

REMARKS ON SOLITARY WAVES IN EQUATIONS WITH NONLOCAL CUBIC TERMS

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ABSTRACT. In this overview paper, we show existence of smooth solitary-wave solutions to the nonlinear, dispersive evolution equations of the form

$$\partial_t u + \partial_x(\Lambda^s u + u\Lambda^r u^2) = 0,$$

where Λ^s, Λ^r are Bessel-type Fourier multipliers. The linear operator may be of low fractional order, $s > 0$, while the operator on the nonlinear part is assumed to act slightly smoother, $r < s - 1$. The problem is related to the mathematical theory of water waves; we build upon previous works on similar equations, extending them to allow for a nonlocal nonlinearity. Mathematical tools include constrained minimization, Lion's concentration–compactness principle, spectral estimates, and product estimates in fractional Sobolev spaces.

1. INTRODUCTION

Many one-dimensional model equations for water waves can be written as

$$\partial_t u + \partial_x(Lu - n(u)) = 0, \tag{1.1}$$

where u is a real-valued function of time and space, L is a Fourier multiplier and n is a local, nonlinear function. This class of equations includes many well-known models, such as the Korteweg–de Vries and Benjamin–Ono equations when the operator L is a positive-order operator, and Whitham-type equations when L has negative order [10, 16]. We restrict our attention here to operators L of positive order, which physically corresponds to situations with surface tension and stronger dispersion [15].

In the present work, we consider a way to include frequency interaction also in the nonlinear term. This is natural in the context of water waves, since exact modeling may give rise to nonlocal nonlinearities [14, 15]. Recent years have seen an increased interest in these types of equations, see e.g. [6, 7, 20] concerning traveling waves.

There has been extensive research on whether or not eq. (1.1) under different assumptions on L and n admits a type of solutions called *solitary waves*. These are localized solutions that propagate at constant speed ν while retaining their original shape:

$$(x, t) \mapsto u(x - \nu t) \quad \text{and} \quad u(x - \nu t) \rightarrow 0 \text{ as } |x - \nu t| \rightarrow \infty.$$

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While their existence may be demonstrated in several ways [3, 5, 13, 22], many constructions of solitary waves are based upon Weinstein’s argument in [23], where solutions are found by L^2 -constrained minimization with the help of Lions’ concentration–compactness method [17], see e.g. [18] and references therein. Building upon [2] and [18], we shall investigate how this technique may be modified to accommodate nonlocal nonlinearities. A variation of Weinstein’s argument adapted to negative s , see [9], has later been developed and applied to bidirectional systems with negative-order Fourier multipliers in nonlinear terms [6, 7]. Nonlocal nonlinearities also arise in works where solitary waves are constructed using a local variational reduction [8, 12]. In this paper, however, we will study nonlocal nonlinearities in the one-dimensional setting for positive s .

Solitary-wave solutions to eq. (1.1) arise when dispersive effects due to the term Lu and nonlinear effects due to $n(u)$ are balanced [15, 16]. Typically, one can consider operators L of order s , and nonlinearities with leading-order power p , and show that solitary-wave solutions to eq. (1.1) exist for a range of parameters s, p . For positive-order operators L , existence is shown for a given p if s is sufficiently high, $s > \frac{p-1}{p+1}$ [2, 18]. Contrariwise, the fractional KdV equation does not admit solitary-wave solutions for s below this limit [16]. Thus, if one wishes to find solitary waves when the dispersion is weak, the nonlinearity must be correspondingly weak.

In our case, instead of considering a range of powers p , we fix a cubic nonlinearity and apply an operator of order r . Let Λ^α be the Bessel Fourier multiplier in space defined by

$$\widehat{\Lambda^\alpha u}(\xi) = (1 + \xi^2)^{\frac{\alpha}{2}} \hat{u}(\xi).$$

For real numbers s, r , we shall consider the equation

$$\partial_t u + \partial_x(\Lambda^s u - u\Lambda^r u^2) = 0. \quad (1.2)$$

It is also possible to replace Λ^s, Λ^r by more general operators, see Section 3. A motivation for studying the cubic nonlinearity $u\Lambda^r u^2$ is that a variational formulation is readily available. This is not the case for the quadratic equivalent $u\Lambda^r u$. To use a similar approach for an equation with a quadratic nonlocal term, one needs to consider pseudo-products. This was recently done by the author in [19]. The author is grateful to Douglas Svensson Seth for pointing out helpful relationships in the variational formulation.

Observe that when $r = 0$, eq. (1.2) coincides with eq. (1.1) with $L = \Lambda^s$ and $n(u) = u^3$. Note that the introduction of an operator in the nonlinear term affects the interplay between dispersive and nonlinear effects. Just as the power of the nonlinearity p is bounded above by the order of the dispersive operator s in [2, 18], r must be bounded above by s in our work. We shall assume that $s, r \in \mathbb{R}$ satisfy

$$s > 0, \quad r < s - 1. \quad (1.3)$$

Here, r may be positive or negative. When $r = 0$, the lower bound for s that follows agrees with $p < 2s + 1$ from [2]. However, if r is sufficiently negative, we will see that there are solitary-wave solutions to eq. (1.2) for all $s > 0$. Thus the existence of solitary waves in this type of equation does not depend directly on the power of the nonlinearity, but rather on the strength of the nonlinearity in a general sense. Under the solitary-wave

ansatz, eq. (1.2) becomes

$$-\nu u + \Lambda^s u - u \Lambda^r u^2 = 0. \quad (1.4)$$

Theorem 1.1. *Let $s > 0$, $r < s - 1$. For every $\mu > 0$, there is a $u \in H^\infty(\mathbb{R})$ satisfying $\frac{1}{2}\|u\|_{L^2}^2 = \mu$ that solves the solitary-wave equation eq. (1.4). The corresponding solitary waves are subcritical, that is, the wave speed satisfies $\nu < 1$. Furthermore, for any fixed $\mu_0 > 0$, the estimates*

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^{\frac{s}{2}}} \approx \mu^{\frac{1}{2}}, \quad 1 - \nu \approx \mu^2. \quad (1.5)$$

hold uniformly in $\mu \in (0, \mu_0)$.

We show existence of smooth solitary waves of any size, small or large. Here, the space $H^\infty(\mathbb{R}) = \bigcap_{t \in \mathbb{R}} H^t(\mathbb{R})$, where $H^t(\mathbb{R})$ are fractional Sobolev spaces, see Section 2.1. In eq. (1.5), the implicit constants may depend on μ_0 , but μ_0 does not need to be small.

While the upper bound on r in eq. (1.3) appears in the proof through embedding and interpolation theorems, it is in fact related to properties of the equation and we cannot expect to find solutions below $s > r + 1$. When $r = 0$ we get the special case of eq. (1.1) under the solitary-wave ansatz that is most similar to our equation,

$$-\nu u + \Lambda^s u - u^3 = 0. \quad (1.6)$$

In this case, our assumptions on r, s imply that $s > 1$. For the generalized fKdV-equation with $p = 3$, the value $s = 1$ is the critical exponent below which solitary-wave solutions are no longer stable [4], and the method we use yields a set of minimizers that form stable sets of solutions [1, 2]. As mentioned, existence of solutions to eq. (1.6) for $s > \frac{p-1}{p+1} = \frac{1}{2}$ was shown in [2], but then by appealing to a scaling argument at the cost of stability.

To keep the exposition clear and highlight what is new, our strategy in proving Theorem 1.1 will be to outline the method for eq. (1.6) and discuss the changes that are made to accommodate the nonlocal nonlinearity in eq. (1.4). This constitutes Section 2. Equation (1.6) is treated in [2] and we mostly follow that paper. We will, however, also refer to specific lemmas in [18] where some arguments have been considerably simplified. We prove rigorously only the parts that are new for eq. (1.4). In Section 3, we discuss possible generalizations.

2. SOLITARY-WAVES SOLUTIONS TO EQS. (1.4) AND (1.6)

2.1. Formulation as a constrained minimization problem. In the classical case, existence of solutions to eq. (1.6) is established via a related constrained minimization problem. Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions, and let $L^p(\mathbb{R})$, $1 \leq p < \infty$ be the standard Lebesgue spaces with norm $\|u\|_{L^p}^p = \int_{\mathbb{R}} |u|^p dx$. We search for minimizers in fractional Sobolev spaces,

$$H^t(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}) : \|u\|_{H^t} = \|\Lambda^t u\|_{L^2} < \infty\}.$$

Define the functionals $\mathcal{E}, \mathcal{Q}, \mathcal{L}, \mathcal{N}: H^{\frac{s}{2}}(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\mathcal{E}(u) = \underbrace{\frac{1}{2} \int_{\mathbb{R}} u \Lambda^s u \, dx}_{=\mathcal{L}(u)} - \underbrace{\frac{1}{4} \int_{\mathbb{R}} u^4 \, dx}_{=\mathcal{N}(u)} \quad \text{and} \quad \mathcal{Q}(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 \, dx.$$

Due to the Lagrange multiplier principle, minimizers of the constrained minimization problem

$$\Gamma_{\mu} = \inf\{\mathcal{E}(u) : u \in H^{\frac{s}{2}}(\mathbb{R}) \text{ and } \mathcal{Q}(u) = \mu\}$$

will solve eq. (1.6) with wave speed ν being the Lagrange multiplier.

To formulate the corresponding minimization problem for eq. (1.4), we simply replace \mathcal{N} with $\tilde{\mathcal{N}}$ incorporating the nonlocal operator:

$$\tilde{\mathcal{E}}(u) = \underbrace{\frac{1}{2} \int_{\mathbb{R}} u \Lambda^s u \, dx}_{=\mathcal{L}(u)} - \underbrace{\frac{1}{4} \int_{\mathbb{R}} u^2 \Lambda^r u^2 \, dx}_{=\tilde{\mathcal{N}}(u)},$$

$$\tilde{\Gamma}_{\mu} = \inf\{\tilde{\mathcal{E}}(u) : u \in H^{\frac{s}{2}}(\mathbb{R}) \text{ and } \mathcal{Q}(u) = \mu\}$$

It is easily verified that the Fréchet derivative of $\tilde{\mathcal{N}}$ is $u \Lambda^r u^2$, and so a minimizer of $\tilde{\Gamma}_{\mu}$ solves eq. (1.4) with Lagrange multiplier ν :

$$\mathcal{L}'(u) - \tilde{\mathcal{N}}'(u) - \nu \mathcal{Q}'(u) = -\nu u + \Lambda^s u - u \Lambda^r u^2 = 0.$$

2.1.1. *The infimum is well defined.* To show that Γ_{μ} is bounded below, the strategy is to bound \mathcal{N} in terms of \mathcal{L} to a power $\gamma < 1$. Then we would have

$$\mathcal{E}(u) \geq \mathcal{L}(u) - \mathcal{N}(u) \geq \mathcal{L}(u) - C\mathcal{L}(u)^{\gamma} > -\infty,$$

since $\mathcal{L}(u)$ is positive and the first term dominates the last as $\mathcal{L}(u) \rightarrow \infty$. This is straightforward since $\mathcal{N} \approx \|u\|_{L^4}^4$ and $\mathcal{L}(u) \approx \|u\|_{H^{\frac{s}{2}}}^2$ and Sobolev embedding and interpolation yields

$$\|u\|_{L^4} \lesssim \|u\|_{H^{\frac{1}{4}}} \lesssim \|u\|_{L^2}^{1-\frac{1}{2s}} \|u\|_{H^{\frac{s}{2}}}^{\frac{1}{2s}}, \quad (2.1)$$

and $\frac{1}{2s} < \frac{1}{2}$ as $s > 1$.

The same argument is directly applicable also to $\tilde{\Gamma}_{\mu}$ as soon as we establish the inequality

$$\underbrace{\|u^2\|_{H^{\frac{s}{2}}}^2}_{\approx \tilde{\mathcal{N}}(u)} \lesssim \|u\|_{L^2}^{4-2\gamma} \underbrace{\|u\|_{H^{\frac{s}{2}}}^{2\gamma}}_{\approx \mathcal{L}(u)^{\gamma}} \quad (2.2)$$

for some $0 < \gamma < 1$. The nonlocal operator makes the argument more involved but has the added benefit of "weakening" the nonlinear part $\tilde{\mathcal{N}}$ when r is sufficiently negative, allowing us to go below the limit in the classical case, $s > 1$. To show eq. (2.2), we rely on product estimates in fractional Sobolev spaces, see Proposition A.1 in Appendix A.

Lemma 2.1. *For $u \in H^{\frac{s}{2}}(\mathbb{R})$, there is a $\gamma < 1$ such that*

$$\|u^2\|_{H^{\frac{r}{2}}} \lesssim \|u\|_{L^2}^{2-\gamma} \|u\|_{H^{\frac{s}{2}}}^{\gamma}.$$

Proof. We treat the cases $s > 1$, $s \leq 1$ separately and apply Proposition A.1.

Suppose $s > 1$. We assume that $r > 0$. Otherwise, we can apply the same method to any \tilde{r} such that $0 < \tilde{r} < s - 1$ and use that $\|u^2\|_{H^{\frac{\tilde{r}}{2}}} \leq \|u^2\|_{H^{\frac{r}{2}}}$. We pick a $\tau \in \mathbb{R}$ satisfying

$$1 < \tau < s - r$$

and apply Proposition A.1 (i) with $t_1 = \frac{\tau}{2}$ and $t_2 = \frac{\tau}{2}$:

$$\|u^2\|_{H^{\frac{\tau}{2}}} \lesssim \|u\|_{H^{\frac{\tau}{2}}} \|u\|_{H^{\frac{\tau}{2}}}$$

Interpolation then yields the desired result,

$$\|u^2\|_{H^{\frac{\tau}{2}}} \lesssim \|u\|_{L^2}^{2-\frac{r+\tau}{s}} \|u\|_{H^{\frac{\tau}{2}}}^{\frac{r+\tau}{s}},$$

with $\gamma = \frac{r+\tau}{s} < \frac{r+s-r}{s} = 1$.

Suppose now that $s \leq 1$. If $s = 1$, we can pick \tilde{s} satisfying $r + 1 < \tilde{s} < 1$, apply the same method to \tilde{s} instead of s . Hence we can assume that $s < 1$. If $r > -1$, applying Proposition A.1 (ii) with $t_1 = 0$ and $t_2 = \frac{r+1}{2}$ yields:

$$\begin{aligned} \|u^2\|_{H^{\frac{r}{2}}} &\lesssim \|u\|_{L^2} \|u\|_{H^{\frac{r+1}{2}}} \\ &\lesssim \|u\|_{L^2}^{2-\frac{r+1}{s}} \|u\|_{H^{\frac{r+1}{2}}}^{\frac{r+1}{s}}, \end{aligned}$$

which shows the result with $\gamma = \frac{r+1}{s} < 1$. If $r \leq -1$, we pick \tilde{r} satisfying $-1 < \tilde{r} < s - 1$ and use \tilde{r} instead of r . \square

2.1.2. *Upper bound for Γ_μ .* To be able to estimate the sizes of the functionals $\mathcal{L}, \mathcal{N}, \tilde{\mathcal{N}}$, we also need an upper bound for $\Gamma_\mu, \tilde{\Gamma}_\mu$. In [2], the author shows that $\Gamma_\mu < 0$. However, as remarked in a later version of that paper, that estimate only holds for large solutions or if the symbol of the dispersive operator is homogeneous. Fortunately, only $\Gamma_\mu < \mu$ is needed to obtain the necessary estimates in the rest of the proof. We show this bound using a long-wave ansatz for $\tilde{\Gamma}_\mu$, which of course implies $\Gamma_\mu < \mu$ since that is a special case with $r = 0$.

Lemma 2.2. *The infimum satisfies $\tilde{\Gamma}_\mu < \mu$.*

Proof. For $\phi \in \mathcal{S}(\mathbb{R})$ satisfying $\mathcal{Q}(\phi) = \mu$ and $0 < \theta < 1$, let $\phi_\theta(x) = \sqrt{\theta}\phi(\theta x)$. Then $\mathcal{Q}(\phi_\theta) = \mu$ and by properties of the Fourier transform,

$$\mathcal{L}(\phi_\theta) = \mu + \frac{1}{2} \int_{\mathbb{R}} (\langle \xi \rangle^s - 1) \left| \widehat{\phi_\theta}(\xi) \right|^2 d\xi = \mu + \frac{1}{2} \int_{\mathbb{R}} (\langle \theta \xi \rangle^s - 1) \left| \widehat{\phi}(\xi) \right|^2 d\xi.$$

If $s < 2$, then $\langle \theta \xi \rangle^s - 1 \lesssim |\theta \xi|^2$, while if $s \geq 2$, then $\langle \theta \xi \rangle^s - 1 \lesssim |\theta \xi|^2 + |\theta \xi|^s$. Since $0 < \theta < 1$ and $\phi \in \mathcal{S}(\mathbb{R})$, this implies that there is a constant $C_1 > 0$ such that

$$\mathcal{L}(\phi_\theta) \leq \mu + C_1 \theta^2.$$

Noting that $\langle \theta \xi \rangle^r \geq \min(\langle \xi \rangle^r, 1)$ for all $\xi \in \mathbb{R}$, we can similarly conclude that

$$\tilde{\mathcal{N}}(\phi_\theta) = \frac{1}{4} \int_{\mathbb{R}} \langle \xi \rangle^r \left| \widehat{\phi_\theta^2}(\xi) \right|^2 d\xi = \frac{1}{4} \theta \int_{\mathbb{R}} \langle \theta \xi \rangle^r \left| \widehat{\phi^2}(\xi) \right|^2 d\xi \geq C_2 \theta$$

for some $C_2 > 0$. Thus

$$\mathcal{E}(\phi_\theta) \leq \mu + C_1\theta^2 - C_2\theta,$$

and picking $\theta > 0$ small enough ensures that

$$\Gamma_\mu \leq \mathcal{E}(\phi_\theta) < \mu.$$

□

2.2. Concentration–compactness. Although $\Gamma_\mu, \tilde{\Gamma}_\mu > \infty$, minimizing sequences of $\Gamma_\mu, \tilde{\Gamma}_\mu$ do not necessarily converge as \mathbb{R} is not compact. To overcome this, we use Lions’ concentration–compactness principle [17]. Informally, any bounded sequence will either *vanish* (the mass spreads out), *dichotomize* (the mass separates in space) or *concentrate* (most of the mass remains in a bounded domain). The concentration–compactness principle is stated in a form suitable for our purpose in [2, Lemma 2.6], but we explain what needs to be shown in the next sections. We will apply the result to the sequence $\{\frac{1}{2}u_n^2\}_{n \in \mathbb{N}}$, where $\{u_n\}_{n \in \mathbb{N}}$ is a minimizing sequence for Γ_μ or $\tilde{\Gamma}_\mu$. The strategy will be to show that any minimizing sequence will have a subsequence for which we can exclude both vanishing and dichotomy. Then we show that concentration leads to convergence.

2.2.1. Excluding vanishing. By definition, the sequence $\{\frac{1}{2}u_n^2\}_{n \in \mathbb{N}}$ *vanishes* if for all $r < \infty$

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} u_n^2 dx = 0.$$

It turns out that if $\|u\|_{H^{\frac{s}{2}}} < \infty$ and $\|u\|_{L^q} \geq \delta > 0$ for some suitable q , then $\{\frac{1}{2}u_n^2\}_{n \in \mathbb{N}}$ cannot vanish, due to the following result.

Lemma 2.3 ([2, Lemma 3.4]). *Let $v \in H^{\frac{s}{2}}(\mathbb{R})$ and suppose q satisfies $q > 2$ if $s \geq 1$ and $2 < q < \frac{2}{1-s}$ if $s < 1$. Given $\delta > 0$, suppose that $\|v\|_{H^{\frac{s}{2}}}^{-1}, \|v\|_{L^q} \geq \delta$. Then there is $\varepsilon > 0$ such that*

$$\sup_{j \in \mathbb{Z}} \int_{j-2}^{j+2} |v|^q dx \geq \varepsilon. \quad (2.3)$$

In [2] it is then shown that eq. (2.3) and $\|v\|_{H^{\frac{s}{2}}}^{-1} > \delta$ implies

$$\sup_{j \in \mathbb{Z}} \int_{j-2}^{j+2} |v|^2 dx \geq \varepsilon,$$

which of course excludes vanishing when applied to u_n .

In the classical case, $\|u_n\|_{H^{\frac{s}{2}}} < \infty$ follows easily from $\Gamma_\mu < \mu$ and eq. (2.1) since

$$\|u_n\|_{H^{\frac{s}{2}}}^2 \approx \mathcal{L}(u_n) = \mathcal{E}(u_n) + \mathcal{N}(u_n) < \mu + \mu^{1-\gamma} \|u_n\|_{H^{\frac{s}{2}}}^{2\gamma}, \quad \gamma < 1.$$

Using only that $\mathcal{L}(u) \geq \mu$, $\Gamma_\mu < \mu$ one can also easily deduce that there is a $\delta > 0$ such that $\mathcal{N}(u_n) \geq \delta$, see e.g. [2, Lemma 4.2], passing to a subsequence if necessary. In the classical case, this is all that is needed to apply Lemma 2.3, as $\|u_n\|_{L^4}^4 \approx \mathcal{N}(u_n) \geq \delta$ and $q = 4$ satisfies the assumptions (recall that $s > 1$ in the classical case).

For the modified equation eq. (1.4), $\|u_n\|_{H^{\frac{s}{2}}} < \infty$ and $\tilde{\mathcal{N}} \geq \delta$ hold by the same argument, since the arguments only relied on bounds on the functionals $\mathcal{E}, \mathcal{L}, \mathcal{N}$ and Γ_μ . The same bounds hold for the modified versions, so one may simply replace $\mathcal{E}, \mathcal{N}, \Gamma_\mu$ with $\tilde{\mathcal{E}}, \tilde{\mathcal{N}}, \tilde{\Gamma}_\mu$ and obtain the same results.

When attempting to go from a lower bound on $\tilde{\mathcal{N}}(u_n)$ to a lower bound on $\|u_n\|_{L^q}$, the introduction of the nonlocal nonlinearity gives rise to two new problems. The first is that the estimate $\|u_n\|_{L^4}^4 \gtrsim \tilde{\mathcal{N}}(u_n)$ does not hold in general. The second, and bigger problem is that $q = 4$ only satisfies the assumptions of Lemma 2.3 when $s > \frac{1}{2}$, while a sufficiently negative r allows the minimization problem to be well-defined for $s > 0$. To overcome this, we need the following lemma.

Lemma 2.4. *Let $u \in H^{\frac{s}{2}}(\mathbb{R})$. When $0 < r < s - 1$,*

$$\|u^2\|_{H^{\frac{r}{2}}} \leq \|u\|_{L^4}^{2(1-\frac{r}{s})} \|u\|_{H^{\frac{s}{2}}}^{\frac{2r}{s}}.$$

Furthermore, when $s < 1$, there is a q satisfying $2 < q < \frac{2}{1-\frac{s}{2}}$ such that

$$\|u^2\|_{H^{\frac{r}{2}}} \lesssim \|u\|_{L^q} \|u\|_{H^{\frac{s}{2}}}. \quad (2.4)$$

Proof. We show the estimate when $0 < r < s - 1$ first. Interpolating, we find that

$$\|u^2\|_{H^{\frac{r}{2}}} \leq \|u^2\|_{L^2}^{1-\frac{r}{s}} \|u^2\|_{H^{\frac{s}{2}}}^{\frac{r}{s}}.$$

Since $s > 1$, $H^{\frac{s}{2}}(\mathbb{R})$ is a Banach algebra. Hence $\|u^2\|_{H^{\frac{s}{2}}} \lesssim \|u\|_{H^{\frac{s}{2}}}^2$, and we conclude that

$$\|u^2\|_{H^{\frac{r}{2}}} \leq \|u^2\|_{L^2}^{1-\frac{r}{s}} \|u\|_{H^{\frac{s}{2}}}^{\frac{2r}{s}} = \|u\|_{L^4}^{2-\frac{2r}{s}} \|u\|_{H^{\frac{s}{2}}}^{\frac{2r}{s}}.$$

The estimate when $s < 1$ is shown by a direct calculation, by appealing to Sobolev embedding and Proposition A.2. If $r > \frac{s}{2} - 1$, set $\tilde{r} = r$. Otherwise, pick $\tilde{r} \in (\frac{s}{2} - 1, s - 1)$. Let

$$p = \frac{2}{1-\tilde{r}} \quad \text{and} \quad q = \frac{2}{\frac{s}{2}-\tilde{r}}.$$

Since $s < 1 < 2$ and $\tilde{r} < s - 1$ the denominator $\frac{s}{2} - \tilde{r} > 0$. Furthermore, $\tilde{r} > \frac{s}{2} - 1$ guarantees that $q > 2$. Clearly also $q < \frac{2}{1-\frac{s}{2}}$. By Sobolev embedding, valid since $p < \frac{2}{2-\frac{s}{2}} < 2$ and $\frac{\tilde{r}}{2} - \frac{1}{2} = -\frac{1}{p}$,

$$\|u^2\|_{H^{\frac{\tilde{r}}{2}}} \lesssim \|u^2\|_{H^{\frac{\tilde{r}}{2}}} \lesssim \|u^2\|_{L^p}.$$

Now applying Proposition A.2 with $t = \frac{s}{2}$ and p, q as above gives eq. (2.4). \square

When $s \geq 1$, $q = 4$ satisfies the assumptions of Lemma 2.3. In that case, we can bound $\|u_n\|_{L^4}$ from below in terms of $\tilde{\mathcal{N}}(u_n)$ using the first part of Lemma 2.4 when $r > 0$ or the simpler estimate $\|u^2\|_{H^{\frac{r}{2}}}^2 \leq \|u_n\|_{L^4}^2$ when $r \leq 0$. When $s < 1$, we can instead use the second part of Lemma 2.4 to find a lower bound for $\|u_n\|_{L^q}$, $2 < q < \frac{2}{1-\frac{s}{2}} < \frac{2}{1-s}$. In any case, we can conclude that there is a $\delta > 0$ and a suitable q such that $\|u_n\|_{L^q} \geq \delta$ when $\tilde{\mathcal{N}}(u_n) \geq \tilde{\delta} > 0$ and use Lemma 2.3 to conclude that vanishing does not occur.

2.2.2. *Excluding dichotomy.* The map $\mu \mapsto \Gamma_\mu$ is strictly subadditive:

$$\Gamma_{\mu_1+\mu_2} < \Gamma_{\mu_1} + \Gamma_{\mu_2}. \quad (2.5)$$

Subadditivity follows from subhomogeneity, $\Gamma_{t\mu} < t\Gamma_\mu, t > 1$ [2, Lemma 3.3], which in turn follows directly from a scaling argument since $\mathcal{L}(u), \mathcal{N}(u)$ are both homogeneous and strictly greater than zero [2, Lemma 4.3]. For $\tilde{\Gamma}_\mu$, nothing changes.

The idea is now to show that dichotomy contradicts eq. (2.5). Here, we follow [18], where the assumptions on the linear, dispersive operator allow for a considerably simpler proof. If $\{u_n^2\}_{n \in \mathbb{N}}$ dichotomizes, then it is possible to find sequences $u_n^{(1)}, u_n^{(2)}$ with $(u_n^{(1)})^2 + (u_n^{(2)})^2 = u_n^2$ such that

$$\mathcal{Q}(u_n^{(1)}) \rightarrow \lambda, \quad \mathcal{Q}(u_n^{(2)}) \rightarrow (\mu - \lambda), \quad \int_{\text{supp } u_n^{(1)} \cap \text{supp } u_n^{(2)}} u_n^2 dx \rightarrow 0 \quad (2.6)$$

for some $\lambda \in (0, \mu)$ as $n \rightarrow \infty$ [18, Proposition 7]. In particular, one can pick two smooth functions $\phi, \psi: \mathbb{R} \rightarrow [0, 1]$ with $\phi(x) = 1$ when $|x| \leq 1$ and $\phi(x) = 0$ when $|x| \geq 2$ and $\phi^2 + \psi^2 = 1$, and find sequences $\{x_n\}_{n \in \mathbb{N}}, \{R_n\}_{n \in \mathbb{N}} \subset \mathbb{R}, R_n \rightarrow \infty$ such that eq. (2.6) holds for $u_n^{(1)} = \phi_n u_n$ and $u_n^{(2)} = \psi u_n = \sqrt{1 - \phi_n^2} u_n$, where $\phi_n(x) = \phi(\frac{x-x_n}{R_n})$. We can assume without loss of generality that $x_n = 0$ for all n . Showing that $\mathcal{E}(u_n)$ eventually also decomposes as $u_n^{(1)}, u_n^{(2)}$ separate in space,

$$\lim_{n \rightarrow \infty} (\mathcal{E}(u_n)) = \lim_{n \rightarrow \infty} (\mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)})), \quad (2.7)$$

gives the desired contradiction, since then

$$\Gamma_\mu = \liminf_{n \rightarrow \infty} \mathcal{E}(u_n) \geq \liminf_{n \rightarrow \infty} \mathcal{E}(u_n^{(1)}) + \liminf_{n \rightarrow \infty} \mathcal{E}(u_n^{(2)}) = \Gamma_{\bar{\mu}} + \Gamma_{(\mu - \bar{\mu})}. \quad (2.8)$$

In the classical case, $\mathcal{N}(u_n)$ is local and the decomposition for this part is automatic,

$$\mathcal{N}(u_n) \rightarrow \mathcal{N}(u_n^{(1)}) + \mathcal{N}(u_n^{(2)}) \quad \text{as } n \rightarrow \infty. \quad (2.9)$$

Since Λ^s is nonlocal, the same decomposition is not automatically true for $\mathcal{L}(u_n)$. However, it holds because of the regularity of the symbol $\langle \xi \rangle^s$ (here, the symbol is smooth, but continuously differentiable suffices). The key is the following commutator estimate.

Lemma 2.5 ([18, Lemma 6.1]). *Let $u, v \in H^{\frac{s}{2}}(\mathbb{R})$ and let $\rho \in \mathcal{S}(\mathbb{R})$ be a non-negative Schwartz function. Define $\rho_R(x) = \rho(x/R)$. Then*

$$\left| \int_{\mathbb{R}} v(\rho_R \Lambda^s u - \Lambda^s(\rho_R u)) dx \right| \rightarrow 0$$

as $R \rightarrow \infty$.

With this estimate, it is an easy matter, see e.g. [18, Proposition 7], to show that

$$\mathcal{L}(u_n) \rightarrow \mathcal{L}(u_n^{(1)}) + \mathcal{L}(u_n^{(2)}) \quad \text{as } n \rightarrow \infty. \quad (2.10)$$

Since $\tilde{\mathcal{N}}$ contains a nonlocal operator, eq. (2.9) but for $\tilde{\mathcal{N}}$ is not automatic. We will show it using a similar strategy as for \mathcal{L} , beginning with a version of Lemma 2.5 with Λ^r and u^2 . Since r can be negative and u^2, v^2 is not necessarily in $H^{\frac{s}{2}}(\mathbb{R})$ or even $L^2(\mathbb{R})$, the proof is more involved than for Lemma 2.5.

Lemma 2.6. *Let $u, v \in H^{\frac{s}{2}}(\mathbb{R})$ and let $\rho \in \mathcal{S}(\mathbb{R})$ be a non-negative Schwartz function. Define $\rho_R(x) = \rho(x/R)$. Then*

$$\left| \int_{\mathbb{R}} v^2(\rho_R \Lambda^r u^2 - \Lambda^r(\rho_R u^2)) dx \right| \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Proof. By Lemma 2.1, $u^2 \in H^{\frac{r}{2}}(\mathbb{R})$. Using Plancherel's and Fubini's theorems,

$$\begin{aligned} & \left| \int_{\mathbb{R}} v^2(\rho_R \Lambda^r u^2 - \Lambda^r(\rho_R u^2)) dx \right| \\ &= \left| \int_{\mathbb{R}} \widehat{v^2}(\xi) ((\widehat{\rho_R} * \langle \cdot \rangle^r \widehat{u^2})(\xi) - \langle \xi \rangle^r (\widehat{\rho_R} * \widehat{u^2})(\xi)) d\xi \right| \\ &\leq \int_{\mathbb{R}} |\widehat{\rho_R}(t)| \int_{\mathbb{R}} \left| \widehat{v^2}(\xi) \right| \left| \widehat{u^2}(\xi - t) \right| \underbrace{|\langle \xi - t \rangle^r - \langle \xi \rangle^r|}_D d\xi dt := I. \end{aligned} \quad (2.11)$$

We split the domain of integration of the outer integral into $|t| \geq R^{-\frac{1}{2}}$ and $|t| < R^{-\frac{1}{2}}$ and show that each part approaches zero as $R \rightarrow \infty$. For the first part, $|t| \geq R^{-\frac{1}{2}}$, we apply the triangle inequality for a coarse upper bound for the difference D : $|\langle \xi - t \rangle^r - \langle \xi \rangle^r| \leq \langle \xi - t \rangle^r + \langle \xi \rangle^r$. Observe that $\langle \xi + t \rangle^{\frac{r}{2}} \lesssim \langle \xi \rangle^{\frac{r}{2}} \langle t \rangle^{\frac{|r|}{2}}$, so that $\|u(\cdot - t)\|_{H^{\frac{r}{2}}} \lesssim \langle t \rangle^{\frac{|r|}{2}} \|u\|_{H^{\frac{r}{2}}}$. Combining these estimates, we find that

$$\begin{aligned} & \int_{|t| \geq R^{-\frac{1}{2}}} |\widehat{\rho_R}(t)| \int_{\mathbb{R}} \left| \widehat{v^2}(\xi) \right| \left| \widehat{u^2}(\xi - t) \right| |\langle \xi - t \rangle^r - \langle \xi \rangle^r| d\xi dt \\ &\leq \int_{|t| \geq R^{-\frac{1}{2}}} |\widehat{\rho_R}(t)| \int_{\mathbb{R}} \left| \widehat{v^2}(\xi) \langle \xi - t \rangle^{\frac{r}{2}} \right| \left| \widehat{u^2}(\xi - t) \langle \xi - t \rangle^{\frac{r}{2}} \right| + \left| \widehat{v^2}(\xi) \langle \xi \rangle^{\frac{r}{2}} \right| \left| \widehat{u^2}(\xi - t) \langle \xi \rangle^{\frac{r}{2}} \right| d\xi dt \\ &\lesssim \|v^2\|_{H^{\frac{r}{2}}} \|u^2\|_{H^{\frac{r}{2}}} \int_{|t| \geq R^{-\frac{1}{2}}} \langle t \rangle^{\frac{|r|}{2}} R |\widehat{\rho}(Rt)| dt = \|v^2\|_{H^{\frac{r}{2}}} \|u^2\|_{H^{\frac{r}{2}}} \int_{|t| \geq R^{\frac{1}{2}}} \langle \frac{t}{R} \rangle^{\frac{|r|}{2}} |\widehat{\rho}(t)| dt. \end{aligned} \quad (2.12)$$

Since $\rho \in \mathcal{S}(\mathbb{R})$, this expression approaches zero as $R \rightarrow \infty$. For the second part, $|t| < R^{-\frac{1}{2}}$, we can assume that $|t| \leq 1$ since $R^{-\frac{1}{2}} \rightarrow 0$ as $R \rightarrow \infty$. With the mean value theorem, we find a refined upper bound for D :

$$|\langle \xi - t \rangle^r - \langle \xi \rangle^r| \leq \sup_{\theta \in (\min(\xi - t, \xi), \max(\xi - t, \xi))} |t| \langle \theta \rangle^{r-1} \lesssim |t| \langle \xi \rangle^{r-1}.$$

Thus

$$\begin{aligned} & \int_{|t| < R^{-\frac{1}{2}}} |\widehat{\rho_R}(t)| \int_{\mathbb{R}} \left| \widehat{v^2}(\xi) \right| \left| \widehat{u^2}(\xi - t) \right| |\langle \xi - t \rangle^r - \langle \xi \rangle^r| d\xi dt \\ &\leq \int_{|t| < R^{-\frac{1}{2}}} R |\widehat{\rho}(Rt)| |t| \int_{\mathbb{R}} \left| \widehat{v^2}(\xi) \langle \xi \rangle^{\frac{r-1}{2}} \right| \left| \widehat{u^2}(\xi - t) \langle \xi \rangle^{\frac{r-1}{2}} \right| d\xi dt \\ &\lesssim \|v^2\|_{H^{\frac{r-1}{2}}} \|u^2\|_{H^{\frac{r-1}{2}}} \frac{1}{R} \int_{\mathbb{R}} |\widehat{\rho}(t)| |t| dt \end{aligned} \quad (2.13)$$

which also approaches zero as $R \rightarrow \infty$. Combining eqs. (2.12) and (2.13) with eq. (2.11), clearly $I \rightarrow 0$ as $R \rightarrow \infty$. \square

With this lemma, we can show that $\mathcal{N}(u_n)$ decomposes as $n \rightarrow \infty$, that is, eq. (2.9) holds for $\tilde{\mathcal{N}}$. Since $(u_n^{(1)})^2 + (u_n^{(2)})^2 = u_n^2$,

$$\tilde{\mathcal{N}}(u_n) = \tilde{\mathcal{N}}(u_n^{(1)}) + \tilde{\mathcal{N}}(u_n^{(2)}) + \frac{1}{2} \int_{\mathbb{R}} (1 - \phi_n^2) u_n^2 \Lambda^r \phi_n^2 u_n^2 dx,$$

and we will show that the last term approaches zero as $n \rightarrow \infty$. First, observe that Lemma 2.6 still holds if we replace ρ_R with $1 - \rho_R$ since

$$\int_{\mathbb{R}} v^2 ((1 - \rho_R) \Lambda^r u^2 - \Lambda^r ((1 - \rho_R) u^2)) dx = - \int_{\mathbb{R}} v^2 (\rho_R \Lambda^r u^2 - \Lambda^r (\rho_R u^2)) dx.$$

Applying this and Lemma 2.6 with ϕ^2 as ρ and $\sqrt{1 - \phi^2}$ as $1 - \rho$ respectively, we find that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} (1 - \phi_n^2) u_n^2 \Lambda^r \phi_n^2 u_n^2 dx = \lim_{n \rightarrow \infty} \tilde{\mathcal{N}}(\phi_n \sqrt{1 - \phi_n^2} u_n).$$

By eq. (2.2), for some $\gamma < 2$,

$$\tilde{\mathcal{N}}(\phi_n \sqrt{1 - \phi_n^2} u_n) \lesssim \|\phi_n \sqrt{1 - \phi_n^2} u_n\|_{L^2}^{4-2\gamma} \|\phi_n \sqrt{1 - \phi_n^2} u_n\|_{H^{\frac{s}{2}}} \rightarrow 0$$

where the limit as $n \rightarrow \infty$ follows from eq. (2.6). Thus we conclude that

$$\tilde{\mathcal{N}}(u_n) \rightarrow \tilde{\mathcal{N}}(u_n^{(1)}) + \tilde{\mathcal{N}}(u_n^{(2)}) \quad \text{as } n \rightarrow \infty.$$

Combining this with eq. (2.10), we see that eq. (2.7) and eq. (2.8) must hold for $\tilde{\mathcal{E}}, \tilde{\Gamma}_\mu$ and we have the desired contradiction.

2.2.3. Convergence from concentration. Having excluded vanishing and dichotomy, we conclude that a subsequence of $\{\frac{1}{2}u_n^2\}_{n \in \mathbb{N}}$ *concentrates*. We can then show that it converges in $H^{\frac{s}{2}}(\mathbb{R})$ to a minimizer of Γ_μ . Since a minimizing sequence of Γ_μ is bounded in $H^{\frac{s}{2}}(\mathbb{R})$, it converges weakly – up to a subsequence – in $H^{\frac{s}{2}}(\mathbb{R})$ to some $\omega \in H^{\frac{s}{2}}(\mathbb{R})$. From this, one can use standard arguments to show that $\{u_n\}_{n \in \mathbb{N}}$ converges – up to subsequences and translations – in $L^2(\mathbb{R})$ to ω , see e.g. [2, Lemma 4.6] or [18, Proposition 8]. In the classical case, eq. (2.1) tells us that $u_n \rightarrow \omega$ in $L^4(\mathbb{R})$ as well so that $\mathcal{N}(u_n) \rightarrow \mathcal{N}(\omega)$. Using this and the weak lower semi-continuity of the $H^{\frac{s}{2}}(\mathbb{R})$ -norm,

$$\Gamma_\mu \leq \mathcal{L}(\omega) - \mathcal{N}(\omega) \leq \liminf_{n \rightarrow \infty} (\mathcal{L}(u_n) - \mathcal{N}(u_n)) = \Gamma_\mu.$$

Thus $\mathcal{E}(u_n) \rightarrow \mathcal{E}(\omega)$. Since $\mathcal{N}(u_n) \rightarrow \mathcal{N}(\omega)$, then $\mathcal{L}(u_n) \rightarrow \mathcal{L}(\omega)$, which together with the weak $H^{\frac{s}{2}}$ -convergence implies that $u_n \rightarrow \omega$ in $H^{\frac{s}{2}}(\mathbb{R})$.

For $\tilde{\mathcal{N}}(u)$, eq. (2.1) does not necessarily hold, so we need to show that $\tilde{\mathcal{N}}(u_n) \rightarrow \tilde{\mathcal{N}}(\omega)$ if $u_n \rightarrow \omega$ in $L^2(\mathbb{R})$. Since $\tilde{\mathcal{N}}(u) \approx \|u^2\|_{H^{\frac{s}{2}}}^2$, it suffices to show that $u_n^2 \rightarrow \omega^2$ in $H^{\frac{s}{2}}(\mathbb{R})$.

Lemma 2.7. *Let $u, v \in H^{\frac{s}{2}}$. Then there exists $0 < \alpha \leq 1$ such that*

$$\|u^2 - v^2\|_{H^{\frac{s}{2}}} \lesssim \|u - v\|_{L^2}^\alpha \|u - v\|_{H^{\frac{s}{2}}}^{1-\alpha} \|u + v\|_{H^{\frac{s}{2}}}.$$

Proof. For $\bar{u}, \bar{v} \in H^{\frac{s}{2}}(\mathbb{R})$, there is always $0 < \alpha \leq 1$ such that

$$\|\bar{u}\bar{v}\|_{H^{\frac{s}{2}}} \lesssim \|\bar{u}\|_{L^2}^\alpha \|\bar{u}\|_{H^{\frac{s}{2}}}^{1-\alpha} \|\bar{v}\|_{H^{\frac{s}{2}}}.$$

This can be seen by applying Proposition A.1 to $\bar{u}\bar{v}$ in the same manner as in the proof of Lemma 2.1, using the lowest regularity t_1 for \bar{u} and then interpolating only that part.

Noting that $u^2 - v^2 = (u - v)(u + v)$, we can apply the above equation to $\bar{u} = u - v$, $\bar{v} = u + v$ to obtain the result. \square

Since $u_n, \omega \in H^{\frac{s}{2}}(\mathbb{R})$, Lemma 2.7 and the already established convergence in $L^2(\mathbb{R})$ implies that u_n^2 converges strongly to ω^2 in $H^{\frac{r}{2}}(\mathbb{R})$. Since a minimizing sequence for $\tilde{\Gamma}_\mu$ satisfies the same properties as a minimizing sequence for Γ_μ , the other arguments are unchanged. This concludes the proof of existence of solitary-waves solutions to eq. (1.4) for any $\mu > 0$.

2.3. Wave speed and regularity for $\mu > 0$. We estimate the wave speed and regularity for minimizers of $\tilde{\Gamma}_\mu$ directly, based on similar results in [2, 18]. We show that the wave speed ν is subcritical by multiplying eq. (1.4) with the solution u , integrating and reshuffling:

$$\nu = \frac{\int_{\mathbb{R}} u \Lambda^s u \, dx - \int_{\mathbb{R}} u^2 \Lambda^r u^2 \, dx}{\int_{\mathbb{R}} u^2 \, dx} = \frac{\tilde{\mathcal{E}}(u) - \tilde{\mathcal{N}}(u)}{\mathcal{Q}(u)} < \frac{\tilde{\mathcal{E}}(u)}{\mathcal{Q}(u)} < 1. \quad (2.14)$$

Here, we used that $\tilde{\mathcal{N}}(u) > 0$ and $\tilde{\Gamma}_\mu = \tilde{\mathcal{E}}(u) < \mu$.

A consequence of this is that the operator $(\Lambda^s - \nu): H^t(\mathbb{R}) \rightarrow H^{t-s}(\mathbb{R})$ is invertible and we can write eq. (1.4) as

$$u = (\Lambda^s - \nu)^{-1}(u \Lambda^r u^2). \quad (2.15)$$

We only know that the solution u is in $H^{\frac{s}{2}}(\mathbb{R})$, but using the formulation above we can show that the solution inherits regularity from the equation itself. We need the following lemma.

Lemma 2.8. *Let $t \geq \frac{s}{2}$ and suppose $u \in H^t(\mathbb{R})$. Then $u \Lambda^r u^2 \in H^{\min(t, t-r)}(\mathbb{R})$.*

Proof. Suppose first that $t > \frac{1}{2}$. Then $H^t(\mathbb{R})$ is a Banach algebra and $u^2 \in H^t(\mathbb{R})$. This implies in turn that $\Lambda^r u^2 \in H^{t-r}(\mathbb{R})$. If $r \leq 0$, then $t - r \geq t$ and we conclude that $u \Lambda^r u^2 \in H^t(\mathbb{R})$. If on the other hand $r > 0$, we use Proposition A.1 with $t_1 = t - r, t_2 = t$ to conclude that $u \Lambda^r u^2 \in H^{t-r}(\mathbb{R})$.

If $t < \frac{1}{2}$, then Proposition A.1 with $t_1 = t_2 = t$ implies that $u^2 \in H^{2t-\frac{1}{2}}(\mathbb{R})$. Then $\Lambda^r u^2 \in H^{2t-\frac{1}{2}-r}(\mathbb{R})$. Since by assumption $s \leq t$, then $2t - \frac{1}{2} - r > 2t - s + 1 - \frac{1}{2} \geq \frac{1}{2} > t$ and $u \Lambda^r u^2 \in H^t(\mathbb{R})$. Finally, if $t = \frac{1}{2}$, pick $\frac{s}{2} < \tilde{t} < t$ and do as above. \square

Now assume that $u \in H^t(\mathbb{R}), t \geq \frac{s}{2}$. Then by the above lemma, $u \Lambda^r u^2 \in H^{\min(t, t-r)}(\mathbb{R})$ which implies that $u \in H^{\min(t, t-r)+s}(\mathbb{R})$ due to eq. (2.15). Since s is positive and larger than r , this gives an improved regularity estimate for u . We start with $t_0 = \frac{s}{2}$ and find $u \in H^{t_1}(\mathbb{R}), t_1 = \min(t_0, t_0 - r) + s$, then $u \in H^{t_2}(\mathbb{R}), t_2 = \min(t_1, t_1 - r) + s$ and so on.

Repeating this procedure gives increasingly better regularity estimates for $u \in H^{\frac{s}{2}}$. Since we can continue indefinitely, we conclude that $u \in H^\infty(\mathbb{R}) = \cap_{t \in \mathbb{R}} H^t(\mathbb{R})$.

With this, we have shown the first part of Theorem 1.1 which holds for all $\mu > 0$.

2.4. Estimates for norms and wave speed when $\mu \in (0, \mu_0)$. It remains to show the estimates in eq. (1.5), which hold uniformly in $\mu \in (0, \mu_0)$ for any fixed $\mu_0 > 0$. The argument is similar to that in Propositions 4, 9, and 10 in [18]. In the following, we fix $\mu_0 > 0$.

To show that $\|u\|_{H^{\frac{s}{2}}} \approx \mu^{\frac{1}{2}}$, we first find a crude upper bound for $\tilde{\mathcal{N}}(u)$. Using eq. (2.2) and Lemma 2.2,

$$\tilde{\mathcal{N}}(u) \lesssim \mu^{2-\gamma} \|u\|_{H^{\frac{s}{2}}}^{2\gamma} \leq \mu^{2-\gamma} (\tilde{\mathcal{N}}(u) + \mu)^\gamma \lesssim \mu^2 + \mu^{2-\gamma} \tilde{\mathcal{N}}(u)^\gamma,$$

for some $\gamma \in (0, 1)$. This implies that

$$\tilde{\mathcal{N}}(u) \lesssim \mu^2, \quad \text{and} \quad \mu \lesssim \|u\|_{H^{\frac{s}{2}}}^2 \approx \mathcal{L}(u) \leq \mu + \tilde{\mathcal{N}}(u) \lesssim \mu. \quad (2.16)$$

for $\mu \in (0, \mu_0)$ with implicit constants depending on μ_0 .

We proceed to show the estimate of the wave speed in eq. (1.5). To do so, we need an improved estimate for $\tilde{\mathcal{N}}(u)$ in terms of μ . As in [18], we find an improved upper bound for the infimum $\tilde{\Gamma}_\mu$ and decompose u into high- and low-frequency components to estimate $\tilde{\mathcal{N}}(u)$ precisely.

Lemma 2.9. *Let $\mu_0 > 0$ and $\mu \in (0, \mu_0)$. A minimizer u of $\tilde{\Gamma}_\mu$ satisfies*

$$\tilde{\mathcal{N}} \approx \mu^3.$$

Proof. The first step is to show that there is a $\kappa > 0$ such that

$$\tilde{\Gamma}_\mu < \mu - \kappa\mu^3 \quad (2.17)$$

Pick $\phi \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{Q}(\phi) = 1$, and define $\phi_{\mu,t}(x) = \sqrt{\mu t} \phi(tx)$. Then $\mathcal{Q}(\phi_{\mu,t}) = \mu$. In a similar manner as in the proof of Lemma 2.2, one may show that there are constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \mathcal{L}(\phi_{\mu,t}) &\leq \mu + C_1 t^2 \mu, \\ \tilde{\mathcal{N}}(\phi_{\mu,t}) &\geq C_2 \mu^2 t. \end{aligned}$$

Set $t = C_3 \mu$, with $C_3 < \frac{1}{\mu_0}$, such that $t < 1$ for all $\mu \in (0, \mu_0)$. Then

$$\tilde{\mathcal{E}}(\phi_{\mu,t}) \leq \mu - \mu^3 (C_2 C_3 - C_1 C_3^2) = \mu - \kappa \mu^3,$$

where we can guarantee that $\kappa > 0$ by picking C_3 sufficiently small.

The lower bound on $\tilde{\mathcal{N}}(u)$ follows immediately from eq. (2.17),

$$\tilde{\mathcal{N}}(u) = \mathcal{L}(u) - \tilde{\mathcal{E}}(u) \geq \kappa \mu^3.$$

To bound $\tilde{\mathcal{N}}(u)$ from above, let χ denote the characteristic function equal to 1 on $[-1, 1]$ and 0 otherwise. Set $u = u_1 + u_2$, where $\hat{u}_1 = \chi \hat{u}$, $\hat{u}_2 = \hat{u}$. Then $\|u_2\|_{H^{\frac{s}{2}}} \approx \|u_2\|_{\dot{H}^{\frac{s}{2}}}$ and

$$\mathcal{L}(u) - \mu = \frac{1}{2} \int_{\mathbb{R}} (\langle \xi \rangle^s - 1) |\hat{u}_1 + \hat{u}_2|^2 d\xi \approx \|u_1\|_{\dot{H}^1}^2 + \|u_2\|_{\dot{H}^{\frac{s}{2}}}^2, \quad (2.18)$$

since $\langle \xi \rangle^s - 1$ grows like $|\xi|^s$ or $|\xi|^2$ when $|\xi|$ is large or small respectively, and \hat{u}_1, \hat{u}_2 have disjoint supports. Now,

$$\tilde{\mathcal{N}}(u) \approx \|(u_1 + u_2)^2\|_{H^{\frac{r}{2}}}^2 \lesssim \|u_1^2\|_{H^{\frac{r}{2}}}^2 + \|u_2^2\|_{H^{\frac{r}{2}}}^2 + \|u_1 u_2\|_{H^{\frac{r}{2}}}^2, \quad (2.19)$$

and we estimate the three terms separately. Using Lemma 2.1 and eqs. (2.16) and (2.18),

$$\|u_2^2\|_{H^{\frac{r}{2}}}^2 \lesssim \|u_2\|_{H^{\frac{s}{2}}}^4 \approx \|u_2\|_{\dot{H}^{\frac{s}{2}}}^4 \lesssim (\mathcal{L}(u) - \mu)^2 \lesssim \tilde{\mathcal{N}}(u)^2 \lesssim \mu^{\frac{3}{2}} \tilde{\mathcal{N}}(u)^{\frac{1}{2}}. \quad (2.20)$$

Similarly,

$$\|u_1 u_2\|_{H^{\frac{r}{2}}}^2 \lesssim \|u_1\|_{H^{\frac{s}{2}}}^2 \|u_2\|_{\dot{H}^{\frac{s}{2}}}^2 \lesssim \mu (\mathcal{L}(u) - \mu) \lesssim \mu \tilde{\mathcal{N}}(u) \lesssim \mu^{\frac{3}{2}} \tilde{\mathcal{N}}(u)^{\frac{1}{2}}. \quad (2.21)$$

Estimating $\|u_1^2\|_{H^{\frac{r}{2}}}^2$ requires a little more care. If $r \leq 0$, then $\|u_1^2\|_{H^{\frac{r}{2}}}^2 \lesssim \|u_1\|_{L^4}^4$, whereas if $r > 0$, then

$$\|u_1^2\|_{H^{\frac{r}{2}}}^2 = \left\| \langle \cdot \rangle^{\frac{r}{2}} \hat{u}_1 * \hat{u}_1 \right\|_{L^2}^2 \lesssim \left\| \Lambda^{\frac{r}{2}} u_1 \right\|_{L^4}^4,$$

where we used that $\langle \xi \rangle \lesssim \langle \xi - \eta \rangle \langle \eta \rangle$. Since $\|u_1\|_{L^4}^4 \lesssim \left\| \Lambda^{\frac{r}{2}} u_1 \right\|_{L^4}^4$ for $r > 0$, it suffices to estimate the latter. We can use the Gagliardo-Nirenberg-Sobolev interpolation inequality, eq. (2.16) and eq. (2.18) to find that

$$\left\| \Lambda^{\frac{r}{2}} u_1 \right\|_{L^4}^4 \lesssim \left\| \Lambda^{\frac{r}{2}} u_1 \right\|_{L^2}^3 \left\| \Lambda^{\frac{r}{2}} u_1 \right\|_{\dot{H}^1} \lesssim \|u_1\|_{H^{\frac{s}{2}}}^3 \|u_1\|_{\dot{H}^1} \lesssim \mu^{\frac{3}{2}} (\mathcal{L} - \mu)^{\frac{1}{2}} \lesssim \mu^{\frac{3}{2}} \tilde{\mathcal{N}}(u)^{\frac{1}{2}}. \quad (2.22)$$

Combining eqs. (2.19) to (2.22) we conclude that

$$\tilde{\mathcal{N}}(u) \lesssim \mu^3.$$

□

The estimate of the wave speed in eq. (1.5) is an immediate consequence of Lemma 2.9. Using eq. (2.14), $\mathcal{L}(u) \geq \mu$, $\tilde{\mathcal{E}}(u) < \mu$ and $\tilde{\mathcal{N}}(u) \approx \mu^3$,

$$1 - \nu = \frac{\mu - \tilde{\mathcal{E}}(u) + \tilde{\mathcal{N}}(u)}{\mu} \geq \frac{\tilde{\mathcal{N}}(u)}{\mu} \approx \mu^2 \approx \frac{2\tilde{\mathcal{N}}(u)}{\mu} \geq \frac{\mu - \mathcal{L}(u) + 2\tilde{\mathcal{N}}(u)}{\mu} = 1 - \nu.$$

Finally, we want to estimate $\|u\|_{L^\infty}$, finishing the proof of eq. (1.5). This can be done by estimating $\|u\|_{L^\infty} \lesssim \|u\|_{H^1}$ and then using a similar bootstrap argument as in Section 2.3. However, we need to ensure that the estimate $\|u\|_{H^1} \lesssim \|u\|_{H^{\frac{s}{2}}}$ is uniform in $\mu \in (0, \mu_0)$. Increasing powers of μ in each iteration does not cause problems as we can always estimate $\mu \leq \mu_0$, but we want to make sure that we do not lose any powers of μ when we estimate u by $u\Lambda^r u^2$. We used that the operator $(\Lambda^s - \nu)$ was invertible, but the operator norm of $(\Lambda^s - \nu)^{-1}$ will blow up as $\mu \rightarrow 0$. Therefore, we instead rewrite eq. (1.4) as

$$(\Lambda^s - \nu + 1)u = u\Lambda^r u^2 + u, \quad (2.23)$$

and note that the operator norm of $(\Lambda^s - \nu + 1)^{-1}: H^\alpha(\mathbb{R}) \rightarrow H^{\alpha+t}(\mathbb{R})$ is independent of $\mu \in (0, \mu_0)$,

$$\left\| (\Lambda^s - \nu + 1)^{-1} \right\|_{H^\alpha \rightarrow H^{\alpha+t}} = \sup_{\xi \in \mathbb{R}} \frac{\langle \xi \rangle^s}{\langle \xi \rangle^s - \nu + 1} \leq 1,$$

since $1 - \nu \approx \mu^2$ (see also Proposition 10 in [18]). Lemma 2.8 clearly also holds with $u\Lambda^r u^2 + u$ instead of $u\Lambda^r u^2$, and so the argument is exactly the same as in Section 2.3, only using eq. (2.23) instead of eq. (2.15). We conclude that

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^1} \lesssim \|u\|_{H^{\frac{s}{2}}} \approx \mu.$$

3. SOME COMMENTS ON POSSIBLE GENERALIZATIONS

In [2], [18], existence of solitary waves to eq. (1.1) is proved for a more general operator than $L = \Lambda^s$. Likewise, the modifications due to the nonlocal nonlinearity presented in this paper do not rely heavily on special properties of Λ^r . In fact, for the equation

$$\partial_t u + \partial_x(Lu - uNu^2) = 0,$$

the proof of Theorem 1.1 will go through under the following assumptions on L and N .

(A1) The symbol $m \in C(\mathbb{R})$ of the operator L is real-valued, positive, even and satisfies the growth bounds

$$\begin{aligned} m(\xi) - m(0) &\approx |\xi|^s && \text{for } |\xi| \geq 1, \\ m(\xi) - m(0) &\approx |\xi|^{s'} && \text{for } |\xi| < 1, \end{aligned}$$

with $s > 1/2, s' > 1$. Furthermore, we require that $\xi \mapsto m(\xi)/\langle \xi \rangle^s$ be uniformly continuous on \mathbb{R} .

(A2) The symbol $n \in C^1(\mathbb{R})$ of the operator N is real-valued, even, and satisfies the growth bounds

$$\begin{aligned} n(\xi) &\approx \langle \xi \rangle^r && \text{for } \xi \in \mathbb{R}, \\ \left| \frac{\partial n}{\partial \xi}(\xi) \right| &\lesssim \langle \xi \rangle^{r-1} && \text{for } \xi \in \mathbb{R}. \end{aligned}$$

where $r < s - 1$.

The assumptions on L are as in [18]. One may observe that $m(\xi) = \langle \xi \rangle^s$ satisfies these assumptions with $s' = 2$. We refer to [18] for a detailed discussion and comment only on a few important changes. Since m can be homogenous, we use $\|u\|_{H^{\frac{s}{2}}}^2 \approx \mathcal{Q}(u) + \mathcal{L}(u)$ instead of $\|u\|_{H^{\frac{s}{2}}}^2 \approx \mathcal{L}(u)$, and the upper bound for the infimum becomes $\Gamma_\mu < m(0)\mu$ instead of $\Gamma_\mu < \mu$. The uniform continuity of $m(\xi)/\langle \xi \rangle^s$ is used to exclude dichotomy.

For the symbol n , a similar regularity assumption to the one on m would allow us to prove Lemma 2.6 and exclude dichotomy. However, while a Fourier multiplier is continuous on $L^2(\mathbb{R})$ as long as it is uniformly bounded, a stronger assumption is needed to ensure continuity on $L^p(\mathbb{R}), p \neq 2$. There are different criteria that guarantee this; continuous differentiability and the bound on $\partial n/\partial \xi$ suffices [11, Theorem 5.2.7]. Unlike m , the symbol n must be inhomogeneous to establish the upper bound on Γ_μ by a long-wave ansatz as

is done here. A physical interpretation is that solitary waves arise when dispersive and nonlinear effects are balanced, and inhomogeneity of n ensures that uNu^2 is nonlinear in the long-wave limit.

Under assumptions A1 and A2, the wave-speed estimate in eq. (1.5) would instead be

$$m(0) - \nu \approx \mu^\beta, \quad \text{where} \quad \beta = \frac{s'}{s' - 1}.$$

This is the same estimate that was shown for an equation like eq. (1.1) with a local nonlinearity with a cubic leading-order term in [18].

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APPENDIX A. TECHNICAL TOOLS

Proposition A.1 (Products in $H^t(\mathbb{R})$ [21, Theorem 4.6.1/1]). *Assume that $t_1, t_2 \in \mathbb{R}$ satisfy*

$$t_1 \leq t_2 \quad \text{and} \quad t_1 + t_2 > 0.$$

Furthermore, let $f \in H^{t_1}(\mathbb{R}), g \in H^{t_2}(\mathbb{R})$.

(i) *Let $t_2 > 1/2$. Then*

$$\|fg\|_{H^{t_1}} \lesssim \|f\|_{H^{t_1}} \|g\|_{H^{t_2}}.$$

(ii) *Let $t_2 < 1/2$. Then*

$$\|fg\|_{H^{t_1+t_2-1/2}} \lesssim \|f\|_{H^{t_1}} \|g\|_{H^{t_2}}.$$

Proposition A.2 (Products in $L^p(\mathbb{R})$ [21, Theorem 4.4.4/3]). *Assume that $t > 0$ and that $1 < p, q < \infty$ satisfy*

$$\frac{1}{p} < \frac{1}{q} + \frac{1}{2}, \quad \frac{1}{p} > \frac{1}{q} + \left[\frac{1}{2} - t \right]^+, \quad t > \frac{1}{q} - \frac{1}{2},$$

where $[a]^+ = \max(a, 0)$. Let $f \in L^q(\mathbb{R}), g \in H^t(\mathbb{R})$. Then

$$\|fg\|_{L^p} \lesssim \|f\|_{L^q} \|g\|_{H^t}.$$

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