ROBUST DUALITY AND SADDLE POINT CHARACTERIZATIONS FOR NONCONVEX MULTIOBJECTIVE OPTIMIZATION WITH DATA UNCERTAINTY

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ABSTRACT. This paper is devoted to investigate the robust duality and saddle point characterizations of nonconvex multiobjective optimization with data uncertainty in both the objective and constraints. Based on the robust necessary optimality conditions, we introduce a mixed type robust dual model of the uncertain multiobjective optimization problem, which covers the Wolfe type dual model and Mond-Weir type dual model as special cases. The weak robust duality, strong robust duality and converse robust duality between the robust dual model and the robust counterpart of original problem are established under some suitable conditions. Moreover, we also obtain the robust saddle point type sufficient and necessary optimality conditions for the uncertain multiobjective optimization problem under the generalized convexity assumptions.

1. Introduction

Multiobjective optimization is also called multicriteria optimization which has multiple objectives that are generally in conflict simultaneously. Multiobjective optimization has been received an increasingly attention and extensively applied in machine learning, engineering, economy, management and sircraft design; see [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. However, the existing optimization theory and algorithms on multiobjective optimization were mainly established in the sense of accurate data.

In most practical applications, the data of parameters in optimization problems are not known exactly, and solutions to optimization problems can be exhibit remarkable sensitivity to perturbations in the parameters of the problem. So, it is worthy to study multiobjective optimization with uncertainty. It is well-known that robust optimization method is an important method to deal with uncertain optimization problems in the worst case. Robust optimization problems were first introduced by Soyster [9], and have been extensively studied in robust optimality, duality, error bounds and algorithms on various robust solutions for uncertain optimization problems; see, e.g., [10, 11, 12, 13] and reference therein.

Chuong [14] studied necessary/sufficient optimality conditions for robust (weakly) Pareto solutions of the a robust nonsmooth multiobjective optimization problem in terms of multipliers and limiting subdifferentials of the related functions, and explored weak/strong duality relations between the primal one and its dual robust

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problem under the (strictly) generalized convexity assumptions. Chen et al. [15] investigated a nonsmooth/nonconvex multiobjective optimization problems with data uncertainty by using robust approach. The robust necessary and sufficient optimality conditions as well as dualties for weakly robust efficient solution and properly robust efficient solution of the uncertain multiobjective optimization problem are established under the convex-like and generalized convexity assumptions. It is naturally proposed a question whether the robust necessary optimality conditions of uncertain multi-criteria optimization problem can be established without the convex-like assumption in more general case. Ou et al. [16] studied the robust optimality conditions as well as saddle point optimality conditions for uncertain multiobjective optimization problems by using image space analysis. Recently, Chai [17] introduced three kinds of robust dual problems, such as the robust augmented Lagrange dual, the robust weak Fenchel dual and the robust weak Fenchel-Lagrangge dual problem, of the primal optimization problem by employing this weak conjugate function. By using Lagrangian functions, strong robust duality was given in robust convex optimization in which all involved functions are convex-concave functions in [18, 19, 20]. Thereafter, the robust duality and robust saddle point results of uncertain multiobjective optimization problems were studied without convexity; see [21, 22]. Very recently, Wang, Li and Chen [23] generalized the results in [24] form differentiable cases to nondifferentiable cases under weaker assumptions, and obtained KKT type robust optimality conditions for multiobjective optimization problem with infinitely many uncertain constraints under the generalized convexity assumptions. However, robust duality of the multiobjective optimization problem with infinitely many uncertain constraints are not considered in [23]. Inspired by the above works, this paper aims to study robust duality and robust saddle point characterizations of the nonconvex multiobjective optimization problems with infinitely many uncertain constraints.

The rest of this paper is organized as follows. In Section 2, we recall some basic notations and several auxiliary results. Section 3 introduces a mixed-type robust dual problem of uncertain optimization problems and deal with duality relations between the primal-dual problems. A vector-valued Lagrangian function is constructed, and then we discuss saddle point results in Section 4. Finally, we give some conclusions in Section 5.

2. Preliminaries

Throughout of this paper, without special statements, let X be a Banach space with its topological dual space X^*, Y be a metric space, $\mathbb{R}^l_+$ be the nonnegative orthant of the l-dimensional Euclidean space $\mathbb{R}^l$. The symbol $\overset{w^*}{\to}$ means the convergence in the weak*-topology of X^*. For each $S \subseteq X$, the topological interior, the closure hull and the convex hull of S are denoted by $\mathrm{int}S$, $\mathrm{cl}S$ and $\mathrm{co}S$, respectively, while cl^*B denotes by the weak* topological closure of $B \subseteq X^*$, and $\mathrm{cl}^*\mathrm{co}B$ is weak*-closed convex hull of $B \subseteq X^*$. The closed ball with the center x and the radius ε is denoted by $B(x, \varepsilon)$.

We now consider the following uncertain constrained multiobjective optimization problem (UMP):

$$\begin{aligned} \min \quad & f(x,u) \\ \text{s.t.} \quad & x \in X,\ g_t(x,v_t) \le 0,\ \forall\, t \in T, \end{aligned}$$

where $f(x,u) := (f_1(x,u_1),\dots,f_l(x,u_l))$, $f_k : X \times U_k \to \mathbb{R}, k \in K := \{1,2,\dots,l\}$, $g_t : X \times V_t \to \mathbb{R}$, $u := (u_1,u_2,\dots,u_l)$ is an uncertain parameter with $u_k \in U_k$ a compact subset of Y for each $k \in K$ and v_t is an uncertain parameter which belongs to the compact set $V_t \subseteq Y, t \in T$ an arbitrary index set.

Since robust optimization is an effective method for treating the uncertain problem, we adopt the robust optimization method proposed in [25] to deal with (UMP) in the worst-case. The robust counterpart of (UMP) is defined as follows:

$$\text{(RMP)} \qquad \min_{x\in C} \left(\sup_{u_1\in U_1} f_1(x,u_1),\dots, \sup_{u_l\in U_l} f_l(x,u_l) \right),$$

where $C := \{x \in X : g_t(x,v_t) \le 0,\ \forall\, v_t \in V_t,\ \forall t \in T\}$ is the so-called the robust feasible region of (UMP).

For the simplicity, we denote $F_k(x) := \sup_{u_k\in U_k} f_k(x,u_k)$, $F(x) := (F_1(x),\dots,F_l(x))$ for $x \in X$, $U_k(x) := \{u_k \in U_k : f_k(x,u_k) = F_k(x)\}$ and $v := (v_t)_{t\in T} \in V := \prod_{t\in T} V_t$, and for each $t \in T$, $G_t(x) := \sup_{v_t\in V_t} g_t(x,v_t)$, $V_t(x) := \{v_t \in V_t : g_t(x,v_t) = G_t(x)\}$, $G(x) := \sup_{t\in T} G_t(x)$ and $T(x) := \{t \in T : G_t(x) = G(x)\}$.

We also define a set-valued mapping $\mathcal{V} : T \rightrightarrows Y$ as $\mathcal{V}(t) := V_t$ for all $t \in T$, and the graph of the mapping $\mathcal{V}$ is denoted by $\text{gph}\mathcal{V} := \{(t,v_t) : v_t \in V_t,\ t \in T\}$.

In order to deal with (RMP), we next recall some basic definitions and facts.

Let T be an arbitrary index set and $|T|$ mean the hypervolume of T. $\mathbb{R}^{|T|}$ is defined as:

$$\mathbb{R}^{|T|} := \{\lambda = (\lambda_t)_{t\in T} : \text{ there only finite } t \in T, \lambda_t \neq 0\},$$

and the nonnegative orthant of $\mathbb{R}^{|T|}$ is defined by

$$\mathbb{R}^{|T|}_+ := \left\{\lambda \in \mathbb{R}^{|T|} : \lambda_t \ge 0, \forall t \in T\right\}.$$

Let the function $\phi : X \to \mathbb{R}$ is locally Lipschitz at $\bar{x} \in X$. The generalized Clarke directional derivative of ϕ at $\bar{x}$ in the direction $d \in X$ is defined by

$$\phi^0(\bar{x};d) := \limsup_{\substack{x\to\bar{x} \\ \tau\to 0}} \frac{\phi(x+\tau d) - \phi(x)}{\tau},$$

and the one-side directional derivative of ϕ at $\bar{x}$ in the direction $d \in X$ is defined by

$$\phi'(\bar{x};d) := \lim_{\tau\to 0} \frac{\phi(\bar{x}+\tau d) - \phi(\bar{x})}{\tau}.$$

The function ϕ is called regular at $\bar{x}$ if $\phi^0(\bar{x};\cdot) = \phi'(\bar{x};\cdot)$.

The Clarke subdifferential of ϕ at $\bar{x}$ is defined by

$$\partial\phi(\bar{x}) := \left\{x^* \in X^* : \langle x^*, d\rangle \le \phi^0(\bar{x};d),\ \forall\, d \in X\right\}.$$

Definition 2.1. Let X be a Banach space, W be a compact metric space and a function $\phi : X \times W \to \mathbb{R}$. The multifunction $(x,\omega) \rightrightarrows \partial_x \phi(x,\omega) \subseteq X^*$ is said to be weak* closed at $(\bar{x},\bar{\omega})$ if $x_n^* \in \partial_x \phi(x_n,\omega_n)$ with $(x_n,\omega_n) \to (\bar{x},\bar{\omega})$, and $x_n^* \xrightarrow{w^*} x^*$ implies that $x^* \in \partial_x \phi(\bar{x},\bar{\omega})$.

Definition 2.2. $\bar{x} \in C$ is called a local robust weakly efficient solution of problem (UMP) if, there exists a neighborhood O of $\bar{x}$ such that

$$F(x) - F(\bar{x}) \notin -\operatorname{int} \mathbb{R}^l_+, \ \forall\, x \in C \cap O.$$

In particular, if $C \subseteq O$, the local robust weakly efficient solution notion reduces to the robust weakly efficient solution notion of (UMP).

Definition 2.3. [23] The robust Mangasarian-Fromowitz constraint qualification (RMFCQ) holds at $\bar{x} \in C$ if

$$0 \notin \mathrm{cl}^* \mathrm{co} \left\{ \partial_x g_t(\bar{x}, v_t) : v_t \in V_t(\bar{x}), t \in T(\bar{x}) \right\}.$$

Definition 2.4. [23] $(f_k, g_t)_{k\in K, t\in T}$ is said to be generalized convex at $\bar{x} \in X$ if for any $x \in X$, there exist $d \in X$ such that

$$f_k(x,u_k) - f_k(\bar{x},u_k) \geq \langle \xi_k, d\rangle, \ \forall \xi_k \in \partial_x f_k(\bar{x},u_k), \forall u_k \in U_k(\bar{x}), k \in K,$$
$$g_t(x,v_t) - g_t(\bar{x},v_t) \geq \langle \eta_t, d\rangle, \ \forall \eta_t \in \partial_x g_t(\bar{x},v_t), \forall v_t \in V_t(\bar{x}), \forall t \in T(\bar{x}).$$

It is worth noting that if for each $u_k \in U_k, k = 1,2,\ldots,l, v_t \in V_t, t \in T, f_k(\cdot,u_k)$ and $g_t(\cdot,v_t)$ are convex, then $(f_k, g_t)_{k\in K,t\in T}$ is generalized convex. Besides, if for each $u \in U$ and $v \in V$, f is $\mathbb{R}^l_+$-generalized convex and $g := (g_t)_{t\in T}$ is $\mathbb{R}^{|T|}_+$-generalized convex defined as [15, Definition 2.6], then $(f_k, g_t)_{k\in K,t\in T}$ is generalized convex.

Lemma 2.5. [23, Theorem 3.1] *Let $x \in X$. Suppose that the following conditions hold:*

(i) *For any given $t \in T, g_t(x,v_t)$ is upper semi-continuous (u.s.c) in $v_t \in V_t$, and for any given $v \in V$, $g_t(x,v_t)$ is u.s.c in $t \in T$.*
(ii) *For any given $t \in T, g_t(x,v_t)$ is locally Lipschitz in x uniformly for $v_t \in V_t$, and for any given $v \in V$, $g_t(x,v_t)$ is locally Lipschitz in x uniformly for $t \in T$.*
(iii) *For any given $t \in T, \partial_x g_t(x,v_t)$ is weak* closed in (x,v_t) for each $v_t \in V_t(x)$, and for any given $v \in V$, $\partial_x g_t(x,v_t)$ is weak* closed in (x,t) for each $t \in T(x)$.*
(iv) *$g_t(x,v_t)$ is regular in x for each $v_t \in V_t$ and $t \in T$.*

Then $\partial G(x) = cl^* co \{\partial_x g_t(x,v_t) : v_t \in V_t(x), t \in T(x)\}$.

For the sake of brevity, we give the blanket hypotheses (see, e.g., [23, 25]):

(H1) For each $k \in K$, $f_k(x,u_k)$ is u.s.c in $u_k \in U_k$.
(H2) For each $k \in K$, $f_k(x,u_k)$ is locally Lispschitz in x uniformly for $u_k \in U_k$, namely, there exists $L > 0$ such that

$$\|f_k(x_1,u_k) - f_k(x_2,u_k)\| \leq L \|x_1 - x_2\|, \ \forall\, x_1,x_2 \in B(x,\delta), \ u_k \in U_k.$$

(H3) For each $k \in K$, the subdifferential $\partial_x f_k(x,u_k)$ is weak* closed in (x,u_k) for each $u_k \in U_k(x)$.

(H4) For each $k \in K$, $f_k(x, u_k)$ is regular in x for each $u_k \in U_k$, i.e., $f_{kx}^0(x, u_k; \cdot) = f_{kx}'(x, u_k; \cdot)$.

The following necessary conditions for local robust weakly efficient solutions of problem (UMP) can be obtained from [23, Theorem 4.1].

Lemma 2.6. *(Necessary optimality conditions) Let $\bar{x}$ be a local robust weakly efficient solution of (UMP) and the (RMFCQ) hold at $\bar{x}$. Suppose that $f_k(x, u_k)$ satisfies (H1)-(H4) for each $k \in K$ and the assumptions of Lemma 2.5 are fulfilled. Then there exist $\theta_k \geq 0, k \in K$ not all zero, and $\lambda \geq 0$ such that*

$$0 \in \sum_{k \in K} \theta_k cl^* co \{\partial_x f_k(\bar{x}, u_k) : u_k \in U_k(\bar{x})\} + \lambda cl^* co \{\partial_x g_t(\bar{x}, v_t) : v_t \in V_t(\bar{x}), t \in T(\bar{x})\},$$

and $\lambda \sup_{v_t \in V_t, t \in T} g_t(\bar{x}, v_t) = 0$.

3. Robust dualities

In this section, based on the robust Karush-Kuhn-Tucker optimality conditions, we formulate a Mixed type robust dual model of the uncertain multiobjective optimization problem. Then we investigate the weak, strong and converse robust duality results between the robust dual model and the original problem.

We first present the following robust mixed dual model (RMD) of (RMP):

$$\begin{aligned} &\max L(z, \lambda, \beta, v) = F(z) + \sum_{t \in T} \lambda_t g_t(z, v_t) e. \\ &\text{s.t. } \beta_t \sup_{v_t \in V_t, t \in T} g_t(z, v_t) \geq 0, \\ &\quad 0 \in \sum_{k \in K} \theta_k \text{cl}^* \text{co}\{\partial_z f_k(z, u_k) : u_k \in U_k(z)\} \\ &\quad\quad + \sum_{t \in T} (\lambda_t + \beta_t) \text{cl}^* \text{co}\{\partial_z g_t(z, v_t) : v_t \in V_t(z), t \in T(z)\}, \\ &\quad \theta_k \geq 0, k \in K, \lambda_t \geq 0, \beta_t \geq 0, v_t \in V_t, t \in T, \end{aligned}$$

where $\lambda := (\lambda_t)_{t \in T} \in \mathbb{R}_+^{|T|}$, $\beta := (\beta_t)_{t \in T} \in \mathbb{R}_+^{|T|}$, $v := (v_t)_{t \in T} \in V, e := (1, 1, \ldots, 1) \in \mathbb{R}^l$.

We also denote the feasible set of (RMD) by C_D.

In particular, if $\beta_t = 0$, $t \in T$, (RMD) reduces to the following Wolfe type robust dual problem (RMDw):

$$\begin{aligned} &\max L(z, \lambda, \beta, v) = F(z) + \sum_{t \in T} \lambda_t g_t(z, v_t) e. \\ &\text{s.t. } 0 \in \sum_{k \in K} \theta_k \text{cl}^* \text{co}\{\partial_z f_k(z, u_k) : u_k \in U_k(z)\} \\ &\quad\quad + \sum_{t \in T} \lambda_t \text{cl}^* \text{co}\{\partial_z g_t(z, v_t) : v_t \in V_t(z), t \in T(z)\}, \\ &\quad \theta_k \geq 0, k \in K, \lambda_t \geq 0, v_t \in V_t, t \in T. \end{aligned}$$

Besides, if $\lambda_t = 0$, $t \in T$, (RMD) reduces to the following Mond-Weir type robust dual problem (RMDm):

$$
\begin{aligned}
&\max L(z,\lambda,\beta,v) = F(z).\\
&\text{s.t.}\; 0 \in \sum_{k\in K} \theta_k \mathrm{cl}^*\mathrm{co}\{\partial_z f_k(z,u_k) : u_k \in U_k(z)\}\\
&\qquad + \sum_{t\in T} \beta_t \mathrm{cl}^*\mathrm{co}\{\partial_z g_t(z,v_t) : v_t \in V_t(z), t \in T(z)\},\\
&\quad \beta_t \sup_{v_t\in V_t, t\in T} g_t(z,v_t) \geq 0,\\
&\quad \theta_k \geq 0, k \in K, \beta_t \geq 0, v_t \in V_t, t \in T.
\end{aligned}
$$

So, (RMD) unifies the Wolfe type robust dual model and Mond-Weir type robust dual model.

The local robust weakly efficient solution of (RMD) is defined similarly as in Definition 2.2. In the rest of this paper, we use the following notations: for $u, v \in \mathbb{R}^l$,

$$u \prec v \Leftrightarrow v - u \in \mathrm{int}\,\mathbb{R}^l_+,\; u \not\prec v \Leftrightarrow v - u \notin \mathrm{int}\,\mathbb{R}^l_+.$$

Definition 3.1. $(\bar{z}, \bar{\lambda}, \bar{\beta}, \bar{v}) \in C_D$ is said to be a local weakly efficient solution of (RMD) if, there exists a neighborhood O of $(\bar{z}, \bar{\lambda}, \bar{\beta}, \bar{v})$ such that there is no $(z, \lambda, \beta, v) \in C_D \cap O$ satisfying

$$L(\bar{z}, \bar{\lambda}, \bar{\beta}, \bar{v}) \not\prec L(z, \lambda, \beta, v).$$

In particular, if $O := X$, then the above local weakly efficient solution notion reduces to the weakly efficient solution notion. We denote S^{lw}_D and S^w_D by the set of local weakly efficient solutions and the set of weakly efficient solutions of (RMD), respectively.

The next theorem gives the robust weak duality between (RMP) and (RMD).

Theorem 3.2. (Weak robust duality) *Let $(f_k, g_t)_{k\in K, t\in T}$ be generalized convex at z. Then*

$$F(x) \not\prec L(z, \lambda, \beta, v),\; \forall\, x \in C,\; (z, \lambda, \beta, v) \in C_D. \tag{3.1}$$

Proof. For any $(z, \lambda, \beta, v) \in C_D$, there exist $\theta_k \geq 0, k \in K$ not all zero, $\lambda_t \geq 0$, $\beta_t \geq 0$, and

$$z^*_k \in \mathrm{cl}^*\mathrm{co}\{\partial_z f_k(z, u_k) : u_k \in U_k(z)\},\; k \in K, \tag{3.2}$$

and

$$x^* \in \mathrm{cl}^*\mathrm{co}\{\partial_z g_t(z, v_t) : v_t \in V_t(z), t \in T(z)\}, \tag{3.3}$$

such that

$$0 \in \sum_{k\in K} \theta_k z^*_k + \sum_{t\in T} (\lambda_t + \beta_t) x^*, \tag{3.4}$$

$$\beta_t \sup_{v_t\in V_t, t\in T} g_t(z, v_t) \geq 0. \tag{3.5}$$

Since $\theta_k \geq 0, k \in K$ are not all zero, without loss of generality, we assume that $\sum_{k\in K} \theta_k = 1$ and set $\theta = (\theta_1, \theta_2, \ldots, \theta_l)$.

Suppose to the contrary that there exists $x \in C$ such that

$$F(x) \prec L(z, \lambda, \beta, v).$$

Then

$$\langle \theta, F(x) - L(z, \lambda, \beta, v) \rangle < 0.$$

Moreover, one has

$$\sum_{k \in K} \theta_k (F_k(x) - F_k(z)) - \sum_{t \in T} \lambda_t g_t(z, v_t) < 0. \tag{3.6}$$

From the generalized convexity of $(f_k, g_t)_{k \in K, t \in T}$, it follows that there exists $d \in X$ such that

$$f_k(x, u_k) - f_k(z, u_k) \geq \langle \xi_k, d \rangle, \ \forall \xi_k \in \partial_z f_k(z, u_k), \forall u_k \in U_k(z), k \in K, \tag{3.7}$$

$$g_t(x, v_t) - g_t(z, v_t) \geq \langle \eta_t, d \rangle, \ \forall \eta_t \in \partial_z g_t(z, v_t), \forall v_t \in V_t(z), \forall t \in T(z). \tag{3.8}$$

We conclude form (3.2) that there is a net

$$\{z_r^*\}_{r \in \Lambda} \subseteq \text{co}\{\partial_z f_k(z, u_k) : u_k \in U_k(z)\}.$$

such that $z_r^* \overset{w*}{\to} z_k^*$, where Λ stands for the directed set of this net. Then, for each $r \in \Lambda$, there exist $\alpha_{rj} \geq 0, z_{rj}^* \in \partial_z f_k(z, u_{rj}), u_{rj} \in U_k(z), j = 1, \ldots, k^{'}, k^{'} \in \mathbb{N}$, and $\sum_{j=1}^{k^{'}} \alpha_{rj} = 1$, such that

$$z_r^* = \sum_{j=1}^{k^{'}} \alpha_{rj} z_{rj}^*. \tag{3.9}$$

This together with (3.7) ensures that

$$\langle z_r^*, d \rangle = \sum_{j=1}^{k^{'}} \alpha_{rj} \langle z_{rj}^*, d \rangle \leq \sum_{j=1}^{k^{'}} \alpha_{rj} [f_k(x, u_{rj}) - f_k(z, u_{rj})]. \tag{3.10}$$

Since $u_{rj} \in U_k(z)$, $f_k(z, u_{rj}) = F_k(z)$ for $j = 1, \ldots, k^{'}$. By the definition of F_k, $f_k(x, u_{rj}) \leq F_k(x)$ for $j = 1, \ldots, k^{'}$. Using (3.10) yields that

$$\langle z_r^*, d \rangle \leq F_k(x) - F_k(z).$$

Passing to the limit with respect to r, one has

$$\langle z_k^*, d \rangle \leq F_k(x) - F_k(z).$$

Similarly, there is a net

$$\{x_s^*\}_{s \in \Lambda} \subseteq \text{co}\{\partial_z g_t(z, v_t) : v_t \in V_t(z), t \in T(z)\}.$$

such that $x_s^* \overset{w*}{\to} x^*$. For each $s \in \Lambda$, there exist $\iota_{s_i}, \tau_{s_j} \geq 0, i = 1, 2, \ldots, i^{'}, j = 1, 2, \ldots, j^{'}, i^{'}, j^{'} \in \mathbb{N}$, $\sum_{i=1}^{i^{'}} \iota_{s_i} = \sum_{j=1}^{j^{'}} \tau_{s_j} = 1, t_{s_i} \in T(z)$, $v_{t_{s_{ij}}} \in V_{t_{s_{ij}}}(z)$ and $x_{t_{s_{ij}}}^* \in \partial_z g_{t_{s_i}}(z, v_{t_{s_{ij}}})$ such that $x_s^* = \sum_{i=1}^{i^{'}} \iota_{s_i} \sum_{j=1}^{j^{'}} \tau_{s_j} x_{t_{s_{ij}}}^*$.

Form (3.8), we deduce that

$$\begin{aligned}\langle x_s^*, d\rangle &= \sum_{i=1}^{i'} \iota_{s_i} \sum_{j=1}^{j'} \tau_{s_j} \langle x^*_{t_{s_{ij}}}, d\rangle \\ &\le \sum_{i=1}^{i'} \iota_{s_i} \sum_{j=1}^{j'} \tau_{s_j} \left(g_{t_{s_i}}(x, v_{t_{s_{ij}}}) - g_{t_{s_i}}(z, v_{t_{s_{ij}}}) \right) \\ &\le \sum_{i=1}^{i'} \iota_{s_i} \left(G_{t_{s_i}}(x) - G_{t_{s_i}}(z) \right) \\ &\le G(x) - G(z).\end{aligned}$$

Taking the limit with respect to $s \in \Lambda$, we have

$$\langle x^*, d\rangle \le G(x) - G(z) \le -G(z), \tag{3.11}$$

because of $x \in C$. Together (3.4), (3.5) with the definition of $G(z)$, we have

$$\begin{aligned} 0 &= \sum_{k\in K} \theta_k \langle z_k^*, d\rangle + \sum_{t\in T} (\lambda_t + \beta_t)\langle x^*, d\rangle \\ &\le \sum_{k\in K} \theta_k (F_k(x) - F_k(z)) - \sum_{t\in T} (\lambda_t + \beta_t) G(z) \\ &\le \sum_{k\in K} \theta_k (F_k(x) - F_k(z)) - \sum_{t\in T} \lambda_t G(z) \\ &\le \sum_{k\in K} \theta_k (F_k(x) - F_k(z)) - \sum_{t\in T} \lambda_t g_t(z, v_t),\end{aligned}$$

where contradicts with (3.6). It therefore implies that (3.1) holds. □

It should be pointed out that the generalized convexity of $(f_k, g_t)_{k\in K, t\in T}$ in Theorem 3.2 is indispensable; see Example 3.3.

Example 3.3. Consider the following uncertain biobjective problem:

$$\begin{aligned} &\min\ \left(-|x| - u_1, -x^2 - u_2\right) \\ &\text{s.t. } t^2|x| - v_t \le 0,\ \forall\, t \in T,\end{aligned}$$

where $u_1, u_2 \in U_2$, $U_1 = U_2 := [0, 1]$, and $v_t \in V_t := [1, 1 + t]$ for $t \in T := [0, 1]$.

The robust counterpart of the uncertain biobjective problem is as follows:

$$\begin{aligned} &\min\ \left(-|x|, -x^2\right) \\ &\text{s.t. } t^2|x| - v_t \le 0,\ \ \forall\, v_t \in V_t,\ \ \forall\, t \in T.\end{aligned}$$

After calculation, we obtain that $G_t(x) = t^2|x| - 1, G(x) = |x| - 1, V_t(x) = \{1\}, U_1(x) = U_2(x) = \{0\},\ T(x) = \{1\},\ C = \mathbb{R},\ \partial_x f_1(x, u_1) = [-1, 1],\ \partial_x f_2(x, u_2) = -2x$ for

$u_1, u_2 \in \{0\}$, and

$$\partial_x g_t(x, v_t) = \begin{cases} [-1,1], & x = 0, \\ \{1\}, & x > 0, \\ \{-1\}, & x < 0. \end{cases}$$

Taking $\bar{z} := 0$, $\bar{\theta}_1 = \bar{\theta}_2 := \frac{1}{2}$, $\bar{\beta}_t = 0$ and $\bar{\lambda}_t = \bar{v}_t := 1$ for all $t \in T$, we have $(\bar{z}, \bar{\lambda}, \bar{\beta}, \bar{v}) \in C_D$ and

$$L(\bar{z}, \bar{\lambda}, \bar{\beta}, \bar{v}_t) = (-1, -1).$$

However, there exists $\bar{x} := 2 \in C$ such that

$$F(\bar{x}) = (-2, -4) \prec (-1, -1) = L(\bar{z}, \bar{\lambda}, \bar{\beta}, \bar{v}).$$

As a matter of fact, $(f_k, g_t)_{k \in K, t \in T}$ is not generalized convex at $\bar{z}$. For $u_1, u_2 \in \{0\}$, taking $(\bar{\xi}_1, \bar{\xi}_2) = (0, 0) \in \partial_z f_1(\bar{z}, u_1) \times \partial_z f_2(\bar{z}, u_2)$, we get that for any $d \in \mathbb{R}$,

$$f_1(\bar{x}, u_1) - f_1(\bar{z}, u_1) = -2 < 0 = \langle \bar{\xi}_1, d \rangle$$

and

$$f_2(\bar{x}, u_2) - f_1(\bar{z}, u_2) = -4 < 0 = \langle \bar{\xi}_2, d \rangle.$$

The following theorem declares strong robust duality relations (RMP) and (R-MD).

Theorem 3.4. (Strong robust duality) *Let $\bar{x}$ be a local robust weakly efficient solution of (UMP) and (RMFCQ) hold at $\bar{x}$. Assume that $f_k(x, u_k)$ satisfies (H1)-(H4) for each $k \in K$ and the assumptions of Lemma 2.5 are satisfied. Then there exist $(\bar{\lambda}, \bar{\beta}, \bar{v}) \in \mathbb{R}_+^{|T|} \times \mathbb{R}_+^{|T|} \times V$ such that*

$$\big(\bar{x}, \bar{\lambda}, 0, \bar{v}\big) \in C_D, \;\; F(\bar{x}) = L(\bar{x}, \bar{\lambda}, 0, \bar{v}), \;\; \text{and} \;\; \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \; t \in T.$$

Furthermore, if $(f_k, g_t)_{k \in K, t \in T}$ is generalized convex, then $(\bar{x}, \bar{\lambda}, 0, \bar{v}) \in S_D^w$.

Proof. It follows from Lemma 2.6 that there exist $\bar{\theta}_k \geq 0, k \in K$ not all zero, and $\tilde{\lambda} \geq 0$ such that

(3.12)
$$0 \in \sum_{k \in K} \bar{\theta}_k \mathrm{cl}^* \mathrm{co}\{\partial_x f_k(\bar{x}, u_k) : u_k \in U_k(\bar{x})\} + \tilde{\lambda} \mathrm{cl}^* \mathrm{co}\{\partial_x g_t(\bar{x}, v_t) : v_t \in V_t(\bar{x}), t \in T(\bar{x})\}$$

and

$$\tilde{\lambda} \sup_{v_t \in V_t, t \in T} g_t(\bar{x}, v_t) = 0. \tag{3.13}$$

If $\tilde{\lambda} > 0$, then $\sup_{v_t \in V_t, t \in T} g_t(\bar{x}, v_t) = 0$ and $T(\bar{x}) \neq \emptyset$. For $t \in T$, we let $\beta = (\beta_t)_{t \in T} = 0$, $\bar{\lambda} = (\bar{\lambda}_t) \in \mathbb{R}_+^{|T|}$ and

$$\bar{\lambda}_t := \begin{cases} \dfrac{\tilde{\lambda}}{|T|}, & \text{if } \tilde{\lambda} = 0, \\ \dfrac{\tilde{\lambda}}{|T(\bar{x})|}, & \text{if } t \in T(\bar{x}), \; \tilde{\lambda} > 0, \\ 0, & \text{if } t \in T \setminus T(\bar{x}), \; \tilde{\lambda} > 0. \end{cases} \tag{3.14}$$

Then $\sum_{t\in T}\bar\lambda_t=\tilde\lambda$ and

$$0\in\sum_{k\in K}\bar\theta_k\mathrm{cl}^*\mathrm{co}\{\partial_x f_k(\bar x,u_k):u_k\in U_k(\bar x)\}$$
$$+\sum_{t\in T}\bar\lambda_t\mathrm{cl}^*\mathrm{co}\{\partial_x g_t(\bar x,v_t):v_t\in V_t(\bar x),t\in T(\bar x)\},$$

which implies that there exists $\bar v\in V$ meeting $\bar v_t\in V_t(\bar x),t\in T(\bar x)$ such that $\big(\bar x,\bar\lambda,0,\bar v\big)\in C_D$ and so,

$$g_t(\bar x,\bar v_t)=G_t(\bar x)=G(\bar x)=\sup_{v_t\in V_t,t\in T}g_t(\bar x,v_t).$$

From (3.13), one has

$$\bar\lambda_t\sup_{v_t\in V_t,t\in T}g_t(\bar x,v_t)=\bar\lambda_t g_t(\bar x,\bar v_t)=0,\ t\in T(\bar x).$$

It therefore follows from (3.14) that

$$\bar\lambda_t g_t(\bar x,\bar v_t)=0,\ t\in T,$$

and

$$L(\bar x,\bar\lambda,0,\bar v)=F(\bar x)+\sum_{t\in T}\bar\lambda_t g_t(\bar x,\bar v_t)e=F(\bar x).$$

Since $(f_k,g_t)_{k\in K,t\in T}$ is generalized convex and $\bar x\in C$, we conclude from Theorem 3.2 that

$$L(\bar x,\bar\lambda,0,\bar v)=F(\bar x)\not< L(z,\lambda,\beta,v),\ \forall\,(z,\lambda,\beta,v)\in C_D.$$

Consequently, one has $\big(\bar x,\bar\lambda,0,\bar v\big)\in S_D^w$. □

Theorem 3.5. (Converse robust duality) *Let $(\bar x,\bar\lambda,\bar\lambda,\bar v)\in C_D$ be a weak efficient solution of (RMD) with $\bar x\in C$ and $\bar\lambda_t=0$ for $t\in T\setminus T(\bar x)$. If $(f_k,g_t)_{k\in K,t\in T}$ is generalized convex at $\bar x$, then $\bar x$ is a robust weak efficient solution of (UMP).*

Proof. Since $(\bar x,\bar\lambda,\bar\lambda,\bar v)\in C_D$ and $\bar\lambda_t=0$ for $t\in T\setminus T(\bar x)$, one has

$$\bar\lambda_t g_t(\bar x,\bar v_t)=\bar\lambda_t\sup_{v_t\in V_t,t\in T}g_t(\bar x,v_t)\ge 0,\ \ t\in T,$$

and so, $\sum_{t\in T}\bar\lambda_t g_t(\bar x,\bar v_t)\ge 0$. It follows from Theorem 3.2 that

$$F(x)\not< L(\bar x,\bar\lambda,\bar\lambda,\bar v),\ \forall\,x\in C. \tag{3.15}$$

Taking $x=\bar x$ in above formula, we have

$$0\not<\sum_{t\in T}\bar\lambda_t g_t(\bar x,\bar v_t)e,$$

i.e., $\sum_{t\in T}\bar\lambda_t g_t(\bar x,\bar v_t)e\notin\mathrm{int}\,\mathbb{R}^l_+$. Due to $e=(1,1,\dots,1)\in\mathbb{R}^l_+$, one has

$$\sum_{t\in T}\bar\lambda_t g_t(\bar x,\bar v_t)\le 0.$$

Consequently, we obtain $\sum_{t\in T}\bar\lambda_t g_t(\bar x,\bar v_t)=0$ and $F(\bar x)=L(\bar x,\bar\lambda,\bar\lambda,\bar v)$. Then (3.15) implies that

$$F(x)\not< F(\bar x),\ \forall\,x\in C,$$

i.e., $F(x) - F(\bar{x}) \notin -\mathrm{int}\,\mathbb{R}^l_+$ for all $x \in C$. Thus, $\bar{x} \in C$ is a robust weak efficient solution of (UMP). □

4. Robust saddle point characterizations

In this section, we present robust saddle point characterizations of (UMP) in the sense of vector-valued Lagrangian function.

We now define the vector-valued Lagrangian function $\mathcal{L} : \mathbb{R}^n \times V \times \mathbb{R}^{|T|}_+ \to \mathbb{R}^l$ as follows

$$\mathcal{L}(x, v, \lambda) = F(x) + \sum_{t \in T} \lambda_t g_t(x, v_t) e,$$

where $\lambda := (\lambda_t)_{t \in T} \in \mathbb{R}^{|T|}_+$, $v := (v_t)_{t \in T} \in V$ and $e := (1, 1, \ldots, 1) \in \mathbb{R}^l$.

Definition 4.1. A point $(\bar{x}, \bar{v}, \bar{\lambda}) \in X \times V \times \mathbb{R}^{|T|}_+$ is called a robust weak saddle point of $\mathcal{L}(x, v, \lambda)$ if,

$$\mathcal{L}(x, \bar{v}, \bar{\lambda}) \not< \mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}) \not< \mathcal{L}(\bar{x}, v, \lambda), \ \forall\, (x, v, \lambda) \in X \times V \times \mathbb{R}^{|T|}_+.$$

The following theorem presents the saddle point type necessary robust optimality conditions of (UMP).

Theorem 4.2. *Let $\bar{x}$ be a robust weakly efficient solutions of (UMP) and (RMFCQ) hold at $\bar{x}$. Assume that the assumptions of Lemma 2.6 are satisfied, and $(f_k, g_t)_{k \in K, t \in T}$ is generalized convex at $\bar{x}$. Then there exist $\bar{\lambda} := (\bar{\lambda}_t)_{t \in T} \in \mathbb{R}^{|T|}_+$ and $\bar{v} := (\bar{v}_t)_{t \in T} \in V$ such that $(\bar{x}, \bar{v}, \bar{\lambda})$ is a robust weak saddle point of $\mathcal{L}(x, v, \lambda)$.*

Proof. It follows from Lemma 2.6 that there exist $\bar{\theta}_k \geq 0, k \in K$ not all zero, and $\tilde{\lambda} \geq 0$ such that

$$0 \in \sum_{k \in K} \bar{\theta}_k \mathrm{cl}^* \mathrm{co}\{\partial_x f_k(\bar{x}, u_k) : u_k \in U_k(\bar{x})\} + \tilde{\lambda} \mathrm{cl}^* \mathrm{co}\{\partial_x g_t(\bar{x}, v_t) : v_t \in V_t(\bar{x}), t \in T(\bar{x})\},$$

and

$$\tilde{\lambda} \sup_{v_t \in V_t, t \in T} g_t(\bar{x}, v_t) = 0. \tag{4.1}$$

Without loss of generality, assume that $\sum_{k \in K} \bar{\theta}_k = 1$.

If $\tilde{\lambda} = 0$, (4.1) implies that there exist $\bar{v}_t \in V_t$, $t \in T$ such that

$$\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \tag{4.2}$$

where $\bar{\lambda}_t := \frac{\tilde{\lambda}}{|T|}$.

If $\tilde{\lambda} > 0$, (4.1) implies that $T(\bar{x}) \neq \emptyset$ and for each $t \in T(\bar{x})$, there exists $\bar{v}_t \in V_t(\bar{x})$ such that

$$g_t(\bar{x}, \bar{v}_t) = \sup_{v_t \in V_t, t \in T} g_t(\bar{x}, v_t) = 0. \tag{4.3}$$

For $t \in T$, we let

$$\bar{\lambda}_t := \begin{cases} \dfrac{\tilde{\lambda}}{|T(\bar{x})|}, \text{ if } t \in T(\bar{x}), \\ 0, \text{ if } t \in T \setminus T(\bar{x}). \end{cases} \tag{4.4}$$

Clearly, $\sum_{t\in T} \bar{\lambda}_t = \tilde{\lambda}$. Therefore, from (4.1) and (4.3), we derive

$$\tilde{\lambda} g_t(\bar{x}, \bar{v}_t) = \sum_{t\in T(\bar{x})} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = \tilde{\lambda} \sup_{v_t \in V_t, t\in T} g_t(\bar{x}, v_t) = 0, \ t \in T(\bar{x}).$$

Moreover, one has

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0. \tag{4.5}$$

Since $\bar{x} \in C$, then $g_t(\bar{x}, v_t) \leq 0$, $v_t \in V_t$ and

$$\sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) \leq 0, \ \forall\, (v, \lambda) := (v, (\lambda_t)_{t\in T}) \in V \times \mathbb{R}^{|T|}_+. \tag{4.6}$$

Combined (4.2), (4.5) with (4.6), we have

$$\sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) \leq \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t).$$

So, one has

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) e - \sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) e \in \mathbb{R}^l_+,$$

which implies that

$$F(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) e - \left(F(\bar{x}) + \sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) e \right) \notin -\operatorname{int} \mathbb{R}^l_+.$$

Then we have

$$\mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}) \not\prec \mathcal{L}(\bar{x}, v, \lambda), \ \forall\, (v, \lambda) \in V \times \mathbb{R}^{|T|}_+. \tag{4.7}$$

By the proof of Theorem 3.4, one has $\big(\bar{x}, \bar{\lambda}, 0, \bar{v}\big) \in C_D$ and

$$\mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}) = F(\bar{x}) = L(\bar{x}, \bar{\lambda}, 0, \bar{v}).$$

Since $(f_k, g_t)_{k\in K, t\in T}$ is generalized convex at $\bar{x}$, we deduce from Theorem 3.2 that

$$F(x) \not\prec \mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}), \ \forall\, x \in C,$$

and so,

$$F(x) + \delta_C(x) e \not\prec \mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}), \ \forall\, x \in X, \tag{4.8}$$

where $\delta_C(x)$ is the indicator function defined on C.

Suppose that there exist $\hat{x} \in X$ such that $\mathcal{L}(\hat{x}, \bar{v}, \bar{\lambda}) \prec \mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda})$. Then

$$F(\hat{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) e \prec F(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) e = F(\bar{x}). \tag{4.9}$$

Observed that

$$F(\hat{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t)e = F(\hat{x}) + \delta_C(\hat{x})e + \left(\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) - \delta_C(\hat{x})\right) e.$$

Combined (4.8) and (4.9), we have $\left(\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) - \delta_C(\hat{x})\right) e \in \operatorname{int} \mathbb{R}^l_+$ and so, $\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) > \delta_C(\hat{x})$. Thus, $\hat{x} \in C$ and $\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) > 0$, which contradicts the fact that $\sum_{t\in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) \leq 0$. Therefore, one has

$$\mathcal{L}(x, \bar{v}, \bar{\lambda}) \not< \mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}),\ \forall\, x \in X. \tag{4.10}$$

Consequently, it follows from (4.7) and (4.10) that there exist $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}^{|T|}_+$ and $\bar{v} := (\bar{v}_t)_{t\in T} \in V$ such that $(\bar{x}, \bar{v}, \bar{\lambda}) \in X \times V \times \mathbb{R}^{|T|}_+$ is a robust weak saddle point of $\mathcal{L}(x, v, \lambda)$. □

The following result presents the saddle point type sufficient robust optimality conditions of (UMP).

Theorem 4.3. *If $(\bar{x}, \bar{v}, \bar{\lambda}) \in X \times V \times \mathbb{R}^{|T|}_+$ is a robust weak saddle point of $\mathcal{L}(x, v, \lambda)$, then $\bar{x} \in C$ is a local robust weakly efficient solution of (UMP).*

Proof. Since $(\bar{x}, \bar{v}, \bar{\lambda}) \in X \times V \times \mathbb{R}^{|T|}_+$ is a robust weak saddle point of $\mathcal{L}(x, v, \lambda)$, we have

$$\mathcal{L}(x, \bar{v}, \bar{\lambda}) \not< \mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}),\ \forall\, x \in X \tag{4.11}$$

and

$$\mathcal{L}(\bar{x}, \bar{v}, \bar{\lambda}) \not< \mathcal{L}(\bar{x}, v, \lambda),\ \forall\, (v, \lambda) = (v, (\lambda_t)_{t\in T}) \in V \times \mathbb{R}^{|T|}_+.$$

So, one has

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t)e \not< \sum_{t\in T} \lambda_t g_t(\bar{x}, v_t)e,\ \forall\, (v, \lambda) = (v, (\lambda_t)_{t\in T}) \in V \times \mathbb{R}^{|T|}_+,$$

i.e., for any $(v, \lambda) = (v, (\lambda_t)_{t\in T}) \in V \times \mathbb{R}^{|T|}_+$, we get

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq \sum_{t\in T} \lambda_t g_t(\bar{x}, v_t). \tag{4.12}$$

For each $t \in T$, set $v_t = \bar{v}_t, \lambda_t = 0$ and $\lambda_t = 2\bar{\lambda}_t$ in (4.12), respectively, we obtain $0 \geq \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq 0$, i.e., $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0$. Thus, we have

$$\sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) \leq 0,\ \forall\, (v, \lambda) = (v, (\lambda_t)_{t\in T}) \in V \times \mathbb{R}^{|T|}_+, \tag{4.13}$$

which implies that $g_t(\bar{x}, v_t) \leq 0$ for all $v_t \in V_t$ and $t \in T$ and so, $\bar{x} \in C$.

Note that for any $x \in C$, $g_t(x, v_t) \leq 0$ for all $t \in T$ and $v_t \in V_t$. Then

$$\sum_{t\in T} \bar{\lambda}_t g_t(x, v_t) \leq 0,\ \forall\, (x, v) \in C \times V. \tag{4.14}$$

Using (4.11) yields that

$$F(x) - F(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(x, \bar{v}_t)e \notin -\operatorname{int} \mathbb{R}^l_+,\ \forall\, x \in X. \tag{4.15}$$

Suppose that for any neighborhood O of $\bar{x}$, there exists $\hat{x} \in C \cap O$ such that $F(\hat{x}) - F(\bar{x}) \in -\mathrm{int}\,\mathbb{R}^l_+$. This together with (4.14) yields that

$$F(\hat{x}) - F(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t g_t(\hat{x}, \bar{v}_t) e \in -\mathrm{int}\,\mathbb{R}^l_+ - \mathbb{R}^l_+ \subseteq -\mathrm{int}\,\mathbb{R}^l_+,$$

which contradicts with (4.15). Therefore, there exists a neighborhood O of $\bar{x}$ such that

$$F(x) - F(\bar{x}) \notin -\mathrm{int}\,\mathbb{R}^l_+, \ \forall\, x \in C \cap O,$$

that is, $\bar{x}$ is a local robust weakly efficient solution of (UMP). □

5. Conclusions

Based on the Karush-Kuhn-Tucker type robust necessary optimality conditions, the mixed type robust dual model of uncertain multiobjective optimization problem is proposed. The weak, strong and converse robust duality results between (RMP) and (RMD) are derived. The robust saddle point type sufficient and necessary optimality conditions for the (UMP) are presented under the generalized convexity assumptions. For the future work, it is interesting to study robust optimality, duality of (UMP) by the image space analysis like in [16] as well as robust stability.

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