Taylor & Francis
Taylor & Francis Group

Optimization

A Journal of Mathematical Programming and Operations Research

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/gopt20

Convexificators for nonconvex multiobjective optimization problems with uncertain data: robust optimality and duality

J. W. Chen, R. Yang, E. Köbis & X. Ou

To cite this article: J. W. Chen, R. Yang, E. Köbis & X. Ou (14 Dec 2023): Convexificators for nonconvex multiobjective optimization problems with uncertain data: robust optimality and duality, Optimization, DOI: 10.1080/02331934.2023.2293061

To link to this article: https://doi.org/10.1080/02331934.2023.2293061

Published online: 14 Dec 2023.

OPTIMIZATION
https://doi.org/10.1080/02331934.2023.2293061

Convexificators for nonconvex multiobjective optimization problems with uncertain data: robust optimality and duality

J. W. Chen[a], R. Yang[a], E. Köbis[b] and X. Ou[a,c]

[a]School of Mathematics and Statistics, Southwest University, Chongqing, People's Republic of China; [b]Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), Norway; [c]College of Management, Chongqing College of Humanities, Science & Technology, Chongqing, People's Republic of China

ABSTRACT

In this paper, we investigate robust optimality conditions and duality for a class of nonconvex multiobjective optimization problems with uncertain data in the worst case by the upper semi-regular convexificator. The Fermat principle for a locally Lipschitz function is presented in terms of the upper semi-regular convexificator. We establish robust necessary optimality conditions of the Fritz-John type and KKT type for the uncertain nonconvex multiobjective optimization problems. In addition, robust sufficient optimality conditions as well as saddle point conditions are derived under the generalized $\hat{\partial}^*$-pseudoquasiconvexity and generalized convexity, respectively. The robust duality relations between the original problem and its mixed robust dual problem are obtained under a generalized pseudoconvexity assumption.

ARTICLE HISTORY
Received 13 June 2023
Accepted 1 December 2023

KEYWORDS
Multiobjective optimization; robust optimality condition; robust duality; saddle point; upper semi-regular convexificator

MATHEMATICS SUBJECT CLASSIFICATIONS
90C29; 49J53

1. Introduction

It is well-known that multiobjective optimization problems, which arise from economics, optimal control, machine learning, engineering and game theory, are very important models in operations research; see [1–4]. In multiobjective optimization, one assumes that there are multiple conflicting objectives that have to be solved simultaneously. Generally, there does not exist a point optimizing all objective functions simultaneously, and one has to find a whole set of points for which no objective can be optimized without worsening another objective. However, such a set can be difficult to find. Therefore, tailored to the different applications and needs, various optimality notions, such as Pareto efficient solution, weak efficient solution and proper efficient solution, were introduced for multiobjective optimization. Theory and applications of multiobjective optimization have made a great development in the past 20 years in terms

CONTACT E. Köbis elisabeth.kobis@ntnu.no

of optimality conditions, duality, penalization, robustness, stability, constraint qualifications, numerical algorithms and applications in machine learning, engineering and economy; see, e.g. [5–9] and the references therein. However, the most achievements on multiobjective optimization problems were established under the condition of a lack of uncertainty. In fact, uncertainty is everywhere, for example because of prediction errors, measurement errors and the lack of complete information. In most real world applications, the coefficient parameters in optimization problems are not known exactly, and solutions to optimization problems can exhibit remarkable sensitivity to perturbations in the parameters of the problem. Therefore, it is necessary to investigate multiobjective optimization problems under uncertainty.

Two well-known mathematical methods for dealing with uncertain problems are stochastic and robust optimization. A key assumption in stochastic optimization is that the decision maker has complete information on the distribution of the uncertainty through empirical data. However, in some circumstances, this might turn out to be difficult if not impossible when a strategic decision has to be made well in advance of the realization of the uncertainty. In particular, robust optimization is a very useful tool to study uncertain problems when the probability distribution of the uncertainty is unknown. For this, robust optimization was applied to study uncertain multiobjective optimization problems, in which the uncertain parameters are described by some deterministic set, under the assumption that finding a solution is feasible for any possible uncertain cases, so that in the worst case, feasibility can still be maintained. Robust optimization has been growing rapidly over the past two decades, see [10–17] and the references therein. Kuroiwa and Lee [18] extended the robust counterpart of single objective uncertain optimization to uncertain multiobjective optimization, and established necessary optimality conditions for robust weakly efficient solutions and properly efficient solutions of the uncertain multiobjective optimization. Ide and Köbis [19] introduced various concepts of efficiency for uncertain multiobjective optimization problems based on set order relations, analysed the resulting concepts of efficiency and presented numerical results on the occurrence of the various concepts. Goberna et al. [20] gave numerically tractable optimality conditions for minmax robust weakly efficient solutions and highly robust weakly efficient solutions of multiobjective linear programming problems with data uncertainty in both the objective function and the constraints, and derived a formula for the radius of robust feasibility guaranteeing constraint feasibility for all possible scenarios within a specified uncertainty set under affine data parametrization and the lower bounds for the radius of highly robust efficiency guaranteeing the existence of highly robust weakly efficient solutions under affine and rank-1 objective data uncertainty. Klamroth et al. [21] studied uncertain optimization problems with infinite scenario sets, and presented a unified characterization of different concepts of robust optimization and stochastic programming by the existing methods arising from vector optimization in general spaces, set optimization as well as scalarization techniques. Bokrantz and Fredriksson [22] obtained necessary and sufficient conditions for robust efficiency to multiobjective optimization problems that depend on uncertain parameters by using a scalarization method. Chuong [23] considered necessary/sufficient optimality conditions for robust (weakly) Pareto solutions of a robust nonsmooth multiobjective optimization problem in terms of multipliers and limiting subdifferentials of the related functions, and explored weak/strong duality relations between the primal one and its dual robust problem under the (strictly) generalized convexity assumptions. Optimality conditions and duality theorems of a robust nonsmooth multiobjective optimization were established in [24]. By using the well-known ϵ-constraint scalarization method and image space analysis, Chen et al. [25] obtained sufficient and necessary optimality conditions of the robust efficient solutions for convex multiobjective optimization problems with data uncertainty. Sun et al. [9] investigated robust optimality necessary and sufficient conditions of a class of uncertain multiobjective fractional semi-infinite optimization problems via robust optimization and scalarization methods, and obtained relationships of a mixed-type dual problem and a corresponding robust optimization problem. It is worth pointing out that robust sufficient optimality conditions and robust duality for uncertain multiobjective optimization problems were established under various generalized convexity notions in the sense of subdifferentials such as Clarke subdifferential, limiting subdifferential and so on; see e.g. [26].

Convexity and its generalization play an important role in establishing optimality conditions and duality theorems for optimization problems. In 1994, the notion of convexificator was introduced by Demyanov [27] as a generalization of the notion of upper convex and lower concave approximations. Demyanov regarded a convexificator as a convex and compact set. Thereafter, Jeyakumar and Luc [28] suggested that one may use a closed and nonconvex set instead of a convex and compact one to define a convexificator. Various properties of convexificators and some chain rules were presented as well as a notion of ∂-pseudoconvex function, and some optimality conditions for vector minimization problems were obtained in terms of convexificators in [29, 30]. Necessary optimality conditions of locally Lipschitz continuous optimization problem were derived under certain constraint qualification and the upper and lower convexificator in [31]. Further, necessary and sufficient optimality conditions for nonsmooth semi-infinite multiobjective programming problems were also presented by using convexificators in [32]. The sufficient optimality conditions of interval-valued programming problem and the dual relations between the original problem and its Mond-Weir type dual model and Wolfe type dual model were also established by means of convexificators in [33]. Convexificators were also applied to study optimality conditions and duality of nonsmooth minimax programming problem and bilevel multiobjective optimization problem; see e.g. [34–36]. And yet, to the best of our knowledge, *there are no papers that* *concentrate on uncertain nonconcex multiobjective optimization problems by using an upper semi-regular convexificator*.

Motivated and inspired by the above works, this paper aims to investigate a class of nonconvex multiobjective optimization problems with uncertain data (UNMOP) by using an upper semi-regular convexificator. The Fermat's lemma for a locally Lipschitz function is obtained in terms of the upper semi-regular convexificator. The robust optimality necessary conditions of the Fritz-John type and KKT type for (UNMOP) are established under certain assumptions. We also obtain the robust optimality sufficient condition including saddle point type sufficient conditions. Further, we introduce a mixed robust dual model of (UNMOP), and explore the robust duality relations between (UNMOP) and the mixed robust dual problem.

This paper is organized as follows. We recall some basic notions and well-known results in Section 2. Optimality conditions for robust weakly efficient solution of (UNMOP) are discussed under suitable conditions in Section 3. In Section 4, a mixed robust dual model of (UNMOP) is presented, and the robust weak/strong/converse robust dual results between the mixed robust dual model and (UNMOP) are derived. Finally, we give some conclusions.

The highlights of this paper are listed as follows:

- The Fermat's lemma for a locally Lipschitz function is derived in terms of the upper semi-regular convexificator.
- We firstly give the robust optimality necessary conditions or sufficient conditions for (UNMOP) in terms of the upper semi-regular convexificator.
- We present the robust duality results between (UNMOP) and its mixed robust dual problem such as robust strong duality and robust converse duality.

2. Preliminaries

Let $\mathbb{R}^m_+$ and $\mathbb{R}^m_{++}$ be respectively the nonnegative orthant and the positive orthant of m-dimensional Euclidean space $\mathbb{R}^m$ with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a vector-valued function. Set $\inf\emptyset := +\infty$ and $\sup\emptyset := -\infty$. For a subset A of $\mathbb{R}^m$, the convex hull, closed hull, and interior of A are denoted by co$A$, cl$A$ and int$A$, respectively.

The nonconvex multiobjective optimization problem with uncertain data is given as follows:

$$\begin{aligned}(\text{UNMOP})\quad & \min f(x)\\ & \text{s.t.}\quad g_j(x,w_j)\le 0,\ j=1,2,\dots,l,\end{aligned}$$

where $x\in\mathbb{R}^n$ is the decision variable, $w_j\in\Omega_j$ are uncertain parameters, Ω_j are nonempty compact convex subsets of $\mathbb{R}^{n_j}, j\in J:=\{1,2,\dots,l\}$, the objective function $f(x):=(f_1(x),\dots,f_m(x))^\top, f_i:\mathbb{R}^n\to\mathbb{R}, i\in I:=\{1,2,\dots,m\}$ are locally Lipschitz continuous, and $g_j : \mathbb{R}^n \times \Omega_j \to \mathbb{R}, j \in J$ are real-valued functions.

For the sake of brevity, set $g(x, w) := (g_1(x, w_1), \ldots, g_l(x, w_l))^\top$, where $w := (w_1, \ldots, w_l)^\top \in \Omega := \prod_{j\in J} \Omega_j$. We adopt the classical robust optimization approach to deal with (UNMOP) in the worst-case. The robust multiobjective optimization model associated with (UNMOP) is defined in the following:

$$\text{(RMP)} \quad \min f(x)$$
$$\text{s.t.} \quad g_j(x, w_j) \le 0, \quad \forall\, w_j \in \Omega_j, j \in J.$$

The feasible set of (RMP) is denoted by

$$C := \{x \in \mathbb{R}^n : g_j(x, w_j) \le 0, \quad \forall\, w_j \in \Omega_j, j \in J\}.$$

For the sake of brevity, we set $G_j(x) := \max_{w_j\in\Omega_j} g_j(x, w_j)$ and

$$\Omega_j(x) := \left\{w_j \in \Omega_j : G_j(x) = g_j(x, w_j)\right\},$$

for $j \in J$. For each $j \in J$, the function $G_j : \mathbb{R}^n \to \mathbb{R}$ and the set-valued function $\Omega_j : \mathbb{R}^n \rightrightarrows \Omega_j$ are called marginal function and active set, respectively. It is easy to see that

$$C = \{x \in \mathbb{R}^n : G_j(x) \le 0, \ \forall\, j \in J\}.$$

We next recall some basic notions and well-known results.

Definition 2.1: $\bar{x} \in C$ is called a robust weakly efficient solution of problem (UNMOP) if,

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_{++}, \quad \forall\, x \in C.$$

Definition 2.2: Let $h : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. The upper Dini directional derivative and lower Dini directional derivative of h at $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ are respectively defined by

$$h_d^+(x, v) = \limsup_{t\downarrow 0} \frac{h(x + tv) - h(x)}{t},$$

and

$$h_d^-(x, v) = \liminf_{t\downarrow 0} \frac{h(x + tv) - h(x)}{t}.$$

For a real-valued function, the upper Dini directional derivative and lower Dini directional derivative may be finite as well as infinite. It is well-known that if $h : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, then the upper Dini directional derivative and lower Dini directional derivative of h at $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ are finite and locally Lipschitz in the direction v.

Definition 2.3 ([28]): A real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ admits:

(i) an upper convexificator $\partial^* h(x)$ at $x \in \mathbb{R}^n$ if $\partial^* h(x) \subset \mathbb{R}^n$ is a closed set and

$$h_d^-(x, v) \leq \sup_{x^* \in \partial^* h(x)} \langle x^*, v \rangle, \quad \forall\, v \in \mathbb{R}^n.$$

(ii) a lower convexificator $\partial_* h(x)$ at $x \in \mathbb{R}^n$ if $\partial_* h(x) \subset \mathbb{R}^n$ is a closed set and

$$h_d^+(x, v) \geq \inf_{x^* \in \partial_* h(x)} \langle x^*, v \rangle, \quad \forall\, v \in \mathbb{R}^n.$$

(iii) a convexificator $\partial h(x)$ at $x \in \mathbb{R}^n$ if it is both upper and lower convexificator of h at x.

Definition 2.4 ([28]): A real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ admits:

(i) an upper semi-regular convexificator $\hat{\partial}^* h(x)$ at $x \in \mathbb{R}^n$ if $\hat{\partial}^* h(x) \subset \mathbb{R}^n$ is a closed set and

$$h_d^+(x, v) \leq \sup_{x^* \in \hat{\partial}^* h(x)} \langle x^*, v \rangle, \quad \forall\, v \in \mathbb{R}^n. \tag{1}$$

The set $\hat{\partial}^* h(x)$ is said to be an upper regular convexificator of h at $x \in \mathbb{R}^n$ if (1) holds with equality.

(ii) a lower semi-regular convexificator $\hat{\partial}_* h(x)$ at $x \in \mathbb{R}^n$ if $\hat{\partial}_* h(x) \subset \mathbb{R}^n$ is a closed set and

$$h_d^-(x, v) \geq \inf_{x^* \in \hat{\partial}_* h(x)} \langle x^*, v \rangle, \quad \forall\, v \in \mathbb{R}^n. \tag{2}$$

The set $\hat{\partial}_* h(x)$ is said to be a lower regular convexificator of h at $x \in \mathbb{R}^n$ if (2) holds with equality.

(iii) a regular convexificator $\hat{\partial} h(x)$ at $x \in \mathbb{R}^n$ if it is both upper regular convexificator and lower regular convexificator of h at x.

Remark 2.1: (i) It is easy to check that $h_d^+(x, \cdot)$ is the support function of the upper regular convexificator $\hat{\partial}^* h(x)$ and the regular convexificator $\hat{\partial} h(x)$; $-h_d^-(x, \cdot)$ is the support function of $-\hat{\partial}_* h(x)$. So, $h_d^-(x, \cdot)$ is not the support function of the lower regular convexificator $\hat{\partial}_* h(x)$.

(ii) According to $h_d^-(x, v) \leq h_d^+(x, v)$, an upper (lower) semi-regular convexificator is also an upper (lower) convexificator. If $h : \mathbb{R}^n \to \mathbb{R}$ has a directional derivative at x in each direction v, then $h_d^-(x, v) = h_d^+(x, v)$. The converse is still true; see [29].

(iii) An upper (lower) regular convexificator is a convexificator of h at x. Moreover, a regular convexificator is also a convexificator of h at x. The converse is not true; see [28].

(iv) If $h : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, then the Clarke subdifferential [37], the Michel-Penot subdifferential [38], the Ioffe-Morduchovich subdifferential [39] and the Treiman subdifferential [40] are convexificators of h at each $x \in \mathbb{R}^n$. Moreover, if h is regular in the Clarke sense, then the Clarke subdifferential is an upper regular convexificator and the Michel-Penot subdifferential is an upper semi-regular convexificator; see [28].

The following results give the properties of the regular convexificator and the upper (lower) semi-regular convexificator.

Lemma 2.5: (i) *If $\hat{\partial} h(x)$ is a regular convexificator of h at x, then for any $\alpha \in \mathbb{R}, \alpha \hat{\partial} h(x)$ is a regular convexificator of αh at x.*

(ii) *If $\hat{\partial}^* h(x)$ is an upper semi-regular convexificator of h at x, then for any $\alpha > 0, \alpha \hat{\partial}^* h(x)$ is an upper semi-regular convexificator of αh at x, and for any $\alpha < 0, \alpha \hat{\partial}^* h(x)$ is a lower convexificator of αh at x.*

(iii) *If $\hat{\partial}_* h(x)$ is a lower semi-regular convexificator of h at x, then for any $\alpha > 0, \alpha \hat{\partial}_* h(x)$ is a lower semi-regular convexificator of αh at x, and for any $\alpha < 0, \alpha \hat{\partial}_* h(x)$ is an upper convexificator of αh at x.*

Proof: It directly follows from Definitions 2.2 and 2.4. ■

Theorem 2.6: *Let $h_i : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function that admits an upper semi-regular convexificator $\hat{\partial}^* h_i(x_0)$ at $x_0 \in \mathbb{R}^n$ for all $i \in I$. Then $h(x) := \max\{h_1(x), \dots, h_m(x)\}$ admits an upper semi-regular convexificator which is convex and is given as*

$$\hat{\partial}^* h(x_0) := \operatorname{co}\left\{ \cup_{i \in I(x_0)} \hat{\partial}^* h_i(x_0) \right\},$$

where $I(x_0) := \{i \in I : h_i(x_0) = h(x_0)\}$ and $I := \{1, 2, \dots, m\}$.

Proof: From the definitions of h and $I(x_0)$, we deduce that $I(x_0) \neq \emptyset$, and that $h(x_0) = h_i(x_0)$ for $i \in I(x_0)$ and $h(x_0) > h_j(x_0)$ for $j \in I \setminus I(x_0)$. Then

$$h_i(x_0) - h_j(x_0) = h(x_0) - h_j(x_0) > 0, \quad \forall\, i \in I(x_0),\ j \in I \setminus I(x_0).$$

Since $h_i : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions for all $i \in I$, for any $v \in \mathbb{R}^n$, there exists a sufficiently small $\bar{t} > 0$ such that

$$h_i(x_0 + tv) > h_j(x_0 + tv), \quad \forall\, i \in I(x_0),\ j \in I \setminus I(x_0),\ \forall\, t \in (0, \bar{t}),$$

Then, we have

$$\begin{aligned}
h_d^+(x_0, v) &= \limsup_{t \searrow 0} \frac{h(x_0 + tv) - h(x_0)}{t} \\
&= \limsup_{t \searrow 0} \frac{\max\{h_1(x_0 + tv), \ldots, h_m(x_0 + tv)\} - h(x_0)}{t} \\
&= \limsup_{t \searrow 0} \frac{\max\{h_1(x_0 + tv) - h(x_0), \ldots, h_m(x_0 + tv) - h(x_0)\}}{t} \\
&= \limsup_{t \searrow 0} \frac{\max\{h_i(x_0 + tv) - h_i(x_0) : i \in I(x_0)\}}{t} \\
&= \max_{i \in I(x_0)} \limsup_{t \searrow 0} \frac{h_i(x_0 + tv) - h_i(x_0)}{t} \\
&\leq \max_{i \in I(x_0)} \sup_{x^* \in \hat{\partial}^* h_i(x_0)} \langle x^*, v \rangle \\
&\leq \sup_{z^* \in \mathrm{co}\left\{\cup_{i \in I(x_0)} \hat{\partial}^* h_i(x_0)\right\}} \langle z^*, v \rangle,
\end{aligned}$$

where the fifth equality holds because $h_i, i \in I$ are locally Lipschitz functions. Moreover, one has

$$h_d^+(x_0, v) \leq \sup_{z^* \in \mathrm{co}\left\{\cup_{i \in I(x_0)} \hat{\partial}^* h_i(x_0)\right\}} \langle z^*, v \rangle.$$

It therefore yields that

$$\hat{\partial}^* h(x_0) = \mathrm{co}\left\{\cup_{i \in I(x_0)} \hat{\partial}^* h_i(x_0)\right\}$$

is convex and an upper semi-regular convexificator of h at x_0. ■

We next present a Fermat principle in terms of the upper semi-regular convexificatior.

Lemma 2.7: *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function that admits an upper semi-regular convexificatior $\hat{\partial}^* h(x)$ at x. If h attains its minimum at x, then $\mathbf{0} \in \hat{\partial}^* h(x)$.*

Proof: Let h attain its minimum at x. Then, we have

$$h(x + tv) - h(x) \geq 0, \quad \forall\, t \in \mathbb{R}, v \in \mathbb{R}^n.$$

So, one has $h_d^-(x,v) \geq 0$ and

$$\sup_{x^* \in \hat{\partial}^* h(x)} \langle x^*, v\rangle \geq h_d^+(x,v) \geq h_d^-(x,v) \geq 0, \quad \forall\, v \in \mathbb{R}^n,$$

i.e.

$$\sup_{x^* \in \hat{\partial}^* h(x)} \langle x^*, v\rangle \geq 0, \quad \forall\, v \in \mathbb{R}^n,$$

which implies that $\mathbf{0} \in \hat{\partial}^* h(x)$. ■

We next introduce some generalized convex functions by the upper semi-regular convexificators.

Definition 2.8: A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be $\hat{\partial}^*$-pseudoconvex at $x^* \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$,

$$h(x) < h(x^*) \Rightarrow \langle \xi, x - x^*\rangle < 0, \quad \forall\, \xi \in \hat{\partial}^* h(x^*),$$

or equivalently,

$$\langle \xi, x - x^*\rangle \geq 0 \Rightarrow h(x) \geq h(x^*), \quad \forall\, \xi \in \hat{\partial}^* h(x^*).$$

Definition 2.9: (f, g) is said to be generalized $\hat{\partial}^*$-pseudoquasiconvex at $\bar{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$, $\xi_i \in \hat{\partial}^* f_i(\bar{x}), i \in I$, and $\zeta_j \in \hat{\partial}^*_{\bar{x}} g_j(\bar{x}, \bar{w}_j), \bar{w}_j \in \Omega_j(\bar{x}), j \in J$, there exists $d \in \mathbb{R}^n$ such that

$$f_i(x) < f_i(\bar{x}) \Rightarrow \langle \xi_i, d\rangle < 0, \quad \forall\, i \in I,$$
$$g_j(x, \bar{w}_j) \leq g_j(\bar{x}, \bar{w}_j) \Rightarrow \langle \zeta_j, d\rangle \leq 0, \quad \forall\, \bar{w}_j \in \Omega_j(\bar{x}), j \in J,$$

where $\hat{\partial}^*_{\bar{x}} g_j(\bar{x}, \bar{w}_j)$ is the upper semi-regular convexificator of $g_j(\cdot, \bar{w}_j)$ at $\bar{x}$.

Remark 2.2: If the upper semi-regular convexificator is replaced by the Clarke subdifferential in Definition 2.9, then the generalized $\hat{\partial}^*$-pseudoquasiconvexity is reduced to the usual generalized $\hat{\partial}^*$-pseudoquasiconvexity, i.e. (f, g) is said to be generalized pseudoquasiconvex at $\bar{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$, $\xi_i \in \partial f_i(\bar{x}), i \in I$, and $\zeta_j \in \partial_{\bar{x}} g_j(\bar{x}, \bar{w}_j), \bar{w}_j \in \Omega_j(\bar{x}), j \in J$, there exists $d \in \mathbb{R}^n$ such that

$$f_i(x) < f_i(\bar{x}) \Rightarrow \langle \xi_i, d\rangle < 0, \quad \forall\, i \in I,$$
$$g_j(x, \bar{w}_j) \leq g_j(\bar{x}, \bar{w}_j) \Rightarrow \langle \zeta_j, d\rangle \leq 0, \quad \forall\, \bar{w}_j \in \Omega_j(\bar{x}), j \in J,$$

where $\partial f_i(\bar{x})$ and $\partial_{\bar{x}} g_j(\bar{x}, \bar{w}_j)$ are respectively the Clarke subdifferentials of f_i and $g_j(\cdot, \bar{w}_j)$ at $\bar{x}$ for $i \in I, j \in J$.

3. Robust optimality conditions

In this section, we study the necessary and sufficient conditions of robust weakly efficient solution of (UNMOP) in terms of upper semi-regular convexificators without boundedness.

We first present the Fritz-John type robust optimality necessary conditions of (UNMOP) under some mild conditions.

Theorem 3.1: *Let $\bar{x} \in C$ and let $f_i, i \in I, g_j(\cdot, w_j), j \in J$ be locally Lipschitz continuous and admit respectively upper semi-regular convexificators $\hat{\partial}^* f_i(\bar{x})$ and $\hat{\partial}^*_x g_j(\bar{x}, w_j)$ for all $w_j \in \Omega_j, i \in I, j \in J$. If $\bar{x} \in C$ is a robust weakly efficient solution of (UNMOP), then there exist $\bar{\tau} := (\bar{\tau}_1, \dots, \bar{\tau}_m) \in \mathbb{R}^m_+$, $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \mathbb{R}^l_+$ with $\|(\bar{\tau}, \bar{\lambda})\|_1 = 1$ and $\bar{\bar{w}}_j \in \Omega_j(\bar{x})$ such that*

$$\mathbf{0} \in \sum_{i \in I} \bar{\tau}_i \hat{\partial}^* f_i(\bar{x}) + \sum_{j \in J} \bar{\lambda}_j \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), \tag{3}$$

$$\bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j) = 0, \quad j \in J. \tag{4}$$

Proof: Let $\bar{x}$ be a robust weakly efficient solution of (UNMOP). Then $\bar{x}$ is a weakly efficient solution of the following problem:

$$\begin{aligned} &\min \quad f(x) \\ &\text{s.t.} \quad G_j(x) \le 0,\ j = 1, 2, \dots, l. \end{aligned}$$

Besides, one has

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_{++}, \quad \forall\, x \in C.$$

We now define a function $\varphi(x) := \max_{i \in I, j \in J}\{f_i(x) - f_i(\bar{x}), G_j(x)\}$. Then

$$\varphi(x) \ge 0 = \varphi(\bar{x}), \quad \forall\, x \in C,$$

i.e. $\bar{x}$ is a minimizer for φ. It therefore follows from Lemma 2.7 that $\mathbf{0} \in \hat{\partial}^* \varphi(\bar{x})$. In turn, we conclude from Theorem 2.6 that

$$\hat{\partial}^* \varphi(\bar{x}) = \mathrm{co}\left\{ \cup_{i \in I} \hat{\partial}^* f_i(\bar{x}) \cup \left\{ \cup_{j \in \hat{J}} \hat{\partial}^* G_j(\bar{x}) \right\} \right\},$$

where $\hat{J} = \{j \in J : G_j(\bar{x}) = 0\}$. It yields that there exist $\bar{\tau} = (\bar{\tau}_1, \bar{\tau}_2, \dots, \bar{\tau}_m)^\top \in \mathbb{R}^m_+$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_l)^\top \in \mathbb{R}^l_+$ with $\|(\bar{\tau}, \bar{\lambda})\|_1 = \sum_{i \in I} \bar{\tau}_i + \sum_{j \in J} \bar{\lambda}_j = 1$ such that

$$\mathbf{0} \in \sum_{i \in I} \bar{\tau}_i \hat{\partial}^* f_i(\bar{x}) + \sum_{j \in J} \bar{\lambda}_j \hat{\partial}^* G_j(\bar{x}), \tag{5}$$

and $\bar{\lambda}_j = 0$ for $j \notin \hat{J}$.

Let us prove that

$$\hat{\partial}^* G_j(\bar{x}) = \left\{ \hat{\partial}^*_x g_j(\bar{x}, \bar{w}_j) : \bar{w}_j \in \Omega_j(\bar{x}) \right\} = \bigcup_{\bar{w}_j \in \Omega_j(\bar{x})} \hat{\partial}^*_x g_j(\bar{x}, \bar{w}_j). \tag{6}$$

By the definition of $\hat{\partial}^* G_j(\bar{x})$, we have

$$\begin{aligned}
&\limsup_{t \searrow 0} \frac{G_j(\bar{x} + tv) - G_j(\bar{x})}{t} \\
&\quad = \limsup_{t \searrow 0} \frac{\sup_{w_j \in \Omega_j} g_j(\bar{x} + tv, w_j) - \sup_{w_j \in \Omega_j} g_j(\bar{x}, w_j)}{t} \\
&\quad \leq \sup_{u^* \in \hat{\partial}^* G_j(\bar{x})} \langle u^*, v \rangle, \quad \forall\, v \in \mathbb{R}^n.
\end{aligned} \tag{7}$$

For $\bar{w}_j \in \Omega_j(\bar{x})$, we obtain

$$\sup_{w_j \in \Omega_j} g_j(\bar{x} + tv, w_j) \geq g_j(\bar{x} + tv, \bar{w}_j), \qquad \sup_{w_j \in \Omega_j} g_j(\bar{x}, w_j) = g_j(\bar{x}, \bar{w}_j),$$

and so,

$$\begin{aligned}
&\limsup_{t \searrow 0} \frac{g_j(\bar{x} + tv, \bar{w}_j) - g_j(\bar{x}, \bar{w}_j)}{t} \\
&\quad \leq \limsup_{t \searrow 0} \frac{\sup_{w_j \in \Omega_j} g_j(\bar{x} + tv, w_j) - \sup_{w_j \in \Omega_j} g_j(\bar{x}, w_j)}{t} \\
&\quad \leq \sup_{u^* \in \hat{\partial}^* G_j(\bar{x})} \langle u^*, v \rangle.
\end{aligned} \tag{8}$$

According to the definition of upper semi-regular convexificator, (6) is implied by (7) and (8). Moreover, (5) and (6) yield that there exists $\bar{\bar{w}}_j \in \Omega_j(\bar{x})$ such that

$$\mathbf{0} \in \sum_{i \in I} \bar{\tau}_i \hat{\partial}^* f_i(\bar{x}) + \sum_{j \in J} \bar{\lambda}_j \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j).$$

Due to $\bar{\lambda}_j = 0$ for $j \notin \hat{J}$, one has

$$\bar{\lambda}_j G_j(\bar{x}) = \bar{\lambda}_j \sup_{w_j \in \Omega_j} g_j(\bar{x}, w_j) = \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) = 0, \quad \bar{w}_j \in \Omega_j(\bar{x}),\ j \in J,$$

and so, $\bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j) = 0$. Therefore the statements (3) and (4) are true. ■

Remark 3.1: The difference between Theorem 3.1 and the corresponding results of [33, 34] is that Theorem 3.1 is established by upper semi-regular convexificator instead of the convexificator. It is noted that an upper semi-regular convexificator may be weaker than a convexificator. It is also true that $\bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) = 0$, $\bar{w}_j \in \Omega_j(\bar{x}), j \in J$ in Theorem 3.1.

In order to get the KKT type robust optimality necessary conditions of (UNMOP), we introduce the Slater-type weak constraint qualification inspired by Mangasarian [41].

Definition 3.2: The Slater-type weak constraint qualification is satisfied at $\bar{x} \in C$ if, for each $\bar{w}_j \in \Omega_j(\bar{x}), j \in J$, $g_j(\cdot, \bar{w}_j)$ is $\hat{\partial}^*$-pseudoconvex at $\bar{x}$, and there exist an $x_0 \in \mathbb{R}^n$ and a $j_0 \in J$ such that $g_{j_0}(x_0, \bar{w}_{j_0}) < 0$ whenever $g_{j_0}(\bar{x}, \bar{w}_{j_0}) = 0, \bar{w}_{j_0} \in \Omega_{j_0}(\bar{x})$.

Remark 3.2: If the functions $g_j, j \in J$ are differentiable at $\bar{x}$ and uncertainty-free, then $g_j, j \in J$ admit upper regular convexificators at $\bar{x}$ and so, the Slater-type weak constraint qualification in Definition 3.2 reduces to the Slater weak constraint qualification given by Mangasarian [41].

We now present the KKT type robust necessary optimality conditions of (UNMOP).

Theorem 3.3: *Assume that the conditions of Theorem 3.1 hold and the Slater-type weak constraint qualification is satisfied at $\bar{x}$. If $\bar{x} \in C$ is a robust weakly efficient solution of* (UNMOP), *then there exist* $\bar{\tau} := (\bar{\tau}_1, \ldots, \bar{\tau}_m) \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_l) \in \mathbb{R}^l_+$ *with* $\|(\bar{\tau}, \bar{\lambda})\|_1 = 1$ *and* $\bar{\bar{w}}_j \in \Omega_j(\bar{x})$ *such that* (3) *and* (4) *hold.*

Proof: It follows from Theorem 3.1 that there exist $\bar{\tau} := (\bar{\tau}_1, \ldots, \bar{\tau}_m) \in \mathbb{R}^m_+$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_l) \in \mathbb{R}^l_+$ with $\|(\bar{\tau}, \bar{\lambda})\|_1 = 1$ such that (3) and (4) hold. Suppose to the contrary that $\bar{\tau} = \mathbf{0}$. Then $\bar{\lambda} \in \mathbb{R}^l_+ \setminus \{\mathbf{0}\}$ and so, (3) implies that there exist $\zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), j \in J$ such that $\sum_{j\in J} \bar{\lambda}_j \zeta_j = \mathbf{0}$. Since the Slater-type weak constraint qualification is satisfied at $\bar{x}$, then $j \in J, g_j(\cdot, \bar{\bar{w}}_j)$ is $\hat{\partial}^*$-pseudoconvex at $\bar{x}$, and there exist an $x_0 \in \mathbb{R}^n$ and a $j_0 \in J$ such that

$$g_{j_0}(x_0, \bar{\bar{w}}_{j_0}) < 0 = g_{j_0}(\bar{x}, \bar{\bar{w}}_{j_0}).$$

By the $\hat{\partial}^*$-pseudoconvexity of $g_j(\cdot, \bar{\bar{w}}_j), \bar{\bar{w}}_j \in \Omega(\bar{x}), j \in J$, we obtain

$$\langle \zeta_{j_0}, x_0 - \bar{x} \rangle < 0, \quad \forall\, \zeta_{j_0} \in \hat{\partial}^*_x g_{j_0}(\bar{x}, \bar{\bar{w}}_{j_0}).$$

Clearly, if $j \neq j_0, j \in J, g_j(\bar{x}, \bar{\bar{w}}_j) < 0$, then (4) yields $\bar{\lambda}_j = 0$. Therefore, we have

$$\left\langle \sum_{j\in J} \bar{\lambda}_j \zeta_j, x_0 - \bar{x} \right\rangle = \left\langle \sum_{j\in J\setminus \bar{J}} \bar{\lambda}_j \zeta_j, x_0 - \bar{x} \right\rangle < 0,$$
$$\forall\, \zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), \bar{w}_j \in \Omega_j(\bar{x}), j \in J,$$

where $\bar{J} := \{j \in J : \bar{\lambda}_j = 0\}$, which contradicts the fact that $\sum_{j\in J} \bar{\lambda}_j \zeta_j = \mathbf{0}$. Consequently, we get $\bar{\tau} \neq \mathbf{0}$. ■

We next give robust optimality sufficient conditions for robust weakly efficient solutions of (UNMOP) under the $\hat{\partial}^*$-pseudoquasiconvexity.

Theorem 3.4: *Let* $(\bar{x}, \bar{\bar{w}}, \bar{\tau}, \bar{\lambda})$ *satisfy* (3) *and* (4) *with* $\bar{x} \in C, \bar{\bar{w}} = (\bar{\bar{w}}_1, \ldots, \bar{\bar{w}}_l)$, $\bar{\bar{w}}_j \in \Omega_j(\bar{x}), j \in J, \bar{\lambda} \in \mathbb{R}^l_+, \bar{\tau} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ *and* $\|(\bar{\tau}, \bar{\lambda})\|_1 = 1$. *Assume that* $(f, \bar{\lambda}^\top g)$ *is generalized* $\hat{\partial}^*$*-pseudoquasiconvex at* $\bar{x} \in C$. *Then* $\bar{x}$ *is a robust weakly efficient solution of* (UNMOP).

Proof: Suppose that $\bar{x}$ is not a robust weakly efficient solution of (UNMOP). Then there exists a feasible point $\tilde{x} \in C$ such that

$$f(\tilde{x}) - f(\bar{x}) \in -\mathbb{R}^m_{++}.$$

From the feasibility of $\tilde{x}$ and $\bar{\lambda}_j \geq 0$, we have $\bar{\lambda}_j g_j(\tilde{x}, \bar{\bar{w}}_j) \leq 0$, $j \in J$. It follows from (4) that

$$\bar{\lambda}_j g_j(\tilde{x}, \bar{\bar{w}}_j) \leq \bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j) = 0, j \in J,$$

and so,

$$\bar{\lambda}^\top g(\tilde{x}, \bar{\bar{w}}) \leq \bar{\lambda}^\top g(\bar{x}, \bar{\bar{w}}) = 0.$$

Using the $\hat{\partial}^*$-pseudoquasiconvexity of $(f, \bar{\lambda}^\top g)$, we obtain

$$\langle \xi_i, \tilde{x} - \bar{x} \rangle < 0, \quad \forall\, \xi_i \in \hat{\partial}^* f_i(\bar{x}), i \in I,$$

and

$$\langle \tilde{\zeta}_j, \tilde{x} - \bar{x} \rangle \leq 0, \quad \forall\, \tilde{\zeta}_j \in \hat{\partial}^*_x (\bar{\lambda}_j g_j)(\bar{x}, \bar{\bar{w}}_j), j \in J. \tag{9}$$

Since $\bar{\tau} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, one has

$$\left\langle \sum_{i \in I} \bar{\tau}_i \xi_i, \tilde{x} - \bar{x} \right\rangle < 0, \quad \forall\, \xi_i \in \hat{\partial}^* f_i(\bar{x}), i \in I. \tag{10}$$

From Lemma 2.5, the inequality (9) yields

$$\langle \bar{\lambda}_j \zeta_j, \tilde{x} - \bar{x} \rangle \leq 0, \quad \forall\, \zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), j \in J.$$

Moreover, we get

$$\left\langle \sum_{j \in J} \bar{\lambda}_j \zeta_j, \tilde{x} - \bar{x} \right\rangle \leq 0, \quad \forall\, \zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), j \in J. \tag{11}$$

Summing up the inequalities (10) and (11), we have

$$\left\langle \sum_{i \in I} \bar{\tau} \xi_i + \sum_{j \in J} \bar{\lambda}_j \zeta_j, \tilde{x} - \bar{x} \right\rangle < 0,$$

for all $\xi_i \in \hat{\partial}^* f_i(\bar{x}), i \in I$ and $\zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), j \in J$, which contradicts (3). ■

The following example shows that the generalized $\hat{\partial}^*$-pseudoquasiconvexity condition in Theorem 3.4 is indispensable.

Example 3.5: Let $\Omega := [-1, 0]$, $f(x) := (x_1 + x_2^3, x_1^3 + x_2)$ and $g(x, w) := w(x_1^2 + x_2^2)$ for $x \in \mathbb{R}^2$ and $w \in \Omega$. After calculation, we have $C = \mathbb{R}^2$. Let $\bar{x} := (0, 0) \in \mathbb{R}^2$. Then

$$\hat{\partial}^* f_1(\bar{x}) = [-1, 1] \times [-1, 1], \quad \hat{\partial}^* f_2(\bar{x}) = [-1, 1] \times [-1, 1],$$

and $\bar{\bar{w}} \in \Omega(\bar{x}) = [-1, 0]$, $\hat{\partial}_x^* g(\bar{x}, \bar{\bar{w}}) = [-1, 1] \times [-1, 1]$. Set $\bar{\tau}_1 = \bar{\tau}_2 := \frac{1}{4}$ and $\bar{\eta} := \frac{1}{2}$. It is easy to verify that $\mathbf{0} \in \bar{\tau}_1 \hat{\partial}^* f_1(\bar{x}) + \bar{\tau}_2 \hat{\partial}^* f_2(\bar{x}) + \bar{\eta} \hat{\partial}_x^* g(\bar{x}, \bar{\bar{w}})$, and $\bar{\eta} g(\bar{x}, \bar{w}) = 0$. However, for $\tilde{x} := (-1, -1) \in C = \mathbb{R}^2$, one has

$$f(\tilde{x}) - f(\bar{x}) = (-2, -2)^\top - (0, 0)^\top = (-2, -2)^\top \in -\mathbb{R}^2_{++}.$$

This implies that $\bar{x}$ is not a robust weakly efficient solution of (UNMOP). Actually, the generalized $\hat{\partial}^*$-pseudoquasiconvexity condition of Theorem 3.4 is not satisfied at $\bar{x} \in C$. For any $x \in -\mathbb{R}^2_{++}$, $f_i(x) < f_i(\bar{x})$ for $i = 1, 2$. However, there exists $\xi = (0, 0)^\top \in \hat{\partial}^* f_1(\bar{x}) = \hat{\partial}^* f_2(\bar{x})$ such that $\langle \xi, d \rangle = 0$ for all $d \in \mathbb{R}^2$.

In the end of this section, we give saddle point type sufficient conditions for robust weak efficient solution of (UNMOP). For $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, we define the function $\mathcal{L}_\tau : \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega \to \mathbb{R}$ as follows

$$\mathcal{L}_\tau(x, \lambda, w) := \tau^\top f(x) + \lambda^\top g(x, w) = \sum_{i \in I} \tau_i f_i(x) + \sum_{j \in J} \lambda_j g_j(x, w_j),$$

for all $(x, \lambda, w) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega$.

Definition 3.6: $(\bar{x}, \bar{\lambda}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega$ is a saddle point of (UNMOP) with respect to $\bar{\tau} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ if it holds that

$$\mathcal{L}_{\bar{\tau}}(\bar{x}, \lambda, w) \le \mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{w}) \le \mathcal{L}_{\bar{\tau}}(x, \bar{\lambda}, \bar{w}), \quad \forall\, (x, \lambda, w) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega.$$

Theorem 3.7: *Let* $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\bar{w}}) \in C \times \mathbb{R}^m_+ \setminus \{\mathbf{0}\} \times \mathbb{R}^l_+ \times \Omega$ *satisfy* (3) *and* (4). *If* $\sum_{i \in I} \bar{\tau}_i f_i(\cdot) + \sum_{j \in J} \bar{\lambda}_j g_j(\cdot, \bar{\bar{w}}_j)$ *is* $\hat{\partial}^*$*-pseudoconvex at* $\bar{x}$, *then* $(\bar{x}, \bar{\lambda}, \bar{\bar{w}})$ *is a saddle point of* (UNMOP) *with respect to* $\bar{\tau}$.

Proof: Suppose that $(\bar{x}, \bar{\lambda}, \bar{\bar{w}})$ is not a saddle point of (UNMOP) with respect to $\bar{\tau}$. Then there exists $(\hat{x}, \hat{\lambda}, \hat{w}) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega$ such that

$$\mathcal{L}_{\bar{\tau}}(\hat{x}, \bar{\lambda}, \bar{\bar{w}}) < \mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{\bar{w}}), \tag{12}$$

or,

$$\mathcal{L}_{\bar{\tau}}(\bar{x}, \hat{\lambda}, \hat{w}) > \mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{\bar{w}}). \tag{13}$$

If (12) holds, then

$$\sum_{i\in I} \bar{\tau}_i f_i(\hat{x}) + \sum_{j\in J} \bar{\lambda}_j g_j(\hat{x}, \bar{\bar{w}}_j) < \sum_{i\in I} \bar{\tau}_i f_i(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j).$$

By the $\hat{\partial}^*$-pseudoconvexity of $\sum_{i\in I} \bar{\tau}_i f_i(\cdot) + \sum_{j\in J} \bar{\lambda}_j g_j(\cdot, \bar{\bar{w}}_j)$ at $\bar{x}$, we get

$$\left\langle \sum_{i\in I} \bar{\tau}_i \xi_i + \sum_{j\in J} \bar{\lambda}_j \zeta_j, \hat{x} - \bar{x} \right\rangle < 0, \tag{14}$$

for all $\xi_i \in \hat{\partial}^* f_i(\bar{x}), i \in I$ and $\zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), j \in J$. Since $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\bar{w}}) \in C \times \mathbb{R}^m_+ \setminus \{\mathbf{0}\} \times \mathbb{R}^l_+ \times \Omega$ satisfy (3) and (4), one has

$$\mathbf{0} \in \sum_{i\in I} \bar{\tau}_i \hat{\partial}^* f_i(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j),$$

and $\bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j) = 0, \;\; j \in J$. Then there exist $\zeta_j \in \hat{\partial}^*_x g_j(\bar{x}, \bar{\bar{w}}_j), j \in J$ and $\xi_i \in \hat{\partial}^* f_i(\bar{x}), i \in I$ such that $\sum_{i\in I} \bar{\tau}_i \xi_i + \sum_{j\in J} \bar{\lambda}_j \zeta_j = \mathbf{0}$, which contradicts with (14).

If (13) holds, then

$$\sum_{j\in J} \hat{\lambda}_j g_j(\bar{x}, \hat{w}_j) > \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j). \tag{15}$$

Since $\bar{x} \in C$, $g_j(\bar{x}, \hat{w}_j) \leq 0$ and $\lambda_j \geq 0, j \in J$, we have $\sum_{j\in J} \lambda_j g_j(\bar{x}, \hat{w}_j) \leq 0$. This together with (15) yields that $\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j) < 0$, which contradicts the fact that $\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{\bar{w}}_j) = 0$. So, $(\bar{x}, \bar{\lambda}, \bar{\bar{w}})$ is a saddle point of (UNMOP) with respect to $\bar{\tau}$. ■

Theorem 3.8: *If* $(\bar{x}, \bar{\lambda}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega$ *is a saddle point of* (UNMOP) *with respect to* $\bar{\tau} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$*, then* $\bar{x}$ *is a robust weakly efficient solution of* (UNMOP).

Proof: Assume that $(\bar{x}, \bar{\lambda}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega$ is a saddle point of (UNMOP) with respect to $\bar{\tau}$. Then we have

$$\mathcal{L}_{\bar{\tau}}(\bar{x}, \lambda, w) \leq \mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{w}) \leq \mathcal{L}_{\bar{\tau}}(x, \bar{\lambda}, \bar{w}), \quad \forall\, (x, \lambda, w) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \Omega.$$

Note that

$$\begin{aligned}\sum_{i\in I} \bar{\tau}_i f_i(\bar{x}) + \sum_{j\in J} \lambda_j g_j(\bar{x}, w_j) &= \mathcal{L}_{\bar{\tau}}(\bar{x}, \lambda, w) \\ &\leq \mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{w}) \\ &= \sum_{i\in I} \bar{\tau}_i f_i(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j).\end{aligned}$$

So, we have

$$\sum_{j\in J} \lambda_j g_j(\bar{x}, w_j) \leq \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j), \quad \forall\, (\lambda, w) \in \mathbb{R}^l_+ \times \Omega. \tag{16}$$

Taking $\lambda_j := 0$ and $\lambda_j := 2\bar{\lambda}_j, j \in J$ in (16), we obtain

$$\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) = 0. \tag{17}$$

In turn, it follows from (16) that

$$\sum_{j\in J} \lambda_j g_j(\bar{x}, w_j) \leq 0, \quad \forall\, \lambda_j \in \mathbb{R}_+,\ w_j \in \Omega_j,\ j \in J, \tag{18}$$

and so, $g_j(\bar{x}, w_j) \leq 0$ for all $w_j \in \Omega_j$ and $j \in J$. Consequently, one has $\bar{x} \in C$.

Suppose that $\bar{x}$ is not a robust weakly efficient solution of (UNMOP). Then there exists $\hat{x} \in C$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\mathbb{R}^m_{++},$$

and $\sum_{j\in J} \bar{\lambda}_j g_j(\hat{x}, \bar{w}_j) \leq 0$. According to $\bar{\tau} \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, we have

$$\sum_{i\in I} \bar{\tau}_i f_i(\hat{x}) < \sum_{i\in I} \bar{\tau}_i f_i(\bar{x}), \tag{19}$$

From (17) and (19), we deduce that

$$\sum_{i\in I} \bar{\tau}_i f_i(\hat{x}) + \sum_{j\in J} \bar{\lambda}_j g_j(\hat{x}, \bar{w}_j) < \sum_{i\in I} \bar{\tau}_i f_i(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j),$$

i.e. $\mathcal{L}_{\bar{\tau}}(\hat{x}, \bar{\lambda}, \bar{w}) < \mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{w})$, which contradicts the fact that $\mathcal{L}_{\bar{\tau}}(\bar{x}, \bar{\lambda}, \bar{w}) \leq \mathcal{L}_{\bar{\tau}}(x, \bar{\lambda}, \bar{w})$ for all $x \in \mathbb{R}^n$. Therefore, $\bar{x}$ is a robust weakly efficient solution of (UNMOP). ■

4. Robust duality

In this final section, we present a mixed robust dual model (MRD) for (UNMOP), and discuss the robust weak (strong and converse) duality properties between (UNMOP) and (MRD). Let $e := (1, 1, \ldots, 1)^\top \in \mathbb{R}^l$. The problem (MRD) in terms of upper semi-regular convexificator is formulated as follows:

$$\begin{aligned}
\text{(MRD)} \quad \max \quad & L(y, w, \tau, \lambda, \beta) := f(y) + \sum_{j\in J} \lambda_j g_j(y, w_j)e \\
\text{s.t.} \quad & \mathbf{0} \in \sum_{i\in I} \tau_i \hat{\partial}^* f_i(y) + \sum_{j\in J} (\lambda_j + \beta_j)\hat{\partial}^*_y g_j(y, w_j), \qquad (20) \\
& \beta_j g_j(y, w_j) \geq 0, \\
& w_j \in \Omega_j(y), \tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}, \lambda, \beta \in \mathbb{R}^l_+, i \in I, j \in J. \qquad (21)
\end{aligned}$$

Denote by $F_{(MRD)}$ the feasible set of (MRD).

Remark 4.1: If $\lambda_j = 0, j \in J$, (MRD) reduces to the Mond-Weir type dual problem (MWD) as follows:

$$\begin{aligned}
\text{(MWD)} \quad \max \quad & L(y, w, \tau, \beta) := f(y) \\
\text{s.t.} \quad & \mathbf{0} \in \sum_{i\in I} \tau_i \hat{\partial}^* f_i(y) + \sum_{j\in J} \beta_j \hat{\partial}^*_y g_j(y, w_j), \\
& \beta_j g_j(y, w_j) \geq 0, \\
& w_j \in \Omega_j(y), \tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}, \beta \in \mathbb{R}^l_+, i \in I, j \in J.
\end{aligned}$$

If $\beta_j = 0, j \in J$, (MRD) reduces to the Wolfe type dual problem (WD) as follows:

$$\begin{aligned}
\text{(WD)} \quad \max \quad & L(y, w, \tau, \lambda) := f(y) + \sum_{j\in J} \lambda_j g_j(y, w_j)e \\
\text{s.t.} \quad & \mathbf{0} \in \sum_{i\in I} \tau_i \hat{\partial}^* f_i(y) + \sum_{j\in J} \lambda_j \hat{\partial}^*_y g_j(y, w_j), \\
& w_j \in \Omega_j(y), \tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}, \lambda \in \mathbb{R}^l_+, i \in I, j \in J.
\end{aligned}$$

In what follows, for $u_1, u_2 \in \mathbb{R}^m$, we define the order relation as follows:

$$u_1 \prec u_2 \Leftrightarrow u_1 - u_2 \in -\mathbb{R}^m_{++}, \quad u_1 \nprec u_2 \Leftrightarrow u_1 - u_2 \notin -\mathbb{R}^m_{++}.$$

Definition 4.1: $(\bar{y}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\beta}) \in F_{(MRD)}$ is said to be a weakly efficient solution of problem (MRD) if

$$L(\bar{y}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\beta}) \nprec L(y, w, \tau, \lambda, \beta), \quad \forall\, (y, w, \tau, \lambda, \beta) \in F_{(MRD)}.$$

We now study the robust weak, strong, converse duality results between (MRD) and (UNMOP) under some convexity assumptions with respect to the upper semi-regular convexificator.

Theorem 4.2 (Robust weak duality): *Let $x \in C$ and $(y, w, \tau, \lambda, \beta) \in F_{(MRD)}$. Assume that $\sum_{i\in I} \tau_i f_i(\cdot) + \sum_{j\in J}(\lambda_j + \beta_j) g_j(\cdot, w_j)$ is $\hat{\partial}^*$-pseudoconvex at y and $w_j \in \Omega_j(y)$. Then it holds that $f(x) \not\prec L(y, w, \tau, \lambda, \beta)$.*

Proof: Suppose that

$$f(x) \prec L(y, w, \tau, \lambda, \beta), \quad \forall\, x \in C, \quad \forall\, (y, w, \tau, \lambda, \beta) \in F_{(MRD)}. \tag{22}$$

Due to $\tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, without loss of generality, one can set $\tau^\top e = 1$. Then (22) yields that

$$\left\langle \tau, f(x) - \left(f(y) + \sum_{j\in J} \lambda_j g_j(y, w_j) e \right) \right\rangle < 0, \quad \forall\, x \in C, \tag{1}$$

$$\forall\, (y, w, \tau, \lambda, \beta) \in F_{(MRD)}, \tag{23}$$

and so,

$$\sum_{i\in I} \tau_i \left(f_i(x) - f_i(y) \right) - \sum_{j\in J} \lambda_j g_j(y, w_j) < 0. \tag{24}$$

Since $(y, w, \tau, \lambda, \beta) \in F_{(MRD)}$, there exist $w_j \in \Omega_j(y), \tau \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}, \lambda, \beta \in \mathbb{R}^l_+$, $\xi_i \in \hat{\partial}^* f_i(y), i \in I$, and $\zeta_j \in \hat{\partial}^*_y g_j(y, w_j), j \in J$ such that

$$\mathbf{0} = \sum_{i\in I} \tau_i \xi_i + \sum_{j\in J} (\lambda_j + \beta_j) \zeta_j, \tag{25}$$

and

$$\beta_j g_j(y, w_j) \geq 0, \quad j \in J. \tag{26}$$

Taking into account that $x \in C, \lambda, \beta \in \mathbb{R}^l_+$, (24) and (26) yield that

$$\sum_{i\in I} \tau_i f_i(x) + \sum_{j\in J} (\lambda_j + \beta_j) g_j(x, w_j) < \sum_{i\in I} \tau_i f_i(y) + \sum_{j\in J} (\lambda_j + \beta_j) g_j(y, w_j).$$

Since $\sum_{i\in I} \tau_i f_i(\cdot) + \sum_{j\in J}(\lambda_j + \beta_j) g_j(\cdot, w_j)$ is $\hat{\partial}^*$-pseudoconvex at y, we get

$$\left\langle \sum_{i\in I} \tau_i \xi_i + \sum_{j\in J} (\lambda_j + \beta_j) \zeta_j, x - y \right\rangle < 0, \tag{27}$$

which contradicts with (25). Consequently, $f(x) \not\prec L(y, w, \tau, \lambda, \beta)$. ■

The following example shows that the $\hat{\partial}^*$-pseudoconvexity condition in Theorem 4.2 is indispensable.

Example 4.3: Let $\Omega := [-1, 0]$ and let $f(x) := x^3$ and $g(x,w) := wx^2$ for $x \in \mathbb{R}$. Obviously, $C = \mathbb{R}$. Let $\bar{x} := -1$. Considering the dual problem (MRD), for any $y \in \mathbb{R}$ and $\bar{w} := 0 \in \Omega_j(y)$, one has

$$f_d^+(y, v) = 3vy^2, \quad g(y, w) = 0.$$

Taking $\bar{y} := 0, \bar{\tau} := \frac{1}{2}, \bar{\lambda} = \bar{\beta} := \frac{1}{4}$, we have $(\bar{y}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\beta}) \in F_{(MRD)}$, $\hat{\partial}^* f(\bar{y}) = [-1, 1]$, $\hat{\partial}_x^* g(\bar{y}, \bar{w}) = [-1, 1]$, $0 \in \bar{\tau}\hat{\partial}^* f(\bar{y}) + (\bar{\lambda} + \bar{\beta})\hat{\partial}_x^* g(\bar{y}, \bar{w})$ and $\bar{\beta} g(\bar{y}, \bar{w}) = 0$. Moreover, one has

$$L(\bar{y}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\beta}) := f(\bar{y}) + \sum_{j \in J} \bar{\lambda}_j g_j(\bar{y}, \bar{w}_j) e = 0 > -1 = f(\bar{x}),$$

which implies that the assertion of Theorem 4.2 is not true. As a matter of fact, the $\hat{\partial}^*$-pseudoconvexity condition in Theorem 4.2 does not hold. Notice that

$$\bar{\tau} f(\bar{x}) + (\bar{\lambda} + \bar{\beta}) g(\bar{x}, \bar{w}) = -\frac{1}{2} < 0 = \bar{\tau} f(\bar{y}) + (\bar{\lambda} + \bar{\beta}) g(\bar{y}, \bar{w}).$$

Taking $\xi = \zeta := -1$, we have

$$\left\langle \frac{1}{2} \times (-1) + (\frac{1}{4} + \frac{1}{4}) \times (-1), (-1) - 0 \right\rangle = 1 > 0.$$

Theorem 4.4 (Robust strong duality): *Let $\bar{x} \in C$ be a robust weakly efficient solution of* (UNMOP). *Assume that all conditions of Theorem 3.1 hold and the Slater-type weak constraint qualification is satisfied at $\bar{x}$. Then there exist $\bar{w} \in \Omega(\bar{x}), \bar{\tau} \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}, \bar{\lambda} = \mathbf{0}$ and $\bar{\beta} \in \mathbb{R}_+^l$ such that $(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta})$ is feasible for* (MRD) *and $f(\bar{x}) = L(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta})$. Moreover, if the conditions of Theorem 4.2 hold, then $(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta})$ is a weakly efficient solution of* (MRD).

Proof: It follows from Theorem 3.3 that there exist $\bar{w}_j \in \Omega_j(\bar{x}), \bar{\tau} \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}, \bar{\beta} \in \mathbb{R}_+^l$ such that

$$\mathbf{0} \in \sum_{i \in I} \bar{\tau}_i \hat{\partial}^* f_i(\bar{x}) + \sum_{j \in J} \bar{\beta}_j \hat{\partial}_x^* g_j(\bar{x}, \bar{w}_j), \tag{28}$$

$$\bar{\beta}_j g_j(\bar{x}, \bar{w}_j) = 0, j \in J. \tag{29}$$

So, $(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta})$ is feasible for (MRD) and

$$f(\bar{x}) = L(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta}).$$

This together with Theorem 4.2 yields that

$$L(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta}) = f(\bar{x}) \not< L(y, w, \tau, \lambda, \beta), \quad \forall\, (y, w, \tau, \lambda, \beta) \in F_{(MRD)}.$$

Thus $(\bar{x}, \bar{w}, \bar{\tau}, \mathbf{0}, \bar{\beta})$ is a weakly efficient solution of (MRD). ■

Theorem 4.5 (Robust converse duality): *Let* $(\bar{x}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\lambda}) \in F_{(MRD)}$ *be a weakly efficient solution of* (MRD) *with* $\bar{x} \in C$. *If* $\sum_{i\in I} \tau_i f_i(\cdot) + \sum_{j\in J}(\lambda_j + \beta_j) g_j(\cdot, w_j)$ *is* $\hat{\partial}^*$*-pseudoconvex at* $\bar{x}$, *then* $\bar{x}$ *is a robust weakly efficient solution of* (UNMOP).

Proof: Since $(\bar{x}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\lambda}) \in F_{(MRD)}$, one has

$$\bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) \geq 0, \bar{w}_j \in \Omega_j(\bar{x}),$$

and so, $\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) \geq 0, \bar{w}_j \in \Omega_j(\bar{x})$. It follows from Theorem 4.2 that

$$f(x) \nless L(\bar{x}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\lambda}), \quad \forall\, x \in C. \tag{30}$$

Taking $x := \bar{x}$ in (30), we have

$$\mathbf{0} \nless \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) e.$$

Recalling that e is the vector of all ones, one has

$$\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) \leq 0.$$

Consequently, we obtain $\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}, \bar{w}_j) = 0$ and $f(\bar{x}) = L(\bar{x}, \bar{w}, \bar{\tau}, \bar{\lambda}, \bar{\lambda})$. It therefore follows from (30) that

$$f(x) \nless f(\bar{x}), \quad \forall\, x \in C,$$

i.e. $f(x) - f(\bar{x}) \notin -\mathbb{R}^m_{++}$ for all $x \in C$. Therefore, $\bar{x} \in C$ is a robust weakly efficient solution of (UNMOP). ■

5. Conclusions

By using the robust approach, we studied robust optimality conditions and duality of (UNMOP). The Fritz-John type and KKT type robust optimality necessary conditions of (UNMOP) are derived via upper semi-regular convexificators. Robust sufficient optimality conditions including saddle point conditions are obtained under a generalized convexity assumptions. The robust weak duality, strong duality and converse duality between the original problem and its mixed robust dual problem are derived under a generalized pseudoconvexity assumption. For future research, it would be interesting to consider conjugate duality of (UNMOP) via the convexificators. We are also interested in designing numerical algorithms for calculating robust solutions of (UNMOP) by a discretization method as a future research aspect.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The authors would like to express their sincere thanks to anonymous referees for their valuable suggestions, which helped to improve the paper. This paper was partially supported by the the Natural Science Foundation of China [No. 12071379], the Natural Science Foundation of Chongqing [No. cstc2021jcyj-msxmX0925, cstc2022ycjh-bgzxm0097], and the Youth Project of Science and Technology Research Program of Chongqing Education Commission of China [No. KJQN202201802].

References

[1] Luc DT. Theory of vector optimization. Berlin: Springer; 1989. (Lecture Notes in Economics and Mathematical Systems).
[2] Ansari QH, Yao JC. Recent advances in vector optimization. Berlin: Springer-Verlag; 2012.
[3] Ansari QH, Köbis E, Yao JC. Vector variational inequalities and vector optimization: theory and applications. Cham: Springer; 2018.
[4] Jahn J. Vector optimization theory, applications and extensions. Berlin: Springer; 2004.
[5] Chen J, Huang L, Li S. Separations and optimality of constrained multiobjective optimization via improvement sets. J Optim Theory Appl. 2018;178:794–823. doi: 10.1007/s10957-018-1325-2
[6] Chen J, Dai Y. Multiobjective optimization with least constraint violation: optimality conditions and exact penalization. J Glob Optim. doi: 10.1007/s10898-022-01158-8
[7] Ansari QH, Köbis E, Sharma PK. Characterizations of multiobjective robustness via oriented distance function and image space analysis. J Optim Theory Appl. 2019;181:817–839. doi: 10.1007/s10957-019-01505-y
[8] Huong NTT, Yao JC, Yen ND. New results on proper efficiency for a class of vector optimization problems. Appl Anal. 2021;100:3199–3211. doi: 10.1080/00036811.2020.1712373
[9] Sun XK, Feng X, Teo KL. Robust optimality, duality and saddle points for multiobjective fractional semi-infinite optimization with uncertain data. Optim Lett. 2022;16:1457–1476. doi: 10.1007/s11590-021-01785-2
[10] Ben-Tal A, Nemirovski A. Robust optimization-methodology and applications. Math Program. 2002;92:453–480. doi: 10.1007/s101070100286
[11] Ben-Tal A, Nemirovski A. A selected topics in robust convex optimization. Math Program. 2008;112:125–158. doi: 10.1007/s10107-006-0092-2
[12] Beck A, Ben-Tal A. Duality in robust optimization: primal worst equals dual best. Oper Res Lett. 2009;37:1–6. doi: 10.1016/j.orl.2008.09.010
[13] Ben-Tal A, Ghaoui LE, Nemirovski A. Robust optimization. Princeton: Princeton University Press; 2009. (Princeton series in applied mathematics).
[14] Jeyakumar V, Li G. Strong duality in robust convex programming: complete characterizations. SIAM J Optim. 2010;20:3384–3407. doi: 10.1137/100791841
[15] Jeyakumar V, Lee GM, Li G. Characterizing robust solutions sets convex programs under data uncertainty. J Optim Theory Appl. 2015;164:407–435. doi: 10.1007/s10957-014-0564-0
[16] Lee GM, Yao JC. On solution sets for robust optimization problems. J Nonlinear Convex Anal. 2015;17:957–966.
[17] Sun X, Teo KL, Long XJ. Characterizations of robust ϵ-quasi optimal solutions for nonsmooth optimization problems with uncertain data. Optim. 2021;70:847–870. doi: 10.1080/02331934.2021.1871730

[18] Kuroiwa D, Lee GM. On robust multiobjective optimization. J Nonlinear Convex Anal. 2012;40:305–317.
[19] Ide J, Köbis E. Concepts of efficiency for uncertain multiobjective problems based on set order relations. Math Meth Oper Res. 2014;80:99–127. doi: 10.1007/s00186-014-0471-z
[20] Goberna MA, Jeyakumar V, Li G, et al. Robust solutions to multiobjective linear programs with uncertain data. Eur J Oper Res. 2015;242:730–743. doi: 10.1016/j.ejor.2014.10.027
[21] Klamroth K, Köbis E, Schöbel A, et al. A unified approach to uncertain optimization. Eur J Oper Res. 2017;260:403–420. doi: 10.1016/j.ejor.2016.12.045
[22] Bokrantz R, Fredriksson A. Necessary and sufficient conditions for pareto efficiency in robust multiobjective optimization. Eur J Oper Res. 2017;262:682–692. doi: 10.1016/j.ejor.2017.04.012
[23] Chuong TD. Optimality and duality for robust multiobjective optimization problems. Nonlinear Anal. 2016;134:127–143. doi: 10.1016/j.na.2016.01.002
[24] Chen JW, Köbis E, Yao JC. Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints. J Optim Theory Appl. 2019;181:411–436. doi: 10.1007/s10957-018-1437-8
[25] Chen JW, Huang L, Lv YB, et al. Optimality conditions of robust convex multiobjective optimization via ϵ-constraint scalarization and image space analysis. Optim. 2020;69:1849–1879. doi: 10.1080/02331934.2019.1658760
[26] Sun X, Tan W, Teo KL. Characterizing a class of robust vector polynomial optimization via sum of squares conditions. J Optim Theory Appl. 2023;197:737–764. doi: 10.1007/s10957-023-02184-6
[27] Demyanov VF. Convexification and concavification of a positively homogenous function by the same family of linear functions. Universita di Pisa. Report3, 208, 802; 1994.
[28] Jeyakumar V, Luc DT. Nonsmooth calculus, minimality, and monotonicity of convexificators. J Optim Theory Appl. 1999;101:599–621. doi: 10.1023/A:1021790120780
[29] Dutta J, Chandra S. Convexifactors, generalized convexity and optimality conditions. J Optim Theory Appl. 2002;113:41–64. doi: 10.1023/A:1014853129484
[30] Dutta J, Chandra S. Convexifactors, generalized convexity and vector optimization. Optim. 2004;53:77–94. doi: 10.1080/02331930410001661505
[31] Li XF, Zhang JZ. Necessary optimality conditions in terms of convexificators in Lipschitz optimization. J Optim Theory Appl. 2006;131:429–452. doi: 10.1007/s10957-006-9155-z
[32] Alireza K, Majid SD. Characterization of (weakly/properly/robust) efficient solutions in nonsmooth semi-infinite multiobjective optimization using convexificators. Optim. 2018;67:217–235. doi: 10.1080/02331934.2017.1393675
[33] Anurag J, Ioan SM, Jonaki B. Optimality conditions and duality for interval-valued optimization problems using convexifactors. Rend Del Circ Mat Di Palermo. 2016;65(1):17–32. doi: 10.1007/s12215-015-0215-9
[34] Ahmad I, Kummari K, Singh V, et al. Optimiality and duality for nonsmooth minimax programming problems using convexifactor. Filomat. 2017;31:4555–4570. doi: 10.2298/FIL1714555A
[35] Dempe S, Gadhib N, El idrissib M. Optimality conditions in terms of convexificators for a bilevel multiobjective optimization problem. Optim. 2020;69:1811–1830. doi: 10.1080/02331934.2020.1750610
[36] Su TV, Hang DD, Dieu NC. Optimality conditions and duality in terms of convexificators for multiobjective bilevel programming problem with equilibrium constraints. Comput Appl Math. 2021;40:26–37. doi: 10.1007/s40314-021-01625-0
[37] Clarke FH. Optimization and nonsmooth analysis. New York: Wiley-Interscience; 1983.

[38] Michel P, Penot JP. A generalized derivative for calm and stable functions. Diff Integral Equ. 1992;5:433–454.
[39] Ioffe AD. Approximate subdifferentials and applications II. Math. 1986;33:111–128.
[40] Treiman JS. The linear nonconvex generalized gradient and lagrange multipliers. SIAM J Optim. 1995;5:670–680. doi: 10.1137/0805033
[41] Mangasarian OL. Nonlinear programming. New York: McGraw-Hill; 1969.