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Analysis of the Wave Equation on the Disk

Bachelor's thesis in Mathematical Sciences
Supervisor: Stine Marie Berge
June 2024

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Contents

1 Introduction

Musical analogy has been responsible for lots of general math concepts, such as the wave equation and spectral decomposition. In this bachelor thesis I focus on how the theory of unbounded operators help understand the wave equation on a circular membrane. Also vice versa, the modeling of sound from a drum head is used as an introductory step into the theory of unbounded operators.
The vibrations of a string is something we all have studied. Calculus (including differential equations) and concepts of eigenvalues and eigenvectors in finite dimensional linear algebra is well understood.
Concepts in Lebesgue integration theory is introduced in the preliminaries but a complete understanding requires a background outside of this thesis. Hilbert space notation is also covered.

With the purpose of foreshadowing what is to come, this thesis starts off by outlining the solution process of the wave equation with a single spatial variable. Following this is four sections of material.

- Preliminaries and Notation
  This section is mostly directed towards setting up the function spaces in order to find a solution space. Some known definitions are stated later in the text.

- Sobolev Space
  Sobolev spaces are widely used in the theory of partial differential equations in particular when the wave equations is combined with Dirichlet conditions. The section is also dedicated to the notion of weak derivatives.

- Unbounded Operators
  Whereas operator theory should be understood to a certain extent by the reader, unbounded operators is likely a new area of study. While theory of unbounded operators is quite vast this section only scratches the surface of such theory.

- Solving the Wave Equation
  The last section is allocated to solving the wave equation on a disk with Dirichlet boundary conditions. This is done in a general sense as the focus is not on a particular solution but to show a connection of theory and application.

1.1 Simple Sketch

In this simple sketch one may imagine hearing a guitar string.

- The wave equation is derived from Newtonian mechanics. This derivation is outside the scope of this thesis. The wave equation is as follows.

$$\begin{cases} \Delta u(x,t) - u_{tt}(x,t) = 0 & \Omega \times [0,\infty) \\ u|_{\partial\Omega} = 0 \\ u(x,0) = f(x) \end{cases}$$

Where Δ is the Laplacian, i.e. the second derivative. (Note that the one dimensional Laplacian is used in examples before defining this operator properly.) The domain Ω is a string (i.e. line segment) and $u(x,t)$ is the height of the string at each point in time.
The strings end points are fixed. This is denoted as $u|_{\partial\Omega} = 0$.
To find a particular solution one needs to know the initial displacement $f(x)$, though this does not affect the fundamental frequency (i.e. lowest frequency).

- The solution process uses the separation of variables assumption ($u(x,t) = H(x)T(t)$). This means we are in a separable hilbert space which you will find stated in definition 3.8.
  Two ordinary differential equations are obtained where the focus is on the spacial equation $\Delta H_{\lambda_n}(x) = -\lambda_n^2 H_{\lambda_n}(x)$. This has the general solution $H(x) = \sum_{n=1}^{\infty} a_n H_{\lambda_n}(x)$. We denote λ_n^2 as the eigenvalues and λ_n as the angular frequency we hear (up to a constant). Since we assumed solutions on the separable form $u(x,t) = H(x)T(t)$ it is to be expected that the eigenfunctions form an orthonormal basis, as in theorem 3.9. It is $a_1 H_{\lambda_1}(x)$ the wave with the biggest length, which has the fundamental frequency and tone. The angular frequency turns out to be inversely proportional with string length. Halving the string increases pitch by octave. The drum is not as well behaved as the string (because of Bessel functions, 2.1) but the eigenvalues of a particular single variable system(i.e. a fixed string being played) are spaced evenly as scalar multiples of the fundamental frequency.

2 Preliminaries and Notation

2.1 Bessel Functions

The Bessel functions are a type of special functions often described using power series. They appear in the Helmholtz equation in the radial part of the solution. We will only work with Bessel functions of the first kind with integer order. The periodicity of the wave equation will assert m (in the equation below) to be a positive integer.

These Bessel functions are stated in [3, p. 189]. Bessel functions

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m}}{2^{2m+n} m!(n-m)!}$$

converge for all x and are solutions to the Bessel equation shown in [3, p. 187].

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \text{ for an integer } n$$

where y is a function of a variable x, as shown in [3, p. 187].
Let $x = \lambda_{nm}r$, then the change of variables gives

$$(\lambda_{nm}r)^2y'' + \lambda_{nm}ry' + ((\lambda_{nm}r)^2 - n^2)y = 0 \text{ for an integer } n.$$

Which of course then have the solutions

$$J_n(\lambda_{nm}r) = (\lambda_{nm}r)^n \sum_{m=0}^{\infty} \frac{(-1)^m(\lambda_{nm}r)^{2m}}{2^{2m+n}m!(n-m)!}.$$

2.2 Hilbert Space

Hilbert spaces provide a rigorous mathematical concept in which many of today's differential equations are modeled. Proving that all the spaces used are in fact Hilbert spaces is out of the scope of this bachelor thesis. Nonetheless Hilbert spaces are of vital importance and an introduction is in order.

2.2.1 Inner Product Space

A complex inner product space is a complex vector space V equipped with an inner product, i.e. a linear functional $< \cdot, \cdot >: V \times V \to \mathbb{C}$ which obeys the following properties;

- The inner product is equal to its complex conjugate when the elements are interchanged

$$< x, y >= \overline{< y, x >} \text{ for all } x, y \in V.$$

- It is positive definite

$$0 \leq \ < x, x > \ \text{ for all } x \in V \text{ and } \ < x, x >= 0 \text{ only when } x = 0.$$

- There must be linearity in the first argument;

$$< ax + by, z >= a < x, z > +b < y, z > .$$

2.2.2 Complete Vector Spaces

The inner product space is called a Hilbert space if it is complete. Completeness of an inner product space is defined as all Cauchy sequences converging (in norm) within the space.Thus we firstly introduce the notions of norm and Cauchy sequences.

Definition 2.1. Norm

A norm (of a vector) is a linear functional denoted as $||x||$ and is defined on a complex or real vector space X and has the real numbers as its range

$$\|x\| : X \to \mathbb{R}.$$

The norm is defined by the following properties which hold for all $a \in \mathbb{C}$ or $\mathbb{R}$ and x, y in a vector space over the field of complex or real numbers.

- satisfying the triangle inequality

$$||x + y|| \leq ||x|| + ||y||$$

- being positive definite

$$0 \leq ||x|| \text{ and only if } x = 0 \text{ we have } ||x|| = 0$$

- having homogeneity

$$||ax|| = |a|||x||.$$

Definition 2.2. Cauchy

A Cauchy sequence $\{x_i\}_{i=0}^{\infty}$ is a sequence satisfying

$$\limsup_{i,j} ||x_i - x_j|| = 0.$$

Alternatively one can define that a sequence is Cauchy if

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}; ||x_i - x_j|| < \epsilon \text{ for all } i, j > N.$$

The inner products admits a natural norm

$$||x|| = \sqrt{< x, x >}$$

by taking the square root the inner product.

A Hilbert space can now be defined in terms of the previous definitions.

Definition 2.3. Hilbert space

A Hilbert space X is an inner product space where all its Cauchy sequences converge (i.e. has a limit contained in the space). For all Cauchy sequences $\{x_i\} \in X$ there exists $x \in X$ such that $\lim_{i \to \infty}\{x_i\} = x$. This means that

$$\lim_{i \to \infty} ||x_i - x|| = 0$$

Lets first see if n-tuples over $\mathbb{C}^n$ with the dot product is a Hilbert space.

Example 2.4. First lets pick two arbitrary vectors with elements from $\mathbb{C}^n$ $z = \{z_i\}_{i=1}^n$ and $w = \{w_j\}_{j=1}^n$. Its know that these vectors indeed form a vector space over $\mathbb{C}^n$. We begin by checking if the complex dot, $< z, w >= \sum_{k=1}^n z_i \overline{w_j}$ where $j = i$ (i.e. component-wise multiplication), is in fact an inner product:

1. $< z, w >= \sum_{k=1}^{n} z_i \overline{w_j} = \sum_{k=1}^{n} \overline{z_i \overline{w_j}} =< w, z >$ (i.e. conjugate symmetry).

2. When a complex number is multiplied with its own conjugate we indeed get positive real number, hence $< z, z >= \{z_i \overline{z_i}\} = \{(a_i^2 + b_i^2)_i\}$for $z = a + bi$ and $< z, z >= 0$ only when $z = 0$ (i.e. positive definite)

3. Let $c = c_r + ic_t$ be a complex number.
Then $< cz, w > \{cz_i \overline{w_j}\} = c\{z_i \overline{w_i}\} = c < z, w >$ thus the complex dot product is linear in the first slot.
And of course a factor in the second slot of the dot product would be factorized as the complex conjugate, i.e. $< z, cw >= \overline{c} < z, w >$.

Now we will check if this inner product space is complete. For every Cauchy sequence of n-tuples,

$$\{z_k\}_{k=1}^{\infty} = \{(\{z_i\}_{i=1}^{n})_k\}_{k=1}^{\infty}$$

we have that

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} : ||z_{k_n} - z_{k_m}|| < \epsilon\ \forall n, m > N.$$

This is understood component-wise for every element in each z_k, meaning that for each z_k we have n Cauchy sequences where

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}; ||z_{i_n} - z_{i_m}|| < \epsilon\ \forall n, m > N$$

for each z_i.
So we have that as k becomes increasingly large each component in z_k become arbitrarily close to one another in norm ($\sqrt{< z_{i_n} - z_{i_m}, z_{i_n} - z_{i_m} >}$). Lets think about the limit $lim_{k \to \infty} z_k$. Well, we have that $\|z_{k_n} - z_{k_m}\| \leq \epsilon_l$ for all $n, m > N \in \mathbb{N}$ for a fixed ϵ_l. This means that every Cauchy sequence is bounded since each element in z_k is in $\mathbb{C}$, the $\epsilon_l \in \mathbb{R}$ and

$$\|z_{k_m}\| = \|z_{k_m} + z_{k_N} - z_{k_N}\| \leq \|z_{k_N}\| + \|z_{k_m} - z_{k_N}\| \leq \|z_{k_N}\| + \epsilon_l$$

for all $m > N \in \mathbb{N}$.

The supremum of $\|z_{k_m}\| = \|z_{k_N}\| + \epsilon_l$ can only decrease as $N \to \infty$ (and $m > N$) as is the case for each component as well. We also have the infimum of $\|z_{k_N}\| = \|z_{k_m}\| - \epsilon_l$ can only increase as $N \to \infty$. Since this is true for all $\epsilon > 0$ (for example $\{\epsilon_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$) we can conclude that

$$lim_{N \to \infty} \{\inf \|z_{k_m}\| + \epsilon_l\} = lim_{N \to \infty} \{\sup \|z_{k_N}\| + \epsilon_l\}$$

$\iff$

$$lim_{N \to \infty} \{\inf \|z_{k_N}\|\} = lim_{N \to \infty} \{\sup \|z_{k_m}\|\} := L.$$

So the limit $lim_{k \to \infty} z_k = L = \{L_i\}_{i=1}^{n}$ exists within our vector space $\mathbb{C}^n$. So then both component wise $\lim_{i \to \infty} \|z_{n_i} - L_i\| = 0$ and as a whole vector $\lim_{n \to \infty} \|z_n - L\| = 0$. Thus the conclusion is that the vectors of n-tuples over $\mathbb{C}^n$ with the complex dot product is a Hilbert space.

The next example is more relevant to this thesis and will prove that the $L^2(\Omega)$ from definition 2.12 in the next subsection with $p = 2$ and some domain Ω is complete.

Example 2.5. Assuming that we have some Ω where the inner product

$$< f, g >= \int_\Omega f\overline{g}$$

is valid. We move on to checking completeness. Again, we have that a sequence of functions $\{f_n(\Omega)\}_{n=1}^{\infty}$ is a Cauchy sequence if

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N};\, ||f_n - f_m|| < \epsilon\ \forall n, m > N\ .$$

This means that as N becomes increasingly large, each $f_n(\omega)$ in f_n and $f_m(\omega)$ in f_m for $\omega \in \Omega$ become arbitrarily close. The norm is $\sqrt{< f_n - f_m, f_n - f_m > = \int_\Omega f_n - f_m\overline{f_n - f_m}}$

$$= \int_\Omega |f_n - f_m|^2 < \epsilon \text{ for } n, m > N \in \mathbb{N}$$

.
Well, we have that $\|f_n - f_m\| \leq \epsilon_f$ for all $n, m > N \in \mathbb{N}$ for a fixed ϵ_f. This means that every Cauchy sequence is bounded since

$$\|f_m\| = \|f_m + f_N - f_N\| \leq \|f_N\| + \|f_m - f_N\| \leq \|f_N\| + \epsilon_f$$

for all $m > N \in \mathbb{N}$. Boundedness means the limit $lim_{k\to\infty} f_k = L_f$ either within our function space $L^2(\Omega)$ or as a limit. The supremum of $\|f_m\| = \|f_N\| + \epsilon_f$ can only decrease as $N \to \infty$ (and $m > N$) analogous to the previous example. The infimum of $\|f_N\| = \|f_m\| - \epsilon_f$ can only increase as $N \to \infty$. Since this is true for all $\epsilon > 0$ we can conclude that

$$lim_{N\to\infty}\{\inf\|f_m\| + \epsilon_f\} = lim_{N\to\infty}\{\sup\|f_N\| + \epsilon_f\}$$

$\iff$

$$lim_{N\to\infty}\{\inf\|f_N\|\} = lim_{N\to\infty}\{\sup\|f_m\|\} := L_f.$$

So the limit $lim_{k\to\infty} f_k = L_f(\Omega)$ exists within our vector space $L^2(\Omega)$. Thus $\lim_{n\to\infty} \|f_n - L_f\| = 0$. Thus the conclusion is that the function space (i.e. vector space of functions) $L^2(\Omega)$ with the $L^2(\Omega)$ norm is a Hilbert space.

2.3 Lebesgue Space

The solution space for the wave equation on a drum head turns out to be a subspace of a Lebesgue space. The introduction to Lebesgue space will be quite brief and mostly for introducing notation and basic concepts. This space revolves around Lebesgue integration, which is an alternative to Riemann integration. Lebesgue integration was introduced into mathematics for a number of reasons, among others an expansion of what can be considered integrable.
Firstly, the notion of a measure space is introduced for the purpose of defining Lebesgue space.

2.3.1 Measure Space

A measure space is defined in [1, p. 42] and is a triplet (X, A, μ) containing a set X, a σ-algebra A of X, and a measure μ. This text will not define the notion of a set X but the other elements of this triplet are as follows.
The σ in σ-algebra alludes to the property that the algebras are closed under *countable* infinite unions. This means that if we have a countable sequence$\{A_i\}_{i=1}^{\infty}$ where all $A_i \in A$ then their union $\cup_{i=1}^{\infty} A_i \in$ as well.

Definition 2.6. σ-algebra, [1, p. 26]
Let A denote a σ-algebra, which is defined as a set of subsets. This set of subsets must contain the empty set, complements of sets and infinite unions of sets in the σ -algebra. Meaning A contains

- $$\emptyset \in A$$
- $$B \in A \text{ implies } C \in A \text{ where } C \text{ is the complement of } B \text{ in } A$$
- $$\bigcup_{i=1}^{\infty} A_i \in A$$
  for a countable sequence $\{A_i\}_{i=1}^{\infty} \subset A$ where all A_i is in A.

Example 2.7. An example of a σ-algebra A of the set $X = \{1, 2, 3\}$ is

$$A = \{\{1, 2, 3\}, \{\emptyset\}, \{1\}, \{2, 3\}, \{2\}, \{1, 3\}, \{3\}\ \{1, 2\}\}.$$

The last element in the triplet μ is defined as follows.

Definition 2.8. Measure, [1, p. 41]

A measure is a function

$$\mu : A \to [0, \infty]$$

which is countably additive, meaning that if we have a countable set of disjoint subsets $\{A_i\}_{i=1}^{n} \subset A$ then we have that

$$\mu(\bigcup_{i=1}^{n} A_i) = \Sigma_{i=1}^{n} \mu(A_i) \text{ where either } n \in \mathbb{N} \text{ or } n = \infty.$$

A measure function has the following properties;

$$\mu(\emptyset) = 0$$

$$\mu(B) \leq \mu(C)$$

$$\mu(C/B) = \mu(C) - \mu(B) \text{ when } \mu(B) \neq \infty.$$

These properties hold for $B \subseteq C$ when $B, C \in A$.

Example 2.9. A measure on the measurable space (X, A) from example 2.7.
Define a measure $\nu : A \to [0, \infty]$ such that
$\nu(A_i) = \sum_j (e_j)$
where e_j is an element in A_i and the $\leq j \leq J$ where J is the number of elements in set A_i.
Thus the properties of measure hold. Note that we also have that

$$\nu(\bigcup_{i=1}^{n} A_i) = \Sigma_{i=1}^{n} \nu(A_i) \text{ where either } n \in \mathbb{N} \text{ or } n = \infty.$$

There are many possible choices of measure for which definition2.8 hold, one of the most common is the Lebesgue measure. To describe the Lebesgue measure one uses the notion of what is called an outer measure.

Definition 2.10. Lebesgue outer measure
A Lebesgue outer measure of a set is the infimum of a measure of an infinite union of open countable sets covering that set. That is to say, the Lebesgue outer measure of a set $S \subset \mathbb{R}^n$ is
$m^*(S) = |S| = \Sigma_{i=1}^{\infty} |B_i|$ where $\cup_{i=1}^{\infty} B_i$ is the infimum of sets $\{B_j\}$ such that $S \subseteq B_j$. Here $|B_i|$ denotes the elementary measure of the cube $B_i \subset \mathbb{R}^n$ (i.e. for $R \subset \mathbb{R}^3$ this is the common volume or R).

Definition 2.11. Lebesgue measure, [1, p. 51]
Lebesgue measure on $(\mathbb{R}^n, \Sigma)$ is the Lebesgue outer measure of Lebesgue measurable subsets or $\mathbb{R}^n$.
The σ-algebra Σ is that of Lebesgue measurable subsets. This means that the Lebesgue measure of $S \subset \mathbb{R}^n$ exists if there is an open set $Z \subset \mathbb{R}^n$ such that

$$m^*(Z \setminus S) \leq \epsilon$$

This should hold for every $\epsilon > 0$.
In that case the Lebesgue measure is $\mu(S) = m^*(S)$.

In this text, if not stated explicitly, the measure being used will be the Lebesgue measure. The Borel algebra (the smallest σ-algebra containing all open subsets) will be used as the $\sigma-$algebra in the measure spaces considered forthwith and throughout this text.

With this notion of a measure space and with the Lebesgue integral (which one can read about in [1] and [2]) one can finally define the Lebesgue spaces.

Definition 2.12. L^P norm and space
Denote a measure space by (X, A, μ) , then the L^p norm of a (μ-)measurable function $f : X \to \mathbb{C}$ is defined as

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

for each $p \in [1, \infty)$.

The L^p spaces are defined as ;

$$L^p(X, d\mu) := \{f \text{ measurable } X \to \mathbb{C} : \|f\|_p < \infty\}.$$

The definition of measurable functions is found in [1, p. 31]. Essentially these are functions of which their inverse image of Borel subsets is in A. If $f : X \to Y$ then $f^-1(C_i) \in A$ if $C_i \subset C$ and A and C are the Borel algebra of their respective sets x and Y.

In order to see how this Lebesgue theory can be exemplified, look at one of the simplest functions.

Example 2.13. A function in L^2
Claim: $f(x) = |x|$ where $x \in [-a, b]$ in $L^2[-a, b]$.

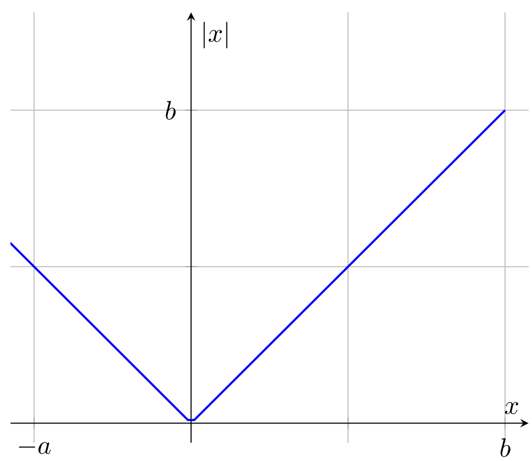

Figure 1: The graph of $f(x) = |x|$.

Let $X = [a, b]$ represent a set (also let all letters to used be positive, arbitrary members of the real number line bounded above by b and below by $-a$).

Let the σ-algebra, A, be the collection of Borel subsets, S_i.
Take the Lebesgue measure to be $\mu(j, k) = dist(j, k) = k - (j)$, $k > j$ where (j, k) is an interval in X

This defines a measure space (X, A, μ).
Since the pre-image of each Borel set is in A, we have that $f(x) = |x|$ is a measurable function. For example $(0, b)$ is indeed in A when looking at $|x|$ when $x \in (-a, b)$. Then we have that $f(x) = |x|$ is a measurable function with respect to (X, A, μ).

Next we check that $|x|$ is in $L^2[-a,b]$ by integrating with respect to a suitable disjoint collection of subsets of A

$$\int_{(-a,b)} |x|^2 d\mu = \frac{b^3 - a^3}{3} < \infty$$

meaning that $||x||_{L^2(a,b)} < \infty$.
Thus we have found a measure space (X, A, μ), that $f(x)$ is measurable here and that $||x||_{L^2(a,b)} < \infty$. So $f(x) = |x| \in L^2(-a,b)$.

The (strong)derivative of $f(x)$ is not well defined at $x = 0$. We see this in the calculation $|x|' = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{(|x+h|-|x|)(|x+h|+|x|)}{h(|x+h|+|x|)} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h(|x+h|+|x|)} = \lim_{h\to 0} \frac{2xh+h^2}{h(|x+h|+|x|)} = \frac{2x}{(|x|+|x|)} = \frac{x}{|x|} = \pm 1, x \neq 0$.

While this derivative is completely valid (for $x \neq 0$), there are many functions in L^2 with many such points where the directional derivatives don't all match up. To lessen the need to specify each non differentiable point in an otherwise differentiable function, the L^2 space is defined such that we take this into account. Functions are seen as equivalent if they differ only on sets of measure zero. For example the set where the derivative of $f'(x)$ is not defined has measure zero, hence the derivative exist *almost everywhere*(a.e.). Also $f(x)$ is differentiable a.e.

2.3.2 Locally Integrable Functions

Some functions, for example $f(x) = \frac{1}{x}$, are "locally integrable".

Definition 2.14. Locally absolutely integrable functions
The space of locally absolutely integrable functions is denoted as $L^1_{\text{loc}}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. This means that if $u \in L^1_{\text{loc}}(\Omega)$ then $|u|$ is integrable over compact subsets of Ω [2, page 19].

Example 2.15. Take $\Omega = (-1,0)\cup(0,1)$ then $f(x) = \frac{1}{x} \in L^1_{\text{loc}}(\Omega)$ but not in $L^1(\Omega)$ because of the singularity at $x = 0$.

3 Sobolev Space

If we where to differentiate $f(x) = |x|$ from example 2.13 twice, the result, $2\delta_0$ would not be in $L^2(-a,b)$ (where δ is the Dirac delta function). Sobolev spaces are subspaces of Lebesgue spaces. One such Sobolev space would contain all functions in $L^2(-a,b)$ which also have first and second order derivatives in $L^2(-a,b)$. Then $f(x) = |x| \in L^2([a,b])$ would not be in this particular Sobolev space.

3.1 Weak Derivatives

The notion of differentiabillity has previously been that a function must be differentiable at every point in order to be deemed differentiable on the larger set. For use in Lebesgue space a more general definition is called for (one with expanded domain). Note that the weak derivative (also an operator) uses differential operators within its definition. For the definition of an operator see definition 4.1.

Definition 3.1. Weak derivative
The multi-index $\alpha \in (\mathbb{N}_0)^n$ yields a shorthand for the differential operator

$$D_\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = u^{(\alpha)},$$

of order $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If both u and $u^{(\alpha)} \in L^1_{\text{loc}}(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$, that is to say, locally absolutely integrable. And if

$$\int_\Omega u^{(\alpha)} \psi \ \ d^n x = (-1)^{|\alpha|} \int_\Omega u D_\alpha \psi \ \ d^n x$$

for all $\psi \in C_0^\infty(\Omega)$, then we say that u admits a weak derivative $D_\alpha u := u^{(\alpha)}$. (The subscript on C_0^∞ indicates compact support inside the interior of Ω as denoted in [2, p. 19].)

Lets use a couple of examples to familiarize ourselves with this new notion of the derivative.

Example 3.2. Finding the weak derivative
Let $\alpha = (2, 2)$ and $u(x, y) = x^2 y^3$ on $\Omega = (1, 2)^2$ then

$$D_\alpha u = u^{(\alpha)} = \frac{\partial^4 u}{\partial x^2 \partial y^2} = 12y.$$

Since u and $u^{(\alpha)} \in L^1_{\text{loc}}(\Omega)$ we can calculate the weak derivative. Lets pick $\psi \in C_0^\infty((1, 2)^2)$ and use integration by parts four times, we get

$$\int_1^2 \int_1^2 (12y)\psi dx dy = \int_1^2 \int_1^2 x^2 y^3 D_{(2,2)} \psi dx dy.$$

Example 3.3. Exemplifying the weak derivative

It is simple to check that if $u(x) = 2x$ and $\psi \in C_0^\infty(-a, a)$ for $a \in \mathbb{R}$ then the weak definition coincide with the strong definition

$$\int_{-a}^a \psi(x) 2 \mathrm{d}x = [\psi(x)2]_{-a}^a - \int_{-a}^a 2x D\psi(x) \mathrm{d}x = - \int_{-a}^a 2x D\psi(x) \mathrm{d}x.$$

Thus we see that $(2x)' = 2$ on $\Omega = (-a, a)$ and

$$\int_{-a}^a 2\psi(x) \mathrm{d}x = (-1)^1 \int_{-a}^a 2x D\psi(x) \mathrm{d}x$$

.

Example 3.4. When analyzing $f(x) = |x|$ on $(-1, 2)$ we need to split the domain into $\Omega = (-1, 0) \cap (0, 2)$, then both $sgn(x)$ and $f(x)$ are in$L^1_{\text{loc}}(\Omega)$. We pick a $\phi \in C_0^\infty(\Omega)$ and and check that $\int_\Omega sgn(x)\phi dx = -\int_\Omega |x|\phi' dx$. The way to do this is by integration by parts
$\int_\Omega sgn(x)\phi dx = \int_{(-1,0)} -1\phi' dx + \int_{(0,1)} 1\phi' dx = -\int_{(-1,0)} -x'\phi dx - \int_{(0,1)} x'\phi dx = -\int_\Omega |x|\phi' dx$ where the boundary terms implicitly vanished.

After one knows the definition of the weak derivative one may go on to defining Sobolev space.

3.1.1 Defining Sobolev Spaces

The Sobolev spaces are encountered in [2, p. 21] and centered on L^2 although there are definitions for the more general L^p spaces.

Definition 3.5. Sobolev space
For an open set $\Omega \subset R^n$ the Sobolev spaces are defined for $m \in \mathbb{N}$

$$H^m(\Omega) := \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \le m \right\}.$$

One can choose the Sobolev inner products

$$< u, v >_{H^m} := \sum_{|\alpha| \le m} < D^\alpha u, D^\alpha v >_{L^2}$$

and Sobolev norms

$$\|u\|_{H^m} := \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^2_{L^2} \right)^{1/2}$$

to go with these spaces.

In order to clarify how this notation acts out the example below shows how the norm of H^2 is written as a combination of L^2 norms. It is of course domain dependent but there is no need to exemplify that.

Example 3.6. To clarify the notation of the norm defined above we write it out for m=2

$$H^2(\Omega) := \left\{ u(x, y) \in L^2(\Omega) : D^\alpha u(x, y) \in L^2(\Omega) \text{ for } |\alpha| \le 2 \right\}.$$

$$\|u\|_{H^2} = \sqrt{< u, u > + < D^{(1,0)}u, D^{(0,1)}u > + < D^{(2,0)}u, D^{(0,2)}u > + < D^{(1,1)}u, D^{(1,1)}u >}$$

Note that every possible way of obtaining $|\alpha| \le 2$ is used.

Now is the time to distinguish Sobolev space and Lebesgue space with a concrete example. We will show that $f(x) = \sqrt{|x|}e^{-x^2}$ is in $L^2(\mathbb{R})$ but not in $H^1(\mathbb{R})$.

Example 3.7. Claim; $f(x) = \sqrt{|x|}e^{-x^2}$ is in $L^2(\mathbb{R})$ but not in $H^1(\mathbb{R})$.

Since the pre-image of $f(K)$ where K is an arbitrary set in the Borel-σ-algebra of $\mathbb{R}$ is itself in a σ-algebra of $\mathbb{R}$ chosen appropriately,it is measurable. Since $f(x)$ is measurable we can indeed compute the $L^2(\mathbb{R})$ norm $\int |x|\, e^{-2x^2}\mathrm{d}x$. Let us simplify with a change of variable $u(x) = 2x^2$ then $du = 4x\mathrm{d}x$.
The norm is now simply

$$||f(x)|| = (\int |x|\, e^{-2x^2}\mathrm{d}x)^{\frac{1}{2}}$$

$$= (2\int_0^\infty |x|e^{-2x^2}\mathrm{d}x)^{\frac{1}{2}} = (2\int_0^\infty \frac{1}{4}e^{-u}du)^{\frac{1}{2}}$$

$$(= \frac{1}{2})^{\frac{1}{2}} < \infty.$$

Thus we conclude that $f(x) \in L^2(\mathbb{R})$

The weak derivative of the function

$$g(x) = \frac{xe^{-x^2}}{|x|2\sqrt{|x|})} - 2xe^{-x^2}\sqrt{|x|}$$

still measurable.When we square this we get

$$g^2(x) = \frac{e^{-2x^2}}{4|x|} - 2xe^{-2x^2} + 4x^3e^{-2x^2}.$$

Because of the linearity of integration operators we check the first term. This would be

$$\int_{\mathbb{R}} \frac{e^{-2x^2}}{4|x|}dx$$

which, close to zero, behaves like

$$\int_{\mathbb{R}} \frac{1}{4|x|}dx = 2\int_0^\infty |\frac{1}{4|x|}|dx.$$

Thus we have that $g(x) \notin L^2(\mathbb{R})$ because when

$$\epsilon \to 0 \text{ we have that } 2\int_\epsilon^\infty \frac{1}{4x}dx \to \infty.$$

So we see that $f(x) = \sqrt{|x|}e^{-x^2}$ is in $L^2(\mathbb{R})$ but not in $H^1(\mathbb{R})$. We see this from the function $g(x)$ in figure 2 as well. Note that we would not want a function that has similar shape as $f(x)$ which is shown in figure 3 in our solution space to the wave equation on a membrane.

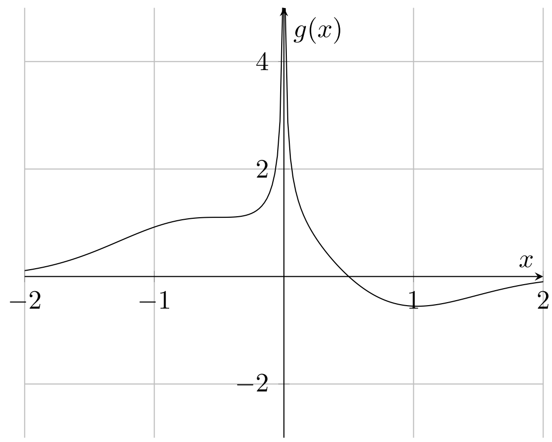

Figure 2: The graph of $g(x)$.

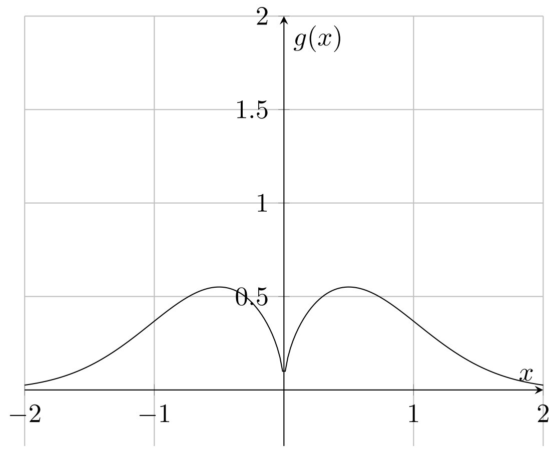

Figure 3: The graph of $f(x)$.

3.2 Separable Hilbert Spaces

Separation of variables used in solving the wave equation relies on our solution space being separable, i.e. having a countable orthonormal basis.
A Hilbert space is in [2, p. 27] called separable if the complete inner product space admits a countable dense subset.
An alternative definition given in [1, p. 245] is as follows.

Definition 3.8. Separable
A separable normed vector space is one where the whole space can be expressed as the closure of a countable subset.

Theorem 3.9. *Basis of separable space*
Every separable Hilbert space admits an orthonormal basis. The proof is in [2, p. 28]

Example 3.10. Example of a separable Hilbert space

The Sobolev space $H^2(\Omega)$ where $\Omega = (0, 2\pi)$ is a separable Hilbert space.

From example 2.32 in [2, p. 28] we have that $\{f_k(\theta)\}_{k\in\mathbb{Z}} = \left\{\frac{1}{\sqrt{2\pi}}e^{ik\theta}\right\}_{k\in\mathbb{Z}}$ is a basis for $L^2(\Omega)$. Each basis element f_k has its first and second derivatives in L^2. Furthermore these derivatives can also be described by the same set $\{f_k(\theta)\}_{k\in\mathbb{Z}}$. This Fourier basis which is know to be complete and orthonormal in $L^2(\Omega)$ needs an adjustment of normallity;

$$\|f_k(\theta)\|^2_{H^2(\Omega)} = \int_0^{2\pi} |f_k|^2 + |f_k'|^2 + |f_k''|^2 d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} |e^{ik\theta}|^2 + |ike^{ik\theta}|^2 + |-k^2e^{ik\theta}|^2 d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} |e^{ik\theta}|^2 + |ike^{ik\theta}|^2 + |-k^2e^{ik\theta}|^2 d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} 1^2 + k^2 + k^4 d\theta$$

$$1 + k^2 + k^4 := c^2$$

This means $\{\frac{f_k(\theta)}{c}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $H^2(\Omega)$. Lets check by calculating the $H^2(0, 2\pi)$ inner product;

$$< \frac{f_k(\theta)}{c}, \frac{f_k(\theta)}{c} >_{L^2(0,2\pi)} + < \frac{f_k'(\theta)}{c}, \frac{f_k'(\theta)}{c} >_{L^2(0,2\pi)} + < \frac{f_k''(\theta)}{c}, \frac{f_k''(\theta)}{c} >_{L^2(0,2\pi)}$$

$$\frac{1}{c^2} + \frac{k^2}{c^2} + \frac{k^4}{c^2}$$

$$\frac{1}{1+k^2+k^4} + \frac{k^2}{1+k^2+k^4} + \frac{k^4}{1+k^2+k^4}$$

$$\frac{1+k^2+k^4}{1+k^2+k^4} = 1.$$

This shows that $H^2(\Omega)$ is a separable Hilbert space.

Until this point we have mostly been looking at spaces. Lets look at linear maps that get us from X to Y.

4 Unbounded Operators

In definition 3.1 the weak derivatives was defined using differential operators. The Laplacian is such an operator. Depending on the domain the Laplacian may or may not be bounded. The notion of bounded versus unbounded operators is useful when comparing norms of functions before and after applying an operator. Furthermore one may draw conclusion about operators by combining several point of view by considering different domains.

Definition 4.1. Operator, [2, p. 36]
A linear map $T : D(T) \subset X \to Y$ where X and Y are a Hilbert space and $D(T)$ is dense in X is called an operator.

The succeeding example shows how the second derivative of one function may be difficult to compare with respect to the function before differentiating. We then go on to define operator norms.

Example 4.2. Single variable example

Lets look at the second derivative of $f(x) = cos(nx)$ and see if it has a larger norm in $L^2(a, b)$ than $cos(nx)$ itself.
Let $T : C^2(a, b) \subset L^2(a, b) \to L^2(a, b)$, where $Tf = f''$.
Then the question becomes is $(Tf)(x)$ larger then $f(x)$ in some sense. While the choice of the domain (a, b) will be made after the calculation, it is clear that this may affect the L^2 norm and hence the boundedness of the differential operator. This is a simple example preceding the definition of operator norm.

Firstly lets calculate the norm of $f(x) = \cos(nx)$

$$\|cos(nx)\|_{L^2(a,b)} = \int_a^b |cos(nx)|^2 dx$$

$$\int_a^b \frac{1 + cos(2nx)}{2} dx = [\frac{x}{2} + \frac{sin(2nx)}{4n}]_a^b$$

$$\| - n^2 cos(nx)\|_{L^2(a,b)} = \int_a^b | - n^2 cos(nx)|^2 dx$$

$$n^2 \int_a^b \frac{1 + cos(2nx)}{2} dx = [\frac{n^2 x}{2} + \frac{nsin(2nx)}{4}]_a^b$$

When $(a, b) = (0, 2\pi)$ we see that

$$\|f(x)\|_{L^2(a,b)} = \pi$$

and

$$\|f''(x)\|_{L^2(a,b)} = \| - n^2 cos(nx)\|_{L^2(a,b)} = n^2\pi$$

For large n these norms become vastly different.

In order to quantify this difference we use the notion of an operator norm.

Definition 4.3. Operator norm
An operator $T : D(T) \subset X \to Y$ has operator norm defined by

$$||T|| = \sup_{x \in D \setminus \{0\}} \frac{||Tx||_Y}{||x||_X}.$$

An operator is called bounded when $||T|| \ < \infty$.

The operator norm, when bounded, satisfies the norm conditions in definition 2.1.

As we can imagine, after applying an operator, such as a differential operator, the norm may have increased enough, leading to the operator not being bounded. The possible unboundedness of differential operators on L^2 comes into play with the sinusoids in the wave equation. Since the wave equation has differential operators on infinite dimensional space we will develop an understanding of unbounded differential operators. Boundedness is often a desired property of operators in physics and depends on the choice of norm. As we have seen, the Sobolev norm in definition 3.5 is different (not smaller but more domain restrictive) from the Lebesgue norm. To be more specific than alluding to unboundedness meaning " not bounded" we write the definition of an unbounded operator.

Definition 4.4. Unbounded
For $T : D(T) \subset X \longrightarrow Y$ and any $x \in D \setminus \{0\}$ and $Tx \in Y$, the operator, T, is unbounded whilst

$$||T|| = \sup_{x \in D \setminus \{0\}} \frac{||Tx||_Y}{||x||_X} = \infty$$

The next example shows the combination of a domain and norm where the Laplacian is unbounded.

Example 4.5. An unbounded Laplacian
The Laplacian is an unbounded operator on $D(\Delta) = C^2[0, 1] \subset L^2[0, 1]$. In this single variable case $\Delta u = \frac{d^2 u}{\mathrm{d}x^2}$. Since $L^2[0, 1]$ consists of all functions where the square has (Lebesgue) integral of finite value, and it's know that x^n has $\mathbb{R}$ as its pre-image in the corresponding σ -algebra for all natural numbers n. The function $f_n(x) = x^n$ Thus $f_n(x)$ is a counter example to the boundedness of the Laplacian in this case. Using the definiton above we get

$$\frac{\|\Delta f_n\|}{\|f_n\|} = \frac{(\int_0^1 \ (n(n-1)x^{n-2})^2 \mathrm{d}x)^{\frac{1}{2}}}{(\int_0^1 (x^n)^2 \mathrm{d}x)^{\frac{1}{2}}}$$

$$= \frac{(\int_0^1 \ (n^4 - n^2)x^{2n-4}) \mathrm{d}x)^{\frac{1}{2}}}{(\int_0^1 x^{2n} \mathrm{d}x)^{\frac{1}{2}}}$$

$$= \frac{(\frac{n^4-n^2}{2n-3}(1)^{2n-3})^{\frac{1}{2}}}{(\frac{1^{2n+1}}{2n+1})^{\frac{1}{2}}}$$
$$= (\frac{2n+1}{2n-3}n^2)^{\frac{1}{2}}$$

This fraction behaves like n for large n.

For a sequence $\{f_n\}_{n\in\mathbb{N}}$ there does not exist a constant c such that one cannot find an $f_n = v$ (v as in the following) . Thus the following definition does not hold.

Definition 4.6. Bonuded away from zero
An operator $T : D(T) \subset X \to Y$ is bounded away from zero if $c\|v\| \leq \|Tv\|$ where c is a positive constant and v is in $D(T) \setminus \{0\}$.

Since $C^2[0,1] \subset H^2[0,1]$ its interesting to see if the Sobolev norm makes sense to use in the example 4.5 . Of course functions in $C^2[0,1]$ need not remain in $C^2[0,1]$ after applying the Laplacian.

Example 4.7. Changing norm of domain
We have $D(\Delta) = C^2[0,1] \subset H^2[0,1]$ and $\Delta u = \frac{d^2u}{\mathrm{d}x^2}$. Recall that the Sobolev norm from definition 3.5. The $H^2[0,1]$ norm has the Lebesgue norm of the second derivative as a term among other positive terms (inside a square root). Thus the fraction can cover the whole square root and its easy to see that $\Delta : C^2[0,1] \subset H^2[0,1] \to L^2[0,1]$ is bounded.

Since we know, that in the finite case, symmetric operators have matrix representation which in turn are diagonalizable. Lets turn our attention to symmetric operators in the infinite dimensional case.

Definition 4.8. Symmetric operator
An operator O is symmetric when $< Ox, y >_A = < x, Oy >_A$ for all $x, y \in D(O)$ if $O : D(O) \subset A \to A$.Where $D(O)$ is dense in A.

The connecting of symmetric operators the notion of weak derivatives in definition 3.1 is interesting.One can at least draw the conclusion that for an operator to be symmetric it needs to account for the $(-1)^{|\alpha|}$ in definition 3.1. Note that for a symmetric operator T the following is a fact

$$< x, Ty >_{L^2(\Omega)} = < Tx, y >_{L^2(\Omega)} \iff \int_\Omega x\overline{Ty}d\mu = \int_\Omega Tx\overline{y}d\mu$$

In the next example one use partial integration to establish that T is symmetric and also see that the factor $(-1)^{|\alpha|}$ is accounted for.

Example 4.9. Example which combines symmetry and derivatives, see [.]Borthwick2020
The operator in this example is $T = -i\frac{d}{dx}$ where $T : D(T) \subset L^2[0,1] \to L^2[0,1]$ and $D(T) = \{f \in AC[0,1] : f(0) = f(1), f' \in L^2[0,1]\}$ where AC denotes the set of absolutely continuous functions. Note that $f, f' \in AC[0,1] \subset L^2(\Omega) \subset$

$L^1_{loc}(\Omega)$. That means if one removes the complex conjugation in the calculations below (or the factor i), one would see yet another verification that the weak derivative is a correct description. That T is a symmetric operator is apparent

$$\int_0^1 T(f(x))\overline{g(x)}d\mu = \int_0^1 -i\frac{f(x)}{dx}\overline{g(x)}d\mu$$

$$= -i\int_0^1 \frac{df(x)}{dx}\overline{g(x)}d\mu = -i([f(x)\overline{g(x)}]_0^1 - \int_0^1 f(x)\overline{\frac{dg(x)}{dx}}d\mu$$

$$= i\int_0^1 f(x)\overline{\frac{dg(x)}{dx}}d\mu = \int_0^1 f(x)\overline{-i\frac{dg(x)}{dx}}d\mu$$

$$= \int_0^1 f(x)\overline{Tg(x)}d\mu.$$

We may conclude that T is symmetric in this case

$$\int_0^1 Tf(x)\overline{g(x)}d\mu =< Tf, g >=< f, Tg >= \int_0^1 f(x)\overline{Tg(x)}d\mu.$$

These symmetric operators have $D(O) \subset D(O^*)$ (since $D(O^*) = A$ in the symmetric operator definition 4.8). We now go on to define adjoint operators (O^*).

Definition 4.10. Adjoint and Self-adjoint, [2, p. 38]
The adjoint of an unbounded operator is pretty much the same as for bounded operators except for domain considerations.
Let $A : D(A) \subseteq B \to B$ and $< Au, v >_B =< u, A^*v >_B$ for $u \in D(A)$ and $v \in D(A^*)$.
Where $D(A^*) = \{u \in B : v \to< u, Av >_B$ is a bounded linear functional in D(A) $\}$.
Then A has an adjoint. Further more if $A = A^*$ its called self-adjoint.

We can continue the use of example 4.9 to show that its self-adjoint as well as symmetric.

Example 4.11. Sequel of example 4.9
We check if $D(A^*) = \{u \in B : v \to< u, Av >$ is a bounded linear functional in D(A) $\}$, where $A = T$ and $B = L^2[0, 1]$ from example 4.9.
Take $u \in L^2[0, 1]$ then $\| < u, Tv > \|_{L^2[0,1]} \leq c\|u\|_{L^2[0,1]}$ for some constant c. Since $\| < u, Tv > \|_{L^2[0,1]} = \| < Tu, v > \|_{L^2[0,1]}$ from example 4.9, we must have that the inequality above holds. This means that u must be in $D(T)$, then we also have that $D(T^*) = D(T)$. Combined with the fact that T is symmetric, we have found that T is self-adjoint.

We have now scratched the surface of unbounded operators and seen some properties like what it for an operator to be self-adjoint. Before defining the Laplacian operator we briefly will have a look at polar coordinates.

4.0.1 Polar

Polar coordinates use one radial unit vector and one that is tangential to the radial one. The gradient in polar coordinates is analogous to derivative in the fundamental theorem of calculus(single variable), in the following way;

$$\int_p \nabla f(qq)dq = F(b) - F(a)$$

where f is differentiable on a curve p which has endpoint a and b. The measure q falls in line with what we indeed know about measures. The specific property to maintain in polar is translation invariance. The change in area coming from a nudge (so to speak) in θ is proportional with the radius r (we don't want that in our measure). One could go use the Jacobian here but this setting is simple enough to do without. So we fill inn the specifics

$$\int_p [\frac{\partial}{\partial r}, \frac{\partial}{r\partial\theta}]^T f(r,\theta) r d\theta dr = F(b) - F(a).$$

Divergence is a differential operator, a functional is also relates to the fundamental theorem of calculus. But is is also defined as

$$\nabla\mathbb{F} = \Delta = \frac{1}{r}(\frac{\partial r F_r}{\partial r} + \frac{\partial F_\theta}{\partial\theta}).$$

Where $F_r = \frac{\partial}{\partial r}, F_\theta = \frac{\partial}{r\partial\theta}$ are the components of some continuously differentiable field.
Here is where the Laplacian surfaces

$$\Delta = (\nabla)^2 = \frac{1}{r}(\frac{\partial r F_r}{\partial r} + \frac{\partial F_\theta}{\partial\theta}) = \frac{1}{r}(\frac{\partial r \frac{\partial}{\partial r}}{\partial r} + \frac{\partial \frac{\partial}{r\partial\theta}}{\partial\theta}) = \frac{1}{r}\frac{\partial}{\partial r}\frac{r\partial}{\partial r} + \frac{\partial}{r^2\partial\theta^2}.$$

This in fact is the Laplace operator (in polar coordinates) can be defined as the divergence of the gradient.

4.1 Laplacian on a drum head

The theory of unbounded operators will now be applied to the Laplacian with domain suitable for the Helmholtz equation on a circular membrane.
The spacial factor of $u(t,r,\theta)$ (defined here as the height of the membrane relative to the horizontal)
is initially written by Herman von helmholtz in the context of electrodynamics and has been made a reference while analyzing waves it reads in [2, p. 125].

The Laplacian can be defined as
$\Delta u = -\text{div}(\nabla u) = -\sum_{i=1}^{2} u_{x_i,x_i} = -\frac{1}{r}\frac{\partial}{\partial r}\frac{r\partial u}{\partial r} - \frac{\partial^2 u}{r^2\partial\theta^2}$, where $u \in D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$. Then we have

$$\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega).$$

Let the function domain Ω be the open disk of radius ρ and $\partial\Omega$ be the clamped boundary. Then $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$ are the functions which are zero on the boundary and where derivatives up to order two exist in $L^2(\Omega)$ i.e.

$$D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \le m \text{ and } u|_{\partial\Omega} = 0\right\}.$$

Then we have that by integration by parts;

$$\int_\Omega \Delta u(r,\theta)v(r,\theta)d\mu(r,\theta) = \nabla u \nabla v|_{\partial\Omega} - \int_\Omega \nabla u \nabla v d\mu(r,\theta)$$

$$= u\nabla v|_{\partial\Omega} - \int_\Omega u\Delta v d\mu(r,\theta) = \int_\Omega u\Delta v d\mu(r,\theta).$$

Thus Δ is symmetric in $L^2(\Omega)$, $< \Delta u, v >_{L^2(\Omega)} = < u, \Delta v >_{L^2(\Omega)}$

Furthermore, analoguos to what we found in example 4.7

$$\|\Delta\| = \sup_{x \in H_0^1 \cap H^2(\Omega) \setminus \{0\}} \frac{\|\Delta u\|_{L^2(\Omega)}}{\|u\|_{H_0^1 \cap H^2(\Omega)}} < \infty.$$

For bounded operators we have that

$$| < \Delta u, v >_{L^2} | \le \|\Delta u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|\Delta u\|_{L^2(\Omega)} \|v\|_{H_0^1 \cap H^2(\Omega)}.$$

Recall that $D(A^*) = \{u \in B : v \to < u, Av > \text{ is a bounded linear functional in D(A) }\}$. Since $< \Delta u, v >_{L^2(\Omega)} = < u, \Delta v >_{L^2(\Omega)}$, we have that

$$D(\Delta^*) = \{u \in B : v \to < \Delta u, v > \text{ is a bounded linear functional in } D(\Delta)\}.$$

Then $D(\Delta^*) \subset D(\Delta)$ and hence $D(\Delta^*) = D(\Delta)$.

Since we have $D(\Delta^*) = D(\Delta)$ then Δ is its self-adjoint if it is symmetric, by definition. This is the case with when $H_0^1 \cap H^2 = D(T^*) = D(T)$ because $< u, Tv >_{L^2} < \infty$ because both Tv and u is in L^2.

4.2 Spectral Theorem

So we know that for a finite dimensional Hilbert space the spectral theorem for self adjoint operators, A, states that $Ax_i = \lambda_i x_i$, where the eigenvalues λ_i are real. This implies that

$$Ax = \sum_j AP_j x$$

for all x in the Hilbert space where P_j is the projection operator which projects x onto the j^th basis element. Thus

$$A = \sum_j AP_j$$

For the infinite dimensional case, the spectrum is not necessarily only the point spectrum (i.e. eigenvalues). Recall that eigenvalues can be defined as elements in the spectrum.

Definition 4.12. Spectrum, [2, page 68]
For an operator T, the spectrum $\sigma(T)$ is the set of points $\lambda \in \mathbb{C}$ for which $T - \lambda$ fails to have a bounded inverse. The complement of the spectrum is the resolvent set, denoted by $\rho(T)$. The bounded operator $(T - z)^{-1}$ is called the resolvent of T at $z \in \rho(T)$.

For finite dimensions we do have ideas such as singular value decomposition but only square matrices are invertible and self-adjoint. These finite dimensional matrices A (which are bounded) have only one way for which $M = (A - \lambda)$ fails to have an inverse. If M is injective, its surjective and vice versa. Thus for self-adjoint matrices the eigenbasis, which is the set of eigenvectors corresponding to the eigenvalues, always span the whole space. In the infinite dimensional case an operator $O = (T - \lambda)$ may not be invertible even if the null space is trivial. For an operator to be self-adjoint we need to consider domain specifications. Even if an operator is self-adjoint and we do have eigenvectors, the domain might be to restrictive for the eigenvectors to span the whole space. Recall that for an operator from the Sobolev space to the corresponding Lebesgue space, the operator is bounded.

Theorem 4.13. *Bounded operator's inverse, [2, p. 46]*
A bounded operator has a bounded inverse if and only if it is bijective.

Using the L^2 norm in the Sobolev space, as we do when $\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$ does not however, imply boundedness.

We go onward to the study the spectrum of the Laplacian on an open disk in L^2.

Theorem 4.14. *Theorem 6.8 in [2, p. 135]*
For a bounded open set Ω on R^n,
the eigen functions e_k of $-\Delta$ with domain,$D = H_0^1(\Omega) \cap H^2(\Omega)$ are real-valued and form a basis on $L^2(\Omega)$. Furthermore the real eigenvalues, λ_k, accumulate at ∞ .
Remark; $H_0^1(\Omega) := \overline{C_0^\infty} \subset H^1$. H^2 is defined in example 3.6

Note that this definition is as in the book [2] and has an explicit minus sign, in comparison to the definition in subsection 4.1.

Take the basis from the preceding theorem to provide a countable dense subset for our Hilbert space $H_0^1 \cap H^2(\Omega)$. When Ω is the open disk with radius ρ we have a separable Hilbert space.

5 Solving the Wave Equation

The wave equation is derived by classical mechanics and certain simplifications. Without loss of generality one may choose to regard the constant c to be one, but it really depends on the physicality of the material, for example the drum head. The wave equation (with an implicit negative Laplacian) is

$$u_{tt} + c^2 \Delta u = 0$$

though we let $c = 1$ this simplifies to

$$u_{tt} + \Delta u = 0.$$

Assume $u = T(t)H(r, \theta)$, then since differential operators are linear we have

$$HT'' = T \Delta H$$
$$\frac{T''}{T} = \frac{\Delta H}{H}.$$

Firstly, to model the drum head oscillating in time we choose
$T(t) = Acos(\lambda t) + Bsin(\lambda t)$, with $A, B \in R^+$ which corresponds to a negative constant, say, $K = -\lambda^2$ in ;

$$T'' = KT.$$

The shared (implicitly negative) constant K leads to the famous Helmholtz equation $\Delta H = -KH$. Furthermore, as we have seen, the solution space u depends on the boundary conditions;

$$\begin{cases} u|_{\partial D} = 0 \\ \Delta H(r, \theta) = \lambda^2 H \end{cases}$$

Recalling that the negative Laplacian is a non-negative operator, and looking for solutions on the form $H(r, \theta) = R(r)\Theta(\theta)$ with $R(\rho) = 0$, ρ being radius of the drum head. We now calculate the Helmholtz equation

$$\Delta R(r)\Theta(\theta) = \lambda^2 R(r)\Theta(\theta)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial \theta^2} = -\lambda^2 R\Theta$$

$$\frac{\partial^2 R}{\partial r^2}\Theta + \frac{1}{r}\frac{\partial R}{\partial r}\Theta + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2}R + \lambda^2 R\Theta = 0$$

$$-\left(\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + \lambda^2 r^2\right) + \left(\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}\right) = 0$$

$$-\left(\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + \lambda^2 r^2\right) = \left(\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}\right) = -n^2.$$

This is a separable partial differential equation, thus;

1. The function of θ Θ has the same form as the time-factor part of the wave equation (also periodic in the simplified model)

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = -n^2$$

Implying $\Theta(\theta) = C_n\cos(n\theta) + D_n\sin(n\theta)$
n must be an (non negative) integer to satisfy periodicity.

2. R has similarities to Bessel equation,

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + \lambda^2 r^2 = n^2$$

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0.$$

This has the solution

$$H(r,\theta) = J_n(\lambda_{mn}r)(A\cos(n\theta) + B\sin(n\theta))$$

which can be seen as the amplitude of the time function. Solutions u have the following form

$$u(t,r,\theta) = T(t)H(r,\theta) = (sin(\lambda_{nm}t)+cos(\lambda_{nm}t))J_n(\lambda_{mn}r)(Acos(n\theta)+Bsin(n\theta)).$$

5.1 Checking Solutions

Is this type of solution actually sound?

- Firstly the u_{tt} term is calculated checked, indeed the double derivative of these sinusoids is such that

$$\frac{\partial^2 T(t)H(r,\theta)}{\partial t^2} = H(r,\theta)\frac{\partial^2 T(t)}{\partial t^2} = -\lambda_{nm}^2 H(r,\theta)T(t).$$

  Next we want the Laplacian to be

$$\Delta(T(t)H(r,\theta)) = T(t)\Delta H(r,\theta) = \lambda_{nm}^2 H(r,\theta)T(t).$$

  If this is the case we see from the right than side that these cancle each other out

$$\frac{\partial^2 T(t)H(r,\theta)}{\partial t^2} + \Delta(T(t)H(r,\theta)) = 0.$$

- Furthermore the calculation of the Laplacian will be somewhat streamlined;

$$\Delta H(r,\theta) = \Delta(J_n(\lambda_{nm}r)(Acos(n\theta)+Bsin(n\theta)))$$
$$= -\frac{1}{r}\frac{\partial}{\partial r}\frac{r\partial(J_n(\lambda_{nm}r)(Acos(n\theta)+Bsin(n\theta)))}{\partial r} - \frac{\partial^2(J_n(\lambda_{nm}r)(Acos(n\theta)+Bsin(n\theta)))}{r^2\partial\theta^2}$$
$$= -\frac{(Acos(n\theta)+Bsin(n\theta))}{r}\frac{\partial}{\partial r}\frac{r\partial(J_n(\lambda_{nm}r))}{\partial r} - \frac{(J_n(\lambda_{nm}r)\partial^2(Acos(n\theta)+Bsin(n\theta)))}{r^2\partial\theta^2}$$
$$= -\frac{(Acos(n\theta)+Bsin(n\theta))}{r}\frac{\partial}{\partial r}\frac{r\partial(J_n(\lambda_{nm}r))}{\partial r} + \frac{n^2J_n(\lambda_{nm}r)(Acos(n\theta)+Bsin(n\theta))}{r^2}$$
$$= -\frac{(Acos(n\theta)+Bsin(n\theta))}{r}\frac{\partial}{\partial r}(r(\lambda_{nm}J_n'(\lambda_{nm}r))) + \frac{n^2J_n(\lambda_{nm}r)(Acos(n\theta)+Bsin(n\theta))}{r^2}$$
$$= -\frac{(Acos(n\theta)+Bsin(n\theta))}{r}\lambda_{nm}J_n'(\lambda_{nm}r) - (Acos(n\theta)+Bsin(n\theta))(\lambda_{nm}^2J_n''(\lambda_{nm}r)))$$
$$+ \frac{n^2J_n(\lambda_{nm}r)(Acos(n\theta)+Bsin(n\theta))}{r^2}.$$

A pit stop is made in order to deal with the derivatives of the Bessel functions. We solve the Bessel equation from subsection 2.1 with the bessel functions and get

$$x^2J_n''(x) + xJ_n'(x) + (x^2-n^2)J_n(x) = 0.$$

Let $x = \lambda_{nm}r$ and exercise some algebra;

$$(\lambda_{nm}r)^2J_n''(\lambda_{nm}r) + \lambda_{nm}rJ_n'(\lambda_{nm}r) + ((\lambda_{nm}r)^2-n^2)J_n(\lambda_{nm}r) = 0$$
$$\lambda_{nm}^2r^2J_n''(\lambda_{nm}r) + \lambda_{nm}rJ_n'(\lambda_{nm}r) + ((\lambda_{nm}r)^2-n^2)J_n(\lambda_{nm}r) = 0$$
$$\lambda_{nm}^2r^2J_n''(\lambda_{nm}r) = -\lambda_{nm}rJ_n'(\lambda_{nm}r) - ((\lambda_{nm}r)^2-n^2)J_n(\lambda_{nm}r)$$
$$\lambda_{nm}^2J_n''(\lambda_{nm}r) = -\lambda_{nm}\frac{J_n'(\lambda_{nm}r)}{r} - \frac{((\lambda_{nm}r)^2-n^2)}{r^2}J_n(\lambda_{nm}r).$$

We slowly get going again after the pit stop by focusing then on the term of J_n''

$$-(Acos(n\theta)+Bsin(n\theta))(\lambda_{nm}^2J_n''(\lambda_{nm}r))$$
$$= -(Acos(n\theta)+Bsin(n\theta))(-\lambda_{nm}\frac{J_n'(\lambda_{nm}r)}{r} - \frac{((\lambda_{nm}r)^2-n^2)}{r^2}J_n(\lambda_{nm}r)))$$
$$= (Acos(n\theta)+Bsin(n\theta))(\lambda_{nm}\frac{J_n'(\lambda_{nm}r)}{r} + (Acos(n\theta)+Bsin(n\theta))\frac{(\lambda_{nm}r)^2-n^2}{r^2}J_n(\lambda_{nm}r)).$$

Having now substituted the J_n'' we finally get

$$\Delta H(r,\theta) = \Delta(J_n(\lambda_{nm}r)(Acos(n\theta) + Bsin(n\theta)))$$
$$= -\frac{(Acos(n\theta) + Bsin(n\theta))}{r}\lambda_{nm}J_n'(\lambda_{nm}r) + (Acos(n\theta) + Bsin(n\theta))(\lambda_{nm}\frac{J_n'(\lambda_{nm}r)}{r}$$
$$+ (Acos(n\theta) + Bsin(n\theta))\frac{((\lambda_{nm}r)^2 - n^2)}{r^2}J_n(\lambda_{nm}r)))$$
$$+ \frac{n^2 J_n(\lambda_{nm}r)(Acos(n\theta) + Bsin(n\theta))}{r^2}.$$
$$= (Acos(n\theta) + Bsin(n\theta))\frac{((\lambda_{nm}r)^2)}{r^2}J_n(\lambda_{nm}r)))$$
$$= \lambda_{nm}^2(Acos(n\theta) + Bsin(n\theta))J_n(\lambda_{nm}r) = \lambda_{nm}^2 H(r,\theta).$$

, which implies that;

$$u_{tt} + \Delta u = 0$$
$$-\lambda_{nm}^2 H(r,\theta)T(t) + T(t)\lambda_{nm}^2 H(r,\theta) = 0$$

Thus our solution was indeed sound.

Lets shortly consider the spectrum in our case as well as the Dirichlet conditions. We have

$$\Delta H(r,\theta) - \lambda_{nm}^2 H(r,\theta) = 0$$
$$(\Delta - \lambda_{nm}^2)H(r,\theta) = 0.$$

The bijectivity (and more precisely, injectivity) of $(\Delta - \lambda_{nm}^2)$ fails exactly at the nontrivial solutions, where $\lambda_{nm}^2$are the eigenvalues. Recall that we have the following general solutions

$$T(t)H(r,\theta) = (sin(\lambda_{nm}t) + cos(\lambda_{nm}t))J_n(\lambda_{nm}r)(Acos(n\theta) + Bsin(n\theta))$$

When the radius $\rho = 1$ we get from the boundary condition that $H(1,\theta) = 0$. Then λ_{nm} are the zeros of the Bessel functions.
From the table on in [3, p. 590] we have that the first zero for is 2.40483 meaning $J_0(\lambda_{01}) = J_0(2.40483) = 0$. Note that
$T(t)H(r,\theta) = (sin(2.40483t)+cos(2.40483t))J_0(2.40483r)(Acos(0\theta)+Bsin(0\theta))$
$= AJ_0(2.40483r)(sin(2.40483t)+cos(2.40483t))$ is the corresponding eigenfunction. And that in general the set of eigenfunctions are when the Bessel functions have zeros, which one can look up in tables.
So the first eigenvalue for a membrane of radius 1 with the constant from the wave equation $c = 1$, would be 2.40483^2, which is the the lowest eigenvalue. The angular frequency 2.40483 would be the fundamental pitch and is a normal mode. We see that the Bessel function makes the amplitude decrease with increasing r.

5.2 Normal Modes

Some wave configurations do not depend on θ. Solving the Helmholtz equation with that regard;

$$\Delta R(r) = \lambda^2 R(r)$$
$$\frac{1}{r}\frac{\partial}{\partial r}\frac{r\partial R}{\partial r} = -\lambda^2 R$$
$$\frac{1}{r}(\frac{r\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r}) = -\lambda^2 R$$
$$\frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{r\partial r} = -\lambda^2 R$$
$$\frac{r\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r} + r\lambda^2 R = 0.$$

To get this in the form of the Bessel equation (with n =0) multiply with r

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + r^2\lambda^2 R = 0.$$

Thus the wave equation in the axisymmetric case are similar, differing only with n=0

$$T(t)H(r,\theta) = (sin(\lambda_{nm}t) + cos(\lambda_{nm}t))J_0(\lambda_{nm}r).$$

One of the first things my percussion teacher showed me was to not hit the drum directly in the center, because it sounds dull. We can see that a initial condition corresponding to striking the membrane dead center leads to only exiting the normal modes. This initial condition $u(0,r,\theta)$ could in a simplified case be the the first two normal modes, say $T(0)(H_1(r) + H_2(r)$.

The fundamental frequency (having $n = 0$ and $m = 1$ to be clear) is such that there are no nodes, or stationary points if you wish, disregarding the boundary. To satisfy the boundary condition with ρ not necessarily equal to one, we let the Bessel function, when $r = \rho$ be $J_0(\lambda_{0m}\rho) = 0$. This also implies that for $m = 1$ we have $J_0(\lambda_{01}\rho) = 0$. Let $\lambda_{01} = \frac{c_{01}}{\rho}$ such that $J_0(\frac{c_{01}}{\rho}\rho) = 0$. Since the first zero of $J_0(x) = 0$ is always the same, 2.40483, We find that the fundamental frequency is inversely proportional to the radius of the drum. Recall (c in $u_{tt} + c^2\Delta u = 0$ is set to 1). The larger the drum, the lower the frequency (and minimum eigenvalue), of course.
A nice representation of the vibrational modes on a circular membrane is found at [4]. Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State

References

[1] Sheldon Axler. *Measure, Integration Real Analysis*. eng. Cham, 2020.

[2] David Borthwick. *Spectral Theory : Basic Concepts and Applications*. 1st ed. 2020. Vol. 284. Graduate Texts in Mathematics. Springer International Publishing : Imprint: Springer, 2020.

[3] Erwin Kreyszig. *Advanced engineering mathematics*. eng. Hoboken, N.J, 2011.

[4] Dan Russell. *Membrane Circle Modes*. Accessed: 2024-06-01 ,Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State. 2024. URL: `https://www.acs.psu.edu/drussell/demos/membranecircle/circle.html`.

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