Consistent DCF methods for constant-growth annuities à la Modigliani & Miller or Miles & Ezzell

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Abstract

In this paper, we develop two complete discounted-cash flow (DCF) frameworks for the valuation of constant-growth annuities and perpetuities. By 'complete' we mean that these frameworks allow the valuation of a firm or project by means of different DCF methods, particularly, the equity method, the free-cash-flow (FCF) method, the adjusted-present-value-method, and the capital-cash-flow method. This also requires the derivation of formulas that allow the translation between different required returns, like the required return on unlevered and levered equity, the discount rate in the FCF method, and the required return on the tax-shield. Our paper departs from the two most advocated and mutually exclusive frameworks when dealing with DCF. The first is based on Modigliani and Miller (M&M), where the FCF at different points in time are independently distributed. The second framework rests on the analysis of Miles and Ezzell (M&E) who presume a first-order autoregressive cash-flow process. Some elements of a 'complete' framework exist in the literature, but in our opinion, a complete picture has not been developed yet.

The contributions of this paper are the following: (1) We develop (or expand) the set of formulas that are required for the valuation of constant-growth annuities and perpetuities; (2) The formulas we develop in this paper are based on a backward-iteration process, which in itself represents a suitable tool for firm valuation; (3) Using a numerical example, we show that the two mutually exclusive frameworks of M&M or M&E achieve very different valuation results; (4) It turns out that the expected returns and the growth rate of the FCF are partly linked, but this relationship is different in the two frameworks; (5) In our numerical examples, we show how the constant-growth annuity or perpetuity, can be integrated with an explicitly planned FCF.

Keywords: Discounted Cash Flow, Annuity with constant growth, Modigliani-Miller Model, Miles-Ezzell Model, Firm Valuation

1 Introduction

Discounted cash flow (DCF) methods belong to the most widely used approaches to evaluate firms and investment projects in practice and academia (Mukhlynina & Nyborg, 2016). Within these methods, the value of annuities and perpetuities with constant growth play a particular role, either for the valuation of the continuation value at the end of some explicit planning horizon (Damodaran, 2006, chapters 5 and 6; Koller et al., 2010, chapter 6; Mukhlynina & Nyborg, 2016, p. 20, Berk & DeMarzo, 2019, chapter 19; Cornell et al., 2021) or for studying relationships between different methods like the free-cash-flow method, the adjusted-present-value-method, capital-cash-flow method, etc. (Arzac & Glosten, 2005, Massari, et al. 2007, Barbi, 2012). As reasons for adding growth to the DCF models, the literature mentions inflation in nominal terms (Bradley & Jarrel, 2003) or that companies aim at growing by means of attractive investment opportunities (Miller & Modigliani, 1961; O'Brien, 2003).

The main purpose of this paper is to show how we can consistently evaluate a firm's cash flow by means of different discounted cash flow (DCF) methods, when these cash flows are modeled as either constant-growth annuities or perpetuities. By "consistently" we mean that the different DCF methods mentioned above must give the same firm value if they are based on the same assumptions.

Our analysis focuses on two of the most commonly advocated frameworks when dealing with DCF. The first is based on Modigliani and Miller (1958, 1963, abbreviated as M&M), and the second has been introduced by Miles and Ezzell (1980, 1985, abbreviated as M&E). These two frameworks are different with respect to one particular assumption that regards the stochastic behavior of the cash flow. In the M&M framework, it is necessary that the free cash flow follows a strictly stationary process, while M&E require the FCF to be a particular auto-regressive process to achieve their results (Becker, 2021 and 2022). Both frameworks do originally not consider any growth in the free cash flow. Growth has been introduced to the M&M framework in Stapleton (1972), and with respect to the model of M&E growth is considered in Arzac & Glosten (2005) or Barbi (2012).

In this paper, we are interested in developing a complete framework for the valuation of constant-growth annuities and perpetuities. This means that we can value firms by both backward iteration, direct mathematical formulas, and transition formulas between levered and unlevered returns. Some elements exist in the literature, but in our opinion a complete picture has not been developed yet. However, formulas for annuities without growth are shown in Becker (2022).

An additional takeaway from our analysis is an understanding about the mechanics between returns and growth-rates, which is not taken care of in the literature.

The sequel of this paper is structured as follows: In section 2 we briefly introduce the relevant DCF methods for this paper. Sections 3 and 0 are devoted to the mutually exclusive frameworks of Modigliani & Miller and Miles & Ezzell, respectively. Each of these sections is structured as follows: Subsection 3.1 (4.1) shows how to compute the value of the unlevered and levered firm by means of a backward iteration process followed by a numerical example. In subsections 3.2 (4.2) we derive the mathematical formulas that allow the direct firm valuation followed by a continuation of the numerical examples. Finally, subsection 3.3 (4.3) discuss the takeaways for the respective framework. Section 5 concludes the paper with some practical advice and open issues for further research.

2 Brief overview of relevant DCF methods

In this section we give a brief overview of existing DCF methods. In all methods we assume that the free cash flow (FCF) is given exogenously. The FCF corresponds to the cash flow to the unlevered firm.

Direct valuation of debt and equity: Here we compute the firm value FV_{L} as the sum of the levered equity value EV_{L} and the debt value DV:

$$FV_{\mathrm{L}} = EV_{\mathrm{L}} + DV$$

The value of levered equity is based on the flow to equity FtE discounted with the required return of the equity holders r_{E}, and the debt value is calculated by discounting the flow to debt FtD with the required return r_{D} of the debt holders. The valuation formulas are as follows:

Perpetuity: *Recursive:*

$$EV_{\mathrm{L}} = \frac{FtE}{r_{\mathrm{E}} - g} \text{ and } DV = \frac{FtD}{r_{\mathrm{D}} - g} \qquad EV_{\mathrm{L},t} = \frac{FtE_{t+1} + EV_{\mathrm{L},t+1}}{1 + r_{\mathrm{E},t}} \text{ and } DV_t = \frac{FtD_{t+1} + DV_t}{1 + r_{\mathrm{D},t}} \qquad (1)$$

The flow to the equity holders corresponds to the free cash flow plus the tax shield minus the flow to the debt holders: $FtE = FCF + TS - FtD$.

Capital-cash-flow method (CCF method): In this method we discount the total flow to the capital holders $CCF = FtE + FtD$ by means of the corresponding required return r_{CCF}:

Perpetuity: *Recursive:*

$$FV_{\mathrm{L}} = \frac{CCF}{r_{\mathrm{CCF}} - g} \qquad FV_{\mathrm{L},t} = \frac{CCF_{t+1} + FV_{\mathrm{L},t+1}}{1 + r_{\mathrm{CCF},t}} \qquad (2)$$

The capital cash flow also equals the free cash flow plus the tax shield: $CCF = FCF + TS$.

Free-cash flow method (FCF method): in this method we retrieve the value of the levered firm by means of discounting the flow to the unlevered firm by means of r_{FCF}.

Perpetuity: *Recursive:*

$$FV_{\mathrm{L}} = \frac{FCF}{r_{\mathrm{FCF}} - g} \qquad FV_{\mathrm{L},t} = \frac{FCF_{t+1} + FV_{\mathrm{L},t+1}}{1 + r_{\mathrm{FCF},t}} \qquad (3)$$

The discount rate in the FCF method (also referred to as after-tax weighted average cost of capital) is usually computed as $r_{\mathrm{FCF}} = q \cdot r_{\mathrm{E}} + (1-q) \cdot (1-\tau) \cdot r_{\mathrm{D}}$, and the required return in the CFC-method (also referred to as before-tax weighted average cost of capital) is calculated as $r_{\mathrm{CCF}} = q \cdot r_{\mathrm{E}} + (1-q) \cdot r_{\mathrm{D}}$ (Arditti, 1973, p. 1002 or p. 1006; McConnell & Sandberg, 1975, p. 885; Harris & Pringle, 1985, p. 237; Ruback, 2002, p. 85 and p. 89). Hence, the required return in the CCF-method can also be directly calculated from the discount rate in the FCF-method: $r_{\mathrm{CCF}} = r_{\mathrm{FCF}} + (1-q) \cdot r_{\mathrm{D}} \cdot \tau$.

Adjusted-present-value-method (APV method): In this method we compute the levered firm value as the sum of the unlevered firm value and the tax shield value: $FV_{\mathrm{L}} = FV_{\mathrm{U}} + TSV$. The value of the unlevered firm is calculated by discounting the FCF with the required return on unlevered equity (firm) r_{U}, and the tax shield TS is discounted with its appropriate discount rate r_{TS}.

Perpetuity: *Recursive:*

$$FV_{\mathrm{U}} = \frac{FCF}{r_{\mathrm{U}} - g} \text{ and } TSV = \frac{TS}{r_{\mathrm{TS}} - g} \qquad FV_{\mathrm{U},t} = \frac{FCF_{t+1} + FV_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}}, TSV_t = \frac{TS_{t+1} + TSV_{t+1}}{1 + r_{\mathrm{TS},t}} \tag{4}$$

In this paper we focus on the two mutually exclusive frameworks of M&M and M&E, that are linked to several assumptions. Most importantly, the leverage of the firm is held constant throughout the lifetime of the firm's or project's cash flow, i. e. $EV_{\mathrm{L},t} = q \cdot FV_{\mathrm{L},t}$ and $DV_t = (1 - q) \cdot FV_{\mathrm{L},t}$ where q is the constant equity-to-firm-value ratio. There exist alternative frameworks, where interest payments and down payments on debt are specified in advance and independent of the firm value. In such cases leverage can vary over time (Inselbag & Kaufold, 1997; Becker, 2020).

Furthermore, it is assumed that the outstanding amount of debt equals the value of debt, and that debt financing is risk-free, which implies that the required return on debt equals the risk-free rate: $r_{\mathrm{D}} = r_{\mathrm{f}}$. Moreover, there are no costs of financial distress, there exists only corporate taxation, whereas wealth taxes or personal taxes are outside this analysis. For a more detailed overview of assumptions, the reader is referred to Becker (2022).

3 Framework based on Modigliani and Miller

3.1 Backward iteration for stationary cash flows (the case of M&M)

The original model of Modigliani & Miller (1958, 1963) is valid for non-growing perpetuities only (See also Brusov et al., 2021, p. 39). To the best of our knowledge a complete and consistent framework for the valuation of growing annuities does not exist; we therefore develop it here.

Although Modigliani & Miller (1958, 1963) have never explicitly made any assumption about the cash flow process, their results can only be obtained for a stochastic cash flow that is independently distributed or stationary (see Becker, 2021). When considering cash flows of different size or cash flows with constant growth, such a process can be described by the following equations:

$$\widetilde{FCF}_t = \tilde{\varepsilon}_t \cdot a_t \quad \text{or} \quad \widetilde{FCF}_t = \tilde{\varepsilon}_t \cdot (1+g)^t \tag{5}$$

where $\tilde{\varepsilon}_t$ is a stochastic input-parameter that is drawn from the same time-invariant distribution $\mathcal{D}$ (this means $\tilde{\varepsilon}_t \sim \mathcal{D}$ for all t), a_t is a factor that determines the size of the cash flow at point in time t, and g is a constant growth rate. Because of the requirement that present values are additive (principle of arbitrage-freeness), this implies that the one-period required return r_{A} for discounting a cash flow $\widetilde{FCF}_t$ from point in time t to $t - 1$ is constant (it does not depend on t).

Furthermore, this means that the stochastic cash flow $\widetilde{FCF}_t$ observed at point t does not depend on the history of cash flows prior to this point in time, i. e.

$$\widetilde{FCF}_t \mid \mathcal{F}_0 = \widetilde{FCF}_t \mid \mathcal{F}_1 = \cdots = \widetilde{FCF}_t \mid \mathcal{F}_{t-1} \tag{6}$$

where $\mathcal{F}_t$ denotes the information (or state of the world) at point in time t.

With other words, the expected cash flow $\overline{FCF}_{t+1}$ is independent of the realized cash flow FCF_t at point in time t. This is different in DCF methods based on Miles & Ezzell (1980 and 1985) who assume an autoregressive FCF process (see section 0).

3.1.1 Value of unlevered firm

We will now derive the formula for the valuation of the unlevered firm. Here we need to start with a maturity (remaining lifetime) of one year, and then we increase the outstanding maturity step by step. In what follows, we apply the following notation. Whenever we write X_t, we refer to some value or cash flow X that appears at point in time $t \in \{0, \dots, T\}$. Alternatively, we use the notation $X_{[v]}$ (notice the brackets in the subscript) to refer to a value or cash flow when the remaining lifetime of the firm/investment is $v \in \{T, \dots, 0\}$. At the end of the maturity (lifetime), where $v = 1$, we can calculate the value of the unlevered firm as follows:

$$FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[1]} = \frac{\mathbb{E}\left[\widetilde{FCF}_{[0]} \mid \mathcal{F}_{[1]}\right]}{1 + r_{\mathrm{A}}}$$

Let us now go backwards in time from $v = 1$ to $v = 2$:

$$FV_{\mathrm{U},[2]} \mid \mathcal{F}_{[2]} = \frac{\mathbb{E}\left[\widetilde{FCF}_{[1]} \mid \mathcal{F}_{[2]}\right] + \mathbb{E}\left[FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{\mathrm{U}}}$$

We do not yet know the riskiness (stochasticity) of $FV_{\mathrm{U},[1]}$. However, we can write this expression component-wise as follows:

$$FV_{\mathrm{U},[2]} \mid \mathcal{F}_{[2]} = \frac{\mathbb{E}\left[\widetilde{FCF}_{[1]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{\mathrm{A}}} + \frac{\mathbb{E}\left[FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{?}}$$

where $r_?$ denotes the unknown discount rate. For the second term on the right side, we can write:

$$\frac{\mathbb{E}\left[FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{?}} = \frac{\mathbb{E}\left[\mathbb{E}\left[FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[1]}\right] \mid \mathcal{F}_{[2]}\right]}{1 + r_{?}} = \frac{\frac{\mathbb{E}\left[\mathbb{E}\left[\widetilde{FCF}_{[0]} \mid \mathcal{F}_{[1]}\right] \mid \mathcal{F}_{[2]}\right]}{1 + r_{\mathrm{A}}}}{1 + r_{?}}$$

From (6) we know that:

$$\widetilde{FCF}_{[0]} \mid \mathcal{F}_{[1]} = \widetilde{FCF}_{[0]} \mid \mathcal{F}_{[2]}$$

Therefore, the right term becomes:

$$\frac{\mathbb{E}\left[FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{?}} = \frac{\mathbb{E}\left[\frac{\mathbb{E}\left[\widetilde{FCF}_{[0]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{\mathrm{A}}}\right] \mid \mathcal{F}_{[2]}}{1 + r_{?}}$$

Here the expression $\frac{\mathbb{E}\left[\widetilde{FCF}_{[0]} \mid \mathcal{F}_{[2]}\right]}{1+r_{\mathrm{A}}}$ is deterministic (non-stochastic), and deterministic terms need to be discounted with the risk-free rate. This means, we are left with:

$$FV_{\mathrm{U},[2]} \mid \mathcal{F}_{[2]} = \frac{\mathbb{E}\left[\widetilde{FCF}_{[1]} \mid \mathcal{F}_{[2]}\right]}{1 + r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},[1]}}{1 + r_{\mathrm{f}}}$$

By going backwards in time from $v = 2$ to $v = 3$, we can observe the same relationship. This means, we retrieve the following general pattern (we now omit the $\mathcal{F}$-notation):

$$FV_{\mathrm{U},[v]} = \frac{\overline{FCF}_{[v-1]}}{1 + r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},[v-1]}}{1 + r_{\mathrm{f}}} \tag{7}$$

The same relationship has been shown in a numerical example in Becker (2021). To discount the continuation value at point in time $t + 1$ (equivalently $v - 1$) to point in time t (equivalently v) can be seen with some discomfort. However, it is a result of the stochasticity of the cash flow defined in (5) and the requirement of arbitrage-free pricing. To understand this phenomenon better, let us look at Figure 1.

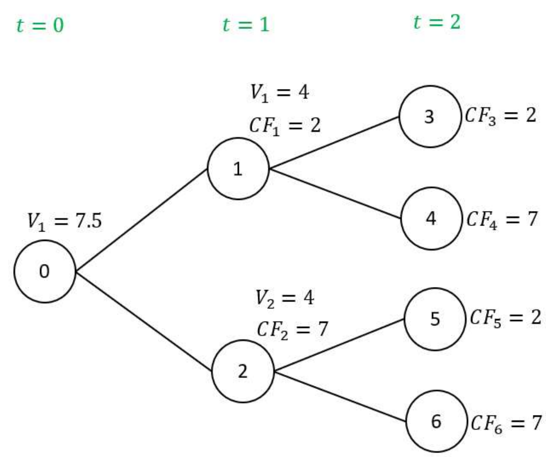

Figure 1: Deterministic continuation values when cash flows are identically and independently distributed

We see that the stochastic future cash flow (at $t = 2$) seen from node 1 is the same as seen from node 2. Hence, the continuation values in nodes 1 and 2 at point in time $t = 1$ are the same. It is the principle of arbitrage-free valuation in finance that dictates us to discount the non-stochastic value at $t = 2$ by means of the risk-free rate. In section 0 we will discuss the framework of Miles & Ezzell (1980, 1985). In their approach continuation values will never be discounted with the risk-free rate, which may seem more tempting to the firm evaluator.

3.1.2 Value of levered firm

In what follows, we derive the formula for the valuation of the levered firm. For this purpose, we derive the FCF method based on the valuation of levered equity, which can be stated as follows:

$$EV_{\mathrm{L},[v]} = \frac{\mathbb{E}\left[\widetilde{FCF}_{[v-1]}\right]}{1 + r_{\mathrm{A}}} + \frac{DV_{[v]} \cdot r_{\mathrm{f}} \cdot \tau - DV_{[v]} \cdot r_{\mathrm{f}} + \Delta DV_{[v-1]} + EV_{\mathrm{L},[v-1]}}{1 + r_{\mathrm{f}}} \tag{8}$$

Here $DV_{[v]} \cdot r_{\mathrm{f}} \cdot \tau$ represents the interest tax shield, $DV_{[v]} \cdot r_{\mathrm{f}}$ is the interest payment, $\Delta DV_{[v-1]} = DV_{[v-1]} - DV_{[v]}$ is the change of debt. $DV_{[v]} \mid \mathcal{F}_{[v]}$ has a known deterministic value at the same point in time; It is therefore discounted with the risk-free rate. $EV_{\mathrm{L},[v-1]} \mid \mathcal{F}_{[v]}$ and $DV_{[v-1]} \mid \mathcal{F}_{[v]}$ are deterministic for exactly the same reason why $FV_{\mathrm{U},[v-1]} \mid \mathcal{F}_{[v]}$ is deterministic. Therefore, they need to be discounted with the risk-free rate r_{f}. Furthermore, both the value of equity and debt are assumed proportional to the firm value, i. e.:

$$EV_{\mathrm{L},[v]} = q \cdot FV_{\mathrm{L},[v]} \quad \text{and} \quad DV_{[v]} = (1-q) \cdot FV_{\mathrm{L},[v]} \tag{9}$$

After substituting (9) into (8) and after solving for the levered firm value, we obtain:

$$FV_{\mathrm{L},[v]} = \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \frac{\mathbb{E}\left[\widetilde{FCF}_{[v-1]}\right]}{1+r_{\mathrm{f}}-(1-q)\cdot r_f \cdot \tau} + \frac{FV_{\mathrm{L},[v-1]}}{1+r_{\mathrm{f}}-(1-q)\cdot r_{\mathrm{f}} \cdot \tau} \tag{10}$$

We denote the discount rate applied in this expression as $\alpha = r_{\mathrm{f}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau$.

We are now able to fill this framework with numerical data. This is shown in the next subsection.

3.1.3 Numerical Example for the case of M&M – Part 1

In what follows, we apply the approaches from the previous subsections to a numerical example. As mentioned before, when evaluating companies, the lifetime is often split into several intervals. In our example, we apply three intervals: In the first interval, we imagine some kind of a start-up company that begins with an initially negative cash flow that later turns positive. For simplicity, we work with only three points in time ($t = 0$ to $t = 3$). The second interval from $t = 4$ to $t = 10$ is modelled as an annuity with a constant growth rate of 8 %. We assume that the firm continues afterwards. Textbooks commonly suggest some perpetual model for this purpose. At this moment, we have not yet discussed, how this perpetual model looks like. This discussion will be part of section 3.2. We therefore assume an interval that starts at $t = 11$ and ends at $t = 250$ (This length was chosen such that the perpetual formulas presented later and the backward iteration in this section deliver the same numerical values with an accuracy of two decimals). We assume a growth rate of 2 %.

Table 1 shows all relevant information concerning the free cash flow in the three intervals. For illustration purposes, we have also specified three states of the world (normally we presume that there are many more). It is important to understand that our framework is based on the critical assumption that stochastic cash flows are independently distributed and that there is a time-invariant parameter $\tilde{\varepsilon}$, such that $\widetilde{FCF}_t = \tilde{\varepsilon} \cdot a_t$. I.e. all states in point of time $t+1$ can be reached from all states at point of time t, and the stochastic cash flows in all points in time are proportional to each other. For example, in Table 1 we see that $\widetilde{FCF}_2 = 0.25 \cdot \widetilde{FCF}_1$ or $\widetilde{FCF}_3 = -5 \cdot \widetilde{FCF}_2$. The proportionality of the cash flows allows us to apply a constant one-period discount r_{A} rates on the cash flows.

After planning the explicit cash flows in the first time interval, the annuity and perpetuity in the following intervals are determined in the following forward manner: $\overline{FCF}_{t+1} = \overline{FCF}_t \cdot (1+g_{\mathrm{F}})$. For example:

$$\overline{FCF}_4 = 100 \cdot (1 + 8\,\%) = 108, \qquad \overline{FCF}_5 = 108 \cdot (1+8\,\%) = 116.64, \qquad \text{etc.}$$

Furthermore, we use the following additional information:

$$r_{\mathrm{f}} = 7\,\%,\;\; \tau = 30\,\%,\;\; r_{\mathrm{A}} = 25\,\%,\;\; q = 60\,\%$$

where r_{f} is the risk-free rate, τ is the tax rate, r_{A} is the one-period required return on the FCF, and q is the equity-to-firm-value ratio.

At this point it is important to recognize that the type of information that is given in our valuation context determines the method that can be applied in this context. We can neither apply the APV-method, equity method, nor the CCF method directly. This is because instead of a given debt value or cash flows to the debt holders, the parameter q is given, and this parameter determines the value of debt and cash flows to debt holders after we computed the firm-value.

In what follows, we apply the backward iteration methods (7) and (10). For example, the last unlevered firm values are determined as follows:

$$FV_{\mathrm{U},249} = \frac{19{,}861}{1+25\ \%} = 15{,}899, \qquad FV_{\mathrm{U},248} = \frac{19{,}472}{1+25\ \%} + \frac{15{,}899}{1+7\ \%} = 30{,}427$$

The corresponding levered firm values are:

$$FV_{\mathrm{L},249} = \frac{1+7\ \%}{1+25\ \%} \cdot \frac{19{,}861}{1+6.16\ \%} = 16{,}015$$

$$FV_{\mathrm{L},248} = \frac{1+7\ \ \%}{1+25\ \%} \cdot \frac{19{,}472}{1+6.16\ \%} + \frac{16{,}015}{1+6.16\ \%} = 30{,}786$$

where we have applied $\alpha = r_{\mathrm{f}} - (1-q) \cdot r_f \cdot \tau = 7\ \ \% - (1-60\ \%) \cdot 7\ \% \cdot 30\ \% = 6.16\ \%$.

Finally, the values in $t = 0$ are:

$$FV_{\mathrm{U},0} = \frac{\overline{FCF}_1}{1+r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},1}}{1+r_{\mathrm{f}}} = \frac{-80}{1+25\ \%} + \frac{2{,}229.91}{1+7\ \%} = 2{,}020.02$$

$$FV_{\mathrm{L},0} = \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \frac{\overline{FCF}_1}{1+\alpha} + \frac{FV_{\mathrm{L},1}}{1+\alpha} = \frac{1+7\ \%}{1+25\ \%} \cdot \frac{-80}{1+6.16\ \%} + \frac{2{,}729.91}{1+6.16\ \%} = 2{,}507.00$$

Once, the values of the levered and unlevered firm are determined, the values of equity, debt and the tax shield can be calculated by:

$$DV_t = FV_{\mathrm{L},t} \cdot (1-q), \qquad EV_{\mathrm{L},t} = FV_{\mathrm{L},t} \cdot q, \qquad VTS_t = FV_{\mathrm{L},t} - FV_{\mathrm{U},t}$$

Note, that before this point it is not possible to apply the methods (1) to (4) directly because we do not know the discount rates r_{E}, r_{FCF}, and r_{CCF}. Furthermore, neither the direct valuation of equity and debt nor the CCF-method can be directly applied since these methods require knowledge about the flow to the debt holders. However, we can calculate these values as a result of the backward iteration by the following expressions:

Flow to debt:	$FtD_{t+1} = DV_t \cdot (1+r_{\mathrm{f}}) - DV_{t+1}$	(11)
Tax shield:	$TS_{t+1} = DV_t \cdot \tau \cdot r_{\mathrm{f}}$	(12)
Flow to levered equity:	$FtE_{t+1} = FCF_{t+1} + TS_t - FtD_{t+1}$	(13)
Capital cash flow:	$CCF_{t+1} = FtE_{t+1} + FtD_{t+1} = FCF_{t+1} + TS_t$	(14)
Required return on equity:	$r_{\mathrm{E},t} = \dfrac{FtE_{t+1} + EV_{\mathrm{L},t+1}}{EV_{\mathrm{L},t}}$	(15)
Required return on CCF:	$r_{\mathrm{CCF},t} = q \cdot r_{\mathrm{E},t} + (1-q) \cdot r_{\mathrm{f}}$	(16)
Discount rate in FCF method:	$r_{\mathrm{FCF},t} = q \cdot r_{\mathrm{E},t} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)$	(17)

The cash flows to the equity and debt holders and the returns are shown in Tables 2 and 3. In the following section we will turn to formulas that allow the direct valuation of the levered and unlevered firm.

		Explicit	Explicit	Explicit	Annuity	Annuity	...	Annuity	Perpetuity	Perpetuity	...	Perpetuity	Perpetuity	Perpetuity
Point in time	t=0	t=1	t=2	t=3	t=4	t=5	...	t=10	t=11	t=12	...	t=248	t=249	t=250
Growth rate		n.a.	n.a.	n.a.	8 %	8 %		8 %	2 %	2 %		2 %	2 %	2 %
State 1		-132.00	-33.00	165.00	178.20	192.46		282.78	288.44	294.21		31,499	32,129	32,771
State 2		-80.00	-20.00	100.00	108.00	116.64		171.38	174.81	178.31		19,090	19,472	19,861
State 3		-28.00	-7.00	35.00	37.80	40.82		59.98	61.18	62.41		6,682	6,815	6,951
Expected CF		-80.00	-20.00	100.00	108.00	116.64		171.38	174.81	178.31		19,090	19,472	19,861
Unlevered Firm Value	2,020.02	2,229.91	2,403.12	2,485.74	2,567.29	2,647.16		2,992.72	3,052.57	3,113.62		30,427	15,889	0
Levered Firm Value	2,507.00	2,729.91	2,915.20	3,009.17	3,102.09	3,193.34		3,596.81	3,668.73	3,742.10		30,786	16,015	0
Equity Value	1,504.20	1,637.95	1,749.12	1,805.50	1,861.25	1,916.00		2,158.09	2,201.24	2,245.26		18,472	9,609	0
Debt Value	1,002.80	1,091.97	1,166.08	1,203.67	1,240.84	1,277.33		1,438.72	1,467.49	1,496.84		12,314	6,406	0
Tax-shield value	486.98	500.01	512.08	523.44	534.80	546.18		604.09	616.16	628.48		359	126	0

Table 1: FCF, growth and values in our numerical example (M&M)

		Explicit	Explicit	Explicit	Annuity	Annuity	…	Annuity	Perpetuity	Perpetuity	…	Perpetuity	Perpetuity	Perpetuity
Point in time	t=0	t=1	t=2	t=3	t=4	t=5	…	t=10	t=11	t=12	…	t=248	t=249	t=250
Expected FCF		-80.00	-20.00	100.00	108.00	116.64	…	171.38	174.81	178.31	…	19,090	19,472	19,861
Expected CF to Equity		-39.97	0.61	80.45	86.19	92.34	…	130.47	133.08	135.74	…	12,777	12,960	13,142
Expected CF to Debt		-18.97	2.32	44.04	47.09	50.36	…	70.53	71.94	73.38	…	6,686	6,771	6,854
Expected CCF		-58.94	2.93	124.49	133.28	142.70	…	201.00	205.02	209.12	…	19,463	19,730	19,996
Expected tax shield		21.06	22.93	24.49	25.28	26.06	…	29.62	30.21	30.82	…	373	259	135

Table 2: Cash Flows in our numerical example (M&M)

		Explicit	Explicit	Explicit	Annuity	Annuity	…	Annuity	Perpetuity	Perpetuity	…	Perpetuity	Perpetuity	Perpetuity
Point in time	t=0	t=1	t=2	t=3	t=4	t=5	…	t=10	t=11	t=12	…	t=248	t=249	t=250
Return unlevered firm	6.430 %	6.871 %	7.599 %	7.626 %	7.654 %	7.685 %	…	7.841 %	7.841 %	7.841 %	…	16.215 %	25.000 %	
Return on equity	6.234 %	6.824 %	7.823 %	7.861 %	7.902 %	7.947 %	…	8.166 %	8.166 %	8.166 %	…	22.180 %	36.764 %	
Return on debt	7.000 %	7.000 %	7.000 %	7.000 %	7.000 %	7.000 %	…	7.000 %	7.000 %	7.000 %	…	7.000 %	7.000 %	
Return on tax shield	7.000 %	7.000 %	7.000 %	7.000 %	7.000 %	7.000 %	…	7.000 %	7.000 %	7.000 %	…	7.000 %	7.000 %	
Discount rate in FCF method	5.700 %	6.055 %	6.654 %	6.677 %	6.701 %	6.728 %	…	6.860 %	6.860 %	6.860 %	…	15.268 %	24.019 %	
Discount rate in CCF method	6.540 %	6.895 %	7.494 %	7.517 %	7.541 %	7.568 %	…	7.700 %	7.700 %	7.700 %	…	16.108 %	24.859 %	

Table 3: Required returns/ discount rates in our numerical example (M&M)

3.2 Direct Valuation formulas for stationary cash flows (the case of M&M)

3.2.1 Value of unlevered firm

Although, a time-varying return r_{U} does not allow the application of the standard annuity factor of the form $\frac{1}{r_{\mathrm{U}}-g_{\mathrm{F}}} \cdot \left(1-\frac{(1+g_{\mathrm{F}})^v}{(1+r_{\mathrm{U}})^v}\right)$, the expression (7) allows the straight-forward derivation of another annuity formula, because each cash flow is discounted once with r_{A} and the remaining periods of time with r_{f}. Hence, we can write the value of the unlevered firm as a sum as follows (Please note that we now use the time index t instead of the remaining lifetime v), i.e.

$$FV_{\mathrm{U},0} = \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \sum_{t=1}^{T} \frac{\overline{FCF}_t}{(1+r_{\mathrm{f}})^t} \tag{18}$$

We now observe the constant risk-free rate in the sum of the discounted cash flows. Hence, we can apply the standard formula for an annuity with constant growth to this sum. This brings us to:

$$FV_{\mathrm{U},[v]} = \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \overline{FCF}_{[v-1]} \cdot \varphi_{\mathrm{U},[v]} \tag{19}$$

with the following annuity factor:

$$\varphi_{\mathrm{U},[v]} = \begin{cases} \frac{1}{r_{\mathrm{f}}-g_{\mathrm{F}}} \cdot \left(1-\frac{(1+g_{\mathrm{F}})^v}{(1+r_{\mathrm{f}})^v}\right) & \text{if} \quad r_{\mathrm{f}} \neq g_{\mathrm{F}} \\ \frac{v}{1+r_{\mathrm{f}}} & \text{if} \quad r_{\mathrm{f}} = g_{\mathrm{F}} \end{cases} \tag{20}$$

This expression also allows us to develop a constant-growth perpetuity formula. From expression (20) we learn that we must analyze the value of the unlevered firm for different magnitudes of the growth rate. If $g_{\mathrm{F}} < r_{\mathrm{f}}$ then the term $\frac{(1+g_{\mathrm{F}})^v}{(1+r_{\mathrm{f}})^v}$ in the upper part of (20) tends to Zero if the maturity tends to infinity. In the case that $g_{\mathrm{F}} = r_{\mathrm{f}}$, the term $\frac{v}{1+r_{\mathrm{f}}}$ (20) tends to infinity if the maturity tends to infinity. Finally, if $g_F > r_{\mathrm{f}}$, we can see that the term $1-\frac{(1+g_{\mathrm{F}})^v}{(1+r_{\mathrm{f}})^v}$ in (20) tends to $-\infty$, while $r_{\mathrm{f}} - g_{\mathrm{F}} < 0$. Based on this behavior, we can state the value of a perpetual unlevered cash flow as follows:

$$FV_{\mathrm{U},[v\to\infty]} = \begin{cases} \overline{FCF}_{\mathrm{t=1}} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \frac{1}{r_{\mathrm{f}}-g_{\mathrm{F}}} & \text{if} \quad g_{\mathrm{F}} < r_{\mathrm{f}} \\ \infty & g_{\mathrm{F}} \geq r_{\mathrm{f}} \end{cases} \tag{21}$$

We can use expressions (20) and (21) to derive the formula for the required return on the unlevered firm. To do this, we can substitute (19) into the following recursive formula:

$$FV_{\mathrm{U},[v]} = \frac{\overline{FCF}_{[v-1]} + FV_{\mathrm{U},[v-1]}}{1+r_{\mathrm{U},[v]}}$$

By using $\overline{FCF}_{[v-2]} = \overline{FCF}_{[v-1]} \cdot (1+g_{\mathrm{F}})$, we obtain the following relationship:

$$r_{\mathrm{U},[v]} = \begin{cases} r_{\mathrm{f}} + \dfrac{1}{\varphi_{\mathrm{U},[v]}} \cdot \left(\dfrac{r_{\mathrm{A}} - r_{\mathrm{f}}}{1 + r_{\mathrm{f}}}\right) & \text{if} \quad r_{\mathrm{f}} \neq g_{\mathrm{F}} \\ r_{\mathrm{U},[v]} = r_{\mathrm{f}} + \dfrac{r_{\mathrm{A}} - r_{\mathrm{f}}}{v} & \text{if} \quad r_{\mathrm{f}} = g_{\mathrm{F}} \end{cases} \tag{22}$$

where $\varphi_{\mathrm{U},[v]}$ is given by (20). If the maturity v tends to infinity, we obtain:

$$r_{\mathrm{U},[v\to\infty]} = \begin{cases} r_{\mathrm{f}} + (r_{\mathrm{f}} - g_{\mathrm{F}}) \cdot \left(\dfrac{r_{\mathrm{A}} - r_{\mathrm{f}}}{1 + r_{\mathrm{f}}}\right) & \text{if} \quad r_{\mathrm{f}} < g_{\mathrm{F}} \\ r_{\mathrm{f}} & \text{if} \quad r_{\mathrm{f}} \geq g_{\mathrm{F}} \end{cases} \tag{23}$$

3.2.2 Value of levered firm

Like for the unlevered firm, we also have a time-varying discount rate r_{FCF} when calculating the levered firm value. However, expression (10) allows to write the present value as a sum of discounted cash flows, because each cash flow is discounted once with r_{A} and the remaining periods with α.

$$FV_{\mathrm{L},0} = \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \sum_{t=1}^{T} \frac{\overline{FCF}_t}{1 + \alpha} \tag{24}$$

Assuming a constant growth annuity, we can furthermore apply the standard-textbook formula to the sum of discounted cash flows, where we use a constant α. This brings us to the following expression:

$$FV_{\mathrm{L},[v]} = \overline{FCF}_{[v-1]} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \varphi_{\mathrm{L},[v]} \tag{25}$$

where we apply the annuity factor:

$$\varphi_{\mathrm{L},[v]} = \begin{cases} \dfrac{1}{\alpha - g_{\mathrm{F}}} \cdot \left(1 - \dfrac{(1 + g_{\mathrm{F}})^v}{(1 + \alpha)^v}\right) & \text{if} \quad \alpha \neq g_{\mathrm{F}} \\ \varphi_{\mathrm{L},[v]} = \dfrac{v}{1 + \alpha} & \text{if} \quad \alpha = g_{\mathrm{F}} \end{cases} \tag{26}$$

From expression (26) we learn that we must analyze the value of the levered firm for different magnitudes of the growth rate. As long as $g_{\mathrm{F}} < \alpha$, the term $\frac{(1+g_{\mathrm{F}})^v}{(1+\alpha)^v}$ in the upper part of (26) tends to Zero if the maturity v tends to infinity. In case of $g_{\mathrm{F}} = \alpha$, the term $\frac{v}{1+\alpha}$ (26) tends to infinity if the maturity tends to infinity. Finally, if $g_F > \alpha$, we can see that the term $1 - \frac{(1+g_{\mathrm{F}})^v}{(1+\alpha)^v}$ in (26) tends to $-\infty$, while $\alpha - g_{\mathrm{F}} < 0$. Summarizing, we obtain:

$$FV_{\mathrm{L},[v\to\infty]} = \begin{cases} \overline{FCF}_{\mathrm{t}=1} \cdot \dfrac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \dfrac{1}{\alpha - g_{\mathrm{F}}} & \text{if} \quad g_{\mathrm{F}} < \alpha \\ \infty & \quad g_{\mathrm{F}} \geq \alpha \end{cases} \tag{27}$$

Let us now turn to the discount rate that needs to be applied in the FCF method. Let us start with substituting (25) into the following recursive formula:

$$FV_{\mathrm{L},[v]} = \frac{\overline{FCF}_{[v-1]} + FV_{\mathrm{L},[v-1]}}{1 + r_{\mathrm{FCF},[v]}}$$

By considering $\overline{FCF}_{[v-2]} = \overline{FCF}_{[v-1]} \cdot (1 + g_{\mathrm{F}})$, we obtain the relationship:

$$r_{\mathrm{FCF},[v]} = \alpha + \frac{1}{\varphi_{\mathrm{L},[v]}} \cdot \left(\frac{r_{\mathrm{A}} - r_{\mathrm{f}}}{1 + r_{\mathrm{f}}}\right) \tag{28}$$

where $\varphi_{\mathrm{L},[v]}$ is defined given by (26). If the growth rate of the FCF equals the rate α then this expression reduces to:

$$r_{\mathrm{FCF},[v]} = \alpha + \frac{r_{\mathrm{A}} - r_{\mathrm{f}}}{v} \quad \text{if} \quad \alpha = g_{\mathrm{F}}$$

Accordingly, if the maturity tends to infinity we obtain:

$$r_{\mathrm{FCF},[v\to\infty]} = \begin{cases} \alpha + (\alpha - g_{\mathrm{F}}) \cdot \left(\frac{r_{\mathrm{A}} - r_{\mathrm{f}}}{1 + r_{\mathrm{f}}}\right) & \text{if} \quad g_{\mathrm{F}} < \alpha \\ \alpha & \text{if} \quad g_{\mathrm{F}} \geq \alpha \end{cases} \tag{29}$$

Based on these derivations we can furthermore establish the relationships between different required returns:

Relationship between r_{U} and r_{FCF}: This relationship is easily obtained by using (22) and (28). Particularly, we obtain:

$$r_{\mathrm{FCF},[v]} = \alpha + \left(r_{\mathrm{U},[v]} - r_{\mathrm{f}}\right) \cdot \frac{\varphi_{\mathrm{U},[v]}}{\varphi_{\mathrm{L},[v]}} \tag{30}$$

If we let $v \to \infty$, we obtain the following possibilities. First we look at the case where $g_{\mathrm{F}} < \alpha < r_{\mathrm{f}}$. This gives:

$$r_{\mathrm{FCF},[\infty]} = \alpha + \left(r_{\mathrm{U},[\infty]} - r_{\mathrm{f}}\right) \cdot \frac{\alpha - g_{\mathrm{F}}}{r_{\mathrm{f}} - g_{\mathrm{F}}} \tag{31}$$

After substituting α, this can be rearranged to:

$$r_{\mathrm{FCF},[\infty]} = r_{\mathrm{U},[\infty]} - \frac{r_{\mathrm{U},[\infty]} - g_{\mathrm{F}}}{r_{\mathrm{f}} - g_{\mathrm{F}}} \cdot r_{\mathrm{f}} \cdot \tau \cdot (1 - q) \tag{32}$$

This formula has also been derived by Copeland et al. (2000, appendix A) and appears in Massari et al. (2007).[1]

In the second case we have $g_{\mathrm{F}} \geq \alpha$ which yields $r_{\mathrm{FCF},[\infty]} = \alpha$ which is independent of r_{U}.

[1] Copeland et al. (2000) and Massari et al. (2007) use the cost of debt r_{D} in their formula. However, they define the interest payment I_{t+1} and tax shield TS_{t+1} as follows: $I_{t+1} = DV_t \cdot r_{\mathrm{D}}$ and $TS_{t+1} = DV_t \cdot r_{\mathrm{D}} \cdot \tau$. Furthermore, they discount the tax shield (and equivalently the interest) by means of r_{D}. This leads to debt being risk-free.

Relationship between r_{U} and r_{E}: Based on relationships (31) or (32) and by applying the after-tax weighted average cost of capital ($r_{\mathrm{FCF}} = q \cdot r_{\mathrm{E}} + (1-q) \cdot (1-\tau) \cdot r_{\mathrm{f}}$) we can also establish the relation between the required return on levered equity and unlevered equity (firm):

$$r_{\mathrm{E},[v]} = r_{\mathrm{f}} + \frac{\left(r_{\mathrm{U},[v]} - r_{\mathrm{f}}\right)}{q} \cdot \frac{\varphi_{\mathrm{U},[v]}}{\varphi_{\mathrm{L},[v]}} \tag{33}$$

If we let $v \to \infty$, then we obtain the following possibilities:

(a) $g_{\mathrm{F}} < \alpha < r_{\mathrm{f}}$: $\quad r_{\mathrm{E},[\infty]} = r_{\mathrm{f}} + \frac{\left(r_{\mathrm{U},[\infty]} - r_{\mathrm{f}}\right)}{q} \cdot \frac{\alpha - g_{\mathrm{F}}}{r_{\mathrm{f}} - g_{\mathrm{F}}}$

(b) $g_{\mathrm{F}} \geq \alpha$: $\quad r_{\mathrm{E},[\infty]} = r_{\mathrm{f}}$ which is independent of r_{U}

In what follows, we will apply the direct valuation formulas to our numerical example.

3.2.3 Numerical Example for the case of M&M – Part 2

In this section we use the mathematical formulas from the previous section to verify the values that we have obtained by means of the backward iteration. Let us start with the third interval which we now approximate by a perpetuity. The superscript in the following expressions indicates that we deal with the values of the third interval. We calculate the unlevered and levered firm value for point in time $t = 10$ as following:

$$FV_{\mathrm{U},10}^{3} = \overline{FCF}_{11} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \frac{1}{r_{\mathrm{f}} - g_{\mathrm{F}}} = \overline{FCF}_{11} \cdot \frac{1}{r_{\mathrm{U},[v\to\infty]} - g_{\mathrm{F}}}$$

$$= 174.81 \cdot \frac{1 + 7\,\%}{1 + 25\,\%} \cdot \frac{1}{7\,\% - 2\,\%} = 174.81 \cdot \frac{1}{7.8411\,\% - 2\,\%}$$

$$= 2{,}992.72$$

Where we have applied:

$$r_{\mathrm{U},[v\to\infty]} = r_{\mathrm{f}} + (r_{\mathrm{f}} - g_{\mathrm{F}}) \cdot \left(\frac{r_{\mathrm{A}} - r_{\mathrm{f}}}{1 + r_{\mathrm{f}}}\right) = 7\,\% + (7\,\% - 2\%) \cdot \left(\frac{25\,\% - 7\,\%}{1 + 7\,\%}\right) \approx 7.8411\,\%$$

The levered firm value becomes:

$$FV_{\mathrm{L},10}^{3} = \overline{FCF}_{11} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \frac{1}{\alpha - g_{\mathrm{F}}} = \overline{FCF}_{11} \cdot \frac{1}{r_{\mathrm{FCF},[v\to\infty]} - g_{\mathrm{F}}}$$

$$= 174.81 \cdot \frac{1 + 7\,\%}{1 + 25\,\%} \cdot \frac{1}{6.16\,\% - 2\,\%} = 174.81 \cdot \frac{1}{6.8598\,\% - 2\,\%}$$

$$= 3597.05$$

where

$$\alpha = r_\mathrm{f} - (1-q)\cdot r_\mathrm{f}\cdot\tau = 7\,\% - (1-60\,\%)\cdot 7\,\%\cdot 30\,\% = 6.16\,\%$$

and

$$r_\mathrm{FCF} = \alpha + (\alpha - g_\mathrm{F})\cdot\left(\frac{r_\mathrm{A}-r_\mathrm{f}}{1+r_\mathrm{f}}\right) = 6.16\,\% + (6.16\,\% - 2\,\%)\cdot\left(\frac{25\,\%-7\,\%}{1+7\,\%}\right) = 6.8598\,\%$$

Note that the firm values are related to $t = 10$, and they need to be further discounted to $t = 0$ as follows:

$$FV_{\mathrm{U},0}^3 = \frac{FV_{\mathrm{U},10}}{(1+r_\mathrm{f})^{10}} = \frac{2{,}992.72}{(1+7\,\%)^{10}} = 1{,}521.36$$

$$FV_{\mathrm{L},0}^3 = \frac{FV_{\mathrm{L},10}}{(1+\alpha)^{10}} = \frac{3597.05}{(1+6.16\,\%)^{10}} = 1{,}978.51$$

In the second interval we have an annuity with constant growth. This annuity has a lifetime of $v = 7$. Let us therefore calculate the annuity factors (20) and (26):

$$\varphi_{\mathrm{U},[7]} = \frac{1}{r_\mathrm{f}-g_\mathrm{F}}\cdot\left(1-\frac{(1+g_\mathrm{F})^v}{(1+r_\mathrm{f})^v}\right) = \frac{1}{7\,\%-8\,\%}\cdot\left(1-\frac{(1+8\,\%)^7}{(1+7\,\%)^7}\right) = 6.7284$$

$$\varphi_{\mathrm{L},[7]} = \frac{1}{\alpha-g_\mathrm{F}}\cdot\left(1-\frac{(1+g_\mathrm{F})^v}{(1+\alpha)^v}\right) = \frac{1}{6.16\,\%-8\,\%}\cdot\left(1-\frac{(1+8\,\%)^7}{(1+6.16\,\%)^7}\right) = 6.9468$$

Now we can compute the values of the growing annuity in the second FCF-interval:

$$FV_{\mathrm{U},3}^2 = \overline{FCF}_4\cdot\frac{1+r_\mathrm{f}}{1+r_\mathrm{A}}\cdot\varphi_{\mathrm{U},[7]} = 108\cdot\frac{1+7\,\%}{1+25\,\%}\cdot 6.7284 = 622.02$$

$$FV_{\mathrm{L},3}^2 = \overline{FCF}_4\cdot\frac{1+r_\mathrm{f}}{1+r_\mathrm{A}}\cdot\varphi_{\mathrm{L},[7]} = 108\cdot\frac{1+7\,\%}{1+25\,\%}\cdot 6.9468 = 642.21$$

Also, these values need to be discounted to point in time $t = 0$:

$$FV_{\mathrm{U},0}^2 = \frac{FV_{\mathrm{U},3}}{(1+r_\mathrm{f})^3} = \frac{622.02}{(1+7\,\%)^3} = 507.76$$

$$FV_{\mathrm{L},0}^2 = \frac{FV_{\mathrm{L},3}}{(1+\alpha)^3} = \frac{642.21}{(1+6.16\,\%)^3} = 536.78$$

Finally, we need to compute the value of the explicit planning interval, where we apply the expressions (18) and (24). We obtain:

$$FV_{\mathrm{U},0}^1 = \frac{1+7\,\%}{1+25\,\%}\cdot\left(\frac{-80}{1+7\,\%}+\frac{-20}{(1+7\,\%)^2}+\frac{-100}{(1+7\,\%)^3}\right) = -9.08$$

$$FV_{\mathrm{L},0}^1 = \frac{1+7\,\%}{1+25\,\%}\cdot\left(\frac{-80}{1+6.16\,\%}+\frac{-20}{(1+6.16\,\%)^2}+\frac{-100}{(1+6.16\,\%)^3}\right) = -8.15$$

Finally, we can add all present values, such that we have the total unlevered and levered firm values:

$$FV_{\mathrm{U},0} = FV_{\mathrm{U},0}^1 + FV_{\mathrm{U},0}^2 + FV_{\mathrm{U},0}^3 = -9.08 + 507.76 + 1{,}521.36 = 2{,}020.04$$

$$FV_{\mathrm{L},0} = FV_{\mathrm{L},0}^{1} + FV_{\mathrm{L},0}^{2} + FV_{\mathrm{L},0}^{3} = -8.15 + 536.78 + 1{,}978.51 = 2{,}507.14$$

3.3 Preliminary Take-aways and practical advice for the M&M model

(1) Without knowing the direct valuation formulas derived in section 3.2, and based on the initially given information, we cannot directly apply the methods (1) to (4) introduced in section 2. Instead, we must rely on the backward iteration process to determine the unlevered and levered firm values. Backward iteration is simple and allows the calculation of all values: Nowadays, spreadsheet software is available to literally everybody. This allows to iterate quickly through many periods, by which we can approximate perpetual cash flows that are often assumed in continuation values at the end of some planning horizon. Strictly speaking, the derivation of more complicated direct-valuation formulas is not necessary. However, we have developed these formulas in section 3.2, because we obtain insights into the mechanics of return and growth. Once we have determined the unlevered and levered firm values, we can apply expressions (11) to (17) to calculate all the flows and discount rates that can be used in the recursive calculations (1) to (4).

(2) The required returns or discount rates r_{U}, r_{CCF}, r_{FCF}, and r_{E} depend on the lifetime of the free cash flow. Particularly, they decrease if the remaining lifetime of the cash flow becomes longer. We could observe this in the final interval, when going backwards from $t = 249$ to $t = 10$. This is because the deterministic continuation value takes a larger share in the total value that consists of both the value of one-period-ahead stochastic cash flow and deterministic continuation value. For example, the discount rate in the FCF method has decreased from $r_{\mathrm{FCF},249} = 24.019\ \%$ to $r_{\mathrm{FCF},10} = 7.86\ \%$, and it is now lower than the risk-free rate. The decay of the required returns happens very sharply even for $g_{\mathrm{F}} = 0\ \%$. Hence, even for short annuities, constant required returns are never a good approximation.

This behavior is depicted in Figure 2 for the required return on unlevered equity that converges towards the risk-free rate. The longer the remaining lifetime of the FCF, the faster the convergence occurs.

This implies that we cannot apply the standard annuity formula from textbooks, because this formula requires a constant discount rate. However, we can apply the alternative annuity factors (20) and (26) which allow the valuation of M&M annuities with constant growth.

(3) From expression (30) and (33) we also learn, that the relationship between r_{U} and the discount rates r_{CCF}, r_{FCF}, or r_{E} depends on the remaining lifetime.

(4) The larger the growth rate the more decrease the required returns or discount rates r_{U}, r_{CCF}, r_{FCF}, and r_{E}. However, r_{U}, r_{CCF}, and r_{E} can never decrease below the risk-free rate r_{f}. The discount rate r_{FCF} in the FCF-method can never fall below α. This can be seen in expressions (23) and (32).

(5) Textbooks often suggest constant discount rates in all valuation methods. This is generally not correct, although some of the rates can be constant: In the M&M approach the required return on the tax shield is constant since it is linked to deterministic continuation values. The required return on debt is constant by assumption.

(6) The perpetual model based on M&M does not allow for a growth rate $g_{\mathrm{F}} \geq \alpha$ because then the levered firm value becomes infinitely large. Note that α is also less than the risk-free rate.

(7) Because of the uncertainty in the input factors, we often use a sensitivity analysis to see how the change in particular input parameters affect the firm value. Because of the preceding bullet points, it is important to recognize the relationship between growth rate and required return. Assuming the M&M framework it is essential that this relationship is not neglected, i. e. if an analyst changes the growth rate of the FCF then a simultaneous change in the discount rates is necessary.

(8) Looking again at the discount rates in the final interval, we observe some convergence if the remaining lifetime of the cash flow is very long. E. g. the discount rates at $t = 10, 11, 12$, etc. are the same. This implies that we can apply the standard perpetuity formulas in (1) to (4) as long as we know the correct discount rates. For this purpose, we have derived expressions (23) and (32) that allow the calculation of r_{U} and r_{FCF}, respectively.

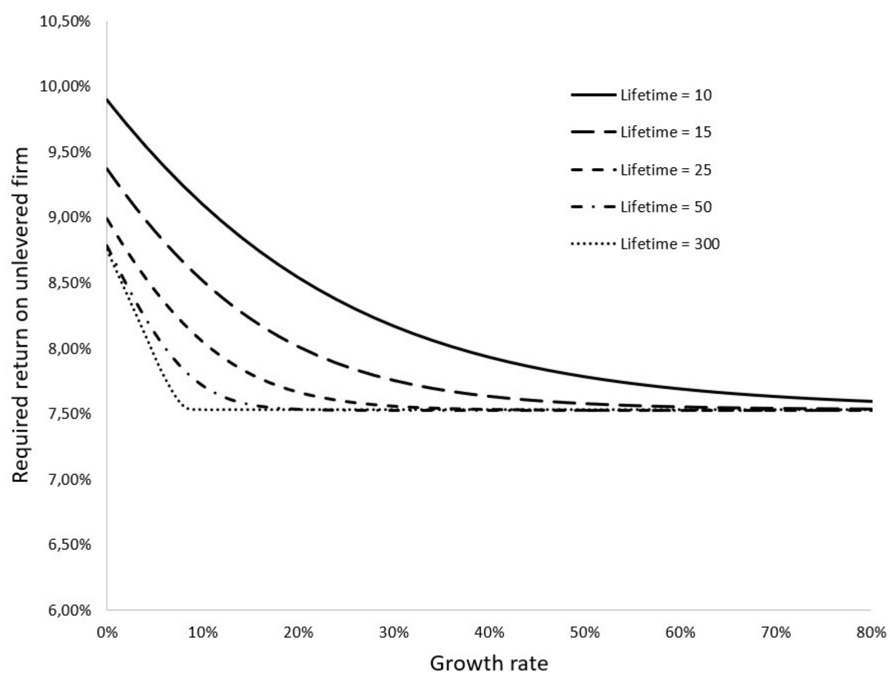

Figure 2: Behavior of required return on unlevered equity dependent on growth rate and lifetime of FCF

4 Framework based on Miles & Ezzell

4.1 Backward iteration for autoregressive cash flows

In this section, we look at the DCF model according to M&E. Contrary to M&M, in M&E the sum of the free cash flow and continuation value ($\widetilde{FCF}_t + \tilde{V}_t$) is discounted with the same one-period discount rate r_{A}:

$$V_{t-1} = \frac{\widetilde{FCF}_t + \tilde{V}_t}{1 + r_{\mathrm{A}}} = \frac{\widetilde{FCF}_t + \tilde{V}_t}{1 + r_{\mathrm{U}}} \quad r_{\mathrm{U}} = r_{\mathrm{A}}$$

To discount the cash flow and continuation value with the same required return r_{U} requires a particular auto-regressive cash flow. This can be stated as follows:

$$\widetilde{FCF}_{[v]} \,\big|\, \mathcal{F}_{[v+1]} = (1 + g_{\mathrm{F}}) \cdot \tilde{\varepsilon}_t \cdot FCF_{[v+1]} \,\big|\, \mathcal{F}_{[v+1]}, \qquad \mathbb{E}[\tilde{\varepsilon}_t] = 1 \tag{34}$$

The expected value of the cash flow is then:

$$\mathbb{E}\big[\widetilde{FCF}_{[v]} \mid \mathcal{F}_{[v+1]}\big] = (1 + g_{\mathrm{F}}) \cdot FCF_{[v+1]} \mid \mathcal{F}_{[v+1]} \tag{35}$$

In what follows, we derive the framework for computing the value of the unlevered and levered firm by means of backward-iteration.

4.1.1 Value of unlevered firm

At the end of the maturity ($v = 0$) when we do not observe any continuation value, the value of the unlevered firm at $v = 1$ can be calculated as:

$$FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[1]} = \frac{\mathbb{E}\big[\widetilde{FCF}_{[0]} \mid \mathcal{F}_{[1]}\big]}{1 + r_{\mathrm{A}}}$$

Using (35), we can then write the value of the unleveled firm as follows:

$$FV_{\mathrm{U},[1]} \mid \mathcal{F}_{[1]} = \frac{(1 + g_{\mathrm{F}}) \cdot FCF_{[1]} \mid \mathcal{F}_{[1]}}{1 + r_{\mathrm{A}}} \tag{36}$$

We now go one period backwards in time from $v = 1$ to $v = 2$. Here the value of the unlevered firm is:

$$FV_{\mathrm{U},[2]} \mid \mathcal{F}_{[2]} = \frac{\mathbb{E}\big[\widetilde{FCF}_{[1]} \mid \mathcal{F}_{[2]}\big] + \mathbb{E}\big[\widetilde{FV}_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\big]}{1 + r_{\mathrm{U},[2]}}$$

We do not yet know the riskiness (stochasticity) of $FV_{\mathrm{U},[1]}$, therefore we cannot blindly apply r_{A} as the discount rate. Therefore, we used $r_{\mathrm{U},[v]}$. However, we can write this expression component-wise as follows:

$$FV_{\mathrm{U},[2]} \mid \mathcal{F}_{[2]} = \frac{\mathbb{E}\big[\widetilde{FCF}_{[1]} \mid \mathcal{F}_{[2]}\big]}{1 + r_{\mathrm{A}}} + \frac{\mathbb{E}\big[\widetilde{FV}_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\big]}{1 + r_{?}}$$

Note that $FV_{\mathrm{U},[1]}$ in equation (36) depends on the realization of the cash flow $FCF_{[1]}$. From the perspective of $v = 2$ the cash flow $\widetilde{FCF}_{[1]} \mid \mathcal{F}_{[2]}$ is stochastic, and therefore also the value $\widetilde{FV}_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}$ is stochastic. More precisely this means:

$$\frac{\mathbb{E}\left[\widetilde{FV}_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\right]}{1+r_{?}} = \frac{\mathbb{E}\left[\frac{\mathbb{E}\left[\left(1+g_{\mathrm{F}}\right)\cdot FCF_{[1]} \mid \mathcal{F}_{[1]}\right]}{1+r_{\mathrm{A}}} \mid \mathcal{F}_{[2]}\right]}{1+r_{?}} = \frac{\mathbb{E}\left[\frac{\mathbb{E}\left[\left(1+g_{\mathrm{F}}\right)^{2}\cdot \tilde{\varepsilon}\cdot FCF_{[2]} \mid \mathcal{F}_{[2]}\right]}{1+r_{\mathrm{A}}} \mid \mathcal{F}_{[2]}\right]}{1+r_{?}}$$

Here, the term $\frac{\mathbb{E}(1+g_{\mathrm{F}})^{2}\cdot\tilde{\varepsilon}\cdot FCF_{[2]} \mid \mathcal{F}_{[2]}}{1+r_{\mathrm{A}}}$ is stochastic. Furthermore, the term is proportional to $\widetilde{FCF}_{[1]} = (1+g_{\mathrm{F}})\cdot\tilde{\varepsilon}\cdot FCF_{[2]}$. Hence, the continuation value needs to be discounted with the same return as the free cash flow, i. e. $r_{\mathrm{U}[v]} = r_{\mathrm{A}}$

$$FV_{\mathrm{U},[2]} \mid \mathcal{F}_{[2]} = \frac{\mathbb{E}\left[\widetilde{FCF}_{[1]} + \widetilde{FV}_{\mathrm{U},[1]} \mid \mathcal{F}_{[2]}\right]}{1+r_{\mathrm{A}}}$$

By going backwards in time from $v=2$ to $v=3$, we can observe the same relationship. This means, we have the following general pattern (we now omitt the $\mathcal{F}$-notation):

$$FV_{\mathrm{U},[v]} = \frac{\overline{FCF}_{[v-1]} + \overline{FV}_{\mathrm{U},[v-1]}}{1+r_{\mathrm{U}}} \qquad r_{\mathrm{U}} = r_{\mathrm{A}} \tag{37}$$

The same relationship has been shown in a numerical example in Becker (2021), which treated non-growing annuities.

4.1.2 Value of levered firm

In this subsection we compute the value of the levered firm. It can be derived from the equity method as follows:

$$\begin{aligned} EV_{\mathrm{L},[v]} &= \frac{\mathbb{E}\left[\widetilde{FCF}_{[v-1]}\right] + DV_{[v]}\cdot r_{\mathrm{f}}\cdot\tau - DV_{[v]}\cdot r_{\mathrm{f}} + \mathbb{E}\left[\Delta\widetilde{DV}_{[v-1]} + \widetilde{EV}_{\mathrm{L},[v-1]}\right]}{1+r_{\mathrm{EL},[v]}} \\ &= \frac{\mathbb{E}\left[\widetilde{FCF}_{[v-1]}\right]}{1+r_{\mathrm{A}}} + \frac{DV_{[v]}\cdot r_{f}\cdot\tau - DV_{[v]}\cdot(1+r_{\mathrm{f}})}{1+r_{\mathrm{f}}} + \frac{\mathbb{E}\left[\Delta\widetilde{DV}_{[v-1]} + \widetilde{EV}_{\mathrm{L},[v-1]}\right]}{1+r_{\mathrm{A}}} \end{aligned} \tag{38}$$

In this expression $DV_{[v]}\cdot r_{\mathrm{f}}\cdot\tau$ reflects the tax shield, $DV_{[v]}\cdot r_{\mathrm{f}}$ is the interest payment, and $\Delta\widetilde{DV}_{[v-1]} = \widetilde{DV}_{[v-1]} - DV_{[v]}$ is the change of debt. $DV_{[v]} \mid \mathcal{F}_{[v]}$ has a known deterministic value at the same point in time, and therefore it is discounted with the risk-free rate. $\widetilde{EV}_{\mathrm{L},[v-1]} \mid \mathcal{F}_{[v]}$ and $\widetilde{DV}_{[v-1]} \mid \mathcal{F}_{[v]}$ are stochastic for exactly the same reason why $\widetilde{FV}_{\mathrm{U},[v-1]} \mid \mathcal{F}_{[v]}$ is stochastic. They are also proportional to $\widetilde{FCF}_{[v-1]} \mid \mathcal{F}_{[v]}$ and need to be discounted with the same required return r_{A}.

Furthermore, both the value of equity and debt are assumed proportional to the firm value, i. e. $EV_{\mathrm{L},[v]} = q\cdot FV_{\mathrm{L},[v]}$ and $DV_{[v]} = (1-q)\cdot FV_{\mathrm{L},[v]}$. Therefore, expression (38) becomes:

$$FV_{\mathrm{L},[v]} = \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}}\cdot\frac{\mathbb{E}\left[\widetilde{FCF}_{[v-1]} + \widetilde{FV}_{\mathrm{L},[v-1]}\right]}{1+\alpha}, \qquad \alpha = r_{\mathrm{f}} - (1-q)\cdot r_{\mathrm{f}}\cdot\tau \tag{39}$$

We see that the required discount rate in the FCF-method is independent of the remaining lifetime (maturity). This formula has also been derived by Myers (1974, p. 13) for single-period cash flows, by

Miles and Ezzell (1980, p. 726) for constant-growth perpetuities, and by Becker (2022, p. 487) for constant-growth annuities.

4.1.3 Numerical Example for the case of M&E – Part 1

In this section we apply the approach from the previous section to a numerical example, where we use the same input parameters as in M&M, i. e. we have the same three intervals, the same explicit expected cash flows in the beginning, the same growth rates in the second and third time interval, and we apply the same parameters r_{f}, r_{A}, q, and τ. Table 4 shows the evolution of the expected free cash flow. Contrary to M&M, we assume an autoregressive cash-flow process, which implies that the number of possible states gets larger with the remaining lifetime. Hence, we cannot show the complete evolution of all states in Table 4. However, for points in time $t = 1$ and $t = 2$ all achievable states are shown in Figure 3. The expected cash flows are the same like in M&M for all points in time.

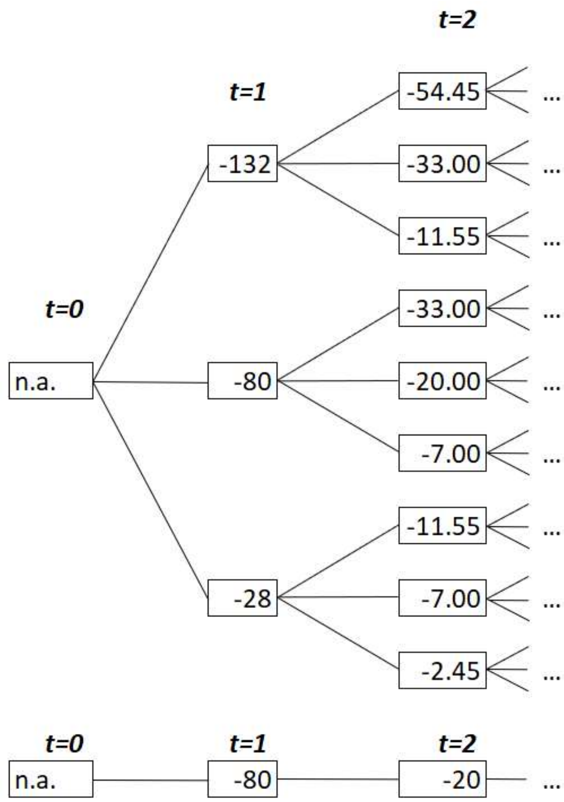

Figure 3: The evolution of an autoregressive free cash flow (M&E)

Let us start with the calculation of the unlevered values, which need to be determined in a backward manner according to expression (37), for example:

$$FV_{\mathrm{U},249} = \frac{19{,}861}{1 + 24.0187\ \%} = 15{,}899, \qquad FV_{\mathrm{U},248} = \frac{19{,}472 + 15{,}899}{1 + 24.0187\ \%} = 28{,}289$$

The levered values are determined according to expression (39):

$$FV_{\mathrm{L},249} = \frac{19{,}861}{1 + 24.0187\ \%} = 16{,}015, \qquad FV_{\mathrm{L},248} = \frac{19{,}472 + 16{,}015}{1 + 24.0187\ \%} = 28{,}614$$

Here we have applied $r_{\text{FCF}} = \frac{(1+r_{\text{A}})\cdot(1+\alpha)}{1+r_{\text{f}}} - 1 = \frac{(1+25\,\%)\cdot(1+6.16\,\%)}{1+7\,\%} = 24.0187\,\%$, and α takes the previously calculated value.

Finally, the values in $t = 0$ are:

$$FV_{\text{U},0} = \frac{\overline{FCF}_1 + \overline{FV}_{\text{U},1}}{1 + r_{\text{U}}} = \frac{-80 + 410.47}{1 + 24.0187\,\%} = 264.37$$

$$FV_{\text{L},0} = \frac{\overline{FCF}_1 + \overline{FV}_{\text{L},1}}{1 + r_{\text{FCF}}} = \frac{-80 + 435.15}{1 + 24.0187\,\%} = 286.36$$

Once, the values of the levered and unlevered firm are determined, the values of equity, debt and the tax shield can be calculated by:

$$DV_t = FV_{\text{L},t} \cdot (1 - q), \qquad EV_{\text{L},t} = FV_{\text{L},t} \cdot q, \qquad VTS_t = FV_{\text{L},t} - FV_{\text{U},t}$$

Note, that before this point neither the capital-cash-flow method nor the equity method could be applied directly, since these methods require knowledge about the flow to the debt holders and the required returns $r_{\text{CCF},t}$ and $r_{\text{E},t}$. Also, the free-cash flow method was not yet accessible because its discount rate $r_{\text{FCF},t}$ requires $r_{\text{E},t}$ as an ingredient. However, we can calculate these values with expressions (11) to (17). The results of these calculations are shown in Tables 5 and 6.

		Explicit	Explicit	Explicit	Annuity	Annuity	...	Annuity	Perpetuity	Perpetuity	...	Perpetuity	Perpetuity	Perpetuity
Point in time	t=0	t=1	t=2	t=3	t=4	t=5	...	t=10	t=11	t=12	...	t=248	t=249	t=250
Growth rate		n.a.	n.a.	n.a.	8 %	8 %		8 %	2 %	2 %		2 %	2 %	2 %
Expected CF		-80.00	-20.00	100.00	108.00	116.64		171.38	174.81	178.31		19,090	19,472	19,861
Unlevered Firm Value	264.37	410.47	533.08	566.35	599.94	633.29		760.04	775.24	790.75		28,289	15,889	0
Levered Firm Value	286.36	435.15	559.66	594.09	628.78	663.16		793.92	809.80	825.99		28,614	16,015	0
Equity Value	171.82	261.09	335.80	356.45	377.27	397.90		476.35	485.88	495.59		17,168	9,609	0
Debt Value	114.55	174.06	223.86	237.63	251.51	265.26		317.57	323.92	330.40		11,446	6,406	0
Tax-shield value	21.99	24.68	26.58	27.73	28.84	29.88		33.87	34.55	35.24		325	126	0

Table 4: FCF, growth and values in our numerical example (M&E)

		Explicit	Explicit	Explicit	Annuity	Annuity	...	Annuity	Perpetuity	Perpetuity	...	Perpetuity	Perpetuity	Perpetuity
Point in time	t=0	t=1	t=2	t=3	t=4	t=5	...	t=10	t=11	t=12	...	t=248	t=249	t=250
Expected FCF		-80.00	-20.00	100.00	108.00	116.64	...	171.38	174.81	178.31	...	19,090	19,472	19,861
Expected CF to Equity		-26.10	21.28	102.80	110.23	118.07	...	162.35	165.60	168.91	...	14,396	13,871	13,142
Expected CF to Debt		-51.49	-37.62	1.90	2.76	3.85	...	15.57	15.88	16.20	...	5,018	5,841	6,854
Expected CCF		-77.59	-16.34	104.70	112.99	121.92	...	177.92	181.48	185.11	...	19,413	19,712	19,996
Expected tax shield		2.41	3.66	4.70	4.99	5.28	...	6.54	6.67	6.80	...	323	240	135

Table 5: Cash Flows in our numerical example (M&E)

		Explicit	Explicit	Explicit	Annuity	Annuity	...	Annuity	Perpetuity	Perpetuity	...	Perpetuity	Perpetuity	Perpetuity
Point in time	t=0	t=1	t=2	t=3	t=4	t=5	...	t=10	t=11	t=12	...	t=248	t=249	t=250
Return unlevered firm	25.000 %	25.000 %	25.000 %	25.000 %	25.000 %	25.000 %	...	25.000 %	25.000 %	25.000 %	...	25.000 %	25.000 %	
Return on equity	36.764 %	36.764 %	36.764 %	36.764 %	36.764 %	36.764 %	...	36.764 %	36.764 %	36.764 %	...	36.764 %	36.764 %	
Return on debt	7.000 %	7.000 %	7.000 %	7.000 %	7.000 %	7.000 %	...	7.000 %	7.000 %	7.000 %	...	7.000 %	7.000 %	
Return on tax shield	23.160 %	22.509 %	22.025 %	21.973 %	21.919 %	21.863 %	...	21.688 %	21.688 %	21.688 %	...	12.567 %	7.000 %	
Discount rate in FCF method	24.019 %	24.019 %	24.019 %	24.019 %	24.019 %	24.019 %	...	24.019 %	24.019 %	24.019 %	...	24.019 %	24.019 %	
Discount rate in CCF method	24.859 %	24.859 %	24.859 %	24.859 %	24.859 %	24.859 %	...	24.859 %	24.859 %	24.859 %	...	24.859 %	24.859 %	

Table 6: Required returns/ discount rates in our numerical example (M&E)

4.2 Direct Valuation formulas for auto-regressive cash flows (the case of M&E)

4.2.1 Value of unlevered firm

From expression (37), we can easily derive the sum of discounted cash flows as:

$$FV_{\mathrm{U},0} = \sum_{t=1}^{T} \frac{\overline{FCF}_{[v-1]}}{1+r_{\mathrm{U}}} \tag{40}$$

Furthermore, we can directly apply the standard constant-growth annuity formula. This means we have:

$$\overline{FV}_{\mathrm{U},[v]} = \overline{FCF}_{[v-1]} \cdot \theta_{\mathrm{U},[v]} \tag{41}$$

with the annuity factor defined as:

$$\theta_{\mathrm{U},[v]} = \begin{cases} \dfrac{1}{r_{\mathrm{U}} - g_{\mathrm{F}}} \cdot \left(1 - \dfrac{(1+g_{\mathrm{F}})^{v}}{(1+r_{\mathrm{U}})^{v}}\right) & \text{if} \quad r_{\mathrm{U}} \neq g_{\mathrm{F}} \\ & \\ \dfrac{v}{1+r_{\mathrm{U}}} & \text{if} \quad r_{\mathrm{U}} = g_{\mathrm{F}} \end{cases} \tag{42}$$

From expression (42) we learn that we must analyze the value of the unlevered firm for different magnitudes of the growth rate. If $g_{\mathrm{F}} < r_{\mathrm{U}}$ then the term $\frac{(1+g_{\mathrm{F}})^{v}}{(1+r_{\mathrm{U}})^{v}}$ in the upper part of (42) tends to Zero if the maturity tends to infinity. In the case of $g_{\mathrm{F}} = r_{\mathrm{U}}$, the term $\frac{v}{1+r_{\mathrm{U}}}$ (42) tends to infinity if the maturity tends to infinity. Finally, if $g_F > r_{\mathrm{U}}$, we can see that the term $1 - \frac{(1+g_{\mathrm{F}})^{v}}{(1+r_{\mathrm{U}})^{v}}$ in (42) tends to $-\infty$, and at the same time we have $r_{\mathrm{U}} - g_{\mathrm{F}} < 0$. Based on these observations, we see that the value of the unlevered firm can be calculated according to the standard text-book perpetuity formula:

$$FV_{\mathrm{U},[v\rightarrow\infty]} = \begin{cases} \overline{FCF} \cdot \dfrac{1}{r_{\mathrm{U}} - g_{\mathrm{F}}} & \text{if} \quad g_{\mathrm{F}} < r_{\mathrm{U}} \\ & \\ \infty & \quad g_{\mathrm{F}} \geq r_{\mathrm{U}} \end{cases} \tag{43}$$

4.2.2 Value of levered firm

Expression (39) allows to write the levered firm value as a sum of discounted cash flows:

$$FV_{\mathrm{L},0} = \sum_{t=1}^{T} \frac{\mathbb{E}[\widetilde{FCF}_t]}{\left((1+\alpha) \cdot \dfrac{1+r_{\mathrm{A}}}{1+r_{\mathrm{f}}}\right)^{t}} \quad \text{where} \quad \alpha = r_{\mathrm{f}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau \tag{44}$$

Obviously, the term under the fraction is constant, and corresponds to the discount rate in the FCF-method, i. e. $1 + r_{\mathrm{FCF}} = \frac{(1+r_{\mathrm{A}})\cdot(1+\alpha)}{1+r_{\mathrm{f}}}$. Substituting $\alpha = r_{\mathrm{f}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau$ we obtain:

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1+r_{\mathrm{U}}}{1+r_{\mathrm{f}}} \tag{45}$$

We also see that we can directly apply the standard constant-growth annuity formula to (44):

$$\overline{FV}_{\mathrm{L},[v]} = \overline{FCF}_{[v-1]} \cdot \theta_{\mathrm{L},[v]} \tag{46}$$

with the annuity factor:

$$\theta_{\mathrm{L},[v]} = \begin{cases} \dfrac{1}{r_{\mathrm{FCF}} - g_{\mathrm{F}}} \cdot \left(1 - \dfrac{(1+g_{\mathrm{F}})^{v}}{(1+r_{\mathrm{FCF}})^{v}}\right) & \text{if} \quad g_{\mathrm{F}} \neq r_{\mathrm{FCF}} \\ \dfrac{v}{1+r_{\mathrm{FCF}}} & \text{if} \quad g_{\mathrm{F}} = r_{\mathrm{FCF}} \end{cases} \tag{47}$$

From expression (47) we learn that we must analyze the value of the levered firm for different magnitudes of the growth rate. As long as $g_{\mathrm{F}} < r_{\mathrm{FCF}}$, the term $\frac{(1+g_{\mathrm{F}})^{v}}{(1+r_{\mathrm{FCF}})^{v}}$ in the upper part of (47) tends to Zero if the maturity v tends to infinity. In the case that $g_{\mathrm{F}} = r_{\mathrm{FCF}}$, the term $\frac{v}{1+r_{\mathrm{FCF}}}$ (47) tends to infinity if the maturity tends to infinity. Finally, if $g_{F} > r_{\mathrm{FCF}}$, we can see that the term $1 - \frac{(1+g_{\mathrm{F}})^{v}}{(1+r_{\mathrm{FCF}})^{v}}$ in (47) tends to $-\infty$. At the same time, we have $r_{\mathrm{FCF}} - g_{\mathrm{F}} < 0$. Summarizing, we obtain:

$$FV_{\mathrm{L},[v\to\infty]} = \begin{cases} \overline{FCF} \cdot \dfrac{1}{r_{\mathrm{FCF}} - g_{\mathrm{F}}} & \text{if} \quad g_{\mathrm{F}} < r_{\mathrm{FCF}} \\ \infty & \quad g_{\mathrm{F}} \geq r_{\mathrm{FCF}} \end{cases} \tag{48}$$

4.2.3 Value of tax shield

Contrary to M&M, the value of the tax shield requires some extra attention, because in M&E the tax shield cannot be discounted with the risk-free rate. Let us depart from the following one-period formula for discounting the tax shield and continuing tax-shield value:

$$TSV_{[v]} = \frac{TS_{[v-1]} + TSV_{[v-1]}}{1 + r_{\mathrm{TS},[v]}}$$

We substitute the following expressions:

- $TSV_{[v]}$ as difference between $FV_{\mathrm{L},[v]}$ and $FV_{\mathrm{U},[v]}$: $TSV_{[v]} = \overline{FCF}_{[v-1]} \cdot \left[\theta_{\mathrm{L},[v]} - \theta_{\mathrm{U},[v]}\right]$
- $TSV_{[v-1]}$ as difference between $FV_{\mathrm{L},[v-1]}$ and $FV_{\mathrm{U},[v-1]}$: $TSV_{[v-1]} = \overline{FCF}_{[v-2]} \cdot \left[\theta_{\mathrm{L},[v-1]} - \theta_{\mathrm{U},[v-1]}\right]$
- The equation of cash-flow growth: $\overline{FCF}_{[v-2]} = \overline{FCF}_{[v-1]} \cdot (1 + g_{\mathrm{F}})$
- The tax-shield: $TS_{[v-1]} = DV_{[v]} \cdot r_{\mathrm{f}} \cdot \tau$
- The debt value as part of the firm value: $DV_{[v]} = (1-q) \cdot FV_{\mathrm{L},[v]} = (1-q) \cdot \overline{FCF}_{[v-1]} \cdot \theta_{\mathrm{L},[v]}$

After solving for the required return of the tax shield, we obtain:

$$r_{\mathrm{TS},[v]} = r_{\mathrm{U}} + (1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{r_{\mathrm{f}} - r_{\mathrm{U}}}{1 + r_{\mathrm{f}}} \cdot \frac{\theta_{\mathrm{L},[v]}}{\theta_{\mathrm{L},[v]} - \theta_{\mathrm{U},[v]}} \tag{49}$$

For $v \to \infty$, the required return on the tax shield becomes:

$$r_{\mathrm{TS},[\infty]} = \begin{cases} r_{\mathrm{U}} + (r_{\mathrm{f}} - r_{\mathrm{U}}) \cdot \left(\dfrac{r_{\mathrm{U}} - g_{\mathrm{F}}}{1 + r_{\mathrm{U}}}\right) & \text{if} \quad g_{\mathrm{F}} < r_{\mathrm{FCF}} \\ r_{\mathrm{U}} + (1 - q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \dfrac{r_{\mathrm{f}} - r_{\mathrm{U}}}{1 + r_{\mathrm{f}}} & \text{if} \quad g_{\mathrm{F}} \geq r_{\mathrm{FCF}} \end{cases} \tag{50}$$

The upper part of this expression is compatible with the result obtained by Arzac and Glosten (2005, equation 13) and Barbi (2012, equation 15).

4.2.4 Numerical Example for the case of M&E – Part 2

In this section we use the mathematical formulas from the previous section to verify the values that we have obtained by means of the backward iteration. Let us start with the third interval which we now represent as a perpetuity. The superscript in the following expressions indicates that we deal with the values of the third interval. We calculate the unlevered and levered firm value for point in time $t = 10$. The discount rates for the valuation of the unlevered and levered firm are given by:

$$FV_{\mathrm{U},10}^{3} = \overline{FCF}_{11} \cdot \frac{1}{r_{\mathrm{U}} - g_{\mathrm{F}}} = 174.81 \cdot \frac{1}{25\,\% - 2\,\%} = 760.04 \quad \text{where} \quad r_{\mathrm{U}} = r_{\mathrm{A}}$$

$$FV_{\mathrm{L},10}^{3} = \overline{FCF}_{11} \cdot \frac{1}{r_{\mathrm{FCF}} - g_{\mathrm{F}}} = 174.81 \cdot \frac{1}{24.019\,\% - 2\,\%} = 793.92$$

where

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - (1 - q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1 + r_{\mathrm{U}}}{1 + r_{\mathrm{f}}} = 25\,\% - (1 - 60\,\%) \cdot 7\,\% \cdot 30\,\% \cdot \frac{1 + 25\,\%}{1 + 7\,\%} = 24.019\,\%$$

Note that the firm values are related to $t = 10$, and they need to be transferred to $t = 0$.

$$FV_{\mathrm{U},0}^{3} = \frac{FV_{\mathrm{U},10}}{(1 + r_{\mathrm{U}})^{10}} = \frac{760.04}{(1 + 25\,\%)^{10}} = 81.61$$

$$FV_{\mathrm{L},0}^{3} = \frac{FV_{\mathrm{L},10}}{(1 + r_{\mathrm{FCF}})^{10}} = \frac{793.92}{(1 + 24.019\,\%)^{10}} = 92.24$$

In the second interval we have an annuity with constant growth. This annuity has a lifetime of $v = 7$. Let us therefore calculate the annuity factors (42) and (47):

$$\theta_{\mathrm{U},[7]} = \frac{1}{r_{\mathrm{U}} - g_{\mathrm{F}}} \cdot \left(1 - \frac{(1 + g_{\mathrm{F}})^{v}}{(1 + r_{\mathrm{U}})^{v}}\right) = \frac{1}{25\,\% - 8\,\%} \cdot \left(1 - \frac{(1 + 8\,\%)^{7}}{(1 + 25\,\%)^{7}}\right) = 3.7681$$

$$\theta_{\mathrm{L},[7]} = \frac{1}{r_{\mathrm{FCF}} - g_{\mathrm{F}}} \cdot \left(1 - \frac{(1 + g_{\mathrm{F}})^{v}}{(1 + r_{\mathrm{FCF}})^{v}}\right) = \frac{1}{24.019\,\% - 8\,\%} \cdot \left(1 - \frac{(1 + 8\,\%)^{7}}{(1 + 24.019\,\%)^{7}}\right) = 3.8717$$

Then the unlevered and levered values at point in time $t = 3$ of the growing annuities in the second CF-interval are:

$$FV_{\mathrm{U},3}^{2} = \overline{FCF}_{4} \cdot \theta_{\mathrm{U},[7]} = 108 \cdot 3.7681 = 406.96$$

$$FV_{\mathrm{L},3}^{2} = \overline{FCF}_{4} \cdot \theta_{\mathrm{L},[7]} = 108 \cdot 3.8717 = 418.15$$

Also, these values need to be discounted to point in time $t = 0$:

$$FV_{\mathrm{U},0}^{2} = \frac{FV_{\mathrm{U},3}}{(1 + r_{\mathrm{U}})^{3}} = \frac{406.96}{(1 + 25\,\%)^{3}} = 208.36$$

$$FV_{\mathrm{L},0}^{2} = \frac{FV_{\mathrm{L},3}}{(1 + r_{\mathrm{FCF}})^{3}} = \frac{418.15}{(1 + 24.019\,\%)^{3}} = 219.21$$

Finally, we need to compute the value of the explicit planning interval, where we apply the expressions (40) and (44). We obtain:

$$FV_{\mathrm{U},0}^{1} = \frac{-80}{1 + 25\,\%} + \frac{-20}{(1 + 25\,\%)^{2}} + \frac{-100}{(1 + 25\,\%)^{3}} = -25.60$$

$$FV_{\mathrm{L},0}^{1} = \frac{-80}{1 + 24.019\,\%} + \frac{-20}{(1 + 24.019\,\%)^{2}} + \frac{-100}{(1 + 24.019\,\%)^{3}} = -25.08$$

Finally, we can add all present values, such that we have the total unlevered and levered firm values:

$$FV_{\mathrm{U},0} = FV_{\mathrm{U},0}^{1} + FV_{\mathrm{U},0}^{2} + FV_{\mathrm{U},0}^{3} = -25.60 + 208.36 + 81.61 = 264.37$$

$$FV_{\mathrm{L},0} = FV_{\mathrm{L},0}^{1} + FV_{\mathrm{L},0}^{2} + FV_{\mathrm{L},0}^{3} = -25.08 + 219.21 + 92.24 = 286.36$$

4.3 Preliminary Take-aways for the practical use of the M&M model

We can now summarize the lessons that we learn from this example. Let us start with the takeaways that are different to the M&M case:

(1) Because $r_{\mathrm{U}} = r_{\mathrm{A}}$ in all points in time, we can directly calculate the value of the unlevered firm by backward iteration, summation of discounted cash flows, or by standard constant-growth annuity and perpetuity formulas.

(2) Contrary to the case of M&M, the discount rate in the FCF method and the required returns on equity and the capital cash flow are constant throughout time.

(3) The preceding point implies that we have an all-times fixed and maturity-independent relationship between r_{U} and r_{FCF}, r_{CCF} or r_{E}. These relationships are described by (45) and the expressions for r_{FCF} and r_{CCF} in section 2. After calculating r_{FCF} based on (45), we can directly calculate the value of the levered firm by backward iteration, summation of discounted cash flows or by standard constant-growth annuity and perpetuity formulas. Like in the setting of M&M, we cannot apply the equity and CCF method directly, since we do not know the value of debt and therefore the cash flow to the debt holders.

(4) None of the discount rates r_{U}, r_{FCF}, r_{CCF} and r_{E} depend on the growth of the free cash flow.

(5) The required return on the tax shield increases with the remaining lifetime. In the very end of the cash flow's lifetime, the tax shield needed to be discounted with the risk-free rate. In the beginning of the third time interval this rate was 21.688 %. The required return on the tax shield does not only depend on the maturity but also on the growth rate g_{F} as can be seen in (49).

(6) The perpetual model based on M&E does not allow for a growth rate $g_{\mathrm{F}} \geq r_{\mathrm{FCF}} = r_{\mathrm{U}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1+r_{\mathrm{U}}}{1+r_{\mathrm{f}}}$ because then the levered firm value becomes infinitely large.

(7) Performing a sensitivity analysis in the M&E setting is simpler than for M&M, because the discount rates r_{U}, r_{FCF}, r_{CCF} and r_{E} do not depend on the growth rate g_{F}, an analyst can change the growth rate without being concerned about the change in the aforementioned discount rates.

(8) All preceding conclusions are different from the conclusions in the M&M model. However, like in the M&M model, we can apply the standard perpetuity formulas in (1) to (4) as long as we know the correct relationships between these rates. This applies also to the valuation of the tax shield, since its required return converges according to expression (50).

5 Conclusions & Practical Advise

Most of the conclusions that concern either M&M or M&E have already been drawn in section 3.3 and section 4.3, respectively. Comparing these mutually exclusive frameworks, we notice that they lead to very different results. One essential difference is that r_{U}, r_{FCF}, r_{CCF}, and r_{E} are much higher in the M&E setting than in the M&M setting, even for shorter lifetimes of the cash flow. This implies that the levered and unlevered firm values in the M&M setting are larger than in the M&E setting. This difference becomes more pronounced the longer the lifetime of the free cash flow is, the higher the growth and the higher the one-period required return r_{A} applied to the FCF. In our numerical example the values according to M&M are almost 8 to 9 times higher than the values according to M&E.

Another important difference between the two approaches is that discount rates depend differently on the growth rate of the free cash flow. While the discount rates r_{U}, r_{FCF}, r_{CCF} and r_{E} are constant in M&E, they depend on growth and maturity in M&M. The required return on the tax shield r_{TS}, however, is constant in M&M, but depends on growth and maturity in M&E. This also implies that the two approaches require different translation formulas between r_{U} and r_{FCF}, r_{CCF} or r_{E}.

M&M and M&E are based on the same assumptions, except for one assumption that concerns the stochastic behavior of the FCF. Hence, an important question is: Which approach should be chosen by the practitioner? This is a somewhat difficult decision, because there does not seem to exist any literature that empirically tests the stochastic long-term behavior of free cash flows based on financial statements. However, economists and finance theorists seem to prefer auto-regressive processes, as such processes are used in the pricing of options (Rubinstein, 2000), modeling of demand (Whitt, 1981; Ryan, 2004), modeling of costs (Nembhard, 2003), among others. Also, time series analysts commonly apply auto-regressive processes of different orders and types to high-frequency financial data. We have also noticed that continuation values in the M&M framework need to be discounted with the risk-free rate. To avoid this rather strange phenomenon, an autoregressive process can be preferred.

At this point we can only give the following advice for valuation in practice and academia:

(1) Apply both the frameworks of M&M and M&E to recognize the spread between the valuation results.

(2) Assuming a particular framework (like M&M or M&E), all methods (1) to (4) need to give the same firm values if these methods are based on the same assumptions concerning the constancy of leverage, the type of the stochastic FCF process, the riskiness of debt, the growth in the FCF, etc. However, not all methods are meant to be applied directly. For example, in both the M&M and M&E settings, the unlevered and levered firm values can be found by means of the backward iteration.

Only then can the equity, debt, and tax-shield value be calculated. The M&E setting also allows the direct (or one-step) calculation of the unlevered firm and the application of the FCF method, presuming the knowledge of the r_{FCF} according to formula (45). In the M&M setting this required the development of special annuity formulas (see section 3.2). However, the equity method, CCF method, and the APV method, would not be directly accessible before we have calculated the unlevered and levered firm values.

(3) Cash flows, values or their constituents should be discounted with the appropriate discount rate. The application of the backward-iteration as described in sections 3.1 and 4.1 can help to discount different elements of the cash flow properly.

Assuming a constant level of debt financing, there are essentially no alternative settings to M&M and M&E. However, it can be shown that auto-regressive processes of first order or stationary cash flows are special cases of Markov chains. From this perspective, future research will show how the mutually exclusive models discussed in this paper can be unified under a more general framework.

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