SPECTRAL SYNTHESIS FOR EXPONENTIALS AND LOGARITHMIC LENGTH

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Abstract. We study hereditary completeness of systems of exponentials on an interval such that the corresponding generating function G is small outside of a lacunary sequence of intervals I_k. We show that, under some technical conditions, an exponential system is hereditarily complete if and only if the logarithmic length of the union of these intervals is infinite, i.e., $\sum_k \int_{I_k} \frac{dx}{1+|x|} = \infty$.

1. Introduction

Let $\{v_n\}_{n\in\mathbb{N}}$ be a complete and minimal system of vectors in a separable Hilbert space $\mathcal{H}$, that is, $\overline{\operatorname{Span}}\{v_n\} = \mathcal{H}$ and $\overline{\operatorname{Span}}\{v_n\}_{n\neq m} \neq \mathcal{H}$ for any m. For any such sequence there exists its unique biorthogonal system $\{w_n\}_{n\in\mathbb{N}}$ such that $\langle v_n, w_m\rangle = \delta_{nm}$. In general, the system $\{w_n\}_{n\in\mathbb{N}}$ needs not be complete (e.g., consider $v_n = e_1 + e_{n+1}$ in $\mathcal{H} = \ell^2(\mathbb{N})$), but even if it is complete, it is possible that for some partition $\mathbb{N} = A\cup B$, $A\cap B = \emptyset$, the "mixed" system $\{v_n\}_{n\in A} \cup \{w_n\}_{n\in B}$ is incomplete. If it is not the case for any partition $\mathbb{N} = A\cup B$, then we call the system $\{v_n\}_{n\in\mathbb{N}}$ *hereditarily complete*. Hereditary completeness can be understood as a weakest form of reconstruction of a vector f from its generalized Fourier series

$$\sum_{n\in\mathbb{N}} \langle f, w_n\rangle v_n,$$

since it is equivalent to the fact that each vector $f \in \mathcal{H}$ can be approximated by linear combinations of the partial sums of its Fourier series. Clearly, if the Fourier series with respect to the biorthogonal pair (v_n, w_n) admit a linear summation method, then the system $\{v_n\}$ is hereditarily complete.

Hereditarily complete systems are also known as strong M-bases or systems which admit *spectral synthesis* due to the relation of this property to the structure of invariant subspaces for certain classes of linear operators discovered by A. Markus [14]. Various geometrical

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aspects of abstract hereditary complete systems were considered in [9, 10] while in [1, 12] some interesting relations with operator algebras can be found.

1.1. **Exponential systems.** We are interested in the case when $\mathcal{H} = L^2(-\pi,\pi)$ and v_n is a system of exponentials, $v_n = e^{i\lambda_n t}$ for some set $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. R. Young [19] proved that in this case the biorthogonal system is always complete and there has been a number of papers establishing hereditary completeness and existence of a linear summation method for nonharmonic Fourier series under some additional hypothesis about the set Λ (see, e.g., [7] or [17, 18]). Nevertheless, in [3] an example was constructed which shows that in general hereditary completeness for exponential systems does not necessarily hold. This result was extended to other functional systems (reproducing kernels in de Branges spaces of entire functions, Gaussian Gabor systems) in [4, 5] (see also a survey paper [2]). It should be mentioned that the synthesis for exponential systems fails with one-dimensional defect only: each mixed system has codimension at most one [3].

The construction in [3] was ingenious, but it had very few "degrees of freedom", i.e., free parameters. Therefore, the structure of such examples remained rather mysterious. Our aim is to give a larger class of examples. Moreover, under some regularity conditions we are able to arrive to a certain qualitative characterization (*finite logarithmic length*) which we believe is intrinsic for the phenomenon of nonhereditary completeness of exponential systems.

It is well-known that if $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is a complete and minimal system, then the following canonical product converges in the sense of principal value, see, e.g., [13, Lecture 18, Theorem 4],

$$G(z) = p.v.\prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right) = \lim_{R\to\infty}\prod_{\lambda\in\Lambda,|\lambda|<R}\left(1-\frac{z}{\lambda}\right).$$

The function G is called *the generating function* of the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$. Numerous properties of exponential systems can be expressed in terms of G, see, e.g., [7, 16].

1.2. **Logarithmic length.** We are interested in the case when the function G is small outside some *lacunary* sequence of intervals $\{I_k\}_{k=1}^{\infty}$,

$$I_k = [\rho_k - d_k, \rho_k + d_k], \qquad 2\rho_k \le \rho_{k+1}, \quad 1 < d_k \le 0.1\rho_k. \tag{1.1}$$

We prove that under some additional restrictions the system of exponentials (reproducing kernels of PW_π) is hereditarily complete if and only if the total logarithmic length of these intervals is infinite, that is,

$$\sum_{k=1}^{\infty} \frac{d_k}{\rho_k} = \infty,$$

see Theorems 2.1, 4.1. Theorem 2.1 gives sufficient conditions for the failure of hereditary completeness, whereas Theorem 4.1 gives sufficient conditions for hereditary completeness; these two results are in a sense converse to each other.

To illustrate this we present one example which immediately follows from Theorems 2.1 and 4.1.

Example. *Let $\{\lambda_n\} = \Lambda$ be a locally dense real sequence, i.e., $\sup_n |\lambda_{n+1} - \lambda_n| < \infty$ such that the generating function G is of exponential type π, and*

$$\frac{|G(z)|}{\operatorname{dist}(z,\Lambda)} \asymp \max_k \frac{1}{\sqrt{|I_k|}(\operatorname{dist}^2(z, I_k) + 1)}, \qquad |\operatorname{Im} z| < 1, \tag{1.2}$$

where $\{I_k\}$ is a lacunary system of intervals satisfying (1.1) *and $d_k/\rho_k < 1/k$. Then the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is hereditarily complete if and only if $\sum_{k=1}^{\infty} \frac{d_k}{\rho_k} = \infty$.*

The existence of sequences Λ satisfying (1.2) can be deduced via standard atomization technique, see, e.g., [6]. From [13, Lecture 18, Theorem 4] it follows that the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is always complete and minimal in $L^2(-\pi, \pi)$.

In Section 3 we apply these results to give an example of a nonhereditarily complete (i.e., complete and minimal but not hereditarily complete) exponential system which partially answers a problem posed in [2]: which perturbations of integers can produce complete and minimal systems of exponentials which are not hereditarily complete? Let $\lambda_n \in \mathbb{R}$ and

$$\delta = \sup_{n\in\mathbb{Z}} |\lambda_n - n|. \tag{1.3}$$

By the results of Kadets and Ingham any sequence with $\delta < 1/4$ generates a Riesz basis of exponentials (see, e.g., [15, Part D, Chapter 4]). One can ask, however, for which δ any complete and minimal system $\{e^{i\lambda_n t}\}$ satisfying (1.3) is automatically hereditarily complete.

Question. *Find δ_{crit} which is the infimum of $\delta > 0$ such that there exists nonhereditarily complete system $\{e^{i\lambda_n t}\}$ with $|\lambda_n - n| < \delta$.*

The exact value of the *synthesis constant* δ_{crit} is not known. Theorem 3.1 shows that such δ_{crit} cannot exceed 1/2. Therefore,

$$\frac{1}{4} \le \delta_{crit} \le \frac{1}{2}.$$

1.3. **Paley–Wiener space.** The classical approach to the study of the properties of exponential systems is to consider Fourier transform $\mathcal{F}$ of our system: in this case the Hilbert space becomes the Paley–Wiener space $PW_\pi = \mathcal{F}L^2(-\pi,\pi)$ (the space of all entire functions of exponential type at most π which belong to $L^2(\mathbb{R})$) and the functions $e^{i\lambda_n t}$ are mapped to the reproducing kernels of PW_π (cardinal sines)

$$K_{\lambda_n}(z) = \frac{\sin \pi(z-\lambda_n)}{\pi(z-\lambda_n)}$$

corresponding to the points $\bar{\lambda}_n$. In the case when the exponential system is complete and minimal, its biorthogonal system $\{w_n\}$ is mapped to the functions $G_n \in PW_\pi$ which vanish on $\{\lambda_m, m \neq n\}$. It is easy to see that

$$G_n(z) = \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}$$

where G is the generating function of the system $\{e^{i\lambda_n t}\}$ (or, simply put, of the set Λ). The function G vanishes on Λ and has no other zeros, it is of exponential type π (with the diagram $[-\pi i, \pi i]$). Clearly, $G \notin PW_\pi$, however $G \in L^2\left(\mathbb{R}, \frac{dx}{1+x^2}\right)$. Thus, the spectral synthesis problem for exponentials is equivalent to the same problem for systems of reproducing kernels in the Paley–Wiener spaces. This equivalence will be frequently used.

Organization of the paper. In Section 2 we give a sufficient condition for an exponential system to be nonhereditarily complete (Theorem 2.1, the case of finite logarithmic length). In Section 3 we apply this result to give an explicit example of a nonhereditarily complete system of exponentials whose frequencies are sufficiently small perturbations of integers. In Section 4 we prove a converse result (Theorem 4.1) establishing hereditary completeness in the case of infinite logarithmic length. Finally, in Section 5, we show that for an incomplete mixed system its exponential part must be always sufficiently irregular and, in particular, cannot be a part of Riesz basis of exponentials with some additional regularity.

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2. Case of finite logarithmic length

In this section we will need the following assumptions on Λ and G:

(a) $\operatorname{dist}(\mathbb{Z}, \Lambda) > 0$;

(b) Λ is locally dense on $\mathbb{R}$, i.e., there exists some $C > 0$ such that any interval $I \subset \mathbb{R}$, $|I| \geq C$, contains at least one element of Λ;

(c) $|G(iy)| = o(e^{\pi|y|}), |y| \to \infty$.

Recall that the family $\{K_n\}_{n\in\mathbb{Z}} = \big\{\frac{\sin\pi(z-n)}{\pi(z-n)}\big\}_{n\in\mathbb{Z}}$ is an orthonormal basis in PW_π and, therefore, $\sum_{n\in\mathbb{Z}} |g(n)|^2 = \|g\|^2_{L^2(\mathbb{R})} < \infty$ for any $g \in PW_\pi$. We will often use this fact in what follows.

On the other hand, note that the series $\sum_{n\in\mathbb{Z}} |G(n)|^2$ diverges for the generating function G of any complete and minimal system satisfying condition (c). Indeed, otherwise we can find a function $F \in PW_\pi$ such that $F(n) = G(n)$, and by the Phragmén–Lindelöf principle we have $\frac{F(z)-G(z)}{\sin\pi z} \equiv 0$. This implies that $F \equiv G$, thus $G \in PW_\pi$, which contradicts completeness.

For our construction of nonhereditarily complete systems to work, we just need $G(n)$ to be an ℓ^2-sequence on the most part of $\mathbb{Z}\setminus\bigcup I_k$. But in some neighborhood of I_k we want $G(n)$ to be slightly better than ℓ^2 (see condition (ii) below).

Theorem 2.1. *Let G be the generating function of some complete and minimal system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ satisfying* (a)–(c). *Let I_k be a system of intervals of the form* (1.1) *such that $\operatorname{dist}(\rho_k, \mathbb{Z}) \ge \frac{1}{3}$ for each k and $G(\rho_k) \ne 0$. Put $g_k = \sum\limits_{n\in I_k} G^2(n)$, $s_k = \sqrt{d_k g_k \rho_k}$ and*

$$J_k = J_k^- \cup J_k^+ = [\rho_k - d_k - 2s_k, \rho_k - d_k - s_k] \cup [\rho_k + d_k + s_k, \rho_k + d_k + 2s_k].$$

Assume that the function G satisfies the following conditions:

(i) $\{G(n) :\ n \in \mathbb{Z}\setminus\bigcup\limits_k I_k\} \in \ell^2$;

(ii) $\sum\limits_k s_k \sum\limits_{n\in J_k} G^2(n) < \infty$;

(iii) $s_k \le 0.1\rho_k$;

(iv) $\sum\limits_k \dfrac{d_k}{\rho_k} < \infty$.

Then the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is not hereditarily complete.

Proof. First, we note that to prove that the system $\{v_n\} = \{e^{i\lambda_n t}\}$ is not hereditarily complete it is enough to find a partition $\mathbb{N} = A \cup B$ and two vectors $\eta, \nu \in L^2(-\pi,\pi)$ such that $\eta \perp \{v_n\}_{n\in A}$ and $\nu \perp \{w_n\}_{n\in B}$, but η and ν are not orthogonal. Indeed, if the system $\{v_n\}_{n\in A} \cup \{w_n\}_{n\in B}$ were complete, then the vector ν would lie in the span of $\{v_n\}_{n\in A}$ and η would lie in the span of $\{w_n\}_{n\in B}$ and so they would be orthogonal.

Since the Fourier transform $\mathcal{F}$ is a unitary operator from $L^2(-\pi,\pi)$ to PW_π, we can pass to the equivalent problem and search for

$$f = \mathcal{F}\eta, \qquad g = \mathcal{F}\nu$$

such that for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$ we have $f \perp \{K_\lambda\}_{\lambda \in \Lambda_1}$ and $g \perp \left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda\in\Lambda_2}$, but f and g are not orthogonal.

Step 1. Construction of f. We will construct a function f as a small perturbation of the function G so that they will share most of their zeros and therefore f is orthogonal to the most of K_{λ_n}, and then construct function g by the fixed point theorem so that it is orthogonal to the remaining functions G_n.

It follows from (1.1) and (iii) that all intervals I_k, J_k are pairwise disjoint. Also we can assume without loss of generality that $s_k \ge 3C$, where C is the constant from the local density condition (b), and so there exists at least one zero of G on each $J_K^\pm$. Indeed, if $s_k < 3C < 3Cd_k$, then $g_k \le 9C^2\frac{d_k}{\rho_k}$. Since the series $\sum\limits_{k=1}^{\infty}\frac{d_k}{\rho_k}$ converges, we can pass to a smaller collection of intervals I_k by throwing away the intervals with $s_k < 3C$ and condition (i) still will be satisfied. Later we will throw away some extra finite set of intervals – it also will not break our assumptions.

Let $t_k \in J_k$ be a zero of G whose choice will be specified later. Put

$$f(z) = G(z)m(z),$$

where

$$m(z) = \prod_{k=1}^{\infty}\frac{1-z/\rho_k}{1-z/t_k}.$$

It is easy to see from the lacunarity of ρ_k that this product converges locally uniformly on $\mathbb{C}\backslash\{t_k\}$, and since $G(t_k) = 0$ we conclude that the function f is entire.

Step 2. $f \in PW_\pi$. We will select two candidates $t_k^\pm \in \frac{1}{2}J_k^\pm$ for t_k (as always, by the half of the interval we mean the interval with the same center and twice smaller length). Subsequently we choose one of them in such a way that

$$0.001 \le \prod_{k=1}^{N}\frac{t_k}{\rho_k} \le 1000 \tag{2.1}$$

for all N (we can always do so by (iii)). Note that in this case we have

$$|m(z)| \asymp \left|\frac{z-\rho_k}{z-t_k}\right|, \qquad \frac{\rho_{k-1}+\rho_k}{2} \le |z| \le \frac{\rho_k+\rho_{k+1}}{2}. \tag{2.2}$$

Therefore the function f is of exponential type at most π and $|f(iy)| = o(e^{\pi|y|})$. Thus, to prove that $f \in PW_\pi$, it is enough to show that $\{f(n)\} \in \ell^2(\mathbb{Z})$.

Trivially, $\{f(n)\} \in \ell^2(\mathbb{Z}\setminus \bigcup_k (I_k \cup J_k))$ since $|m(n)| \lesssim 1$ for those n. For $n \in I_k$ we have $\frac{|x-\rho_k|}{|x-t_k|} \le \frac{d_k}{s_k}$ and, therefore,

$$\sum_{k=1}^{\infty} \sum_{n\in I_k} |f(n)|^2 \lesssim \sum_{k=1}^{\infty} \frac{d_k^2}{s_k^2} g_k = \sum_{k=1}^{\infty} \frac{d_k}{\rho_k} < \infty.$$

For $n \in J_k$ we have $|m(n)| \asymp \frac{s_k}{n-t_k}$. Let us divide $\frac{1}{2}J_k^{\pm}$ into $r_k \asymp s_k$ intervals of length C and choose one root of G from each of them (there is always at least one by our assumption (b)). Denote these roots in $\frac{1}{2}J_k^{+}$ by λ_j, $j = 1, \dots r_k$. Since $\operatorname{dist}(\Lambda, \mathbb{Z}) > 0$, we have

$$\sum_{j=1}^{r_k} \sum_{n\in J_k} \frac{s_k^2 G^2(n)}{(n-\lambda_j)^2} \lesssim s_k^2 \sum_{n\in J_k} G^2(n),$$

whence there exists λ_j such that

$$\sum_{n\in J_k} |f(n)|^2 \lesssim \sum_{n\in J_k} \frac{s_k^2 G^2(n)}{(n-\lambda_j)^2} \lesssim s_k \sum_{n\in J_k} G^2(n)$$

with constants in $\lesssim$ independent on k. We put $t_k^+ = \lambda_j$. Similarly, one can choose $t_k^- \in \frac{1}{2}J_k^-$. By condition (ii), $\sum_k \sum_{n\in J_k} |f(n)|^2 < \infty$ regardless of which of t_k^- or t_k^+ we choose to satisfy (2.1).

Step 3. Construction of the function g**.** Put $a_n = (-1)^n f(n)$. We are going to construct a real sequence $\{b_n\} \in \ell^2(\mathbb{Z})$ such that $\sum_{n\in\mathbb{Z}} a_n b_n \neq 0$ and the function

$$S(z) = \sum_{n\in\mathbb{Z}} \frac{a_n b_n}{z-n}$$

has zeros at each ρ_k.

Let us show that once such system $\{b_n\}$ is constructed, the functions f and

$$g(z) = \frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}} \frac{b_n}{z-n} = \sum_{n\in\mathbb{Z}} (-1)^n b_n K_n(z)$$

will achieve our goals. By construction, f is orthogonal to all K_{λ_n} except for t_k and $\langle f, g\rangle_{PW_\pi} = \sum_{n\in\mathbb{Z}} a_n b_n \neq 0$. It remains to prove that g is orthogonal to $\frac{G(z)}{z-t_k}$ for all k whence the mixed system

$$\{K_\lambda\}_{\lambda\in\Lambda_1} \cup \left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda\in\Lambda_2}$$

with $\Lambda_2 = \{t_k\}_{k\ge1}$, $\Lambda_1 = \Lambda\setminus\Lambda_2$, is incomplete. Thus, we need to prove that

$$\Big\langle \frac{G(z)}{z-t_k}, g\Big\rangle_{PW_\pi} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n b_n G(n)}{n-t_k} = 0. \tag{2.3}$$

We are going to prove that

$$\frac{G(z)S(z)}{f(z)} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n b_n G(n)}{z-n}.$$

If we do so, then substituting $z = t_k$ we get (2.3) (note that $f(t_k)\ne 0$). Consider the function

$$H(z) = \frac{G(z)S(z)}{f(z)} - \sum_{n\in\mathbb{Z}} \frac{(-1)^n b_n G(n)}{z-n}.$$

Trivial computation shows that its residues at $\mathbb{Z}$ are zero and since GS vanish in all zeros of f we conclude that H is entire. On the other hand, by comparing indicator diagrams of corresponding functions we see that H is of minimal exponential type. Finally, from $|m(iy)|\asymp 1, y\to\infty$ and the definitions of f and S we see that $|H(iy)| = o(1), |y|\to\infty$. Therefore, by the Phragmén–Lindelöf principle, $H\equiv 0$.

It remains to construct a sequence $\{b_n\}_{n\in\mathbb{Z}}$ with the desired properties.

Step 4. Construction of the sequence $\{b_n\}$. Put $b_0 = \frac{1}{a_0}$, $b_n = 0$ for $n\in\mathbb{Z}\setminus\bigcup I_k, n\ne 0$, and $b_n = c_k\frac{a_n}{\rho_k - n}$ for $n\in I_k$ for some c_k. We want to construct a sequence c_k such that $S(\rho_k) = 0$ for all k.

Consider the Banach space $\mathcal{B}$ of sequences $\{c_k\}_{k\ge1}$ with the norm $\|c\|_{\mathcal{B}} = \sup\limits_{k\ge1}\frac{|c_k|}{d_k}$, and consider the following operator on it:

$$\begin{aligned}(Tc)_k &= \Big(\sum_{n\in I_k}\frac{a_n^2}{(\rho_k-n)^2}\Big)^{-1}\Big(-\frac{1}{\rho_k} - \sum_{j\ne k}\sum_{n\in I_j}\frac{a_n b_n}{\rho_k - n}\Big)\\ &= \Big(\sum_{n\in I_k}\frac{a_n^2}{(\rho_k-n)^2}\Big)^{-1}\Big(-\frac{1}{\rho_k} - \sum_{j\ne k} c_j\sum_{n\in I_j}\frac{a_n^2}{(\rho_k - n)(\rho_j-n)}\Big).\end{aligned}$$

It is easy to see that if $\{c_k\}$ is a fixed point of this operator then $S(\rho_k) = 0$ for all k. Thus it remains to prove that T is contractive.

Recall that $(-1)^n a_n = f(n) = G(n)/m(n)$. We have

$$\sum_{n\in I_k}\frac{a_n^2}{(\rho_k-n)^2} = \sum_{n\in I_k}\frac{G^2(n)}{m^2(n)(\rho_k-n)^2} \asymp \sum_{n\in I_k}\frac{G^2(n)}{(t_k-n)^2} \asymp \frac{g_k}{s_k^2} \asymp \frac{1}{d_k\rho_k}. \tag{2.4}$$

On the other hand, for $j \neq k$, by (2.2),

$$\sum_{n\in I_j} \frac{a_n^2}{|(\rho_k - n)(\rho_j - n)|} \lesssim \frac{1}{\rho_k} \sum_{n\in I_j} \frac{a_n^2}{|\rho_j - n|} \lesssim \frac{1}{\rho_k} \sum_{n\in I_j} \frac{G^2(n)|\rho_j - n|}{|t_j - n|^2} \leq \frac{g_j d_j}{\rho_k s_j^2} = \frac{1}{\rho_k \rho_j}.$$

Clearly, the operator T is the sum of a constant vector and some linear operator. Moreover, by (2.4), this vector is in $\mathcal{B}$. So it remains to prove that the linear part of T is contractive. Denoting it by T_{lin} we have

$$\frac{|(T_{lin}c)_k|}{d_k} \lesssim \frac{d_k\rho_k}{d_k} \sum_{j\neq k} \frac{|c_j|}{\rho_k\rho_j} \leq \|c\|_{\mathcal{B}} \sum_j \frac{d_j}{\rho_j}.$$

As we mentioned in the beginning of the proof, we can safely throw away any finite number of intervals. Thus, we can assume that $\sum\limits_j \frac{d_j}{\rho_j}$ is as small as we like and so $\|T_{lin}c\|_{\mathcal{B}} \leq \frac{\|c\|_{\mathcal{B}}}{2}$. Therefore, T has a fixed point c.

It remains to prove that $\{b_n\} \in \ell^2(\mathbb{Z})$ and $\sum\limits_{n\in\mathbb{Z}} a_n b_n \neq 0$. We have, by (2.4),

$$\sum_{n\in\mathbb{Z}} |b_n|^2 = |b_0|^2 + \sum_k |c_k|^2 \sum_{n\in I_k} \frac{a_n^2}{(\rho_k - n)^2} \lesssim |b_0|^2 + \|c\|_{\mathcal{B}}^2 \sum_k \frac{d_k}{\rho_k} < \infty.$$

and, again using (2.4),

$$\sum_n a_n b_n = 1 + \sum_k c_k \sum_{n\in I_k} \frac{a_n^2}{\rho_k - n} \geq 1 - \|c\|_{\mathcal{B}} \sum_k d_k \sum_{n\in I_k} \frac{a_n^2}{|\rho_k - n|} \geq 1 - A\|c\|_{\mathcal{B}} \sum_k \frac{d_k}{\rho_k}$$

for some absolute constant A. We can once again throw away some intervals I_k so that the expression in the right-hand side be positive (note that since $\|T_{lin}\| \leq 1/2$, we can give uniform upper bound for $\|c\|_{\mathcal{B}}$ so it is enough to make $\sum\limits_k \frac{d_k}{\rho_k}$ sufficiently small). $\square$

Remark 2.2. We can replace condition (iii) by $s_k \leq C\rho_k$ – just replace s_k with εs_k for sufficiently small ε.

Remark 2.3. Note that in the proof of Theorem 2.1 we actually need only that we have a locally dense subset of the zeroes of G on $\cup J_k \subset \mathbb{R} \setminus \cup I_k$ which has positive distance from $\mathbb{Z}$.

We will now use the Theorem 2.1 to construct a completely explicit example of a function G which gives us a nonhereditarily complete system.

Theorem 2.4. *Let Λ be the set of zeros of the entire function*

$$G(x) = \cos \pi x \left(\frac{1}{x - 1/2} + \sum_{k=10}^{\infty} \left(\frac{1}{x - 2^k + 1/2} - \frac{1}{x - 2^k - 1/2} \right) \right). \tag{2.5}$$

Then the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ *is not hereditarily complete.*

Proof. It is easy to see that $G \notin PW_\pi$, but $G \in PW_\pi + zPW_\pi$. Moreover, $|G(z)| \gtrsim |z|^{-1}e^{\pi|\mathrm{Im} z|}$ for $|z| = 2^k + 2^{k-1}$ and $k \in \mathbb{N}$ is sufficiently large. Therefore, one cannot multiply G by an entire function and remain in PW_π. By [13, Lecture 18, Theorem 4] G is the generating function of some complete and minimal system of exponentials.

Put $\rho_k = 2^k$ and $d_k = \rho_k^{1/5}$. Conditions (a)–(c) and (iii), (iv) are easy to verify (for conditions (a), (b) see Remark 2.3). Let us now verify conditions (i) and (ii). We begin with the condition (i).

For $|n| \in [\frac{\rho_{k-1}+\rho_k}{2}, \frac{\rho_k+\rho_{k+1}}{2}]$ we have from (2.5)

$$|G(n)| \lesssim \frac{1}{|n-\rho_k|^2} + \frac{1}{2^k} + \frac{k}{4^k} \lesssim \frac{1}{|n-\rho_k|^2} + \frac{k}{2^k}. \tag{2.6}$$

Therefore

$$\sum_{n\notin\cup I_k} G^2(n) \lesssim \sum_{k=1}^{\infty}\left(\frac{1}{d_k^3} + \frac{k^2}{2^k}\right) < \infty.$$

To prove (ii) note that $g_k = \sum_{n\in I_k} G^2(n) \asymp 1$ and so $s_k \asymp \rho_k^{3/5}$. By the bound (2.6) we get

$$\sum_{k=1}^{\infty} s_k \sum_{n\in J_k} G^2(n) \lesssim \sum_{k=1}^{\infty} s_k\left(\frac{1}{s_k^3} + \frac{k^2 s_k}{4^k}\right) < \infty. \tag{2.7}$$

□

Note that the minus sign in (2.5) is absolutely essential to verify the condition (ii), similar cancelation can be observed implicitly in the example from [3]. Moreover, if we enumerate zeros of the function G in increasing order then for all $n \in \mathbb{Z}$ we would have (after shifting by $\frac{1}{2}$) $|\lambda_n - n| \le 1$ just as in [3] (one can check that all the roots of the function G are real). The drawback of these examples is that they do not use the full potential of the Theorem 2.1 – we could have chosen d_k as any positive power of ρ_k and the analysis would still work. In the following section we will construct a more advanced example which will break this barier and give us $|\lambda_n - n| < \frac{1}{2} + \varepsilon$.

3. Example of a nonhereditarily complete sequence

In this section we give another explicit example of a sequence Λ satisfying conditions of Theorem 2.1. Moreover, this system will be a sufficiently small perturbation of integers.

Theorem 3.1. *For any* $\delta > \frac{1}{2}$ *there exist* G *and* Λ *satisfying all conditions of Theorem 2.1 and such that* Λ *satisfies* (1.3).

Proof. We will start with the following auxiliary function. Let $\delta_0 \in [1/2, 1)$. Consider the function

$$G_0(z) = (z - 1/2) \prod_{n\in\mathbb{N}} \left(1 - \frac{z^2}{(n+\delta_0)^2}\right).$$

It is well known that

$$|G_0(x)| \asymp (|x|+1)^{-2\delta_0} \mathrm{dist}(x, \mathcal{Z}_{G_0})$$

(here and in what follows we denote by $\mathcal{Z}_F$ the zero set of an entire function F), whence $G_0 \in PW_\pi$ and, in particular, $|G_0(iy)| = o(e^{\pi|y|})$, $|y| \to \infty$.

Now let $\delta \in (\delta_0, 1)$. The idea is to shift a part of the zeros of G_0 which belong to some lacunary sequence of intervals $\Delta_k = [\rho_k - d_k, \rho_k + d_k]$ back to the origin. Let ρ_k be an arbitrary sequence such that $\rho_{k+1} > 2\rho_k > 0$ and choose d_k so that

$$d_k^{\delta_0+\delta} = \rho_k^{2\delta_0}. \tag{3.1}$$

Of course, we may assume that $d_k \le \rho_k/100$ for all k. Now put

$$G(z) = G_0(z) \prod_k \prod_{n\in\Delta_k} \frac{z-(n-\delta)}{z-(n+\delta_0)} = G_0(z) \prod_k \prod_{n\in\Delta_k} \left(1 + \frac{\delta+\delta_0}{z-(n+\delta_0)}\right).$$

Let $x \in \left[\frac{\rho_{k-1}+\rho_k}{2}, \frac{\rho_k+\rho_{k+1}}{2}\right]$. Then

$$\sum_{m\ne k} \sum_{n\in\Delta_m} \frac{1}{|x-(n+\delta_0)|} \lesssim \frac{1}{\rho_k} \sum_{m<k} d_m + \sum_{m>k} \frac{d_m}{\rho_m}.$$

Since $\sum_k \frac{d_k}{\rho_k} < \infty$, the corresponding product converges and, moreover,

$$\left| \prod_{m\ne k} \prod_{n\in\Delta_m} \left(1 + \frac{\delta+\delta_0}{x-(n+\delta_0)}\right)\right| \asymp 1, \qquad x \in \left[\frac{\rho_{k-1}+\rho_k}{2}, \frac{\rho_k+\rho_{k+1}}{2}\right].$$

Also, let $n_0 + \delta_0$ and $n_1 - \delta$, $n_0, n_1 \in \Delta_k$, be respectively the zeros of G_0 and G closest to x. Then

$$\begin{aligned}\log\left| \prod_{n\in\Delta_k} \left(1 + \frac{\delta+\delta_0}{x-(n+\delta_0)}\right)\right| &= \log \frac{|x-(n_1-\delta)|}{|x-(n_0+\delta_0)|} + \sum_{n\in\Delta_k, n\ne n_0,n_1} \frac{\delta+\delta_0}{x-(n+\delta_0)} + O(1) \\ &= \log \frac{|x-(n_1-\delta)|}{|x-(n_0+\delta_0)|} + (\delta+\delta_0) \ln \frac{|x-(\rho_k-d_k)|+1}{|x-(\rho_k+d_k)|+1} + O(1).\end{aligned}$$

Thus, for $x \in \left[\frac{\rho_{k-1}+\rho_k}{2}, \frac{\rho_k+\rho_{k+1}}{2}\right]$, we have

$$|G(x)| \asymp |G_0(x)| \cdot \frac{\mathrm{dist}(x, \mathcal{Z}_G)}{\mathrm{dist}(x, \mathcal{Z}_{G_0})} \cdot \left(\frac{|x-(\rho_k-d_k)|+1}{|x-(\rho_k+d_k)|+1}\right)^{\delta_0+\delta}. \tag{3.2}$$

We will show that G satisfies all conditions of Theorem 2.1 with intervals

$$I_k = [\rho_k - 2d_k, \rho_k + 2d_k]$$

(note that there is a small change in notations since the length of I_k is $4d_k$ in place of $2d_k$).

Obviously, G is an entire function of exponential type π (with the diagram $[-\pi i, \pi i]$) and $|G(iy)| = o(e^{\pi|y|})$, $y \to \infty$, since $|G(iy)| \asymp |G_0(iy)|$. Thus, all conditions (a)–(c) are satisfied.

Let us show that G is the generating function of some complete and minimal system of reproducing kernels in PW_π. It is clear from (3.2) that $|G(x)| \gtrsim (|x|+1)^{-K}\mathrm{dist}(x, \mathcal{Z}_G)$, $x \in \mathbb{R}$, for some $K > 0$. Thus, if $GT \in PW_\pi$ for some entire function T of zero exponential type, then T is a polynomial. However, by (3.1), for any $n \in [\rho_k + d_k, \rho_k + d_k + 2]$ we have $|G(n)| \asymp \rho_k^{-2\delta_0} d_k^{\delta_0+\delta} = 1$. Thus, $GT \notin PW_\pi$ for any polynomial T. By (3.2) we also have

$$|G(x)| \asymp (|x|+1)^{-2\delta_0}\mathrm{dist}(x, \mathcal{Z}_G), \qquad x \notin \cup_k I_k,$$

and so $\{G(n) : n \in \mathbb{Z} \setminus \cup_k I_k\} \in \ell^2$. Also,

$$1 \lesssim g_k = \sum_{n\in I_k} |G(n)|^2 \lesssim \frac{d_k^{\delta_0+\delta}}{\rho_k^{2\delta_0}} \sum_{n \in I_k} \frac{1}{(|n-(\rho_k+d_k)|+1)^{\delta_0+\delta}} \lesssim 1.$$

It follows that $\frac{G}{z-\lambda} \in PW_\pi$ for any $\lambda \in \mathcal{Z}_G$ and so G is the generating function of some complete and minimal system which satisfies (i) of Theorem 2.1.

Since $g_k = \sum_{n\in I_k} G^2(n) \asymp 1$ we have $s_k = \sqrt{2d_k g_k \rho_k} \asymp \sqrt{\rho_k d_k} < \rho_k/100$ for sufficiently large k. It remains to verify (ii). Let $J_k = [\rho_k - 2d_k - 2s_k, \rho_k - 2d_k - s_k] \cup [\rho_k + 2d_k + s_k, \rho_k + 2d_k + 2s_k]$. Then

$$\sum_k s_k \sum_{n\in J_k} G^2(n) \lesssim \sum_k s_k^2 \rho_k^{-4\delta_0} = \sum_k \rho_k^{1+\frac{2\delta_0}{\delta+\delta_0}-4\delta_0} < \sum_k \rho_k^{\frac{2\delta_0}{\delta+\delta_0}-1}.$$

Since $\delta > \delta_0$, we conclude that the above sum converges.

Note that the constants $\delta > \delta_0 \geq 1/2$ were arbitrary and so Theorem 3.1 is proved. It is clear from the last step of the proof that the condition $\delta_0 \geq 1/2$ is essential for this construction. □

4. Case of infinite logarithmic length

Throughout this section the symbols I_k, ρ_k, d_k and g_k will have the same meaning as in Section 2 (note that J_k will denote a different object).

For a statement which will be in a sense converse to Theorem 2.1 we need to somehow formalize the statement "G is big on the intervals I_k". This must include the following two ingredients: first of all we do not want, for some k, the sum $g_k = \sum\limits_{n\in I_k} |G(n)|^2$ to be significantly larger than the same sum for its neighbours because otherwise we will not "feel" them in G and the total logarithmic length may become finite; secondly, we do not want the main contribution to g_k be due to the values of G on a small part of I_k because in that case we will only "feel" this small part of I_k and logarithmic length may again become finite. These two parts corresponds to the assumptions (i) and (ii) of the following theorem.

Theorem 4.1. *Let G be the generating function of some complete and minimal system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ such that $\Lambda \subset \mathbb{C}$, $\Lambda \cap \mathbb{Z} = \emptyset$ and*

$$\sum_{n\in\mathbb{Z}} \frac{|G(n)|^2 + |G(n)|}{|n|+1} < \infty.$$

Assume that there exists a constant $C > 0$ such that for all k we have

(i) $\sum\limits_{n\notin I_k} \dfrac{|G(n)|^2}{|n-\rho_k|} \le C\dfrac{g_k}{d_k}$;

(ii) $\sqrt{\dfrac{g_k}{d_k}} \le C|G(x)|, \quad x \in I_k$;

(iii) $\sum\limits_k \dfrac{d_k}{\rho_k} = \infty$.

Then the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is hereditarily complete.

Remark 4.2. Formally, Theorems 2.1 and 4.1 apply to different classes of functions G, since condition (ii) of Theorem 4.1 is incompatible with the existence of any roots of G on I_k (condition (b) in Theorem 2.1). However, for the Theorem 2.1 we do not need condition (b) in full, but only its weaker form indicated in the Remark 2.3 which is consistent with the assumptions of Theorem 4.1. Alternatively, we can weaken condition (b) and assume that for every $x \in \mathbb{R}$ there exists $\lambda \in \Lambda$ with $|x-\lambda| < C$. For example, a locally dense subset Λ of $(\mathbb{R}+i) \cup (\mathbb{R}-i)$ is compatible with the condition (ii). Thus, there exists a class of functions G for which hereditary completeness depends only on finiteness of the logarithmic length of the intervals I_k.

For the proof of Theorem 4.1 we need the following proposition.

Proposition 4.3. *Let $t_k \in \mathbb{R}$ be an increasing sequence (one-sided or two-sided) which is separated, i.e., $t_{k+1} - t_k \ge \delta$ for some $\delta > 0$, and let $\mu_k > 0$, $\{\mu_k\} \in \ell^1(\mathbb{Z})$. Let $\{\gamma_n\}$ be an* *increasing separated sequence such that* $\mathrm{dist}(\{\gamma_n\},\{t_k\}) = d > 0$ *and* $\sum_n \frac{1}{\gamma_n} = \infty$. *Then, if for the function*

$$f(z) = \sum_k \frac{\mu_k}{z - t_k}$$

we have $\{f(\gamma_n)\} \in \ell^1(\mathbb{N})$, *then* $\mu_k \equiv 0$.

Proof of Proposition 4.3. Let us assume that $\sup\{t_k\} = \infty$. Otherwise, $f(x) \asymp x^{-1}$, $x \to \infty$, and the statement is trivial.

Clearly, f has a unique zero s_k in each interval (t_k, t_{k+1}). It is known and not difficult to show (see [3, Proposition 5.4]) that these zeros in a sense approach the "outer" ends of the intervals (t_k, t_{k+1}), namely,

$$\sum_{t_k>0} \frac{t_{k+1} - s_k}{s_k} < \infty, \qquad \sum_{t_k<0} \frac{s_k - t_k}{|s_k|} < \infty.$$

Put $\tilde{f}(z) = f(z) - \frac{\mu_0}{2(z-t_0)}$ and denote by $\tilde{s}_k$ the unique zero of $\tilde{f}$ in (t_k, t_{k+1}). Then we also have

$$\sum_{t_k>0} \frac{t_{k+1} - \tilde{s}_k}{\tilde{s}_k} < \infty. \tag{4.1}$$

Consider first those γ_n which belong to $\cup_k E_k$, where $E_k = (\tilde{s}_k, t_{k+1})$. Since $\{\gamma_n\}$ is separated and $\mathrm{dist}(\{\gamma_n\},\{t_k\}) = d > 0$, we conclude that for a fixed k the number of points $\gamma_n \in E_k$ does not exceed $C(t_{k+1} - \tilde{s}_k)$ for some $C > 0$ independent on k. In particular, the interval E_k contains no points γ_n if $t_{k+1} - \tilde{s}_k < d$. Hence,

$$\sum_{t_k>0} \sum_{\gamma_n \in E_k} \frac{1}{\gamma_n} \lesssim \sum_{t_k>0} \frac{t_{k+1} - \tilde{s}_k}{\tilde{s}_k} < \infty.$$

Thus, we may assume without loss of generality that $\gamma_n \notin \cup_{t_k>0}(t_k, \tilde{s}_k)$ for all n. Since $\tilde{f}$ is decreasing on each interval (t_k, t_{k+1}) we have $\tilde{f}(\gamma_n) \ge f(\tilde{s}_k) = 0$ whenever $\gamma_n \in (t_k, t_{k+1})$, and so

$$f(\gamma_n) = \frac{\mu_0}{2(\gamma_n - t_0)} + \tilde{f}(\gamma_n) \gtrsim \frac{1}{\gamma_n}.$$

This contradicts the assumption that $\{f(\gamma_n)\} \in \ell^1$. □

In the proof of Theorem 4.1 we will need some auxiliary Hilbert space of meromorphic functions in $\mathbb{C} \setminus \mathbb{Z}$. Put

$$\mathcal{H} = \left\{ \sum_{n\in\mathbb{Z}} \frac{b_n G(n)}{z-n} : \{b_n\} \in \ell^2 \right\}, \qquad \left\langle \sum_{n\in\mathbb{Z}} \frac{b_n G(n)}{z-n}, \sum_{n\in\mathbb{Z}} \frac{c_n G(n)}{z-n} \right\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} b_n \overline{c_n}.$$

Recall that $\Lambda \cap \mathbb{Z} = \emptyset$. Therefore, $G(n) \neq 0$, $n \in \mathbb{Z}$, and the inner product in $\mathcal{H}$ is well defined. It is easy to see that $\mathcal{H}$ is a Hilbert space whose reproducing kernel at $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ is given by $\mathcal{K}_\lambda(z) = \sum\limits_{n\in\mathbb{Z}} \frac{|G(n)|^2}{(z-n)(\bar{\lambda}-n)}$ and, in particular,

$$\|\mathcal{K}_\lambda\|_{\mathcal{H}}^2 = \sum_{n\in\mathbb{Z}} \frac{|G(n)|^2}{|\lambda - n|^2}. \tag{4.2}$$

Next, consider the function

$$M(t) = \sum_n \frac{|G(n)|^2}{n-t}. \tag{4.3}$$

Each interval $(n, n+1)$ contains exactly one root of the equation $M(t) = 0$. Pick those roots which lie in $J_k = \frac{1}{2} I_k = [\rho_k - \frac{d_k}{2}, \rho_k + \frac{d_k}{2}]$ for some k and denote the resulting sequence by $\{t_n\}$.

Note that, for real $z \neq w$, we have

$$\langle \mathcal{K}_w, \mathcal{K}_z \rangle = \mathcal{K}_w(z) = \frac{M(z) - M(w)}{z - w}.$$

Since $M(t_n) = 0$ for all n, the functions $\mathcal{K}_{t_n}$ form an orthogonal system in $\mathcal{H}$.

Lemma 4.4. *There exist sets $N_k \subset \mathbb{N} \cap J_k$ and a constant $C > 0$ (independent of k) such that $|N_k| \geq d_k/2$ and*

$$\|\mathcal{K}_{n+\frac{1}{2}}\|_{\mathcal{H}}^2 \leq C \frac{g_k}{d_k}, \qquad \Big|M\Big(n + \frac{1}{2}\Big)\Big| \leq C \frac{g_k}{d_k}, \qquad n \in N_k. \tag{4.4}$$

Proof. Throughout the proof, symbol C will denote different constants independent on k. We have

$$\sum_{n\in J_k} \|\mathcal{K}_{n+\frac{1}{2}}\|_{\mathcal{H}}^2 = \sum_{n\in J_k} \sum_{m\in\mathbb{Z}} \frac{|G(m)|^2}{(n-m+\frac{1}{2})^2}.$$

By condition (i) of Theorem 4.1,

$$\sum_{n\in J_k} \sum_{m\notin I_k} \frac{|G(m)|^2}{(n-m+\frac{1}{2})^2} \leq C g_k,$$

while

$$\sum_{n\in J_k} \sum_{m\in I_k} \frac{|G(m)|^2}{(n-m+\frac{1}{2})^2} = \sum_{m\in I_k} |G(m)|^2 \sum_{n\in J_k} \frac{1}{(n-m+\frac{1}{2})^2} \leq C g_k.$$

We conclude that $\sum\limits_{n\in J_k} \|\mathcal{K}_{n+\frac{1}{2}}\|_{\mathcal{H}}^2 \leq C g_k$, whence, for any $\varepsilon > 0$, we have $\|\mathcal{K}_{n+\frac{1}{2}}\|_{\mathcal{H}}^2 \leq C\varepsilon^{-1} \frac{g_k}{d_k}$ for $n \in N_k$, where N_k is a subset of $\mathbb{N} \cap J_k$ with $|N_k| \geq (1-\varepsilon) d_k$.

Estimate for $|M(n+\frac{1}{2})|$ is more delicate. First, we split it into the sums over I_k and $\mathbb{Z}\backslash I_k$

$$M\left(n+\frac{1}{2}\right)=\sum_{m\in I_k}\frac{|G(m)|^2}{m-n-\frac{1}{2}}+\sum_{m\notin I_k}\frac{|G(m)|^2}{m-n-\frac{1}{2}}=S_1(n)+S_2(n). \tag{4.5}$$

For $n\in J_k$ as above we can deduce from the assumption (i) that $|S_2(n)|\le C\frac{g_k}{d_k}$. Therefore it remains to estimate $S_1(n)$.

For a sequence $x=(x_m)\in\ell^1(\mathbb{Z})$ consider the operator T defined as

$$(Tx)_n=\sum_{m\in\mathbb{Z}}\frac{x_m}{m-n-\frac{1}{2}}. \tag{4.6}$$

It is well known that the discrete Hilbert transform T satisfies a weak-type $(1,1)$ bound

$$|\{n\in\mathbb{Z}:|(Tx)_n|>\tau\}|\le C\frac{||x||_{\ell^1}}{\tau},\qquad \tau>0, \tag{4.7}$$

where C is an absolute constant.

Applying this bound to the sequence $x_n=|G(n)|^2\chi_{I_k}(n)$ with $\tau=100Cg_k/d_k$ we get

$$|\{n\in\mathbb{Z}:|S_1(n)|>100Cg_k/d_k\}|\le\frac{d_k}{100}. \tag{4.8}$$

Therefore for $n\in J_k$ and outside of this exceptional set we get the desired estimate. Since $|J_k|-\frac{1}{100}d_k=\frac{99}{100}d_k>\frac{d_k}{2}$ the lemma is proved. □

Lemma 4.5. *Let N_k be the sets from Lemma 4.4. Then there exists $\varepsilon>0$ such that for any k and for any $n\in N_k$ the zero t of the function M in the interval $(n,n+1)$ belongs to $(n+\varepsilon,n+1-\varepsilon)$.*

Proof. Assume that $M(n+\frac{1}{2})>0$. We have

$$M'(t)=\sum_{m\in\mathbb{Z}}\frac{|G(m)|^2}{(t-m)^2}\ge\frac{|G(n)|^2}{(t-n)^2}.$$

Since $|G(n)|^2\ge C\frac{g_k}{d_k}$ and $M\big(n+\frac{1}{2}\big)\le C\frac{g_k}{d_k}$ there exists $\varepsilon>0$ (depending on C but not on k and n) such that $M(n+\varepsilon)\le M\big(n+\frac{1}{2}\big)-|G(n)|^2\int_{n+\varepsilon}^{n+\frac{1}{2}}\frac{dt}{(t-n)^2}<0$. Thus, for some $t\in(n+\varepsilon,n+\frac{1}{2})$ we have $M(t)=0$.

In the case when $M(n+\frac{1}{2})<0$, one shows by the same argument that the root of M will lie in $(n+\frac{1}{2},n+1-\varepsilon)$. □

Proof of Theorem 4.1. Assume that the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ is not hereditarily complete. Then there exists a nonzero function $f \in PW_\pi$,

$$f(z) = \frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n \bar{a}_n}{z-n},$$

and a set $\Lambda_1 \subset \Lambda$ such that f is orthogonal to K_λ for all $\lambda \in \Lambda_1$, and to $G_\lambda = \frac{G(z)}{G'(\lambda)(z-\lambda)}$ for all $\lambda \in \Lambda_2 = \Lambda \backslash \Lambda_1$. Then

$$\Big\langle \frac{G(z)}{z-\lambda}, f \Big\rangle_{PW_\pi} = \sum_{n\in\mathbb{Z}} \frac{a_n G(n)}{n-\lambda} = 0, \qquad \lambda \in \Lambda_2.$$

Our first step is to prove the following equality:

$$f(z) \sum_n \frac{a_n G(n)}{z-n} = G(z) \sum \frac{|a_n|^2}{z-n}. \tag{4.9}$$

Indeed, note that $f(n) = \bar{a}_n$ and so the residues at $\mathbb{Z}$ coincide. Since the left-hand side of (4.9) vanishes at $\lambda \in \Lambda$, there is an entire function T such that

$$f(z) \sum_n \frac{a_n G(n)}{z-n} - G(z) \sum \frac{|a_n|^2}{z-n} = G(z)T(z). \tag{4.10}$$

It is clear that T is of zero exponential type. Recall that $\frac{G}{z-\lambda} \in PW_\pi$ for any zero λ of G. Hence, the left-hand side of (4.10) is in the class $PW_\pi + zPW_\pi$. If T has at least one zero ζ, we conclude that $G \cdot \frac{T}{z-\zeta} \in PW_\pi$, a contradiction to the fact that G is the generating function of a complete sequence of reproducing kernels. Thus, $T = c \in \mathbb{C}$ and

$$f(z) \sum_n \frac{a_n G(n)}{z-n} = G(z) \Big(c + \sum \frac{a_n^2}{z-n} \Big).$$

It remains to exclude the case when $c \neq 0$. Put $E_k = J_k \setminus \cup_{n\in\mathbb{Z}} (n - 1/10, n + 1/10)$. Then, for $x \in E_k$,

$$\Big| \sum_n \frac{a_n G(n)}{x-n} \Big|^2 \le \sum_{n\notin I_k} \frac{|G(n)|^2}{|x-n|} \sum_{n\notin I_k} \frac{|a_n|^2}{|x-n|} + \sum_{n\in I_k} |G(n)|^2 \sum_{n\in I_k} \frac{|a_n|^2}{(x-n)^2} \lesssim g_k$$

by (i). Thus,

$$\int_{E_k} \Big| f(x) \sum_n \frac{a_n G(n)}{x-n} \Big|^2 dx \lesssim g_k \int_{E_k} |f(x)|^2 dx = o(g_k),$$

as $k \to \infty$. Note also that $\sum\limits_n \frac{a_n^2}{x-n} \to 0$ when $x \to \infty$, $x \in E_k$. Therefore, if $c \neq 0$, then

$$\int_{E_k} \Big| G(x) \Big(c + \sum \frac{a_n^2}{x-n} \Big) \Big|^2 dx \gtrsim \int_{E_k} |G(x)|^2 dx \gtrsim \frac{g_k}{d_k} \cdot d_k = g_k$$

by (ii). This contradiction shows that $c = 0$ and (4.9) is proved.

Now, put

$$H(z) = \sum_n \frac{a_n G(n)}{z-n}, \qquad h(z) = \sum_n \frac{a_n^2}{z-n}.$$

Since $H \in \mathcal{H}$ and $\{\mathcal{K}_{t_n}\}$ is an orthogonal system in $\mathcal{H}$, we have $\left\{\frac{H(t_n)}{\|\mathcal{K}_{t_n}\|_{\mathcal{H}}}\right\} \in \ell^2$. Also, $\{f(t_n)\} \in \ell^2$ by the classical Plancherel–Pólya inequality. Equality (4.9) yields

$$h(t_n)\frac{G(t_n)}{\|\mathcal{K}_{t_n}\|_{\mathcal{H}}} = \frac{H(t_n)}{\|\mathcal{K}_{t_n}\|_{\mathcal{H}}} f(t_n) \in \ell^1.$$

Denote by $\{\tilde{t}_j\}$ the sequence of all zeros of M which belong to the intervals $(n, n+1)$ for $n \in N_k$, where the sets N_k are constructed in Lemma 4.4. By Lemma 4.5 we have $\mathrm{dist}(\{\tilde{t}_j\}, \mathbb{Z}) \ge \varepsilon > 0$. Hence, if $\tilde{t}_j \in (n, n+1)$, then $\|\mathcal{K}_{\tilde{t}_j}\|^2_{\mathcal{H}}/\|\mathcal{K}_{n+\frac12}\|^2_{\mathcal{H}}$ is bounded from above and from below by some positive constants depending on ε, since this is true for all the summands in their definitions (see (4.2)). By Lemma 4.4 and (iii) we have

$$|G(\tilde{t}_j)| \ge C\sqrt{\frac{g_k}{d_k}}, \qquad \|\mathcal{K}_{\tilde{t}_j}\|_{\mathcal{H}} \le C\sqrt{\frac{g_k}{d_k}}$$

for any $\tilde{t}_j \in J_k$. We conclude that $h(\tilde{t}_j) \in \ell^1$. Also, since $|N_k| \ge d_k/2$ we have

$$\sum_j \frac{1}{\tilde{t}_j} = \sum_k \sum_{\tilde{t}_j \in J_k} \frac{1}{\tilde{t}_j} \asymp \sum_k \frac{|N_k|}{\rho_k} \asymp \sum_k \frac{d_k}{\rho_k} = \infty.$$

Applying Proposition 4.3 to $h(z) = \sum\limits_{n\in\mathbb{Z}:\ a_n \ne 0} \frac{|a_n|^2}{z-n}$ and $\tilde{t}_j$ in place of γ_n we conclude that $a_n \equiv 0$ for all n. This contradiction proves the theorem. $\square$

Remark 4.6. Theorem 4.1 can be extended to a wider class of generating functions. Namely, we can allow some functions G with divergent sum $\sum_{n\in\mathbb{Z}} \frac{G^2(n)}{n}$ with the following modification of the above method: we consider the function $M(x) = \sum_{n\in\mathbb{Z}} G^2(n)\left(\frac{1}{x-n} + \frac{n}{n^2+1}\right)$ and on the interval I_k we will consider the points $\lambda_n \in (n, n+1)$ which are the solution to the equation

$$M(x) = \sum_{n \le \rho_k + d_k} \frac{nG^2(n)}{n^2+1}. \tag{4.11}$$

Although normalized reproducing kernels at the points λ_n are not an orthonormal sequence anymore, they come in big groups of pairwise orthogonal kernels corresponding to one interval I_k. Thus, one can still prove that they form a Riesz sequence by examining the Gram matrix and Riesz sequence is sufficient for our proof. This, in particular, allows us to consider $|G| \asymp 1$ on I_k and (with slight modifications) even non-lacunary case $|G| \asymp 1$ on $\mathbb{R}$. We leave the details to the interested reader.

5. A remark on exponential parts of incomplete mixed system

Assume that $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is a nonhereditary complete system of exponentials and the system $\{K_\lambda\}_{\lambda\in\Lambda_1}\cup\{\frac{G(z)}{z-\lambda}\}_{\lambda\in\Lambda_2}$ is incomplete for some partition $\Lambda = \Lambda_1\cup\Lambda_2$. We have seen in Section 3 that Λ can be a sufficiently small perturbation of integers. However, the system must also have a certain irregularity. We will show that the exponential part $\{e^{i\lambda t}\}_{\lambda\in\Lambda_1}$ cannot be a part of a Riesz basis of exponentials with some additional properties. E.g., Λ_1 cannot be a subset of $\mathbb{Z}$.

Theorem 5.1. *Let $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ be a complete and minimal system in $L^2(-\pi,\pi)$, $\Lambda\subset\mathbb{R}$. Assume that $\Lambda_1\subset\Lambda$ and there exists $\tilde\Lambda_2\subset\mathbb{R}$ such that $\{e^{i\lambda t}\}_{\lambda\in\Lambda_1\cup\tilde\Lambda_2}$ is a Riesz basis in $L^2(-\pi,\pi)$ whose generating function F satisfies $|F'(\zeta)|\lesssim 1$, $\zeta\in\mathcal{Z}_F$. Let $\{w_\lambda\}_{\lambda\in\Lambda}$ be the system biorthogonal to $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$. Then the system*

$$\{e^{i\lambda t}\}_{\lambda\in\Lambda_1}\cup\{w_\lambda\}_{\lambda\in\Lambda_2},$$

where $\Lambda_2=\Lambda\setminus\Lambda_1$, is complete in $L^2(-\pi,\pi)$.

Proof. Since Riesz bases of exponentials are stable under small perturbations (even in Euclidean metric) we can perturb slightly $\tilde\Lambda_2$ so that $\Lambda_2\cap\tilde\Lambda_2=\emptyset$ and still $|F'(\zeta)|\lesssim 1$, $\zeta\in\mathcal{Z}_F$. We also assume in what follows that Λ_2 and $\tilde\Lambda_2$ are infinite (otherwise it is well known that the corresponding mixed system is complete).

We pass again to the equivalent formulation for the Paley–Wiener space PW_π. Put $\mathcal{Z}=\mathcal{Z}_F=\Lambda_1\cup\tilde\Lambda_2$. Since $\{K_\zeta\}_{\zeta\in\mathcal{Z}}$ is a Riesz basis, its biorthogonal system $\big\{\frac{F(z)}{F'(\zeta)(z-\zeta)}\big\}_{\zeta\in\mathcal{Z}}$ also is a Riesz basis. Consider the Hilbert space

$$\mathcal{H}=\Big\{f(z)=\sum_{\zeta\in\mathcal{Z}}c_\zeta\frac{F(z)}{F'(\zeta)(z-\zeta)}:(c_\zeta)\in\ell^2\Big\}$$

with the norm $\|f\|_{\mathcal{H}}=\|(c_\zeta)\|_{\ell^2}$. Then $\mathcal{H}$ coincides with PW_π with equivalence of norms and the system $\big\{\frac{F(z)}{F'(\zeta)(z-\zeta)}\big\}_{\zeta\in\mathcal{Z}}$ is an orthonormal basis of reproducing kernels in $\mathcal{H}$ (note that $c_\zeta=f(\zeta)$, $f\in\mathcal{H}$).

Assume now that the system

$$\{K_\lambda\}_{\lambda\in\Lambda_1}\cup\{G_\lambda\}_{\lambda\in\Lambda_2},\qquad G_\lambda(z)=\frac{G(z)}{G'(\lambda)(z-\lambda)},$$

is not complete in PW_π. This means that there exists a nonzero function $f \in PW_\pi$ such that $f|_{\Lambda_1} = 0$ and

$$f \notin \overline{\mathrm{Span}}_{PW_\pi}\big\{G_\lambda:\ \lambda \in \Lambda_2\big\} = \overline{\mathrm{Span}}_{\mathcal{H}}\big\{G_\lambda:\ \lambda \in \Lambda_2\big\}.$$

Note that the system $\{G_\lambda\}_{\lambda\in\Lambda}$ is biorthogonal also to the system of reproducing kernels $\{\tilde{K}_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{H}$. Thus, there exists a function $f \in \mathcal{H} = PW_\pi$ such that

$$f \perp \{\tilde{K}_\lambda\}_{\lambda\in\Lambda_1} \cup \{G_\lambda\}_{\lambda\in\Lambda_2}$$

with respect to the inner product of $\mathcal{H}$. Let $f(z) = \sum_{\zeta\in\mathcal{Z}} c_\zeta \frac{F(z)}{F'(\zeta)(z-\zeta)}$. Recall that $\mathcal{Z} = \Lambda_1 \cup \tilde{\Lambda}_2$ whence $c_\zeta = f(\zeta) = 0$, $\zeta \in \Lambda_1$. Also,

$$\big\langle G_\lambda, f\big\rangle_{\mathcal{H}} = \frac{1}{G'(\lambda)} \sum_{\zeta\in\tilde{\Lambda}_2} \frac{G(\zeta)\overline{c}_\zeta}{\zeta - \lambda} = 0, \qquad \lambda \in \Lambda_2, \tag{5.1}$$

and so

$$F(z) \sum_{\zeta\in\mathcal{Z}} \frac{G(\zeta)\overline{c}_\zeta}{z - \zeta} = G(z)T(z)$$

for some entire function T. Comparing the indicator diagrams of the left and right hand sides we see that T of zero exponential type. Since $\big\{\frac{G(\zeta)}{|\zeta|+1}\big\} \in \ell^2$, the left-hand side of the above equality belongs to $PW_\pi + zPW_\pi$. If T has at least one zero, say z_0, then $G\frac{T}{z-z_0} \in PW_\pi$, a contradiction to the fact that G is the generating function of a complete sequence. Thus, $T = c$ for some $c \in \mathbb{C}$. Comparing the values at $\zeta \in \tilde{\Lambda}_2$ we get $F'(\zeta)G(\zeta)\overline{c}_\zeta = cG(\zeta)$ and so $|c_\zeta| = |F'(\zeta)|^{-1}|c| \gtrsim 1$, $\zeta \in \tilde{\Lambda}_2$, a contradiction. □

Condition $|F'(\zeta)| \lesssim 1$, $\zeta \in \mathcal{Z}_F$, is essential and the result is no longer true as soon as this condition is not satisfied. Note that the following example gives yet another method to construct nonhereditarily complete systems of exponentials. As an example of a Riesz basis of exponentials with growing $|F'(\zeta)|$ one can take the system corresponding to $\mathcal{Z} = \{n - \delta\,\mathrm{sign}\, n\}_{n\in\mathbb{Z}}$ with $0 < \delta < \frac{1}{4}$, which is a Riesz basis in $L^2(-\pi, \pi)$, e.g., by the Kadets 1/4 theorem [15, Part D, Chapter 4].

Example. Let F be the generating function of an exponential Riesz basis $\{e^{i\lambda t}\}_{\lambda\in\mathcal{Z}}$ with $\mathcal{Z} \subset \mathbb{R}$ and $\sup_{\zeta\in\mathcal{Z}} |F'(\zeta)| = \infty$. We show that there exists a partition $\mathcal{Z} = \Lambda_1 \cup \tilde{\Lambda}_2$ and a set $\Lambda = \Lambda_1 \cup \Lambda_2$ such that $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is a complete and minimal system in $L^2(-\pi, \pi)$, but the mixed system $\{e^{i\lambda t}\}_{\lambda\in\Lambda_1} \cup \{w_\lambda\}_{\lambda\in\Lambda_2}$ is incomplete.

Choose $\tilde{\Lambda}_2 \subset \mathcal{Z}$, $\tilde{\Lambda}_2 = \{\zeta_{n_k}\}_{k\in\mathbb{N}}$, such that $\{|F'(\zeta)|^{-1}\}_{\zeta\in\tilde{\Lambda}_2} \in \ell^2$. Then

$$f(z) = \sum_{\zeta\in\tilde{\Lambda}_2} \frac{F(z)}{|F'(\zeta)|^2(z - \zeta)} \in PW_\pi$$

(in this expansion $c_\zeta = 1/\overline{F'(\zeta)}$, $\zeta \in \tilde{\Lambda}_2$, and $c_\zeta = 0$ otherwise). Without loss of generality let $\{\zeta_{n_k}\}_{k\in\mathbb{N}}$ be positive, increasing and $\sum_k \zeta_{n_k}^{-1} < \infty$. We construct Λ_2 as a perturbation of $\tilde{\Lambda}_2$:

$$\Lambda_2 = \{\zeta_{n_k} + \alpha_k\}_{k\in\mathbb{N}}, \qquad 0 \le \alpha_k \le \zeta_{n_{k+1}} - \zeta_{n_k}.$$

Let G be the generating function of $\Lambda = \Lambda_1 \cup \Lambda_2$, i.e.,

$$G(z) = F(z) \prod_k \frac{1 - z/(\zeta_{n_k} + \alpha_k)}{1 - z/\zeta_{n_k}}.$$

Note that if $\alpha_k = 0$, then $G = F$, while for $\alpha_k = \zeta_{n_{k+1}} - \zeta_{n_k}$ we have $G(z) = F(z)/(z - \zeta_{n_1}) \in PW_\pi$. Therefore, by a certain "continuity" argument one can always find perturbations α_k such that

$$\lim_{y\to\infty} \left| \frac{G(iy)}{F(iy)} \right| = 0, \qquad \sum_{\zeta\in\mathcal{Z}} |G(\zeta)|^2 = \infty, \qquad \sum_{\zeta\in\mathcal{Z}} \frac{|G(\zeta)|^2}{|\zeta|^2 + 1} < \infty.$$

We omit an elementary but tedious proof of this fact. From the two last conditions one easily deduces that G is the generating function of a complete and minimal system, while the first one guarantees that the interpolation formula

$$\frac{G(z)}{F(z)} = \sum_{\zeta\in\mathcal{Z}} \frac{G(\zeta)}{F'(\zeta)(z-\zeta)} = \sum_{\zeta\in\tilde{\Lambda}_2} \frac{G(\zeta)}{F'(\zeta)(z-\zeta)} \tag{5.2}$$

holds (the difference of the left and right hand sides is an entire function of zero exponential type who tends to zero along the imaginary axis).

Now consider the Hilbert space $\mathcal{H}$ constructed from F as in the proof of Theorem 5.1. By construction, f is orthogonal to reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda_1}$ in PW_π and to $\{\tilde{K}_\lambda\}_{\lambda\in\Lambda_1}$ in $\mathcal{H}$. It remains to show that $f \perp \{G_\lambda\}_{\lambda\in\Lambda_2}$ in $\mathcal{H}$, which follows immediately from (5.2) and (5.1).

References

[1] S. Argyros, M. Lambrou, W. E. Longstaff, *Atomic Boolean subspace lattices and applications to the theory of bases*, Mem. Amer. Math. Soc. **91** (1991), no. 445, 94 pp.

[2] A. Baranov, Yu. Belov, *Spectral synthesis in Hilbert spaces of entire functions*, European Congress of Mathematics, 2018, 203–218.

[3] A. Baranov, Yu. Belov, A. Borichev, *Hereditary completeness for systems of exponentials and reproducing kernels*, Adv. Math. **235** (2013) 525–554.

[4] A. Baranov, Yu. Belov, A. Borichev, *Spectral synthesis in de Branges spaces*, Geom. Funct. Anal. **25** (2015) 2, 417–452.

[5] A. Baranov, Yu. Belov, A. Borichev, *Summability properties of Gabor expansions*, J. Funct. Anal. **274** (2018), 9, 2532–2552.

[6] Y.Belov, *Model functions with an almost prescribed modulus*, (Russian) Algebra i Analiz **20** (2008), 2, 3–18; English transl.: in St. Petersburg Math. J. **20** (2009), 2, 163–174.

[7] Yu. Belov, Yu. Lyubarskii, *On summation of nonharmonic Fourier series*, Constr. Approx. **43** (2016), 2, 291–309.

[8] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice Hall, Engelwood Cliffs, 1968

[9] L. Dovbysh, N. Nikolski, *Two methods for avoiding hereditary completeness*, Zap. Nauchn. Semin. LOMI **65** (1976), 183–188; English transl.: J. Soviet Math. **16** (1981), 3, 1175–1179.

[10] L. Dovbysh, N. Nikolski, V. Sudakov, *How good can a nonhereditary family be?* Zap. Nauchn. Semin. LOMI **73** (1977), 52–69; English transl.: J. Soviet Math. **34** (1986), 6, 2050–2060.

[11] G. Gubreev, A. Kovalenko, *Tests for completeness of root subspaces of a differentiation operator with abstract boundary conditions*, Mat. Zametki, **30** (1981) 4, 543–552 (in Russian); English transl.: Math. Notes **30** (1981) 4, 765–770.

[12] A. Katavolos, M. Lambrou, M. Papadakis, *On some algebras diagonalized by M-bases of ℓ^2*, Integr. Equat. Oper. Theory **17** (1993), 1, 68–94.

[13] B. Ya. Levin *Lectures on Entire Functions* Transl. Math. Monogr. Vol. 150, AMS, Providence, RI, 1996.

[14] A. Markus, *The problem of spectral synthesis for operators with point spectrum*, Izv. Akad. Nauk SSSR **34** (1970), 3, 662–688 (Russian); English transl.: Math. USSR-Izv. **4** (1970), 3, 670–696.

[15] N. Nikolski, *Operators, Functions, and Systems: an Easy Reading. Vol. 2*, Math. Surveys Monogr., Vol. 93, AMS, Providence, RI, 2002.

[16] B. Pavlov, *Basic property of a system of exponentials and the condition of Muckenhoupt*, Soviet Math. Dokl., **21**(1) (1979), 37–40.

[17] A. M. Sedletskii, *On the summability and convergence of nonharmonic Fourier series*, Izv. RAN. Ser. Mat. **64** (2000), 3, 151–168; English transl.: in Izv. Math. **64** (2000), 3, 583–600.

[18] A. M. Sedletskii, *Nonharmonic analysis*, Functional analysis, Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz. **96**, VINITI, Moscow, 2006, 106–211; English transl.: in J. Math. Sci. (N.Y.), **116** (2003), 5, 3551–3619.

[19] R. Young, *On complete biorthogonal system*, Proc. Amer. Math. Soc. **83** (1981) 3, 537–540.

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