Master's thesis

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Norwegian University of Science and Technology
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Department of Mathematical Sciences

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# The reproducing kernel of the Fourier symmetric Sobolev space

Master's thesis in Mathematics (MSMNFMA)
Supervisor: Kristian Seip
April 2024

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# Abstract

By showing the unitarity of the Bargmann transform between the Fourier symmetric Sobolev space $\mathcal{H}$ consisting of functions $f \in L^2(\mathbb{R})$ such that $\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} |f(x)|^2(1+x^2)dx + \int_{\mathbb{R}} |\hat{f}(\xi)|^2(1+\xi^2)d\xi < \infty$ and the corresponding Fock space, we find an orthonormal basis of $\mathcal{H}$. This allows us to find the reproducing kernel of $\mathcal{H}$, which is expected to be useful in e.g. the area of Fourier interpolation.

# Acknowledgements

I would like to thank my supervisor Kristian Seip for his guidance and for drawing my attention to the topic of Fourier interpolation. I am looking forward to continuing working on this topic as a PhD-student.

# Sustainability

Due to the wide applications of reproducing kernel Hilbert spaces in complex analysis, harmonic analysis, and quantum mechanics, but also in the field of statistical learning theory via the representer theorem, this thesis aligns with the United Nations' Sustainable Development Goals by contributing to Goal 4: Quality Education and Goal 8: Decent Work and Economic Growth.

# Contents

# Chapter 1

# Results

We define the Fourier symmetric Sobolev space $\mathcal{H}$ to be $\mathcal{H} = \{f : f, \hat{f} \in H_1, \|f\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}_1}^2 + \|\hat{f}\|_{\mathcal{H}_1}^2\}$, where $\mathcal{H}_1$ is the Sobolev space of functions $f \in L^2(\mathbb{R})$ such that

$$\|f\|_{\mathcal{H}_1}^2 = \int_{\mathbb{R}} (1+\xi^2)|\hat{f}(\xi)|^2 d\xi < \infty,$$

and where $\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i t x} dx$. Note that the eigenfunctions for the Fourier transform are given by the scaled Hermite functions $e^{-\pi x^2} H_n(\sqrt{2\pi} x)$, where $H_n$ denotes the $n$th Hermite polynomial.

**Lemma 1.** *$\mathcal{H}$ is a reproducing kernel Hilbert space.*

*Proof.* The proof can be found in [1], but is repeated here for the reader's convenience. For any $x \in \mathbb{R}$ we define $E_x : \mathcal{H} \to \mathbb{R}, \quad E_x(f) = f(x)$. Using the Fourier inversion and the Cauchy–Schwarz inequality we see that

$$\begin{aligned} |E_x f| = |f(x)| &\leq \int_{\mathbb{R}} |\hat{f}(\xi)| d\xi = \int_{\mathbb{R}} \sqrt{1+\xi^2} |\hat{f}(\xi)| \frac{d\xi}{\sqrt{1+\xi^2}} \\ &\leq \sqrt{\pi} \|f\|_{\mathcal{H}_1} \leq \sqrt{\pi} \|f\|_{\mathcal{H}}. \end{aligned}$$

This proves that $E_x$ is bounded for any $x \in \mathbb{R}$, and so $\mathcal{H}$ is a reproducing kernel Hilbert space by the Riesz representation theorem. □

Although the space $\mathcal{H}$ has occurred in some papers before (see e.g. [1], [2]), the author did not find the reproducing kernel of this space in the literature. Thus, this is the topic of this thesis.

Since a reproducing kernel can be expressed via an orthonormal basis, henceforth abbreviated ONB, we want to find a suitable ONB of $\mathcal{H}$. We will show that such a basis consists of scaled Hermite functions. We start with the main observation of the thesis, i.e. orthogonality of Hermite functions in $\mathcal{H}$. For that, we first need the following lemma.

**Lemma 2.** *If $n > m+2$, $n, m \geq 0$, then*

$$\int_{\mathbb{R}} H_n(\sqrt{2\pi}x)H_m(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx = 0.$$

*Proof.* The lemma follows by using $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ and $n$ integrations by parts:

$$\begin{aligned}
&\int_{\mathbb{R}} H_n(\sqrt{2\pi}x)H_m(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx \\
&\qquad = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}\left(1+\frac{x^2}{2\pi}\right)dx \\
&\qquad = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (-1)^n\left(\frac{d^n}{dx^n}e^{-x^2}\right)H_m(x)\left(1+\frac{x^2}{2\pi}\right)dx \\
&\qquad = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(\frac{d^n}{dx^n}H_m(x)\left(1+\frac{x^2}{2\pi}\right)\right)e^{-x^2}dx \\
&\qquad = 0,
\end{aligned}$$

as $H_m(x)$ is a polynomial of degree $m$, and therefore $\frac{d^n}{dx^n}H_m(x)\left(1+\frac{x^2}{2\pi}\right) = 0$ for $n > m+2$. □

**Lemma 3.** *The Hermite functions $H_n(\sqrt{2\pi}x)e^{-\pi x^2}$ are pairwise orthogonal in $\mathcal{H}$.*

*Proof.* We will use the fact that the Hermite functions are eigenvalues of the Fourier transform: $\widehat{H_n(\sqrt{2\pi}x)e^{-\pi x^2}}(\xi) = (-i)^n H_n(\sqrt{2\pi}\xi)e^{-\pi\xi^2}$. For any $n, m \geq 0$ we have

$$\begin{aligned}
&\langle H_n(\sqrt{2\pi}x)e^{-\pi x^2}, H_m(\sqrt{2\pi}x)e^{-\pi x^2}\rangle_{\mathcal{H}} \\
&\quad = \int_{\mathbb{R}} H_n(\sqrt{2\pi}x)e^{-\pi x^2}\overline{H_m(\sqrt{2\pi}x)e^{-\pi x^2}}(1+x^2)dx \\
&\quad + \int_{\mathbb{R}} \widehat{H_n(\sqrt{2\pi}x)e^{-\pi x^2}}(\xi)\overline{\widehat{H_m(\sqrt{2\pi}x)e^{-\pi x^2}}(\xi)}(1+\xi^2)d\xi \\
&\quad = \int_{\mathbb{R}} H_n(\sqrt{2\pi}x)H_m(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx + \\
&\quad + \int_{\mathbb{R}} (-i)^n H_n(\sqrt{2\pi}\xi)e^{-\pi\xi^2}(\xi)\overline{(-i)^m}H_m(\sqrt{2\pi}\xi)e^{-\pi\xi^2}(1+\xi^2)d\xi \\
&\quad = \int_{\mathbb{R}} H_n(\sqrt{2\pi}x)H_m(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx(1+(-i)^n i^m).
\end{aligned}$$

Note that if $n, m$ differ by one, then $H_n H_m$ is an odd function, and so the integral is zero. If $n, m$ differ by two, then $(1+(-i)^n i^m) = 0$. If $n, m$ differ by more than two, then the integral is zero by Lemma 2. Thus, if $n \neq m$, then $\langle H_n(\sqrt{2\pi}x)e^{-\pi x^2}, H_m(\sqrt{2\pi}x)e^{-\pi x^2}\rangle_{\mathcal{H}} = 0$.

□

As a next natural step, we find the norms of $H_n(\sqrt{2\pi}x)e^{-\pi x^2}$.

**Lemma 4.** *For every $n \geq 0$,*

$$\int_{\mathbb{R}} H_n^2(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx = 2^{n-\frac{3}{2}}n!\left(n+2\pi+\frac{1}{2}\right)\frac{1}{\pi}.$$

*Thus*

$$\|H_n(\sqrt{2\pi}x)e^{-\pi x^2}\|_{\mathcal{H}}^2 = 2^{n-\frac{1}{2}}n!\left(n+2\pi+\frac{1}{2}\right)\frac{1}{\pi}.$$

*Proof.* Notice first, that using again $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ and $n$ integrations by parts we obtain

$$\begin{aligned}\int_{\mathbb{R}} x^n H_n(\sqrt{2\pi}x)e^{-2\pi x^2}dx &= (2\pi)^{-\frac{n+1}{2}}\int_{\mathbb{R}} x^n H_n(x)e^{-x^2}dx \\ &= (2\pi)^{-\frac{n+1}{2}}\int_{\mathbb{R}} x^n(-1)^n\frac{d^n}{dx^n}e^{-x^2}dx \\ &= (2\pi)^{-\frac{n+1}{2}}n!\int_{\mathbb{R}} e^{-x^2}dx \\ &= (2\pi)^{-\frac{n+1}{2}}n!\sqrt{\pi},\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}} x^{n+2}H_n(\sqrt{2\pi}x)e^{-2\pi x^2}dx &= (2\pi)^{-\frac{n+3}{2}}\int_{\mathbb{R}} x^{n+2}H_n(x)e^{-x^2}dx \\ &= (2\pi)^{-\frac{n+3}{2}}\int_{\mathbb{R}} x^{n+2}(-1)^n\frac{d^n}{dx^n}e^{-x^2}dx \\ &= (2\pi)^{-\frac{n+3}{2}}\frac{(n+2)!}{2!}\int_{\mathbb{R}} x^2 e^{-x^2}dx \\ &= (2\pi)^{-\frac{n+3}{2}}\frac{(n+2)!}{2}\frac{\sqrt{\pi}}{2},\end{aligned}$$

for any $m < n$

$$\int_{\mathbb{R}} x^m H_n(\sqrt{2\pi}x)e^{-2\pi x^2}dx = 0.$$

Combining this with the fact that $H_n$ can be written as

$$H_n(\sqrt{2\pi}x) = n!\sum_{m=0}^{\lfloor n/2\rfloor}\frac{(-1)^m}{m!(n-2m)!}(2\sqrt{2\pi}x)^{n-2m},$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}} H_n^2(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx \\
&= \int_{\mathbb{R}} H_n(\sqrt{2\pi}x)H_n(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx \\
&= \int_{\mathbb{R}} \left((2\sqrt{2\pi})^n x^n - \frac{n!}{(n-2)!}(2\sqrt{2\pi}x)^{n-2}\right)H_n(\sqrt{2\pi}x)e^{-2\pi x^2}(1+x^2)dx \\
&= (2\sqrt{2\pi})^n \int_{\mathbb{R}} x^n H_n(\sqrt{2\pi}x)e^{-2\pi x^2}dx + (2\sqrt{2\pi})^n \int_{\mathbb{R}} x^{n+2}H_n(\sqrt{2\pi}x)e^{-2\pi x^2}dx \\
&- \frac{(2\sqrt{2\pi})^{n-2}n!}{(n-2)!}\int_{\mathbb{R}} x^n H_n(\sqrt{2\pi}x)e^{-2\pi x^2}dx \\
&= (2\sqrt{2\pi})^n(2\pi)^{-\frac{n+1}{2}}n!\sqrt{\pi} + (2\sqrt{2\pi})^n(2\pi)^{-\frac{n+3}{2}}\frac{(n+2)!}{2}\frac{\sqrt{\pi}}{2} \\
&- \frac{(2\sqrt{2\pi})^{n-2}n!}{(n-2)!}(2\pi)^{-\frac{n+1}{2}}n!\sqrt{\pi} \\
&= (2\sqrt{2\pi})^n(2\pi)^{-\frac{n+3}{2}}n!\sqrt{\pi}\left(2\pi + \frac{(n+1)(n+2)}{4} - \frac{n(n-1)}{4}\right) \\
&= 2^n(2\pi)^{-\frac{3}{2}}n!\sqrt{\pi}\left(n + 2\pi + \frac{1}{2}\right).
\end{aligned}
$$

□

Thus, we know already that $\left\{\sqrt{\frac{\pi}{2^{n-\frac{1}{2}}n!\left(n+2\pi+\frac{1}{2}\right)}}H_n(\sqrt{2\pi}x)e^{-\pi x^2}, n \geq 0\right\}$ is an orthonormal set in $\mathcal{H}$. It remains to show that it is also complete.

The idea of doing so is based on [2]. We link functions from our space to the space of entire functions by the Bargmann transform:

$$\mathfrak{B} : f \to F(z) = (\mathfrak{B}f)(z) = \frac{2^{1/4}}{\pi^{3/2}}\int_{\mathbb{R}} f\left(\frac{t}{\sqrt{2\pi}}\right)e^{2tz-z^2-t^2/2}dt.$$

If we let $\mathcal{B}_\beta$ be the space of all entire functions satisfying

$$||F||^2_{\mathcal{B}_\beta} = \int_{\mathbb{C}} |F(z)|^2 e^{-2|z|^2}\left(2\pi - \frac{1}{2} + 2|z|^2\right)^\beta dA,$$

where $dA$ denote the Lebesgue area measure on $\mathbb{C}$, then by [2] $\mathfrak{B}$ is a unitary linear operator from $\frac{1}{\sqrt{\pi}}$-weighted $L^2(\mathbb{R})$ (meaning $||f||^2 = \frac{1}{\pi}\int_{\mathbb{R}}|f(x)|^2dx$) onto $\mathcal{B}_0$, and a bounded invertible mapping from $\mathcal{H}$ onto $\mathcal{B}_1$. We will actually improve this result by showing that $\mathfrak{B}$ is also a unitary operator from $\mathcal{H}$ onto $\mathcal{B}_1$. Proving that it sends an ONB of $\mathcal{B}_1$ to the orthonormal set of Hermite functions mentioned above will conclude the proof.

**Lemma 5.** *The system of functions* $\{\sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}}z^n, n \geq 0\}$ *form an ONB of* $\mathcal{B}_1$.

*Proof.* Orthogonality is a trivial computation, while completeness follows by analyticity of functions in $\mathcal{B}_1$ (any $f \in \mathcal{B}_1$ can be written as $f(z) = \sum_{n\geq 0} a_n z^n$, and so $\langle f, z^m \rangle_{\mathcal{B}_1} = 0$ for every $m$ implies that $a_m = 0$ for every $m$). Now,

$$\begin{aligned}
\|z^n\|^2_{\mathcal{B}_1} &= \int_{\mathbb{C}} |z|^{2n} e^{-2|z|^2}\left(2\pi - \frac{1}{2} + 2|z|^2\right) dA \\
&= 2\pi \int_0^\infty r^{2n+1} e^{-2r^2}\left(2\pi - \frac{1}{2} + 2r^2\right) dr \\
&= 2\pi \int_0^\infty (t/2)^{n+\frac{1}{2}} e^{-t}(2\pi - \frac{1}{2} + t)\frac{dt}{4\sqrt{t/2}} \\
&= \frac{2\pi}{4\cdot 2^n}\int_0^\infty t^n e^{-t}(2\pi - \frac{1}{2} + t)dt \\
&= \frac{\pi}{2^{n+1}}\left(\left(2\pi - \frac{1}{2}\right)n! + (n+1)!\right) \\
&= \frac{\pi n!}{2^{n+1}}(n + 2\pi + 1/2),
\end{aligned}$$

which concludes the proof. □

**Lemma 6.**

$$\mathfrak{B}(H_n(\sqrt{2\pi}x)e^{-\pi x^2})(z) = \frac{\sqrt{2}}{\pi^{1/4}}(2z)^n,$$

*and thus*

$$\mathfrak{B}^{-1}\left(\sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}}z^n\right) = \sqrt{\frac{\pi}{2^{n-\frac{1}{2}}n!\left(n+2\pi+\frac{1}{2}\right)}}H_n(\sqrt{2\pi}x)e^{-\pi x^2}.$$

*Proof.* The first part follows by $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ and $n$ integrations by parts:

$$\begin{aligned}
\mathfrak{B}(H_n(\sqrt{2\pi}x)e^{-\pi x^2})(z) &= \frac{2^{1/4}}{\pi^{3/2}}\int_{\mathbb{R}} H_n(x)e^{-x^2/2}e^{2xz-z^2-x^2/2}dx \\
&= \frac{2^{1/4}}{\pi^{3/2}}\int_{\mathbb{R}}\left((-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\right)e^{2xz-z^2-x^2}dx \\
&= \frac{2^{1/4}}{\pi^{3/2}}\int_{\mathbb{R}}(-1)^n\left(\frac{d^n}{dx^n}e^{-x^2}\right)e^{2xz-z^2}dx \\
&= \frac{2^{1/4}}{\pi^{3/2}}\int_{\mathbb{R}} e^{-x^2}(2z)^n e^{2xz-z^2}dx \\
&= \frac{2^{1/4}}{\pi^{3/2}}(2z)^n\int_{\mathbb{R}} e^{-(x-z)^2}dx \\
&= \frac{2^{n+1/4}}{\pi}z^n.
\end{aligned}$$

This means that
$$\mathfrak{B}^{-1}(z^n) = \frac{\pi}{2^{n+1/4}} e^{-\pi x^2} H_n(\sqrt{2\pi} x),$$
and thus
$$\mathfrak{B}^{-1}\left(\sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}} z^n\right) = \sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}} \frac{\pi}{2^{n+1/4}} e^{-\pi x^2} H_n(\sqrt{2\pi} x)$$
$$= \sqrt{\frac{\pi}{2^{n-\frac{1}{2}} n! \left(n+2\pi+\frac{1}{2}\right)}} H_n(\sqrt{2\pi} x) e^{-\pi x^2}.$$

□

**Theorem 1.** *The Bargmann transform is a unitary operator from $\mathcal{H}$ onto $\mathcal{B}_1$. Thus the system of functions $\left\{\sqrt{\frac{\pi}{2^{n-\frac{1}{2}} n!\left(n+2\pi+\frac{1}{2}\right)}} H_n(\sqrt{2\pi} x) e^{-\pi x^2}, n \geq 0\right\}$ forms an ONB of $\mathcal{H}$.*

*Proof.* Our lemmas prove that for any $n, m \geq 0$,
$$\left\langle \sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}} z^n, \sqrt{\frac{2^{m+1}}{m!(m+2\pi+1/2)\pi}} z^m \right\rangle_{\mathcal{B}_1}$$
$$= \left\langle \mathfrak{B}^{-1}\left(\sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}} z^n\right), \mathfrak{B}^{-1}\left(\sqrt{\frac{2^{m+1}}{m!(m+2\pi+1/2)\pi}} z^m\right)\right\rangle_{\mathcal{H}}.$$

Since $\left\{\sqrt{\frac{2^{n+1}}{n!(n+2\pi+1/2)\pi}} z^n, n \geq 0\right\}$ form an ONB of $\mathcal{B}_1$, this proves that $\mathfrak{B}^{-1}$ is a unitary transformation $\mathcal{B}_1 \to \mathcal{H}$. By lemma 6, as a unitary map sends an ONB to ONB, $\left\{\sqrt{\frac{\pi}{2^{n-\frac{1}{2}} n!\left(n+2\pi+\frac{1}{2}\right)}} H_n(\sqrt{2\pi} x) e^{-\pi x^2}, n \geq 0\right\}$ is an ONB for $\mathcal{H}$. □

**Remark.** Referring to [2], one can easily show that it is the only other case different than $L^2(\mathbb{R}) \to \mathcal{B}_0$ when the Bargmann transform is a unitary operator between corresponding Bargmann and Schwartz scales. Thus, the methods of this thesis would not work for the spaces with different powers of $x$ in the norm (although one could work with norms defined by $\int_{\mathbb{R}} |f(x)|^2(1+x^2)dx + \int_{\mathbb{R}} |\hat{f}(\xi)|^2(1+c\xi^2)d\xi$, in which case appropriately scaled Hermite functions will still be orthogonal, but the norm is no longer Fourier symmetric when $c \neq 1$).

Having obtained an ONB we can now turn our attention to finding the reproducing kernel of $\mathcal{H}$.

**Theorem 2.** *The reproducing kernel $K_x(y)$ for $\mathcal{H}$ is given by*
$$K_x(y) = \sqrt{2}\pi e^{-\pi(x^2+y^2)} \int_0^1 \frac{t^{2\pi-1/2}}{\sqrt{1-t^2}} e^{2\pi \frac{2xyt-(x^2+y^2)t^2}{1-t^2}} dt.$$

*Proof.* Using the ONB of $\mathcal{H}$ we find that

$$K_x(y) = \sum_{n=0}^{\infty} \frac{H_n(\sqrt{2\pi}x)e^{-\pi x^2} H_n(\sqrt{2\pi}y)e^{-\pi y^2}\pi}{2^{n-\frac{1}{2}} n! \left(n + 2\pi + \frac{1}{2}\right)}$$
$$= \sqrt{2}\pi e^{-\pi(x^2+y^2)} \sum_{n=0}^{\infty} \frac{H_n(\sqrt{2\pi}x)H_n(\sqrt{2\pi}y)}{2^n n!(n+2\pi+1/2)}.$$

To get the integral representation we will use Mehler's Hermite polynomial formula [3],

$$\sum_{n=0}^{\infty} \frac{H_n(\sqrt{2\pi}x)H_n(\sqrt{2\pi}y)}{2^n n!} t^n = (1-t^2)^{-1/2} e^{2\pi \frac{2xyt-(x^2+y^2)t^2}{1-t^2}}.$$

Multiplying both sides by $t^{2\pi-1/2}$ and then integrating from 0 to 1 gives

$$\sum_{n=0}^{\infty} \frac{H_n(\sqrt{2\pi}x)H_n(\sqrt{2\pi}y)}{2^n n!(n+2\pi+1/2)} = \int_0^1 \frac{t^{2\pi-1/2}}{\sqrt{1-t^2}} e^{2\pi \frac{2xyt-(x^2+y^2)t^2}{1-t^2}} dt,$$

which concludes the proof. □

The Fourier transform of the kernel can be found in a similar way and is given below for the sake of completeness.

**Theorem 3.** *The Fourier transform of the reproducing kernel $K_x(y)$ for $\mathcal{H}$ is given by*

$$\widehat{K_x}(y) = \sqrt{2}\pi e^{-\pi(x^2+y^2)} \int_0^1 \frac{t^{2\pi-1/2}}{\sqrt{1+t^2}} e^{2\pi \frac{-2xyti+(x^2+y^2)t^2}{1+t^2}} dt.$$

*Proof.* First, we will again use the facts that

$$K_x(y) = \sqrt{2}\pi e^{-\pi(x^2+y^2)} \sum_{n=0}^{\infty} \frac{H_n(\sqrt{2\pi}x)H_n(\sqrt{2\pi}y)}{2^n n!(n+2\pi+1/2)},$$

and that

$$\widehat{e^{-\pi x^2} H_n(\sqrt{2\pi}x)}(\xi) = (-i)^n e^{-\pi\xi^2} H_n(\sqrt{2\pi}\xi).$$

This gives

$$\widehat{K_x}(y) = \sqrt{2}\pi e^{-\pi(x^2+y^2)} \sum_{n=0}^{\infty} (-i)^n \frac{H_n(\sqrt{2\pi}x)H_n(\sqrt{2\pi}y)}{2^n n!(n+2\pi+1/2)}.$$

Multiplying the Mehler's Hermite polynomial formula by $t^{2\pi-1/2}$ as before, but now integrating from 0 to $-i$, we obtain

$$\widehat{K_x}(y) = \sqrt{2}\pi e^{-\pi(x^2+y^2)} (-i)^{-2\pi-1/2} \int_0^{-i} \frac{t^{2\pi-1/2}}{\sqrt{1-t^2}} e^{2\pi \frac{2xyt-(x^2+y^2)t^2}{1-t^2}} dt$$
$$= \sqrt{2}\pi e^{-\pi(x^2+y^2)} \int_0^1 \frac{t^{2\pi-1/2}}{\sqrt{1+t^2}} e^{2\pi \frac{-2xyti+(x^2+y^2)t^2}{1+t^2}} dt,$$

where in the last equality we used the change of variable $t \to -it$.

□

As mentioned in the abstract, we expect this result to be of relevance in the area of Fourier interpolation. In a forthcoming paper, inspired by [4], we plan to use this result to state and prove density theorems for Fourier interpolation for the Fourier symmetric Sobolev space $\mathcal{H}$ and look for other possible applications.

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