g = p(t)^{-1} \circ p(t') = p(t^{-1} \circ t')$. As t is an isomorphism, the dashed arrow is clearly unique. This shows that t is cartesian.

Conversely, assume that t is cartesian. We create two diagrams; first the left one, and then the right:



To see that this makes sense, we have to show that the morphism s from the first diagram is cartesian. Notice that as $t \circ s = \mathrm{id}_d$ and id_d is cartesian by the other direction of this lemma, we get that s is cartesian by Corollary 2.2.4. Thus, we can form the diagram on the right, and get a morphism u as shown. Now, we calculate that $t = t \circ \mathrm{id}_e = t \circ s \circ u = \mathrm{id}_d \circ u = u$. So, $s \circ t$ and $t \circ s$ are identities, and hence t is an isomorphism. ☐

Remark 2.2.6. So far, we have considered being a Grothendieck fibration as a *property* of a functor. That is, a functor is a fibration is there *exist* enough cartesian morphisms. However, in some situations it is convenient to have the structure of a *specified* collection of cartesian morphisms. In more detail, a *cleavage* for a functor $p \colon E \to B$ is a choice of a cartesian morphism $t \colon e \to d$ such that $p(t) = f$, for every morphism $f \colon a \to b$ in B and every object $d \in p^{-1}(b)$. A functor equipped with a cleavage will be called a *cloven* Grothendieck fibration. Of course, we can turn any Grothendieck fibration into a cloven fibration by non-canonically choosing cartesian morphisms. We can always choose an identity morphism to be the chosen cartesian morphism lying over an identity morphism; a cleavage with this property is called *normalized* A cleavage is called *normalized*. We can also impose a stronger condition on a cleavage, namely that the collection of chosen cartesian morphisms is closed under composition. Such a cleavage is called *split*, and a functor having such a cleavage is called a *split* Grothendieck fibration. Not all fibrations are split.

2.3 Examples of Grothendieck fibrations

Example 2.3.1. Given categories A and B, one can consider the projection functor $\pi_B \colon A \times B \to B$. This is a cartesian fibration *and* a cocartesian fibration: for any object (a, b) in $A \times B$ and any morphisms $f \colon b' \to b$, or $g \colon b \to b''$, we have the following:

$$\begin{array}{ccccccc} (a,b') & \xrightarrow{(\mathrm{id}_a, f)} & (a,b) & \qquad & (a,b) & \xrightarrow{(\mathrm{id}_a, g)} & (a,b'') \\ & & & & & & \\ b' & \xrightarrow{\quad f \quad} & b & & b & \xrightarrow{\quad g \quad} & b'' \end{array}$$

One can easily check that the top morphisms are cartesian and cocartesian, respectively. This shows that π_B is a Grothendieck bifibration.

Example 2.3.2. Consider a diagram of categories and functors as follows:

$$A \xrightarrow{\quad F \quad} C \xleftarrow{\quad G \quad} B$$

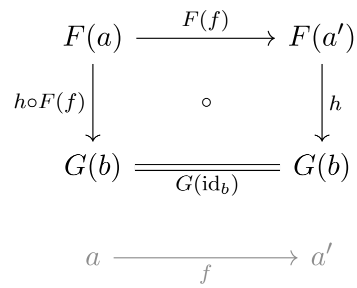
In this scenario we can form the *comma category* $F \downarrow G$, where objects are triples (a, b, h), where a and b are objects of A and B, respectively, and $h \colon F(a) \to G(b)$ is a morphism in C. A morphism from (a, b, h) to (a', b', h') is defined to be a pair (f, g), where $f \colon a \to a'$ and $g \colon b \to b'$ are morphisms in A and B respectively, such that the following square commutes.

$$\begin{array}{ccc} F(a) & \xrightarrow{\quad F(f) \quad} & F(a') \\ {\scriptstyle h}\big\downarrow & \circ & \big\downarrow{\scriptstyle h'} \\ G(b) & \xrightarrow{\quad G(g) \quad} & G(b') \end{array}$$

We have two evident "projection" functors from $A \downarrow B$: the domain projection $\mathrm{dom} \colon F \downarrow G \to A$, and the codomain projection $\mathrm{cod} \colon F \downarrow G \to B$, with the following action on objects and morphisms:

$$\begin{array}{ccccccc} & & F(a) & \xrightarrow{F(f)} & F(a') & & \\ b \xrightarrow{\ g\ } b' & \xleftarrow{\mathrm{cod}} & {\scriptstyle h}\big\downarrow & \circ & \big\downarrow{\scriptstyle h'} & \xrightarrow{\mathrm{dom}} & a \xrightarrow{\ f\ } a' \\ & & G(b) & \xrightarrow[G(g)]{} & G(b') & & \end{array}$$

One can check that dom is a cartesian fibration: given a morphism $f \colon a \to a'$ in A and an object $h \colon F(a) \to G(b)$ in $\mathrm{dom}^{-1}(a')$, we have that



is a cartesian morphism. Dually, one can show that cod is Grothendieck opfibration.

The case when F is the identity functor going to be important for us later on, so we give it the special notation $C \downarrow G$, instead of the usual $\mathrm{id}_C \downarrow G$. Another special case worth mentioning is a *slice category*, where F is the identity functor, and B is the terminal category, so that $G\colon * \to C$ merely picks out an object $c \in C$. In this case we denote the comma category as $C_{/c}$. We have already seen a slice category in Example 1.0.5.

Example 2.3.3. This is a very contrived example, but it connects Grothendieck (op)fibrations of categories with the likely more familiar notion of fibrations in topology.

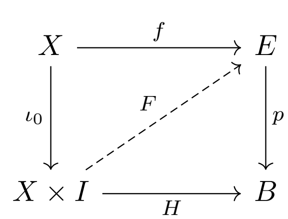
Let X be a topological space. A classical construction lets us define the *fundamental groupoid* of X, to be the category $\Pi(X)$ with

$$\operatorname{ob}\Pi(X) = \{\text{points of } X\},$$
$$\mathrm{Hom}_{\Pi(X)}(x, y) = \{\text{paths from } x \text{ to } y \text{ in X}\}/\{\text{homotopy rel. endpoints}\}.$$

Composition is induced by the evident composition of paths. As the name suggests, this is a groupoid, meaning that all morphisms are isomorphisms.

Any continuous map $f\colon X \to Y$ of topological spaces induces a functor $\Pi(f)\colon \Pi(X) \to \Pi(Y)$ between fundamental groupoids, which is given by postcomposing paths with f. One can ask when this induced functor is a Grothendieck (op)fibration. As luck would have it, being a fibration in the topological sense is sufficient! We recall the definition below:

A continuous function $p\colon E \to B$ of topological spaces is called a *Hurewicz fibration*, or simply *fibration*, if it has the *homotopy lifting property* with respect to all topological spaces X. Spelled out, this means the following:



For any continuous map $f\colon X \to E$ and homotopy $H\colon X \times I \to B$ such that $p \circ f = H(-, 0)$, there should exist a homotopy $F\colon X \times I \to E$ such that $F(-, 0) = f$ and $p \circ F = H$.

For the purposes of this example, we actually only need a very weak version of this called the *path lifting property*, which means that for any path $\gamma\colon I \to B$ starting at $b \in B$ and any $e \in p^{-1}(b)$, there should exist a path $\Gamma\colon I \to E$ starting at e such that $p \circ \Gamma = \gamma$. By letting $X = *$, the one-point space, in the diagram above, we see that the homotopy lifting property implies the path lifting property.

We claim that every Hurewicz fibration f induces a Grothendieck bifibration $\Pi(f)$.

Because we are dealing with groupoids, Proposition 2.2.5 tells us $\Pi(f)$ is a Grothendieck fibration as soon as there merely exists a lift of every morphism in the base with a given codomain. Unwinding definitions, this amounts to having the path lifting property *up to homotopy*, i.e. that for a path γ in B there exists a path Γ in E starting at a given point such that $p \circ \Gamma$ is homotopic (rel. endpoints) to γ. Clearly the path lifting property implies this! Using a nearly identical argument, we see that that $\Pi(f)$ is an opfibration as well.

3 The equivalence of fibrations and pseudofunctors

3.1 The case of sets

Before getting into the general version of what is to follow, let us warm up by considering a decategoried version, where instead of categories and functors, we deal with sets and functions.

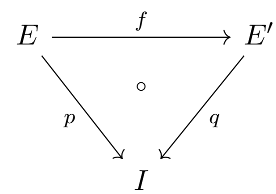
Fix a set I, to serve as a "base" set. Consider an I-indexed set $\{A_i\}_{i \in I}$. This can be considered as a functor $I \to \mathbf{Set}$, where we treat the set I as a discrete category. For some purposes, indexed sets are unsatisfactory; for each $i \in I$, we are given a set A_i, but there is no connection between the different A_i's, and they do not all live in one common "place". Is there an alternative viewpoint we can look at indexed sets from?

The answer is yes. One approach is to "package up" all of the sets A_i into a single set, the disjoint union $\coprod_{i \in I} A_i$. This way there is only one set to keep track of: the disjoint union, instead of an entire family of sets.

Of course, this by itself as a set is not a suitable substitute for indexed sets: with only the disjoint union available, there is no way to keep track of *which* A_i an element of $\coprod_{i \in I} A_i$ is a member of (at least, if we do not assume a particular implementation of disjoint unions in our foundations). The crucial ingredient is the projection function $\pi\colon \coprod_{i \in I} A_i \to I$, sending an element a to the unique $i \in I$ for which $a \in A_i$. From this *pair* $(\coprod_{i \in I} A_i, \pi)$, one can recover the indexed set by identifying A_i with the fiber $\pi^{-1}(i)$.

We can also go the other way. Given any pair (E, p) where E is a set and $p\colon E \to I$ is a function, we can define an indexed set $\{X_i\}_{i \in I}$ by setting $X_i = p^{-1}(i)$.

These two constructions can be seen as a blueprint for an equivalence of categories between I-indexed sets and functions with codomain I. Of course, we need to describe the morphisms in these two categories. A morphism of I-indexed sets $\{A_i\}_{i \in I} \to \{C_i\}_{i \in I}$ is simply a natural transformation of the corresponding $\mathbf{Set}$-valued functors, in other words an I-indexed family of functions $\eta_i\colon A_i \to C_i$. So one of the categories is $\mathrm{Func}(I, \mathbf{Set})$. A morphism of tuples $(E, p) \to (E', q)$ is defined to be a function $f\colon E \to E'$ such that the triangle



commutes, i.e. such that $p = q \circ f$. This definition is set up such that f restricts to a well-defined function between fibers $p^{-1}(i) \to q^{-1}(i)$, and hence gives a natural transformation between the corresponding I-indexed sets. In other words, the second category is the slice category $\mathbf{Set}_{/I}$ (see Example 2.3.2). Now we can state the result:

Proposition 3.1.1. *Let I be a set. Then there is an equivalence of categories of I-indexed sets and functions with codomain I:*

$$\mathbf{Set}_{/I} \simeq \mathrm{Func}(I, \mathbf{Set}).$$

3.2 The Grothendieck construction and the case of categories

We will now generalize the preceding discussion to categories. Let us first attempt to naively "categorify" every item in Proposition 3.1.1. Instead of a set I, we should have a base *category* B. The slice category $\mathbf{Set}_{/I}$ becomes the slice category $\mathbf{Cat}_{/B}$. And for the functor category, we get $\mathrm{Func}(B, \mathbf{Cat})$, consisting of strict **Cat**-valued functors. One might then hope that the proposition remains true when categorified, i.e. we have an equivalence of categories

$$\mathbf{Cat}_{/B} \simeq \mathrm{Func}(B, \mathbf{Cat}).$$

This turns out not to be true. We highlight the main problems, and what can be done to solve them:

1. Not all functors with codomain B will give rise to **Cat**-valued functors. The difficulty lies in the fact that there may be nontrivial morphisms in B, which we did not have in the case of sets. Even if we could define a functor $F\colon B \to \mathbf{Cat}$ on objects by setting $F(b)$ equal to an appropriate notion of fiber over b as in the case of sets, we still have to define F on morphisms as well, and there is no natural way to do this in general. Luckily, if our functor $p\colon E \to B$ is a Grothendieck fibration, we can use the existence of enough cartesian morphisms to define the **Cat**-valued functor on morphisms.

2. We have to take the 2-categorical nature of **Cat** into account. Instead of working with strict **Cat**-valued functors, we should use *pseudofunctors* instead. We view B as a 2-category with trivial 2-morphisms in order for this to make sense. Furthermore, one can show that the collection of pseudofunctors forms a 2-category, and by Example 1.0.5, so does the slice category in question. With this in mind, we should ask for an appropriate notion of equivalence between 2-categories.

3. For reasons which will become clear during our proof, we should work with functors from the *opposite category* B^{op} if we start with a Grothendieck fibration. A set considered as a category is equal to its opposite, so the distinction was invisible when we were in the case of sets. Alternatively, we can use Grothendieck opfibrations instead of fibrations in order to not have to take the opposite category of B.

It turns out that taking these three points into account, we do in fact get a generalization of 3.1.1 which reads as follows:

Theorem 3.2.1. *Let B be a category. Then, there is a 2-equivalence of 2-categories (see Definition 1.0.13)*

$$S : \mathrm{Fib}^{\mathrm{cart}}(B) \simeq \mathrm{PsFunc}(B^{\mathrm{op}}, \mathbf{Cat}).$$

As a first step towards this theorem, let us focus on a key step: a categorification of the process of turning a functor $A \colon I \to \mathbf{Set}$ (an I-indexed family of sets) into a function $\pi \colon \coprod_{i \in I} A_i \to I$. For categories and functors, this is known as the *Grothendieck construction* associated to a pseudofunctor $F \colon B^{\mathrm{op}} \to \mathbf{Cat}$. For simplicity, we only consider the case where F is a 2-functor; in this situation F can be seen as an ordinary functor where **Cat** has forgotten its 2-categorical structure.

Definition 3.2.2. Let $F \colon B^{\mathrm{op}} \to \mathbf{Cat}$ be a 2-functor. The *Grothendieck construction* of S is the category $\int_B F$ determined by the following data:

- objects are pairs (b, x) where b is an object of B and x is an object of $F(b)$,
- morphisms from (b, x) to (b', x') are pairs (f, g) where $f \colon b \to b'$ is a morphism in B and $g \colon x \to F(f)(x')$ is a morphism in $F(b)$,
- composition of a composable pair of morphisms $(f, g) \colon (b, x) \to (b', x')$ and $(f', g') \colon (b', x') \to (b'', x'')$ is defined as $(f', g') \circ (f, g) = (f' \circ f, F(f)(g') \circ g)$.

The composition operation makes sense because $F(f)(g')$ has codomain $F(f)(F(f')(x'')) = (F(f) \circ F(f'))(x'') = F(f' \circ f)(x'')$, where we used that F was a strict functor. One readily checks that composition is associative and that identities are given by $\mathrm{id}_{(b,x)} = (\mathrm{id}_b, \mathrm{id}_x)$. Thus this is a well-defined category.

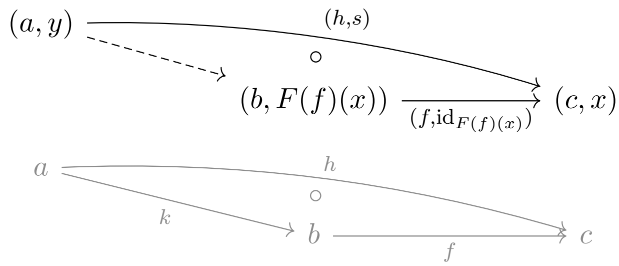
Furthermore, there is an evident projection functor $\pi \colon \int_B F \to B$ with the following action on objects and morphisms:

$$(b, x) \xrightarrow{(f,g)} (b', x') \quad \overset{\pi}{\rightsquigarrow} \quad b \xrightarrow{f} b'$$

As one would hope from the name, this is indeed a fibration.

Proposition 3.2.3. *Let $F \colon B^{op} \to$ **Cat** be a (strict) functor. Then $\pi \colon \int_B F \to B$ is a (split) Grothendieck fibration.*

Proof. Pick a morphism $f \colon b \to c$ in the base category B, and any any object (c, x) lying in the fiber $\pi^{-1}(c)$. We want to find a cartesian morphism $(b, y) \to (c, x)$ lying over f. This amounts to a morphism $y \to F(f)(x)$ in $F(b)$. We simply choose $\mathrm{id}_{F(f)(x)} \colon F(f)(x) \to F(f)(x)$. To show it is cartesian, consider a diagram as follows:



Finding a dashed arrow in the diagram amounts to finding a morphism $y \to F(k)(F(f)(x))$. We see that $F(k)(F(f)(x)) = (F(k) \circ F(f))(x) = F(f \circ k)(x) = F(h)(x)$. So, we can use the morphism $s\colon y \to F(h)(x)$ for this purpose. It is easily seen that (k, s) satisfies the diagram. To see that it is the unique such morphism, let $(k, t)\colon (a, y) \to (b, F(f)(x))$ be another morphism making the diagram commute. Then $s = F(k)(\mathrm{id}_{F(f)(x)}) \circ t = \mathrm{id}_{F(k)(F(f)(x))} \circ t = \mathrm{id}_{F(h)(x)} \circ t = t$. So, it is unique. Hence $(f, \mathrm{id}_{F(f)(x)})$ is cartesian.

It is also clear from the definition of these cartesian morphisms that they are closed under composition and include the identities. Hence, π is a split Grothendieck fibration. □

Remark 3.2.4. If one starts with a *pseudofunctor* F, the Grothendieck construction still works with one minor change: the second component of the composition $(f', g') \circ (f, g)$ is defined to be $F(f)(g') \circ g$ postcomposed with a component of the natural isomorphism $F(f' \circ f) \cong F(f) \circ F(f')$ given as part of the data of a pseudofunctor. One can check that $\int_B F$ is still a category in the usual sense, but this time it involves using the coherence conditions on the pseudofunctor, rather than being a simple verification. Furthermore, the projection functor $\pi\colon \int_B F \to B$ is still a Grothendieck fibration, but no longer necessarily split.

Now, let us continue onwards to proving Theorem 3.2.1. A full proof of this theorem unfortunately involves more 2-categorical constructions and verifications than we are willing to explore. Reference for such proofs include [Bor94, Vol. 2, Sect. 8.3] and [JY20, Ch. 10]. For our purposes, we will provide a proof sketch that attempts to balance providing enough insight about Grothendieck fibrations, while avoiding unnecessary 2-categorical complications.

Proof sketch of Theorem 3.2.1. We begin by listing the reasons why this will merely be a proof sketch:

1. On objects, we will only define S on functors p which are *split, cloven* Grothendieck fibrations. This will ensure that $S(p)$ is a *strict* functor. On 1-morphisms, that is Cartesian functors, we will only define S in the case where the Cartesian functor H in question sends the chosen cleavage in the domain to the chosen cleavage in the codomain. By doing this, $S(H)$ will be a (strict) 2-natural transformation, instead of a pseudonatural transformation.

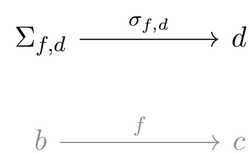
2. We will not define S on 2-morphisms, being vertical natural transformations. This is because S applied to a vertical natural transformation would have to be a *modification* between pseudonatural transformations of pseudofunctors. In other words, a 2-morphism in the category of pseudofunctors, which is something we will not go into. Modifications are explained in [JY20, Ch. 4.4].

3. Because of the preceding item, we will not show that S induces equivalences of hom-categories, as we will not have defined S on their morphisms. We will merely show that S is essentially surjective on objects that are *strict* **Cat**-valued presheaves, in the sense that for every strict presheaf there is some split Grothendieck fibration such that applying S to it gives a presheaf that is 2-naturally isomorphic to it.

We feel that these simplifications do not take away from too much of the understanding of the equivalence. However, it is understandable that some readers will be left unsatisfied with so many missing details. To accommodate for this, we have sprinkled in some remarks in the course of the proof sketch indicating how one can remove these simplifications. With all that said, let us begin.

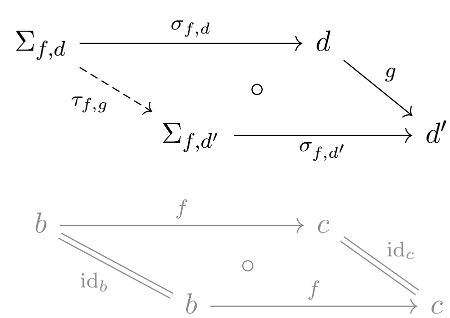
We define the functor S on (our choice of) objects as follows. Given a split fibration $p\colon E \to B$ in $\mathrm{Fib}^{\mathrm{splitcart}}(B)$, we must produce a functor $S(p)\colon B^{\mathrm{op}} \to \mathbf{Cat}$.

Taking inspiration from the analogous construction for sets, we set $S(p)(b) = p^{-1}(b)$, the fiber of p over the object $b \in B$ as in Definition 2.1.1. The action of $S(p)$ on morphisms is more elaborate. Given a morphism $f\colon b \to c$ in B, $S(p)(f)$ must itself be a functor from $p^{-1}(c)$ to $p^{-1}(b)$. This is where we use the fact that p is a Grothendieck fibration. By assumption we can find a cartesian morphism as shown:



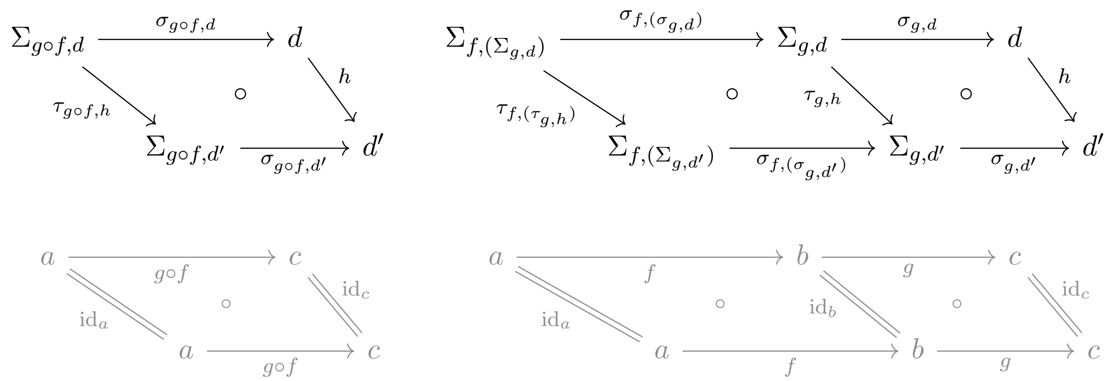
Furthermore, as p is a *split* fibration, we can assume that the collection of all these cartesian morphisms assemble into a split cleavage. We define $S(p)(f)(d) = \Sigma_{f,d}$, the domain of the chosen cartesian morphism.

Given a morphism $g\colon d \to d'$ in $p^{-1}(c)$, we can form the diagram below:



This diagram makes sense since g, being in $p^{-1}(c)$, is vertical. The morphism $\tau_{f,g}$ satisfying the diagram exists and is unique because $\sigma_{f,d'}$ is cartesian. We define $S(p)(f)(g) = \tau_{f,g}$. This assignment turns $S(p)(f)$ into a functor: by uniqueness in the diagram above, it is easy to see that it respects identities and composition.

Next, we show that $S(p)$ is a functor. Pick objects a, b, c and composable morphisms $f\colon a \to b$, $g\colon b \to c$ in B. We will show that $S(p)(g \circ f) = S(p)(f) \circ S(p)(g)$ using the following two diagrams, where $h\colon d \to d'$ is an arbitrary morphism in $p^{-1}(c)$:



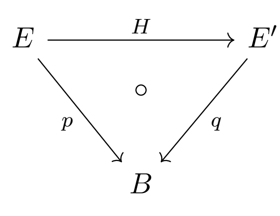
By unwinding definitions, we see that

1. $\Sigma_{g\circ f,d} = S(p)(g\circ f)(d)$ and $\Sigma_{f,(\Sigma_{g,d})} = S(p)(f)(S(p)(g)(d)) = (S(p)(f)\circ S(p)(g))(d)$,
2. $\tau_{g\circ f,h} = S(p)(g\circ f)(h)$ and $\tau_{f,(\tau_{g,h})} = S(p)(f)(S(p)(g)(h)) = (S(p)(f)\circ S(p)(g))(h)$.

Because we specifically chose a split cleavage, the two objects in the first point are equal. For the same reason, the two morphisms in the second point are equal. So these functors agree on objects and on morphisms, and so they are the same, as wanted. Thus $S(p)$ respects composition. The proof that it preserves identities is an easier version of the preceding argument using the same techniques, so we omit it.

Remark 3.2.5. If we do not want to restrict ourselves to split fibrations, we can do the following. First, we notice that both morphisms $\sigma_{g,d}\circ\sigma_{f,(\Sigma_{g,d})}$ and $\sigma_{g\circ f,d}$ are cartesian by Lemma 2.2.4, and they project to the same morphism in B. By Lemma 2.2.1, there is a canonical vertical isomorphism $\Sigma_{f,(\Sigma_{g,d})} \cong \Sigma_{g\circ f,d}$. One then shows that the collection of these isomorphisms assemble into a natural isomorphism $S(p)(g\circ f)\cong S(p)(f)\circ S(p)(g)$. Similarly one obtains a canonical natural isomorphism $S(p)(\mathrm{id}_b)\cong \mathrm{id}_{S(p)(b)}$. After checking that these isomorphisms satisfy the relevant coherence conditions, one deduces that $S(p)$ is a pseudofunctor.

Next, we will define S on (1)-morphisms. Pick a morphism H in $\mathrm{Fib}^{\mathrm{splitcart}}(B)$. This is a cartesian functor, so in particular the diagram below commutes.

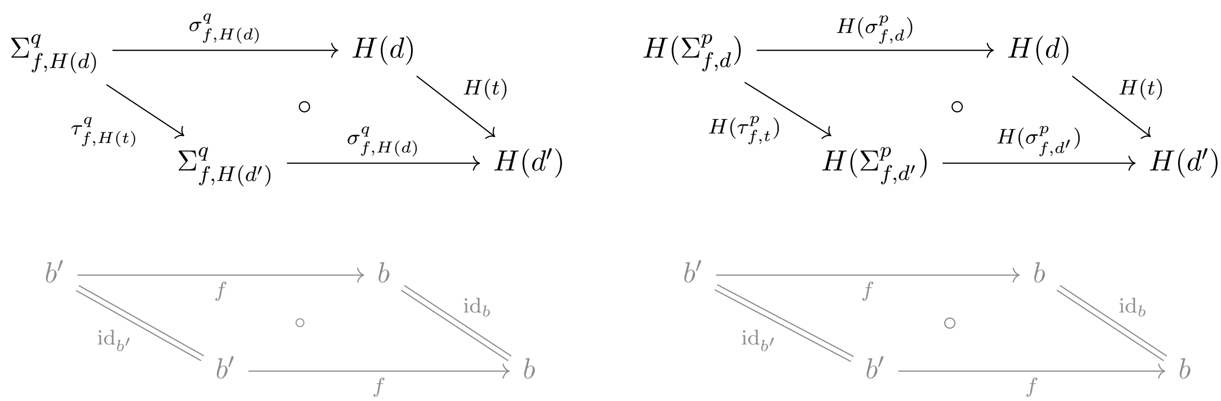


We will define a 2-natural transformation $S(H)\colon S(p)\Rightarrow S(q)$ (pseudonatural transformations are the 1-morphisms in the category of pseudofunctors). This amounts to a natural family of functors $S(H)_b\colon p^{-1}(b)\to q^{-1}(b)$, parameterized by objects $b\in B$. Notice that if e is an object in $p^{-1}(b)$, then $q(H(e)) = p(e) = b$, so $H(e)\in q^{-1}(b)$. Similarly, H sends morphisms in $p^{-1}(b)$ to morphisms in $q^{-1}(b)$. This means that we can define $S(H)_b$ simply by restricting the domain and codomain of H to $p^{-1}(b)$ and $q^{-1}(b)$, respectively. This definition makes it clear that $S(H)_b$ is a functor.

All that remains to show is 2-naturality of $S(H)$. Choose a morphism $f\colon b' \to b$ in B. The naturality square is:

$$\begin{array}{ccc} p^{-1}(b) & \xrightarrow{S(p)(f)} & p^{-1}(b') \\ {\scriptstyle S(H)_b}\downarrow & & \downarrow{\scriptstyle S(H)_{b'}} \\ q^{-1}(b) & \xrightarrow[S(q)(f)]{} & q^{-1}(b') \end{array}$$

This is a diagram of functors, so we check its commutativity on objects and morphisms. Let d, d' be objects, and $t\colon d \to d'$ be a morphism in $p^{-1}(b')$. Consider the following diagrams, where we have labeled the chosen cartesian morphisms with superscripts indicating which fibration they belong to:



We once again make two observations:

1. $\Sigma^q_{f,H(d)}$ and $H(\Sigma^p_{f,d})$ are the two composite functors of the naturality square applied to d,
2. $\tau^q_{f,H(t)}$ and $H(\tau^p_{f,t})$ are the two composite functors applied to t.

We use our assumption that H sends the chosen cleavage of p to the chosen cleavage of q to conclude that the two objects in the first point are equal, and that the top horizontal morphisms of the black squares are equal to each other, similarly for the bottom horizontal morphisms. This means that the two black squares are identical except possibly at the left diagonal morphisms, being the morphisms in the second point. But these are both vertical morphisms making the squares commute, and $\sigma^q_{f,H(d)}$ is cartesian, so they are equal. This shows that the two composite functors are equal on objects and on morphisms. Thus $S(H)$ is 2-natural (we still technically have to check that the hidden condition in Definition 1.0.9 is satisfied, but we omit this).

Remark 3.2.6. What if we do not assume that H sends cleavages to cleavages on the nose? Notice that since H is a cartesian functor, $H(\sigma^p_{f,d})$ is a cartesian morphism. It has the same codomain and projects down to the same morphism as $\sigma^q_{f,H(d)}$. Thus, we once again obtain a canonical vertical isomorphism $H(\Sigma^p_{f,d}) \cong \Sigma^q_{f,H(d)}$. These assemble into a natural isomorphism $S(H)_{b'} \circ S(p)(f) \cong S(p)(f) \circ S(H)_b$. Finally, the collection of these natural isomorphisms make $S(H)$ into a pseudonatural transformation $S(H)\colon S(p) \Rightarrow S(q)$, which one verifies by checking the coherence conditions.

In order to prove essential surjectivity on (our choice of) objects, we will use the Grothendieck construction (Definition 3.2.2). Pick any 2-functor $F\colon B^{\mathrm{op}} \to \mathbf{Cat}$. We claim that $S(\int_B F)$ is 2-naturally isomorphic to F.

A 2-natural isomorphism, in this context, consists of family of functors, each being isomorphisms of categories, indexed by objects $b \in B$. We will define these natural isomorphisms componentwise, as usual. Pick an object $b \in B$. One can easily check that the assignment

$$x \xrightarrow{\;g\;} y \qquad \rightsquigarrow \qquad (b,x) \xrightarrow{(\mathrm{id}_b, g)} (b,y)$$

defines a functor $F(b) \to \pi^{-1}(b) = S(\int_B F)(b)$, which is bijective on objects and on all hom-sets, and hence is an isomorphism of categories. This is the component of the 2-natural isomorphism at b.

It remains to show 2-naturality. Pick a morphism $f\colon b \to b'$ in B. The relevant naturality square is

$$\begin{array}{ccc} F(b) & \xrightarrow{\;F(f)\;} & F(b') \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ S(\int_B F)(b) & \xrightarrow[S(\int_B F)(f)]{} & S(\int_B F)(b') \end{array}$$

where the vertical arrows are the functors we just defined. This is a diagram of functors, so we must show that the composites are equal on objects and on morphisms. So, pick objects x, y and a morphism $g\colon x \to y$ in $F(b)$. A quick calculation shows that the right-then-down composite applied to this is $(\mathrm{id}_{b'}, F(f)(g))\colon (b', F(f)(x)) \to (b', F(f)(y))$. The down-then-right composite applied to the same thing is defined to be the unique dashed arrow in the following diagram:

$$\begin{array}{ccccc} (b', F(f)(x)) & & \xrightarrow{(f, \mathrm{id}_{F(f)(x)})} & & (b,x) \\ & \overset{S(\int_B F)(f)((\mathrm{id}_b, g))}{\searrow} & \circ & & \searrow{\scriptstyle (\mathrm{id}_b, g)} \\ & & (b', F(f)(y)) & \xrightarrow[(f, \mathrm{id}_{F(f)(y)})]{} & (b,y) \end{array}$$

$$\begin{array}{ccccc} b' & & \xrightarrow{\;f\;} & & b \\ & {\scriptstyle \mathrm{id}_{b'}} \Vert & \circ & & \Vert {\scriptstyle \mathrm{id}_b} \\ & & b' & \xrightarrow[f]{} & b \end{array}$$

Here we are using the split cleavage for π provided by Proposition 3.2.3. We immediately see that the two composite functors agree on objects. Next, we see that setting the dashed arrow equal to $(\mathrm{id}_{b'}, F(f)(g))$ satisfies the diagram above, hence by uniqueness the two composites also agree on morphisms. So, they are equal, and hence (after forgetting about the hidden compatibility condition in the definition of the 2-natural transformation) the family of isomorphisms of categories assembles into a 2-natural isomorphism $F \cong S(\int_B F)$. Hence, S is essentially surjective on (some) objects.

Remark 3.2.7. If F is a merely a pseudofunctor, we have remarked that π is not necessarily a split fibration. Still, we can use similar arguments to the ones found in the previous remarks to see that we obtain natural isomorphisms between the two composite functors in the naturality square, which assemble into a *pseudonatural* isomorphism $F \cong S(\int_B F)$.

$\square$

4 The free Grothendieck fibration

4.1 Motivation

In the previous section, we have seen (or sketched) how to turn a Grothendieck fibration $p\colon E \to B$ into a pseudofunctor $S(p)\colon B^{\mathrm{op}} \to \mathbf{Cat}$. Looking back, we see that our construction relied heavily on the fact that p is a fibration. Without this property, it is unclear how to define the pseudofunctor $S(p)$ on morphisms of B.

Still, suppose that we wanted to turn an arbitrary functor $F\colon A \to B$ into a **Cat**-valued presheaf. Is this dream unattainable? Not quite! One can do the following: first, replace the functor F by a suitable Grothendieck fibration p_F, and then apply the construction of the previous section to the replacement p_F. If this process of replacement is done in a sufficiently canonical and controlled way, the results may be satisfactory.

This situation also pops up in homotopy theory, where it may be more familiar. Given a continuous map of topological spaces $f\colon X \to Y$, some constructions in the field require f to be a fibration, as defined in Example 2.3.3, for best results. There is a systematic way of replacing the function f with a fibration: the replacement is the projection of the *mapping path space* P_f associated to f. As a set, P_f consist of triples (y, x, γ) where $y \in Y$, $x \in X$ and $\gamma\colon I \to Y$ is a path in Y from y to $f(x)$. The projection $\pi\colon P_f \to Y$ is given by sending (y, x, γ) to y, and this can be shown to be a fibration and a suitable replacement for f.

Let's now try to do something analogous for a functor of categories $F\colon A \to B$. By interpreting paths as morphisms, our analogue of a mapping path space should be a category whose objects are triples (b, a, k) where $a \in A$ and $b \in B$ are objects, and k is a morphism in B from b to $F(a)$. It just so happens that we already know of a category like this: the comma category $B \downarrow F$, whose objects and morphisms are

$$
\begin{array}{c}
b \\
\Big\downarrow{\scriptstyle k} \\
F(a)
\end{array}
\qquad\qquad
\begin{array}{ccc}
b & \xrightarrow{\;f\;} & b' \\
{\scriptstyle k}\Big\downarrow & \circ & \Big\downarrow{\scriptstyle k'} \\
F(a) & \xrightarrow[F(g)]{} & F(a')
\end{array}
$$

This intuition turns out to be correct! But first, we must specify precisely what we mean when we say "controlled and specified replacement". For us, it will be the *free* fibration on F, in the sense of being the value of a left adjoint of the forgetful functor (subcategory inclusion) $U\colon \mathrm{Fib}^{\mathrm{cart}}(B) \to \mathbf{Cat}_{/B}$.

...well, not quite. The notion of an ordinary left adjoint to U turns out to be too strong. Luckily, we have seen that both $\mathbf{Cat}_{/B}$ and $\mathrm{Fib}^{\mathrm{cart}}(B)$ are 2-categories, and it is easily verified that U is a 2-functor. So, instead, we can ask for a left *pseudoadjoint* to U, as in Definition 1.0.11.

Taking all this into account, below, we will construct a 2-functor

$$\mathrm{Free}\colon \mathbf{Cat}_{/B} \to \mathrm{Fib}^{\mathrm{cart}}(B),$$

together with a 2-natural transformation

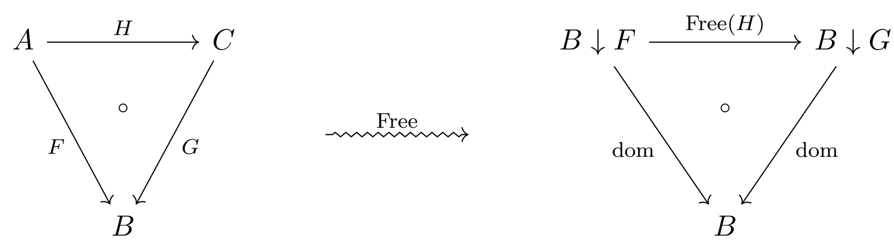
$$\eta\colon \mathrm{id}_{\mathbf{Cat}_{/B}} \Rightarrow U \circ \mathrm{Free},$$

such that precomposition with η induces a natural equivalence of hom-categories

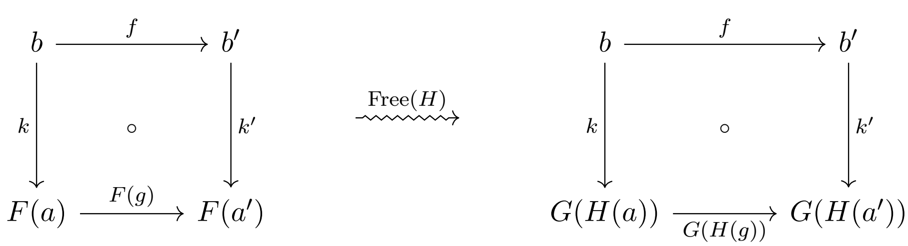
$$\mathrm{Hom}_{\mathrm{Fib}^{\mathrm{cart}}(B)}(\mathrm{Free}(-), -) \simeq \mathrm{Hom}_{\mathbf{Cat}_{/B}}(-, U(-)).$$

4.2 Construction of free functor and unit

Let's get started. First, the action on objects: given a functor $F\colon A \to B$ in $\mathbf{Cat}_{/B}$, the free fibration is defined to be $\mathrm{dom}\colon B \downarrow F \to B$, as indicated in the previous discussion. By Example 2.3.2 this is a Grothendieck fibration. Next, the action on 1-morphisms. We need to transform commutative triangles on the left to commutative triangles on the right of the following diagram:



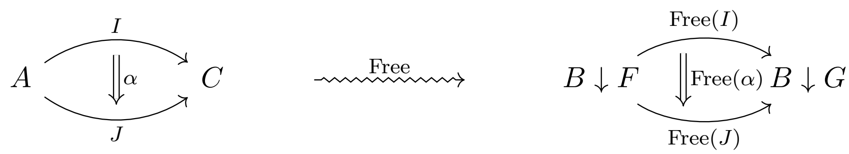
Of course, $\mathrm{Free}(H)$ is to be a functor itself, so we must specify *its* action on objects and morphisms in $B \downarrow F$, which we define below.



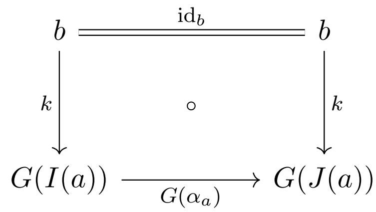
Notice, we haven't really done anything; only used the fact that $F = G \circ H$. This evidently respects composition and identities, so it is functorial. It also clearly commutes with the domain projections.

The last thing we need to check for $\mathrm{Free}(H)$ to be a morphism in $\mathrm{Fib}^{\mathrm{cart}}(B)$ is that it is a cartesian functor. It is enough to check $\mathrm{Free}(H)$ sends all cartesian morphisms in the cleavage of Example 2.3.2 to cartesian morphisms; looking at how they are defined, it is clear that $\mathrm{Free}(H)$ sends them to morphisms of the same form, hence to cartesian morphisms.

Finally, we need to define the action of Free on 2-morphisms:



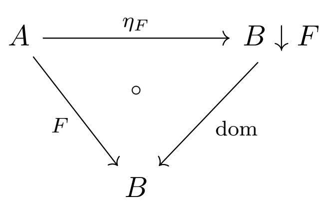
So, suppose we are given a vertical natural transformation α between two functors I and J in $\mathrm{Hom}_{\mathbf{Cat}_{/B}}(A, C)$. We define $\mathrm{Free}(\alpha)\colon \mathrm{Free}(I) \Rightarrow \mathrm{Free}(J)$ as follows: given an object $k\colon b \to F(a)$ in $B \downarrow F$, $\mathrm{Free}(\alpha)_k$ is the square



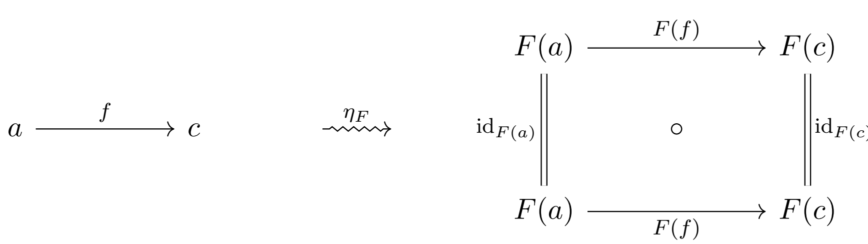
This makes sense: as α is vertical and I and J commute with F and G, we get that $G(\alpha_a) = \mathrm{id}_{F(a)} = \mathrm{id}_{G(I(a))} = \mathrm{id}_{G(J(a))}$, so the square is commutative. We omit checking that $\mathrm{Free}(\alpha)$ is a natural transformation, but this follows straightforwardly from the fact that α is natural. Clearly $\mathrm{Free}(\alpha)$ is vertical.

We also omit the verification that Free respects the two types of compositions of natural transformations; this is a lengthy but routine verification. Thus, we see that Free is a 2-functor.

Next up, we define the unit $\eta\colon \mathrm{id}_{\mathbf{Cat}_{/B}} \Rightarrow U \circ \mathrm{Free}$. Given an object $F\colon A \to B$ in $\mathbf{Cat}_{/B}$, η_F should be a 1-morphism in $\mathbf{Cat}_{/B}$ from F to $U(\mathrm{Free}(F))$. In other words, η_F should be a functor such that the following triangle commutes:



Its action on objects and morphisms in A is straightforward:



This is clearly functorial and makes the above triangle commute. We omit checking that η is a 2-natural transformation, as this is unenlightening, but it is once again a routine verification.

4.3 Showing adjointness

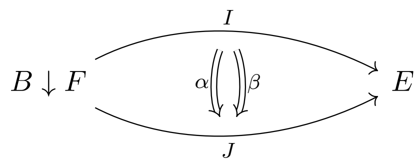
Pick objects $F\colon A \to B$ and $p\colon E \to B$ in $\mathbf{Cat}_{/B}$ and $\mathrm{Fib}^{\mathrm{cart}}(B)$, respectively. We will show precomposition by η_F induces an equivalence of hom-categories as shown:

$$\eta_F{}^*\colon \mathrm{Hom}_{\mathrm{Fib}^{\mathrm{cart}}(B)}(\mathrm{Free}(F), p) \simeq \mathrm{Hom}_{\mathbf{Cat}_{/B}}(F, U(p)).$$

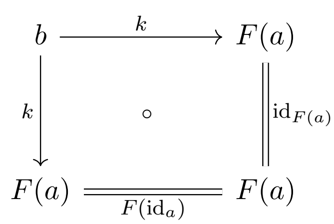
Before we start, we should mention the action of $\eta_F{}^*$ on natural transformations (morphisms in the hom-categories). It is given by *whiskering*: given a natural transformation α, we get $\eta_F{}^*(\alpha)_a = \alpha_{\eta_F(a)}$ for all objects $a \in A$.

We will omit checking 2-naturality of this precomposition map . The reason is that it follows formally from the fact that η is a 2-natural transformation. In other words, it is simply a 2-categorical result that would not help us understand the subject matter at hand.

We begin with faithfulness. Pick cartesian functors, and parallel vertical natural transformations as shown:



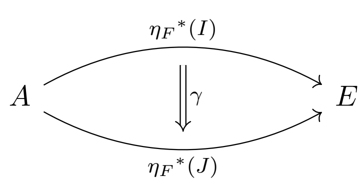
Assume $\eta_F{}^*(\alpha) = \eta_F{}^*(\beta)$. Then $\alpha_{\eta_F(a)} = \beta_{\eta_F(a)}$ for all $a \in A$. We must show that $\alpha = \beta$. To this end, pick any object $k\colon b \to F(a)$ in $B \downarrow F$. Notice that



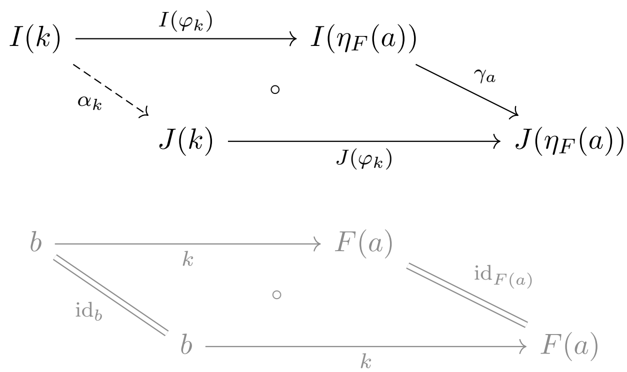
is a cartesian morphism in $B \downarrow F$, by Example 2.3.2. We will denote it by $\varphi\colon k \to \mathrm{id}_{F(a)}$.

As J is a cartesian functor, $J(\varphi)$ is a cartesian morphism in E. Now, by naturality we get that $\alpha_{\eta_F(a)} \circ I(\varphi) = J(\varphi) \circ \alpha_k$ and $\beta_{\eta_F(a)} \circ I(\varphi) = J(\varphi) \circ \beta_k$. By assumption we get that $J(\varphi) \circ \alpha_k = J(\varphi) \circ \beta_k$, and we also have that $p(\alpha_k) = p(\beta_k) = \mathrm{id}_b$ by verticality of the natural transformations. Hence, we deduce that $\alpha_k = \beta_k$. As k was an arbitrary object in $B \downarrow F$, we get that $\alpha = \beta$, like we wanted to show.

Next, we show fullness. Pick any cartesian functors I and J, and an arbitrary vertical natural transformation as shown:

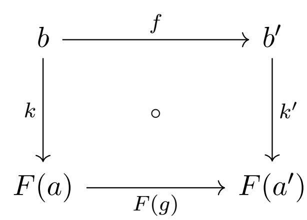


We must produce a vertical natural transformation $\alpha\colon I \Rightarrow J$ such that ${\eta_F}^*(\alpha) = \gamma$. We will define α componentwise as follows. Pick any object $k\colon b \to F(a)$ in $B \downarrow F$. We will make use of the morphism φ from before, but we will write it as φ_k to indicate the dependence on k. We define α_k to be the unique morphsim such that the following diagram is satisfied:

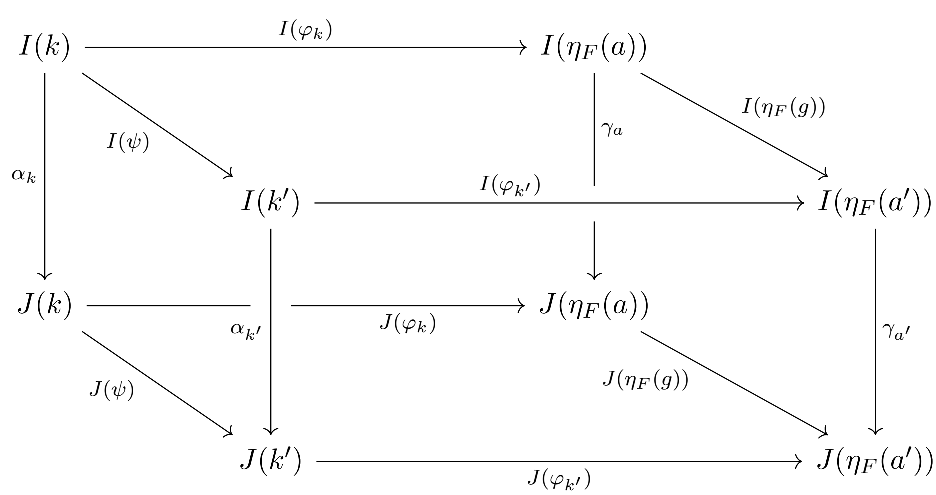


This works because J is a cartesian functor, so $J(\varphi_k)$ is a cartesian morphism, hence there is a unique dashed morphism as shown.

We need to show that α is natural. Verticality is clear from the definition. Pick any morphism



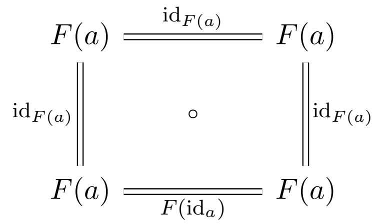
in $B \downarrow F$, and denote it by ψ. We will show that $\alpha_{k'} \circ I(\psi) = J(\psi) \circ \alpha_k$, by contemplating the following magical cube:



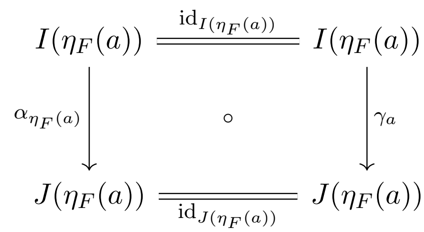
First, notice that the back and front faces are the same ones as in the definition of α, hence they commute by definition. The top and bottom faces commute by the fact that the square defining ψ is commutative and by functoriality of I and J. And, the right face commutes by naturality of γ.

The remaining face is precisely the relevant naturality square for α. By chasing the cube around, we see that the left face commutes after postcomposing with $J(\varphi_{k'})$. A quick calculation yields that $p(J(\varphi) \circ \alpha_k) = p(\alpha_{k'} \circ I(\varphi)) = f$. As J is a cartesian functor and $\varphi_{k'}$ is a cartesian morphism, we get that $J(\varphi_{k'})$ is a cartesian morphism. Thus, we conclude that the left face commutes. As ψ was an arbitrary morphism in $B \downarrow F$, we deduce that α is a vertical natural transformation.

Finally, we show that precomposing α with the unit gives us what we sought. For any $a \in A$, consider the following commutative square:



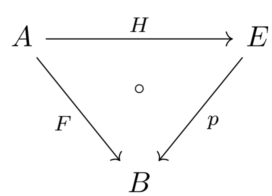
This is an identity morphism in $B \downarrow F$, and so by definition



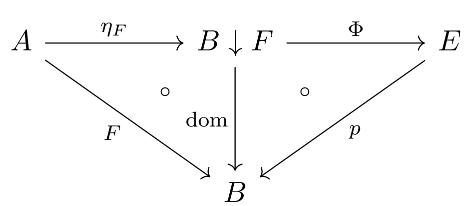
commutes, showing that $\alpha_{\eta_F(a)} = \gamma_a$. As $a \in A$ was arbitrary, we get that ${\eta_F}^*(\alpha) = \gamma$. Hence, ${\eta_F}^*$ is full.

Lastly, we will show that η_F is essentially surjective on objects. In fact, we will show the slightly stronger statement that η_F is strictly surjective on objects.

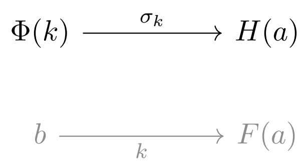
Pick any object $H \in \mathrm{Hom}_{\mathbf{Cat}_{/B}}(F, U(p))$. This is simply a functor such that the following triangle commutes:



We will construct a cartesian functor $\Phi \in \mathrm{Hom}_{\mathrm{Fib}^{\mathrm{cart}}(B)}(\mathrm{Free}(F), p)$ such that ${\eta_F}^*(\Phi) = H$. This amounts to factoring the above triangle as below:

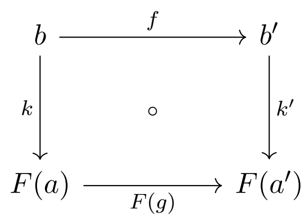


We begin by defining Φ on objects. Pick $k\colon b \to F(a)$ in $B \downarrow F$. We will once again make use of the morphism $\varphi_k\colon k \to \eta_F(a)$ defined earlier. As we have seen, φ_k is a cartesian morphism, so if Φ is to be a cartesian functor, $\Phi(\varphi_k)$ must be a cartesian morphism. Also, from the relations $H = \Phi \circ \eta_F$ and $p \circ \Phi = \mathrm{dom}$, we must have $H(a) = \Phi(\eta_F(a))$ and $p(\Phi(k)) = b$. In addition, we know that E is a cartesian fibration. Therefore, we can pick a cartesian morphism as indicated:

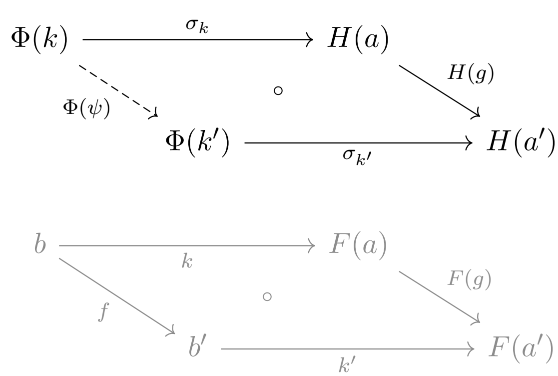


In other words, we define $\Phi(k)$ to be the domain of a cartesian morphism σ_k with codomain $H(a)$ that projects to k. We make sure that whenever the grey morphism k is an identity, so is σ_k. This ensures that $\Phi(\eta_F(a)) = H(a)$.

The action on morphisms is as follows: given a morphism

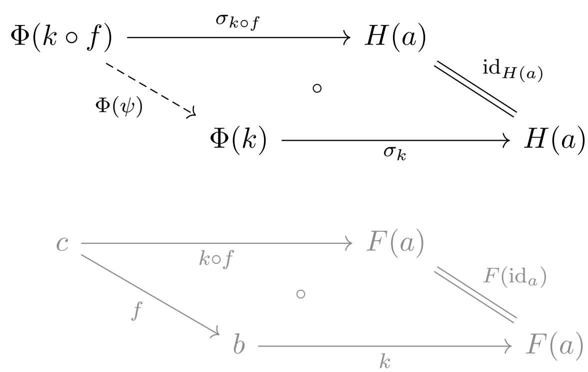


in $B \downarrow F$, which we will again denote by ψ, we define $\Phi(\psi)$ to be the unique morphism satisfying the diagram below:



As $\sigma_{k'}$ is cartesian, this is well-defined. By the uniqueness of this morphism, it is easy to see that this assignment is functorial. It also easily follows that $p \circ \Phi = \mathrm{dom}$ and $\Phi \circ \eta_F = H$.

All that remains is to show that Φ is a cartesian functor. As we have already mentioned, it's enough to check that morphisms in the cleavage given in Example 2.3.2 get sent to cartesian morphisms by Φ. Applying Φ to such a morphism ψ (which will be the bottom grey square below), we get:



Now, by definition σ_k and $\sigma_{k\circ f} = \sigma_k \circ \Phi(\psi)$ are cartesian, so by Proposition 2.2.4, $\Phi(\psi)$ is cartesian. So, Φ is a cartesian functor, and thus we have shown that η_F is surjective on objects, making it an equivalence of hom-categories.

This concludes the proof that the unit η gives rise to a pseudoadjunction between arbitrary functors and Grothendieck fibrations.

References

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