Daria Barjaktarevic

Adams spectral sequences and Toda brackets

Master's thesis in Mathematical Sciences
Supervisor: Marius Thaule
Co-supervisor: Sebastian H. Martensen
June 2024

Master's thesis

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Abstract

The computation of stable homotopy groups of topological spaces, spheres in particular, has been and continues to be one of the driving forces of algebraic topology. Two tools used for this are Toda brackets[19] and Adams spectral sequences[1].

As demonstrated by Miller[11] the Adams spectral sequence can be constructed in any triangulated category $\mathcal{T}$ equipped with projective and injective classes. Christensen and Frankland[6] prove that the differential d_r in an Adams spectral sequence can be expressed as an $(r+1)$-fold Toda bracket.

In this thesis we aim to compile and present some of these results with additional details. The thesis has four parts. In the first part we discuss the construction and properties of projective and injective classes. We then move on to Toda brackets, before returning to the projective and injective classes and Adams spectral sequences with respect to them. In the last part, all the preceding parts come together, and we show how Toda brackets and the general Adams spectral sequences relate.

Contents

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Chapter 1

Introduction

Stable homotopy groups and in general stable classes of maps between topological spaces have been in the center of attention for many algebraic topologists for decades. This thesis is devoted to a relation between two of the most renowned methods of computations, namely the Adams spectral sequence and Toda brackets.

Letting H be mod p singular cohomology and $\mathcal{A}$ the mod p Steenrod algebra, we can apply the Hurewicz homomorphism to stable classes of maps between spaces X and Y:

$$[X, Y] \longrightarrow \mathrm{Hom}(H^*(Y), H^*(X)).$$

This yields a spectral sequence which abuts to $[X, Y]$/non-p torsion, and has

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(Y), H^*(X)),$$

known now as the classical Adams spectral sequence, cf.[1]. Since then, it has been altered and appeared in many variations, for instance for a general cohomology theory E, instead of singular cohomology.

In his PhD thesis, Miller generalized the Adams spectral sequence even further. Injective and projective classes are central here. A projective class is a pair $(\mathcal{P}, \mathcal{E})$, where the first is a collection of objects, the second a collection of morphisms, that model objects that "look projective" from the viewpoint of the morphisms in $\mathcal{E}$. Using these, he constructed what is known as Adams resolutions of an object in a general triangulated category $\mathcal{T}$. Under the functors $\mathcal{T}(-,-)$ they become exact couples, through which we obtain the spectral sequence. This is known as an Adams spectral sequence.

Toda brackets, defined by Toda as "stable secondary composition," were first used to calculate stable homotopy groups of spheres, by exploiting the triangulated structure of the homotopy category of topological spaces. As with Adams spectral sequence, this has found its generalization. For a diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$$

in a triangulated category $\mathcal{T}$ the Toda bracket $\langle f_3, f_2, f_1 \rangle$ is a subset of $\mathcal{T}(\Sigma X_0, X_3)$. It is computed by comparing the sequence to one of the cofiber sequences of f_1, f_2 or f_3, and including all maps that make the comparison possible. Although there are three different ways of doing this comparison, they coincide, which we will prove in Chapter 3. Toda brackets have been generalized further as well, to n-fold Toda brackets, which are applied to sequences of length n instead of length 3.

The aim of this thesis is to show the following theorem, first shown by Christensen and Frankland [6], which we do in Theorem 5.2.2. It is stated slightly differently here, for simplicity.

Theorem. *Consider the Adams spectral sequence with respect to a projective class associated to $\mathcal{T}(X, Y)$. Let $[x]$ be a cycle in $E_1^{s,t}$, and let $d_2[x]$ denote the set of all representatives of $d_2([x])$. Then*

$$d_2[x] = \langle x, d_1, \Sigma^{-1} d_1 \rangle_{fc}^{\alpha}$$

for a fixed α.

The superscript means that a morphism in the construction of the Toda bracket is fixed.

Christensen and Frankland also showed that this result can be generalized to the higher differentials and the higher Toda brackets, with an n-fold Toda bracket representing the $n-1$ differential in the spectral sequence.

The thesis has four chapters, excluding this one, the first three of which explore the different components necessary to state and prove Theorem 5.2.2. We start in Chapter 2 by defining projective and injective classes, and look at how they show up in practice. We also recall some terminology necessary for discussing convergence of the spectral sequences.

In Chapter 3 we look at Toda brackets in a general triangulated category and compute some examples. Then we move on to the general Adams spectral sequences in Chapter 4. We also look at how our definition will be related to the classical Adams spectral sequence, and discuss some aspects of convergence.

Finally, in Chapter 5 we are all set to prove our main result.

This thesis does not relate to the UN sustainable development goals.

1.1 Preliminaries

We will construct the Adams spectral sequence in a general triangulated category. In particular, we will apply the construction and results to the stable homotopy category, denoted throughout this thesis by $\mathcal{SHC}$. Our main reference will be [2], especially chapters 1, 2 and 5.

The stable homotopy category is defined as the homotopy category of a stable model category [2, Chapter 3.2] of **spectra**. There are many different such model categories, each of which have both advantages and disadvantages. A spectrum is defined differently in each of them. However, by Schwede's rigidity theorem [15] all the different stable model categories of spectra that model $\mathcal{SHC}$ are Quillen equivalent [2, Definition A.4.7].

Theorem 1.1.1 (Schwede). *Let $\mathcal{C}$ be a stable model category. If the homotopy category of $\mathcal{C}$ and the homotopy category of spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathcal{C}$ and the model category of spectra.*

Essentially, $\mathcal{SHC}$ captures all the homotopy structure of spectra. Therefore, when we say spectra, we will mean an object in $\mathcal{SHC}$, and leave the model structures be.

To simplify things even further, we will mostly consider the compact objects in $\mathcal{SHC}$. An object X in a triangulated category $\mathcal{T}$ is **compact** if the functor $\mathcal{T}(A,-)_*$ commutes with arbitrary coproducts. By [2, Theorem 5.6.13] the compact objects in $\mathcal{SHC}$ are exactly the objects that are isomorphic (in $\mathcal{SHC}$) to a CW-spectrum with finitely many stable cells, which are known as finite CW-spectra. Furthermore, by [14, Chapter 2.7] the full subcategory of compact objects in $\mathcal{SHC}$ is equivalent to the Spanier–Whitehead category $\mathcal{SW}$ [9, Chapter 1.2]. The objects in $\mathcal{SW}$ are denoted by (X,n), where X is a finite CW-complex, and $n \in \mathbb{Z}$. Hence, for a finite CW-spectrum in $\mathcal{SHC}$ we will often be thinking about it as a finite CW-complex.

We recall some notation and definitions on spectra. Classes of maps between two spectra X and Y are denoted by $[X,Y]$. Every spectrum E defines a cohomology theory (dually a homology theory) through the graded groups $[-,E]_{-*}$. An **Eilenberg–Mac Lane spectrum** HG, where G is a group, is a spectrum which has all zero homotopy groups except for $\pi_0(HG) = G$. Levelwise it consists of the spaces Eilenberg–Mac Lane spaces $K(G,n)$. The Eilenberg–Mac Lane spectra are also the spectra which represent singular cohomology with G coefficients.

Let $\mathbb{S}$ denote the sphere spectrum, and let $\mathbb{S}^n$ denote the n^{th} suspension of $\mathbb{S}$. The homotopy groups of a spectrum X is a homology theory through $[\mathbb{S}^n, X] \cong \pi_n(X)$.

A spectrum X is **connective** if it only has finitely many non-zero negative homotopy groups. It is of **finite type** if $H_*(X)$ is finitely generated in all degrees.

Chapter 2

Projective and injective classes

Projective and injective modules are well known from abstract algebra. In homological algebra projective and injective modules and resolutions are the foundation of many constructions. In this chapter we will provide a more general description of a projective (injective) object, and will define what is called projective (injective) classes. These definitions will yield objects that "look projective" from the viewpoint of certain specified morphisms. The projective class is an important construction for the general Adams spectral sequence, which we will define in Chapter 4.

We first define a projective class in a category with weak kernels, and then we look at how this definition can be altered when the category is triangulated. In the end we dualize and define an injective class. All the definitions and results are due to [5].

In this chapter we will be assuming that all categories are pointed. A category is **pointed** if it has a zero object, that is, an object that is both initial and terminal. Note that all triangulated categories are pointed.

2.1 Projective classes in a category with weak kernels

Let $\mathcal{T}$ be a category. A **weak kernel** for morphism $f : X \longrightarrow Y$ in $\mathcal{T}$ is another morphism $W \longrightarrow X$ such that the sequence

$$\mathcal{T}(V,W) \longrightarrow \mathcal{T}(V,X) \longrightarrow \mathcal{T}(V,Y)$$

is an exact sequence of pointed sets for all objects $V \in \mathcal{T}$. Recall that a set is pointed if it has a basepoint, and that functions between pointed sets preserve the basepoint.

In other words, a morphsim $V \longrightarrow X$ is zero if and only if it factors through W. The object W behaves like a kernel for f, but it is not necessarily unique, hence the name "weak kernel."

Definition 2.1.1. Let $\mathcal{P}$ be a collection of objects in $\mathcal{T}$. A morphism $X \longrightarrow Y$ such that $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is surjective for all $P \in \mathcal{P}$ is called $\mathcal{P}$**-epic**. The collection of all such morphisms is called $\mathcal{P}$**-epi**.

Let $\mathcal{E}$ be a collection of morphisms in $\mathcal{T}$. An object P such that $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is surjective for all morphisms $X \longrightarrow Y$ in $\mathcal{E}$ is called $\mathcal{E}$**-projective**. The collection of all such objects is called $\mathcal{E}$**-proj**.

There is also a notion of $\mathcal{P}$-monic morphism, which induce injective functions under $\mathcal{T}(P,-)$ for all P in $\mathcal{P}$.

Using these two definitons we define a projective class in the context of category with weak kernels.

Definition 2.1.2. Let $(\mathcal{P},\mathcal{E})$ be a pair in a category $\mathcal{T}$ with weak kernels, where $\mathcal{P}$ is a collection of objects and $\mathcal{E}$ is a collection of morphisms. If the pair satisfies $\mathcal{P}$-epi $= \mathcal{E}$ and $\mathcal{E}$-proj $= \mathcal{P}$, and if for each $X \in \mathcal{T}$ we have some morphism $P \longrightarrow X$ in $\mathcal{E}$ with $P \in \mathcal{P}$, we say that it is a **projective class**.

Projective classes mimic the behavior of projective modules. The $\mathcal{P}$-projective objects and the $\mathcal{P}$-epic morphisms determine each other through the following lifting property

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ X & \twoheadrightarrow & Y. \end{array}$$

The double-tipped arrow indicates that the morphism is $\mathcal{P}$-epic.

Example 2.1.3. Let R be a ring. By the diagram above, we see that the projective R-modules along with surjective R-module homomorphisms are a projective class in $\operatorname{Mod} R$.

2.2 Projective classes in a triangulated category

In the following, $\mathcal{T}$ is always assumed to be a triangulated category. The distinguished triangles induce exact sequences under $\mathcal{T}(V,-)$ for all $V \in \mathcal{T}$. Since every morphism $X \longrightarrow Y$ lies in a distinguished triangle $W \longrightarrow X \longrightarrow Y \longrightarrow \Sigma W$, we have that

$$\mathcal{T}(V,W) \longrightarrow \mathcal{T}(V,X) \longrightarrow \mathcal{T}(V,Y) \longrightarrow \mathcal{T}(V,\Sigma W)$$

is an exact sequence of abelian groups for all V. So, in a triangulated category every morphism both has a weak kernel and is a weak kernel.

In fact, we could make the following definition and proposition in a category where we assume all morphisms are weak kernels, in addition to having weak kernels. However, as we will be working mostly with triangulated categories and $\mathcal{SHC}$, we are really only interested in the projective classes in the context of a triangulated category.

Definition 2.2.1. Let $\mathcal{P}$ be any collection of objects in $\mathcal{T}$. A morphism $X \longrightarrow Y$ such that $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is zero for all objects $P \in \mathcal{P}$ is called $\mathcal{P}$**-null**, and the collection of all such morphisms is called $\mathcal{P}$**-nulls**.

Let $\mathcal{N}$ be a collection of morphisms in $\mathcal{T}$. An object P such that the induced morphism $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is zero for all $X \longrightarrow Y$ in $\mathcal{N}$ is called $\mathcal{N}$**-projective**. The collection of all such objects is called $\mathcal{N}$**-proj**.

The following proposition tells us that in a triangulated category we can choose whether we determine a projective class through its $\mathcal{P}$-epic morphisms or its $\mathcal{P}$-null morphisms.

Proposition 2.2.2 ([5, Proposition 2.6])**.** *Let $\mathcal{P}$ be a collection of objects, and $\mathcal{N}$ a collection of morphisms in $\mathcal{T}$, such that $\mathcal{P}$-nulls $= \mathcal{N}$ and $\mathcal{N}$-proj $= \mathcal{P}$. In addition, assume that for each $X \in \mathcal{T}$ there is a morphism $P \longrightarrow X$ with $P \in \mathcal{P}$, that is a weak kernel of a morphism in $\mathcal{N}$. Then $(\mathcal{P}, \mathcal{P}$-epi$)$ is a projective class. Furthermore, every projective class is of this form for a unique pair $(\mathcal{P}, \mathcal{N})$ as above.*

Proof. Let $(\mathcal{P},\mathcal{N})$ be a pair as above. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a diagram that is exact under $\mathcal{T}(V,-)$. Then especially for $P \in \mathcal{P}$ we have the exact sequence

$$\mathcal{T}(P,X) \xrightarrow{f_*} \mathcal{T}(P,Y) \xrightarrow{g_*} \mathcal{T}(P,Z).$$

Here we have that f_* is surjective if and only if g_* is zero. So f is $\mathcal{P}$-epic if and only if g is $\mathcal{P}$-null.

Now, assume that $X \longrightarrow Y$ is a $\mathcal{P}$-epic morphism. Since every morphism is a weak kernel, there is a morphism $Y \longrightarrow Z$ such that $X \longrightarrow Y \longrightarrow Z$ is exact under $\mathcal{T}(V,-)$ for all $V \in \mathcal{T}$. Then the morphism $Y \longrightarrow Z$ is $\mathcal{P}$-null.

Conversely, if $Y \longrightarrow Z$ is $\mathcal{P}$-null, we can use the fact that every morphism has a weak kernel to obtain a morphism $X \longrightarrow Y$ such that $X \longrightarrow Y \longrightarrow Z$ is exact under $\mathcal{T}(V,-)$. Then the morphism $X \longrightarrow Y$ is necessarily $\mathcal{P}$-epic. From this we see that $\mathcal{P}$-epi and $\mathcal{P}$-nulls determine each other.

That every X has a $P \longrightarrow X$ that is a weak kernel of a morphism in $\mathcal{N}$ means that there is a morphism $X \longrightarrow Y$ such that $\mathcal{T}(V,P) \longrightarrow \mathcal{T}(V,X) \longrightarrow \mathcal{T}(V,Y)$ is exact and $\mathcal{T}(P',X) \longrightarrow \mathcal{T}(P',Y)$ is zero for all $P' \in \mathcal{P}$. Hence, this condition is equivalent to demanding that for each X there is a $P \longrightarrow X$ that is $\mathcal{P}$-epic, in a category where every morphism is a weak kernel.

We conclude that every pair $(\mathcal{P},\mathcal{N})$ as described in the proposition corresponds bijectively to a pair $(\mathcal{P},\mathcal{E})$ as in Definition 2.1.2. □

Remark 2.2.3. In [5] a projective class is actually defined simply in a pointed category, using sequences that are exact under $\mathcal{T}(\mathcal{P},-)$. Then it is shown that we can rephrase this definition whenever the category has more structure (e.g. it has weak kernels, or is triangulated). However, we are really only interested in the more structured categories, so we have taken Definition 2.1.2 as our definition of a projective class, even though it reads as Proposition 2.4 in [5].

As we already have mentioned, we will be working mostly with triangulated categories. However, both of the descriptions we have provided for a projective class can be useful. In later chapters we will use whichever of the two that is most fitting.

In the setting of a triangulated category, we can rephrase the definition even more.

Lemma 2.2.4. *In Proposition 2.2.2 we can replace the condition*

"every X has a $P \longrightarrow X$ with $P \in \mathcal{P}$ that is a weak kernel of a morphism in $\mathcal{N}$"

with

"every X lies in a distinguished triangle $P \longrightarrow X \longrightarrow Y \longrightarrow \Sigma P$ where $P \in \mathcal{P}$ and $X \longrightarrow Y$ is in $\mathcal{N}$."

Proof. The direction ($\Leftarrow$) is straightforward: If we have a distinguished triangle as described, then $\mathcal{T}(V,P) \longrightarrow \mathcal{T}(V,X) \longrightarrow T(V,Y)$ is exact for all $V \in \mathcal{T}$. So by definition $P \longrightarrow X$ is the weak kernel of a morphism in $\mathcal{N}$.

($\Rightarrow$): Assume that $P \longrightarrow X$ is the weak kernel of a morphism in $\mathcal{N}$, say $X \longrightarrow Y'$. This means, as we have seen, that $\mathcal{T}(P',P) \longrightarrow \mathcal{T}(P',X)$ is an epimorphism for all $P' \in \mathcal{P}$.

Let $X \longrightarrow Y$ be a cofiber of $P \longrightarrow X$, such that $P \longrightarrow X \longrightarrow Y$ is a distinguished triangle. Then $\mathcal{T}(P',P) \longrightarrow \mathcal{T}(P',X) \longrightarrow \mathcal{T}(P',Y)$ is exact for all $P' \in \mathcal{P}$, which means that $\mathcal{T}(P',X) \longrightarrow \mathcal{T}(P',Y)$ is zero for all $P' \in \mathcal{P}$. □

Example 2.2.5. A simple example of a projective class is $\mathcal{P} = \{0\}$, which makes $\mathcal{N}$ contain all morphisms in $\mathcal{T}$: For all $X \longrightarrow Y$ the induced morphism $\mathcal{T}(0,X) \longrightarrow \mathcal{T}(0,Y)$ is zero, and if P is an object such that $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is zero for all morphisms $X \longrightarrow Y$, then P is necessarily the zero object. Finally, we know that the triangle $X \longrightarrow X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished for all X, which shows that $(\mathcal{P},\mathcal{N})$ is a projective class.

2.3 The ghost projective class

An interesting question is what a projective class looks like outside of module categories. Our focus in this thesis will be on the stable homotopy category, and one example of a projective class here is the projective class generated by the sphere spectrum, known as ghost projective class. It is called "ghost" because the null-maps will be the nullhomotopic maps.

We need to show that this actually is a projective class. We begin by discussing generated projective classes.

Lemma 2.3.1. *The objects in a projective class are closed under retracts and coproducts.*

Proof. Let $(\mathcal{P},\mathcal{N})$ be a projective class and $\mathcal{N}$ the $\mathcal{P}$-null morphisms. We first show that it is closed under coproducts. Let P_1 and P_2 be two objects in $\mathcal{P}$. Their coproduct $P_1 \sqcup P_2$ is in $\mathcal{P}$ if $\mathcal{T}(P_1 \sqcup P_2, X) \longrightarrow \mathcal{T}(P_1 \sqcup P_2, Y)$ is zero for all morphisms $X \longrightarrow Y$ in $\mathcal{N}$. We have $\mathcal{T}(P_1 \sqcup P_2, Z) \cong \mathcal{T}(P_1, Z) \times \mathcal{T}(P_2, Z)$ for all $Z \in \mathcal{T}$, so the following diagram commutes for all morphisms $X \longrightarrow Y$

$$\begin{array}{ccc}
\mathcal{T}(P_1,X) \times \mathcal{T}(P_2,X) & \longrightarrow & \mathcal{T}(P_1,Y) \times \mathcal{T}(P_2,Y) \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{T}(P_1 \sqcup P_2, X) & \longrightarrow & \mathcal{T}(P_1 \sqcup P_2, Y)
\end{array}$$

and so $P_1 \sqcup P_2$ is in $\mathcal{P}$ if and only if P_1 and P_2 are in $\mathcal{P}$.

Next we show that it is closed under retracts. Let $r\colon P \longrightarrow P'$ be a retract, $P \in \mathcal{P}$, with $i\colon P' \hookrightarrow P$ the section. Let $f : X \longrightarrow Y$ be in $\mathcal{N}$. We then get the following diagram of induced morphisms

$$\begin{array}{ccccc}
\mathcal{T}(P',X) & \xrightarrow{r^*} & \mathcal{T}(P,X) & \xrightarrow{i^*} & \mathcal{T}(P',X) \\
\downarrow f_* & & \downarrow 0 & & \downarrow f_* \\
\mathcal{T}(P',Y) & \xrightarrow{r^*} & \mathcal{T}(P,Y) & \xrightarrow{i^*} & \mathcal{T}(P',Y).
\end{array}$$

Since $ri = id$, we get that $i^* r^* f_* = f_* = 0$, so P' is in $\mathcal{P}$. □

Next, we show that for some collection of objects, closure under retracts and coproducts actually produces a projective class.

Proposition 2.3.2. *Let $\mathcal{S}$ be a set of objects in $\mathcal{T}$, and let $\mathcal{P}_\mathcal{S}$ be the set of retracts of coproducts of objects in $\mathcal{S}$. Then $(\mathcal{P}_\mathcal{S}, \mathcal{S}$-epi$)$ is the smallest projective class that contains S.*

Proof. If $(\mathcal{P}_\mathcal{S}, \mathcal{S}$-epi$)$ is a projective class, it is necessarily also the smallest projective class containing $\mathcal{S}$, since all projective classes are closed under coproducts and retracts.

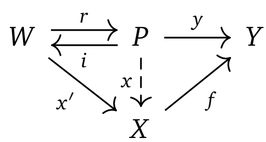
To show that the pair is projective, we need to show that

1) $\mathcal{P}_{\mathcal{S}}$-epi $= \mathcal{S}$-epi.
2) ($\mathcal{S}$-epi)-proj $= \mathcal{P}_{\mathcal{S}}$.
3) For each X there is a morphism $P \longrightarrow X$ that is $\mathcal{S}$-epic, where $P \in \mathcal{P}_{\mathcal{S}}$.

Starting with 1), we have that since $\mathcal{S} \subseteq \mathcal{P}_{\mathcal{S}}$, every morphism $f : X \longrightarrow Y$ in $\mathcal{P}_{\mathcal{S}}$-epi induces a surjective morphism $\mathcal{T}(S,X) \longrightarrow \mathcal{T}(S,Y)$ for all $S \in \mathcal{S}$, so $\mathcal{P}_{\mathcal{S}}$-epi $\subseteq \mathcal{S}$-epi. To see the opposite inclusion, let $f : X \longrightarrow Y$ be $\mathcal{S}$-epic. Let P be in $\mathcal{P}_{\mathcal{S}}$, with $r\colon \sqcup_i S_i \longrightarrow P$ a retract where $\sqcup_i S_i$ is a coproduct of objects in $\mathcal{S}$. Let i be the inclusion of P into the coproduct. We want to show that $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is surjective. Let $W = \sqcup_i S_i$. Since $ri = \mathrm{Id}_P$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{T}(P,X) & \xrightarrow{f_*} & \mathcal{T}(P,Y) \\ \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} \\ \mathcal{T}(W,X) & \xrightarrow{f_*} & \mathcal{T}(W,Y) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \mathcal{T}(P,X) & \xrightarrow{f_*} & \mathcal{T}(P,Y). \end{array}$$

Let $y \in \mathcal{T}(P,Y)$. Since W is ($\mathcal{S}$-epi)-projective, there is an $x'\colon W \longrightarrow X$ such that $fx' = yr$ as illustrated in the diagram below.



Let $x := x'i$. This satisfies $fx = fx'i = yri = y$. Since P was arbitrary in $\mathcal{P}_{\mathcal{S}}$ we can conclude that f is $\mathcal{P}_{\mathcal{S}}$-epic.

Moving on to 2), we can see that the inclusion $\mathcal{P}_{\mathcal{S}} \subseteq$ ($\mathcal{S}$-epi)-proj follows from Lemma 2.3.1: Since the objects that look projective from the point of view of $\mathcal{S}$-epis must be closed under retracts and coproducts, it must contain all retracts of coproducts of objects in $\mathcal{S}$, which is $\mathcal{P}_{\mathcal{S}}$.

To show the opposite inclusion, let X be ($\mathcal{S}$-epi)-projective, that is it looks projective from $\mathcal{S}$-epic morphisms. Let

$$W = \bigsqcup_{A \in \mathcal{S}} \bigsqcup_{f \in \mathcal{T}(A,X)} A \xrightarrow{r} X,$$

that is, the coproduct over all morphisms from objects in $\mathcal{S}$. We claim that this is a retraction. To see this we need to find a section. Since r is $\mathcal{S}$-epic, the induced morphism

$$\mathcal{T}(X,W) \xrightarrow{r_*} \mathcal{T}(X,X)$$

is surjective. Hence, there is an $i\colon X \longrightarrow W$ such that $ri = \mathrm{Id}_X$, making X a retract of a coproduct in $\mathcal{S}$.

Finally, we see that the construction above may be applied to any object of $\mathcal{T}$, which proves 3). □

Thus, taking any collection of objects in $\mathcal{T}$, we can use it to make a projective class. Often we want the projective class to be **stable**, which means that it is closed under suspension. For a set $\mathcal{S}$ let $\mathcal{S}' = \{\Sigma^n S\colon S \in \mathcal{S}, n \in \mathbb{Z}\}$. Then the smallest stable projective class containing $\mathcal{S}$ is $\mathcal{P}_{\mathcal{S}'}$, i.e., the smallest projective class containing $\mathcal{S}'$.

Recall also that in a triangulated category we can describe a projective class equivalently with its null-morphisms. In the stable homotopy category, we could look at the stable projective class generated by the sphere spectrum. Then the null-morphisms in this projective class are necessarily all maps $X \longrightarrow Y$ such that $\mathcal{T}(S^i,X) \longrightarrow \mathcal{T}(S^i,Y)$ are zero for all i, i.e., the maps that induce zero maps of homotopy groups. This leads us to the following definition.

Definition 2.3.3. Consider the stable projective class in the stable homotopy category generated by S^0. This is the projective class that contains retracts of wedges of spheres of all dimensions, and whose null-maps are the nullhomotopic maps. This projective class is known as the **ghost projective class**.

Throughout this thesis, we will denote the objects in the ghost projective class by $\mathcal{S}$.

2.4 Injective classes

We can take everything we have done so far in this chapter, and dualize it. This gives the definition of what we call an **injective class**. The definitions are provided briefly, and then we state the dual results.

Definition 2.4.1. Let $\mathcal{I}$ be a collection of objects and $\mathcal{M}$ be a collection of morphisms in a pointed category $\mathcal{T}$ with weak kernels. We say that a morphism $X \longrightarrow Y$ is **$\mathcal{I}$-monic** if it induces an injective morphism under $\mathcal{T}(-, I)$ for all objects $I \in \mathcal{I}$. An object I such that $\mathcal{T}(-, I)$ induces injective morphisms on all morphisms in $\mathcal{M}$ is called **$\mathcal{M}$-injective**.

An **injective class** is then a pair $(\mathcal{I}, \mathcal{M})$ such that the $\mathcal{I}$-monic morphisms are exactly $\mathcal{M}$, the $\mathcal{M}$-injective objects are exactly $\mathcal{I}$, and for all objects X there is a morphism $X \longrightarrow I$ in $\mathcal{M}$ with $I \in \mathcal{I}$.

The $\mathcal{I}$-injective objects and $\mathcal{I}$-monic morphisms determine each other through the extension property

$$\begin{array}{ccc} I & & \\ \uparrow & \nwarrow & \\ X & \rightarrowtail & Y. \end{array}$$

The tailed arrow indicates that the morphism is $\mathcal{I}$-monic.

The objects in injective classes are closed under products and retracts. If the category $\mathcal{T}$ is triangulated we can describe an injective class with the $\mathcal{I}$-null morphisms instead of the $\mathcal{I}$-monic morphisms. Furthermore, in a triangulated category, the condition "every object X admits an $\mathcal{I}$-monic morphism into an injective object" can be replaced with "every object X lies in a distinguished triangle $I \longrightarrow W \longrightarrow X \longrightarrow \Sigma I$ with $I \in \mathcal{I}$, $W \longrightarrow X$ an $\mathcal{I}$-null morphism." We don't prove these properties, as the proofs are dual to those for projective classes.

Similarly to the projective case, the smallest injective class generated by a set $\mathcal{S}$ is the set of retracts of products of objects in $\mathcal{S}$. If we are in a triangulated category, and want the injective class to be stable, then we include all suspensions as well.

The following is a common (and important) example of an injective class in the stable homotopy category, generated by the mod n Eilenberg–Mac Lane spectrum.

Example 2.4.2. Let p be a prime. Denote the Eilenberg–Mac Lane spectrum of mod p coefficients by $H\mathbb{F}_p$. Then the set $\left\{\prod_i \Sigma^{n_i} H\mathbb{F}_p : n_i \in \mathbb{Z}\right\}$ along with maps that induce zero on mod p singular cohomology is an injective class.

Actually, any spectrum E generates a stable injective class in the stable homotopy category, where the null maps induce zero on E-cohomology.

It is the injective class generated by $H\mathbb{F}_p$ which will eventually lead to the classical Adams spectral sequence.

2.5 Some properties of projective classes

We will need the following terminology when discussing convergence of the Adams spectral sequence. The definitions and results are collected from [5].

Definition 2.5.1. Let $(\mathcal{P},\mathcal{N})$ be a projective class. We say that it **generates** if for all $X \neq 0$ there is a $P \in \mathcal{P}$ such that $\mathcal{T}(P,X) \neq 0$. Equivalently, $\mathcal{N}$ has no non-zero identity morphisms.

Lemma 2.5.2. *An equivalent description of a projective class that generates is that a morphism $X \longrightarrow Y$ is an isomorphism if and only if $\mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is an isomorphsim for all $P \in \mathcal{P}$.*

Proof. Functors preserve isomorphisms, so the direction ($\Rightarrow$) is clear. Conversely, if the projective class generates, then there is at least one P such that $\mathcal{T}(P,X) \neq 0$. If $f_*\colon \mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y)$ is an isomorphism, it has an inverse, and by Yoneda lemma this inverse corresponds to a morphism $g\colon Y \longrightarrow X$ that must necessarily also be an inverse for f. □

Example 2.5.3. We see that the ghost projective class generates: A spectrum X such that $[S^i,X]=0$ for all i is contractible.

For a projective class $(\mathcal{P},\mathcal{N})$, let $\mathcal{N}_n$ be the collection of n-fold compositions of morphisms in $\mathcal{N}$. This becomes a decreasing filtration of $\mathcal{N}$. We can also make an increasing filtration of $\mathcal{P}$. Let $\mathcal{P}_1=\mathcal{P}$, and let $\mathcal{P}_{n+1}$ be the projective class generated by elements Y that lie in a cofiber sequence $X \longrightarrow Y \longrightarrow P$, where $X \in \mathcal{P}_n$ and $P \in \mathcal{P}$. Let $\mathcal{P}_0$ denote the zero-objects of $\mathcal{T}$, and let $\mathcal{N}_0$ denote all morphisms in $\mathcal{T}$.

Theorem 2.5.4 ([5, Theorem 3.5])**.** *The pair $(\mathcal{P}_n,\mathcal{N}_n)$ is a projective class for all $n \geq 0$.*

This follows from a more general idea that certain sets of morphisms are ideals in a triangulated category. More specifically, the null morphisms in a projective class behaves somewhat like an ideal (in the algebraic sense) in relation to other collections of null morphisms in other projective classes, and there are several operations that can be done on these "ideals." The theorem is proven using these operations. For more about ideals in triangulated categories, and a proof of the theorem, see [5, Chapter 3].

Definition 2.5.5. An object X has **length** n with respect to the projective class $(\mathcal{P}, \mathcal{N})$ if it is in $\mathcal{P}_n$, but not in $\mathcal{P}_{n-1}$.

Example 2.5.6. All finite CW-spectra have finite length. Indeed, if X is a CW-spectrum with structure $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, then each X_i lies in a cofiber sequence $\vee S^i \longrightarrow X_{i-1} \longrightarrow X_i$. Since X_0 is a finite set of points, $X_0 \in \mathcal{P}_0$, so inductively each $X_i \in \mathcal{P}_i$, and X has length at most n.

The last thing we need is a generalization of a projective resolution.

Definition 2.5.7. Let X be in $\mathcal{T}$, and let

$$X \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots \longleftarrow P_i \longleftarrow \cdots$$

be a diagram where each $P_i \in \mathcal{P}$. Then it is a $\mathcal{P}$-**projective resolution** of X if for all $P \in \mathcal{P}$ the following is an exact sequence:

$$0 \longleftarrow \mathcal{T}(P, X) \longleftarrow \mathcal{T}(P, P_0) \longleftarrow \mathcal{T}(P, P_1) \longleftarrow \cdots.$$

As before, these definitions can be dualized to an injective class.

Example 2.5.8. Going back to the injective class in Example 2.4.2 we can ask whether this class generates, the way ghost projective class does. The claim is that it does not. Consider the Poincaré homology sphere S_P [18, Example 1.4.4], which has homology groups of a 3-sphere, but a non-trivial fundamental group. Let S'_P be S_P, but with one point removed. Then the reduced homology of S'_P is zero in all degrees, but it is still not contractible. Hence, this is a space which becomes zero under $[-, I]$ for all I injectives in this class, but that is not zero itself.

Chapter 3

Toda brackets

Toda brackets were originally defined by Toda in [19]. There it was defined for maps between spheres of different dimensions, and used to compute stable homotopy groups of spheres. Later, the defition has been generalized and expanded, and we present here a general definition for morphisms in a triangulated (not necessarily topological) category. The definitions and propositions in this chapter are due to [6].

We fix a triangulated category $\mathcal{T}$.

3.1 Three coinciding definitions

Given a diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$$

in $\mathcal{T}$ one might ask "how far" the triangle is from being distinguished. One way of finding an answer to that question is by comparing it to triangles already known to be distinguished. Given the three maps in the triangle, there are three distinguished triangles that it is natural to compare it to, namely the three different cofiber sequences associated to each map. This is what gives us three different definitions of the Toda bracket.

Definition 3.1.1. Let $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ be a diagram in $\mathcal{T}$. Then we can define the following three subsets of $\mathcal{T}(\Sigma X_0, X_3)$:

- The **iterated cofiber Toda bracket** is the set $\langle f_3, f_2, f_1 \rangle_{cc}$ of all maps ψ such that the following diagram with distinguished top row commutes

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{f_1} & X_1 & \longrightarrow & C_{f_1} & \longrightarrow & \Sigma X_0 \\
\| & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\
X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & X_3 .
\end{array} \tag{3.1}$$

- The **fiber-cofiber Toda bracket** is the set $\langle f_3, f_2, f_1 \rangle_{fc}$ of all maps $\beta \circ \Sigma\alpha$ such that the following diagram with distinguished middle row commutes

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{f_1} & X_1 & & & & \\
{\scriptstyle \alpha}\downarrow & & \| & & & & \\
\Sigma^{-1} C_{f_2} & \longrightarrow & X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & C_{f_2} \\
 & & & & \| & & \downarrow{\scriptstyle \beta} \\
 & & & & X_2 & \xrightarrow{f_3} & X_3 .
\end{array} \tag{3.2}$$

- The **iterated fiber Toda bracket** is the set $\langle f_3, f_2, f_1 \rangle_{ff}$ of all maps $\Sigma\delta$ such that the following diagram with distinguished bottom row commutes

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & X_3 \\
{\scriptstyle \delta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \| & & \| \\
\Sigma^{-1} X_3 & \longrightarrow & \Sigma^{-1} C_{f_3} & \longrightarrow & X_2 & \xrightarrow{f_3} & X_3 .
\end{array} \tag{3.3}$$

Note that the Toda brackets depend on the triangulation of $\mathcal{T}$. For instance, the Toda bracket of a distinguished triangle contains the identity: If $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is distinguished, then (3.1) can be completed with

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & C_f & \longrightarrow & \Sigma X \\
\| & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}_{\Sigma X}} \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X
\end{array}$$

where ϕ is an isomorphism.

Proposition 3.1.2. *The three definitions of a Toda bracket coincide.*

Proof. We will only show that $\langle f_3, f_2, f_1\rangle_{cc} = \langle f_3, f_2, f_1\rangle_{fc}$, since the proof that $\langle f_3, f_2, f_1\rangle_{ff} = \langle f_3, f_2, f_1\rangle_{fc}$ is dual.

$\boldsymbol{\langle f_3, f_2, f_1\rangle_{fc} \subseteq \langle f_3, f_2, f_1\rangle_{cc}}$: Let $\beta\Sigma\alpha \in \langle f_3, f_2, f_1\rangle_{fc}$. Then they appear in the following commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{f_1} & X_1 & \longrightarrow & C_{f_1} & \longrightarrow & \Sigma X_0 \\
\downarrow{\scriptstyle \alpha} & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Sigma\alpha} \\
\Sigma^{-1}C_{f_2} & \longrightarrow & X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & C_{f_2} \\
 & & & & \| & & \downarrow{\scriptstyle \beta} \\
 & & & & X_2 & \xrightarrow{f_3} & X_3.
\end{array}$$

Here the map ϕ exists by the morphism axiom, (TR3), as the rows are distinguished triangles. So we have maps ϕ and $\psi = \beta\Sigma\alpha$ that make (3.1) commute. This proves the inclusion of $\beta\Sigma\alpha$ in $\langle f_3, f_2, f_1\rangle_{cc}$.

$\boldsymbol{\langle f_3, f_2, f_1\rangle_{cc} \subseteq \langle f_3, f_2, f_1\rangle_{fc}}$: Let $\psi \in \langle f_3, f_2, f_1\rangle_{cc}$ and let ϕ be the map making (3.1) commute. We need to find β and α with $\beta\Sigma\alpha = \psi$ such that (3.2) commutes.

Let ι_1 be the map $X_1 \longrightarrow C_{f_1}$. We can compare the cofibers of ϕ, ι_1 and $f_2 = \phi\iota_1$ with the octahedron axiom, (TR4), for triangulated categories:

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{\iota_1} & C_{f_1} & \xrightarrow{q_1} & \Sigma X_0 & \xrightarrow{-\Sigma f_1} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Sigma\alpha} & & \| \\
X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{\iota_2} & C_{f_2} & \xrightarrow{q_2} & \Sigma X_1 \\
\downarrow{\scriptstyle \iota_1} & & \| & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \Sigma\iota_1} \\
C_{f_1} & \xrightarrow{\phi} & X_2 & \xrightarrow{\iota} & C_\phi & \xrightarrow{q} & \Sigma C_{f_1}. \\
 & & & & \downarrow{\scriptstyle \eta} & & \\
 & & & & \Sigma^2 X_0 & &
\end{array} \tag{3.4}$$

The vertical row is a distinguished triangle, so the sequence

$$\mathcal{T}(C_{f_2}, X_3) \xrightarrow{\Sigma\alpha^*} \mathcal{T}(\Sigma X_0, X_3) \xrightarrow{(-\Sigma^{-1}\eta)^*} \mathcal{T}(\Sigma^{-1}C_\phi, X_3)$$

is exact. By rotating the bottom row in (3.4) to the right we get the following commutative diagram

$$
\begin{array}{ccc}
\Sigma^{-1}C_{\phi} & = & \Sigma^{-1}C_{\phi} \\
{\scriptstyle -\Sigma^{-1}q}\downarrow & & \downarrow{\scriptstyle -\Sigma^{-1}\eta} \\
C_{f_1} & \xrightarrow{\iota_1} & \Sigma X_0 \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\
X_2 & \xrightarrow[f_3]{} & X_3
\end{array}
$$

where the left column is part of a distinguished triangle. Hence

$$(-\Sigma^{-1}\eta)^*(\psi) = f_3\phi(-\Sigma^{-1}q) = 0,$$

so by exactness there is a $\beta : C_{f_2} \longrightarrow X_3$ such that $\beta\Sigma\alpha = \psi$.

This β might not satisfy $\beta i_2 = f_3$, which is required by the fiber-cofiber definition. We will correct it to a new map which does. Since $(f_3 - \beta\iota_2)\phi = 0$, there is a factorization of the error through the cofiber of ϕ, in other words, we have a map $\theta : C_{\phi} \longrightarrow X_3$ such that $f_3 - \beta\iota_2 = \theta\iota$. Let $\beta' := \beta + \theta\xi$, and note that $\beta'\iota_2 = f_3$. Furthermore, $\beta'\Sigma\alpha = \psi = \beta\Sigma\alpha$, since $\theta\xi\Sigma\alpha = 0$.

We have found the desired factorization of ψ. This concludes the proof. □

From now on we denote the Toda bracket simply by $\langle f_3, f_2, f_1\rangle$.

3.2 Some examples of Toda brackets

Example 3.2.1. We can already compute some simple brackets in a general triangulated category $\mathcal{T}$. Let $X \xrightarrow{0} Y \xrightarrow{1} Y \xrightarrow{0} Z$ be a diagram. The cofiber of $X \xrightarrow{0} Y$ is $Y \oplus \Sigma X$, and hence the map ϕ in (3.1) can be any map $\begin{pmatrix} 1 & b \end{pmatrix}$ as seen below. So by the iterated cofiber definition of the Toda bracket, we must have $\langle 0, 1, 0\rangle = \{0\}$.

$$\begin{array}{ccccccc} X & \xrightarrow{0} & Y & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & Y \oplus \Sigma X & \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle \begin{pmatrix}1 & b\end{pmatrix}} & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{0} & Y & \xrightarrow{1} & Y & \xrightarrow{0} & Z \end{array}$$

If we on the other hand consider the diagram $X \xrightarrow{0} Y \xrightarrow{0} Z \xrightarrow{1} Z$ in $\mathcal{T}$ we get that $\langle 1, 0, 0 \rangle = \mathcal{T}(\Sigma X, Z)$, by looking at the iterated cofiber definition of the Toda bracket.

$$\begin{array}{ccccccc} X & \xrightarrow{0} & Y & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & Y \oplus \Sigma X & \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} & \Sigma X \\ \| & & \| & & \downarrow{\scriptstyle \begin{pmatrix}0 & b\end{pmatrix}} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{0} & Y & \xrightarrow{0} & Z & \xrightarrow{1} & Z \end{array}$$

Here ϕ can be any map $\begin{pmatrix}0 & b\end{pmatrix}$, and so by letting $\psi = b$, we can make the above diagram commute.

Lemma 3.2.2. *For any diagram $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ in $\mathcal{T}$ the Toda bracket $\langle f_3, f_2, f_1 \rangle$ is a coset of the subgroup*

$$(f_3)_* \mathcal{T}(\Sigma X_0, X_2) + (\Sigma f_1)^* \mathcal{T}(\Sigma X_1, X_3) \subseteq \mathcal{T}(\Sigma X_0, X_3).$$

Proof. The iterated cofiber Toda bracket of the diagram above contains maps ψ that make the following diagram commute

$$\begin{array}{ccccccc} X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{\iota} & C_{f_1} & \xrightarrow{q} & \Sigma X_0 \\ \| & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & X_3. \end{array}$$

We want to show that if ψ and ψ' are two elements in $\langle f_3, f_2, f_1 \rangle$, along with maps ϕ and ϕ' that make (3.1) commute, then $\psi - \psi' = f_3 g + h(\Sigma f_1)$ for some $g\colon \Sigma X_0 \longrightarrow X_2$ and some $h\colon \Sigma X_1 \longrightarrow X_3$. Since the top row is a distinguished triangle, we can make the following diagram of long exact sequences:

$$\begin{array}{ccccccc}
\mathcal{T}(\Sigma X_1, X_2) & \xrightarrow{(\Sigma f_1)^*} & \mathcal{T}(\Sigma X_0, X_2) & \xrightarrow{q^*} & \mathcal{T}(C_{f_1}, X_2) & \xrightarrow{\iota^*} & \cdots \\
\Big\downarrow{\scriptstyle f_3^*} & & \Big\downarrow{\scriptstyle f_3^*} & & \Big\downarrow{\scriptstyle f_3^*} & & \\
\mathcal{T}(\Sigma X_1, X_3) & \xrightarrow[(\Sigma f_1)^*]{} & \mathcal{T}(\Sigma X_0, X_3) & \xrightarrow[q^*]{} & \mathcal{T}(C_{f_1}, X_3) & \xrightarrow[\iota^*]{} & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\iota^*} & \mathcal{T}(X_1, X_2) & \xrightarrow{f_1^*} & \mathcal{T}(X_0, X_2) \\
 & & \Big\downarrow{\scriptstyle f_3^*} & & \Big\downarrow{\scriptstyle f_3^*} \\
\cdots & \xrightarrow[\iota^*]{} & \mathcal{T}(X_1, X_3) & \xrightarrow[f_1^*]{} & \mathcal{T}(X_0, X_3)
\end{array}$$

Then

$$\iota^*(\phi) = \iota^*(\phi') = f_2$$

implying that

$$\phi' - \phi \in \ker(\iota^*) = \operatorname{Im}(q^*).$$

Therefore there is a $g \in \mathcal{T}(\Sigma X_0, X_2)$ such that $\phi' = \phi + gq$, and it induces a member of $\langle f_3, f_2, f_1 \rangle$.

Now, suppose ψ and ψ' are in $\langle f_3, f_2, f_1 \rangle$. Then we have

$$(\psi - \psi')q = f_3(\phi - \phi') = f_3 g q$$

which implies that

$$\psi - \psi' - f_3 g \in \ker q = \operatorname{Im}(\Sigma f_1)^*.$$

So there is an $h \in \mathcal{T}(\Sigma X_1, X_3)$ such that $h(\Sigma f_1) + g f_3 = \psi - \psi'$. Furthermore, any $h \in \mathcal{T}(\Sigma X_1, X_3)$ satisfies $q^*(h(\Sigma f_1)) = h(\Sigma f_1)q = 0$, so it induces another element of $\langle f_3, f_2, f_1 \rangle$. □

Remark 3.2.3. The subgroup in Lemma 3.2.2 is called the **indeterminacy** of the Toda bracket $\langle f_3, f_2, f_1 \rangle$. If it is zero we say that the bracket has **no indeterminacy**.

Example 3.2.4. Several relations of Toda brackets are known for spectra. The most famous, perhaps, is that for a map $x\colon \mathbb{S}^n \longrightarrow \mathbb{S}$ with $2x = 0$ we have $x\eta \in \langle 2, x, 2\rangle$, where η is the Hopf fibration $\eta\colon S^3 \longrightarrow S^2$. As a consequence, we have $\eta^2 = \langle 2, \eta, 2\rangle$, since the indeterminacy of this Toda bracket is zero. By η^2 we mean $(\Sigma\eta)\eta$.

One way to prove this is by using the mod 2 Moore spectrum M. A mod n Moore spectrum is a spectrum which has homology $\mathbb{Z}/n$ in degree zero, and zero otherwise. The spectrum M is the cofiber of $\mathbb{S} \xrightarrow{2} \mathbb{S}$, so $\mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{\text{incl}} M \xrightarrow{\text{pinch}} \mathbb{S}^1$ is a distinguished triangle in the stable homotopy category. Central to the proof is the fact that $[M, M] = \mathbb{Z}/4\{\mathrm{Id}_M\}$ (the group $\mathbb{Z}/4$, with generator Id_M), with $2\,\mathrm{Id}_M = \text{incl} \circ \eta \circ \text{pinch} \neq 0$, which is proven in Lemma A.2.

Let $x \in \pi_n(\mathbb{S})$ with $2x = 0$. By (TR3), we get a lifting of x as follows:

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "2"] \arrow[d] & \mathbb{S}^n \arrow[r, "\text{incl}"] \arrow[d, "x"'] & \Sigma^n M \arrow[r, "\text{pinch}"] \arrow[d, dashed, "\bar{x}"] & \mathbb{S}^{n+1} \arrow[d] \\ * \arrow[r] & \mathbb{S} \arrow[r, equal] & \mathbb{S} \arrow[r] & *. \end{tikzcd}$$

Then by the relation $2\,\mathrm{Id}_M = \text{incl} \circ \eta \circ \text{pinch}$ and the commutativity of $\pi_*(\mathbb{S})$ we get that $\eta x \in \langle 2, x, 2\rangle$ as the following diagram commutes

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "2"] \arrow[dd, equal] & \mathbb{S}^n \arrow[r, "\text{incl}"] \arrow[dd, equal] & \Sigma^n M \arrow[r, "\text{pinch}"] \arrow[d, "2"] \arrow[dd, bend right, "\bar{x}"'] & \mathbb{S}^{n+1} \arrow[d, "\eta"] \\ & & \Sigma^n M & \mathbb{S}^n \arrow[l, "\text{incl}"'] \arrow[d, "x"] \\ \mathbb{S}^n \arrow[r, "2"'] & \mathbb{S}^n \arrow[r, "x"'] & \mathbb{S} \arrow[r, "2"'] & \mathbb{S}. \arrow[from=2-3, to=3-4, "\bar{x}"] \end{tikzcd}$$

When Toda defined what we now call Toda brackets, it was for maps between spheres. Assume we have maps

$$S^k \xrightarrow{f} S^l \xrightarrow{g} S^m \xrightarrow{h} S^n$$

where $g \circ f$ and $h \circ g$ are nullhomotopic. Then we have (null)homotopies $F\colon CS^k \longrightarrow S^m$ and $G\colon CS^l \longrightarrow S^n$. We can construct a map $\psi\colon S^{k+1} \longrightarrow S^n$, where $\psi = h \circ F \bigsqcup_{CS^k} G \circ Cf$. This is illustrated in Figure 3.1.

Toda defined the Toda bracket as the set of maps ψ as above. We see that the different maps in the brackets is a consequence of different choices of the homotopies F and G. One can actually show that this set is a coset of $h_* \pi_{l+1}(S^n) + (\Sigma f)^* \pi_{n+1}(S^k)$. So the two definitions coincide for topological spaces/spectra.

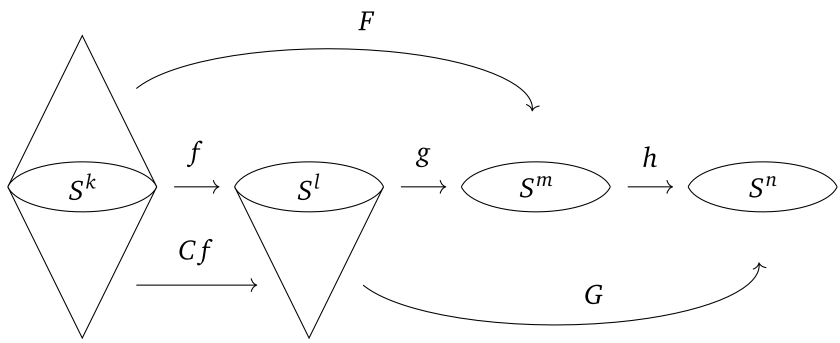


Figure 3.1: The Toda bracket of $\langle h, g, f \rangle$ consists of maps from ΣS^k to S^n.

Note how we here demand that the maps in the Toda bracket pairwise compose to zero (which means nullhomotopic in $\mathcal{SHC}$), but that we did not demand this in Definition 3.1.1. This might seem like an inconsistency, but the Toda bracket of $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ is empty if either $f_2 \circ f_1$ or $f_3 \circ f_2$ is nonzero. Consider the latter case, and look at the iterated cofiber definition of the Toda bracket:

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{i} & C_{f_1} & \xrightarrow{q} & \Sigma X_0 \\
\| & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\
X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & X_3.
\end{array}$$

We need the map ψ to satisfy $\psi \circ q \circ i = f_3 \circ f_2 \neq 0$. Since q and i are in a distinguished triangle, they compose to zero. Therefore, no ψ can satisfy this equation, and the bracket is empty.

3.3 Heller's theorem and higher Toda brackets

Before moving on to the Adams spectral sequence we discuss some interesting aspects of Toda brackets. We first consider a relation between the triangulated structure of the category $\mathcal{T}$ and Toda brackets. Then we take a quick look at a generalization of the Toda bracket for diagrams of morphisms.

The first is a theorem first stated by Heller ([8, Theorem 13.2]). We present it in the form of [6, Theorem B.1] as we find this more accessible.

Theorem 3.3.1 (Heller). *Let $\mathcal{T}$ be a triangulated category. A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is distinguished if and only if the following holds*

- *The sequence following sequence of abelian groups is exact for all $A \in \mathcal{T}$*

$$\mathcal{T}(A, \Sigma^{-1}Z) \xrightarrow{(\Sigma^{-1}h)_*} \mathcal{T}(A, X) \xrightarrow{f_*} \mathcal{T}(A, Y) \xrightarrow{g_*} \mathcal{T}(A, Z) \xrightarrow{h_*} \mathcal{T}(A, \Sigma X).$$

- *The Toda bracket $\langle h, g, f \rangle$ contains the identity of ΣX.*

That the two conditions hold for a triangulated category is straightforward. The converse holds because Z and the mapping cone of f are isomorphic as a consequence of the two conditions. Another way of stating this theorem, is that the triangulated structure of a category $\mathcal{T}$ with a fixed automorphism Σ is determined by the Toda brackets.

The second is the notion of n-fold Toda brackets. The construction is rather comprehensive, so we will not be elaborating it. It was first provided by Shipley [16]. Given a diagram $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_n$ in $\mathcal{T}$, the n-fold Toda bracket $\langle f_n, \cdots, f_1 \rangle$ is a subset of $\mathcal{T}(X_0, \Sigma X_n)$. In [6] it is shown that the n-fold Toda bracket can be calculated inductively, and although there are $(n-2)!$ ways of doing this, all the different subsets coincide up to a sign ([6, Theorem 5.11]).

Chapter 4

Adams spectral sequences

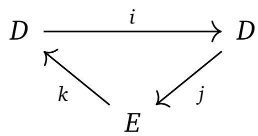
The Adams spectral sequence was first constructed by Adams as a tool to compute stable homotopy groups of spheres [1], where the singular mod p cohomology theories play an important role. Later, more general Adams spectral sequences were presented using general cohomology theories, like the Adams–Novikov spectral sequence [12]. We will provide a general construction of an Adams spectral sequence in a triangulated category, using the projective and injective classes of Chapter 2. They are called Adams spectral sequences because they coincide with the classical Adams spectral sequences in $\mathcal{SHC}$, for certain injective classes. The general construction makes it very versatile, as it can be applied in any triangulated category, not just topological ones.

In this chapter we will first review necessary prerequisites, like exact couples and Adams resolutions, before we move on to the spectral sequences. As we will be constructing the spectral sequences with projective and injective classes, there will be two different definitions.

4.1 Exact couples

Before defining the Adams spectral sequence in the context of a general triangulated category, we recall the definition and properties of an exact couple. The definition and results can be found in [10, Chapter 2.2].

Definition 4.1.1. Let R be a ring. An **exact couple** is a pair (D, E) of R-modules, and homomorphisms i, j and k such that the triangle

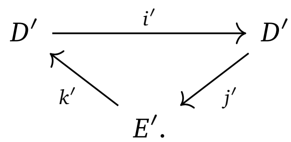


is exact.

From this we can then define a differential $d\colon E \longrightarrow E$ by letting $d := jk$. It then follows by exactness that $d^2 = 0$. Using the differential we can form the **derived exact couple**:

- $D' := \operatorname{Im} i = \ker j$
- $E' := H(E,d) = \frac{\ker d}{\operatorname{Im} d}$
- $i' := i|_{D'}$
- $j'(i(x)) := [j(x)]$
- $k'([y]) := k(y)$

which can be summarized in the following diagram



One can show that this is well-defined. We have the following result about the derived couple:

Lemma 4.1.2. *The derived couple of an exact couple is exact.*

For a proof, see [10, Proposition 2.7].

We can iterate this process and obtain the n^{th} derived couple (E^n, D^n), and we can define the n^{th} differential as $d^n := j^n k^n$. We say that (E^n, d^n) is the **spectral sequence** associated to the exact couple (E, D).

We include the following example to show how exact couples can appear, inspired by [7, Chapter 5.1]

Example 4.1.3. Let $\pi\colon X \longrightarrow B$ be a Serre fibration of CW-complexes, with B path-connected. Let B_p be the p-skeleton of B. Then we have a filtration of X via $X_p := \pi^{-1}(B_p)$. Since (B_p, B_{p-1}) is p-connected, (X, X_p) is p-connected. In particular, the inclusion $X_p \hookrightarrow X$ induces an isomorphism $H_n(X_p) \longrightarrow H_n(X)$ if $n < p$. For the pair (X_p, X_{p-1}) we have a long exact sequence in homology

$$\cdots \to H_n(X_{p-1}) \xrightarrow{i} H_n(X_p) \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{k} H_{n-1}(X_{p-1}) \to \cdots$$

and since $H_n(X_p, X_{p-1}) = 0$ whenever $n < p$, we can let $n = p + q$, and summarize the homologies of the filtration of X with the following exact couple:

$$\begin{array}{ccc} \bigoplus_{p,q} H_{p+q}(X_p) & \xrightarrow{\quad i \quad} & \bigoplus_{p,q} H_{p+q}(X_p). \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & \bigoplus_{p,q} H_{p+q}(X_p, X_{p-1}) & \end{array}$$

This exact couple yields a spectral sequence by the construction above. One can show that this is the Serre spectral sequence associated to the fibration p, with

$$E^2_{p,q} = H_p(B, H_q(F)),$$

where F is the fiber of the fibration.

4.2 Adams resolutions

In this section we fix a triangulated category $\mathcal{T}$, whose suspension functor is an adjoint equivalence of $\mathcal{T}$.

Definition 4.2.1. Let $X \in \mathcal{T}$. An **Adams resolution** of X with respect to a projective class $(\mathcal{P}, \mathcal{N})$ is a diagram of the form

$$\begin{array}{ccccccc} X = X_0 & \xrightarrow{\quad i_0 \quad} & X_1 & \xrightarrow{\quad i_1 \quad} & X_2 & \longrightarrow & \cdots \\ {\scriptstyle p_0}\nwarrow & & \swarrow{\scriptstyle \delta_0}\ \nwarrow{\scriptstyle p_1} & & \swarrow{\scriptstyle \delta_1} & & \\ & P_0 & & P_1 & & & \end{array}$$

where each $P_s \in \mathcal{P}$, each $i_s \in \mathcal{N}$ and the triangle $P_s \xrightarrow{p_s} X_s \xrightarrow{i_s} X_{s+1} \xrightarrow{\delta_s} \Sigma P_s$ is distinguished. The circles indicates that the morphisms are degree-shifting.

Remark 4.2.2. Recalling the equivalent definitions of a projective class in Chapter 2, we see that all the morphisms p_s in the resolution are $\mathcal{P}$-epic, and all the morphisms δ_s are $\mathcal{P}$-monic, since all morphisms i_s are $\mathcal{P}$-null. If we assume that $\mathcal{T}$ has **enough $\mathcal{P}$-projectives**, that is every object X admits a $\mathcal{P}$-epic morphism from a projective into X, then by the properties of a projective class there is an Adams resolution for all objects $X \in \mathcal{T}$.

Example 4.2.3. Recall that the ghost projective class $\mathcal{S}$ in Definition 2.3.3 is a projective class in $\mathcal{SHC}$. We find an Adams resolution for the sphere spectrum.

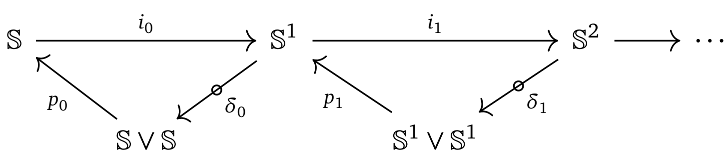
Recall the inductive CW-structure of the sphere: We can build S^{n+1} by gluing together two cones of S^n with the pushout

$$\begin{array}{ccc} S^n \vee S^n & \xrightarrow{p_n} & S^n \\ \downarrow & & \downarrow{\scriptstyle i_n} \\ CS^n \vee CS^n & \longrightarrow & S^{n+1} \end{array}$$

where p_n is the identity when restricted to each sphere. This means that S^{n+1} is the cofiber of p_n, so the triangle

$$\mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{p_n} \mathbb{S}^n \xrightarrow{i_n} \mathbb{S}^{n+1} \xrightarrow{\delta_n} \mathbb{S}^{n+1} \vee \mathbb{S}^{n+1}$$

in $\mathcal{SHC}$ is distinguished. The map δ_n is the pinch map. Using this we can make the following resolution of $\mathbb{S}$ (which implicitly is also a resolution for $\mathbb{S}^n$).



Since all inclusions $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$ are nullhomotopic, this is an Adams resolution with respect to $\mathcal{S}$.

In the above example, we could swap the sphere spectrum with any CW-spectrum on which the inclusion of the n-skeleton into the $(n+1)$-skeleton is nullhomotopic.

Definition 4.2.4. Let Y be an object in $\mathcal{T}$. An **Adams resolution** of Y with respect to the injective class $(\mathcal{I}, \mathcal{N})$ is a diagram of the form

$$\begin{array}{ccccccccc} Y = Y_0 & & \xleftarrow{\quad i_0 \quad} & & Y_1 & & \xleftarrow{\quad i_1 \quad} & & Y_2 \longrightarrow \cdots \\ & \searrow^{p_0} & & \nearrow_{\delta_0} & & \searrow^{p_1} & & \nearrow_{\delta_1} & \\ & & I_0 & & & & I_1 & & \end{array}$$

where each I_s is in $\mathcal{I}$, each morphism i_s is in $\mathcal{N}$, and all triangles $Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{p_s} I_s \xrightarrow{\delta_s} \Sigma Y_{s+1}$ are distinguished.

Remark 4.2.5. From the way injective classes are constructed, we see that all the morphisms p_s in the Adams resolution are $\mathcal{I}$-monic and all the morphisms δ_s are $\mathcal{I}$-epic, since all morphisms i_s are $\mathcal{I}$-null. If the category $\mathcal{T}$ has enough $\mathcal{I}$-injectives (dual to enough $\mathcal{P}$-projectives), then every object admits an Adams resolution.

4.3 A general Adams spectral sequence

We obtain two dual constructions of an Adams spectral sequence. One with respect to a projective class, and one with respect to an injective class.

4.3.1 Adams spectral sequence with respect to a projective class

In this subsection we consider a projective class $(\mathcal{P}, \mathcal{N})$. Let X be an object in $\mathcal{T}$ with a projective Adams resolution as in Definition 4.2.1, and let Y be another object in $\mathcal{T}$. Then we obtain an unraveled exact couple by applying $\mathcal{T}(-, Y)$ to the resolution:

$$\begin{array}{ccccc} \cdots \longrightarrow \mathcal{T}(X_s, Y) & & \xleftarrow{\quad (i_s)^* \quad} & & \mathcal{T}(X_{s+1}, Y) \longrightarrow \cdots \\ & \searrow^{(p_s)^*} & & \nearrow_{(\delta_s)^*} & \\ & & \mathcal{T}(P_s, Y) & & \end{array}$$

which can be summarized in the following exact couple of bigraded groups

$$\begin{array}{ccc} \bigoplus_{s,k} \mathcal{T}(\Sigma^k X_s, Y) & \xleftarrow{\oplus_s (i_s)^*} & \bigoplus_{s,k} \mathcal{T}(\Sigma^k X_s, Y) \\ & \searrow^{\oplus_s (p_s)^*} \quad \nearrow_{\oplus_s (\delta_s)^*} & \\ & \bigoplus_{s,k} \mathcal{T}(\Sigma^k P_s, Y). & \end{array}$$

We usually let $t = k + s$, and $i = \oplus_s (i_s)^*, p = \oplus_s (p_s)^*$ and $\delta = \oplus_s (\delta_s)^*$ to simplify the notation. We are left with the following exact couple

$$\begin{array}{ccc} \mathcal{T}(X_s, \Sigma^{s-t} Y) & \xleftarrow{i} & \mathcal{T}(X_s, \Sigma^{s-t} Y) \\ & \searrow^{p} \quad \nearrow_{\delta} & \\ & \mathcal{T}(P_s, \Sigma^{s-t} Y). & \end{array} \tag{4.1}$$

To this exact couple there is an associated spectral sequence, which is the **Adams spectral sequence** with respect to the projective class $\mathcal{P}$.

We have used the identification $\mathcal{T}(\Sigma^{t-s} Z, Y) \cong \mathcal{T}(Z, \Sigma^{s-t} Y)$, and so we can write $E_1^{s,t} = \mathcal{T}(\Sigma^{t-s} P_s, Y) \cong \mathcal{T}(P_s, \Sigma^{s-t} Y)$. We see that for these spectral sequences, $E_r^{s,t} = 0$ for all $s < 0$ by construction.

We can describe the E_2-page with Ext groups.

Lemma 4.3.1. *For an Adams spectral sequence with respect to a projective class we have* $E_2^{s,t} = \operatorname{Ext}_{\mathcal{P}}^{s,t}(X, Y)$.

Remark 4.3.2. We define

$$\operatorname{Ext}_{\mathcal{P}}^{s,t}(X, Y) := \operatorname{Ext}_{\mathcal{P}}^{s}(X, \Sigma^{-t} Y) = H_{-s}(\mathcal{T}(P_\bullet^X, \Sigma^{-t} Y)),$$

where $P_\bullet^X$ is a $\mathcal{P}$-projective resolution of X.

Proof. Given the projective Adams resolution of X, we can make a $\mathcal{P}$-projective resolution of X:

$$0 \longleftarrow X \xleftarrow{p_0} P_0 \xleftarrow{\Sigma^{-1}\delta_0 \Sigma^{-1} p_1} \Sigma^{-1} P_1 \xleftarrow{\Sigma^{-2}\delta_1 \Sigma^{-2} p_2} \Sigma^{-2} P_2 \longleftarrow \cdots .$$

Applying $\mathcal{T}(-,\Sigma^{-t}Y)$ to this resolution gives us the following sequence

$$0 \to \mathcal{T}(X,\Sigma^{-t}Y) \xrightarrow{p_0^*} \mathcal{T}(P_0,\Sigma^{-t}Y) \xrightarrow{\Sigma^{-1}(p_1\delta_0)^*} \mathcal{T}(\Sigma^{-1}P_1,\Sigma^{-t}Y) \to \cdots$$

and we find that

$$\begin{aligned}
\mathrm{Ext}_{\mathcal{P}}^{s,t}(X,Y) &= H_{-s}(\mathcal{T}(P_\bullet^X,\Sigma^{-t}Y)) \\
&= \frac{\ker(\Sigma^{-s-1}\delta_s\Sigma^{-s-1}p_{s+1})^*}{\mathrm{Im}(\Sigma^{-s}\delta_{s-1}\Sigma^{-s}p_s)^*} \\
&= \frac{\ker(d_1\colon E_1^{s,t} \longrightarrow E_1^{s+1,t})}{\mathrm{Im}(d_1\colon E_1^{s-1,t} \longrightarrow E_1^{s,t})} \\
&= E_2^{s,t}.
\end{aligned}$$

□

Example 4.3.3. We now return to the ghost projective class and consider the projective resolution of the sphere spectrum in Example 4.2.3. Here the maps $p_s\colon \mathbb{S}^n \vee \mathbb{S}^n \longrightarrow \mathbb{S}^n$ are two copies of the identity, while the maps $\delta_s\colon \mathbb{S}^n \longrightarrow \mathbb{S}^n \vee \mathbb{S}^n$ are pinch maps. Then the composition $\delta_s \circ p_{s+1}$ becomes the identity. This means that $\ker(\delta_s p_{s+1})^* = \{0\}$ for all s. By Lemma 4.3.1 we get that $E_2^{s,t} = \mathrm{Ext}_{\mathcal{S}}^{s,t}(\mathbb{S},Y) = \{0\}$ for all s and t for this resolution of the sphere spectrum, no matter what the object Y is.

4.3.2 Adams spectral sequence with respect to an injective class

Let X and Y be objects in our triangulated category $\mathcal{T}$. Let Y have an Adams resolution with respect to an injective class $(\mathcal{I},\mathcal{N})$ as in Definition 4.2.4. We can apply $\mathcal{T}(X,-)$ to the resolution, and get the unraveled exact couple

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{T}(X,Y_s) & & \xleftarrow{(i_s)_*} & & \mathcal{T}(X,Y_{s+1}) & \longrightarrow \cdots \\
& & \searrow^{(p_s)_*} & & \nearrow_{(\delta_s)_*}^{\circ} & & \\
& & & \mathcal{T}(X,I_s) & & &
\end{array}$$

which can be summarized in the following exact couple of bigraded groups

$$\begin{array}{ccc} \bigoplus_{s,k}\mathcal{T}(\Sigma^k X, Y_{s+1}) & \xrightarrow{\quad \oplus_s(i_s)_* \quad} & \bigoplus_{s,k}\mathcal{T}(\Sigma^k X, Y_s) \\ & {\scriptstyle \oplus_s(\delta_s)_*}\nwarrow \qquad \swarrow{\scriptstyle \oplus_s(p_s)_*} & \\ & \bigoplus_{s,k}\mathcal{T}(\Sigma^k X, I_s). & \end{array}$$

The **Adams spectral sequence** with respect to the injective class $(\mathcal{I}, \mathcal{N})$ is the spectral sequence associated to this exact couple. As with the projective case, we usually let $t = k + s$ and define $i = \oplus_s(i_s)_*, p = \oplus_s(p_s)_*, \delta = \oplus_s(\delta_s)_*$ to obtain the notation

$$\begin{array}{ccc} \bigoplus_{s,t}\mathcal{T}(\Sigma^{t-s} X, Y_{s+1}) & \xrightarrow{\quad i \quad} & \bigoplus_{s,t}\mathcal{T}(\Sigma^{t-s} X, Y_s) \\ & {\scriptstyle \delta}\nwarrow \qquad \swarrow{\scriptstyle p} & \\ & \bigoplus_{s,t}\mathcal{T}(\Sigma^{t-s} X, I_s). & \end{array}$$

This spectral sequence has $E_1^{s,t} = \mathcal{T}(\Sigma^{t-s}X, Y_s)$. We see that for these spectral sequences, $E_r^{s,t} = 0$ for all $s < 0$.

The Ext groups play an important role in the injective Adams spectral sequence as well.

Lemma 4.3.4. *The E_2 term of the spectral sequence is given by $E_2^{s,t} = \mathrm{Ext}_{\mathcal{I}}^{s,t}(X, Y)$.*

Remark 4.3.5. We define

$$\mathrm{Ext}_{\mathcal{I}}^{s,t}(X,Y) := \mathrm{Ext}_{\mathcal{I}}^{s}(\Sigma^t X, Y) = H_{-s}(\mathcal{T}(\Sigma^t X, I_\bullet^Y)),$$

where $I_\bullet^Y$ is an $\mathcal{I}$-injective resolution of Y.

Proof. We can construct the following $\mathcal{I}$-injective resolution of Y from its Adams resolution with respect to $\mathcal{I}$

$$0 \longrightarrow Y \xrightarrow{p_0} I_0 \xrightarrow{\Sigma p_1 \delta_0} \Sigma I_1 \xrightarrow{\Sigma^2 p_2 \Sigma \delta_1} \Sigma^2 I_2 \longrightarrow \cdots$$

to which we can apply $\mathcal{T}(\Sigma^t X, -)$

$$0 \longrightarrow \mathcal{T}(\Sigma^t X, Y) \xrightarrow{(p_0)_*} \mathcal{T}(\Sigma^t X, I_0) \xrightarrow{(\Sigma p_1 \delta_0)_*} \mathcal{T}(\Sigma^t X, \Sigma I_1) \longrightarrow \cdots$$

and obtain that

$$\mathrm{Ext}_{\mathcal{I}}^{s,t}(X,Y) = H_{-s}(\mathcal{T}(\Sigma^t X, I_\bullet^Y)) = \frac{\ker(\Sigma^{s+1} p_{s+1} \Sigma^s \delta_s)_*}{\mathrm{Im}(\Sigma^s p_s \Sigma^{s-1} \delta_{s-1})_*}.$$

This is precisely $E_2^{s,t} = H(E_1^{s,t}, d_1)$ since $d_1 = p\delta$. □

4.4 The classical Adams spectral sequence

In this section we focus on the classical Adams spectral sequence, and how it can be obtained from our general Adams spectral sequence. The approach is inspired by [13, Chapter 11.3]. For X and Y two connective spectra (where X is often the sphere spectrum), the classical Adams spectral sequence is the spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(Y), H^*(X)) \Longrightarrow [\Sigma^{t-s} X, Y_p].$$

Here H is the mod p Eilenberg–Mac Lane spectrum, $\mathcal{A}$ the mod p Steenrod algebra, and $[X, Y_p] = [X, Y]/\text{non} - p$ torsion. The groups $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(Y), H^*(X))$ are defined as $\mathrm{Ext}_{\mathcal{A}}^{s}(H^*(Y), H^*(\Sigma^t X))$.

We saw in Section 2.4 that an example of an injective class in the stable homotopy category is the smallest injective class generated by mod p Eilenberg–Mac Lane spectra, along with maps that induce zero on mod p cohomology. Denote this injective class by $\mathcal{H}$, and the mod p Eilenberg–Mac Lane spectrum by H.

In this section we want to show that this injective class leads to the classical Adams spectral sequence. The claim is that the two spectral sequences have the same E_2-page. Recall Lemma 4.3.4 where we saw that the E_2-page of a spectral sequence with respect to an injective class is $\mathrm{Ext}_{\mathcal{I}}^{s,t}(X,Y)$.

We need some technical results before we can prove our claim. First, we show that we can make an injective Adams resolution of Y with respect to $\mathcal{H}$ using only wedge sums of Eilenberg–Mac Lane spectra, i.e., there are no retracts involved in the Adams resolution.

Lemma 4.4.1. *Let Y be a connective spectrum of finite type. Then there exists an injective Adams resolution of Y with respect to $\mathcal{H}$*

$$\begin{array}{ccccccc} \cdots \longleftarrow & Y_S & \xleftarrow{\quad i_s \quad} & & Y_{s+1} & \longleftarrow \cdots \\ & & \searrow^{p_s} & & \nearrow_{\delta_s} & & \\ & & & K_s & & & \end{array}$$

where each K_s is a wedge sum of suspensions of Eilenberg–Mac Lane spectra, and each Y_s is a connective spectrum of finite type.

Proof. We prove this inductively. Assume we have our wanted resolution up until Y_s. Then, let $K_s = H \wedge Y_s$, and let $p_s = \eta \wedge \mathbb{1} \colon \mathbb{S}^0 \wedge Y_s \longrightarrow H \wedge Y_s$, where $\eta \colon \mathbb{S}^0 \longrightarrow H$ is the unit map. From the Künneth formula we have that

$$H^*(H \wedge Y_s) \cong H^*(H) \otimes_{\mathbb{F}_p} H^*(Y_s) \cong \mathcal{A} \otimes_{\mathbb{F}_p} H^*(Y_s).$$

Hence an element $\mathbb{1} \otimes y \in \mathcal{A} \otimes_{\mathbb{F}_p} H^*(Y_s)$, is in correspondence with a map in $[H \wedge Y_s, H]_{-*}$. Let $\{y_i \colon i \in I\}$ be the set of generators of $H^*(Y_s)$. Summing over them we get a wedge sum $\bigvee_I H$ along with projection maps $\pi_i \colon \vee_I H \longrightarrow H$. Then for all i and j we have the diagram

$$\begin{array}{ccccc} & & H \wedge Y_s & & \\ & \swarrow^{1\otimes y_i} & \downarrow{\scriptstyle\psi} & \searrow^{1\otimes y_j} & \\ H & \xleftarrow[\pi_i]{} & \bigvee_I H & \xrightarrow[\pi_j]{} & H \end{array}$$

and so the map ψ exists by the universal property of the product. Furthermore, this map is an isomorphism of homotopy groups [13, Lemma 11.1.4], so by Whitehead's theorem $K_s = H \wedge Y_s \cong \vee_I H$.

The map p_s is $\eta \wedge \mathbb{1}$, which means it induces the Steenrod action $\mathcal{A} \otimes_{\mathbb{F}_2} H^*(Y_s) \xrightarrow{(\eta \wedge \mathbb{1})^*} H^*(Y_s)$, which is surjective. Letting Y_{s+1} be the fiber of p_s, we can expand the Adams resolution with the cofiber sequence

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{p_s} K_s \xrightarrow{\delta_s} \Sigma Y_{s+1}, \tag{4.2}$$

in which i_s is $\mathcal{H}$-null, since p_s is $\mathcal{H}$-epi. To complete the proof we need to see that Y_{s+1} is connective and of finite type. Consider the long exact sequence in homotopy of (4.2):

$$\cdots \longrightarrow \pi_{n+1}(K_s) \longrightarrow \pi_n(Y_{s+1}) \longrightarrow \pi_n(Y_s) \longrightarrow \pi_n(K_s) \longrightarrow \cdots .$$

Since both Y_s and K_s are connective, there is an N such that $\pi_n(K_s) = \pi_n(Y_s) = 0$ for all $n \leq N$. So Y_{s+1} is connective.

To see that it is of finite type, we can look at the long exact sequence in homology:

$$\cdots \longrightarrow H_*(Y_{s+1}) \xrightarrow{(i_s)_*} H_*(Y_s) \xrightarrow{(p_s)_*} H_*(K_s) \xrightarrow{(\delta_s)_*} H_*(\Sigma Y_{s+1}) \longrightarrow \cdots .$$

Since both Y_s and K_s are of finite type, $H_*(Y_s)$ and $H_*(K_s)$ are finitely generated. We can split the long exact sequence into short exact sequences

$$0 \longrightarrow \operatorname{coker}(p_{s-1})_* \longrightarrow H_*(Y_{s+1}) \longrightarrow \operatorname{Im}(i_s)_* \longrightarrow 0$$

in which $\operatorname{coker}(p_{s-1})_*$ and $\operatorname{Im}(i_s)_*$ are finitely generated, so $H_*(Y_{s+1})$ is finitely generated.

Continuing this construction inductively, we obtain the Adams resolution of Y that we wanted. □

The next step is to see that this Adams resolution leads to an $\mathcal{A}$-projective resolution with free $\mathcal{A}$-modules.

Lemma 4.4.2. *For a connective spectrum Y of finite type, there is a projective resolution of $H^*(Y)$ with only free $\mathcal{A}$ modules.*

Proof. From the Adams resolution in Lemma 4.4.1, we have maps

$$\Sigma^{s-1}K_{s-1} \xrightarrow{(\Sigma^s p_s)\circ(\Sigma^{s-1}\delta_{s-1})} \Sigma^s K_s.$$

Let $P_s = H^*(\Sigma^s K_s)$, $f_s = (\Sigma^{s-1}\delta_{s-1})^* \circ (\Sigma^s p_s)^*$ and $f_0 = p_0^*$.

In the long exact sequence in cohomology of

$$Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{p_s} K_s \xrightarrow{\delta_s} \Sigma Y_{s+1}$$

we have that the map $(\Sigma^s i_s)^*$ is zero, so the sequence splits into a short exact sequence

$$0 \longrightarrow H^*(\Sigma^{s+1}Y_{s+1}) \xrightarrow{(\Sigma^s\delta_s)^*} H^*(\Sigma^s K_s) \xrightarrow{(\Sigma^s p_s)^*} H^*(\Sigma^s Y_s) \longrightarrow 0.$$

Varying s we get different short exact sequences which we can splice together using the maps f_s as follows

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^*(\Sigma^s K_s) & & \xrightarrow{\quad f_s \quad} & & H^*(\Sigma^{s-1}K_{s-1}) & \longrightarrow \cdots \\ & & \searrow^{(\Sigma^s p_s)^*} & & \nearrow_{(\Sigma^{s-1}\delta_{s-1})^*} & & \\ & & & H^*(\Sigma^s Y_s) & & & \end{array}$$

and get the following projective resolution of $H^*(Y)$

$$\cdots \longrightarrow P_s \xrightarrow{f_s} P_{s-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{f_0} H^*(Y) \longrightarrow 0$$

where each P_s is a free $\mathcal{A}$-module by construction. □

Finally we need the following result about the Hurewicz homomorphism.

Lemma 4.4.3. *The Hurewicz homomorphism*

$$d\colon [Y,X] \longrightarrow \operatorname{Hom}_{\mathcal{A}}(H^*(X), H^*(Y))$$

given by $d(f) = f^*$ *is an isomorphism if* X *is a wedge sum of suspensions of Eilenberg–Mac Lane spectra.*

Proof. It is sufficient to assume that $X = \Sigma^n H$, because of distribution over the wedge sum. Then we have that

$$\begin{aligned} [Y, \Sigma^n H] &\cong [\Sigma^{-n}Y, H] \cong \operatorname{Hom}_{\mathcal{A}}(H^*(H), H^*(\Sigma^{-n}Y)) \\ &\cong \operatorname{Hom}_{\mathcal{A}}(H^*(\Sigma^n H), H^*(Y)) \end{aligned}$$

The second isomorphism follows from the general fact that for a ring R and an R-module M, $\operatorname{Hom}_R(R, M) \cong M$. □

Proposition 4.4.4. *The Adams spectral sequence with respect to the injective class* $\mathcal{H}$ *is the classical Adams spectral sequence.*

Proof. We know from Lemma 4.3.4 that the E_2-page of the injective Adams spectral sequence is $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{H}}(X,Y) = H_{-s}([\Sigma^t X, I^Y_\bullet])$. For the resolution in Lemma 4.4.1, we are looking for the $(-s)^{\text{th}}$ cohomology of the sequence

$$\cdots \longrightarrow [\Sigma^t X, \Sigma^{s-1} K_s] \xrightarrow{(\Sigma^s p_s \Sigma^{s-1}\delta_{s-1})_*} [\Sigma^t X, \Sigma^s K_s] \longrightarrow \cdots .$$

Let $\mathrm{Hom}^t_{\mathcal{A}}(H^*(Y), H^*(Z)) := \mathrm{Hom}_{\mathcal{A}}(H^*(Y), H^*(\Sigma^t Z))$. Under the isomorphism in Lemma 4.4.3 the above sequence becomes

$$\cdots \to \mathrm{Hom}^t_{\mathcal{A}}(H^*(\Sigma^s K_s), H^*(X)) \xrightarrow{f_s^*} \mathrm{Hom}^t_{\mathcal{A}}(H^*(\Sigma^{s-1} K_{s-1}), H^*(X)) \to \cdots .$$

Using Lemma 4.4.2 we see that the $(-s)^{\text{th}}$ cohomology of this becomes exactly $\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(Y), H^*(X))$. □

4.5 Convergence

In the following section we will be using Cartan–Eilenberg terms of convergence and strongly convergent spectral sequences [4].

Definition 4.5.1. Let A^s be a sequence of graded abelian groups, and let i_s be morphisms $\cdots \longrightarrow A^{s+1} \xrightarrow{i_s} A^s \xrightarrow{i_{s-1}} A^{s-1} \longrightarrow \cdots$.

The **limit** is $A^\infty = \varprojlim A^s$. It comes with morphisms $\epsilon_s \colon A^\infty \longrightarrow A^s$ such that $i_s \circ \epsilon_{s+1} = \epsilon_s$.

The **colimit** is $A^{-\infty} = \varinjlim A^s$. It comes with maps $\eta_s \colon A^s \longrightarrow A^{-\infty}$ such that $\eta_s \circ i_s = \eta_{s+1}$.

The **derived limit** is $RA^\infty = \mathrm{R}\varprojlim A^s$, the first derived functor of $\lim_s$.

Let $\Pi_s A^s$ be the product formed degree-wise, and let $i \colon \Pi_s A^s \longrightarrow \Pi_s A^s$ be the morphism Πi_s. The derived limit is constructed explicitly through the exact sequence

$$0 \longrightarrow A^\infty \longrightarrow \Pi_s A^s \xrightarrow{1-i} \Pi_s A^s \longrightarrow RA^\infty \longrightarrow 0. \tag{4.3}$$

For this and more about the construction of the derived limit, see [17, Section 15.86].

Definition 4.5.2. Let the following be an unraveled exact couple of bigraded groups

$$\begin{array}{ccccccccc} \cdots \to A^{s+1} & \xrightarrow{i_s} & A^s & \xrightarrow{i_{s-1}} & A^{s-1} \to \cdots . \\ {\scriptstyle k_s} \nwarrow & \swarrow {\scriptstyle j_s} & {\scriptstyle k_{s-1}} \nwarrow & \swarrow {\scriptstyle j_{s-1}} & \\ & E^s & & E^{s-1} & \end{array} \tag{4.4}$$

The spectral sequence associated to this exact couple is **conditionally convergent** and converges to the colimit $A^{-\infty}$ if $A^{\infty} = 0$ and $RA^{\infty} = 0$. The spectral sequence is conditionally convergent and converges to the limit A^{∞} if $A^{-\infty} = 0$.

Theorem 4.5.3. *Suppose the spectral sequence associated to* (4.4) *is a half-plane spectral sequence with respect to s, that is, $E^s = 0$ for all $s < 0$. If it is conditionally convergent and collapses ($E^n = E^{\infty}$ for some n) then it is strongly convergent.*

The theorem and its proof can be found in [3, Theorem 7.1].

We now go back to a triangulated category $\mathcal{T}$ and some stable projective class $\mathcal{P}$. All results in the rest of this section are due to [5, Chapter 4].

Proposition 4.5.4. *Let X have an Adams resolution as in Definition 4.2.1 with respect to $\mathcal{P}$. Assume furthermore that the category $\mathcal{T}$ has all small coproducts. The spectral sequence derived from $\mathcal{T}(X, Y)$ is conditionally convergent for all X and Y if the projective class generates.*

Proof. Assume that the projective class generates. Consider the exact couple in (4.1). The exact sequence (4.3) for this couple becomes

$$0 \to \varprojlim \mathcal{T}(X_s, Y)_* \to \Pi\, \mathcal{T}(X_s, Y)_* \xrightarrow{\partial} \Pi\, \mathcal{T}(X_s, Y)_* \to \mathrm{R}\varprojlim \mathcal{T}(X_s, Y)_* \to 0.$$

Here the map ∂ is induced by $\mathbb{1} - \sqcup_s i_s \colon \sqcup_s X_s \longrightarrow \sqcup_s X_s$. By construction each map i_s induces zero maps $\mathcal{T}(P, X_s) \xrightarrow{(i_s)_*} \mathcal{T}(P, X_{s+1})$ for all $P \in \mathcal{P}$, so $(\mathbb{1} - \sqcup_s i_s)_* \colon \mathcal{T}(P, \sqcup X_s) \longrightarrow \mathcal{T}(P, \sqcup X_s)$ is the identity for all P. This means that the map ∂ is an isomorphism by Lemma 2.5.2. Thus

$$\varprojlim \mathcal{T}(X_s, Y)_* = 0 = \mathrm{R}\varprojlim \mathcal{T}(X_s, Y)_*. \qquad \square$$

Remark 4.5.5. One can show that the converse also holds, that if the projective class does not generate, then the spectral sequence is not conditionally convergent.

Proposition 4.5.6. *Let X be an object with length n. Then the spectral sequence derived from $\mathcal{T}(X, Y)$ collapses at E_{n+1} with $E_{n+1} = E_\infty$.*

Proof. Let X be in $\mathcal{P}_n$, with an Adams resolution as in Definition 4.2.1. Since each X_{s+1} is a cofiber of the previous X_s, we see inductively that each X_s is also in $\mathcal{P}_n$. This means that all n-fold composites $i_{s+n} \cdots i_s$ are zero. This again means that the differential d_r is zero for all $r > n$. Hence the spectral sequence collapses at E_{n+1}. □

Proposition 4.5.7. *If the projective class generates, and X is an object with finite length, then the spectral sequence derived from $\mathcal{T}(X, Y)$ is strongly convergent.*

Proof. The Adams resolutions are defined only for $s \geq 0$, so by definition the Adams spectral sequences we can form with respect to a projective class satisfy $E_s = 0$ for $s < 0$. Then by Proposition 4.5.6 and Proposition 4.5.4 along with Theorem 4.5.3 the spectral sequence is strongly convergent. □

Example 4.5.8. We have already seen that the ghost projective class generates. We have also seen that all finite CW-spectra have finite length. Hence for all finite CW-spectra X the spectral sequence derived from $\mathcal{T}(X, Y)$ with respect to the ghost projective class is strongly convergent.

Remark 4.5.9. All of the results above hold, dually, for an injective class.

Example 4.5.10. As we saw in Example 2.5.8, the injective class of Eilenberg–Mac Lane spectra does not generate. Hence we can not expect Adams spectral sequences with respect to this injective class to be strongly convergent in general.

Chapter 5

The differential

When working with spectral sequences, the difficult part is often to compute the differentials. New and more efficient ways of computations are sought after. In this chapter we present results about how the second differential in a general Adams spectral sequence can be found using Toda brackets.

5.1 The differential $\mathbf{d_1}$

Consider an Adams resolution with respect to a projective class, as in Definition 4.2.1. This leads to a spectral sequence in which $d_1^{s,t}$ is a map $\mathcal{T}(P_s, \Sigma^{s-t}Y) \longrightarrow \mathcal{T}(\Sigma^{-1}P_{s+1}, \Sigma^{s-t}Y)$. Then by the Yoneda lemma this is in one-to-one correspondence with a map $\Sigma^{-1}P_{s+1} \longrightarrow P_s$. By definition $d_1^{s,t} = (\Sigma^{-1}\delta_s\Sigma^{-1}p_{s+1})^*$, and so we see that the differential corresponds to the map $\Sigma^{-1}\delta_s\Sigma^{-1}p_{s+1} \colon \Sigma^{-1}P_{s+1} \longrightarrow P_s$. We will therefore use the notation d_1 for both of these maps.

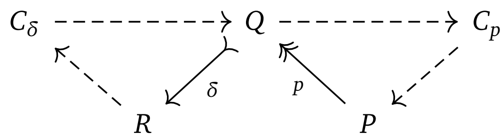
The following proposition is found as a dual version in [6, Proposition 2.17].

Proposition 5.1.1. *Let $\theta \colon P \longrightarrow R$ be a map between projectives. The map θ appears as d_1 in some Adams resolution if and only if it admits a factorization into a $\mathcal{P}$-epic followed by a $\mathcal{P}$-monic.*

Proof. That a differential d_1 in any Adams spectral sequence has such a factorization holds by construction. To see that the condition is sufficient, suppose we have such a factorization

$$P \xrightarrow{p} Q \xrightarrow{\delta} R$$

of θ. Then we can extend the factorization with the cofibers of p and δ and obtain the following diagram



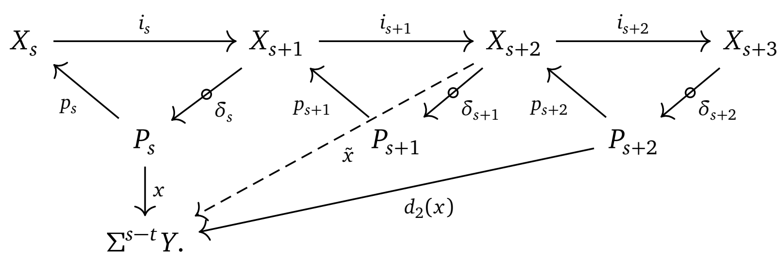
which we again can extend to the right to make an Adams resolution of C_δ, with $d_1 = \delta p$. □

5.2 The differential $\mathbf{d_2}$

We now move on to the differential d_2. We will prove dual results to [6, Proposition 4.1], which applies to Adams spectral sequences with respect to an injective class.

Consider an Adams spectral sequence with respect to a projective class $\mathcal{P}$, and recall the notation in (4.1). Let $[x]$ be a class in $E_2^{s,t}$, represented by a morphism $x \in \mathcal{T}(P_s, \Sigma^{s-t}Y)$. We want to describe the differential d_2 when applied to $[x]$. Let $d_2[x]$ denote the set of all representatives of $d_2([x])$. Note that $d_2([x]) \in E_2^{s+2,t+1}$, so $d_2[x] \subseteq E_1^{s+2,t+1} = \mathcal{T}(\Sigma^{-1}P_{s+2}, \Sigma^{s-t}Y)$.

We display x along with the projective resolution of X, and a representative $d_2(x)$ for $d_2([x])$



Recall that when defining the second differential of a spectral sequence associated to an exact couple, we started by defining two new morphisms:

p' by $[i(y)] \mapsto [p(y)]$ and δ' by $[e] \mapsto [\delta(e)]$. Then for our class $[x]$, $d_2([x]) = p'\delta'([x]) = p'[(x\Sigma^{-1}\delta_s)]$. The map x is a cycle, i.e., it satisfies $x\Sigma^{-1}\delta_s\Sigma^{-1}p_{s+1} = 0$, so we can choose a lift $\tilde{x}\colon \Sigma^{-1}X_{s+2} \longrightarrow \Sigma^{s-t}Y$ of $x\Sigma^{-1}\delta_s$ to the cofiber of $\Sigma^{-1}p_{s+1}$. This follows by the exactness of

$$\mathcal{T}(X_{s+2}, \Sigma^{s-t}Y) \xrightarrow{i^*_{s+1}} \mathcal{T}(X_{s+1}, \Sigma^{s-t}Y) \xrightarrow{p^*_{s+1}} \mathcal{T}(P_{s+1}, \Sigma^{s-t}Y).$$

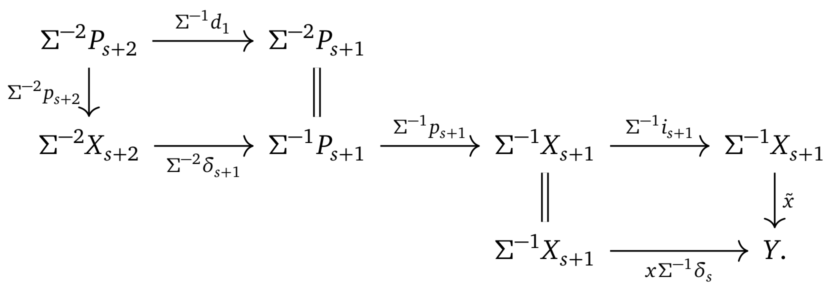
This means precisely that $d_2([x])$ is given by the class $[\tilde{x}\Sigma^{-1}p_{s+2}]$, since $i(\tilde{x}) = (\Sigma^{-1}i_s)^*(\tilde{x}) = x\Sigma^{-1}\delta_s$, i.e., $p'[(x\Sigma^{-1}\delta_s)] = [p(\tilde{x})] = [\tilde{x}\Sigma^{-1}p_{s+2}]$.

Proposition 5.2.1. *Let $d_2[x] \subseteq E_1^{s+2,t+1}$ be the subset of all representatives of $d_2([x]) \in E_2^{s+2,t+1}$. Then we have*

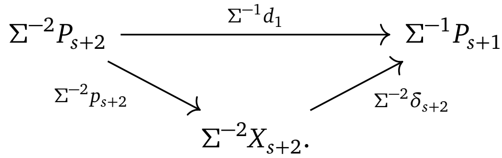
$$d_2[x] = \langle x\Sigma^{-1}\delta_s, \Sigma^{-1}p_{s+1}, \Sigma^{-1}d_1\rangle.$$

Proof. Since t plays no role in the statement we can assume, without loss of generality, that $t = s$.

We have seen that an element of $d_2[x]$ is found by choosing a lift $\tilde{x}$, such that it makes the following diagram commute



The morphism $\Sigma^{-2}p_{s+2}$ plays no role here in finding the different maps in the Toda bracket, as it is unique in the factorization



This is because the morphism δ_{s+1} is $\mathcal{P}$-monic and P_{s+2} is projective, which means that $(\Sigma^{-2}\delta_{s+2})_*\colon \mathcal{T}(\Sigma^{-2}P_{s+2}, \Sigma^{-2}X_{s+2}) \longrightarrow$

$\mathcal{T}(\Sigma^{-2}P_{+2}, \Sigma^{-1}P_{s+1})$ is injective. Thus, varying over all the possible lifts $\tilde{x}$, we get that

$$d_2[x] = \langle x\Sigma^{-1}\delta_s, \Sigma^{-1}p_{s+1}, \Sigma^{-1}d_1\rangle. \qquad \square$$

Theorem 5.2.2. *As sets we have* $d_2[x] = \langle x, d_1, \Sigma^{-1}d_1\rangle_{fc}$, *where the morphism* $\Sigma\alpha$ *in* (3.2) *is fixed as the composition* $\alpha = \tilde{\alpha}\Sigma^{-2}p_{s+2}$, *and* $\tilde{\alpha}$ *is obtained from the octahedron axiom, applied to the composition*

$$d_1 = \Sigma^{-1}\delta_s\Sigma^{-1}p_{s+1}.$$

Proof. The morphism $\tilde{\alpha}$ is obtained through the diagram below

$$\begin{array}{ccccccc}
 & & \Sigma^{-1}X_s & = & \Sigma^{-1}X_s & & \\
 & & {\scriptstyle \Sigma^{-1}i_s}\downarrow & & \downarrow & & \\
\Sigma^{-1}P_{s+1} & \xrightarrow{\Sigma^{-1}p_{s+1}} & \Sigma^{-1}X_{s+1} & \xrightarrow{\Sigma^{-1}i_{s+1}} & \Sigma^{-1}X_{s+2} & \xrightarrow{\Sigma^{-1}\delta_{s+1}} & P_{s+1} \\
\| & & {\scriptstyle \Sigma^{-1}\delta_s}\downarrow & & \downarrow{\scriptstyle \Sigma\tilde{\alpha}} & & \| \\
\Sigma^{-1}P_{s+1} & \xrightarrow{d_1} & P_s & \xrightarrow{i} & C_{d_1} & \xrightarrow{q} & P_{s+1} \\
{\scriptstyle \Sigma^{-1}p_{s+1}}\downarrow & & \| & & \downarrow{\scriptstyle \tilde{\beta}} & & \downarrow{\scriptstyle p_{s+1}} \\
\Sigma^{-1}X_{s+1} & \xrightarrow{\Sigma^{-1}\delta_s} & P_s & \xrightarrow{p_s} & X_s & \xrightarrow{i_s} & X_{s+1}.
\end{array}$$

On the other hand, the morphisms in the bracket where α is fixed consists of $\beta\Sigma\alpha$ as in the diagram below

$$\begin{array}{ccccccc}
\Sigma^{-2}P_{s+2} & \xrightarrow{\Sigma^{-1}d_1} & \Sigma^{-1}P_{s+1} & & & & \\
{\scriptstyle \alpha}\downarrow & & \| & & & & \\
\Sigma^{-1}C_{d_1} & \xrightarrow{\Sigma^{-1}q} & \Sigma^{-1}P_{s+1} & \xrightarrow{d_1} & P_s & \xrightarrow{i} & C_{d_1} \\
 & & & & \| & & \downarrow{\scriptstyle \beta} \\
 & & & & P_s & \xrightarrow{x} & Y.
\end{array} \qquad (5.1)$$

Denote this restricted bracket by $\langle x, d_1, \Sigma^{-1}d_1\rangle^{\alpha}_{fc}$.

We show first that $\langle x, d_1, \Sigma^{-1}d_1\rangle^{\alpha}_{fc} \subseteq d_2[x]$. Let $\beta\Sigma\alpha = \beta\Sigma\tilde{\alpha}\Sigma^{-1}p_{s+1}$ be in the restricted bracket. Then $\beta\Sigma\tilde{\alpha}$ is a valid choice for the lift of $\tilde{x}$ since

$$\beta\Sigma\tilde{\alpha}\Sigma^{-1}i_{s+1} = \beta i\Sigma^{-1}\delta_s = x\Sigma^{-1}\delta_s,$$

so $\beta\Sigma\alpha \in d_2[x]$.

Now we show that $d_2[x] \subseteq \langle x, d_1, \Sigma^{-1}d_1\rangle^{\alpha}_{fc}$. Let $\tilde{x}\Sigma^{-1}p_{s+1}$ be in $d_2[x]$. We want to show that $\tilde{x}$ factors as

$$\Sigma^{-1}P_{s+1} \xrightarrow{\Sigma\tilde{\alpha}} C_{d_1} \xrightarrow{\beta} Y$$

for some β such that

$$\tilde{x}\Sigma^{-1}p_{s+2} = \beta\Sigma\tilde{\alpha}\Sigma^{-1}p_{s+2} = \beta\Sigma\alpha.$$

By construction the morphism $\Sigma^{-1}i_{s+1}\Sigma^{-1}i_s \colon \Sigma^{-1}X_s \longrightarrow \Sigma^{-1}X_{s+2}$ is a fiber of $\Sigma\tilde{\alpha}$. Furthermore $\tilde{x}\Sigma^{-1}i_{s+1}\Sigma^{-1}i_s = x\Sigma^{-1}\delta_s\Sigma^{-1}i_s = 0$. This means that $\tilde{x} \in \mathrm{Im}(\Sigma\alpha)^*$, by the exactness of the sequence

$$\mathcal{T}(C_{d_1}, Y) \xrightarrow{(\Sigma\alpha)^*} \mathcal{T}(\Sigma^{-1}X_{s+1}, Y) \xrightarrow{(\Sigma^{-1}i_{s+1}\Sigma^{-1}i_s)^*} \mathcal{T}(\Sigma^{-1}X_s, Y).$$

This means that there is a $\beta \colon C_{d_1} \longrightarrow Y$ such that $\beta\Sigma\alpha = \tilde{x}$. However, this map β might not satisfy $\beta i = x$ in (5.1), but we correct it so that it does.

The two morphisms coincide under precomposition with $\Sigma^{-1}\delta_s$, since

$$x\Sigma^{-1}\delta_s = \tilde{x}\Sigma^{-1}i_{s+1} = \beta\Sigma\alpha\Sigma^{-1}i_{s+1} = \beta i\Sigma^{-1}\delta_s.$$

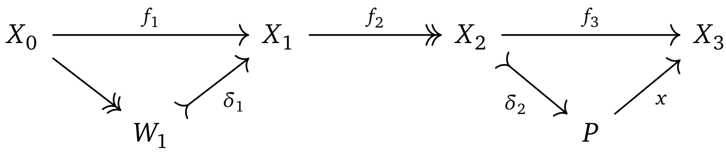
The morphism p_s is a cofiber of $\Sigma^{-1}\delta_s$, so $x - \beta i = \theta p_s$ for some morphism $\theta \colon X_s \longrightarrow Y$. Then we let the correction be

$$\beta' = \beta + \theta\tilde{\beta}.$$

This satisfies $\beta' i = \beta i + \theta\tilde{\beta} i = \beta i + \theta p_s = \beta i + (x - \beta i) = x$. And since $\tilde{\beta}\Sigma\tilde{\alpha} = 0$ it also satisfies $\beta'\Sigma\tilde{\alpha} = (\beta + \theta\tilde{\beta})\Sigma\tilde{\alpha} = \beta\Sigma\tilde{\alpha} = \tilde{x}$. So we have found our desired factorization of $\tilde{x}$. □

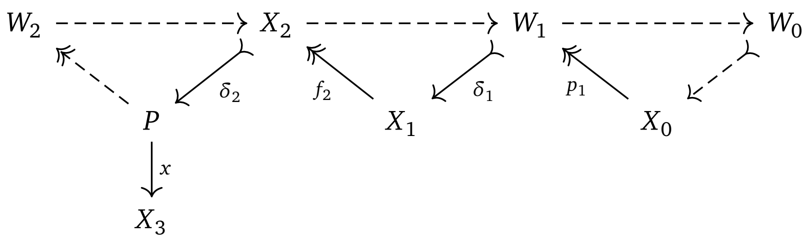
If we want to use the Toda bracket as a tool for computing differentials in an Adams spectral sequence, a natural question to ask is when does a Toda bracket arise as the set $d_2[x]$ for some Adams resolution. Inspired by Proposition 5.2.1, we have the following proposition.

Proposition 5.2.3. *Let $\langle f_3, f_2, f_1\rangle$ be a Toda bracket. Assume f_1 and f_2 are maps from projective objects. Assume also that f_1 factors through a* *projective as a $\mathcal{P}$-epic followed by a $\mathcal{P}$-monic, f_2 is $\mathcal{P}$-epic and f_3 factors as a $\mathcal{P}$-epic followed by some map. This is illustrated in the following diagram:*



Then there is an Adams resolution such that $d_2[x]$ in the induced spectral sequence equals $\langle f_3, f_2, f_1\rangle$ as sets. The converse also holds.

Proof. We complete the triangles in the natural way to make an Adams resolution:



From here we can extend the Adams resolution in both directions, under the assumption that there are enough projectives. That $d_2[x] = \langle f_3, f_3, f_1\rangle$ holds by the assumptions along with Proposition 5.2.1.

The converse holds by construction. □

Example 5.2.4. The Toda bracket $\langle 2, \eta, 2\rangle$ from Example 3.2.4 does not appear as $d_2[x]$ in any Adams spectral sequence with respect to the ghost projective class $\mathcal{S}$. Assume that 2 admits a factorization into a $\mathcal{S}$-epic followed by a $\mathcal{S}$-monic:

$$\mathbb{S} \xrightarrow{p} X \xrightarrow{\delta} \mathbb{S}.$$

Recall that the mod 2 Moore spectrum M is the cofiber of the map $\mathbb{S} \xrightarrow{2} \mathbb{S}$. The octahedron axiom applied to the factorization yields the following diagram

$$\begin{array}{ccccccc}
\mathbb{S} & \overset{p}{\twoheadrightarrow} & X & \xrightarrow{i} & C_p & \xrightarrow{q} & \mathbb{S}^1 \\
\| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \alpha} & & \| \\
\mathbb{S} & \xrightarrow{2} & \mathbb{S} & \xrightarrow{i'} & M & \xrightarrow{q'} & \mathbb{S}^1 \\
\downarrow{\scriptstyle p} & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow \\
X & \overset{\delta}{\rightarrowtail} & \mathbb{S} & \overset{p'}{\twoheadrightarrow} & C_\delta & \longrightarrow & \Sigma X
\end{array}$$

with distinguished rows. Looking at the long exact sequences in homotopy of the two top rows

$$\begin{array}{ccccccc}
\pi_n(X) & \xrightarrow{0} & \pi_n(C_p) & \xrightarrow{q_*} & \pi_{n-1}(\mathbb{S}) & \xrightarrow{p_*} & \pi_{n-1}(X) \\
{\scriptstyle \delta_*}\downarrow & & {\scriptstyle \alpha_*}\downarrow & & \| & & {\scriptstyle \delta_*}\downarrow \\
\pi_n(\mathbb{S}) & \xrightarrow{i'_*} & \pi_n(M) & \xrightarrow{q'_*} & \pi_{n-1}(\mathbb{S}) & \xrightarrow{2} & \pi_{n-1}(\mathbb{S})
\end{array}$$

we see that q_* is injective, so

$$\pi_n(C_p) \cong \operatorname{Im} q_* = \ker p_* = \ker(\delta p)_* = \ker(2) = {}_2\pi_{n-1}(\mathbb{S}).$$

Furthermore, the map $q_*\colon \pi_n(C_p) \longrightarrow \pi_{n-1}(\mathbb{S})$ corresponds to the inclusion ${}_2\pi_{n-1}(\mathbb{S}) \longrightarrow \pi_{n-1}(\mathbb{S})$. Similarly, $\pi_n(C_\delta) \cong \pi_n(\mathbb{S})/2$, and the map $p'_*\colon \pi(\mathbb{S}) \longrightarrow \pi_n(C_\delta)$ corresponds to the quotient map $\pi_n(\mathbb{S}) \longrightarrow \pi_n(\mathbb{S})/2$.

The long exact sequence in homotopy of the Moore cofiber sequence can be split into short exact sequences

$$0 \longrightarrow \pi_n(\mathbb{S})/2 \xrightarrow{i'_*} \pi_n(M) \xrightarrow{q'_*} {}_2\pi_{n-1}(\mathbb{S}) \longrightarrow 0.$$

The map α_* satisfies $q'_*\alpha_* = q_*$, which becomes the identity when considered as a map onto the subgroup ${}_2\pi_{n-1}(\mathbb{S}) \subseteq \pi_{n-1}(\mathbb{S})$. Therefore the short exact sequences split for all n, contradicting the fact that $\pi_2(M) = \mathbb{Z}/4$. This means there can be no such factorization of 2.

Recall the n-fold Toda brackets introduced in Chapter 3. We can use the higher Toda brackets to compute higher differentials, as shown in [6]. In the following theorem we use the notation obtained in Section 4.3.2.

Theorem 5.2.5. *In an Adams spectral sequence with respect to an injective class, we have that*

$$d_r[x] = \langle \Sigma^{r-1} d_1, \cdots, \Sigma^2 d_1, \Sigma d_1, \Sigma p_{s+1}, \delta_s x \rangle$$

where $d_r[x]$ *denotes the subset of* $E_1^{s+r,t+r-1}$ *with all representatives of* $d_r([x])$.

For a proof see [6, Chapter 6].

Proposition 5.2.1, Theorem 5.2.2 and Theorem 5.2.5 look very powerful at first sight, and it is an unexpected connection between two seemingly unrelated topics. However, as illustrated by Example 5.2.4, we can not expect all the known Toda brackets to immediately provide new and exciting spectral sequences. The requirements to the Adams resolutions are rather strict, often not resulting in particularly favorable situations for computing differentials with the Toda brackets. Additionally, computing a Toda bracket can be quite complex, as shown by Example 3.2.4, which raises questions about the efficiency of this method compared to existing methods for computing differentials.

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Appendix A

Some computations of the Moore spectrum

The following lemmas are a necessary part of computing the Toda bracket $\langle 2, \eta, 2\rangle$. The results and proofs are from [2, Chapter 4.6].

Let $\mathbb{S}$ be the sphere spectrum. Recall that the mod 2 Moore spectrum M is the cofiber of $\mathbb{S} \xrightarrow{2} \mathbb{S}$, such that the triangle

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{\text{incl}} M \xrightarrow{\text{pinch}} \mathbb{S}^1 \tag{A.1}$$

is distinguished.

Lemma A.1. *$M \wedge M \ncong \Sigma M \vee M$.*

Proof. To prove this we show that although $H^*(M \wedge M; \mathbb{Z}/2) \cong H^*(M \vee \Sigma M; \mathbb{Z}/2)$ as $\mathbb{Z}/2$-modules, we have $H^*(M \wedge M; \mathbb{Z}/2) \ncong H^*(M \vee \Sigma M; \mathbb{Z}/2)$ as $\mathcal{A}$-modules, where $\mathcal{A}$ denotes the mod 2 Steenrod algebra.

The long exact sequence of cohomology associated to (A.1) is

$$\cdots \longrightarrow H^i(\mathbb{S}) \longrightarrow H^i(M) \longrightarrow H^i(\mathbb{S}) \longrightarrow H^{i+1}(\mathbb{S}) \longrightarrow \cdots$$

so we have

$$\tilde{H}^i(M) = \begin{cases} \mathbb{Z}/2 & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Let x_i denote the generator of $\tilde{H}^i(M)$. To determine the $\mathcal{A}$-module structure of $\tilde{H}^*(M)$ we need only check what happens to $\mathrm{Sq}^1(x_0)$, as all other degrees are trivial.

By the Steenrod axioms we have that Sq^1 is the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{i} \mathbb{Z}/4 \xrightarrow{p} \mathbb{Z}/2 \longrightarrow 0.$$

We can apply the cohomology of M to this, and obtain the long exact sequence

$$\cdots \to \tilde{H}^0(M;\mathbb{Z}/4) \xrightarrow{p^*} \tilde{H}^0(M;\mathbb{Z}/2) \xrightarrow{\beta} \tilde{H}^1(M;\mathbb{Z}/2) \xrightarrow{i^*} \tilde{H}^1(M;\mathbb{Z}/4) \to \cdots . \tag{A.2}$$

Using the Universal Coefficient Theorem we obtain that $\tilde{H}^0(M;\mathbb{Z}/4) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/4) \cong \mathbb{Z}/2$ and $\tilde{H}^1(M;\mathbb{Z}/4) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/4) = 0$, so (A.2) simplifies as

$$\cdots \longrightarrow \mathbb{Z}/2 \xrightarrow{p^*} \mathbb{Z}/2 \xrightarrow{\beta} \mathbb{Z}/2 \xrightarrow{i^*} 0.$$

We saw before that $\tilde{H}^0(M;\mathbb{Z}/4) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/4)$, from which we deduce that the map i is actually the generator of $\tilde{H}^0(M;\mathbb{Z}/4)$. Thus, $p^* = 0$, since $p \circ i = 0$, and we get that $\mathrm{Sq}^1 = \beta : \tilde{H}^0(M;\mathbb{Z}/2) \longrightarrow \tilde{H}^1(M;\mathbb{Z}/2)$ is an isomorphism. So $\mathrm{Sq}^1(x_0) = x_1$.

Knowing the cohomology of M we move on to $M \wedge M$ and $M \vee \Sigma M$. We see that $\tilde{H}^*(M \vee \Sigma M;\mathbb{Z}/2)$ is generated by elements x_0, x_1, y_1 and y_2 in degrees $0, 1, 1$ and 2 respectively. Furthermore, we have $\mathrm{Sq}^1(x_0) = x_1$ and $\mathrm{Sq}^1(y_1) = y_2$. The Steenrod action is trivial in all other cases.

For $M \wedge M$ we have the Künneth formula:

$$\tilde{H}^*(M \wedge M;\mathbb{Z}/2) \cong \tilde{H}^*(M;\mathbb{Z}/2) \otimes \tilde{H}^*(M;\mathbb{Z}/2).$$

This is then generated by $\{x_0 \otimes x_0, x_0 \otimes x_1, x_1 \otimes x_0, x_1 \otimes x_1\}$. So as $\mathbb{Z}/2$-modules, $\tilde{H}^*(M \vee \Sigma M;\mathbb{Z}/2)$ and $\tilde{H}^*(M \wedge M;\mathbb{Z}/2)$ are isomorphic. They are however, not isomorphic as $\mathcal{A}$-modules. The Cartan formula tells us the following:

- $\mathrm{Sq}^1(x_0 \otimes x_0) = x_0 \otimes x_1 + x_1 \otimes x_0$
- $\mathrm{Sq}^1(x_0 \otimes x_1) = x_1 \otimes x_1$
- $\mathrm{Sq}^2(x_0 \otimes x_0) = x_1 \otimes x_1$

As Sq^2 is trivial on $\tilde{H}^*(M \vee \Sigma M; \mathbb{Z}/2)$, we can conclude that $M \vee \Sigma M \not\cong M \wedge M$. □

Lemma A.2. *For the mod 2 Moore spectrum M we have $[M, M] = \mathbb{Z}/4\{\mathrm{Id}_M\}$, in which $2\,\mathrm{Id}_M = \mathrm{incl} \circ \eta \circ \mathrm{pinch} \neq 0$.*

Proof. For this proof we use the long exact Puppe sequence repeatedly. From (A.1) we get the following sequence of homotopy groups

$$\cdots \to \pi_i(\mathbb{S}) \xrightarrow{2_*} \pi_i(\mathbb{S}) \xrightarrow{\mathrm{incl}_*} \pi_i(M) \xrightarrow{\mathrm{pinch}_*} \pi_{i-1}(\mathbb{S}) \xrightarrow{2_*} \pi_{i-1}(\mathbb{S}) \xrightarrow{\mathrm{incl}_*} \cdots$$

which we can split into two short exact sequences at $\pi_1(M)$ and $\pi_0(M)$. Since $\mathrm{coker}(2_*) = \pi_i(\mathbb{S})/2$ and $\mathrm{Im}(\mathrm{pinch}_*) = \ker(2_*)$ we get the short exact sequences

$$0 \longrightarrow \pi_0(\mathbb{S})/2 \xrightarrow{\mathrm{incl}_*} \pi_0(M) \xrightarrow{\mathrm{pinch}_*} {}_2\pi_{-1}(\mathbb{S}) \longrightarrow 0$$

$$0 \longrightarrow \pi_1(\mathbb{S})/2 \xrightarrow{\mathrm{incl}_*} \pi_1(M) \xrightarrow{\mathrm{pinch}_*} {}_2\pi_0(\mathbb{S}) \longrightarrow 0$$

where we have let ${}_2\pi_i(\mathbb{S}) := \{x \in \pi_i(\mathbb{S}) : 2x = 0\}$. Since $\pi_{-1}(\mathbb{S}) = {}_2\pi_0(\mathbb{S}) = 0$, and $\pi_0(\mathbb{S}) = \mathbb{Z}\{\mathrm{Id}\}$, $\pi_1(\mathbb{S}) = \mathbb{Z}/2\{\eta\}$, we get that

$$\pi_0(M) = \mathbb{Z}/2\{\mathrm{incl}\},$$
$$\pi_1(M) = \mathbb{Z}/2\{\mathrm{incl} \circ \eta\}.$$

Next, we can apply $[-, M]$ to (A.1) to obtain the following long exact sequence

$$\cdots \to [\mathbb{S}^1, M] \xrightarrow{2^*} [\mathbb{S}^1, M] \xrightarrow{\mathrm{pinch}^*} [M, M] \xrightarrow{\mathrm{incl}^*} [\mathbb{S}, M] \xrightarrow{2^*} [\mathbb{S}, M] \to \cdots$$

which we again can split into a short exact sequence

$$0 \longrightarrow \pi_1(M)/2 \xrightarrow{\mathrm{pinch}^*} [M, M] \xrightarrow{\mathrm{incl}^*} {}_2\pi_0(M) \longrightarrow 0$$

to obtain two possibilities for $[M, M]$. Either $[M, M] = \mathbb{Z}/2\{\mathrm{Id}_M\} \oplus \mathbb{Z}/2\{\mathrm{incl} \circ \eta\}$ or $[M, M] = \mathbb{Z}/4\{\mathrm{Id}_M\}$, as these are the only groups that fit into the sequence. The generators follow from the generators of $\pi_0(M)$ and $\pi_1(M)$. Note also that ${}_2\pi_0(M) = \pi_0(M) = \mathbb{Z}/2$.

If $[M, M] = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, then we necessarily have $2\,\mathrm{Id}_M = 0$. In this case, we could apply $M \wedge -$ to (A.1), to obtain the triangle

$$0 \longrightarrow M \xrightarrow{2\,\mathrm{Id}_M} M \longrightarrow M \wedge M \longrightarrow \Sigma M \longrightarrow 0$$

which splits if $2\,\mathrm{Id}_M = 0$. So if $[M, M] = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, then $M \wedge M \cong \Sigma M \vee M$, contradicting Lemma A.1. Consequently, $[M, M] = \mathbb{Z}/4\{\mathrm{Id}_M\}$, with $2\,\mathrm{Id}_M = \mathrm{incl} \circ \eta \circ \mathrm{pinch}$. $\square$

NTNU

Norwegian University of
Science and Technology