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Abstract

The goal of this thesis is to lay the foundations for a theory of ∞-categories internal to an ∞-topos $\mathcal{B}$. Our model for such internal ∞-categories is based on the notion of a complete Segal object, but can equivalently be described by sheaves of ∞-categories on $\mathcal{B}$. After setting up the basic framework of this theory, we study internal presheaf ∞-categories: we prove a version of Yoneda's lemma in this context, and we show that internal presheaf ∞-categories can be characterised by a universal property: they provide a model for free cocompletions by internal colimits. As a prerequisite for the latter result, we develop the theory of adjunctions, limits and colimits and Kan extensions for internal ∞-categories.

We then move on to the study of accessibility and presentability of internal ∞-categories, which we use to define and study internal ∞-topoi. One of our main results is a correspondence between these internal ∞-topoi and geometric morphisms into the base ∞-topos $\mathcal{B}$. We use this result to study relative aspects in higher topos theory from an internal point of view: we provide a formula for general pullbacks of ∞-topoi, and we characterise smooth and proper geometric morphisms in terms of properties of the associated internal ∞-topoi. We furthermore use the latter result to compare the notions of smooth and proper maps in topology and in higher topos theory.

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Contents

Introduction

Part of the usefulness of (higher) topoi comes from their ability to serve as a *bridge* between mathematical microcosms that are a priori very different: on the one hand, one can regard the theory of ∞-topoi as an enlargement of the theory of topological spaces. In this interpretation, an ∞-topos $\mathcal{X}$ (as developed in [49]) is thus conceptualised as a *generalised space*. This point of view has been the driving motivation for the development of topos theory by the Grothendieck school [8] and is supported by the observation that most topological features of a space X are visible on the level of its underlying ∞-topos of sheaves $\mathrm{Sh}(X)$. One can therefore do *geometry* with ∞-topoi (see for example [3] for more on this perspective). In another interpretation, ∞-topoi can be viewed as *mathematical universes* for certain *structures*. As such, they allow us to equip mathematical objects with additional data of the kind that is governed by them: if $\mathcal{B}$ is such an ∞-topos, a mathematical object Z can be *enriched* with the structure that is encoded by $\mathcal{B}$ by means of equipping Z with some form of *parametrisation* by $\mathcal{B}$. Usually, this simply means realising Z as a *sheaf* on $\mathcal{B}$, i.e. as a limit-preserving functor from the opposite of $\mathcal{B}$ into the (∞)-category that Z naturally belongs to, although there are cases in which the parametrisation is encoded by more data. An example that has generated a lot of attention in recent years is the ∞-topos $\mathrm{Cond}(\mathrm{Ani})$ of *condensed anima* in the sense of [75, 14] (modulo to some set-theoretic complications). This ∞-topos governs *topological* structures, so that an arbitrary mathematical object can be equipped with a topology through a parametrisation by $\mathrm{Cond}(\mathrm{Ani})$. For example, a topological ring R can be encoded as a sheaf $\mathrm{Cond}(\mathrm{Ani})^{\mathrm{op}} \to \mathrm{Rings}$, where the target denotes the category of (discrete) rings. Another example is the ∞-topos $\mathrm{PSh}(\mathrm{Orb}_G)$ of presheaves on the *orbit category* Orb_G associated with a finite group G. This ∞-topos can be thought of as a mathematical universe governing *G-equivariant* structures [12].

The fact that these a priori very different mathematical concepts come together in the theory of ∞-topoi allows us in particular to *combine* them in a meaningful way. The connection is established through the notion of a *geometric morphism* $f_* : \mathcal{X} \to \mathcal{B}$. If $\mathcal{X}$ is thought of as a spatial object, such a map can encode several things: if $\mathcal{B}$ is to be regarded as a generalised space as well, the map f_* exhibits $\mathcal{X}$ as a *family* of generalised spaces that is *parametrised* by the base space $\mathcal{B}$. If the latter is instead interpreted as a mathematical universe for a certain structure, we can still regard f_* as a parametrisation of $\mathcal{X}$ by $\mathcal{B}$, but its meaning is a different one: in this case, it exhibits $\mathcal{X}$ as being *enriched* with the kind of structure that is encoded by $\mathcal{B}$. As an example of the first kind, if $(X_s)_{s\in S}$ is a family of spaces parametrised by a base space S, then this family determines a continuous map $X \to S$, and by passing to ∞-topoi one obtains a geometric morphism $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(S)$. As an example of the second kind, we may consider the case where the base ∞-topos is given by Cond(Ani). In this situation, a geometric morphism $f_* : \mathcal{X} \to \mathrm{Cond}(\mathrm{Ani})$ tells us that $\mathcal{X}$ is in some way equipped with additional topological structure. As a concrete example from algebraic geometry, if X is a scheme, its associated hypercomplete *proétale* ∞-topos $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ admits a canonical geometric morphism into Cond(Ani), which witnesses the fact that all of the various homotopical invariants that one can extract from X, such as its (pro)étale fundamental groups, can be equipped with an additional topological structure [89].

No matter whether a geometric morphism $f_* : \mathcal{X} \to \mathcal{B}$ exhibits a generalised space $\mathcal{X}$ as a family of spaces or as a single space with additional structure: any kind of information that we may extract from $\mathcal{X}$ to improve our understanding of this space ought to *remember* this parametrisation. For example, any homotopical data associated to $\mathcal{X}$, such as its homotopy groups, should be parametrised by $\mathcal{B}$ as well, which means that the should define sheaves of groups on $\mathcal{B}$. If $\mathcal{B}$ is regarded as a space, this simply means that the homotopy groups remember the fibre-wise information of the family. If $\mathcal{B}$ is viewed as a universe for a mathematical structure, this instead means that the homotopy groups carry this additional structure as well. As an example of the first kind, if $(X_s)_{s\in S}$ is a family of spaces and if $f\colon X \to S$ is the associated continuous map, the nth sheaf cohomology groups $(\mathbf{H}^n(X_s;\mathbb{Z}))_{s\in S}$ can be assembled to a sheaf on S: if $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(S)$

is the geometric morphism induced by f, then this sheaf can be defined as the 0-truncation of $f_* \underline{\mathbf{B}^n \mathbb{Z}}$, where $\underline{\mathbf{B}^n \mathbb{Z}}$ denotes the constant sheaf on X associated with the n-fold delooping of $\mathbb{Z}$. Under certain assumptions on f, the stalk of this sheaf at any point $s \in S$ then precisely recovers the group $\mathbf{H}^n(X_s, \mathbb{Z})$ [49, Chapter 7]. As an example of the second kind, to any scheme X, Barwick-Glasman-Haine associate its *Galois category* $\mathrm{Gal}(X)$, which can be thought of as a global version of the Galois groups of its residue fields [13]. In light of our previous observation that the proétale ∞-topos $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ comes equipped with a geometric morphism into condensed anima, one would expect the Galois category of X to carry a certain topological structure. This is indeed the case: by its very construction, the Galois category is a *profinite* category, and can as such be encoded by a certain sheaf of categories on $\mathrm{Cond}(\mathrm{Ani})$.

In modern days, *categorified* invariants of geometrical objects have come more and more into focus. For example, where classically one would have studied a space through its cohomology groups, one nowadays tends to focus on the entire derived ∞-category instead. This was made possible by the emergence of higher category theory with its multitude of useful and easily applicable methods. Thus, if one were to choose this modern approach in order to study a (generalised) space $\mathcal{X}$ that comes together with a parametrisation by an ∞-topos $\mathcal{B}$ in form of a geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$, any categorified invariant that one associates to $\mathcal{X}$ should come in the form of a *sheaf of ∞-categories* on $\mathcal{B}$.

Categorified invariants do not only play an important role in studying spatial structures, but also when trying to understand *algebraic* objects: for example, it is common practice to try to understand a ring R by means of studying its ∞-category of modules $\mathrm{Mod}(R)$. Now if R carries additional structure that is encoded by exhibiting R as a sheaf of rings on some ∞-topos $\mathcal{B}$ that governs this kind of structure, we would likewise expect the ∞-category of modules on R to inherit this enrichment, so that it ought to come in the form of a sheaf of ∞-categories on $\mathcal{B}$ as well. If $\mathcal{B}$ is the ∞-topos $\mathrm{Cond}(\mathrm{Ani})$ of condensed anima, this for example means that the ∞-category of modules over a topological ring should admit a topological structure as well. Likewise, the key algebraic objects of interest in equivariant homotopy theory are *genuine G-spectra*, which can be thought of as spectra that carry a G-action. Also in this case, the ∞-category of genuine G-spectra (which can be thought of as the ∞-category of modules over the equivariant sphere spectrum) should be considered together with a G-equivariant structure, i.e. an enhancement as a sheaf of ∞-categories on $\mathrm{PSh}(\mathrm{Orb}_G)$.

For many purposes, it is crucial that we take the additional structure of a categorified invariant into account. For example, if X is a scheme, we need to take the profinite structure on its Galois category $\mathrm{Gal}(X)$ into account in order to be able to recover constructible sheaves on X from representations of $\mathrm{Gal}(X)$ [13, Theorem 12.1.6]. Moreover, if one wishes to classify constructible sheaves of R-modules on X when R carries a topology, one has to consider *continuous* representations of $\mathrm{Gal}(X)$ with values in (the subcategory of perfect objects in) $\mathrm{Mod}(R)$, which means in particular that the topological structure of $\mathrm{Mod}(R)$ must also be taken into account.

This raises the question: how does one work with such categorified invariants when they are accompanied by an additional parametrisation by an ∞-topos $\mathcal{B}$? After all, the fact that one now has to work with *sheaves* of ∞-categories instead of just bare ∞-categories appears to constitute a substantial increase in complexity. One possible way out of this problem is provided by the process of *internalisation.*

Parametrisation and internalisation

One interpretation of the notion of an ∞-topos that we have not yet mentioned is that of a *model for constructive logic.* In fact, it was observed by Lawvere and Tierney [48] already shortly after the concept of a topos had been introduced by the Grothendieck school [8] that this very notion can be viewed as an abstraction of the category of sets, so that every topos can be used as an environment for (constructive) set-theoretic reasoning. This means that every topos admits a semantics for first-order logic with respect to which every statement from *intuitionistic* set theory is valid in it[1]. Consequently, every set-theoretic construction can be *interpreted internally in any topos* $\mathcal{B}$. Likewise, every statement about these constructions that does not rely on non-intuitionistic axioms like the law of excluded middle or the axiom of choice will remain valid. For example, when

[1]By a topos, we always mean a *Grothendieck* topos, i.e. one which is a locally presentable category and therefore in particular admits a natural numbers object.

interpreting the theory of rings internal to a topos $\mathcal{B}$, one ends up with the notion of an *internal ring* in $\mathcal{B}$: an object $R \in \mathcal{B}$, together with four maps

$$+ \colon R \times R \to R \qquad \cdot \colon R \times R \to R \qquad 0 \colon 1_{\mathcal{B}} \to R \qquad 1 \colon 1_{\mathcal{B}} \to R$$

in which $+$ is the addition map and $\cdot$ the multiplication map and where 0 and 1 are to be thought of as picking out the neutral elements for addition and multiplication, respectively. Furthermore, there are certain *axioms* that can be expressed in a diagrammatic manner involving these four maps. Similarly, one can interpret the theory of *categories* internally in the topos $\mathcal{B}$, which leads to the notion of an *internal category*: a pair of objects $C_0, C_1 \in \mathcal{B}$ in which one is to think of C_0 as the *object of objects* and of C_1 as the *object of morphisms*, together with maps

$$\begin{tikzcd} C_0 \arrow[r, "e"] & C_1 \arrow[l, bend right, "t"'] \arrow[l, bend left, "s"] \end{tikzcd} \qquad\qquad C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{comp}} C_1$$

where e picks out the identity on an object, s and t are the source and target maps, respectively, and comp is the composition map. Again, there are in addition certain axioms involving these maps that can be expressed in a purely diagrammatic form. Now a key observation is that the datum of an internal ring in $\mathcal{B}$ is tantamount to that of a *sheaf of rings* on $\mathcal{B}$, and likewise the notion of an internal category in $\mathcal{B}$ is entirely equivalent to that of a *sheaf of categories* on $\mathcal{B}$[2]. In other words, *internal* rings and categories in $\mathcal{B}$ correspond precisely to rings and categories *parametrised* by $\mathcal{B}$. Hence, by interpreting intuitionistic first-order logic internally in $\mathcal{B}$, one can study such parametrised rings and categories in exactly the same way as one studies ordinary rings and ordinary categories, essentially without any increase in complexity.

For ∞-topoi, the same heuristic remains valid: every ∞-topos $\mathcal{B}$ can be regarded as a model for *homotopy theory* and as such an abstraction of the ∞-category Ani of ∞-groupoids (or *anima*). Informally, this means that any construction in Ani can be performed internally in an arbitrary ∞-topos $\mathcal{B}$, and every statement about such constructions that is sufficiently *intuitionistic* remains valid internally

[2]Strictly speaking, since Cat is a $(2,1)$-category, a sheaf of categories is a *pseudofunctor* $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ that preserves small limits. Such an object is traditionally called a *stack*.

in $\mathcal{B}$. One way to make this precise is through the emerging research area of *homotopy type theory* [80]. In this approach, the idea is to develop homotopy theory synthetically within an (intuitionistic) dependent type theory. As this type theory has been shown to have models in any ∞-topos [78], it follows that any construction and argument that one can make entirely within homotopy type theory has meaning and is valid internal to any ∞-topos $\mathcal{B}$. In the same way as in the 1-toposic case, one can therefore study (higher) mathematical objects that are parametrised by $\mathcal{B}$ on the same footing as one studies their non-parametrised analogues, as long as one's arguments can be formalised within homotopy type theory. To some degree, this approach can be applied to the theory of ∞-categories themselves: although these kind of objects are beyond the realm of bare homotopy type theory, one can consider *extensions* of this type theory that are sufficiently strong so that they support a notion of *synthetic higher categories*. A candidate for such an extension has been suggested by Riehl-Shulman [72] and further studied by Buchholtz-Weinberger [15] and Weinberger [84, 85, 86, 87, 88]. Thus, by arguing about ∞-categories synthetically within this framework, one can argue simultaneously about parametrised ∞-categories without any additional effort.

However, this approach has a shortcoming: the fact that the theory of synthetic ∞-categories is a nascent field and still requires a few foundational limitations to be resolved before it is powerful enough to support the full weight of higher category theory. Luckily, though, the average practitioner need not necessarily work *syntactically* with ∞-categories in order to reap the benefits of internalisation: instead, one can choose a middle ground by working *semantically* with ∞-categories internal to an ∞-topos. That is, instead of working from within a formal language, one can work directly with those structures internal to an ∞-topos $\mathcal{B}$ that one would end up with if one were to interpret synthetic ∞-category theory in $\mathcal{B}$. In practice, this amounts to working *diagrammatically* with higher categorical structures, i.e. to only make use the abstract formal properties that all ∞-topoi have in common in order to set up the theory of higher categories. This approach still constitutes a reduction in complexity, as it makes no difference whether one works internal to the ∞-topos Ani (which recovers usual ∞-category theory) or any other ∞-topos $\mathcal{B}$ (which recovers the theory of parametrised ∞-categories). At the same time, it circumvents the need for a high-level formal language to develop this theory; it only requires the framework of higher category theory itself, which has by now reached adulthood. We call this approach *internal higher category theory*. The goal of this thesis is to develop such a framework.

Internal higher category theory

$\mathcal{B}$-categories The notion of an *∞-category* C *internal to an ∞-topos* $\mathcal{B}$, hereafter referred to as a *$\mathcal{B}$-category*, is a straightforward generalisation of that of an internal category in a 1-topos as discussed above. The main difference is that (1) one now has an object of n-morphisms C_n for every $n \geq 0$ (where for $n = 0$ one recovers the object of objects and for $n = 1$ the objects of morphisms), and (2) each object of n-morphisms C_n itself contains higher homotopical information, being an object of the ∞-topos $\mathcal{B}$ that need not be truncated. Formally, we can define such a $\mathcal{B}$-category as a certain *simplicial object*

$$\mathsf{C} \colon \Delta^{\mathrm{op}} \to \mathcal{B}$$

in $\mathcal{B}$, where Δ denotes the *simplex category*, i.e. the category of finite non-empty linearly ordered sets. In addition, for this simplicial object C to be a $\mathcal{B}$-category, it has to satisfy two conditions:

(Segal conditions) for every $n \geq 2$ the natural map $\mathsf{C}_n \to \mathsf{C}_1 \times_{\mathsf{C}_0} \cdots \times_{\mathsf{C}_0} \mathsf{C}_1$ is an equivalence;

(Univalence) the map $\mathsf{C}_0 \to (\mathsf{C}_0 \times \mathsf{C}_0) \times_{\mathsf{C}_1 \times \mathsf{C}_1} \mathsf{C}_3$ is an equivalence

(cf. Section 1.2.3 for details on how these maps are defined). The Segal conditions specify that the object of n-morphisms C_n precisely encodes *composable sequences* of 1-morphisms, and univalence makes sure that the subobject of *equivalences* in C_1 (which is precisely what the right-hand side encodes) can be identified with C_0, i.e. that a map in C is an equivalence if and only if it is equivalent to an *identity*. Thus, the Segal conditions make sure that the composition of morphisms is well-defined, and univalence guarantees that the *internal* notion of an equivalence in a $\mathcal{B}$-category is in agreement with the *external* notion of an equivalence as provided by the meta-theory of ∞-categories. In the case where $\mathcal{B} = \mathrm{Ani}$, this definition precisely recovers the notion of a *complete Segal space* as studied by Rezk [70]. By a theorem of Joyal and Tierney [44], these are a model for ∞-categories, so that the theory of $\mathcal{B}$-categories reduces to ordinary higher category theory in that case. Moreover, one can canonically identify $\mathcal{B}$-categories with *sheaves of ∞-categories* on $\mathcal{B}$, i.e. limit-preserving functors $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, so that this notion indeed recovers what we were originally interested in, namely ∞-categories that are parametrised by $\mathcal{B}$.

It is precisely the tight connection between internal and parametrised higher category theory that brings life to this framework. In fact, virtually every single concrete example of a $\mathcal{B}$-category comes in the form of a sheaf of ∞-categories on $\mathcal{B}$. On the other hand, carrying out arguments about these objects is usually much simpler when viewing them from an internal point of view, simply because, as explained above, this allows us to reason about $\mathcal{B}$-categories in the same way as we can reason about ∞-categories.

Another fact that allows for an efficient development of the theory of $\mathcal{B}$-categories is that we allow ourselves to make use of the full strength of the ambient theory of ∞-categories. For example, the framework of localisations in presentable ∞-categories makes it immediate to define the ∞-category $\mathrm{Cat}(\mathcal{B})$ of $\mathcal{B}$-categories and to show that it is presentable and cartesian closed. The latter property means that if C and D are $\mathcal{B}$-categories, we can form the associated *functor $\mathcal{B}$-category* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$. Presentability of $\mathrm{Cat}(\mathcal{B})$ furthermore implies that we can talk about *limits* and *colimits* of $\mathcal{B}$-categories without any further effort.

The universe Since univalence means that the equivalences in a $\mathcal{B}$-category C can be identified with its identities, C has the property that all its morphisms are equivalences precisely if the unique morphism $\mathsf{C}_0 \to \mathsf{C}_1$ is an equivalence (which can be taken as the *definition* of this property). This condition already implies that *all* simplicial maps are equivalences, i.e. that C is in the essential image of the diagonal map $\mathcal{B} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B})$ (which is an embedding as Δ is weakly contractible). Thus, the latter map identifies $\mathcal{B}$ with the full subcategory of $\mathrm{Cat}(\mathcal{B})$ that is spanned by the *$\mathcal{B}$-groupoids*. We now come to our first example of a $\mathcal{B}$-category, which will also be the single most important one: the *universe for $\mathcal{B}$-groupoids* $\mathrm{Grpd}_{\mathcal{B}}$. It is defined as the sheaf of ∞-categories

$$\mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}, \qquad A \mapsto \mathcal{B}_{/A}$$

that carries $A \in \mathcal{B}$ to the slice ∞-category $\mathcal{B}_{/A}$ and that acts on morphisms via pullback. The fact that this defines a sheaf, i.e. a limit-preserving functor, is a consequence (in fact, the definition) of the *descent* property of ∞-topoi. Strictly speaking, this sheaf takes values in the ∞-category of *large* ∞-categories, but this is not a problem: it simply means that $\mathrm{Grpd}_{\mathcal{B}}$ defines a *large* $\mathcal{B}$-category, as was to be expected.

The universe $\mathrm{Grpd}_{\mathcal{B}}$ plays the same role among $\mathcal{B}$-categories that the ∞-category Ani of ∞-groupoids plays among ∞-categories. Thus, we may regard $\mathrm{Grpd}_{\mathcal{B}}$ as the *reflection* of the base ∞-topos $\mathcal{B}$ within itself. Large parts of the first half of this thesis are dedicated to a justification of this heuristic.

Presheaves One of the most fundamental constructions in category theory is that of *presheaf* categories. Using the notion of functor $\mathcal{B}$-categories and the universe for $\mathcal{B}$-groupoids, it is immediate to obtain a $\mathcal{B}$-categorical version of this construction. In fact, if C is an arbitrary $\mathcal{B}$-category, we may define the associated $\mathcal{B}$-category of presheaves as $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}) = \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}})$. Here the opposite C^{op} can be defined as the simplicial object obtained from C by precomposing with the involution $\Delta \simeq \Delta$ that is induced by taking the opposite of a linearly ordered set.

The main reason why presheaves play such a pivotal role is *Yoneda's lemma*. As a first cornerstone of our framework, we will establish a $\mathcal{B}$-categorical analogue of this result. To even be able to formulate the very statement, we need to be able to construct *mapping bifunctors*. As in higher category theory, it can be very hard to directly construct such functors, due to the presence of infinite coherence data. The solution in higher category theory is to model such functors by *left fibrations*, which are usually much easier to construct, and to then use a result commonly referred to as the *straightening equivalence* to obtain the desired functors from the latter [49, Theorem 2.2.1.2]. The same strategy can be employed in internal higher category theory.

We define a notion of *left fibrations* $p\colon \mathsf{P} \to \mathsf{C}$ between $\mathcal{B}$-categories, and we construct a $\mathcal{B}$-category $\mathsf{LFib}_{\mathsf{C}}$ of such left fibrations over C. We then show:

Theorem 2.2.1.1. *There is a canonical equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \simeq \mathsf{LFib}_{\mathsf{C}}$$

that is natural in C.

Using this result, it is now easy to construct mapping bifunctors: for every $\mathcal{B}$-category, we can define a left fibration $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ where $\mathsf{Tw}(\mathsf{C})$ is the $\mathcal{B}$-categorical analogue of the *twisted arrow ∞-category* [52, Tag 03JF], and by straightening this left fibration, we end up with the desired map of $\mathcal{B}$-categories

$$\mathrm{map}_{\mathsf{C}}(-,-)\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}\,.$$

By transposing this map across the adjunction $\mathsf{C}^{\mathrm{op}} \times - \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, -)$, one furthermore obtains the *Yoneda embedding* $h_{\mathsf{C}}\colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Using these constructions, we then prove *Yoneda's lemma*:

Theorem 2.3.2.3. *For any* $\mathcal{B}$*-category* C*, there is a commutative diagram*

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "h\times\mathrm{id}"] \arrow[dr, "\mathrm{ev}"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-,-)"] \\
 & \mathsf{Grpd}_{\mathcal{B}}
\end{tikzcd}$$

of $\mathcal{B}$*-categories, where* ev *denotes the evaluation map.*

Colimits and cocompletions We claimed above that the universe $\mathsf{Grpd}_{\mathcal{B}}$ for $\mathcal{B}$-groupoids is the $\mathcal{B}$-categorical analogue of the ∞-category of ∞-groupoids Ani. So far, we have not yet provided any real evidence for this claim. In higher category theory, the ∞-category Ani is characterised by its *universal property*: the fact that it is the *free cocompletion* of the point. More generally, if $\mathcal{C}$ is an ∞-category, the universal property of the associated Yoneda embedding $h_{\mathcal{C}}\colon \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ asserts that it exhibits $\mathrm{PSh}(\mathcal{C})$ as the free cocompletion of $\mathcal{C}$ [49, Theorem 5.1.5.6]. Thus, to ensure that the universe fulfils its intended role, we need to establish the $\mathcal{B}$-categorical version of this theorem: that the Yoneda embedding $h_{\mathsf{C}} \colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ exhibits $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ as the free cocompletion of C, for every $\mathcal{B}$-category C.

To do so, we first need to establish a notion of *cocompleteness* for $\mathcal{B}$-categories, which is to capture the property that a $\mathcal{B}$-category admits all *internal* colimits. This requires developing a theory of such colimits, which we will do in this thesis and which will be completely parallel to the theory of colimits in ∞-categories: the ∞-category $\mathrm{Cat}(\mathcal{B})$ of $\mathcal{B}$-category admits a canonical $(\infty, 2)$-enhancement, which allows us to define a notion of *adjunctions* between $\mathcal{B}$-category using abstract 2-categorical constructions. Consequently, if I and C are arbitrary $\mathcal{B}$-categories, we can define I-indexed colimits in C via the left adjoint of the diagonal map $\mathrm{diag} \colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$, provided that such an adjoint exists. The condition that all possible such left adjoints exist can then be taken as the evident definition of cocompleteness in internal higher category theory. For the practitioner, though, the most interesting aspect will be that as an outcome of this theory, we will obtain an explicit *sheaf-theoretic* characterisation of the resulting notion of cocompleteness:

Corollary 3.5.4.4. *A $\mathcal{B}$-category C is cocomplete if and only if the following conditions are satisfied:*

1. *For every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ is cocomplete and for any map $s \colon B \to A$ in $\mathcal{B}$ the functor $s^* \colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves colimits.*

2. *For every map $p \colon P \to A$ in $\mathcal{B}$ the functor p^* has a left adjoint $p_!$ such that for every pullback square*

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

   the natural map $q_! t^* \to s^* p_!$ is an equivalence.

Furthermore a functor $f \colon \mathsf{C} \to \mathsf{D}$ of cocomplete $\mathcal{B}$-categories is cocontinuous if and only if for every $A \in \mathcal{B}$ the functor $f(A)$ preserves colimits, and for every map $p \colon P \to A$ in $\mathcal{B}$ the natural map $p_! f(P) \to f(A) p_!$ is an equivalence. □

With respect to this notion of cocompleteness, the Yoneda embedding

$$h_{\mathsf{C}} \colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$$

of a $\mathcal{B}$-category C can now be shown to have the desired universal property:

Theorem 3.5.1.1. *For any $\mathcal{B}$-category C and any cocomplete $\mathcal{B}$-category E, restriction along the Yoneda embedding embedding $h_{\mathsf{C}} \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ induces an equivalence*

$$h_{\mathsf{C}}^{*} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}),$$

where $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$ is the $\mathcal{B}$-category of cocontinuous *functors between $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ and E.*

Presentable $\mathcal{B}$-categories Cocomplete ∞-categories are exceptionally well-behaved, provided that one imposes certain *size constraints*. Under these assumptions, they are called *presentable ∞-categories* [49, § 5.5] and are one of the main reasons that make higher category theory so applicable. Among the features that make them so convenient is the presence of adjoint functor theorems [49, Corollary 5.5.2.9] and the fact that they admit a tensor product [50, § 4.8.1]. We would like to have analogous results in internal higher category theory at our disposal. Therefore, we will need to establish a notion of presentability for $\mathcal{B}$-categories. As before, the development of the theory proceeds completely parallel to that of presentable ∞-categories: having already set up a functioning theory of cocompleteness, we only need one additional element: that of an *accessible* $\mathcal{B}$-category. Recall that in higher category theory, an ∞-category is accessible if it is obtained as the free cocompletion of a (small) ∞-category by κ-filtered colimits, for some regular cardinal κ [49, § 5.4]. Thus, in order to make sense of such a property in $\mathcal{B}$-category theory, we need an internal notion of *filteredness*. Once established, it easily yields the desired concept of accessible $\mathcal{B}$-categories since the universal property of presheaf $\mathcal{B}$-categories that we can already make use of will make it easy to construct free cocompletions by an arbitrary class of internal colimits.

Upon combining this notion of accessibility with cocompleteness, we then end up with the definition of a presentable $\mathcal{B}$-category. We will provide a plethora of equivalent characterisations of this notion, akin to the Lurie-Simpson characterisation of presentable ∞-categories [49, Theorem 5.5.1.1], which will add significantly to the applicability of this concept and which is further evidence of the heuristic that one can work with $\mathcal{B}$-categories in exactly the same way as with ∞-categories. But again, the practitioner will probably be most interested in ways to recognise this property from a sheaf-theoretic perspective. Luckily, there is again an explicit description:

Theorem 5.4.2.5. *A (large) $\mathcal{B}$-category* D *is presentable if and only if*

1. *it is* section-wise *given by presentable ∞-categories, in the sense that the associated sheaf of ∞-categories on $\mathcal{B}$ takes values in the subcategory $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ of presentable ∞-categories;*
2. *it is* $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete*, in the sense that for every pullback square*

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

in $\mathcal{B}$ the induced square of ∞-categories

$$\begin{tikzcd} \mathsf{D}(Q) & \mathsf{D}(P) \arrow[l, "g^*"'] \\ \mathsf{D}(B) \arrow[u, "q^*"] & \mathsf{D}(A) \arrow[l, "f^*"] \arrow[u, "p^*"'] \end{tikzcd}$$

is left adjointable*: both p^* and q^* admit left adjoints $p_!$ and $q_!$, and the natural map $q_! g^* \to f^* p_!$ is an equivalence.*

Presentable $\mathcal{B}$-categories are equally well-behaved as presentable ∞-categories: we will show that they satisfy *adjoint functor theorems*, and we will prove that the ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ of presentable $\mathcal{B}$-categories admits a symmetric monoidal structure $- \otimes -$, i.e. we construct a *tensor product* of presentable $\mathcal{B}$-categories. This symmetric monoidal structure can be used to realise $\mathcal{B}$-*modules* in the ∞-category of presentable ∞-categories as presentable $\mathcal{B}$-categories:

Proposition 5.5.4.6. *There is a fully faithful functor*

$$- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$$

which carries a $\mathcal{B}$-module $\mathcal{M}$ in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ to the presentable $\mathcal{B}$-category $\mathcal{M} \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ given by the sheaf

$$\mathcal{M} \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}, \qquad A \mapsto \mathcal{M} \otimes_{\mathcal{B}} \mathcal{B}_{/A},$$

in which $- \otimes_{\mathcal{B}} -$ denotes the relative tensor product over $\mathcal{B}$, i.e. the symmetric monoidal structure of $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$. Moreover, this functor is strong symmetric monoidal.

As a consequence of this result, we can now define the *$\mathcal{B}$-category of $\mathcal{B}$-categories* $\mathrm{Cat}_{\mathcal{B}}$ as the presentable $\mathcal{B}$-category that is associated with the $\mathcal{B}$-module $\mathrm{Cat}_{\infty} \otimes \mathcal{B}$. By construction, this is precisely the sheaf that carries $A \in \mathcal{B}$ to the ∞-category $\mathrm{Cat}(\mathcal{B}_{/A})$ of $\mathcal{B}_{/A}$-categories.

$\mathcal{B}$-topoi We now come to the core part of our theory of internal higher categories: the development of higher topos theory internal to $\mathcal{B}$. The defining property of ∞-topoi is that of *descent*: the property of an ∞-category $\mathcal{C}$ with finite limits and small colimits that the *slice functor* $\mathcal{C}_{/-} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ preserves small limits [49, § 6.1.3]. In order to achieve our goal of developing a theory of higher topoi internal to an ∞-topos $\mathcal{B}$, we therefore need a $\mathcal{B}$-categorical version of this notion, which requires constructing slice functors for $\mathcal{B}$-categories. The most convenient way to achieve this is by going through *straightening and unstraightening*. We already mentioned above that we can dispose of a straightening equivalence for *left* fibrations, which allows us to construct $\mathrm{Grpd}_{\mathcal{B}}$-valued functors. In the current situation, however, we need to construct a functor that takes values in the $\mathcal{B}$-category $\mathrm{Cat}_{\mathcal{B}}$ of $\mathcal{B}$-categories. Thus, we need a more general version of the straightening equivalence: one for *cartesian* fibrations.

It is immediate to parse the definition of a cartesian fibration of ∞-categories in the world of $\mathcal{B}$-categories, resulting in a notion of cartesian fibrations between $\mathcal{B}$-categories. We then define a $\mathcal{B}$-category $\mathrm{Cart}_{\mathsf{C}}$ of such cartesian fibrations over a fixed $\mathcal{B}$-category C, and we show:

Theorem 4.4.3.1. *For every $\mathcal{B}$-category* C*, there is an equivalence*

$$\mathrm{St}_{\mathsf{C}} \colon \mathsf{Cart}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{B}})$$

that is natural in $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$.

Using this result, the $\mathcal{B}$-categorical notion of descent can be formulated easily: by the universal property of Ani, there is a unique left exact and cocontinuous functor const: Ani $\to \mathcal{B}$. This functor can be naturally extended to a map $\mathrm{Cat}_{\infty} \to \mathrm{Cat}(\mathcal{B})$, so that every ∞-category $\mathcal{C}$ can be regarded as a *constant* $\mathcal{B}$-category. In particular, we can regard the interval Δ^1 as a $\mathcal{B}$-category, so that the codomain fibration $d_0 \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X}) \to \mathsf{X}$ is well-defined for any $\mathcal{B}$-category X. If X admits finite limits, then one can show that d_0 is a cartesian fibration. Hence, by straightening this map, one ends up with a functor

$$\mathsf{X}_{/-} \colon \mathsf{X}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}\,.$$

Now if X is in addition presentable, it is in particular cocomplete. Since $\mathsf{Cat}_{\mathcal{B}}$ is presentable as well and therefore *complete* (i.e. admits all internal limits), it makes perfect sense to ask for this functor to be continuous, i.e. to preserve all internal limits. If this is the case, we say that X satisfies *descent*. We define a $\mathcal{B}$-topos to be a presentable $\mathcal{B}$-category that has this property.

However, having a simple and elegant definition of $\mathcal{B}$-topoi is not enough. To be able to efficiently work with $\mathcal{B}$-topoi, in particular to have a way to *construct* them, we need a characterisation in terms of left exact Bousfield localisations of presheaf $\mathcal{B}$-categories, similar to Lurie's characterisation of ∞-topoi in [49, Theorem 6.1.0.6]. Fortunately, since at this point we have a plethora of useful $\mathcal{B}$-categorical tools at our disposal, proving such a result simply amounts to translating the arguments that lead to its ∞-categorical counterpart into our framework. By doing so, we obtain:

Theorem 6.2.3.1. *A large $\mathcal{B}$-category* X *is a $\mathcal{B}$-topos if and only if there is a $\mathcal{B}$-category* C *such that* X *arises as a left exact and accessible Bousfield localisation of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Here left exact and accessible are understood in the $\mathcal{B}$-categorical sense, using our theory of internal limits and colimits. By Bousfield localisation, we simply mean an adjunction between $\mathcal{B}$-categories in which the right adjoint is fully faithful (which is a condition that can be easily made sense of for complete Segal spaces and therefore for $\mathcal{B}$-categories as well).

With this explicit description of $\mathcal{B}$-topoi, we can now develop their theory: we study limits and colimits of $\mathcal{B}$-topoi, prove a $\mathcal{B}$-categorical version of *Diaconescu's theorem*, and we study *étale* and *subterminal* $\mathcal{B}$-topoi: the former being those that arise as $\mathcal{B}$-categories of presheaves on $\mathcal{B}$-groupoids, and the latter being those that arise as left exact and accessible Bousfield localisation of $\mathsf{Grpd}_{\mathcal{B}}$. Furthermore, we set up a theory of *localic* $\mathcal{B}$-topoi, which are those that arise as $\mathcal{B}$-categories of *sheaves* on what we call a $\mathcal{B}$*-locale*: an internal version of the notion of a locale.

The microcosm principle in higher topos theory

The attentive reader will have noticed that in our discussion of $\mathcal{B}$-topoi, we omitted how they can be recognised and constructed from an *external*, i.e. *sheaf-theoretic* perspective. The reason is that in this case, the situation is somewhat special: we are defining a certain notion, that of a higher topos, internal to an object that is defined by the very same property, just within the meta-theory of ∞-categories. In other words, the theory of higher topoi obeys a certain *microcosm principle* in the sense of Baez-Dolan [11]. This principle is usually understood in the context of higher algebra, where it serves as a heuristic for the observation that certain algebraic structures, such as *algebras*, are defined internal to mathematical objects, in the case of algebras *monoidal ∞-categories*, which are themselves examples of the same algebraic structures: they can be defined as algebras in the monoidal ∞-category of ∞-categories. In this situation, one speaks of an algebra as a *microcosm* and the ambient monoidal ∞-category as the *macrocosm*. Thus, by applying this heuristic to higher topos theory, the base ∞-topos $\mathcal{B}$ can be viewed as a macrocosm and a $\mathcal{B}$-topos as a microcosm.

The microcosm principle in higher topos theory goes beyond its analogue in higher algebra, in that it furthermore comes with an intrinsic *identification* between microcosm and macrocosm: the theory of $\mathcal{B}$-topoi can be regarded as a *reflection* of the theory of ∞-topoi *over* $\mathcal{B}$ inside the latter: when viewing a $\mathcal{B}$-topos X as a sheaf of ∞-categories, we may take its underlying ∞-category of *global sections* (by evaluating the sheaf at the final object $1 \in \mathcal{B}$), which turns out to be an ∞-topos. Moreover, the universal property of $\mathrm{Grpd}_{\mathcal{B}}$ implies that this is the *final* $\mathcal{B}$-topos, so that there is a canonical *geometric morphism* of $\mathcal{B}$-topoi $\Gamma_{\mathsf{X}} \colon \mathsf{X} \to \mathrm{Grpd}_{\mathcal{B}}$. Thus, by passing to global sections, one sees that the underlying ∞-topos of X comes equipped with a geometric morphism into $\mathcal{B}$. In other words, we can associate to every $\mathcal{B}$-topos a geometric morphism of ∞-topoi with codomain $\mathcal{B}$. One of the main results in this thesis is that these two notions are entirely equivalent:

Theorem 6.2.5.1. *Passing from a $\mathcal{B}$-topos X to the associated geometric morphism of ∞-topoi $f_* \colon \mathcal{X} \to \mathcal{B}$ constitutes an equivalence of ∞-categories*

$$\mathrm{Top}^{\mathrm{R}}(\mathcal{B}) \simeq (\mathrm{Top}^{\mathrm{R}}_{\infty})_{/\mathcal{B}},$$

where $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$ is the ∞-category of $\mathcal{B}$-topoi and geometric morphisms and $\mathrm{Top}^{\mathrm{R}}_{\infty}$ is the ∞-category of ∞-topoi and geometric morphisms.

The 1-toposic version of this theorem has been known for a long time; it was shown by Moens in his PhD thesis [60]. The version for higher topoi now allows us to directly relate properties and constructions on the level of geometric morphisms of ∞-topoi with properties and constructions of $\mathcal{B}$-topoi. As the theory of $\mathcal{B}$-topoi is built completely parallel to that of ∞-topoi, this result therefore allows us to reduce every relative notion in higher topos theory (i.e. one concerning maps of ∞-topoi) to an absolute one (i.e. one about individual ∞-topoi). In view of our interpretation of geometric morphisms $f_* \colon \mathcal{X} \to \mathcal{B}$ as ∞-topoi *parametrised* by $\mathcal{B}$, this result thus constitutes the promised mechanism for reducing parametrised problems in higher topos theory to their unparametrised analogues.

It is beyond the scope of this thesis to analyse the full potential of this mechanism. Instead, we will provide a few selected examples to showcase how it can be utilised in practice.

Pullbacks of ∞-topoi The correspondence between geometric morphisms into $\mathcal{B}$ and $\mathcal{B}$-topoi can be used to derive a formula for general pullbacks of

∞-topoi. In fact, since a pullback diagram

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Z} \arrow[r] & \mathcal{B} \end{tikzcd}$$

of ∞-topoi is the same datum as a *binary product* in the slice ∞-category of ∞-topoi over $\mathcal{B}$, such a pullback can be computed as the binary product of $\mathcal{B}$-topoi. As it is the case in the theory of ∞-topoi, such a binary product admits an explicit formula by means of the *tensor product* of the underlying presentable $\mathcal{B}$-categories, so that we are able to show:

Corollary 6.2.7.3. *Let $f_* : \mathcal{X} \to \mathcal{B}$ and $g_* : \mathcal{Z} \to \mathcal{B}$ be geometric morphisms of ∞-topoi, and let X and Z be the associated $\mathcal{B}$-topoi. Then the fibre product $\mathcal{Z} \times_{\mathcal{B}} \mathcal{X}$ of ∞-topoi can be computed as the ∞-topos that is obtained by taking global sections of the $\mathcal{B}$-category $\mathsf{X} \otimes \mathsf{Z}$, where $- \otimes -$ denotes the tensor product of presentable $\mathcal{B}$-categories. Explicitly, the fibre product can be identified with the ∞-category of global sections of the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cont}}(\mathsf{X}^{\mathrm{op}}, \mathsf{Z})$ of continuous functors from X^{op} to Z.*

Smooth and proper geometric morphisms of ∞-topoi Recall that one may regard the theory of ∞-topoi as an enhancement of topology, so that ∞-topoi can be interpreted as generalised spaces and geometric morphisms as continuous maps. But before one can do actual geometry with ∞-topoi, one first needs to establish ∞-toposic analogues of topological properties and constructions. Among the most fundamental of them is the notion of *compactness.* One promising candidate for an ∞-toposic definition of compactness is the property of an ∞-topos $\mathcal{X}$ that its global sections functor $\Gamma_{\mathcal{X}} : \mathcal{X} \to \mathrm{Ani}$ preserves filtered colimits. In fact, this condition naturally arises from the observation that a topological space X is compact if and only if the (geometric) morphism of locales $\mathrm{Open}(X) \to \mathrm{Open}(*)$ from the locales of open subsets in X to the locale of open subsets of the singleton space $*$ preserves filtered colimits. This notion of compactness makes perfect sense for $\mathcal{B}$-topoi as well: as we already have a $\mathcal{B}$-categorical version of filteredness at our disposal, the very same definition can be parsed in $\mathcal{B}$-topos theory.

Concretely, a $\mathcal{B}$-topos X is defined to be compact if the global sections functor $\mathsf{X} \to \mathrm{Grpd}_{\mathcal{B}}$ preserves (internally) filtered colimits.

In view of the correspondence between $\mathcal{B}$-topoi and geometric morphisms into $\mathcal{B}$, the notion of compactness for $\mathcal{B}$-topoi gives rise to a *relative* version of ∞-toposic compactness. However, an a priori very different candidate for such a relative compactness condition in higher topos theory has been suggested by Lurie [49, § 7.3]: that of a *proper* geometric morphism. Properness of a geometric morphism $p_* : \mathcal{X} \to \mathcal{B}$ is defined as the property that in every commutative diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "g'_*"] \arrow[d, "q'_*"'] & \mathcal{Y} \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{A}' \arrow[r, "f'_*"'] & \mathcal{A} \arrow[r, "f_*"'] & \mathcal{B} \end{tikzcd}$$

of ∞-topoi in which both squares are pullbacks, the left square is horizontally left adjointable, in the sense that the mate transformation $(f')^* q_* \to q'_*(g')^*$ is an equivalence. Even in the degenerate case where p_* is simply given by the global sections functor $\Gamma_{\mathcal{X}} : \mathcal{X} \to \mathrm{Ani}$, it is not clear at all whether $\mathcal{X}$ being a compact ∞-topos is equivalent to $\Gamma_{\mathcal{X}}$ being proper. Using the theory of $\mathcal{B}$-topoi, we will show that this is in fact the case, and that more generally properness of geometric morphisms is equivalent to compactness of the associated internal higher topoi:

Theorem 7.2.5.1. *Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism, and let X be the associated $\mathcal{B}$-topos. Then f_* is proper if and only if X is a compact $\mathcal{B}$-topos.*

The 1-toposic analogue of this theorem was shown by Moerdijk-Vermeulen in [61]. Its ∞-categorical version can be used to extend the class of examples of proper maps of ∞-topoi. In fact, it is generally much simpler to show compactness of a certain $\mathcal{B}$-topos than to show properness of a geometric morphism. We will make advantage of this when we show:

Theorem 7.3.2.1. *Let $p : Y \to X$ be a proper and separated map of topological spaces. Then the induced geometric morphism $p_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is proper.*

A version of this result has already been shown by Lurie in the special case where the space Y is *completely regular*, i.e. a subspace of a compact Hausdorff space [49, Theorem 7.3.1.16]. Thus, our result constitutes a generalisation of Lurie's theorem to non-Hausdorff examples, and it furthermore provides a very different proof strategy that circumvents the need for $\mathcal{K}$-sheaves.

The notion of a proper geometric morphism can be formally dualised to that of a *smooth* geometric morphism of ∞-topoi: here smoothness of a geometric morphism $p_* \colon \mathcal{X} \to \mathcal{B}$ is defined as the property that in every commutative diagram

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "g'_*"] \arrow[d, "q'_*"'] & \mathcal{Y} \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\
\mathcal{A}' \arrow[r, "f'_*"'] & \mathcal{A} \arrow[r, "f_*"'] & \mathcal{B}
\end{tikzcd}$$

of ∞-topoi in which both squares are pullbacks, the left square is vertically left adjointable, in the sense that the mate transformation $q^* f'_* \to g'_*(q')^*$ is an equivalence. Again, the natural question arises which property of the $\mathcal{B}$-topos associated with p_* this notion corresponds to. In 1-topos theory, this is well-known [39, Corollary C.3.3.16]: a geometric morphism is smooth precisely if the associated internal topos is *locally connected.* We show the ∞-toposic analogue of this result. To that end, note that if X is a $\mathcal{B}$-topos, then the unique geometric morphism $\Gamma_{\mathsf{X}} \colon \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}$ always admits a left adjoint $\mathrm{const}_{\mathsf{X}} \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{X}$. We say that X is *locally contractible* if $\mathrm{const}_{\mathsf{X}}$ in turn admits a further left adjoint π_{X}. We then prove:

Theorem 7.1.3.1. *Let X be a $\mathcal{B}$-topos and let $f_* \colon \mathcal{X} \to \mathcal{B}$ be the associated geometric morphism. Then X is locally contractible if and only if f_* is smooth.*

As before, we can utilise this result to extend the class of examples of smooth geometric morphisms. In fact, in combination with work of Volpe [82], it immediately yields:

Theorem 7.3.1.5. *If $f \colon Y \to X$ is a topological submersion, then the geometric morphism $f_* \colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is smooth.*

Linear overview

This thesis is organised into seven chapters. We begin in Chapter 1 by setting up the basic language of internal higher category theory. Chapter 2 is dedicated to the study of internal presheaves, containing a treatment of internal left fibrations and their straightening equivalence as well as our proof of Yoneda's lemma for internal higher categories. In Chapter 3 we study internal limits and colimits and internal cocompletions. Here we establish the universal property of internal presheaf categories. In Chapter 4 we develop the theory of internal (co)cartesian fibrations and establish their straightening equivalence. In Chapter 5, we develop the theory of internal accessibility and internal presentability, which we use in Chapter 6 to set up the theory of internal higher topoi. In particular, the latter chapter contains our correspondence between internal higher topoi and geometric morphisms of ∞-topoi. Lastly, we apply this framework in Chapter 7 to characterise smooth and proper morphisms of ∞-topoi.

Related work

The idea of developing category theory internal to a topos is not new; as early as 1963, Lawvere formulated axioms for a theory of categories [47], and in subsequent years Bénabou established the theory of internal categories in a presheaf topos through the notion of fibred categories (see [79]). Internal category theory remained an active research area for several years. Further development was mainly driven by Lawvere and Tierney [48], Giraud [27], Johnstone [38], Diaconescu [20], Moens [60] and Pitts [65]. The interested reader is referred to [39] for an excellent exposition of this theory. More recently, Caramello and Zanfa started developing internal 1-topos theory based on the theory of stacks [16].

In higher higher category theory, Riehl and Shulman [72] proposed a synthetic approach to the theory of ∞-categories based on homotopy type theory, which was further studied by Buchholtz-Weinberger [15] and Weinberger [84, 85, 86, 87, 88]. As every ∞-topos gives a model for homotopy type theory [78], the theory of ∞-categories internal to an ∞-topos as developed in this thesis can be viewed as a model of simplicial homotopy type theory.

On the analytic side of the story, Rasekh previously worked out some aspects of the theory of internal higher categories in [69]. Furthermore, Nardin and Shah [77, 76, 63, 64] developed a theory of ∞-categories that are parametrised by a base ∞-category $\mathcal{C}$. Further aspects of this theory were contributed by

Hilman [35]. In our language, their framework precisely corresponds to internal ∞-category theory in the presheaf ∞-topos $\mathrm{PSh}(\mathcal{C})$ and can thus be regarded as a special case of internal higher category theory. However, the scope of the two projects is very different: while our focus lies on developing higher topos theory internal to an ∞-topos, the main goal for the development of parametrised higher category theory was the study of parametrised phenomena in equivariant homotopy theory. As a result, while the foundational parts of both theories are very similar, they diverge at a certain point.

Lastly, since the development of internal higher category theory proceeds along similar lines as the theory of ∞-categories, our work has been very strongly influenced by the writings of Joyal [41, 43, 42], Lurie [49, 50] and Cisinski [18].

Declaration of originality

The content of this thesis has previously appeared in a paper series on the subject [54, 55, 59, 56, 57, 58]. Four of these papers are based on joint work with Sebastian Wolf. The parts of this thesis that are based on material from this collaboration are Chapter 3, Chapter 5, Chapter 6 and Chapter 7.

1. The language of $\mathcal{B}$-categories

This chapter is intended to introduce the basic language and some core concepts of of internal higher category theory that we will be working with throughout this thesis. The central object of interest will be that of a $\mathcal{B}$*-category*, where $\mathcal{B}$ is an arbitrary ∞-topos. After recalling some conventions and basic constructions in higher category theory in Section 1.1, we will focus most of Section 1.2 on studying various approaches to think of these objects, each coming with its own specific use cases. We furthermore explain how these objects can be regarded as a certain flavour of *categories* by making sense of the notion of objects and morphisms in a $\mathcal{B}$-category. This will lead us to the notion of a *context*, the presence of which being the main feature that distinguishes $\mathcal{B}$-category theory from ∞-categories.

In Section 1.3, we enter more deeply into the world of $\mathcal{B}$-categories by studying certain classes of *functors* between them that will be of importance throughout this thesis. Finally, we introduce our first (and most important) examples of $\mathcal{B}$-categories in Section 1.4: the *universe for* $\mathcal{B}$*-groupoids* and the $\mathcal{B}$*-category of* $\mathcal{B}$*-categories*.

1.1. Preliminaries on higher category theory

The theory of ∞-categories is omnipresent throughout this thesis. The goal of this section is to explain how we intend to use this theory, and to recall some key constructions that will become relevant later on. In Section 1.1.1 and Section 1.1.2, we begin by establishing the basic terminology and the set-theoretic foundations that we will adhere to. In Section 1.1.3 and Section 1.1.4, we recall some basic facts from the theory of ∞-topoi. Lastly, we review the theory of *factorisation systems* in an ∞-category in Section 1.1.5 as these will form the basis for many of our arguments.

1.1.1. General conventions and notation

Throughout this thesis we freely make use of the language of higher category theory. We will generally follow a model-independent approach to higher categories. This means that as a general rule, all statements and constructions that are considered herein will be invariant under equivalences in the ambient ∞-category, and we will always be working within such an ambient ∞-category. For example, this means that all constructions involving ∞-categories and functors between ∞-categories will be assumed to take place in the ∞-category of ∞-categories. In the same vein, a set will be a discrete ∞-groupoid, and a 1-category will be an ∞-category all of whose mapping ∞-groupoids are discrete. These conventions in particular imply that we will understand the adjective *unique* in the homotopical sense, i.e. as the condition that there is a contractible ∞-groupoid of choices.

We denote by Δ the simplex category, i.e. the category of non-empty totally ordered finite sets with order-preserving maps. Every natural number $n \in \mathbb{N}$ can be considered as an object in Δ by identifying n with the totally ordered set $\langle n\rangle = \{0, \dots n\}$. For $i = 0, \dots, n$ we denote by $\delta^i : \langle n-1\rangle \to \langle n\rangle$ the unique injective map in Δ whose image does not contain i. Dually, for $i = 0, \dots n$ we denote by $\sigma^i : \langle n+1\rangle \to \langle n\rangle$ the unique surjective map in Δ such that the preimage of i contains two elements. Furthermore, if $S \subset n$ is an arbitrary subset of $k+1$ elements, we denote by $\delta^S : \langle k\rangle \to \langle n\rangle$ the unique injective map in Δ whose image is precisely S. In the case that S is an interval, we will denote by $\sigma^S : \langle n\rangle \to \langle n-k\rangle$ the unique surjective map that sends S to a single object. If $\mathcal{C}$ is an ∞-category, we refer to a functor $C_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$ as a simplicial object in $\mathcal{C}$. We write C_n for the image of $n \in \Delta$ under this functor, and we write d_i, s_i, d_S and s_S for the image of the maps δ^i, σ^i, δ^S and σ^S under this functor. Dually, a functor $C^\bullet : \Delta \to \mathcal{C}$ is referred to as a cosimplicial object in $\mathcal{C}$. In this case we denote the image of δ^i, σ^i, δ^S and σ^S by d^i, s^i, d^S and σ^S.

The 1-category Δ embeds fully faithfully into the ∞-category of ∞-categories by means of identifying posets with 0-categories and order-preserving maps between posets with functors between such 0-categories. We denote by Δ^n the image of $n \in \Delta$ under this embedding.

1.1.2. Set theoretical foundations

Once and for all we will fix three Grothendieck universes $\mathbf{U} \in \mathbf{V} \in \mathbf{W}$ that contain the first infinite ordinal ω. A set is *small* if it is contained in $\mathbf{U}$, *large* if it is contained in $\mathbf{V}$ and *very large* if it is contained in $\mathbf{W}$. An analogous naming convention will be adopted for ∞-categories and ∞-groupoids. The ∞-categories of small, large and very large ∞-groupoids (or *anima*) are denoted by $\mathrm{Ani}^{\mathbf{U}}$, $\mathrm{Ani}^{\mathbf{V}}$ and $\mathrm{Ani}^{\mathbf{W}}$, respectively. Similarly, we denote the ∞-categories of small, large and very large ∞-categories by $\mathrm{Cat}_\infty^{\mathbf{U}}$, $\mathrm{Cat}_\infty^{\mathbf{V}}$ and $\mathrm{Cat}_\infty^{\mathbf{W}}$, respectively.

As especially the first two layers will appear very often in this thesis, we will use the simplified notation $\mathrm{Ani} = \mathrm{Ani}^{\mathbf{U}}$ and $\widehat{\mathrm{Ani}} = \mathrm{Ani}^{\mathbf{V}}$ as well as $\mathrm{Cat}_\infty = \mathrm{Cat}_\infty^{\mathbf{U}}$ and $\widehat{\mathrm{Cat}}_\infty = \mathrm{Cat}_\infty^{\mathbf{V}}$.

1.1.3. On ∞-topoi

A large ∞-category $\mathcal{B}$ is said to be an ∞-topos if there exists a small ∞-category $\mathcal{C}$ such that $\mathcal{B}$ arises as a left exact and accessible localisation of the presheaf ∞-category $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$, see [49, § 6] for alternative characterisations and the basic theory. An algebraic morphism between two ∞-topoi $\mathcal{A}$ and $\mathcal{B}$ is a functor $f^* : \mathcal{A} \to \mathcal{B}$ that commutes with small colimits and finite limits. Dually, a geometric morphism between ∞-topoi is a functor $f_* : \mathcal{B} \to \mathcal{A}$ that admits a left adjoint f^* which defines an algebraic morphism (i.e. which commutes with finite limits). We let $\mathrm{Top}_\infty^{\mathrm{R}}$ be the subcategory of $\widehat{\mathrm{Cat}}_\infty$ that is spanned by the ∞-topoi and geometric morphisms, and we denote by $\mathrm{Top}_\infty^{\mathrm{L}}$ the subcategory of $\widehat{\mathrm{Cat}}_\infty$ that is spanned by the ∞-topoi and algebraic morphisms. There is an equivalence $(\mathrm{Top}_\infty^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Top}_\infty^{\mathrm{L}}$ that sends an ∞-topos to itself and a geometric morphism to its left adjoint. The ∞-category Ani of small ∞-groupoids is a final object in $\mathrm{Top}_\infty^{\mathrm{R}}$; for any ∞-topos $\mathcal{B}$ we denote by $\Gamma : \mathcal{B} \to \mathrm{Ani}$ the unique geometric morphism and refer to this functor as the *global sections* functor. Explicitly, this functor is given by $\mathrm{map}_{\mathcal{B}}(1, -)$ where $1 \in \mathcal{B}$ denotes a final object. Dually, we denote the unique algebraic morphism from Ani to $\mathcal{B}$ by $\mathrm{const} : \mathrm{Ani} \to \mathcal{B}$ and refer to this map as the *constant sheaf functor*.

1.1.4. Universe enlargement

For any two large ∞-categories $\mathcal{C}$ and $\mathcal{A}$, an $\mathcal{A}$*-valued presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$, and an $\mathcal{A}$*-valued sheaf* an $\mathcal{A}$-valued presheaf that preserves small limits (whenever they exist). We denote the ∞-categories of $\mathcal{A}$-valued presheaves and sheaves on $\mathcal{C}$ by $\mathrm{PSh}_{\mathcal{A}}(\mathcal{C})$ and $\mathrm{Sh}_{\mathcal{A}}(\mathcal{C})$, respectively.

For any ∞-topos $\mathcal{B}$, we define its *universe enlargement* $\mathcal{B}^{\mathbf{V}}$ relative to $\mathbf{V}$ as the very large ∞-category of $\widehat{\mathrm{Ani}}$-valued sheaves on $\mathcal{B}$, i.e. as $\mathcal{B}^{\mathbf{V}} = \mathrm{Sh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$. Again, we will use the simplified notation $\widehat{\mathcal{B}} = \mathcal{B}^{\mathbf{V}}$. By [49, Remark 6.3.5.17] this is an ∞-topos relative to $\mathbf{V}$. Moreover, one can turn the assignment $\mathcal{B} \mapsto \widehat{\mathcal{B}}$ into a functor as follows:

Consider the functor $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-)\colon \mathrm{Top}^{\mathrm{R}}_{\infty} \to \mathrm{Cat}^{\mathbf{W}}_{\infty}$ that acts by sending a map $f_*\colon \mathcal{B} \to \mathcal{A}$ in $\mathrm{Top}^{\mathrm{R}}_{\infty}$ to the map $(-) \circ f^*\colon \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}) \to \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{A})$. Since f^* commutes with small colimits, the functor $(-) \circ f^*$ restricts to a functor $\widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ that we will denote by f_* as well. As a consequence, if $\int \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-) \to \mathrm{Top}^{\mathrm{R}}_{\infty}$ is the cocartesian fibration that is classified by the functor $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-)$, then the full subcategory of $\int \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-)$ that is spanned by pairs $(\mathcal{B}, A)$ with $\mathcal{B} \in \mathrm{Top}^{\mathrm{R}}_{\infty}$ and $A \in \widehat{\mathcal{B}} \subset \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ is stable under cocartesian arrows and therefore defines a cocartesian subfibration of $\int \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-)$ over $\mathrm{Top}^{\mathrm{R}}_{\infty}$. Moreover, by making use the adjunction $f^* \dashv f_*$, one obtains a commutative diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "f_*"] \arrow[d, hook] & \mathcal{A} \arrow[d, hook] \\ \widehat{\mathcal{B}} \arrow[r, "\hat{f}_*"] & \widehat{\mathcal{A}}, \end{tikzcd}$$

hence the same argumentation implies that the full subcategory of $\int \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-)$ spanned by pairs $(\mathcal{B}, A)$ with $\mathcal{B} \in \mathrm{Top}^{\mathrm{R}}_{\infty}$ and $A \in \mathcal{B}$ defines a cocartesian subfibration of $\int \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(-)$ over $\mathrm{Top}^{\mathrm{R}}_{\infty}$ too. Consequently, one obtains a functor

$$\mathrm{Top}^{\mathrm{R}}_{\infty} \to \mathrm{Cat}^{\mathbf{W}}_{\infty}, \quad \mathcal{B} \mapsto \widehat{\mathcal{B}}$$

together with a natural transformation

$$\begin{tikzcd} \mathrm{Top}^{\mathrm{R}}_{\infty} \arrow[r, bend left, "\mathcal{B}\mapsto\mathcal{B}"] \arrow[r, bend right, "\mathcal{B}\mapsto\widehat{\mathcal{B}}"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Cat}^{\mathbf{W}}_{\infty} \end{tikzcd}$$

that is given by the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$.

By [49, Remark 6.4.6.18] the functor $f_* \colon \widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ defines a geometric morphism between ∞-topoi relative to the universe $\mathbf{V}$, and the associated left adjoint f^* is obtained as the restriction of $(f^*)_! \colon \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{A}) \to \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ (the functor of left Kan extension along f^*) to $\widehat{\mathcal{A}}$. Since the functor $\mathcal{B} \mapsto \widehat{\mathcal{B}}$ above therefore takes values in the ∞-category of ∞-topoi relative to the universe $\mathbf{V}$, passing to opposite ∞-categories therefore results in a functor

$$\mathrm{Top}_\infty^{\mathrm{L}} \to \mathrm{Cat}_\infty^{\mathbf{W}}, \quad \mathcal{B} \mapsto \widehat{\mathcal{B}}$$

that sends the geometric morphism $f^* \colon \mathcal{A} \to \mathcal{B}$ to $f^* \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$. By construction, one furthermore obtains a commutative square

$$\begin{tikzcd} \mathcal{B} \arrow[d, hook] & \mathcal{A} \arrow[l, "f^*"'] \arrow[d, hook] \\ \widehat{\mathcal{B}} & \widehat{\mathcal{A}}, \arrow[l, "f^*"'] \end{tikzcd}$$

hence an analogous argument as above shows that the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ defines a natural transformation

$$\begin{tikzcd} \mathrm{Top}_\infty^{\mathrm{L}} \arrow[r, bend left, "\mathcal{B}\mapsto\mathcal{B}"{name=U}] \arrow[r, bend right, "\mathcal{B}\mapsto\widehat{\mathcal{B}}"'{name=D}] & \mathrm{Cat}_\infty^{\mathbf{W}} . \arrow[from=U, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

Note that if f^* admits a further left adjoint $f_!$, then the map $f^* \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ is given by precomposition with $f_!$.

Remark 1.1.4.1. If $\mathcal{B}$ is an ∞-topos, there are a priori two ways to define the universe enlargement $\mathcal{B}^{\mathbf{W}}$ relative to the universe $\mathbf{W}$: either by applying the above construction to the pair $\mathbf{U} \in \mathbf{W}$, i.e. by defining $\mathcal{B}^{\mathbf{W}} = \mathrm{Sh}_{\mathrm{Ani}^{\mathbf{W}}}(\mathcal{B})$, or by applying this construction first to the pair $\mathbf{U} \in \mathbf{V}$ and then to the pair $\mathbf{V} \in \mathbf{W}$, i.e. by setting $\mathcal{B}^{\mathbf{W}} = \mathrm{Sh}_{\mathrm{Ani}^{\mathbf{W}}}(\widehat{\mathcal{B}})$, where the right-hand side now denotes the ∞-categories of functors $\widehat{\mathcal{B}}^{\mathrm{op}} \to \mathrm{Ani}^{\mathbf{W}}$ that commute with $\mathbf{V}$-small limits. It turns out that either approach results in the same object: in fact, upon identifying $\mathbf{U}$ with a regular cardinal in $\mathbf{V}$, we may identify $\widehat{\mathcal{B}}$ with the ∞-category $\mathrm{Ind}^{\mathbf{U}}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$, i.e. the free cocompletion of $\mathcal{B}$ by $\mathbf{V}$-small $\mathbf{U}$-filtered colimits. Consequently, [49, Proposition 5.3.5.10] implies that the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ induces an equivalence

$$\operatorname{Fun}^{\mathrm{Filt}_{\mathbf{U}}\text{-cc}}(\widehat{\mathcal{B}}, (\mathrm{Ani}^{\mathbf{W}})^{\mathrm{op}}) \simeq \operatorname{Fun}(\mathcal{B}, (\mathrm{Ani}^{\mathbf{W}})^{\mathrm{op}})$$

in which the left-hand side denotes the full subcategory of $\operatorname{Fun}(\widehat{\mathcal{B}}, (\mathrm{Ani}^{\mathbf{W}})^{\mathrm{op}})$ that is spanned by those functors that preserve $\mathbf{U}$-filtered colimits. Now [49, Proposition 5.5.1.9] implies that the above equivalence restricts to an equivalence $\mathrm{Sh}_{\mathrm{Ani}^{\mathbf{W}}}(\widehat{\mathcal{B}}) \simeq \mathrm{Sh}_{\mathrm{Ani}^{\mathbf{W}}}(\mathcal{B})$, noting that its proof does not require $(\mathrm{Ani}^{\mathbf{W}})^{\mathrm{op}}$ to be presentable (relative to the universe $\mathbf{V}$) but merely to admit $\mathbf{V}$-small colimits.

Recall that the assignment $A \mapsto \mathcal{B}_{/A}$ (which is defined as the straightening of the cartesian fibration $d_0 \colon \operatorname{Fun}(\Delta^1, \mathcal{B}) \to \mathcal{B}$) defines a fully faithful functor $\mathcal{B} \hookrightarrow (\mathrm{Top}^{\mathrm{R}}_{\infty})_{/\mathcal{B}}$. By [25, Corollary 9.9] there is a functorial equivalence

$$\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}_{/-}) \simeq \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})_{/-}$$

of functors $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}^{\mathbf{W}}_{\infty}$ that is given on each object $A \in \mathcal{B}$ by the left Kan extension of the functor $\mathcal{B}_{/A} \to \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})_{/A}$ that is induced by the Yoneda embedding $\mathcal{B} \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ along the Yoneda embedding $\mathcal{B}_{/A} \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}_{/A})$.

Lemma 1.1.4.2. *For every $A \in \mathcal{B}$, the equivalence $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}_{/A}) \simeq \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})_{/A}$ restricts to an equivalence*

$$\widehat{\mathcal{B}_{/A}} \simeq \widehat{\mathcal{B}}_{/A}.$$

Proof. In the commutative diagram

$$\begin{tikzcd} & \widehat{\mathcal{B}_{/A}} \arrow[r, hook] \arrow[d, dashed, "\phi"] & \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}_{/A}) \arrow[d, "\simeq"] \\ \mathcal{B}_{/A} \arrow[r] \arrow[ur, hook] & \widehat{\mathcal{B}}_{/A} \arrow[r, hook] & \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})_{/A} \end{tikzcd}$$

the task is to find the dashed arrow ϕ that completes the diagram and to show that this functor is an equivalence of ∞-categories. To that end, note that one has $\widehat{\mathcal{B}_{/A}} \simeq \mathrm{Ind}^{\mathbf{U}}_{\widehat{\mathrm{Ani}}}(\mathcal{B}_{/A})$, which by [49, Proposition 5.3.5.10] implies that composition with the Yoneda embedding gives rise to an equivalence

$$\operatorname{Fun}^{\mathrm{Filt}_{\mathbf{U}}\text{-cc}}(\widehat{\mathcal{B}_{/A}}, \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{B}_{/A}, \mathcal{D})$$

for any ∞-category $\mathcal{D}$ which admits $\mathbf{U}$-small filtered colimits. This result implies that the functor ϕ in the diagram above is well-defined and makes the diagram indeed commute. By construction, ϕ must be fully faithful. On the other hand, combining the equivalence of ∞-categories $\widehat{\mathcal{B}} \simeq \mathrm{Ind}^{\mathbf{U}}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ with the fact that the projection $\widehat{\mathcal{B}}_{/A} \to \widehat{\mathcal{B}}$ creates colimits shows that every object in $\widehat{\mathcal{B}}_{/A}$ is obtained as the colimit of a functor $\mathcal{J} \to \mathcal{B}_{/A} \to \widehat{\mathcal{B}}_{/A}$ where $\mathcal{J}$ is a $\mathbf{U}$-filtered ∞-category. Since ϕ commutes with $\mathbf{U}$-filtered colimits, this shows that this functor is essentially surjective. $\square$

Proposition 1.1.4.3. *There is a canonical equivalence*

$$\widehat{\mathcal{B}_{/-}} \simeq \widehat{\mathcal{B}}_{/-}$$

of functors $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\mathbf{W}}$.

Proof. By Lemma 1.1.4.2, the functorial equivalence

$$\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}_{/-}) \simeq \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})_{/-}$$

restricts object-wise to an equivalence on the level of sheaves, hence the result follows. $\square$

We finish this section by discussing the preservation of structure under universe enlargement:

Proposition 1.1.4.4. *For any ∞-topos $\mathcal{B}$ the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ commutes with small limits and colimits and with the internal hom.*

Proof. Since the Yoneda embedding $h\colon \mathcal{B} \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ commutes with small limits and since $\widehat{\mathcal{B}}$ is a Bousfield localisation of $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$, the embedding $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ preserves small limits. The case of small colimits is proved in [49, Remark 6.3.5.17]. Lastly, since the product bifunctor on $\widehat{\mathcal{B}}$ restricts to the product bifunctor on $\mathcal{B}$, it suffices to show that for $A, B \in \mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ their internal hom $\underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}}(A, B)$ is contained in $\mathcal{B}$. Now $\widehat{\mathcal{B}}$ is a left exact localisation of $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ and therefore an exponential ideal in $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$, hence it suffices to show that the internal hom

$\underline{\mathrm{Hom}}_{\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})}(h(A), h(B))$ in $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ is representable. This follows from the computation

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})}(h(A), h(B)) &\simeq \mathrm{map}_{\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})}(h(-)\times h(A), h(B)) \\ &\simeq \mathrm{map}_{\mathcal{B}}(- \times A, B) \\ &\simeq \mathrm{map}_{\mathcal{B}}(-, \underline{\mathrm{Hom}}_{\mathcal{B}}(A, B))\end{aligned}$$

in which we make repeated use of Yoneda's lemma. □

1.1.5. Factorisation systems

Let $\mathcal{C}$ be an ∞-category. Given two maps $f\colon a \to b$ and $g\colon c \to d$ in $\mathcal{C}$, we say that f and g are orthogonal if the commutative square

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(b, c) \arrow[r, "g_*"] \arrow[d, "f^*"] & \mathrm{map}_{\mathcal{C}}(b, d) \arrow[d, "f^*"] \\ \mathrm{map}_{\mathcal{C}}(a, c) \arrow[r, "g^*"] & \mathrm{map}_{\mathcal{C}}(a, d) \end{tikzcd}$$

is cartesian. We denote the orthogonality relation between f and g by $f \perp g$, and we will say that f is left orthogonal to g and g is right orthogonal to f. In particular, f and g being orthogonal implies that any lifting square

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] & c \arrow[d, "g"] \\ b \arrow[r] \arrow[ur, dotted] & d \end{tikzcd}$$

has a unique solution. If $\mathcal{C}$ is cartesian closed, we furthermore say that f and g are *internally* orthogonal if the square

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathcal{C}}(b, c) \arrow[r, "g_*"] \arrow[d, "f^*"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(b, d) \arrow[d, "f^*"] \\ \underline{\mathrm{Hom}}_{\mathcal{C}}(a, c) \arrow[r, "g^*"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(a, d) \end{tikzcd}$$

is cartesian. By definition, this is equivalent to $c \times f \perp g$ for every $c \in \mathcal{C}$. We will denote the internal orthogonality relation between f and g by $f \underline{\perp} g$.

If $\mathcal{C}$ has a terminal object $1 \in \mathcal{C}$, then an object $c \in \mathcal{C}$ is said to be *local* with respect to the map $f\colon a \to b$ in $\mathcal{C}$ if the terminal map $\pi_c\colon c \to 1$ is right orthogonal to f, i.e. if $f \perp \pi_c$ holds. Similarly, c is *internally local* with respect to f if $f \underline{\perp} \pi_c$ holds.

If S is an arbitrary family of maps in $\mathcal{C}$, we will denote by $S^{\perp}$ the collection of maps that are right orthogonal to any map in S, and by ${}^{\perp}S$ the collection of maps that are left orthogonal to any map in S.

Definition 1.1.5.1. Let $\mathcal{C}$ be an ∞-category. A *factorisation system* is a pair $(\mathcal{L}, \mathcal{R})$ of families of maps in $\mathcal{C}$ such that

1. Any map f in $\mathcal{C}$ admits a factorisation $f \simeq rl$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$.

2. $\mathcal{L}^{\perp} = \mathcal{R}$ as well as ${}^{\perp}\mathcal{R} = \mathcal{L}$.

The following proposition summarises some properties of factorisation systems whose proof is a straightforward consequence of the definition:

Proposition 1.1.5.2. *Let $\mathcal{C}$ be an ∞-category and let $(\mathcal{L}, \mathcal{R})$ be a factorisation system in $\mathcal{C}$. Then*

1. *The intersection $\mathcal{L} \cap \mathcal{R}$ is precisely the collection of equivalences in $\mathcal{C}$;*

2. *if $g \in \mathcal{L}$, then $fg \in \mathcal{L}$ if and only $f \in \mathcal{L}$; dually, if $f \in \mathcal{R}$ then $fg \in \mathcal{R}$ if and only if $g \in \mathcal{R}$;*

3. *$\mathcal{R}$ is stable under pullbacks and $\mathcal{L}$ is stable under pushouts;*

4. *both $\mathcal{R}$ and $\mathcal{L}$ are stable under taking retracts.*

5. *$\mathcal{R}$ is stable under all limits that exist in* $\mathrm{Fun}(\Delta^1, \mathcal{C})$*, and dually $\mathcal{L}$ is stable under all colimits that exist in* $\mathrm{Fun}(\Delta^1, \mathcal{C})$. □

If $(\mathcal{L}, \mathcal{R})$ is a factorisation system in an ∞-category $\mathcal{C}$, then $\mathcal{R}$ defines a full subcategory of the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. The factorisation of maps in $\mathcal{C}$ then defines a left adjoint to this inclusion [49, Lemma 5.2.8.19]. More precisely, if $f\colon d \to c$ is a map in $\mathcal{C}$ and if $rl\colon d \to e \to c$ is the factorisation of f into maps $l \in \mathcal{L}$ and $r \in \mathcal{R}$, then the assignment $f \mapsto r$ extends to a functor $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{R}$

that is left adjoint to the inclusion. The unit of this adjunction is then given by the square

$$\begin{tikzcd} d \arrow[r, "l"] \arrow[d, "f"] & e \arrow[d, "r"] \\ c \arrow[r, "\mathrm{id}"] & c. \end{tikzcd}$$

By dualisation, this also shows that the inclusion $\mathcal{L} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ admits a right adjoint.

Note that the fact that the inclusion $\mathcal{R} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ admits a left adjoint moreover proves that for any $c \in \mathcal{C}$ the induced inclusion $\mathcal{R}_{/c} \hookrightarrow \mathcal{C}_{/c}$ is reflective. If $f\colon d \to c$ is an object in $\mathcal{C}_{/c}$ and if $f \simeq rl$ is its factorisation, then the map r is the image of f under the localisation functor and the map l is the component of the counit of the adjunction at f. In particular, this shows that the map $f\colon d \to c$ is contained in $\mathcal{L}$ if and only if it is sent to the terminal object by the localisation functor $\mathcal{C}_{/c} \to \mathcal{R}_{/c}$, and the essential image of the inclusion $\mathcal{R}_{/c} \hookrightarrow \mathcal{C}_{/c}$ is spanned by those objects in $\mathcal{C}_{/c}$ that are local with respect to the class of maps in $\mathcal{C}_{/c}$ that are sent to $\mathcal{L}$ via the projection functor $(\pi_c)_! \colon \mathcal{C}_{/c} \to \mathcal{C}$.

Remark 1.1.5.3. Suppose that $(\mathcal{L}, \mathcal{R})$ is a factorisation system in an ∞-category $\mathcal{C}$ that has a final object $1 \in \mathcal{C}$, and let $f\colon c \to d$ be a map in $\mathcal{C}$. Let $L \colon \mathcal{C} \to \mathcal{R}_{/1}$ be a left adjoint to the inclusion, and consider the commutative diagram

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d] & d \arrow[d] \\ L(c) \arrow[r, "L(f)"] \arrow[d] & L(d) \arrow[d] \\ 1 \arrow[r, "\mathrm{id}"] & 1 \end{tikzcd}$$

in which the two vertical compositions are determined by the factorisation of the two terminal maps $\pi_c \colon c \to 1$ and $\pi_d \colon d \to 1$ into maps in $\mathcal{L}$ and $\mathcal{R}$. If f is contained in $\mathcal{L}$, then item (2) of Proposition 1.1.5.2 implies that $L(f)$ must be contained in $\mathcal{L}$ as well. On the other hand, item (2) of Proposition 1.1.5.2 also implies that $L(f)$ is contained in $\mathcal{R}$. Hence $L(f)$ must be an equivalence. In other words, the functor L sends maps in $\mathcal{L}$ to equivalences in $\mathcal{R}_{/1}$. The converse is however not true in general, i.e. not every map that is sent to an equivalence by L must necessarily be contained in $\mathcal{L}$. A notable exception is the case where $\pi_d : d \to 1$ is already contained in $\mathcal{L}$. In this case, the map $d \to L(d)$ is an equivalence, hence $L(f)$ being an equivalence does imply that f is contained in $\mathcal{L}$.

Lastly, let us discuss how a factorisation system can be *generated* by a set of maps:

Proposition 1.1.5.4 ([49, Proposition 5.5.5.7])**.** *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small set of maps in $\mathcal{C}$. Then there is a factorisation system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ with $\mathcal{R} = S^{\perp}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$.* □

In the situation of Proposition 1.1.5.4, the assignment $S \mapsto \mathcal{L}$ can be viewed as a certain closure operation that is referred to as *saturation*. Recall the definition of a saturated class:

Definition 1.1.5.5. Let $\mathcal{C}$ be a presentable ∞-category and let S be a class of maps in $\mathcal{C}$. Then S is *saturated* if

1. S contains all equivalences in $\mathcal{C}$ and is closed under composition;
2. S is closed under small colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$;
3. S is closed under pushouts.

By Proposition 1.1.5.2, the left class in any factorisation system is saturated. Now if S is a small set of maps and if $(\mathcal{L}, \mathcal{R})$ is the induced factorisation system in $\mathcal{C}$ as provided by Proposition 1.1.5.4, then $\mathcal{L}$ is the *universal* saturated class that contains S, in the following sense:

Proposition 1.1.5.6. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small set of maps in $\mathcal{C}$. Let $(\mathcal{L}, \mathcal{R})$ be the associated factorisation system. Then $\mathcal{L}$ is the smallest saturated class of maps that contains S.*

Proof. To begin with, note that the property of a class being saturated is preserved under taking arbitrary intersections, hence the *smallest* saturated class containing S is well-defined and is explicitly given by the intersection

$$\overline{S} = \bigcap_{S \subset T} T$$

over all saturated classes of maps that contain S. We need to show that any saturated $T \supset S$ contains $\mathcal{L}$ as well. But since $\mathcal{L} = {}^{\perp}\mathcal{R}$ and as $(\mathcal{L}, \mathcal{R})$ is a factorisation system, this is equivalent to $T^{\perp} \subset \mathcal{R}$, which in turn follows immediately from $S \subset T$ and $S^{\perp} = \mathcal{R}$. □

An analogous construction can be carried out when replacing orthogonality by *internal orthogonality* in the case where $\mathcal{C}$ is cartesian closed:

Proposition 1.1.5.7 ([2, Proposition 3.2.9]). *Let $\mathcal{C}$ be a presentable and cartesian closed ∞-category and let S be a small set of maps in $\mathcal{C}$. Then there is a factorisation system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ such that $\mathcal{R} = S^{\perp\!\!\!\perp}$ and $\mathcal{L} = {}^{\perp\!\!\!\perp}\mathcal{R} = {}^{\perp}\mathcal{R}$.* □

Since a map r in a cartesian closed ∞-category $\mathcal{C}$ is internally right orthogonal to a map l if and only if r is right orthogonal to $c \times l$ for any $c \in \mathcal{C}$, Proposition 1.1.5.4 implies:

Proposition 1.1.5.8. *Let $\mathcal{C}$ be a presentable and cartesian closed ∞-category and let S be a small set of maps in $\mathcal{C}$. Let $(\mathcal{L}, \mathcal{R})$ be the factorisation system provided by Proposition 1.1.5.7. Then $\mathcal{L}$ is the smallest saturated class of maps that contains the set $\{c \times f \mid c \in \mathcal{C},\ f \in S\}$.* □

Remark 1.1.5.9. If $(\mathcal{L}, \mathcal{R})$ is an *internal* factorisation system (i.e. if we have $\mathcal{R} = \mathcal{L}^{\perp\!\!\!\perp}$), the class $\mathcal{L}$ is preserved by $- \times c$ for every $c \in \mathcal{C}$. Consequently, this implies that $\mathcal{R}$ is closed under exponentiation, i.e. preserved by $\underline{\mathrm{Hom}}_{\mathcal{C}}(c, -)$ for every $c \in \mathcal{C}$.

Example 1.1.5.10. Let $\mathcal{C}$ be a presentable and cartesian closed ∞-category. We say that a map $f\colon c \to d$ is a *monomorphism* if it is internally right orthogonal to the codiagonal $1 \sqcup 1 \to 1$, where 1 is the final object in $\mathcal{C}$. By construction, a map in $\mathcal{C}$ is a monomorphism if and only if its diagonal is an equivalence. Dually, we say that a map is a *strong epimorphism* if it is internally left orthogonal to every monomorphism. By Proposition 1.1.5.7, one obtains a factorisation system in which the left class are strong epimorphisms and the right class are monomorphisms. If $\mathcal{C}$ is an ∞-topos, then strong epimorphisms are precisely *covers*, i.e. effective epimorphisms in the terminology of [49]. This follows from the fact that covers and monomorphisms form a factorisation system in any

∞-topos [49, Example 5.2.8.16], combined with the fact that the left class of a factorisation system is uniquely determined by the right class.

1.2. $\mathcal{B}$-categories

In this section, we introduce the basic framework of the theory of higher categories internal to an ∞-topos $\mathcal{B}$, hereafter referred to as *$\mathcal{B}$-categories*. We will *define* these objects by mimicking the model for ∞-categories provided by *complete Segal spaces*. Thus, a $\mathcal{B}$-category will be defined as a simplicial object in $\mathcal{B}$ satisfying two conditions: the *Segal conditions* and *univalence*. We begin in Section 1.2.1 by establishing some basic facts about general simplicial objects in an ∞-topos, before we define *$\mathcal{B}$-groupoids* in Section 1.2.2 and $\mathcal{B}$-categories in Section 1.2.3. In Section 1.2.4, we discuss in what way our definitions are *functorial* in the base ∞-topos $\mathcal{B}$, and in Section 1.2.5, we shed light on certain $(\infty, 2)$-categorical aspects of the theory of $\mathcal{B}$-categories.

In Section 1.2.6, we explain an alternative approach to $\mathcal{B}$-categories: that of *sheaves of ∞-categories on $\mathcal{B}$*. Conceptually, this approach is quite different, focussing more on the *parametrised* point of view rather than the *internal* one. Nonetheless, the two perspectives are entirely equivalent to one another, which is a key insight that allows us to reap the benefits of both worlds. In fact, virtually every concrete example of a $\mathcal{B}$-category will come in the form of a sheaf of ∞-categories on $\mathcal{B}$, whereas the development of the theory will mostly take place within the complete Segal model of $\mathcal{B}$-categories.

Lastly, we explain in Section 1.2.7 how $\mathcal{B}$-categories can be thought of as a certain flavour of *categories* by defining *objects* and *morphisms* in a $\mathcal{B}$-category. The key difference from ∞-categories will be that every object or morphism in a $\mathcal{B}$-category will be defined within a *context*, an object in the base ∞-topos $\mathcal{B}$. The presence of contexts is a consequence of the spatial nature of ∞-topoi, which enable us to make definitions and perform constructions *locally*.

1.2.1. Simplicial objects in an ∞-topos

Let $\mathcal{B}$ be an arbitrary ∞-topos and let $\mathcal{B}_\Delta$ denote the ∞-topos of simplicial objects in $\mathcal{B}$. By postcomposition with the adjunction $(\mathrm{const} \dashv \Gamma)\colon \ \mathrm{Ani} \rightleftarrows \mathcal{B}$

one obtains an induced adjunction $(\mathrm{const} \dashv \Gamma)\colon \mathrm{Ani}_\Delta \rightleftarrows \mathcal{B}_\Delta$ on the level of simplicial objects. This defines a functor

$$(-)_\Delta \colon \mathrm{Top}_\infty^{\mathrm{R}} \to (\mathrm{Top}_\infty^{\mathrm{R}})_{/\mathrm{Ani}_\Delta}$$

from the ∞-category of ∞-topoi with geometric morphisms as maps to the slice ∞-category of ∞-topoi over Ani_Δ.

Remark 1.2.1.1. We will often not notationally distinguish between a simplicial ∞-groupoid K and its image along the functor $\mathrm{const}\colon \mathrm{Ani}_\Delta \to \mathcal{B}_\Delta$, if it is clear from the context in which ∞-category our arguments are taking place.

We define the *tensoring* of $\mathcal{B}_\Delta$ over Ani_Δ as the bifunctor

$$- \otimes - \colon \mathrm{Ani}_\Delta \times \mathcal{B}_\Delta \to \mathcal{B}_\Delta$$

that is given by the composition $(- \times -) \circ (\mathrm{const} \times \mathrm{id}_{\mathcal{B}_\Delta})$. Dually, we define the *powering* of $\mathcal{B}_\Delta$ over Ani_Δ as the bifunctor

$$(-)^{(-)} \colon \mathrm{Ani}_\Delta^{\mathrm{op}} \times \mathcal{B}_\Delta \to \mathcal{B}_\Delta$$

that is given by the composite $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(-,-) \circ (\mathrm{const} \times \mathrm{id}_{\mathcal{B}_\Delta})$ where $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(-,-)$ denotes the internal hom in $\mathcal{B}_\Delta$. Let $\mathrm{Hom}_{\mathcal{B}_\Delta}(-,-)\colon \mathcal{B}_\Delta^{\mathrm{op}} \times \mathcal{B}_\Delta \to \mathrm{Ani}_\Delta$ be the bifunctor given by the composition $\Gamma \circ \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(-,-)$. We then obtain equivalences

$$\mathrm{map}_{\mathcal{B}_\Delta}(-,(-)^{(-)}) \simeq \mathrm{map}_{\mathcal{B}_\Delta}(- \otimes -, -) \simeq \mathrm{map}_{\mathrm{Ani}_\Delta}(-, \mathrm{Hom}_{\mathcal{B}_\Delta}(-,-)).$$

Remark 1.2.1.2. We may regard every $A \in \mathcal{B}$ as a simplicial object in $\mathcal{B}$ via the diagonal functor $\mathcal{B} \hookrightarrow \mathcal{B}_\Delta$. Since products in $\mathcal{B}_\Delta$ are computed object-wise, the endofunctor $A \times -$ on $\mathcal{B}_\Delta$ is equivalent to the functor that is given by postcomposing simplicial objects with the product functor $A \times - \colon \mathcal{B} \to \mathcal{B}$. The latter admits a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{B}}(A,-)\colon \mathcal{B} \to \mathcal{B}$, and since postcomposition with an adjunction induces an adjunction on the level of functor ∞-categories, the uniqueness of adjoints implies that the internal hom $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(A,-)\colon \mathcal{B}_\Delta \to \mathcal{B}_\Delta$ is obtained by applying $\underline{\mathrm{Hom}}_{\mathcal{B}}(A,-)$ level-wise to simplicial objects in $\mathcal{B}$. More precisely, the restriction of the internal hom $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(-,-)$ along the inclusion $\mathcal{B}^{\mathrm{op}} \times \mathcal{B}_\Delta \hookrightarrow \mathcal{B}_\Delta^{\mathrm{op}} \times \mathcal{B}_\Delta$ is equivalent to the transpose of the composite functor

$$\mathcal{B}^{\mathrm{op}} \times \mathcal{B}_\Delta \times \Delta^{\mathrm{op}} \xrightarrow{\mathrm{id}_{\mathcal{B}^{\mathrm{op}}} \times \mathrm{ev}_{\Delta^{\mathrm{op}}}} \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{B}}(-,-)} \mathcal{B}$$

in which $\mathrm{ev}_{\Delta^{\mathrm{op}}}$ denotes the evaluation map.

In particular, this argument shows that the internal hom of $\mathcal{B}_\Delta$ restricts to the internal hom of $\mathcal{B}$.

Remark 1.2.1.3. Let $i\colon \mathcal{G} \hookrightarrow \mathcal{B}$ be a small full subcategory such that the left Kan extension $\mathrm{PSh}(\mathcal{G}) \to \mathcal{B}$ of the i is a left exact and accessible Bousfield localisation. Then $\mathcal{B}_\Delta$ is a left exact and accessible Bousfield localisation of $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$. Let $L\colon \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G}) \to \mathcal{B}_\Delta$ be the localisation functor. Then the composition

$$\Delta \times \mathcal{G} \overset{h}{\hookrightarrow} \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G}) \xrightarrow{L} \mathcal{B}_\Delta$$

can be identified with the restriction $- \otimes -|_{\Delta\times\mathcal{G}}$ of the tensoring bifunctor along the inclusion $\Delta^\bullet \times i\colon \Delta \times \mathcal{G} \hookrightarrow \mathrm{Ani}_\Delta \times \mathcal{B}$, where $\Delta^\bullet$ denotes the Yoneda embedding. By [51, § 20.4.1], this means that $- \otimes -|_{\Delta\times\mathcal{G}}$ is a *dense* functor, i.e. that every simplicial object in $\mathcal{B}$ can be canonically obtained as a colimit of objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{G}$, or equivalently that the identity functor on $\mathcal{B}_\Delta$ is a left Kan extension of $- \otimes -|_{\Delta\times\mathcal{G}}$ along itself.

Observe that the diagonal functor $\iota\colon \mathcal{B} \hookrightarrow \mathcal{B}_\Delta$ admits both a right adjoint given by the evaluation functor $(-)_0\colon \mathcal{B}_\Delta \to \mathcal{B}$ and a left adjoint given by the colimit functor $\mathrm{colim}_{\Delta^{\mathrm{op}}}\colon \mathcal{B}_\Delta \to \mathcal{B}$. Restricting the powering bifunctor along the Yoneda embedding $\Delta^\bullet\colon \Delta \hookrightarrow \mathrm{Ani}_\Delta$ then defines a functor $(-)^{\Delta^\bullet}\colon \mathcal{B}_\Delta \to \mathrm{PSh}_{\mathcal{B}_\Delta}(\Delta)$. The computation

$$\begin{aligned}
\mathrm{map}_{\mathcal{B}}(-, ((-)^{\Delta^\bullet})_0) &\simeq \mathrm{map}_{\mathcal{B}_\Delta}(\iota(-), (-)^{\Delta^\bullet}) \\
&\simeq \mathrm{map}_{\mathrm{Ani}_\Delta}(\Delta^\bullet, \mathrm{Hom}_{\mathcal{B}_\Delta}(\iota(-), -)) \\
&\simeq \Gamma\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(\iota(-), (-)_\bullet) \\
&\simeq \mathrm{map}_{\mathcal{B}}(-, (-)_\bullet)
\end{aligned}$$

in which the penultimate equivalence follows from Yoneda's lemma and Remark 1.2.1.2 now shows :

Proposition 1.2.1.4. *The composite functor*

$$\mathcal{B}_\Delta \xrightarrow{(-)^{\Delta^\bullet}} \mathrm{PSh}_{\mathcal{B}_\Delta}(\Delta) \xrightarrow{(-)_0} \mathcal{B}_\Delta$$

in which the second arrow denotes postcomposition with the evaluation functor $(-)_0 \colon \mathcal{B}_\Delta \to \mathcal{B}$ *is equivalent to the identity functor on* $\mathcal{B}_\Delta$. □

Lastly, we mention that we can construct *opposites* on the level of simplicial objects in $\mathcal{B}$: In fact, recall that the autoequivalence $(-)^{\mathrm{op}} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ restricts to an autoequivalence on $\Delta \hookrightarrow \mathrm{Cat}_\infty$. By precomposition, one thus obtains an autoequivalence $(-)^{\mathrm{op}} \colon \mathcal{B}_\Delta \to \mathcal{B}_\Delta$. For any simplicial object C in $\mathcal{B}$, we refer to the simplicial object C^{op} as the *opposite* of C. Note that the restriction of $(-)^{\mathrm{op}}$ along the diagonal embedding $\iota \colon \mathcal{B} \hookrightarrow \mathcal{B}_\Delta$ recovers the identity on $\mathcal{B}$.

1.2.2. $\mathcal{B}$-groupoids

Before we define and study the central notion of a $\mathcal{B}$-category, we first discuss the simpler case of $\mathcal{B}$-*groupoids*:

Definition 1.2.2.1. A simplicial object in $\mathcal{B}$ is said to be a $\mathcal{B}$-*groupoid* if it internally local with respect to the map $s^0 \colon \Delta^1 \to \Delta^0$. We denote by $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ the full subcategory spanned by the $\mathcal{B}$-groupoids.

$\mathcal{B}$-groupoids turn out to be precisely those simplicial objects that are contained in the full subcategory $\iota \colon \mathcal{B} \hookrightarrow \mathcal{B}_\Delta$:

Proposition 1.2.2.2. *For any simplicial object $C \in \mathcal{B}_\Delta$, the following are equivalent:*

1. *C is contained in the essential image of the diagonal functor $\iota \colon \mathcal{B} \hookrightarrow \mathcal{B}_\Delta$.*
2. *C is internally local with respect to the map $s^0 \colon \Delta^1 \to \Delta^0$.*

The proof of Proposition 1.2.2.2 will rely on the following combinatorial lemma:

Lemma 1.2.2.3. *Let S be a saturated class of maps in $\mathcal{B}$ that contains the maps $s^0 \colon \Delta^1 \otimes C \to C$ for every $C \in \mathcal{B}_\Delta$. Then S contains the projection $\Delta^n \otimes A \to A$ for every $n \geq 0$ and every $A \in \mathcal{B}$.*

Proof. We will use induction on n, the case $n = 1$ being true by assumption. Let us therefore assume that for an arbitrary integer $n \geq 1$ the projection $s^0 \colon \Delta^n \otimes A \to A$ is contained in S. Then the composition $(\Delta^1 \times \Delta^n) \otimes A \to \Delta^n \otimes A \to A$ in which the first map is induced by $s^0 \colon \Delta^1 \to \Delta^0$ is contained in S as well. Let

$\alpha\colon \Delta^{n+1} \to \Delta^1 \times \Delta^n$ be the map that is defined by $\alpha(0) = (0,0)$ and $\alpha(k) = (1, k-1)$ for all $1 \leq k \leq n+1$, and let $\beta\colon \Delta^1 \times \Delta^n \to \Delta^{n+1}$ be defined by $\beta(0,k) = 0$ and $\beta(1,k) = k+1$ for all $0 \leq k \leq n$. Then the composition $\beta\alpha$ is equivalent to the identity on Δ^{n+1}, and we therefore obtain a retract diagram

$$\begin{tikzcd}
\Delta^{n+1} \otimes A \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[d] & (\Delta^1 \times \Delta^n) \otimes A \arrow[r, "\beta \otimes \mathrm{id}"] \arrow[d] & \Delta^{n+1} \otimes A \arrow[d] \\
A \arrow[r, "\mathrm{id}"] & A \arrow[r, "\mathrm{id}"] & A
\end{tikzcd}$$

which shows that the map $\Delta^{n+1} \otimes A \to A$ is contained in S as well. $\square$

Proof of Proposition 1.2.2.2. If C is internally local with respect to $s^0 : \Delta^1 \to \Delta^0$, Lemma 1.2.2.3 and Proposition 1.2.1.4 imply that the simplicial maps $C_0 \to C_n$ are equivalences in $\mathcal{B}$ for all $n \geq 0$, which in turn implies that C is in the essential image of the diagonal functor (as Δ^{op} is weakly contractible). Conversely, let $A \in \mathcal{B}$ be an arbitrary object. By making use of the adjunction $\mathrm{colim}_{\Delta^{\mathrm{op}}} \dashv \iota$ and the fact that the functor $\mathrm{colim}_{\Delta^{\mathrm{op}}}$ commutes with finite products as Δ^{op} is a sifted ∞-category, the object $\iota(A)$ is internally local to s^0 whenever A is internally local to $\mathrm{colim}_{\Delta^{\mathrm{op}}}(s^0)\colon\ \mathrm{colim}_{\Delta^{\mathrm{op}}} \Delta^1 \to \mathrm{colim}_{\Delta^{\mathrm{op}}} \Delta^0$ in $\mathcal{B}$. Since the colimit of a representable presheaf on a small ∞-category is always the final object $1 \in \mathrm{Ani}$, the latter map must be an equivalence, hence the result follows. $\square$

1.2.3. $\mathcal{B}$-categories

We now proceed by defining a $\mathcal{B}$-category via the *Segal conditions* and the notion of *univalence*. To that end, let us recall the following combinatorial constructions:

Definition 1.2.3.1. For any $n \geq 1$, let $I^n = \Delta^1 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^0} \Delta^1 \subset \Delta^n$ denote the *spine* of Δ^n, i.e the simplicial subset of Δ^n that is spanned by the inclusions $d^{\{i-1,i\}} : \Delta^1 \subset \Delta^n$ for $i = 1, \ldots, n$. Furthermore, let E^1 be the simplicial set that is defined by the pushout square

$$\begin{tikzcd}
\Delta^1 \sqcup \Delta^1 \arrow[r, "s^0 \sqcup s^0"] \arrow[d, "d^{\{0,2\}} \sqcup d^{\{1,3\}}"'] & \Delta^0 \sqcup \Delta^0 \arrow[d] \\
\Delta^3 \arrow[r] & E^1 \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

in Ani_{Δ}. We refer to E^1 as the *walking equivalence.*

Remark 1.2.3.2. Many authors define the walking equivalence as the simplicial set that arises as the nerve of the category with two objects and a unique isomorphism between them. The simplicial set E^1 from Definition 1.2.3.1 is different in that it is comprised of a map together with *separate* left and right inverses.

Remark 1.2.3.3. In the situation of Definition 1.2.3.1, the colimits that define I^n and E^1 a priori live in Ani_Δ. However, it turns out that they can be computed as the ordinary (i.e 1-categorical) pushouts in the 1-category of simplicial sets. Indeed, colimits in Ani_Δ are computed level-wise and so are ordinary colimits in Set_Δ, hence it suffice to show that for any integer $k \geq 0$ the colimits in Ani of the diagrams in $\mathrm{Set} \hookrightarrow \mathrm{Ani}$ that define I^n_k and E^1_k can be computed by the ordinary colimits of these diagrams in Set. As the Quillen model structure on the 1-category of simplicial sets is left proper, ordinary pushouts along monomorphisms of simplicial sets present homotopy pushouts in Ani. Now I^n is an iterated pushout along monomorphisms in Ani_Δ, hence I^n_k is an iterated pushout along monomorphisms in $\mathrm{Set} \hookrightarrow \mathrm{Ani}$, hence the claim follows for I^n. Regarding the simplicial ∞-groupoid E^1, note that E^1 fits into the commutative diagram

$$\begin{tikzcd}
 & \Delta^1 \arrow[r] \arrow[d, hook, "d^{\{1,3\}}"] & \Delta^0 \arrow[d] \\
\Delta^1 \arrow[r, hook, "d^{\{0,2\}}"] \arrow[d] & \Delta^3 \arrow[r] \arrow[d] \arrow[ur, phantom, "\lrcorner"] & \Delta^3 \sqcup_{\Delta^1} \Delta^0 \arrow[d] \\
\Delta^0 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \Delta^0 \sqcup_{\Delta^1} \Delta^3 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & E^1
\end{tikzcd}$$

of simplicial ∞-groupoids. Since the pushouts in the lower left and in the upper right corner are computed by the ordinary pushouts of simplicial sets, the claim now follows from the straightforward observation that the composition $\Delta^1 \hookrightarrow \Delta^3 \to \Delta^3 \sqcup_{\Delta^1} \Delta^0$ is a monomorphism.

Definition 1.2.3.4. An object $C \in \mathcal{B}_\Delta$ is called a $\mathcal{B}$-*category* if

(Segal conditions) C is internally local with respect to the map $I^2 \hookrightarrow \Delta^2$, and

(univalence) C is internally local with respect to the map $E^1 \to \Delta^0$.

We denote the full subcategory of $\mathcal{B}_\Delta$ that is spanned by the $\mathcal{B}$-categories by $\mathrm{Cat}(\mathcal{B})$.

Lemma 1.2.3.5. *The following sets generate the same saturated class of maps in $\mathcal{B}_\Delta$:*

1. $S = \{I^2 \otimes K \hookrightarrow \Delta^2 \otimes K \mid K \in \mathcal{B}_\Delta\}$;
2. $T = \{I^n \otimes A \hookrightarrow \Delta^n \otimes A \mid n \geq 2,\ A \in \mathcal{B}\}$;
3. $U = \{I^n \otimes K \hookrightarrow \Delta^n \otimes K \mid n \geq 2,\ K \in \mathcal{B}_\Delta\}$.

Proof. Using Remark 1.2.1.3, we may assume without loss of generality that K is of the form $\Delta^k \otimes A$ for some $A \in \mathcal{B}$ and some $k \geq 0$. Moreover, note that if $i\colon K \to L$ is a map in Ani_Δ, a map $f\colon C \to D$ in $\mathcal{B}_\Delta$ is right orthogonal to $i \otimes \mathrm{id}_A$ if and only if $f_*\colon \mathrm{Fun}_{\mathcal{B}}(A, C) \to \mathrm{Fun}_{\mathcal{B}}(A, D)$ is right orthogonal to i. This allows us to reduce to the case where $\mathcal{B} \simeq \mathrm{Ani}$, where $A \simeq 1$ and where $K \simeq \Delta^k$ for an arbitrary $k \geq 0$.

We now claim that a map in Ani_Δ is contained in the saturation $\overline{T}$ of T if and only if it is contained in the saturation $\overline{T}'$ of the set

$$T' = \{\Lambda^n_i \hookrightarrow \Delta^n \mid n \geq 2, 0 < i < n\}$$

of inner horn inclusions. In fact, by [43, Proposition 2.13] the spine inclusions $I^n \hookrightarrow \Delta^n$ are contained in $\overline{T}'$, and the converse is proved in [44, Lemma 3.5]. As a consequence, every inner anodyne map between simplicial sets is contained in $\overline{T}$. Therefore, [49, Proposition 2.3.2.4] implies that for any $n \geq 2$ and any $k \geq 0$, the map $I^n \times \Delta^k \hookrightarrow \Delta^n \times \Delta^k$ is contained in $\overline{T}$ as well. To complete the proof, we therefore only need to show that $\overline{T}$ is contained in the saturation $\overline{S}$ of S. To that end, observe that $\overline{S}$ contains all maps of the form $I^2 \times \partial\Delta^k \hookrightarrow \Delta^2 \times \partial\Delta^k$ and therefore all maps of the form

$$(I^2 \times \Delta^k) \sqcup_{I^2 \times \partial\Delta^k} (\Delta^2 \times \partial\Delta^k) \hookrightarrow \Delta^2 \times \Delta^k$$

as well. By [49, Proposition 2.3.2.1], this implies that T' is contained in $\overline{S}$ as well, hence the claim follows. □

Lemma 1.2.3.6. *Let S be a strongly saturated class of maps in $\mathcal{B}$ that contains the maps $E^1 \otimes A \to A$ and $I^n \otimes A \hookrightarrow \Delta^n \otimes A$ for all $A \in \mathcal{B}$ and all $n \geq 0$. Then S contains the maps $E^1 \otimes C \to C$ for all $C \in \mathcal{B}_\Delta$.*

Proof. Using a similar argumentation as in Lemma 1.2.3.5, we may assume without loss of generality that $\mathcal{B} \simeq \mathrm{Ani}$, that A is the final simplicial ∞-groupoid and furthermore that $C \simeq \Delta^n$. As furthermore by Lemma 1.2.3.5 the map $E^1 \times I^n \hookrightarrow E^1 \times \Delta^n$ is contained in S for all $n \geq 0$, we need only show that the induced map $E^1 \times I^n \to I^n$ is contained in S. Using that I^n is a colimit of a diagram involving only Δ^0 and Δ^1, we may further restrict to the case $n = 1$, in which case the statement was proven by Rezk in [70, Proposition 12.1]. □

Proposition 1.2.3.7. *Let C be a simplicial object in $\mathcal{B}$. The following conditions are equivalent:*

1. *C is a $\mathcal{B}$-category;*
2. *for all $n \geq 2$ the maps*

$$C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1$$

as well as the map

$$C_0 \to (C_0 \times C_0) \times_{C_1 \times C_1} C_3$$

are equivalences.

Proof. A simplicial object C in $\mathcal{B}$ satisfies the Segal condition if and only if it is local with respect to the collection of maps $I^2 \otimes E \hookrightarrow \Delta^2 \otimes E$ for any simplicial object $E \in \mathcal{B}_\Delta$. On the other hand, the first map of condition (2) is an equivalence if and only if C is local with respect to all maps $I^n \otimes A \hookrightarrow \Delta^n \otimes A$ for arbitrary $A \in \mathcal{B}$. By Lemma 1.2.3.5, these two conditions are equivalent. Similarly, Lemma 1.2.3.6 implies that C is univalent if and only if the second map of condition (2) is an equivalence. □

Remark 1.2.3.8. Proposition 1.2.3.7 allows us to make sense of the notion of a $\mathcal{C}$-category for any ∞-category $\mathcal{C}$ with finite limits. That is, we may define a $\mathcal{C}$-category to be a simplicial object $C \in \mathcal{C}_\Delta$ that satisfies the second condition of Proposition 1.2.3.7.

Example 1.2.3.9. Proposition 1.2.3.7 shows that an Ani-category is precisely a *complete Segal space* as developed by Rezk [70]. By a theorem of Joyal and Tierney [44] the ∞-category of complete Segal spaces is a model for the ∞-category of ∞-categories Cat_∞, i.e. the left Kan extension of the inclusion $\Delta \hookrightarrow \mathrm{Cat}_\infty$ determines an equivalence $\mathrm{Cat}(\mathrm{Ani}) \simeq \mathrm{Cat}_\infty$.

Since $\mathrm{Cat}(\mathcal{B})$ is a localisation of $\mathcal{B}_\Delta$ at a small set of morphisms, one finds:

Proposition 1.2.3.10. *The inclusion* $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ *exhibits* $\mathrm{Cat}(\mathcal{B})$ *as an accessible localisation of* $\mathcal{B}_\Delta$*. In particular, the* ∞*-category* $\mathrm{Cat}(\mathcal{B})$ *is presentable.* □

Remark 1.2.3.11. As $\mathrm{Cat}(\mathcal{B})$ is an accessible localisation of $\mathcal{B}_\Delta$, the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ creates small limits. On the other hand, colimits of small diagrams are usually not created (or even preserved) by the inclusion, as these can be computed by applying the localisation functor $\mathcal{B}_\Delta \to \mathrm{Cat}(\mathcal{B})$ to the colimits of the underlying diagrams of simplicial objects in $\mathcal{B}$. *Filtered* colimits, on the other hand, are created by the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$. In fact, as such colimits commute with finite limits, it is immediate from Proposition 1.2.3.7 that the colimit in $\mathcal{B}_\Delta$ of a small filtered diagram of $\mathcal{B}$-categories is contained in $\mathrm{Cat}(\mathcal{B})$ and is therefore the colimit of this diagram in $\mathrm{Cat}(\mathcal{B})$. In particular, as colimits are universal in $\mathcal{B}_\Delta$, this observation implies that *filtered* colimits are universal in $\mathrm{Cat}(\mathcal{B})$.

As $\mathcal{B}$-categories are *internally* local with respect to the maps $I^2 \hookrightarrow \Delta^2$ and $E^1 \to 1$, the ∞-category $\mathrm{Cat}(\mathcal{B})$ is cartesian closed. More precisely, the construction of $\mathrm{Cat}(\mathcal{B})$ implies:

Proposition 1.2.3.12. *The full subcategory* $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ *is an* exponential ideal *and therefore in particular a cartesian closed* ∞*-category. In other words, if* C *is a* $\mathcal{B}$*-category and* D *is a simplicial object in* $\mathcal{B}$*, the internal hom* $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(D, \mathsf{C})$ *is a* $\mathcal{B}$*-category.* □

Corollary 1.2.3.13. *The localisation functor* $\mathcal{B}_\Delta \to \mathrm{Cat}(\mathcal{B})$ *preserves finite products.* □

Definition 1.2.3.14. If C and D are $\mathcal{B}$-categories, we denote their internal hom by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ and refer to it as the $\mathcal{B}$*-category of functors* between C and D.

Recall that the inclusion $\iota\colon \mathcal{B} \hookrightarrow \mathcal{B}_\Delta$ admits a right adjoint $(-)_0$ and a left adjoint $\mathrm{colim}_{\Delta^{\mathrm{op}}}(-)$. It is moreover immediate from Proposition 1.2.3.7 that ι

factors through the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$, i.e. that for any object $A \in \mathcal{B}$ the associated $\mathcal{B}$-groupoid defines a $\mathcal{B}$-category. Therefore, we obtain functors

$$\begin{tikzcd} \mathrm{Grpd}(\mathcal{B}) \arrow[r, hook] \arrow[r, phantom, shift left=3, "\top"] \arrow[r, phantom, shift right=3, "\top"] & \mathrm{Cat}(\mathcal{B}). \arrow[l, bend right, "(-)^{\simeq}"'] \arrow[l, bend left, "(-)^{\mathrm{gpd}}"] \end{tikzcd}$$

The functor $(-)^{\simeq}$ is referred to as the *core $\mathcal{B}$-groupoid* functor, and the functor $(-)^{\mathrm{gpd}}$ is referred to as the *groupoidification* functor. If C is a $\mathcal{B}$-category, the $\mathcal{B}$-groupoid $\mathsf{C}^{\simeq}$ is to be thought of as the maximal $\mathcal{B}$-groupoid that is contained in C (i.e. the subcategory spanned by all objects and equivalences in C), whereas the $\mathcal{B}$-groupoid $\mathsf{C}^{\mathrm{gpd}}$ should be regarded as the result of formally inverting all morphisms in C.

Proposition 1.2.3.15. *The groupoidification functor* $(-)^{\mathrm{gpd}}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Grpd}(\mathcal{B})$ *commutes with small colimits and finite products.*

Proof. As the groupoidification functor is a left adjoint, it commutes with small colimits. Moreover, since the final object $1 \in \mathrm{Cat}(\mathcal{B})$ is given by the constant simplicial object on $1 \in \mathcal{B}$ and since groupoidification is a localisation functor, there is an equivalence $1^{\mathrm{gpd}} \simeq 1$. It therefore suffices to consider the case of binary products, which follows from Δ^{op} being a sifted ∞-category. □

Proposition 1.2.3.16. *The core $\mathcal{B}$-groupoid functor* $(-)^{\simeq}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Grpd}(\mathcal{B})$ *commutes with small filtered colimits and small limits.*

Proof. As the core $\mathcal{B}$-groupoid functor is a right adjoint, it commutes with small limits. By Remark 1.2.3.11, small filtered colimits of $\mathcal{B}$-categories are created by the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$, hence it suffices to show that $(-)_0\colon \mathcal{B}_\Delta \to \mathcal{B}$ commutes with such colimits, which is immediate. □

Lastly, we observe that the involution $(-)^{\mathrm{op}}\colon \mathcal{B}_\Delta \simeq \mathcal{B}_\Delta$ that takes a simplicial object C to its opposite C^{op} restricts to an involution $(-)^{\mathrm{op}}\colon \mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Cat}(\mathcal{B})$. In other words, the *opposite* C^{op} of a $\mathcal{B}$-category C is well-defined.

1.2.4. Functoriality and base change

In this section we discuss how to change the base ∞-topos for internal higher category theory. What makes this possible is the following general lemma:

Lemma 1.2.4.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories that admit finite limits such that F preserves pullbacks and such that for any map $f\colon c \to c'$ in $\mathcal{C}$ the commutative square*

$$\begin{tikzcd} F(c \times c) \arrow[r, "F(f\times f)"] \arrow[d] & F(c' \times c') \arrow[d] \\ F(c) \times F(c) \arrow[r, "F(f)\times F(f)"] & F(c') \times F(c') \end{tikzcd}$$

is cartesian. Then the induced functor $\mathcal{C}_\Delta \to \mathcal{D}_\Delta$ that is given by postcomposition with F sends $\mathcal{C}$-categories to $\mathcal{D}$-categories and therefore restricts to a functor

$$F\colon \ \mathrm{Cat}(\mathcal{C}) \to \mathrm{Cat}(\mathcal{D}).$$

In particular, any left exact functor F between finitely complete ∞-categories induces a functor on the level of categories.

Proof. Since F preserves pullbacks the functor $\mathcal{C}_\Delta \to \mathcal{D}_\Delta$ that is given by post-composition with F preserves the Segal conditions, and by assumption on F the commutative square

$$\begin{tikzcd} F(\mathsf{C}_0) \arrow[r] \arrow[d] & F(\mathsf{C}_3) \arrow[d, "(d_{\{0,2\}}, d_{\{1,3\}})"] \\ F(\mathsf{C}_0) \times F(\mathsf{C}_0) \arrow[r, "(s_0, s_0)"] & F(\mathsf{C}_1) \times F(\mathsf{C}_1) \end{tikzcd}$$

is cartesian for every $\mathcal{C}$-category C, hence the claim follows. □

Since both geometric and algebraic morphisms of ∞-topoi preserve finite limits and therefore satisfy the condition from Lemma 1.2.4.1, one concludes that if $f_*\colon \mathcal{B} \to \mathcal{A}$ is a geometric morphism of ∞-topoi with left adjoint f^*, one obtains an induced adjunction

$$(f^* \dashv f_*)\colon \ \mathrm{Cat}(\mathcal{A}) \rightleftarrows \mathrm{Cat}(\mathcal{B}).$$

In particular, the adjunction $(\mathrm{const} \dashv \Gamma)\colon \mathrm{Ani} \rightleftarrows \mathcal{B}$ gives rise to an adjunction

$$(\mathrm{const} \dashv \Gamma)\colon \mathrm{Cat}_\infty \rightleftarrows \mathrm{Cat}(\mathcal{B}).$$

Hence, we may regard every ∞-category as a *constant* $\mathcal{B}$-category, and every $\mathcal{B}$-category has an underlying *∞-category of global sections.*

Remark 1.2.4.2. We will usually not notationally distinguish between an ∞-category and the associated constant $\mathcal{B}$-category, as long as it is clear from the context in which ∞-category we are working.

The procedure of changing the base ∞-topos can be made functorial in the strongest possible sense. In fact, if we let $\mathcal{Z}$ denote the subcategory of $\mathrm{Cat}_\infty^{\mathbf{W}}$ spanned by the ∞-categories that admit finite limits, together with those functors that satisfy the conditions of Lemma 1.2.4.1, and if $\int(-)_\Delta \to \mathcal{Z}$ is the cocartesian fibration that classifies the functor $(-)_\Delta\colon \mathcal{Z} \to \mathrm{Cat}_\infty^{\mathbf{W}}$, then Lemma 1.2.4.1 implies that the full subcategory of $\int(-)_\Delta$ that is spanned by the pairs $(\mathcal{C}, \mathsf{C})$ with $\mathcal{C} \in \mathcal{Z}$ and C a $\mathcal{C}$-category is stable under taking cocartesian arrows and therefore defines a cocartesian subfibration of $\int(-)_\Delta$ over $\mathcal{Z}$. Hence one obtains a functor

$$\mathrm{Cat}\colon \mathcal{Z} \to \mathrm{Cat}_\infty^{\mathbf{W}}.$$

Since both the forgetful functor $\mathrm{Top}_\infty^{\mathrm{R}} \to \mathrm{Cat}_\infty^{\mathbf{W}}$ and the universe enlargement functor $\mathcal{B} \mapsto \widehat{\mathcal{B}}$ for ∞-topoi factor through $\mathcal{Z}$ and since moreover the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ commutes with small limits, postcomposition with the functor Cat gives rise to functors $\mathcal{B} \mapsto \mathrm{Cat}(\mathcal{B})$ as well as $\mathcal{B} \mapsto \mathrm{Cat}(\widehat{\mathcal{B}})$ together with a natural transformation

$$\begin{tikzcd} \mathrm{Top}_\infty^{\mathrm{R}} \arrow[r, bend left, "\mathcal{B}\mapsto\mathrm{Cat}(\mathcal{B})"] \arrow[r, bend right, "\mathcal{B}\mapsto\mathrm{Cat}(\widehat{\mathcal{B}})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Cat}_\infty^{\mathbf{W}} \end{tikzcd}$$

that is given by the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})$. Analogously, one obtains two functors $\mathrm{Top}_\infty^{\mathrm{L}} \rightrightarrows \mathrm{Cat}_\infty^{\mathbf{W}}$ together with a natural transformation

$$\begin{tikzcd} \mathrm{Top}_\infty^{\mathrm{L}} \arrow[r, bend left, "\mathcal{B}\mapsto\mathrm{Cat}(\mathcal{B})"] \arrow[r, bend right, "\mathcal{B}\mapsto\mathrm{Cat}(\widehat{\mathcal{B}})"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{Cat}_\infty^{\mathbf{W}}. \end{tikzcd}$$

As the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}_\Delta$ and its right adjoint $(-)_0 \colon \mathcal{C}_\Delta \to \mathcal{C}$ are clearly functorial in $\mathcal{C} \in \mathcal{Z}$, we may now conclude:

Proposition 1.2.4.3. *There are two commutative squares*

$$\begin{tikzcd} \mathrm{Grpd}(\mathcal{B}) \arrow[r, hook] \arrow[d, hook, shift right] & \mathrm{Grpd}(\widehat{\mathcal{B}}) \arrow[d, hook, shift right] \\ \mathrm{Cat}(\mathcal{B}) \arrow[r, hook] \arrow[u, shift right, "(-)^{\simeq}"'] & \mathrm{Cat}(\widehat{\mathcal{B}}) \arrow[u, shift right, "(-)^{\simeq}"'] \end{tikzcd}$$

that are functorial in $\mathcal{B}$ both with respect to maps in $\mathrm{Top}^{\mathrm{R}}_\infty$ and maps in $\mathrm{Top}^{\mathrm{L}}_\infty$. □

Let $\mathcal{W}$ be the full subcategory of $\mathcal{Z}$ that is spanned by the ∞-categories that admit colimits indexed by Δ^{op} and the functors that preserve such colimits. Then the map $\mathrm{colim}_{\Delta^{\mathrm{op}}} \colon \mathcal{C}_\Delta \to \mathcal{C}$ is functorial in $\mathcal{C} \in \mathcal{W}$, and as universe enlargement preserves small colimits one finds:

Proposition 1.2.4.4. *There is a commutative square*

$$\begin{tikzcd} \mathrm{Grpd}(\mathcal{B}) \arrow[r, hook] & \mathrm{Grpd}(\widehat{\mathcal{B}}) \\ \mathrm{Cat}(\mathcal{B}) \arrow[r, hook] \arrow[u, "(-)^{\mathrm{gpd}}"'] & \mathrm{Cat}(\widehat{\mathcal{B}}) \arrow[u, "(-)^{\mathrm{gpd}}"'] \end{tikzcd}$$

that is functorial in $\mathcal{B}$ with respect to maps in $\mathrm{Top}^{\mathrm{L}}_\infty$. □

Convention 1.2.4.5. We refer to the objects in $\mathrm{Cat}(\widehat{\mathcal{B}})$ as *large* $\mathcal{B}$-categories (or as $\widehat{\mathcal{B}}$-categories) and to the objects in $\mathrm{Cat}(\mathcal{B})$ as *small* $\mathcal{B}$-categories. If not specified otherwise, every $\mathcal{B}$-category is small. Note, however, that by replacing the universe **U** with the larger universe **V** (i.e. by working internally to $\widehat{\mathcal{B}}$), every statement about $\mathcal{B}$-categories carries over to one about large $\mathcal{B}$-categories as well. Also, we will sometimes omit specifying the relative size of a $\mathcal{B}$-category if it is evident from the context.

Base change along *étale* geometric morphisms is particularly well-behaved, as we will discuss now. Let $\mathrm{Cat}(\mathcal{B})_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ be the functor that classifies the cartesian fibration

$$\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B})) \times_{\mathrm{Cat}(\mathcal{B})} \mathcal{B} \to \mathcal{B}.$$

One now finds:

Proposition 1.2.4.6. *There is an equivalence*

$$\mathsf{Cat}(\mathcal{B}_{/-}) \simeq \mathsf{Cat}(\mathcal{B})_{/-}$$

of functors $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$ that fits into a commutative square

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{B}_{/-}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathsf{Cat}(\widehat{\mathcal{B}_{/-}}) \arrow[d, "\simeq"] \\ \mathsf{Cat}(\mathcal{B})_{/-} \arrow[r, hook] & \mathsf{Cat}(\widehat{\mathcal{B}})_{/-} \end{tikzcd}$$

Proof. On account of the natural equivalence $\mathsf{Fun}(\Delta^1, \mathcal{B}_\Delta) \simeq \mathsf{Fun}(\Delta^1, \mathcal{B})_\Delta$ and the fact that the cartesian fibration $\mathsf{Fun}(\Delta^1, \mathcal{B})_\Delta \times_{\mathcal{B}_\Delta} \mathcal{B} \to \mathcal{B}$ is classified by the functor $(\mathcal{B}_{/-})_\Delta$ (see for example [17, Proposition 2.6.1] for an argument), one obtains an equivalence

$$(\mathcal{B}_{/-})_\Delta \simeq (\mathcal{B}_\Delta)_{/-}$$

of functors $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$. In order to obtain an equivalence $\mathsf{Cat}(\mathcal{B}_{/-}) \simeq \mathsf{Cat}(\mathcal{B})_{/-}$, it therefore suffices to show that this equivalence of functors sends $\mathcal{B}_{/A}$-categories to objects in $(\mathcal{B}_\Delta)_{/A}$ whose underlying simplicial object in $\mathcal{B}$ is a $\mathcal{B}$-category. By construction, the component of the above equivalence at $A \in \mathcal{B}$ sits inside the commutative diagram

$$\begin{tikzcd} (\mathcal{B}_{/A})_\Delta \arrow[rr, "\simeq"] \arrow[dr, "(\pi_A)_! \circ (-)"'] & & (\mathcal{B}_\Delta)_{/A} \arrow[dl, "(\pi_A)_!"] \\ & \mathcal{B}_\Delta, & \end{tikzcd}$$

hence the claim follows from the fact that the forgetful functor $(\pi_A)_! \colon \mathcal{B}_{/A} \to \mathcal{B}$ satisfies the conditions of Lemma 1.2.4.1. Lastly, the existence of a commutative square as in the statement of the proposition follows from the construction of the equivalence $\mathsf{Cat}(\mathcal{B}_{/-}) \simeq \mathsf{Cat}(\mathcal{B})_{/-}$ and Lemma 1.1.4.2. □

1.2.5. The $(\infty, 2)$-categorical structure of $\mathsf{Cat}(\mathcal{B})$

Recall from Definition 1.2.3.14 that we denote by

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-) \colon \mathsf{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathsf{Cat}(\mathcal{B}) \to \mathsf{Cat}(\mathcal{B})$$

the internal hom in $\mathrm{Cat}(\mathcal{B})$. By Proposition 1.2.3.12, this bifunctor is obtained by restricting the internal hom of $\mathcal{B}_\Delta$ to the full subcategory $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$. As the product bifunctor on $\mathrm{Cat}(\mathcal{B})$ is obtained in the same fashion, the three bifunctors that are defined in the beginning of Section 1.2.1 restrict to bifunctors on the level of $\mathcal{B}$-categories and ∞-categories. Explicitly, one obtains a tensoring bifunctor

$$- \otimes - \colon \mathrm{Cat}_\infty \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$$

which is given by $\mathrm{const}(-) \times -$, a powering bifunctor

$$(-)^{(-)} \colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$$

given by $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathrm{const}(-), -)$, and a functor ∞-category bifunctor

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \colon \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$$

which is defined as $\Gamma \underline{\mathrm{Fun}}_{\mathcal{B}}(-,-)$. These functors are equipped with equivalences

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-, (-)^{(-)}) \simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(- \otimes -, -) \simeq \mathrm{map}_{\mathrm{Cat}_\infty}(-, \mathrm{Fun}_{\mathcal{B}}(-,-)).$$

In particular, the second equivalence implies that postcomposing $\mathrm{Fun}_{\mathcal{B}}(-,-)$ with the core ∞-groupoid functor $(-)^{\simeq} \colon \mathrm{Cat}_\infty \to \mathrm{Ani}$ recovers the bifunctor $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-,-)$.

The above constructions are well-behaved with respect to universe enlargement:

Proposition 1.2.5.1. *The internal hom in* $\mathrm{Cat}(\widehat{\mathcal{B}})$ *restricts to the internal hom in* $\mathrm{Cat}(\mathcal{B})$.

Proof. As the product bifunctor on $\mathrm{Cat}(\widehat{\mathcal{B}})$ clearly restricts to the product bifunctor on $\mathrm{Cat}(\mathcal{B})$, it suffices to show that for any two *small* $\mathcal{B}$-categories $\mathsf{C}, \mathsf{D} \in \mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})$ their internal hom $\underline{\mathrm{Fun}}_{\widehat{\mathcal{B}}}(\mathsf{C}, \mathsf{D})$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$ is small as well. It suffices to show this on the level of simplicial objects, i.e. we need to show that the simplicial object $\underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(\mathsf{C}, \mathsf{D}) \in \widehat{\mathcal{B}}_\Delta$ is contained in $\mathcal{B}_\Delta$. Using Proposition 1.2.1.4 one finds

$$\underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(\mathsf{C}, \mathsf{D})_n \simeq \underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(\Delta^n \otimes \mathsf{C}, \mathsf{D})_0,$$

hence it suffices to show that $\underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(\mathsf{C}, \mathsf{D})_0$ is contained in $\mathcal{B}$. By Remark 1.2.1.3, using that the functor $\underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(-, \mathsf{D})_0$ sends colimits in $\widehat{\mathcal{B}}_\Delta$ to limits in $\widehat{\mathcal{B}}$, we may assume without loss of generality $\mathsf{C} \simeq \Delta^n \otimes A$. In this case one computes

$$\underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(\Delta^n \otimes A, \mathsf{D})_0 \simeq \underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}_\Delta}(A, \mathsf{D})_n \simeq \underline{\mathrm{Hom}}_{\widehat{\mathcal{B}}}(A, \mathsf{D}_n)$$

in which the last step follows from Remark 1.2.1.2. Therefore, the claim is a consequence of Proposition 1.1.4.4. □

Combining Proposition 1.2.5.1 with Proposition 1.2.4.3, one now easily deduces:

Corollary 1.2.5.2. *The tensoring, powering and mapping ∞-category bifunctors on* $\mathrm{Cat}(\widehat{\mathcal{B}})$ *restrict to the tensoring, powering and mapping ∞-category bifunctors on* $\mathrm{Cat}(\mathcal{B})$. □

Convention 1.2.5.3. For simplicity, we will usually also denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ the internal hom in $\mathrm{Cat}(\widehat{\mathcal{B}})$, and likewise by $\mathrm{Fun}_{\mathcal{B}}(-,-)$ the mapping ∞-category bifunctor on $\mathrm{Cat}(\widehat{\mathcal{B}})$. By Proposition 1.2.5.1 and Corollary 1.2.5.2, there is no possibility of confusion.

Functor $\mathcal{B}$-categories are moreover preserved by *étale* base change:

Proposition 1.2.5.4. *For any object $A \in \mathcal{B}$ there is a canonical equivalence*

$$\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*(-), \pi_A^*(-))$$

of bifunctors $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}_{/A})$.

Proof. By definition of functor categories in $\mathrm{Cat}(\mathcal{B})$ and in $\mathrm{Cat}(\mathcal{B}_{/A})$, the datum of an equivalence of bifunctors $\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*(-), \pi_A^*(-))$ is equivalent to the datum of an equivalence

$$(\pi_A)_!(- \times_A \pi_A^*(-)) \simeq (\pi_A)_!(-) \times -$$

of bifunctors $\mathrm{Cat}(\mathcal{B}_{/A}) \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$. Since π_A^* commutes with products, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[r, "\pi_A^* \times \pi_A^*"] \arrow[d, "- \times -"'] & \mathcal{B}_{/A} \times \mathcal{B}_{/A} \arrow[d, "- \times_A -"] \\
\mathcal{B} \arrow[r, "\pi_A^*"'] & \mathcal{B}_{/A}.
\end{tikzcd}$$

The (horizontal) *mate* of this square (see for example Definition 3.1.2.5 below) yields a map

$$\phi\colon (\pi_A)_!(- \times_A -) \to (\pi_A)_!(-) \times \pi_A(-),$$

and by combining this map with the adjunction counit $\epsilon\colon (\pi_A)_!\pi_A^* \to \mathrm{id}_{\mathrm{Cat}(\mathcal{B})}$, we can define a map

$$(\pi_A)_!(- \times_A \pi_A^*(-)) \xrightarrow{\phi(\mathrm{id}\times\pi_A^*)} (\pi_A)_!(-) \times (\pi_A)_!\pi_A^*(-) \xrightarrow{\mathrm{id}\times\epsilon} (\pi_A)_!(-) \times -.$$

By construction, when evaluated at a pair $(\mathsf{C} \to A, \mathsf{D})$, this map is given by the morphism ψ in the commutative diagram

$$\begin{tikzcd}
 & \mathsf{C}\times\mathsf{D} \arrow[rr, "\mathrm{pr}_1"] \arrow[dd] & & \mathsf{D} \arrow[dd] \\
\mathsf{C}\times_A(\mathsf{D}\times A) \arrow[ur, "\psi"] \arrow[rr, "\mathrm{pr}_1"] \arrow[dd, "\mathrm{pr}_0"] & & \mathsf{D}\times A \arrow[ur, "\mathrm{pr}_0"] \arrow[dd, "\mathrm{pr}_1"] & \\
 & \mathsf{C} \arrow[rr] & & 1 \\
\mathsf{C} \arrow[ur, "\mathrm{id}"] \arrow[rr] & & A. \arrow[ur, "\pi_A"] &
\end{tikzcd}$$

The square on the left side being cartesian now implies that ψ is an equivalence, hence the desired result follows. $\square$

Remark 1.2.5.5. Note that Proposition 1.2.5.4 also implies that for any two $\mathcal{B}$-categories C and D, there is a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}, \pi_A^*\mathsf{D}) \times \pi_A^*\mathsf{C} \arrow[r, "\simeq"] \arrow[dr, "\mathrm{ev}_{\pi_A^*\mathsf{C}}"'] & \pi_A^*(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \times \mathsf{C}) \arrow[d, "\pi_A^*\mathrm{ev}_{\mathsf{C}}"] \\
 & \pi_A^*\mathsf{D}.
\end{tikzcd}$$

In fact, by the argument in the proof of Proposition 1.2.5.4, the equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}, -) \simeq \pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \pi_A^*(-))$$

is obtained as the (horizontal) mate of the commutative square

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathsf{C}\times-"] \arrow[d, "\pi_A^*"] & \mathcal{B} \arrow[d, "\pi_A^*"] \\
\mathcal{B}_{/A} \arrow[r, "\pi_A^*\mathsf{C}\times_A-"] & \mathcal{B}_{/A}.
\end{tikzcd}$$

Therefore, the claim can be deduced from the functoriality of mates (see Remark 3.1.2.7).

Remark 1.2.5.6. It is clear from the definition that base change along every algebraic morphism of ∞-topoi commutes with the tensoring bifunctor, so that dually base change along every geometric morphism of ∞-topoi commutes with the powering bifunctor. Moreover, Proposition 1.2.5.4 implies that the powering bifunctor is also preserved by the étale base change π_A^* for every $A \in \mathcal{B}$.

Remark 1.2.5.7. The bifunctor $\mathrm{Fun}_{\mathcal{B}}(-,-)$ gives rise to an $(\infty, 2)$-categorical enhancement of $\mathrm{Cat}(\mathcal{B})$. More precisely, on account of $\mathrm{Cat}(\mathcal{B})$ being cartesian closed, this ∞-category is canonically enriched over itself [24, § 7]. The functor $\Gamma\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$ can then be used to change enrichment from $\mathrm{Cat}(\mathcal{B})$ to Cat_∞, so that $\mathrm{Cat}(\mathcal{B})$ becomes a Cat_∞-enriched ∞-category, which is one of the known models for $(\infty, 2)$-categories [31].

1.2.6. Cat_∞-valued sheaves on an ∞-topos

$\mathcal{B}$-categories can be alternatively regarded as sheaves of ∞-categories on $\mathcal{B}$. To see this, first recall from the discussion in Section 1.1.4 that the Yoneda embedding induces a commutative square

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook] \arrow[d, hook] & \widehat{\mathcal{B}} \arrow[d, hook] \\ \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{B}) \arrow[r, hook] & \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B}) \end{tikzcd}$$

that is natural in $\mathcal{B}$ both with respect to maps in $\mathrm{Top}_\infty^{\mathrm{R}}$ and in $\mathrm{Top}_\infty^{\mathrm{L}}$. Postcomposition with the functor $(-)_\Delta$ therefore gives rise to a natural commutative square

$$\begin{tikzcd} \mathcal{B}_\Delta \arrow[r, hook] \arrow[d, hook] & \widehat{\mathcal{B}}_\Delta \arrow[d, hook] \\ \mathrm{PSh}_{\mathrm{Ani}_\Delta}(\mathcal{B}) \arrow[r, hook] & \mathrm{PSh}_{\widehat{\mathrm{Ani}}_\Delta}(\mathcal{B}) \end{tikzcd}$$

in which the essential image of the two vertical maps is spanned by the collection of Ani_Δ-valued and $\widehat{\mathrm{Ani}}_\Delta$-valued sheaves, respectively. Using Proposition 1.2.3.7

it is immediate that this square further restricts to a natural commutative square

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{B}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Cat}(\widehat{\mathcal{B}}) \arrow[d, hook] \\ \mathrm{PSh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \arrow[r, hook] & \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}). \end{tikzcd}$$

As limits in Cat_∞ and in $\widehat{\mathrm{Cat}}_\infty$ are computed on the level of the underlying simplicial ∞-groupoids, the essential image of the two vertical maps is spanned by the collection of Cat_∞-valued and $\widehat{\mathrm{Cat}}_\infty$-valued sheaves, respectively. One therefore obtains:

Proposition 1.2.6.1. *The inclusions*

$$\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \quad \textit{and} \quad \mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$$

induce a commutative square

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{B}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{Cat}(\widehat{\mathcal{B}}) \arrow[d, "\simeq"] \\ \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \arrow[r, hook] & \mathrm{Sh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}) \end{tikzcd}$$

that is natural in $\mathcal{B}$ both with respect to maps in $\mathrm{Top}_\infty^{\mathrm{R}}$ and in $\mathrm{Top}_\infty^{\mathrm{L}}$. □

Remark 1.2.6.2. In what follows, we will often implicitly identify a $\mathcal{B}$-category C with the associated Cat_∞-valued sheaf on $\mathcal{B}$. In particular, if $A \in \mathcal{B}$ is an arbitrary object we will write $\mathsf{C}(A)$ for the ∞-category of *local sections* of C over A.

Remark 1.2.6.3. Recall that straightening and unstraightening yields a natural equivalence $\mathrm{PSh}_{\mathrm{Cat}_\infty}(\mathcal{C}) \simeq \mathrm{Cart}(\mathcal{C})$ between the ∞-category of Cat_∞-valued presheaves on a small ∞-category $\mathcal{C}$ and the ∞-category of cartesian fibrations over $\mathcal{C}$. Hence, the inclusion $\mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$ gives rise to an embedding

$$\mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{Cart}(\mathcal{B})$$

that is natural in $\mathcal{B}$. For any (large) $\mathcal{B}$-category C we will denote the image of C under this functor by $\int \mathsf{C} \to \mathcal{B}$.

The equivalence in Proposition 1.2.6.1 can also be implemented by making use of the bifunctor $\mathrm{Fun}_{\mathcal{B}}(-,-)$: If $\iota\colon \mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ denotes the natural inclusion, the computation

$$\mathrm{map}_{\mathrm{Cat}_\infty}(\Delta^\bullet, \mathrm{Fun}_{\mathcal{B}}(\iota(-),-)) \simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\iota(-),(-)^{\Delta^\bullet}) \simeq \mathrm{map}_{\mathcal{B}}(-,(-)_\bullet)$$

(in which the last equivalence follows from Proposition 1.2.1.4) shows that the transpose of the bifunctor

$$\mathrm{Fun}_{\mathcal{B}}(\iota(-),-)\colon\ \mathcal{B}^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$$

recovers the natural inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ and therefore in particular the equivalence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ from Proposition 1.2.6.1. It is therefore reasonable to define:

Definition 1.2.6.4. For any object $A \in \mathcal{B}$, the *local sections functor over A* is defined as the functor $\mathrm{Fun}_{\mathcal{B}}(A,-)\colon\ \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$.

Remark 1.2.6.5. In the context of Definition 1.2.6.4, the local sections functor over an object $A \in \mathcal{B}$ is equivalently given by the composite

$$\mathrm{Cat}(\mathcal{B}) \xrightarrow{\pi_A^*} \mathrm{Cat}(\mathcal{B}_{/A}) \xrightarrow{\Gamma_{\mathcal{B}_{/A}}} \mathrm{Cat}_\infty\,.$$

In fact, the equivalence of functors $- \times (\pi_A)_!(-) \simeq (\pi_A)_!(\pi_A^*(-) \times_A -)$ gives rise to the following chain of equivalences

$$\begin{aligned}\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(- \otimes (\pi_A)_!(-),-) &\simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}((\pi_A)_!(- \otimes -),-)\\ &\simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B}_{/A})}(- \otimes -, \pi_A^*(-))\end{aligned}$$

that induces an equivalence of functors

$$\mathrm{Fun}_{\mathcal{B}}((\pi_A)_!(-),-) \simeq \mathrm{Fun}_{\mathcal{B}_{/A}}(-,\pi_A^*(-)).$$

Remark 1.2.6.6. By construction, the equivalence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ fits into two commutative squares

$$\begin{tikzcd}
\mathrm{Grpd}(\mathcal{B}) \arrow[r, "\simeq"] \arrow[d, hook, shift right] & \mathrm{Sh}_{\mathrm{Ani}}(\mathcal{B}) \arrow[d, hook, shift right] \\
\mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] \arrow[u, shift right, "(-)^{\simeq}"'] & \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \arrow[u, shift right, "(-)^{\simeq}"']
\end{tikzcd}$$

that are functorial in $\mathcal{B}$ with respect to maps both in $\mathrm{Top}^{\mathrm{R}}_{\infty}$ and in $\mathrm{Top}^{\mathrm{L}}_{\infty}$. Here the two vertical maps on the right are given by postcomposition with the adjunction $(\iota \dashv (-)^{\simeq})\colon \mathrm{Ani} \rightleftarrows \mathrm{Cat}_{\infty}$. One moreover has a commutative square

$$\begin{tikzcd} \mathrm{Grpd}(\mathcal{B}) \arrow[r, "\simeq"] & \mathrm{Sh}_{\mathrm{Ani}}(\mathcal{B}) \\ \mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] \arrow[u, "(-)^{\mathrm{gpd}}"] & \mathrm{Sh}_{\mathrm{Cat}_{\infty}}(\mathcal{B}) \arrow[u, "(-)^{\mathrm{gpd}}"] \end{tikzcd}$$

that is functorial in $\mathcal{B}$ with respect to maps in $\mathrm{Top}^{\mathrm{L}}_{\infty}$, where the vertical map on the right is given by postcomposition with the groupoidification functor $(-)^{\mathrm{gpd}}\colon \mathrm{Cat}_{\infty} \to \mathrm{Ani}$.

Remark 1.2.6.7. The equivalence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Sh}_{\mathrm{Cat}_{\infty}}(\mathcal{B})$ also fits into a commutative square

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] \arrow[d, "(-)^{\mathrm{op}}"] & \mathrm{Sh}_{\mathrm{Cat}_{\infty}}(\mathcal{B}) \arrow[d, "(-)^{\mathrm{op}}"] \\ \mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] & \mathrm{Sh}_{\mathrm{Cat}_{\infty}}(\mathcal{B}) \end{tikzcd}$$

in which the right vertical map is given by postcomposition with the equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat}_{\infty} \simeq \mathrm{Cat}_{\infty}$.

Remark 1.2.6.8. The fact that the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Cat}_{\infty}}(\mathcal{B})$ is obtained by the functor $\mathrm{Fun}_{\mathcal{B}}(\iota(-), -)$ implies that the Cat_{∞}-valued presheaf that underlies the powering $\mathsf{C}^{\mathcal{X}}$ of a $\mathcal{B}$-category C by an ∞-category $\mathcal{X} \in \mathrm{Cat}_{\infty}$ is given by the functor $\mathrm{Fun}(\mathcal{X}, \mathsf{C}(-))$. Moreover, this identification is natural both in $\mathcal{X} \in \mathrm{Cat}_{\infty}$ and in $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$.

By contrast, the Cat_{∞}-valued presheaf that underlies the *tensoring* $\mathcal{X} \otimes \mathsf{C}$ of the $\mathcal{B}$-category C by the ∞-category $\mathcal{X}$ is *not* given by the functor $\mathcal{X} \times \mathsf{C}(-)$. In fact, the latter is in general not a sheaf. However, one can show that the presheaf that is associated with $\mathcal{X} \otimes \mathsf{C}$ is given by the sheafification of the presheaf $\mathcal{X} \times \mathsf{C}(-)$, i.e. by the image of $\mathcal{X} \times \mathsf{C}(-)$ under the left adjoint of the inclusion $\mathrm{Sh}_{\mathrm{Cat}_{\infty}}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Cat}_{\infty}}(\mathcal{B})$.

1.2.7. Objects and morphisms in a $\mathcal{B}$-category

Let C be a $\mathcal{B}$-category, and let $A \in \mathcal{B}$ be an arbitrary object. An *object c of C in context A* is defined to be a local section $c\colon A \to \mathsf{C}$, which is equivalently determined by a map $c\colon A \to \mathsf{C}_0$ since A is a $\mathcal{B}$-groupoid. A *morphism in C in context A* is defined as an object f of C^{Δ^1}, i.e. as a map $f\colon \Delta^1 \otimes A \to \mathsf{C}$, or equivalently as a map $f\colon A \to \mathsf{C}_1$. Similarly one defines the notion of an n-morphism for any $n \geq 1$ as a map $\Delta^n \otimes A \to \mathsf{C}$. Any morphism f has a source and a target which are obtained by precomposing $f\colon \Delta^1 \otimes A \to \mathsf{C}$ with $d^1 : A \to \Delta^1 \otimes A$ and with $d^0 : A \to \Delta^1 \otimes A$, respectively. If c and d are the source and target of such a morphism f, we also use the familiar notation $f\colon c \to d$. For any object c in C in context A, there is a morphism id_c that is defined by the composite $cs^0 : \Delta^1 \otimes A \to A \to \mathsf{C}$.

Remark 1.2.7.1. On account of the adjunction $(\pi_A)_! \dashv \pi_A^*$, specifying an object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ is tantamount to specifying an object $\bar{c} : 1 \to \pi_A^*\mathsf{C}$ in context $1 \in \mathcal{B}_{/A}$. A similar observation can be made for morphisms in a $\mathcal{B}$-category.

Given two objects c and d in C in context $A \in \mathcal{B}$, the *mapping $\mathcal{B}$-groupoid* $\mathrm{map}_{\mathsf{C}}(c,d) \in \mathcal{B}_{/A}$ is defined as the pullback

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}}(c,d) \arrow[r] \arrow[d] & \mathsf{C}_1 \arrow[d, "{(d_1,d_0)}"] \\
A \arrow[r, "{(c,d)}"] & \mathsf{C}_0 \times \mathsf{C}_0.
\end{tikzcd}$$

Equivalently, this object can be defined by the pullback

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}}(c,d) \arrow[r] \arrow[d] & \mathsf{C}^{\Delta^1} \arrow[d, "{(d_1,d_0)}"] \\
A \arrow[r, "{(c,d)}"] & \mathsf{C} \times \mathsf{C},
\end{tikzcd}$$

see Section 2.1.1 below. By construction, sections $A \to \mathrm{map}_{\mathsf{C}}(c,d)$ over A correspond to morphisms $f\colon c \to d$ in C in context A. Two maps $f, g : c \rightrightarrows d$ are said to be *equivalent* if they are equivalent as sections $A \rightrightarrows \mathrm{map}_{\mathsf{C}}(c,d)$ over A, in which case we write $f \simeq g$.

Remark 1.2.7.2. Since for every $A \in \mathcal{B}$ the adjunction counit $(\pi_A)_! \pi_A^* \to \mathrm{id}_{\mathcal{B}}$ carries maps in $\mathcal{B}$ to pullback squares, we deduce that if $c, d \colon A \rightrightarrows \mathsf{C}$ are two objects in context A and if $\bar{c}, \bar{d} \colon 1 \rightrightarrows \pi_A^* \mathsf{C}$ are the transposed objects (see Remark 1.2.7.1), we have a canonical equivalence

$$\mathrm{map}_{\mathsf{C}}(c, d) \simeq \mathrm{map}_{\pi_A^* \mathsf{C}}(\bar{c}, \bar{d})$$

in $\mathcal{B}_{/A}$.

Similarly to the case of two objects, if $c_0, \dots, c_n$ are objects in context A in C, one writes $\mathrm{map}_{\mathsf{C}}(c_0, \dots, c_n)$ for the pullback of $(d_n, \dots, d_0) \colon \mathsf{C}_n \to \mathsf{C}_0^{n+1}$ along $(c_0, \dots, c_n) \colon A \to \mathsf{C}_0^{n+1}$. Using the Segal conditions, one obtains an equivalence

$$\mathrm{map}_{\mathsf{C}}(c_0, \dots, c_n) \simeq \mathrm{map}_{\mathsf{C}}(c_0, c_1) \times_A \cdots \times_A \mathrm{map}_{\mathsf{C}}(c_{n-1}, c_n).$$

By combining this identification with the map $\mathrm{map}_{\mathsf{C}}(c_0, \dots, c_n) \to \mathrm{map}_{\mathsf{C}}(c_0, c_n)$ that is induced by the map $d_{\{0,n\}} \colon \mathsf{C}_n \to \mathsf{C}_1$, one obtains a composition map

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \times_A \cdots \times_A \mathrm{map}_{\mathsf{C}}(c_{n-1}, c_n) \to \mathrm{map}_{\mathsf{C}}(c_0, c_n).$$

Given maps $f_i \colon c_{i-1} \to c_i$ in C for $i = 1, \dots, n$, we write $f_n \cdots f_1$ for their composition. By making use of the simplicial identities, it is straightforward to verify that composition is associative and unital, i.e. that the relations $f(gh) \simeq (fg)h$ and $f\mathrm{id} \simeq f \simeq \mathrm{id} f$ as well as their higher analogues hold whenever they make sense, see [70, Proposition 5.4] for a proof.

A morphism $f \colon c \to d$ in C is an *equivalence* if there are maps $g \colon c \to d$ and $h \colon c \to d$ (all in context A) such that $gf \simeq \mathrm{id}_c$ and $fh \simeq \mathrm{id}_d$. Let $\Delta^1 \to E^1$ be the map that is induced by the inclusion $d^{\{1,2\}} \colon \Delta^1 \hookrightarrow \Delta^3$. One then obtains the following characterisation of equivalences in C:

Proposition 1.2.7.3. *A map $f \colon c \to d$ in C in context A is an equivalence if and only if the map $\Delta^1 \otimes A \to \mathsf{C}$ that is determined by f factors through the map $\Delta^1 \otimes A \to E^1 \otimes A$.*

Proof. Suppose that there are $g, h \colon d \rightrightarrows c$ together with equivalences $gf \simeq \mathrm{id}_c$ and $fh \simeq \mathrm{id}_d$ that witness f as an equivalence in C. The triple (h, f, g) then determines a map $I^3 \otimes A \to \mathsf{C}$ which can be uniquely extended to a map $\Delta^3 \otimes A \to \mathsf{C}$

since C is a $\mathcal{B}$-category. By construction, the restriction of this map along the inclusions $d^{\{0,2\}}\colon \Delta^1 \otimes A \to \Delta^3 \otimes A$ and $d^{\{1,3\}}\colon \Delta^1 \otimes A \to \Delta^3 \otimes A$ are equivalent to id_d and id_c, respectively. By definition, this means that $\Delta^3 \otimes A \to \mathsf{C}$ extends along the map $\Delta^3 \otimes A \to E^1 \otimes A$.

Conversely, if the map $\Delta^1 \otimes A \to \mathsf{C}$ that is determined by f factors through the map $\Delta^1 \otimes A \to E^1 \otimes A$, it in particular determines a map $\Delta^3 \otimes A \to \mathsf{C}$ whose restriction along $d^{\{0,1\}}\colon \Delta^1 \otimes A \to \Delta^3 \otimes A$ and $d^{\{2,3\}}\colon \Delta^1 \otimes A \to \Delta^3 \otimes A$ gives rise to two maps $h, g\colon c \rightrightarrows d$ in C. By construction of E^1 and the definition of composition, the composites fh and gf factor through $d\colon A \to \mathsf{C}$ and $c\colon A \to \mathsf{C}$, respectively, which means that these composites are equivalent to id_c and id_d. □

Corollary 1.2.7.4. *A map $f\colon A \to \mathsf{C}_1$ defines an equivalence in* C *if and only if it factors through the map $s_0\colon \mathsf{C}_0 \to \mathsf{C}_1$.*

Proof. Since C is a $\mathcal{B}$-category, any map $E^1 \otimes A \to \mathsf{C}$ extends uniquely along the projection $E^1 \otimes A \to A$, hence the result follows from Proposition 1.2.7.3. □

Remark 1.2.7.5. We will see in Section 1.3.1 below that the map $s_0\colon \mathsf{C}_0 \to \mathsf{C}_1$ is a monomorphism in $\mathcal{B}$. Therefore, a map $f\colon A \to \mathsf{C}_1$ being an equivalence is a property, and not extra structure.

As a consequence of Corollary 1.2.7.4, given two objects c, d in C in context $A \in \mathcal{B}$, we may define the $\mathcal{B}$-*groupoid of equivalences* $\mathrm{eq}_{\mathsf{C}}(c, d) \in \mathcal{B}_{/A}$ via the pullback square

$$\begin{tikzcd}
\mathrm{eq}_{\mathsf{C}}(c,d) \arrow[r] \arrow[d] & \mathsf{C}_0 \arrow[d, "(d_1,d_0)"] \\
A \arrow[r, "(c,d)"] & \mathsf{C}_0 \times \mathsf{C}_0.
\end{tikzcd}$$

By construction, sections $A \to \mathrm{eq}_{\mathsf{C}}(c, d)$ over A correspond to equivalences $f\colon c \to d$ in C in context A.

We will say that two objects $c, d\colon A \rightrightarrows \mathsf{C}$ are *equivalent* if there is an equivalence $c \simeq d$, i.e. a section $A \to \mathrm{eq}_{\mathsf{C}}(c, d)$ over A. This is equivalent to the condition that c and d are equivalent as objects in $\mathrm{map}_{\mathcal{B}}(A, \mathsf{C}_0)$. Furthermore, we will say that they are *locally* equivalent if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^*(c) \simeq s_i^*(d)$ for every i (where $s_i^*(c)$ is simply given by the composition cs_i, and likewise for $s_i^*(d)$).

Remark 1.2.7.6. Using the same arguments as in Remark 1.2.7.2, for every pair of objects $c, d\colon A \rightrightarrows \mathsf{C}$ in context $A \in \mathcal{B}$ we obtain a canonical equivalence $\mathrm{eq}_{\mathsf{C}}(c,d) \simeq \mathrm{eq}_{\pi_A^*\mathsf{C}}(\bar{c},\bar{d})$ which fits into a commutative square

$$\begin{tikzcd}
\mathrm{eq}_{\mathsf{C}}(c,d) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{eq}_{\pi_A^*\mathsf{C}}(\bar{c},\bar{d}) \arrow[d, hook] \\
\mathrm{map}_{\mathsf{C}}(c,d) \arrow[r, "\simeq"] & \mathrm{map}_{\pi_A^*\mathsf{C}}(\bar{c},\bar{d}).
\end{tikzcd}$$

in which the vertical maps are induced by $s_0\colon \mathsf{C}_0 \hookrightarrow \mathsf{C}_1$ and $s_0\colon \pi_A^*\mathsf{C}_0 \hookrightarrow \pi_A^*\mathsf{C}_1$, respectively.

Remark 1.2.7.7. As a $\mathcal{B}$-category C is determined by the associated sheaf of ∞-categories on $\mathcal{B}$ but not just by the underlying ∞-category $\Gamma(\mathsf{C})$ of global sections, it is crucial that we allow objects and morphisms in C to have arbitrary context $A \in \mathcal{B}$. In other words, we need to allow objects and morphisms to be only *locally* defined. Alternatively, this phenomenon can be viewed as a shadow of the notion of contexts in type theory (hence the name), where they are needed to keep track of the types of the variables that occur in a formula.

Remark 1.2.7.8. At first, the fact that objects and morphisms of a $\mathcal{B}$-category C might have non-global context A appears to complicate things, but in practice this is usually not the case: in fact, by Remark 1.2.7.1, the datum of an object $c\colon A \to \mathsf{C}$ precisely corresponds to that of an object $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$. In other words, upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and C with $\pi_A^*\mathsf{C}$, object in context A are turned into objects in global context, and the same is true for morphisms as well (see Remark 1.2.7.2). Very often, we will make use of this correspondence in order to be able to restrict our attention to objects and morphisms in global context. This will be possible as we make sure that virtually *every* construction that we make is preserved under étale base change, so that the $\mathcal{B}$-category $\pi_A^*\mathsf{C}$ plays the same role among $\mathcal{B}_{/A}$-categories as C plays among $\mathcal{B}$-categories. Furthermore, we make sure that *every* property of an object in a $\mathcal{B}$-category is invariant under the transposition from Remark 1.2.7.1, so that c satisfies a certain condition if and only if $\bar{c}$ does. For example, Remark 1.2.7.6 implies that the property of a map $f\colon c \to d$ in context A to be an equivalence follows this rule.

Remark 1.2.7.9. Observe that every $\mathcal{B}$-category C has a distinguished object $\tau\colon \mathsf{C}_0 \to \mathsf{C}$ that is determined by the counit of the adjunction

$$\iota \dashv (-)_0 \colon \mathcal{B} \rightleftarrows \mathrm{Cat}(\mathcal{B}).$$

We refer to τ as the *tautological* object of C. By definition, *every* object $c\colon A \to \mathsf{C}$ arises as a pullback of τ, in the sense that we have $c \simeq c^*\tau$ (where $c^*\tau$ is simply the composition τc). In that way, many questions about an arbitrary object in a $\mathcal{B}$-category can be reduced to questions about the tautological object.

Remark 1.2.7.10. Even if we are dealing with a *large* $\mathcal{B}$-category C, we will usually only need to consider *small* contexts $A \in \mathcal{B}$ when speaking about objects or maps in C. Essentially, this is possible since $\widehat{\mathcal{B}}$ is generated by $\mathcal{B}$ under large colimits, which implies that every *large* context $A \in \widehat{\mathcal{B}}$ admits a cover by small contexts. Another way of saying this is that C is uniquely specified by its value at small contexts, cf. Remark 1.1.4.1.

1.3. Functors of $\mathcal{B}$-categories

A *functor* between two $\mathcal{B}$-categories C and D is simply defined to be a map $f\colon \mathsf{C} \to \mathsf{D}$ in $\mathrm{Cat}(\mathcal{B})$, or equivalently an object $f\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ in global context $1 \in \mathcal{B}$. More generally, we may call an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ a functor between C and D in context $A \in \mathcal{B}$. By Proposition 1.2.5.4, this is precisely the datum of a map $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ of $\mathcal{B}_{/A}$-categories. A map $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})^{\Delta^1}$ is referred to as a *morphism of functors* or *natural transformation* in context $A \in \mathcal{B}$, By Proposition 1.2.5.4, the datum of such a map is tantamount to that of a map $\Delta^1 \otimes \pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ in $\mathrm{Cat}(\mathcal{B}_{/A})$.

In this section, our goal is to study several important classes of functors between $\mathcal{B}$-categories. We begin in Section 1.3.1 with the study of monomorphisms and strong epimorphisms, and in Section 1.3.2 we discuss the related notion of fully faithful and essentially surjective functors. Finally, we discuss conservative functors and localisations in Section 1.3.3.

1.3.1. Monomorphisms and strong epimorphisms

Recall that a *monomorphism* in $\mathrm{Cat}(\mathcal{B})$ (i.e. a (-1)-truncated map) is a functor that is internally left orthogonal to the map $\Delta^0 \sqcup \Delta^0 \to \Delta^0$, cf. Example 1.1.5.10. In other words, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is a monomorphism if and only if the square

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "(\mathrm{id}_{\mathsf{C}},\mathrm{id}_{\mathsf{C}})"] & \mathsf{D} \arrow[d, "(\mathrm{id}_{\mathsf{D}},\mathrm{id}_{\mathsf{D}})"] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "f\times f"] & \mathsf{D} \times \mathsf{D} \end{tikzcd}$$

is a pullback, or equivalently that the diagonal map $\mathsf{C} \to \mathsf{C} \times_{\mathsf{D}} \mathsf{C}$ is an equivalence. Dually, we say that a functor of $\mathcal{B}$-categories is a *strong epimorphism* if it is (internally) left orthogonal to the class of monomorphisms.

Remark 1.3.1.1. Since the notion of a monomorphism can be phrased entirely in terms of pullbacks, base change both geometric and algebraic morphisms of ∞-topoi preserves monomorphisms. Consequently, base change along algebraic morphisms also preserves strong epimorphisms.

Proposition 1.3.1.2. *A functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is a monomorphism if and only if both f_0 and f_1 are monomorphisms in $\mathcal{B}$. In particular, both the inclusion* $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ *as well as the core $\mathcal{B}$-groupoid functor* $(-)^{\simeq}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Grpd}(\mathcal{B})$ *preserve monomorphisms.*

Proof. Since limits in $\mathrm{Cat}(\mathcal{B})$ are computed level-wise, the map f is a monomorphism precisely if f_n is a monomorphism in $\mathcal{B}$ for all $n \geq 0$. Owing to the Segal conditions, this is automatically satisfied whenever only f_0 and f_1 are monomorphisms. □

Example 1.3.1.3. For any $\mathcal{B}$-category C, the canonical map $\mathsf{C}^{\simeq} \to \mathsf{C}$ is a monomorphism. In fact, using Proposition 1.3.1.2 this is equivalent to the map $s_0\colon \mathsf{C}_0 \to \mathsf{C}_1$ being a monomorphism in $\mathcal{B}$, which in turn follows from the observation that the square

$$\begin{tikzcd} \Delta^1 \arrow[r, "s^0"] \arrow[d, "s^0"] & \Delta^0 \arrow[d] \\ \Delta^0 \arrow[r] & \Delta^0 \end{tikzcd}$$

is a pushout in Cat_∞ (which can be shown by similar arguments as used in the proof of Lemma 1.3.2.8 below).

Proposition 1.3.1.4. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories. Then the following are equivalent:*

1. *f is a monomorphism;*
2. *$f^{\simeq}$ is a monomorphism in $\widehat{\mathcal{B}}$, and for any $A \in \mathcal{B}$ and any two objects $c_0, c_1 \colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$, the morphism*
$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$
that is induced by f is a monomorphism in $\widehat{\mathcal{B}}_{/A}$;
3. *for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ is a monomorphism of ∞-categories;*

Proof. As monomorphisms are defined by a limit condition and as the inclusion $\mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$ creates limits, one easily sees that conditions (1) and (3) are equivalent. Moreover, Proposition 1.3.1.2 implies that f is a monomorphism if and only if both f_0 and f_1 are monomorphisms in $\widehat{\mathcal{B}}$. It therefore suffices to show that f_1 is a monomorphism if and only if for every $A \in \mathcal{B}$ and any two objects $c_0, c_1 \colon A \rightrightarrows \mathsf{C}$ in context A, the morphism
$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$
that is induced by f is a monomorphism in $\widehat{\mathcal{B}}_{/A}$, provided that f_0 is a monomorphism. By definition, the map that f induces on mapping $\mathcal{B}$-groupoids fits into the commutative diagram
$$\begin{tikzcd}
& \mathrm{map}_{\mathsf{C}}(c_0, c_1) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1)) \arrow[dl] \arrow[dd] \\
\mathsf{C}_1 \arrow[rr, "f_1"] \arrow[dd] & & \mathsf{D}_1 \arrow[dd] & \\
& A \arrow[rr, "\mathrm{id}"] \arrow[dl] & & A \arrow[dl] \\
\mathsf{C}_0 \times \mathsf{C}_0 \arrow[rr, "f_0 \times f_0"] & & \mathsf{D}_0 \times \mathsf{D}_0. &
\end{tikzcd}$$
in which the two squares on the left and on the right are pullbacks. As f_0 is a monomorphism, the bottom square is a pullback, which implies that the top square

is a pullback as well. Hence if f_1 is a monomorphism, then the morphism on mapping $\mathcal{B}$-groupoids must be a monomorphism as well. Conversely, suppose that f induces a monomorphism on mapping $\mathcal{B}$-groupoids. Let $P \simeq (\mathsf{C}_0 \times \mathsf{C}_0) \times_{\mathsf{D}_0 \times \mathsf{D}_0} \mathsf{D}_1$ denote the pullback of the front square in the above diagram. Then f_1 factors as $\mathsf{C}_1 \to P \to \mathsf{D}_1$ in which the second arrow is a monomorphism. It therefore suffices to show that the map $\mathsf{C}_1 \to P$ is a monomorphism as well. Note that the map $\mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1)) \to \mathsf{D}_1$ factors through the inclusion $P \hookrightarrow \mathsf{D}_1$ such that the induced map $\mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1)) \to P$ arises as the pullback of the map $P \to \mathsf{C}_0 \times \mathsf{C}_0$ along (c_0, c_1). As the object $C_0 \times C_0$ is obtained as the colimit of the diagram

$$\mathcal{B}_{/C_0 \times C_0} \to \mathcal{B} \hookrightarrow \widehat{\mathcal{B}},$$

we obtain a cover $\bigsqcup_{A \to \mathsf{C}_0 \times \mathsf{C}_0} A \twoheadrightarrow C_0 \times C_0$ in $\widehat{\mathcal{B}}$ and therefore a cover

$$\bigsqcup_{(c_0, c_1)} \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1)) \twoheadrightarrow P.$$

We conclude the proof by observing that there is a pullback diagram

$$\begin{tikzcd}
\bigsqcup_{(c_0,c_1)} \mathrm{map}_{\mathsf{C}}(c_0, c_1) \arrow[r, two heads] \arrow[d, hook] & \mathsf{C}_1 \arrow[d] \\
\bigsqcup_{(c_0,c_1)} \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1)) \arrow[r, two heads] & P
\end{tikzcd}$$

in which the left vertical map is a monomorphism. Thus $\mathsf{C}_1 \to P$ is also a monomorphism by [49, Proposition 6.2.3.17]. $\square$

Remark 1.3.1.5. In light of Proposition 1.3.1.2 it might be tempting to expect that dually a map $f\colon \mathsf{C} \to \mathsf{D}$ in $\mathrm{Cat}(\mathcal{B})$ is a strong epimorphism if and only if f_0 and f_1 are covers in $\mathcal{B}$. In fact, this is a sufficient condition: the Segal conditions imply that f_0 and f_1 being a cover is equivalent to f being a cover in the ∞-topos $\mathcal{B}_\Delta$ (where covers are given by level-wise covers in $\mathcal{B}$). Therefore, this claim follow from the observation that the left adjoint of the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ carries covers to strong epimorphisms as dually this inclusion preserves monomorphisms. The condition that f_0 and f_1 are covers is however not necessary. For example, the functor $(d_2, d_0)\colon \Delta^1 \sqcup \Delta^1 \to \Delta^2$ in Cat_∞ is a strong epimorphism since every subcategory of Δ^2 that contains the image of this functor must necessarily be Δ^2, but this map is not surjective on the level of morphisms.

For any ∞-category $\mathcal{C}$ with finite limits and any object $c \in \mathcal{C}$, we write $\mathrm{Sub}_{\mathcal{C}}(c)$ for the poset of *subobjects* of c, i.e. the full subcategory of $\mathcal{C}_{/c}$ that is spanned by the (-1)-truncated objects. Since a functor $f\colon \mathsf{C} \to \mathsf{D}$ is a monomorphism in $\mathrm{Cat}(\mathcal{B})$ if and only if f is a (-1)-truncated object in $\mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$, it makes sense to define:

Definition 1.3.1.6. Let D be a $\mathcal{B}$-category. A *subcategory* of D is defined to be an object in $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{D})$.

Warning 1.3.1.7. If C is a $\mathcal{B}$-category, not every subobject of C in $\mathcal{B}_\Delta$ need to be a $\mathcal{B}$-category. Therefore, the two posets $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$ and $\mathrm{Sub}_{\mathcal{B}_\Delta}(\mathsf{C})$ are in general different.

Recall from the discussion in Section 1.2.7 that if C is a $\mathcal{B}$-category and A is an object in $\mathcal{B}$, the datum of a map $A \to \mathsf{C}_1$ is equivalent to that of a map $A \to \mathsf{C}^{\Delta^1}$, which is in turn equivalent to that of a map $\Delta^1 \otimes A \to \mathsf{C}$. Hence, the identity $\mathsf{C}_1 \to \mathsf{C}_1$ transposes to a functor $\Delta^1 \otimes \mathsf{C}_1 \to \mathsf{C}$.

Lemma 1.3.1.8. *For any $\mathcal{B}$-category C, the functor $\Delta^1 \otimes \mathsf{C}_1 \to \mathsf{C}$ is a strong epimorphism in $\mathrm{Cat}(\mathcal{B})$.*

Proof. In light of Remark 1.3.1.5, it suffices to show that the functor $\Delta^1 \otimes \mathsf{C}_1 \to \mathsf{C}$ induces a cover on level 0 and level 1. On level 0, the map is given by

$$(d_1, d_0)\colon \mathsf{C}_1 \sqcup \mathsf{C}_1 \to \mathsf{C}_0$$

which is clearly a cover since precomposition with $s_0 \sqcup s_0\colon \mathsf{C}_0 \sqcup \mathsf{C}_0 \to \mathsf{C}_1 \sqcup \mathsf{C}_1$ recovers the diagonal $\mathsf{C}_0 \sqcup \mathsf{C}_0 \to \mathsf{C}_0$ which is always a cover in $\mathcal{B}$. On level 1, one obtains the map

$$(s_0 d_1, \mathrm{id}, s_0 d_0)\colon \mathsf{C}_1 \sqcup \mathsf{C}_1 \sqcup \mathsf{C}_1 \to \mathsf{C}_1$$

which is similarly a cover in $\mathcal{B}$, as desired. □

Proposition 1.3.1.9. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories and let $\mathsf{E} \hookrightarrow \mathsf{D}$ be a subcategory. The following are equivalent:*

1. *f factors through the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$;*

2. *$f^{\simeq}$ factors through $\mathsf{E}^{\simeq} \hookrightarrow \mathsf{D}^{\simeq}$, and for each pair $(c_0, c_1)\colon A \to \mathsf{C}_0 \times \mathsf{C}_0$ of objects in context $A \in \mathcal{B}$, the map*

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

that is induced by f factors through the inclusion

$$\mathrm{map}_{\mathsf{E}}(f(c_0), f(c_1)) \hookrightarrow \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1));$$

3. *for each map $\Delta^1 \otimes A \to \mathsf{C}$ in context $A \in \mathcal{B}$ its image in D is contained in E.*

Proof. It is immediate that (1) implies (2) and that (2) implies (3). Suppose therefore that condition (3) holds. As in the proof of Proposition 1.3.1.4, the collection of all maps $A \to \mathsf{C}_1$ constitutes a cover

$$\bigsqcup_{A \to \mathsf{C}_1} A \twoheadrightarrow \mathsf{C}_1$$

in $\widehat{\mathcal{B}}$. By Remark 1.3.1.5, we may view this map as a strong epimorphism between large $\mathcal{B}$-groupoids. Since strong epimorphisms are *internally* left orthogonal to monomorphisms and therefore closed under products in $\mathrm{Cat}(\widehat{\mathcal{B}})$, we deduce that the induced map

$$\bigsqcup_{A \to \mathsf{C}_1} \Delta^1 \otimes A \to \Delta^1 \otimes \mathsf{C}_1$$

is a strong epimorphism. Together with Lemma 1.3.1.8, we therefore obtain a strong epimorphism $\bigsqcup_{A \twoheadrightarrow \mathsf{C}_1} \Delta^1 \otimes A \to \mathsf{C}$. Using the assumptions, we may now construct a lifting problem

$$\begin{tikzcd}
\bigsqcup_{A \to \mathsf{C}_1} \Delta^1 \otimes A \arrow[r] \arrow[d, twoheadrightarrow] & \mathsf{E} \arrow[d, hookrightarrow] \\
\mathsf{C} \arrow[r, "f"] \arrow[ur, dashed] & \mathsf{D}
\end{tikzcd}$$

which admits a unique solution, hence condition (1) follows. □

Corollary 1.3.1.10. *A functor $f\colon \mathsf{C} \to \mathsf{D}$ of $\mathcal{B}$-categories factors through the inclusion $\mathsf{D}^{\simeq} \hookrightarrow \mathsf{D}$ if and only if f sends all morphisms in C to equivalences in D.* □

Definition 1.3.1.11. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a map in $\mathrm{Cat}(\mathcal{B})$ and let $\mathsf{C} \twoheadrightarrow \mathsf{E} \hookrightarrow \mathsf{D}$ be the factorisation of f into a strong epimorphism and a monomorphism. Then the subcategory $\mathsf{E} \hookrightarrow \mathsf{D}$ is referred to as the 1-*image* of f.

In higher category theory, one can define a subcategory of an ∞-category by rather low-dimensional data, namely by specifying a collection of objects and a collection of morphisms between these objects that are closed under composition and contain all equivalences. Hereafter, our goal is to obtain a similar result for subcategories of $\mathcal{B}$-categories. To that end, note that the functor

$$(-)^{\Delta^1}\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}} \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}}$$

admits a left adjoint that is given by the composition

$$\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}} \xrightarrow{\Delta^1\otimes -} \mathrm{Cat}(\mathcal{B})_{/\Delta^1\otimes\mathsf{C}^{\Delta^1}} \xrightarrow{\mathrm{ev}_!} \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$$

in which ev denotes the evaluation map. Similarly, the functor

$$(-)^{\simeq}\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}} \to \mathcal{B}_{/\mathsf{C}_1}$$

has a left adjoint that is given by the composition

$$\mathcal{B}_{/\mathsf{C}_1} \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}_1} \xrightarrow{i_!} \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}}$$

where $i\colon \mathsf{C}_1 \simeq (\mathsf{C}^{\Delta^1})^{\simeq} \hookrightarrow \mathsf{C}^{\Delta^1}$ denotes the canonical inclusion. By Proposition 1.3.1.2, the functor $(-)_1 = (-)^{\simeq}\circ(-)^{\Delta^1}$ sends a monomorphism $\mathsf{D} \hookrightarrow \mathsf{C}$ to the inclusion $\mathsf{D}_1 \hookrightarrow \mathsf{C}_1$ and therefore restricts to a functor $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \to \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$. As $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ admits a left adjoint that sends a functor $f\colon \mathsf{D} \to \mathsf{C}$ to its 1-image in C, we thus obtain an adjunction

$$(\langle - \rangle \dashv (-)_1)\colon \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1) \rightleftarrows \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$$

in which the left adjoint $\langle - \rangle$ sends a monomorphism $S \hookrightarrow \mathsf{C}_1$ to the 1-image $\langle S \rangle$ of the associated map $\Delta^1 \otimes S \to \mathsf{C}$. Note that for any subcategory $\mathsf{D} \hookrightarrow \mathsf{C}$, the counit $\langle \mathsf{D}_1 \rangle \to \mathsf{D}$ is given by the unique solution to the lifting problem

$$\begin{tikzcd}
\Delta^1 \otimes \mathsf{D}_1 \arrow[r] \arrow[d, two heads] & \mathsf{D} \arrow[d, hook] \\
\langle \mathsf{D}_1 \rangle \arrow[r, hook] \arrow[ur, dashed] & \mathsf{C}
\end{tikzcd}$$

in which the upper horizontal map is the transpose of the identity $\mathsf{D}_1 \to \mathsf{D}_1$. By Lemma 1.3.1.8, this is a strong epimorphism, hence we conclude that the map $\langle \mathsf{D}_1 \rangle \to \mathsf{D}$ must be an equivalence. We have thus shown:

Proposition 1.3.1.12. *For any $\mathcal{B}$-category C, the functor*

$$(-)_1 \colon \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \to \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$$

exhibits the poset $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$ as a reflective subposet of $\mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$. ☐

Remark 1.3.1.13. The inclusion $(-)_1 \colon \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$ is in general not an equivalence. For example, consider $\mathcal{B} = \mathrm{Ani}$ and $\mathsf{C} = \Delta^2$: here the two maps $d^{\{0,1\}} \colon \Delta^1 \to \Delta^2$ and $d^{\{1,2\}} \colon \Delta^1 \to \Delta^2$ determine a proper subobject of Δ^2_1, but the associated subcategory of Δ^2 is nevertheless Δ^2 itself.

As a consequence of Proposition 1.3.1.12, it is now possible to define the subcategory of a $\mathcal{B}$-category that is *generated* by a collection of morphisms:

Definition 1.3.1.14. Let C be a $\mathcal{B}$-category and let $(f_i \colon A_i \to \mathsf{C}_1)_{i\in I}$ be a small family of morphisms in C. Let $E \hookrightarrow \mathsf{C}_1$ be the subobject that is obtained by taking the image of the induced map $\bigsqcup_i A_i \to \mathsf{C}_1$. Then we refer to the induced subcategory $\langle E \rangle \hookrightarrow \mathsf{C}$ as the subcategory of C that is *generated* by the family $(f_i)_{i\in I}$.

Remark 1.3.1.15. In the situation of Definition 1.3.1.14, the condition that $(f_i)_{i\in I}$ is a *small* family is superfluous. In fact, if the family is large, one can simply pass to the universe enlargement $\widehat{\mathcal{B}}$ to make sense of the image $E \hookrightarrow \mathsf{C}_1$ in this case. Since C_1 is contained in $\mathcal{B}$, the same must be true for E, hence the subcategory generated by the family $(f_i)_{i\in I}$ is still well-defined.

Remark 1.3.1.16. Unwinding the definitions, if $(f_i \colon A_i \to \mathsf{C}_1)$ is a family of maps in a $\mathcal{B}$-category C, then the subcategory of C that is generated by this family is given by the 1-image of the associated functor $\bigsqcup_i \Delta^1 \otimes A_i \to \mathsf{C}$.

As Remark 1.3.1.13 exemplifies, one obstruction to

$$(-)_1 \colon \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$$

being an equivalence is that the collection of maps that determine a subobject $S \hookrightarrow \mathsf{C}_1$ need not be stable under composition. In other words, to make sure that a subobject of C_1 arises as the object of morphisms of a subcategory of C, we need to impose a composability condition on this subobject. Altogether, we obtain the following characterisation of the essential image of $(-)_1$:

Proposition 1.3.1.17. *For any $\mathcal{B}$-category C, a subobject $S \hookrightarrow \mathsf{C}_1$ lies in the essential image of the inclusion $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$ if and only if*

1. *it is closed under equivalences, i.e. the map $(s_0 d_1, s_0 d_0)\colon S \sqcup S \to \mathsf{C}_1$ factors through $S \hookrightarrow \mathsf{C}_1$;*
2. *it is closed under composition, i.e. the restriction of $d_1\colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \to \mathsf{C}_1$ along the inclusion $S \times_{\mathsf{C}_0} S \hookrightarrow \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ factors through $S \hookrightarrow \mathsf{C}_1$.*

The remainder of this section is devoted to the proof of Proposition 1.3.1.17. Our strategy is to make use of the intuition that the datum of a subcategory of C should be equivalent to the datum of a collection of objects in C, together with a composable collection of maps between these objects. Our goal hereafter is turn this surmise into a formal statement.

For any integer $k \geq 0$, let $i_k\colon \Delta^{\leq k} \hookrightarrow \Delta$ denote the full subcategory spanned by $\langle n \rangle$ for $n \leq k$, and let $\mathcal{B}_{\Delta}^{\leq k}$ denote the ∞-category of $\mathcal{B}$-valued presheaves on $\Delta^{\leq k}$. The truncation functor $i_k^*\colon \mathcal{B}_{\Delta} \to \mathcal{B}_{\Delta}^{\leq k}$ admits both a left adjoint $(i_k)_!$ and a right adjoint $(i_k)_*$ given by left and right Kan extension. Note that both $(i_k)_!$ and $(i_k)_*$ are fully faithful. We will generally identify $\mathcal{B}_{\Delta}^{\leq k}$ with its essential image in $\mathcal{B}_{\Delta}$ along the *right* Kan extension $(i_k)_*$. We define the associated *coskeleton* functor as $\mathrm{cosk}_k = (i_k)_* i_k^*$ and the *skeleton* functor as $\mathrm{sk}_k = (i_k)_! i_k^*$. The unit of the adjunction $i_k^* \dashv (i_k)_*$ provides a map $\mathrm{id}_{\mathcal{B}_{\Delta}} \to \mathrm{cosk}_k$, and the counit of the adjunction $(i_k)_! \dashv i_k^*$ provides a map $\mathrm{sk}_k \to \mathrm{id}_{\mathcal{B}_{\Delta}}$. We say that $C \in \mathcal{B}_{\Delta}$ is *k-coskeletal* if the map $C \to \mathrm{cosk}_k(C)$ is an equivalence, i.e. if C is contained in $\mathcal{B}_{\Delta}^{\leq k} \subset \mathcal{B}_{\Delta}$. Dually, C is *k-skeletal* if the map $\mathrm{sk}_k(C) \to C$ is an equivalence. Note that the adjunction $\mathrm{sk}_k \dashv \mathrm{cosk}_k$ implies that a simplicial object is k-coskeletal if and only if it is local with respect to the maps $\mathrm{sk}_k(D) \to D$ for every $D \in \mathcal{B}_{\Delta}$.

Definition 1.3.1.18. For any integer $k \geq 0$, a map $f\colon C \to D$ in $\mathcal{B}_{\Delta}$ is said to be *k-coskeletal* if it is right orthogonal to $\mathrm{sk}_k(K) \to K$ for every $K \in \mathcal{B}_{\Delta}$.

Note that by using the adjunction $\mathrm{sk}_k \dashv \mathrm{cosk}_k$ and Yoneda's lemma, one has the following criterion for a map between simplicial objects in $\mathcal{B}$ to be k-coskeletal:

Proposition 1.3.1.19. *For any integer $k \geq 0$, a map $f\colon C \to D$ in $\mathcal{B}_\Delta$ is k-coskeletal precisely if the canonical map $C \to D \times_{\mathrm{cosk}_k(D)} \mathrm{cosk}_k(C)$ is an equivalence.* □

For any $n \geq 1$, denote by $\partial\Delta^n$ the simplicial ∞-groupoid $\mathrm{sk}_{n-1}\Delta^n$ and by $\partial\Delta^n \hookrightarrow \Delta^n$ the natural map induced by the adjunction counit.

For later use, we record the following obvious consequence of the skeletal filtration on simplicial sets:

Lemma 1.3.1.20. *Let $j\colon K \hookrightarrow L$ be a monomorphism of finite simplicial sets and assume that $\mathrm{sk}_k K = \mathrm{sk}_k L$ for some $k \in \mathbb{N}$. Then j is contained in the smallest saturated class containing the maps $\partial\Delta^l \to \Delta^l$ for $k < l < \dim L$.* □

Lemma 1.3.1.21. *Let $k \geq 0$ be an integer. Then the following sets generate the same saturated class of morphisms in $\mathcal{B}_\Delta$:*

1. $\{\mathrm{sk}_k D \to D \mid D \in \mathcal{B}_\Delta\}$;
2. $\{\partial\Delta^n \otimes A \hookrightarrow \Delta^n \otimes A \mid n > k,\ A \in \mathcal{B}\}$.
3. $\{\partial\Delta^{k+1} \otimes D \hookrightarrow \Delta^{k+1} \otimes D \mid D \in \mathcal{B}_\Delta\}$.

Proof. We start by showing that the saturations of (1) and (2) agree. Given $A \in \mathcal{B}$, note that since the truncation functor i_k^* commutes with postcomposition by both the pullback functor $\pi_A^*\colon \mathcal{B} \to \mathcal{B}_{/A}$ and its right adjoint $(\pi_A)_*$, the uniqueness of adjoints implies that the functor sk_k commutes with $- \times A\colon \mathcal{B}_\Delta \to \mathcal{B}_\Delta$. By a similar argument, the functor sk_k commutes with $\mathrm{const}\colon \mathrm{Ani}_\Delta \to \mathcal{B}_\Delta$. We therefore obtain an equivalence $\mathrm{sk}_k(\Delta^m \otimes A) \simeq \mathrm{sk}_k(\Delta^m) \otimes A$ with respect to which the canonical map $\mathrm{sk}_k(\Delta^m \otimes A) \to \Delta^m \otimes A$ corresponds to the map obtained by applying the functor $- \otimes A$ to the map $\mathrm{sk}_k(\Delta^m) \to \Delta^m$. This already implies that the set in (2) is contained in the set in (1), so that the saturation of (2) is contained in the saturation of (1). Conversely, as any $D \in \mathcal{B}_\Delta$ can be written as a colimit of objects of the form $\Delta^n \otimes A$ (see Remark 1.2.1.3), the above argument also shows that every map in (1) is a colimit of maps of the form $\mathrm{sk}_k(\Delta^n) \otimes A \to \Delta^n \otimes A$. Since moreover $\mathrm{const}_\mathcal{B}$ and $- \otimes A$ are colimit-preserving functors, one finds that (1) is contained in the saturation of (2) as soon as we can show that any saturated class S of maps in Ani_Δ which contains $\partial\Delta^n \to \Delta^n$ for all $n > k$ must also contain the maps $\mathrm{sk}_k \Delta^m \to \Delta^m$ for all m. To prove this latter claim, we argue by induction over $n > k$. If $n = k+1$ this is clear by definition. For $n > k+1$ we consider the composite $\mathrm{sk}_k \Delta^n \to \partial\Delta^n \to \Delta^n$. By our induction hypothesis and Lemma 1.3.1.20, the first map is in S and the composite is so by assumption. Using part (2) of Proposition 1.1.5.2, the claim now follows.

Next, to show that the saturation of (2) contains (3), we may again assume $D \simeq \Delta^m \otimes A$. In this case, the inclusion $\partial\Delta^{k+1} \times \Delta^m \hookrightarrow \Delta^{k+1} \times \Delta^m$ can be obtained as an iterated pushout of maps of the form $\partial\Delta^n \hookrightarrow \Delta^n$ for $n > k$ (by Lemma 1.3.1.20), hence the claim follows. For the converse inclusion, we will use induction on n, the case $n = k+1$ being satisfied by definition. Given that for a fixed $n > k$ the inclusion $\partial\Delta^n \otimes A \hookrightarrow \Delta^n \otimes A$ is contained in the saturation of (3), Lemma 1.3.1.20 allows us to build the inclusion $\partial\Delta^n \times \Delta^1 \hookrightarrow \mathrm{sk}_n(\Delta^n \times \Delta^1)$ as an iterated pushout along $\partial\Delta^n \hookrightarrow \Delta^n$. Therefore, the map $\mathrm{sk}_n(\Delta^n \times \Delta^1) \otimes A \hookrightarrow (\Delta^n \times \Delta^1) \otimes A$ is contained in the saturation of (3) (again by part (2) of Proposition 1.1.5.2). Let $\alpha\colon \Delta^{n+1} \to \Delta^n \times \Delta^1$ be defined by $\alpha(i) = (i, 0)$ for $i = 0, \dots, n$ and $\alpha(n+1) = (n+1, 1)$, and let $\beta\colon \Delta^n \times \Delta^1 \to \Delta^{n+1}$ be given by $\beta(i, 0) = i$ and $\beta(i, 1) = n+1$. We then obtain a retract diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "\alpha'"] \arrow[d, hook] & \mathrm{sk}_n(\Delta^n \times \Delta^1) \arrow[r, "\beta'"] \arrow[d, hook] & \partial\Delta^{n+1} \arrow[d, hook] \\
\Delta^{n+1} \arrow[r, "\alpha"] & \Delta^n \times \Delta^1 \arrow[r, "\beta"] & \Delta^{n+1}
\end{tikzcd}$$

in which α' and β' are given by the restriction of α and β, respectively. We therefore conclude that the map $\partial\Delta^{n+1} \otimes A \hookrightarrow \Delta^{n+1} \otimes A$ is in the saturation of (3), as desired. □

As a consequence of Lemma 1.3.1.21, one finds:

Proposition 1.3.1.22. *For any integer $k \geq 0$, a map $f\colon C \to D$ in $\mathcal{B}_\Delta$ is k-coskeletal if and only if it is internally right orthogonal to the map $\partial\Delta^{k+1} \hookrightarrow \Delta^{k+1}$.* □

We can use Proposition 1.3.1.22 to show that every monomorphism between $\mathcal{B}$-categories is 1-coskeletal:

Lemma 1.3.1.23. *Let S be the internal saturation of $\Delta^0 \sqcup \Delta^0 \to \Delta^0$ and $I^2 \hookrightarrow \Delta^2$ in $\mathcal{B}_\Delta$. Then S contains the map $\partial\Delta^2 \hookrightarrow \Delta^2$.*

Proof. Let $f\colon K \to L$ be a map in $\mathcal{B}_\Delta$ that is internally right orthogonal to the maps $\Delta^0 \sqcup \Delta^0 \to \Delta^0$ and the inclusion of the 2-spine $I^2 \hookrightarrow \Delta^2$. Then f is a monomorphism. Now consider the commutative diagram

$$\begin{tikzcd}
K^{\Delta^2} \arrow[dr] \arrow[drrr, bend left] \arrow[dd, bend right] & & & & & \\
& P \arrow[rr] \arrow[dl] \arrow[dd, hook] \arrow[drr, phantom, "\ulcorner", very near start] & & K^{\partial\Delta^2} \arrow[rr] \arrow[dl] \arrow[dd, hook] & & K^{I^2} \arrow[dl, "\mathrm{id}"] \arrow[dd, hook] \\
Q \arrow[rr] \arrow[dr] \arrow[dd, hook] \arrow[dr, phantom, "\ulcorner", very near start] & & R \arrow[rr] \arrow[dd, hook] \arrow[drr, phantom, "\ulcorner", very near start] & & K^{I^2} \arrow[dd, hook] & \\
& L^{\Delta^2} \arrow[rr] \arrow[dl, "\mathrm{id}"] & & L^{\partial\Delta^2} \arrow[rr] \arrow[dl, "\mathrm{id}"] & & L^{I^2} \arrow[dl, "\mathrm{id}"] \\
L^{\Delta^2} \arrow[rr] & & L^{\partial\Delta^2} \arrow[rr] & & L^{I^2} &
\end{tikzcd}$$

in which P, Q and R are defined by the condition that the respective square is a pullback diagram. We need to show that the map $K^{\Delta^2} \to P$ is an equivalence. As by assumption on f the map $K^{\Delta^2} \to Q$ is an equivalence, it suffices to show that $P \to Q$ is an equivalence as well. But this map is already a monomorphism, hence the claim follows from the observation that $P \to Q$ must be a cover as the map $K^{\Delta^2} \to Q$ is one. □

Proposition 1.3.1.24. *Every monomorphism between $\mathcal{B}$-categories is 1-coskeletal.*

Proof. Lemma 1.3.1.23 implies that every monomorphism between $\mathcal{B}$-categories is internally right orthogonal to $\partial\Delta^2 \hookrightarrow \Delta^2$ and therefore 1-coskeletal. □

Let C be a $\mathcal{B}$-category and let $\mathrm{Cat}(\mathcal{B})^{\leq 1}_{/\mathsf{C}}$ be the full subcategory of $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ that is spanned by the 1-coskeletal maps into C. By restricting the inclusion $\mathrm{Cat}(\mathcal{B})^{\leq 1}_{/\mathsf{C}} \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ to (-1)-truncated objects (i.e. to monomorphisms into D), one obtains a full embedding

$$\mathrm{Sub}^{\leq 1}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$$

of partially ordered sets. Proposition 1.3.1.24 now implies:

Corollary 1.3.1.25. *For any $\mathcal{B}$-category C, the inclusion*

$$\mathrm{Sub}^{\leq 1}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$$

is an equivalence. □

For any $\mathcal{B}$-category C, the functor $(\mathrm{cosk}_1)_{/\mathsf{C}} \colon (\mathcal{B}_\Delta)_{/\mathsf{C}} \to (\mathcal{B}^{\leq 1}_\Delta)_{/\mathrm{cosk}_1 \mathsf{C}}$ that is induced by the coskeleton functor on the slice ∞-categories admits a fully faithful right adjoint η^* that is given by base change along the adjunction unit $\eta \colon \mathsf{C} \to \mathrm{cosk}_1 \mathsf{C}$. Upon restricting to subobjects, we therefore obtain an adjunction

$$\mathrm{Sub}_{\mathcal{B}_\Delta}(\mathsf{C}) \underset{\eta^*}{\overset{(\mathrm{cosk}_1)_{/\mathsf{C}}}{\rightleftarrows}} \mathrm{Sub}_{\mathcal{B}^{\leq 1}_\Delta}(\mathrm{cosk}_1 \mathsf{C}).$$

In general, the functor η^* does not take values in $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$, but we may explicitly characterise those subobjects of $\mathrm{cosk}_1 \mathsf{C}$ that do give rise to a $\mathcal{B}$-category. To that end, note that given a subobject $D \hookrightarrow \mathrm{cosk}_1 \mathsf{C}$ in $\mathcal{B}^{\leq 1}_\Delta$, the restriction of $d_1 \colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \to \mathsf{C}_1$ along the inclusion $D_1 \times_{D_0} D_1 \hookrightarrow \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ determines a map $d_1 \colon D_1 \times_{D_0} D_1 \to \mathsf{C}_1$.

Definition 1.3.1.26. Let C be a $\mathcal{B}$-category. A subobject $D \hookrightarrow \mathrm{cosk}_1 \mathsf{C}$ in $\mathcal{B}^{\leq 1}_\Delta$ is said to be *closed under composition* if the map $d_1 \colon D_1 \times_{D_0} D_1 \to \mathsf{C}_1$ factors through $D_1 \hookrightarrow \mathsf{C}_1$. We denote by $\mathrm{Sub}^{\mathrm{comp}}_{\mathcal{B}^{\leq 1}_\Delta}(\mathrm{cosk}_1 \mathsf{C})$ the full subcategory of $\mathrm{Sub}_{\mathcal{B}^{\leq 1}_\Delta}(\mathrm{cosk}_1 \mathsf{C})$ that is spanned by these subobjects.

Lemma 1.3.1.27. *Let $A \in \mathcal{B}$ be an arbitrary object and let S be a saturated set of maps in $\mathcal{B}_\Delta$ that contains the internal saturation of $\partial\Delta^2 \hookrightarrow \Delta^2$ as well as the map $I^2 \otimes A \hookrightarrow \Delta^2 \otimes A$. Then S contains $I^n \otimes A \hookrightarrow \Delta^n \otimes A$ for all $n \geq 2$.*

Proof. We may assume $n > 2$. By [43, Proposition 2.13], it suffices to show that for all $0 < i < n$ the inclusion $\Lambda^n_i \otimes A \hookrightarrow \Delta^n \otimes A$ is contained in S. On account of the factorisation $\Lambda^n_i \hookrightarrow \partial\Delta^n \hookrightarrow \Delta^n$ in which the first map is obtained as a pushout along $\partial\Delta^{n-1} \hookrightarrow \Delta^n$, this is immediate. □

Proposition 1.3.1.28. *Let $D \hookrightarrow \mathrm{cosk}_1 \mathsf{C}$ be a subobject in $\mathcal{B}^{\leq 1}_\Delta$. Then $\eta^* D$ is a $\mathcal{B}$-category if and only if D is closed under composition. In particular, η^* defines an equivalence*

$$\mathrm{Sub}^{\mathrm{comp}}_{\mathcal{B}^{\leq 1}_\Delta}(\mathrm{cosk}_1 \mathsf{C}) \simeq \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}).$$

Proof. If $\eta^* D$ is a $\mathcal{B}$-category, the fact that applying cosk_1 to the inclusion $\eta^* D \hookrightarrow \mathsf{C}$ recovers the subobject $D \hookrightarrow \operatorname{cosk}_1 \mathsf{C}$ implies that D is closed under composition. Conversely, suppose that D is closed under composition. Since $E^1 \to 1$ is a cover in $\mathcal{B}_\Delta$ (where E^1 is the walking equivalence, see Section 1.2.3), every monomorphism of simplicial objects in $\mathcal{B}$ is internally right orthogonal to $E^1 \to 1$. Therefore $\eta^* D$ is univalent. We still need to show that $\eta^* D$ satisfies the Segal conditions. Since $\eta^* D \hookrightarrow \mathsf{C}$ is 1-coskeletal, Lemma 1.3.1.27 implies that we only need to show that $(\eta^* D)_2 \to C_1 \times_{D_0} D_1$ is an equivalence. As this map is a monomorphism, it furthermore suffices to show that it is a cover in $\mathcal{B}$. Note that since the natural map $(-)^{\Delta^2} \to (-)^{\partial\Delta^2}$ induces an equivalence on 1-coskeletal objects, the identification $\partial\Delta^2 \simeq I^2 \sqcup_{\Delta^0 \sqcup \Delta^0} \Delta^1$ gives rise to a commutative square

$$\begin{tikzcd}
& \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[rr] \arrow[ddrr, "d_1"'] & & (\operatorname{cosk}_1 \mathsf{C})_2 \arrow[rr] \arrow[dd] & & \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[dd] \\
D_1 \times_{D_0} D_1 \arrow[ur, hook] \arrow[rr, dashed] \arrow[ddrr, dashed, bend right, "d_1"'] & & D_2 \arrow[ur, hook] \arrow[rr] \arrow[dd, "d_1"'] & & D_1 \times_{D_0} D_1 \arrow[ur, hook] \arrow[dd] & \\
& & & \mathsf{C}_1 \arrow[rr] & & \mathsf{C}_0 \times \mathsf{C}_0 \\
& & D_1 \arrow[ur, hook] \arrow[rr] & & D_0 \times D_0 \arrow[ur, hook] &
\end{tikzcd}$$

in which the two squares in the front and in the back of the cube are pullbacks and where the dashed arrows exist as D is closed under composition. By combining this diagram with the pullback square

$$\begin{tikzcd}
(\eta^* D)_2 \arrow[r] \arrow[d, hook] & D_2 \arrow[d, hook] \\
\mathsf{C}_2 \arrow[r] & (\operatorname{cosk}_1 \mathsf{C})_2,
\end{tikzcd}$$

one concludes that the map $(\eta^* D)_2 \to D_1 \times_{D_0} D_1$ admits a section and is therefore a cover, as desired. Lastly, the claim that that η^* induces an equivalence $\operatorname{Sub}^{\mathrm{comp}}_{\mathcal{B}^{\leq 1}_{\Delta}}(\operatorname{cosk}_1 \mathsf{C}) \simeq \operatorname{Sub}_{\mathsf{Cat}(\mathcal{B})}(\mathsf{C})$ now follows easily with Corollary 1.3.1.25. $\square$

Proof of Proposition 1.3.1.17. It is clear that any subobject $S \hookrightarrow \mathsf{C}_1$ that arises as the object of morphisms of a subcategory of C must necessarily satisfy the two conditions, so it suffices to prove the converse. Let $D_0 \hookrightarrow \mathsf{C}$ be the image of $(d_1, d_0) \colon S \sqcup S \to \mathsf{C}_0$. As S is closed under equivalences in C, the restriction of $s_0 \colon \mathsf{C}_0 \to \mathsf{C}_1$ to D_0 factors through $S \hookrightarrow \mathsf{C}_1$. By setting $D_1 = S$, we thus obtain a subobject $D \hookrightarrow \operatorname{cosk}_1 \mathsf{C}$ in $\mathcal{B}_{\Delta}^{\leq 1}$. By assumption, this subobject is closed under composition in the sense of Definition 1.3.1.26, hence Proposition 1.3.1.28 implies that $\eta^* D$ is a subcategory of C. Hence $S = D_1$ arises as the object of morphisms of $\eta^* D$ and is therefore contained in the essential image of $(-)_1 \colon \operatorname{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \operatorname{Sub}_{\mathcal{B}}(\mathsf{C}_1)$. □

1.3.2. Fully faithful and essentially surjective functors

Definition 1.3.2.1. A functor $\mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is said to be *fully faithful* if it is internally right orthogonal to the map $(d^1, d^0) \colon \Delta^0 \sqcup \Delta^0 \to \Delta^1$. Dually, a functor is *essentially surjective* if is (internally) left orthogonal to the class of fully faithful functors.

Remark 1.3.2.2. A functor $f \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is essentially surjective in the sense of Definition 1.3.2.1 if and only if every object in $\mathcal{D}$ is equivalent to an object in the image of f, i.e. if and only if it is essentially surjective in the usual sense of the term. We will show this in Corollary 1.3.2.15 below.

By Proposition 1.1.5.7, the two classes of essentially surjective and fully faithful functors form an orthogonal factorisation system in $\mathrm{Cat}(\mathcal{B})$. In particular, one obtains:

Proposition 1.3.2.3. *Let $f \colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories. Then f is an equivalence if and only if f is fully faithful and essentially surjective.* □

Let $f \colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories. By definition, f being fully faithful precisely means that the square

$$\begin{tikzcd} \mathsf{C}^{\Delta^1} \arrow[r, "f^{\Delta^1}"] \arrow[d] & \mathsf{D}^{\Delta^1} \arrow[d] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "f \times f"] & \mathsf{D} \times \mathsf{D} \end{tikzcd}$$

is cartesian. Applying the core functor, this implies that the square

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "f_1"] \arrow[d] & \mathsf{D}_1 \arrow[d] \\ \mathsf{C}_0 \times \mathsf{C}_0 \arrow[r, "f_0 \times f_0"] & \mathsf{D}_0 \times \mathsf{D}_0 \end{tikzcd}$$

is cartesian as well. In fact, the latter square being cartesian is even a sufficient criterion for f to be fully faithful. The proof of this statement requires the following combinatorial lemma:

Lemma 1.3.2.4. *Let $A \in \mathcal{B}$ be an arbitrary object and let S be a saturated class of morphisms in $\mathcal{B}_\Delta$ that contains the maps $I^n \otimes A \hookrightarrow \Delta^n \otimes A$ for all $n \geq 0$. If S contains the map $(d^1, d^0) \colon A \sqcup A \to \Delta^1 \otimes A$, then it also contains the map*

$$(d^1, d^0) \colon (\Delta^n \otimes A) \sqcup (\Delta^n \otimes A) \to (\Delta^1 \times \Delta^n) \otimes A$$

for any integer $n \geq 0$.

Proof. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume without loss of generality that $A \simeq 1$. As all maps in the statement of the lemma are contained in the essential image of $\mathrm{const} \colon \mathrm{Ani} \to \mathcal{B}$, we may further assume $\mathcal{B} \simeq \mathrm{Ani}$. Furthermore, as the inclusion $I^n \sqcup I^n \hookrightarrow \Delta^n \sqcup \Delta^n$ is contained in S, it suffices to show that the map $I^n \sqcup I^n \hookrightarrow \Delta^1 \times \Delta^n$ is contained in S as well. By Lemma 1.2.3.5 the map $\Delta^1 \times I^n \hookrightarrow \Delta^1 \times \Delta^n$ is contained in S, hence we need only show that also the map $I^n \sqcup I^n \hookrightarrow \Delta^1 \times I^n$ is an element of S. Now I^n being defined as the colimit $\Delta^1 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^0} \Delta^1$, we can assume without loss of generality $n = 1$. Using the decomposition $\Delta^1 \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$, one easily sees that this map is the pushout in $\mathrm{Fun}(\Delta^1, \mathrm{Ani}_\Delta)$ that is obtained by glueing the two maps

$$(d^{\{0,1\}}, d^{\{2\}}) \colon \Delta^1 \sqcup \Delta^0 \hookrightarrow \Delta^2$$

and

$$(d^{\{0\}}, d^{\{1,2\}}) \colon \Delta^0 \sqcup \Delta^1 \hookrightarrow \Delta^2$$

along the morphism $(d^1, d^0) \colon \Delta^0 \sqcup \Delta^0 \hookrightarrow \Delta^1$. The proof is therefore finished once we show that the two maps above are contained in S. We show this for the first one, the case of the second one is completely analogous. Making use once more of the assumption that the spine inclusion $I^2 \hookrightarrow \Delta^2$ is contained in S, it suffices to show that the map

$$(d^{\{0,1\}}, d^{\{2\}}) \colon \Delta^1 \sqcup \Delta^0 \hookrightarrow I^2$$

is contained in S. This follows from the observation that this map is obtained by glueing the two maps $\Delta^0 \sqcup \Delta^0 \hookrightarrow \Delta^1$ and id_{Δ^1} along id_{Δ^0} in $\mathrm{Fun}(\Delta^1, \mathrm{Ani}_\Delta)$. □

Proposition 1.3.2.5. *A functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is fully faithful if and only if the square*

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "f_1"] \arrow[d] & \mathsf{D}_1 \arrow[d] \\ \mathsf{C}_0 \times \mathsf{C}_0 \arrow[r, "f_0 \times f_0"] & \mathsf{D}_0 \times \mathsf{D}_0 \end{tikzcd}$$

is cartesian.

Proof. We already observed above that f being fully faithful implies that the square is cartesian. Conversely, the square being cartesian is equivalent to f being right orthogonal to the set of maps

$$S = \{(\Delta^0 \sqcup \Delta^0) \otimes A \to \Delta^1 \otimes A \mid A \in \mathcal{B}\}$$

in $\mathrm{Cat}(\mathcal{B})$. By Lemma 1.3.2.4, the saturation of S in $\mathrm{Cat}(\mathcal{B})$ contains the maps

$$(\Delta^n \sqcup \Delta^n) \otimes A \to (\Delta^1 \times \Delta^n) \otimes A$$

for $A \in \mathcal{B}$ and $n \geq 0$, which translates into the statement that the induced square

$$\begin{tikzcd} (\mathsf{C}^{\Delta^1})_n \arrow[r, "f_n^{\Delta^1}"] \arrow[d] & (\mathsf{D}^{\Delta^1})_n \arrow[d] \\ \mathsf{C}_n \times \mathsf{C}_n \arrow[r, "f_n \times f_n"] & \mathsf{D}_n \times \mathsf{D}_n \end{tikzcd}$$

is a pullback square for all $n \geq 0$. As limits in $\mathrm{Cat}(\mathcal{B})$ can be computed on the underlying simplicial objects, this shows that f is fully faithful. □

Remark 1.3.2.6. As a consequence of Proposition 1.3.2.5, base change along both geometric and algebraic morphisms of ∞-topoi preserves fully faithful functors. Consequently, base change along algebraic morphisms also preserves essentially surjective functors.

Proposition 1.3.2.7. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories. Then the following are equivalent:*

1. *The functor f is fully faithful;*

2. *for any $A \in \mathcal{B}$ and any two objects $c_0, c_1 \colon A \to \mathsf{C}$ in context A, the morphism*

$$\operatorname{map}_{\mathsf{C}}(c_0, c_1) \to \operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

   that is induced by f is an equivalence in $\widehat{\mathcal{B}}_{/A}$;

3. *for every $A \in \mathcal{B}$ the functor $f(A) \colon \mathsf{C}(A) \to \mathsf{D}(A)$ of ∞-categories is fully faithful;*

4. *the induced map $\int \mathsf{C} \to \int \mathsf{D}$ of cartesian fibrations over $\mathcal{B}$ is fully faithful.*

Proof. As limits in $\operatorname{Sh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$ are computed object-wise, it is clear that the first and the third condition are equivalent. Moreover, the third and forth conditions are equivalent by (the proof of) [10, Lemma 3.1.9]. Now if f is fully faithful and $c_0, c_1 \colon A \rightrightarrows \mathsf{C}$ are arbitrary objects in context $A \in \mathcal{B}$, the map

$$\operatorname{map}_{\mathsf{C}}(c_0, c_1) \to \operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

is defined by the commutative diagram

$$\begin{tikzcd}
 & \operatorname{map}_{\mathsf{C}}(c_0, c_1) \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1)) \arrow[dl] \arrow[dd] \\
\mathsf{C}_1 \arrow[rr, "f_1"] \arrow[dd] & & \mathsf{D}_1 \arrow[dd] & \\
 & A \arrow[rr, "\mathrm{id}"] \arrow[dl] & & A \arrow[dl] \\
\mathsf{C}_0 \times \mathsf{C}_0 \arrow[rr, "f_0 \times f_0"] & & \mathsf{D}_0 \times \mathsf{D}_0. &
\end{tikzcd}$$

in which the two vertical squares on the left and on the right are cartesian. As f is fully faithful, the square in the front is cartesian, hence the square in the back must be cartesian as well, which implies that the second condition holds. Conversely, if f induces an equivalence on mapping groupoids, the fact that the object $C_0 \times C_0 \in \widehat{\mathcal{B}}$ is obtained as the colimit of the diagram $\mathcal{B}_{/C_0 \times C_0} \to \mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$, we obtain a cover

$$\bigsqcup_{A \to \mathsf{C}_0 \times \mathsf{C}_0} A \twoheadrightarrow \mathsf{C}_0 \times \mathsf{C}_0$$

in $\widehat{\mathcal{B}}$. By assumption, pasting the front square in the above diagram with the pullback square

$$\begin{tikzcd}
\bigsqcup_{A\to \mathsf{C}_0\times\mathsf{C}_0} \mathrm{map}_{\mathsf{C}}(c_0,c_1) \arrow[r, two heads] \arrow[d] & \mathsf{C}_1 \arrow[d] \\
\bigsqcup_{A\to \mathsf{C}_0\times\mathsf{C}_0} A \arrow[r, two heads] & \mathsf{C}_0\times\mathsf{C}_0
\end{tikzcd}$$

results in a pullback square, hence the claim follows from the fact that the étale base change along a cover in an ∞-topos constitutes a conservative algebraic morphism. $\square$

Our next goal is to show that fully faithful functors are monomorphisms in $\mathrm{Cat}(\mathcal{B})$. For this, we will need the following lemma:

Lemma 1.3.2.8. *The map $\Delta^1 \to \Delta^0$ is essentially surjective as a functor of $\mathcal{B}$-categories.*

Proof. Algebraic morphisms preserve essential surjectivity since dually geometric morphisms preserve full faithfulness. We may therefore assume without loss of generality $\mathcal{B} \simeq \mathrm{Ani}$. Let S be the saturated class of maps in Ani_Δ that is generated by $\Delta^0 \sqcup \Delta^0 \hookrightarrow \Delta^1$, the spine inclusions $I^n \hookrightarrow \Delta^n$ for $n \geq 0$ as well as the map $E^1 \to 1$, where E^1 denotes the walking equivalence. It suffices to show that $\Delta^1 \to \Delta^0$ is contained in S. Let $K \hookrightarrow \Delta^3$ be the unique map of simplicial ∞-groupoids that fits into the commutative diagram

$$\begin{tikzcd}
\Delta^1 \arrow[r, "d^{\{0,1\}}"] \arrow[d, "d^{\{1,2\}}"'] & \Delta^2 \arrow[d] \arrow[ddr, bend left, "d^{\{1,2,3\}}"] & \\
\Delta^2 \arrow[r] \arrow[drr, bend right, "d^{\{0,1,2\}}"'] & K \arrow[ul, phantom, "\ulcorner"] \arrow[dr, hook] & \\
& & \Delta^3.
\end{tikzcd}$$

The inclusion $K \hookrightarrow \Delta^3$ is contained in S: in fact, as the inclusion $I^3 \hookrightarrow \Delta^3$ is an element of S and factors through $K \hookrightarrow \Delta^3$, it suffices to observe that the map $I^3 \hookrightarrow K$ can be obtained by glueing two copies of the inclusion $I^2 \hookrightarrow \Delta^2$ along the id_{Δ^1} in $\mathrm{Fun}(\Delta^1, \mathrm{Ani}_\Delta)$. We now obtain a commutative diagram

$$\begin{tikzcd}
\Delta^1 \sqcup \Delta^1 \arrow[r, hook] \arrow[d] & K \arrow[r, hook] \arrow[d] & \Delta^3 \arrow[d] \\
\Delta^0 \sqcup \Delta^0 \arrow[r] & L \arrow[r, hook] & E^1
\end{tikzcd}$$

in which the upper left horizontal map is induced by composing $d^{\{0,2\}} \colon \Delta^1 \hookrightarrow \Delta^2$ with the two maps $\Delta^2 \rightrightarrows K$ that are defined by the pushout square. As $K \hookrightarrow \Delta^3$ is contained in S, we conclude that the induced map $L \hookrightarrow E^1$ must be in S as well. As a consequence, the terminal map $L \to \Delta^0$ is an element of S too. Let $\Delta^1 \to K$ be the composite map in the pushout square that defines K. Postcomposing with the map $K \to L$ from the previous diagram gives rise to a map $\Delta^1 \to L$. We finish the proof by showing that this map is contained in S. Let H be defined by the pushout square

$$\begin{tikzcd}
\Delta^1 \arrow[r] \arrow[d, hook, "d^{\{0,2\}}"'] & \Delta^0 \arrow[d] \\
\Delta^2 \arrow[r] & H.
\end{tikzcd}$$

Then the map $\Delta^1 \to L$ is recovered as the composite map in the cocartesian square

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\beta"] \arrow[d, "\alpha"] & H \arrow[d] \\
H \arrow[r] & L,
\end{tikzcd}$$

in which α and β are given by composing $d^{\{1,2\}} \colon \Delta^1 \hookrightarrow \Delta^2$ and $d^{\{0,1\}} \colon \Delta^1 \hookrightarrow \Delta^2$, respectively, with the map $\Delta^2 \to H$. As a consequence, we only need to verify that $\alpha, \beta \in S$. We will show this for α, the case of the map β is analogous. Consider the commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook, "d^0"] \arrow[d, hook, "d^0"] & \Delta^1 \arrow[r] \arrow[d, hook] & \Delta^0 \arrow[d, hook, "d^0"] \\
\Delta^1 \arrow[r, hook] \arrow[dr, hook, "d^{\{1,2\}}"'] & \Lambda^2_0 \arrow[r] \arrow[d, hook] & \Delta^1 \arrow[d, "\alpha"] \\
& \Delta^2 \arrow[r] & H
\end{tikzcd}$$

in which the composite of the two vertical maps in the middle is given by $d^{\{0,2\}}\colon \Delta^1 \hookrightarrow \Delta^2$. As maps in S are stable under pushouts, we only need to show that the inclusion $\Lambda^2_0 \hookrightarrow \Delta^2$ is contained in S. Consider the commutative square

$$\begin{tikzcd}
\Delta^0 \sqcup \Delta^1 \arrow[r, hook] \arrow[d, hook] & \Lambda^2_0 \arrow[d, hook] \\
I^2 \arrow[r, hook] & \Delta^2
\end{tikzcd}$$

that is uniquely determined by the condition that the composite map is induced by the inclusions $d^{\{0\}}\colon \Delta^0 \hookrightarrow \Delta^2$ and $d^0\colon \Delta^1 \hookrightarrow \Delta^2$. As the lower horizontal map is contained in S by assumption on S, it suffices to show that the two maps from $\Delta^0 \sqcup \Delta^1$ are in S as well. This follows immediately from the observation that both of these maps can be obtained as a pushout of the map $\Delta^0 \sqcup \Delta^0 \hookrightarrow \Delta^1$. □

Proposition 1.3.2.9. *Every fully faithful functor of $\mathcal{B}$-categories is a monomorphism in* $\mathrm{Cat}(\mathcal{B})$.

Proof. By Lemma 1.3.2.8, the map $\Delta^1 \to \Delta^0$ is essentially surjective, hence $\Delta^0 \sqcup \Delta^0 \to \Delta^0$ is essentially surjective as well. Since monomorphisms are precisely those maps that are internally right orthogonal to this map, the claim follows. □

For fully faithful functors between $\mathcal{B}$-*groupoids*, the converse of Proposition 1.3.2.9 is true as well:

Corollary 1.3.2.10. *For a map $f\colon \mathsf{G} \to \mathsf{H}$ between $\mathcal{B}$-groupoids, the following are equivalent:*

1. *f is fully faithful;*
2. *f is a monomorphism in* $\mathrm{Cat}(\mathcal{B})$*;*
3. *f is a monomorphism in* $\mathrm{Grpd}(\mathcal{B})$.

Proof. By Proposition 1.3.2.9, if f is fully faithful, then f is a monomorphism in $\mathrm{Cat}(\mathcal{B})$. Conversely, f being a monomorphism in $\mathrm{Cat}(\mathcal{B})$ precisely means that f is internally right orthogonal to the map $\Delta^0 \sqcup \Delta^0 \to \Delta^0$. As the latter is the image of the map $\Delta^0 \sqcup \Delta^0 \to \Delta^1$ under the groupoidification functor $(-)^{\mathrm{gpd}}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$, the claim follows by adjunction from the assumption that f is internally right orthogonal to $\Delta^0 \sqcup \Delta^0 \to \Delta^0$. Lastly, (2) and (3) are equivalent as inclusion $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ commutes with small limits. □

Definition 1.3.2.11. Let D be a $\mathcal{B}$-category. A *full* subcategory of D is a fully faithful functor $\mathsf{C} \hookrightarrow \mathsf{D}$. The collection of full subcategories of D spans a partially ordered subset of $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{D})$ that we denote by $\mathrm{Sub}_{\mathrm{full}}(\mathsf{D})$.

As in ∞-category theory, a full subcategory of a $\mathcal{B}$-category should be uniquely specified by the collection of objects that are contained in it. Hereafter our goal is to turn this heuristic into a precise statement. To that end, note that the functor $(-)_0\colon \mathcal{B}_\Delta \to \mathcal{B}$ admits a right adjoint $\check{C}(-)$ that sends an object $A \in \mathcal{B}$ to its *Čech nerve* $\check{C}(A)$. Now if D is an arbitrary $\mathcal{B}$-category, the functor $(-)_0\colon (\mathcal{B}_\Delta)_{/\mathsf{D}} \to \mathcal{B}_{/\mathsf{D}_0}$ admits a right adjoint $\langle - \rangle_{\mathsf{D}}$ that is given by the composition

$$\mathcal{B}_{/\mathsf{D}_0} \xrightarrow{\check{C}} (\mathcal{B}_\Delta)_{/\check{C}(\mathsf{D}_0)} \xrightarrow{\eta^*} (\mathcal{B}_\Delta)_{/\mathsf{D}}$$

in which $\eta\colon \mathsf{D} \to \check{C}(\mathsf{D}_0)$ denotes the adjunction unit. As $\check{C}$ is fully faithful, so is the functor $\langle - \rangle_{\mathsf{D}}$.

Lemma 1.3.2.12. *For every $\mathcal{B}$-category D and any monomorphism $P \hookrightarrow \mathsf{D}_0$, the simplicial object $\langle P \rangle_{\mathsf{D}}$ is a $\mathcal{B}$-category, and the functor $\langle P \rangle_{\mathsf{D}} \to \mathsf{D}$ is fully faithful.*

Proof. By construction, the map $\langle P \rangle_{\mathsf{D}} \to \mathsf{D}$ fits into a cartesian square

$$\begin{tikzcd} \langle P \rangle_{\mathsf{D}} \arrow[r] \arrow[d] & \mathsf{D} \arrow[d, "\eta"] \\ \check{C}(P) \arrow[r] & \check{C}(\mathsf{D}_0). \end{tikzcd}$$

To show that $\langle P \rangle_{\mathsf{D}}$ is a $\mathcal{B}$-category, it therefore suffices to show that the map $\check{C}(P) \to \check{C}(\mathsf{D}_0)$ is internally right orthogonal to the two maps $E^1 \to 1$ and $I^2 \hookrightarrow \Delta^2$. This is in turn equivalent to the map $P \hookrightarrow \mathsf{D}_0$ being internally right orthogonal (in $\mathcal{B}$) to the two maps $(E^1)_0 \to 1$ and $(I^2)_0 \hookrightarrow (\Delta^2)_0$. As the first one is a cover in $\mathcal{B}$ and the second one is an equivalence, this is immediate. By the same argumentation, the functor $\langle P \rangle_{\mathsf{D}} \to \mathsf{D}$ is fully faithful precisely if the map $P \hookrightarrow \mathsf{D}_0$ is internally right orthogonal to $(\Delta^0 \sqcup \Delta^0)_0 \to (\Delta^1)_0$, which follows from the observation that this map is an equivalence in $\mathcal{B}$. $\square$

As a consequence of Lemma 1.3.2.12, the functor $\langle - \rangle_{\mathsf{D}}$ restricts to an embedding

$$\langle - \rangle_{\mathsf{D}} \colon \mathrm{Sub}_{\mathcal{B}}(\mathsf{D}_0) \hookrightarrow \mathrm{Sub}_{\mathrm{full}}(\mathsf{D})$$

of partially ordered sets.

Proposition 1.3.2.13. *For any $\mathcal{B}$-category D, the map*

$$\langle - \rangle_{\mathsf{D}} \colon \mathrm{Sub}_{\mathcal{B}}(\mathsf{D}_0) \hookrightarrow \mathrm{Sub}_{\mathrm{full}}(\mathsf{D})$$

is an equivalence.

Proof. It suffices to show that the map is essentially surjective. Let therefore $\mathsf{C} \hookrightarrow \mathsf{D}$ be a full subcategory of D. Using Corollary 1.3.2.10, the induced map $\mathsf{C}_0 \to \mathsf{D}_0$ is a monomorphism in $\mathcal{B}$. We therefore obtain a factorisation

$$\mathsf{C} \hookrightarrow \langle \mathsf{C}_0 \rangle_{\mathsf{D}} \hookrightarrow \mathsf{D}$$

in which the first map is fully faithful since the second map and the composite map are fully faithful. As moreover the map $\mathsf{C} \hookrightarrow \langle \mathsf{C}_0 \rangle_{\mathsf{D}}$ induces an equivalence on level 0, Proposition 1.3.2.5 implies that it must be an equivalence on level 1 as well. Together with the Segal condition, this implies that this map is an equivalence, which completes the proof. $\square$

Proposition 1.3.2.14. *Let $f \colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories and let $\mathsf{E} \hookrightarrow \mathsf{D}$ be a full subcategory. Then the following are equivalent:*

1. *f factors through the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$;*
2. *$f^{\simeq}$ factors through $\mathsf{E}^{\simeq} \hookrightarrow \mathsf{D}^{\simeq}$;*
3. *for every object c in C in context $A \in \mathcal{B}$ its image $f(c)$ is contained in E.*

Proof. Clearly (1) implies (2). By making use of the adjunction $(-)_0 \dashv \langle - \rangle_{\mathsf{D}}$ and Proposition 1.3.2.13, one finds that conversely (2) implies (1). A fortiori (2) implies (3). Conversely, suppose that for any $c\colon A \to \mathsf{C}$ the composite map $A \to \mathsf{C}_0 \to \mathsf{D}_0$ factors through $\mathsf{E}_0 \hookrightarrow \mathsf{D}_0$. As the map

$$\bigsqcup_{A \to \mathsf{C}_0} A \twoheadrightarrow \mathsf{C}_0$$

defines a cover in $\widehat{\mathcal{B}}$, the lifting problem

$$\begin{tikzcd}
\bigsqcup_{A\to\mathsf{C}_0} A \arrow[r] \arrow[d, two heads] & \mathsf{E}_0 \arrow[d, hook] \\
\mathsf{C}_0 \arrow[r, "f"] \arrow[ur, dashed] & \mathsf{D}_0
\end{tikzcd}$$

admits a unique solution, which proves that (2) holds. □

Corollary 1.3.2.15. *A map $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is essentially surjective if and only if $f_0\colon \mathsf{C}_0 \to \mathsf{D}_0$ is a cover in $\mathcal{B}$.*

Proof. Suppose first that f_0 is a cover, and let

$$\mathsf{C} \xrightarrow{p} \mathsf{E} \xrightarrow{i} \mathsf{D}$$

be the factorisation of f into an essentially surjective and a fully faithful functor. We need to show that i is an equivalence. Since i is fully faithful, Proposition 1.3.2.13 implies that this is the case if and only if i_0 is an equivalence. But since f_0 is a cover, the map i_0 is one as well and must therefore be an equivalence as it is already a monomorphism by Proposition 1.3.2.9.

Conversely, suppose that f is essentially surjective, and let

$$\mathsf{C}_0 \xrightarrow{p} P \xrightarrow{i} \mathsf{D}_0$$

be the factorisation of f_0 into a cover and a monomorphism in $\mathcal{B}$. We need to show that i is an equivalence. By Proposition 1.3.2.13, the object P determines a full subcategory $\langle P\rangle_{\mathsf{D}}$ of D and since f_0 factors through P, Proposition 1.3.2.14 implies that f factors through a map $\mathsf{C} \to \langle P\rangle_{\mathsf{D}}$. It suffices to show that this functor is essentially surjective. Let $\mathsf{C} \twoheadrightarrow \mathsf{E} \hookrightarrow \langle P\rangle_{\mathsf{D}}$ be the factorisation of this functor into an essentially surjective and a fully faithful functor. Then $p\colon \mathsf{C}_0 \to P$ factors through a monomorphism $\mathsf{E}_0 \hookrightarrow P$, but since p is a cover this map must be a cover as well and therefore an equivalence. Proposition 1.3.2.13 then implies that the map $\mathsf{E} \hookrightarrow \langle P \rangle_{\mathsf{D}}$ is an equivalence and therefore that the functor $\mathsf{C} \to \langle P \rangle_{\mathsf{D}}$ is essentially surjective, as desired. □

Definition 1.3.2.16. Let $f \colon \mathsf{C} \to \mathsf{D}$ be a map in $\mathrm{Cat}(\mathcal{B})$ and let $\mathsf{C} \twoheadrightarrow \mathsf{E} \hookrightarrow \mathsf{D}$ be the factorisation of f into an essentially surjective and a fully faithful functor. Then the full subcategory $\mathsf{E} \hookrightarrow \mathsf{D}$ is referred to as the *essential image* of f.

Definition 1.3.2.17. Let D be a $\mathcal{B}$-category and let $(d_i \colon A_i \to \mathsf{D})_{i \in I}$ be a small family of objects in D. The essential image of the induced map $\bigsqcup_i A_i \to \mathsf{D}$ is referred to as the full subcategory of D that is *generated* by the family $(d_i)_{i \in I}$.

Remark 1.3.2.18. In the context of Definition 1.3.2.17, let $G \hookrightarrow \mathsf{D}_0$ be the image of the map $(d_i)_{i \in I} \colon \bigsqcup_i A_i \to \mathsf{D}_0$. Then Corollary 1.3.2.15 implies that the full subcategory of D generated by the family $(d_i)_{i \in I}$ is given by $\langle G \rangle_{\mathsf{D}}$. Moreover, an object $d \colon A \to \mathsf{D}$ is contained in $\langle G \rangle_{\mathsf{D}}$ if and only if it *locally* equivalent to one of the d_i, in the sense that there is a cover $(s_j)_{j \in J} \colon \bigsqcup_i B_i \twoheadrightarrow A$, a map $\phi \colon J \to I$ as well as maps $t_j \colon B_j \to A_{\phi(j)}$ such that for each j there is an equivalence $s_j^*(d) \simeq t_j^*(d_{\phi(j)})$. In fact, that this is sufficient follows from the fact that the lifting problem

$$\begin{tikzcd}
\bigsqcup_j B_j \arrow[r] \arrow[d, two heads] & G \arrow[d, hook] \\
A \arrow[r] \arrow[ur, dashed] & \mathsf{D}_0
\end{tikzcd}$$

(in which the upper horizontal arrow is induced by the maps $B_j \to A_{\phi(j)} \to G$) has a unique solution, and that it is also necessary follows from considering the pullback square

$$\begin{tikzcd}
\bigsqcup_i A_i \times_G A \arrow[r, two heads] \arrow[d] & A \arrow[d] \\
\bigsqcup_i A_i \arrow[r, two heads] & G
\end{tikzcd}$$

in $\mathcal{B}$.

Remark 1.3.2.19. By the same argument as in Remark 1.3.1.15, the full subcategory of a $\mathcal{B}$-category D that is generated by a collection of objects $(d_i \colon A_i \to \mathsf{D})_{i \in I}$ is well-defined even if this family is *large*.

1.3.3. Conservative functors and localisations

Definition 1.3.3.1. A functor of $\mathcal{B}$-categories is said to be *conservative* if it is internally right orthogonal to the map $\Delta^1 \to \Delta^0$.

By Proposition 1.2.2.2, if C is a $\mathcal{B}$-category, the unique map $\mathsf{C} \to 1$ in $\mathrm{Cat}(\mathcal{B})$ is conservative if and only if C is a $\mathcal{B}$-groupoid. Consequently, item (2) of Proposition 1.1.5.2 implies that if $A \in \mathcal{B}$ is an arbitrary object, a map $\mathsf{C} \to A$ in $\mathrm{Cat}(\mathcal{B})$ is conservative if and only if C is a $\mathcal{B}_{/A}$-groupoid. In particular, if $f \colon \mathsf{C} \to \mathsf{D}$ is a conservative functor, then the fibre $\mathsf{C}|_d$ of f over any object $d \colon A \to \mathsf{D}$ with $A \in \mathcal{B}$ is a $\mathcal{B}_{/A}$-groupoid This property turns out to *characterise* conservative functors. To show this, we first need the following lemma:

Lemma 1.3.3.2. *Let S be a saturated class of maps in* $\mathrm{Cat}(\mathcal{B})$ *that contains the projections $\Delta^1 \otimes A \to A$ for all $A \in \mathcal{B}$. Then S contains the projection $\Delta^1 \otimes \mathsf{C} \to \mathsf{C}$ for any $\mathcal{B}$-category C.*

Proof. As every $\mathcal{B}$-category C is a colimit of $\mathcal{B}$-categories of the form $\Delta^n \otimes A$ for some $n \geq 0$ and some $A \in \mathcal{B}$, we may assume without loss of generality $\mathsf{C} \simeq \Delta^n$. Since $\Delta^n \simeq I^n$ in $\mathrm{Cat}(\mathcal{B})$, we may furthermore assume $n = 1$. In light of the decomposition $\Delta^1 \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$, the projection $\Delta^1 \times \Delta^1 \to \Delta^1$ is equivalent to the composition

$$\Delta^2 \sqcup_{\Delta^1} \Delta^2 \xrightarrow{s^1 \sqcup_{\Delta^1} \mathrm{id}} \Delta^2 \xrightarrow{s^0} \Delta^1.$$

It will therefore be enough to show that the two maps $s^0, s^1 \colon \Delta^2 \rightrightarrows \Delta^1$ are contained in S, which follows immediately from the observation that these two maps are given by $s^0 \sqcup_{\Delta^0} \mathrm{id}$ and $\mathrm{id} \sqcup_{\Delta^0} s^0$ in light of the decomposition $\Delta^2 \simeq \Delta^1 \sqcup_{\Delta^0} \Delta^1$. □

Lemma 1.3.3.2 immediately implies:

Proposition 1.3.3.3. *A functor $f \colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is conservative if and only if the square*

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, "f_0"] \arrow[d, "s_0"] & \mathsf{D}_0 \arrow[d, "s_0"] \\ \mathsf{C}_1 \arrow[r, "f_1"] & \mathsf{D}_1 \end{tikzcd}$$

is cartesian. □

Remark 1.3.3.4. Proposition 1.3.3.3 implies that base change along both algebraic and geometric morphisms of ∞-topoi preserves conservative functors.

Corollary 1.3.3.5. *A functor* $f\colon \mathsf{C} \to \mathsf{D}$ *between* $\mathcal{B}$*-categories is conservative if and only if the commutative square*

$$\begin{tikzcd} \mathsf{C}^{\simeq} \arrow[r, "f^{\simeq}"] \arrow[d] & \mathsf{D}^{\simeq} \arrow[d] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

is cartesian.

Proof. On account of the Segal conditions, a cartesian square of $\mathcal{B}$-categories is cartesian if and only if it is cartesian on level 0 and level 1, hence the claim follows from the observation that the square in the statement of the corollary is trivially cartesian on level 0 and recovers the square from Proposition 1.3.3.3 on level 1. □

Corollary 1.3.3.6. *A functor* $\mathsf{C} \to \mathsf{D}$ *between large* $\mathcal{B}$*-categories is conservative if and only if for any object* $d\colon A \to \mathsf{D}$ *in context* $A \in \mathcal{B}$ *the fibre* $\mathsf{C}|_d = \mathsf{C} \times_A \mathsf{D}$ *is a* $\widehat{\mathcal{B}}_{/A}$*-groupoid.*

Proof. If f is conservative, then for any object $d\colon A \to \mathsf{D}$ the map $\mathsf{C}|_d \to A$ is conservative as well, hence $\mathsf{C}|_d$ is internally local with respect to $\Delta^1 \to \Delta^0$ and therefore a $\widehat{\mathcal{B}}_{/A}$-groupoid. Conversely, if $\mathsf{C}|_d$ is a $\widehat{\mathcal{B}}_{/A}$-groupoid for any object $d\colon A \to \mathsf{D}$, the fact that $\mathsf{C}_0 \simeq \operatorname{colim}_{A \to \mathsf{C}_0} A$ and descent in $\widehat{\mathcal{B}}_{\Delta}$ imply that the fibre of f over the map $\mathsf{C}_0 \to \mathsf{C}$ is contained in $\widehat{\mathcal{B}}$. On account of Corollary 1.3.3.5, the claim now follows. □

We now turn to the left complement of the factorisation system that is generated by conservative functors:

Definition 1.3.3.7. A functor between $\mathcal{B}$-categories is an *iterated localisation* if it is left orthogonal to every conservative functor.

Remark 1.3.3.8. As base change along geometric morphisms preserves conservative functors (see Remark 1.3.3.4), base change along algebraic morphisms dually preserves iterated localisations.

The saturated class of iterated localisations in $\mathrm{Cat}(\mathcal{B})$ is internally generated by $\Delta^1 \to \Delta^0$. Since this map is a strong epimorphism by Remark 1.3.1.5, we deduce:

Proposition 1.3.3.9. *Every iterated localisation in* $\mathrm{Cat}(\mathcal{B})$ *is a strong epimorphism and therefore in particular essentially surjective. Dually, every monomorphism is conservative.* □

Definition 1.3.3.10. Let C be a $\mathcal{B}$-category and let $\mathsf{S} \to \mathsf{C}$ be a functor. The *localisation* of C at S is the $\mathcal{B}$-category $\mathsf{S}^{-1}\mathsf{C}$ that fits into the pushout square

$$\begin{tikzcd} \mathsf{S} \arrow[r] \arrow[d] & \mathsf{S}^{\mathrm{gpd}} \arrow[d] \\ \mathsf{C} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathsf{S}^{-1}\mathsf{C}. \end{tikzcd}$$

We refer to the map $\mathsf{C} \to \mathsf{S}^{-1}\mathsf{C}$ as the *localisation functor* that is associated with the map $\mathsf{S} \to \mathsf{C}$. More generally, a functor $\mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is said to be a localisation if there is a functor $\mathsf{S} \to \mathsf{C}$ and an equivalence $\mathsf{D} \simeq \mathsf{S}^{-1}\mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})_{\mathsf{C}/}$.

Remark 1.3.3.11. The above definition is a direct analogue of the construction of localisations of ∞-categories, see [18, Proposition 7.1.3].

By definition, the groupoidification functor $\mathsf{S} \to \mathsf{S}^{\mathrm{gpd}}$ in Definition 1.3.3.10 is an iterated localisation. One therefore finds:

Proposition 1.3.3.12. *For any* $\mathcal{B}$*-category* C *and any functor* $\mathsf{S} \to \mathsf{C}$*, the localisation functor* $\mathsf{C} \to \mathsf{S}^{-1}\mathsf{C}$ *is an iterated localisation.* □

Lemma 1.3.3.13. *Let* G *be a* $\mathcal{B}$*-groupoid and let* $\mathsf{G} \to \mathsf{C}$ *be a strong epimorphism in* $\mathrm{Cat}(\mathcal{B})$*. Then* C *is a* $\mathcal{B}$*-groupoid as well.*

Proof. Since G is a $\mathcal{B}$-groupoid, Corollary 1.3.1.10 implies that the functor $\mathsf{G} \to \mathsf{C}$ factors through the inclusion $\mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$. We may therefore construct a lifting problem

$$\begin{tikzcd} \mathsf{G} \arrow[r] \arrow[d] & \mathsf{C}^{\simeq} \arrow[d, hook] \\ \mathsf{C} \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \mathsf{C} \end{tikzcd}$$

which admits a unique solution. Hence the identity on C factors through $\mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$, which evidently implies that $\mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$ is already an equivalence. □

Lemma 1.3.3.14. *For any strong epimorphism $f\colon \mathsf{C} \to \mathsf{D}$ in* $\mathrm{Cat}(\mathcal{B})$*, the commutative square*

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[d, "f"] & \mathsf{C}^{\mathrm{gpd}} \arrow[d, "f^{\mathrm{gpd}}"] \\ \mathsf{D} \arrow[r] & \mathsf{D}^{\mathrm{gpd}} \end{tikzcd}$$

is cocartesian.

Proof. If $\mathsf{P} = \mathsf{D} \sqcup_{\mathsf{C}} \mathsf{C}^{\mathrm{gpd}}$ denotes the pushout, we need to show that the induced functor $g\colon \mathsf{P} \to \mathsf{D}^{\mathrm{gpd}}$ is an equivalence. Since iterated localisations are stable under pushout, the map $\mathsf{D} \to \mathsf{P}$ is an iterated localisation, which (by the left cancellation property) implies that g must be an iterated localisation as well. We therefore only need to show that g is conservative. Since $\mathsf{D}^{\mathrm{gpd}}$ is a $\mathcal{B}$-groupoid, this is equivalent to P being a $\mathcal{B}$-groupoid as well (Corollary 1.3.3.6). But since strong epimorphisms are also preserved by pushouts, the map $\mathsf{C}^{\mathrm{gpd}} \to \mathsf{P}$ is a strong epimorphism, hence Lemma 1.3.3.13 implies the claim. □

Proposition 1.3.3.15. *Let $f\colon \mathsf{S} \to \mathsf{T}$ and $g\colon \mathsf{T} \to \mathsf{C}$ be functors in* $\mathrm{Cat}(\mathcal{B})$*, and suppose that f is a strong epimorphism. Then the induced functor* $\mathsf{S}^{-1}\mathsf{C} \to \mathsf{T}^{-1}\mathsf{C}$ *is an equivalence.*

Proof. This is an immediate consequence of the pasting lemma for pushout squares, together with Lemma 1.3.3.14. □

Remark 1.3.3.16. Proposition 1.3.3.15 implies that when considering localisations of a $\mathcal{B}$-category C, we may restrict our attention to *subcategories* $\mathsf{S} \hookrightarrow \mathsf{C}$ instead of general functors, as we can always factor a functor $\mathsf{S} \to \mathsf{C}$ into a strong epimorphism followed by a monomorphism. Alternatively, by making use of the strong epimorphism $\Delta^1 \otimes \mathsf{S}_0 \to \mathsf{S}$ from Lemma 1.3.1.8, we can always assume that S is of the form $\Delta^1 \otimes A$ for some $A \in \mathcal{B}$.

Our next goal is to derive an explicit construction of the unique factorisation of a functor into an iterated localisation and a conservative map. To that end, let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories. Let $f^{-1}\mathsf{D}^{\simeq} \hookrightarrow \mathsf{C}$ be the subcategory that is defined by the pullback square

$$\begin{tikzcd}
f^{-1}\mathsf{D}^{\simeq} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{D}^{\simeq} \arrow[d, hook] \\
\mathsf{C} \arrow[r, "f"] & \mathsf{D}.
\end{tikzcd}$$

Since $\mathsf{D}^{\simeq}$ is a $\mathcal{B}$-groupoid, the map $f^*\mathsf{D}^{\simeq} \to \mathsf{D}^{\simeq}$ factors through the morphism $f^{-1}\mathsf{D}^{\simeq} \to (f^{-1}\mathsf{D}^{\simeq})^{\mathrm{gpd}}$. Consequently, one obtains a factorisation of f into the composition

$$\mathsf{C} \to (f^{-1}\mathsf{D}^{\simeq})^{-1}\mathsf{C} \xrightarrow{f_1} \mathsf{D}.$$

Let us set $\mathsf{C}_1 = (f^{-1}\mathsf{D}^{\simeq})^{-1}\mathsf{C}$. By replacing C by C_1 and f by f_1 and iterating this procedure, we obtain an $\mathbb{N}$-indexed diagram in $\mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$. Let $f_\infty\colon \mathsf{E} \to \mathsf{D}$ denote the colimit of this diagram. By construction, the map f factors into the composition $\mathsf{C} \to \mathsf{E} \to \mathsf{D}$ in which the first map is a countable composition of localisations and therefore an iterated localisation in the sense of Definition 1.3.3.7. We claim that the map f_∞ is conservative. To see this, consider the cartesian square

$$\begin{tikzcd}
f_\infty^{-1}\mathsf{D}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{D}^{\simeq} \arrow[d] \\
\mathsf{E} \arrow[r, "f_\infty"] & \mathsf{D}.
\end{tikzcd}$$

On account of filtered colimits being universal in $\mathrm{Cat}(\mathcal{B})$ (see Remark 1.2.3.11), we obtain an equivalence $f_\infty^{-1}\mathsf{D}^{\simeq} \simeq \mathrm{colim}_n f_n^{-1}\mathsf{D}^{\simeq}$. By construction, the categories $f_n^{-1}\mathsf{D}^{\simeq}$ sit inside the $\mathbb{N}$-indexed diagram

$$\cdots \to f_{n-1}^{-1}\mathsf{D}^{\simeq} \to (f_{n-1}^{-1}\mathsf{D}^{\simeq})^{\mathrm{gpd}} \to f_n^{-1}\mathsf{D}^{\simeq} \to (f_n^{-1}\mathsf{D}^{\simeq})^{\mathrm{gpd}} \to f_{n+1}^{-1}\mathsf{D}^{\simeq} \to \cdots$$

such that the functor $\cdot 2\colon \mathbb{N} \to \mathbb{N}$ that is given by the inclusion of all even natural numbers recovers the $\mathbb{N}$-indexed diagram $n \mapsto f_n^{-1}\mathsf{D}^{\simeq}$ that is defined by the cartesian square above. As both the inclusion of all even natural numbers and that of all odd natural numbers define final functors $\mathbb{N} \to \mathbb{N}$, we conclude that $f_\infty^{-1}\mathsf{D}^{\simeq}$ is obtained as the colimit of the diagram $n \mapsto (f_n^{-1}\mathsf{D}^{\simeq})^{\mathrm{gpd}}$ and is therefore a $\mathcal{B}$-*groupoid*. Applying Corollary 1.3.3.5, we thus conclude that f_∞ is conservative. Therefore the factorisation of f into the composite $\mathsf{C} \to \mathsf{E} \to \mathsf{D}$ as constructed above is the unique factorisation of f into an iterated localisation and a conservative functor. Applying this construction when f is already an iterated localisation, one in particular obtains:

Proposition 1.3.3.17. *Every iterated localisation between $\mathcal{B}$-categories is obtained as a countable composition of localisation functors.* □

We conclude this section by establishing that localisation functors admit a universal property. To that end, given any two $\mathcal{B}$-categories C and D and any functor $\mathsf{S} \to \mathsf{C}$, we shall denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ the full subcategory that is spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ which encode functors $\pi_A^* \mathsf{C} \to \pi_A^* \mathsf{D}$ with the property that the composition $\pi_A^* \mathsf{S} \to \pi_A^* \mathsf{C} \to \pi_A^* \mathsf{D}$ factors through the inclusion $\pi_A^* \mathsf{D}^{\simeq} \hookrightarrow \pi_A^* \mathsf{D}$. We denote by $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}}$ for the underlying ∞-category of global sections.

Remark 1.3.3.18. Note that a functor $f \colon \pi_A^* \mathsf{S} \to \pi_A^* \mathsf{D}$ factors through $\pi_A^* \mathsf{D}^{\simeq}$ if and only if the transposed map $A \times \mathsf{S} \to \mathsf{D}$ factors through $\mathsf{D}^{\simeq}$. As the map $\mathsf{D}^{\simeq} \hookrightarrow \mathsf{D}$ is a monomorphism by Example 1.3.1.3, this condition is *local*, in the sense that for every cover $(s_i) : \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, the functor f factors through $\pi_A^* \mathsf{D}^{\simeq}$ if and only if each of the functors $s_i^*(f)$ factors through $\pi_{A_i}^* \mathsf{D}^{\simeq}$. In fact, this is certainly a necessary condition, and it is also sufficient as the lifting diagram

$$\begin{tikzcd} \bigsqcup_i A_i \times \mathsf{S} \arrow[r] \arrow[d, two heads] & \mathsf{D}^{\simeq} \arrow[d, hook] \\ A \times \mathsf{S} \arrow[r] \arrow[ur, dotted] & \mathsf{D} \end{tikzcd}$$

admits a unique solution (on account of the left vertical arrow being a strong epimorphism, see Remark 1.3.1.5). Consequently, an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}}$ if and only if it encodes a functor $\pi_A^* \mathsf{C} \to \pi_A^* \mathsf{D}$ whose restriction along $\pi_A^* \mathsf{S} \to \pi_A^* \mathsf{C}$ factors through $\pi_A^* \mathsf{D}^{\simeq}$ (cf. Remark 1.3.2.18). In conjunction with Proposition 1.2.5.4, this observation furthermore implies that there is a canonical equivalence $\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}, \pi_A^* \mathsf{D})_{\pi_A^* \mathsf{S}}$ for every $A \in \mathcal{B}$.

Remark 1.3.3.19. By Corollary 1.3.1.10 and Remark 1.3.3.18, for any $A \in \mathcal{B}$ a functor $\pi_A^* \mathsf{C} \to \pi_A^* \mathsf{D}$ defines an object in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}}$ precisely if its restriction along $\pi_A^* \mathsf{S} \to \pi_A^* \mathsf{C}$ sends every map in $\pi_A^* \mathsf{S}$ to an equivalence in $\pi_A^* \mathsf{C}$.

Proposition 1.3.3.20. *Let* C *be a* $\mathcal{B}$*-category and let* $\mathsf{S} \to \mathsf{C}$ *be a functor. Then precomposition with the localisation functor* $L\colon \mathsf{C} \to \mathsf{S}^{-1}\mathsf{C}$ *induces an equivalence*

$$L^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C}, \mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}}$$

for any $\mathcal{B}$*-category* D.

Proof. By applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{D})$ to the pushout square that defines the localisation of C at S, one obtains a pullback square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C}, \mathsf{D}) \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \arrow[d] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{\mathrm{gpd}}, \mathsf{D}) \arrow[r] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}, \mathsf{D}).
\end{tikzcd}$$

We claim that the two horizontal functors are fully faithful. To see this, it suffices to consider the lower horizontal map. This is a fully faithful functor precisely if it is internally right orthogonal to the map $\Delta^0 \sqcup \Delta^0 \to \Delta^1$, and by making use of the adjunction between tensoring and powering in $\mathsf{Cat}(\mathcal{B})$, one sees that this is equivalent to the induced functor $\mathsf{D}^{\Delta^1} \to \mathsf{D} \times \mathsf{D}$ being internally right orthogonal to the map $\mathsf{S} \to \mathsf{S}^{\mathrm{gpd}}$. Hence it suffices to show that the functor $\mathsf{D}^{\Delta^1} \to \mathsf{D} \times \mathsf{D}$ is conservative, i.e. internally right orthogonal to $\Delta^1 \to \Delta^0$. Making use of the adjunction between tensoring and powering in $\mathsf{Cat}(\mathcal{B})$ once more, this is seen to be equivalent to D being internally local with respect to the map $K \to \Delta^1$ that is defined by the commutative diagram

$$\begin{tikzcd}
\Delta^1 \sqcup \Delta^1 \arrow[r, "(d_1\times\mathrm{id}, d_0\times\mathrm{id})"] \arrow[d, "s_0\sqcup s_0"'] & \Delta^1 \times \Delta^1 \arrow[d] \arrow[ddr, bend left, "\mathrm{pr}_1"] & \\
\Delta^0 \sqcup \Delta^0 \arrow[r] \arrow[drr, bend right, "(d_1, d_0)"'] & K \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr] & \\
& & \Delta^1
\end{tikzcd}$$

in which pr_1 denotes the projection onto the second factor. By the same reasoning as in the proof of Lemma 1.3.2.8, the map $K \to \Delta^1$ is an equivalence in $\mathsf{Cat}(\mathcal{B})$, hence the claim follows.

Since for any $A \in \mathcal{B}$ a functor $\pi_A^* \mathsf{S} \to \pi_A^* \mathsf{D}$ factors through $\pi_A^* \mathsf{D}^{\simeq}$ if and only if it factors through the map $\pi_A^* \mathsf{S} \to \pi_A^* \mathsf{S}^{\mathrm{gpd}}$, one obtains a commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}} \arrow[r, hook] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \arrow[d] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{\mathrm{gpd}}, \mathsf{D}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}, \mathsf{D}).
\end{tikzcd}$$

and therefore a map $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C}, \mathsf{D})$. Since every object of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C}, \mathsf{D})$ by definition gives rise to an object in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})_{\mathsf{S}}$, this map must also be essentially surjective and is thus an equivalence. $\square$

1.4. The universe and the $\mathcal{B}$-category of $\mathcal{B}$-categories

In this section, we discuss our first examples of $\mathcal{B}$-categories: the *universe for $\mathcal{B}$-groupoids* (Section 1.4.1) and the *$\mathcal{B}$-category of $\mathcal{B}$-categories* (Section 1.4.2). These two $\mathcal{B}$-categories will take a central place within our theory. The universe for $\mathcal{B}$-groupoids will be the $\mathcal{B}$-categorical analogue of the ∞-category Ani of ∞-groupoids; it can be regarded as a *reflection* of the base ∞-topos $\mathcal{B}$ within itself and is therefore a (categorical incarnation of an) *object classifier* of $\mathcal{B}$. The $\mathcal{B}$-category of $\mathcal{B}$-categories, on the other hand, allows us to argue globally about $\mathcal{B}$-categories within the theory of $\mathcal{B}$-categories.

1.4.1. The universe for $\mathcal{B}$-groupoids

Recall that for any large ∞-category $\mathcal{C}$ that admits pullbacks one can define a $\widehat{\mathrm{Cat}}_{\infty}$-valued presheaf $\mathcal{C}_{/-}$ as the functor that classifies the codomain fibration $d_0 \colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. If $\mathcal{C}$ is an ∞-topos, then this presheaf is a *sheaf* [49, Proposition 6.1.3.10]. One may therefore define:

Definition 1.4.1.1. The *universe for $\mathcal{B}$-groupoids* is defined to be the large $\mathcal{B}$-category $\mathsf{Grpd}_{\mathcal{B}}$ that corresponds to the $\widehat{\mathrm{Cat}}_{\infty}$-valued sheaf $\mathcal{B}_{/-}$ on $\mathcal{B}$.

Remark 1.4.1.2. For any object $A \in \mathcal{B}$ there is a canonical equivalence

$$\pi_A^* \mathsf{Grpd}_{\mathcal{B}} \simeq \mathsf{Grpd}_{\mathcal{B}_{/A}}.$$

In fact, the $\widehat{\mathrm{Cat}}_\infty$-valued sheaf associated with $\pi_A^* \operatorname{Grpd}_{\mathcal{B}}$ can be identified with $\operatorname{Fun}_{\mathcal{B}}((\pi_A)_!(-), \operatorname{Grpd}_{\mathcal{B}})$, hence the claim follows from the observation that since $(\pi_A)_!$ is a right fibration, the commutative square

$$\begin{tikzcd}
\operatorname{Fun}(\Delta^1, \mathcal{B}_{/A}) \arrow[r] \arrow[d, "d_0"] & \operatorname{Fun}(\Delta^1, \mathcal{B}) \arrow[d, "d_0"] \\
\mathcal{B}_{/A} \arrow[r, "(\pi_A)_!"] & \mathcal{B}
\end{tikzcd}$$

is cartesian.

The universe for small $\mathcal{B}$-groupoids $\operatorname{Grpd}_{\mathcal{B}}$ is supposed to be regarded as the $\mathcal{B}$-categorical analogue of the ∞-category Ani of ∞-groupoids. It can therefore be viewed as a reflection of the ∞-topos $\mathcal{B}$ within itself. This is supported by the observation that there is an equivalence

$$\operatorname{Fun}_{\mathcal{B}}(A, \operatorname{Grpd}_{\mathcal{B}}) \simeq \mathcal{B}_{/A}$$

for every $A \in \mathcal{B}$, which in particular shows that objects $A \to \operatorname{Grpd}_{\mathcal{B}}$ correspond to objects in the slice ∞-topos $\mathcal{B}_{/A}$, i.e. to $\mathcal{B}_{/A}$-groupoids. We can also identify the mapping $\mathcal{B}$-groupoids of $\operatorname{Grpd}_{\mathcal{B}}$:

Proposition 1.4.1.3. *For any two objects $P, Q \in \mathcal{B}_{/A}$, viewed as objects of $\operatorname{Grpd}_{\mathcal{B}}$ in context $A \in \mathcal{B}$, there is an equivalence*

$$\operatorname{map}_{\operatorname{Grpd}_{\mathcal{B}}}(P, Q) \simeq \underline{\operatorname{Hom}}_{\mathcal{B}_{/A}}(P, Q)$$

in $\mathcal{B}_{/A}$, where the right-hand side denotes the internal hom in $\mathcal{B}_{/A}$.

We will not prove Proposition 1.4.1.3 at this point, as it will follow quite easily once we have the theory of adjunctions between $\mathcal{B}$-categories at our disposal. The impatient reader is referred to Proposition 3.2.5.11.

Remark 1.4.1.4. Let $\operatorname{Grpd}_{\widehat{\mathcal{B}}}$ denote the universe for large $\mathcal{B}$-groupoids, i.e. the very large $\mathcal{B}$-category that corresponds to the sheaf $\widehat{\mathcal{B}}_{/-}$ on $\widehat{\mathcal{B}}$. By the discussion in Section 1.2.4, the inclusions $\mathcal{B}_{/A} \hookrightarrow \widehat{\mathcal{B}}_{/A}$ for $A \in \mathcal{B}$ define an embedding of presheaves $\mathcal{B}_{/-} \hookrightarrow \widehat{\mathcal{B}}_{/-}$ on $\mathcal{B}$. Since moreover restriction along the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ defines an equivalence

$$\operatorname{Fun}^{\lim}(\widehat{\mathcal{B}}^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathbf{W}}) \simeq \operatorname{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathbf{W}})$$

(see the argument in Remark 1.1.4.1), we obtain a fully faithful functor

$$\mathrm{Grpd}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\widehat{\mathcal{B}}}$$

in $\mathrm{Cat}(\mathcal{B}^{\mathbf{W}})$. Explicitly, an object $A \to \mathrm{Grpd}_{\widehat{\mathcal{B}}}$ in context $A \in \widehat{\mathcal{B}}$ that corresponds to a map $P \to A$ in $\widehat{\mathcal{B}}$ is contained in $\mathrm{Grpd}_{\mathcal{B}}$ precisely if for every map $B \to A$ in $\widehat{\mathcal{B}}$ where B is small, the pullback $B \times_A P$ is small as well.

The goal for the remainder of this section is to study the poset $\mathrm{Sub}_{\mathrm{full}}(\mathrm{Grpd}_{\mathcal{B}})$ of full subcategories of the universe. To that end, let us recall the notion of a *local class* in an ∞-topos [49, § 6.1.3]:

Definition 1.4.1.5. Let S be a collection of maps in $\mathcal{B}$ that is stable under pullbacks. Then the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{B})$ spanned by the maps in S forms a cartesian subfibration of the codomain fibration $d_0\colon \mathrm{Fun}(\Delta^1, \mathcal{B}) \to \mathcal{B}$ that is classified by a $\widehat{\mathrm{Cat}}_\infty$-valued presheaf $S_{/-}$ on $\mathcal{B}$. The class S is said to be *local* if $S_{/-}$ is a sheaf and *bounded* if $S_{/-}$ takes values in Cat_∞.

In the situation of Definition 1.4.1.5, [49, Lemma 6.1.3.7] implies that the presheaf $S_{/-}$ is a sheaf if and only if $(S_{/-})^{\simeq}$ is an $\widehat{\mathrm{Ani}}$-valued sheaf, and since the latter takes values in Ani if and only if $S_{/-}$ takes values in Cat_∞, one obtains:

Proposition 1.4.1.6. *Let S be a collection of maps in $\mathcal{B}$ that is stable under pullbacks. Then the following are equivalent:*

1. *S is a (bounded) local class.*
2. *$S_{/-}$ is a (Cat_∞-valued) sheaf.*
3. *$(S_{/-})^{\simeq}$ is an (Ani-valued) sheaf.* □

The set of local classes in $\mathcal{B}$ can be identified with a subset of the partially ordered set $\mathrm{Sub}_{\mathrm{full}}(\mathrm{Fun}(\Delta^1, \mathcal{B}))$ and therefore inherits a partial order. One now finds:

Proposition 1.4.1.7. *There is an equivalence between the partially ordered set of local classes in $\mathcal{B}$ and $\mathrm{Sub}_{\mathrm{full}}(\mathrm{Grpd}_{\mathcal{B}})$ with respect to which bounded local classes in $\mathcal{B}$ correspond to* small *full subcategories of $\mathrm{Grpd}_{\mathcal{B}}$.*

Proof. If S is a local class, Proposition 1.4.1.6 shows that $S_{/\_}$ is a $\widehat{\mathrm{Cat}}_\infty$-valued sheaf and therefore corresponds to a large $\mathcal{B}$-category Grpd_S. By Proposition 1.3.2.7, this is a full subcategory of the universe of $\mathcal{B}$. Conversely, if $\mathsf{C} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ exhibits C as a full subcategory of $\mathrm{Grpd}_{\mathcal{B}}$, Proposition 1.4.1.6 implies that the set of objects contained in the essential image of the associated inclusion $\int \mathsf{C} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{B})$ of cartesian fibrations over $\mathcal{B}$ (see Proposition 1.3.2.7) defines a local class. Clearly these operations are inverse to each other and order-preserving. Applying Proposition 1.4.1.6 once more, one moreover sees that this equivalence restricts to an equivalence between the poset of bounded local classes and the poset of small full subcategories of $\mathrm{Grpd}_{\mathcal{B}}$. □

Definition 1.4.1.8. A full subcategory $\mathsf{C} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ of the universe is said to be a *subuniverse* in $\mathcal{B}$.

Example 1.4.1.9. For every factorisation system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{B}$ (in the sense of Section 1.1.5), the right class $\mathcal{R}$ is local provided that the left class $\mathcal{L}$ is closed under pullbacks in $\mathcal{B}$ (see for example [2, Proposition 3.6.1]). Such a factorisation system is referred to as a *modality*. Hence any such modality gives rise to a subuniverse $\mathrm{Grpd}_{\mathcal{R}}$.

In the situation of Proposition 1.4.1.7, the case where S is a bounded local class deserves a more careful discussion. In this case, the sheaf $S_{/\_}$ is represented by the $\mathcal{B}$-category Grpd_S, hence $(S_{/\_})^{\simeq}$ is representable by the object $\mathrm{Grpd}_S^{\simeq} \in \mathcal{B}$ which by Yoneda's lemma implies that the full subcategory $S \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{B})$ admits a final object $\phi_S \colon (\mathrm{Grpd}_S)_*^{\simeq} \to \mathrm{Grpd}_S^{\simeq}$ that is referred to as the *universal morphism* in S. Hereafter, our goal is to reverse this discussion: Suppose that $p \colon P \to A$ is an arbitrary morphism in $\mathcal{B}$, and denote by $\langle p \rangle$ the class of morphisms in $\mathcal{B}$ that arise as a pullback of p. Since $\langle p \rangle$ is stable under pullbacks, the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{B})$ spanned by the maps in $\langle p \rangle$ defines a cartesian fibration over $\mathcal{B}$ and is therefore classified by a $\widehat{\mathrm{Cat}}_\infty$-valued presheaf $\langle p \rangle_{/\_}$ on $\mathcal{B}$. We would like to investigate the conditions that ensure $\langle p \rangle$ to be a bounded local class in $\mathcal{B}$, with p as a universal morphism.

Definition 1.4.1.10. A map $p \colon P \to A$ in $\mathcal{B}$ is *univalent* if $\langle p \rangle$ is a bounded local class.

Remark 1.4.1.11. Let S be a bounded local class of morphisms in $\mathcal{B}$ and let

$$\phi_S \colon (\mathrm{Grpd}_S)_*^{\simeq} \to \mathrm{Grpd}_S^{\simeq}$$

denote the associated universal morphism in S. Then a map in $\mathcal{B}$ arises as a pullback of ϕ_S if and only if it is contained in S, hence the map ϕ_S is univalent.

By Proposition 1.4.1.6 the notion of univalence admits the following equivalent characterisation:

Proposition 1.4.1.12. *For a map $p \colon P \to A$ in $\mathcal{B}$, the following conditions are equivalent:*

1. *p is univalent.*
2. *$\langle p \rangle_{/\_}$ is representable by a $\mathcal{B}$-category*
3. *$(\langle p \rangle_{/\_})^{\simeq}$ is representable by a $\mathcal{B}$-groupoid.* □

Lemma 1.4.1.13. *Let $\mathcal{B}$ be an ∞-topos and let $\mathcal{C}$ be a small ∞-category. A map $f \colon Y \to X$ in the ∞-topos* $\mathrm{Fun}(\mathcal{C}, \mathcal{B})$ *is a cover if and only if $f(c) \colon Y(c) \to X(c)$ is a cover for every $c \in \mathcal{C}$.*

Proof. For every $c \in \mathcal{C}$, evaluation at c defines an algebraic morphism

$$\mathrm{ev}_c \colon \mathrm{Fun}(\mathcal{C}, \mathcal{B}) \to \mathcal{B},$$

and since equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{B})$ are determined object-wise, the induced algebraic morphism $\mathrm{Fun}(\mathcal{C}, \mathcal{B}) \to \prod_{c \in \mathcal{C}} \mathcal{B}$ is conservative. Now f is a cover if and only if the inclusion $\mathrm{Im}(f) \hookrightarrow X$ is an equivalence. Since algebraic morphisms preserve the image factorisation of a map and since conservative functors reflect equivalences, the claim follows. □

Suppose that $p \colon P \to A$ is a map in $\mathcal{B}$, viewed as an object $p \colon A \to (\mathrm{Grpd}_{\mathcal{B}})_0$. By definition of $\langle p \rangle$ and the fact that a map in $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ is a cover if and only if it is object-wise given by a cover in $\widehat{\mathrm{Ani}}$ (cf. Lemma 1.4.1.13), the image factorisation of p in $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ is given by $A \twoheadrightarrow (\langle p \rangle_{/\_})^{\simeq} \hookrightarrow (\mathrm{Grpd}_{\mathcal{B}})_0$. Therefore p is univalent if and only if the cover $A \twoheadrightarrow (\langle p \rangle_{/\_})^{\simeq}$ is a monomorphism, which is the case if and only if $p \colon A \to (\mathrm{Grpd}_{\mathcal{B}})_0$ itself is a monomorphism. We therefore conclude:

Proposition 1.4.1.14 ([26, Proposition 3.8]). *A map $p\colon P \to A$ in $\mathcal{B}$ is univalent if and only if $p\colon A \to \mathrm{Grpd}_{\mathcal{B}}$ is a monomorphism.* □

Corollary 1.4.1.15. *Let $p\colon P \to A$ be a map in $\mathcal{B}$, viewed as an object in $\mathrm{Grpd}_{\mathcal{B}}$ in context A, and let $\mathrm{pr}_i\colon A \times A \to A$ be the projection onto the ith factor for $i \in \{0, 1\}$. Then p is univalent if and only if the canonical map $\phi\colon A \to \mathrm{eq}_{\mathrm{Grpd}_{\mathcal{B}}}(\mathrm{pr}_0^* p, \mathrm{pr}_1^* p)$ in $\mathcal{B}_{/A\times A}$ is an equivalence.*

Proof. By Proposition 1.4.1.14, the morphism p is univalent precisely if the map $p\colon A \to (\mathrm{Grpd}_{\mathcal{B}})_0$ is a monomorphism in $\widehat{\mathcal{B}}$, which is equivalent to the commutative square

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "(\mathrm{id},\mathrm{id})"] & (\mathrm{Grpd}_{\mathcal{B}})_0 \arrow[d, "(\mathrm{id},\mathrm{id})"] \\
A \times A \arrow[r, "p\times p"] & (\mathrm{Grpd}_{\mathcal{B}})_0 \times (\mathrm{Grpd}_{\mathcal{B}})_0
\end{tikzcd}$$

being cartesian. On account of the cartesian square

$$\begin{tikzcd}
\mathrm{eq}_{\mathrm{Grpd}_{\mathcal{B}}}(\mathrm{pr}_0^* p, \mathrm{pr}_1^* p) \arrow[r] \arrow[d] & (\mathrm{Grpd}_{\mathcal{B}})_0 \arrow[d, "(\mathrm{id},\mathrm{id})"] \\
A \times A \arrow[r, "p\times p"] & (\mathrm{Grpd}_{\mathcal{B}})_0 \times (\mathrm{Grpd}_{\mathcal{B}})_0,
\end{tikzcd}$$

we see that this is the case if and only if the map ϕ is an equivalence. □

The object of morphisms in the $\mathcal{B}$-category $\mathrm{Grpd}_{\langle p\rangle}$ that is associated with a univalent map $p\colon P \to A$ in $\mathcal{B}$ admits an explicit description as well:

Proposition 1.4.1.16. *Let $p\colon P \to A$ be a univalent morphism in $\mathcal{B}$ and let $\mathrm{Grpd}_{\langle p\rangle}$ be the associated $\mathcal{B}$-category. Then $(\mathrm{Grpd}_{\langle p\rangle})_1$ is equivalent to the internal hom $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A\times A}}(\mathrm{pr}_0^* P, \mathrm{pr}_1^* P)$ in $\mathcal{B}_{/A\times A}$.*

Proof. By construction there is a fully faithful functor $\mathrm{Grpd}_{\langle p\rangle} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$, which means that the square

$$\begin{tikzcd}
(\mathrm{Grpd}_{\langle p\rangle})_1 \arrow[r] \arrow[d] & (\mathrm{Grpd}_{\mathcal{B}})_1 \arrow[d] \\
(\mathrm{Grpd}_{\langle p\rangle})_0 \times (\mathrm{Grpd}_{\langle p\rangle})_0 \arrow[r] & (\mathrm{Grpd}_{\mathcal{B}})_0 \times (\mathrm{Grpd}_{\mathcal{B}})_0
\end{tikzcd}$$

is cartesian. On the other hand, Proposition 1.4.1.3 identifies the pullback of the above diagram with $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A\times A}}(\mathrm{pr}_0^* P, \mathrm{pr}_1^* P)$, which finishes the proof. $\square$

Remark 1.4.1.17. The theory of univalent maps in an ∞-topos has been previously worked out by Gepner and Kock in [26] and by Rasekh in [68], using slightly different methods.

1.4.2. The $\mathcal{B}$-category of $\mathcal{B}$-categories

The goal in this section is to define the large $\mathcal{B}$-category of $\mathcal{B}$-categories. What makes this possible is the following general construction:

Construction 1.4.2.1. Lurie's tensor product of presentable ∞-categories introduced in [50, § 4.8.1] defines a functor

$$-\otimes -\colon \mathrm{Pr}^{\mathrm{R}} \times \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Pr}^{\mathrm{R}}, \quad (\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Sh}_{\mathcal{D}}(\mathcal{C})$$

that preserves limits in each variable. Since the functor $\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ factors through the inclusion $\mathrm{Pr}^{\mathrm{R}} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ we may consider the composite

$$\mathrm{Pr}^{\mathrm{R}} \times \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times \mathcal{B}_{/-}} \mathrm{Pr}^{\mathrm{R}} \times \mathrm{Pr}^{\mathrm{R}} \xrightarrow{-\otimes -} \mathrm{Pr}^{\mathrm{R}} \to \widehat{\mathrm{Cat}}_\infty.$$

Its transpose defines a functor $\mathrm{Pr}^{\mathrm{R}} \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$. It follows from [49, Theorem 5.5.3.18] that this map factors through the full subcategory spanned by the limit-preserving functors and thus defines a functor

$$-\otimes \mathrm{Grpd}_{\mathcal{B}}\colon \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Cat}(\widehat{\mathcal{B}}).$$

By the explicit description of the tensor product in $\mathrm{Pr}^{\mathrm{R}}_\infty$, this functor is equivalently given by $\mathrm{Sh}_{-}(\mathcal{B}_{/-})$. In other words, given any presentable ∞-category $\mathcal{E}$, the associated large $\mathcal{B}$-category $\mathcal{E} \otimes \mathrm{Grpd}_{\mathcal{B}}$ is given by the composition

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}_{/-}} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \xrightarrow{\mathrm{Sh}_{\mathcal{E}}(\mathcal{B}_{/-})} \widehat{\mathrm{Cat}}_\infty.$$

Let us now consider the above construction in the special case $\mathcal{E} = \mathrm{Cat}_\infty$. By construction, $\mathrm{Cat}_\infty \otimes \mathrm{Grpd}_{\mathcal{B}}$ is given by the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}_{/-}} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \xrightarrow{\mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}_{/-})} \widehat{\mathrm{Cat}}_\infty$$

and thus agrees with the presheaf of ∞-categories $\mathrm{Cat}(\mathcal{B}_{/-})$ defined in Section 1.2.4 (see also the discussion in Section 1.2.6). In particular, it follows that the latter is indeed a sheaf. Therefore we feel inclined to make the following definition:

Definition 1.4.2.2. The *large $\mathcal{B}$-category* $\mathrm{Cat}_{\mathcal{B}}$ *of (small) $\mathcal{B}$-categories* is defined via the formula $\mathrm{Cat}_{\mathcal{B}} = \mathrm{Cat}_\infty \otimes \mathrm{Grpd}_{\mathcal{B}}$, i.e. as the large $\mathcal{B}$-category that corresponds to the sheaf $\mathrm{Cat}(\mathcal{B}_{/-})$.

Remark 1.4.2.3. There is a small subtlety in the definition of $\mathrm{Cat}_{\mathcal{B}}$: the claim that there is a *functorial* equivalence between $\mathrm{Cat}_\infty \otimes \mathrm{Grpd}_{\mathcal{B}}$ and the presheaf $\mathrm{Cat}(\mathcal{B}_{/-})$ is not as innocent as it may seem. In fact, the identification

$$\mathcal{D} \otimes \mathcal{E} \simeq \mathrm{Sh}_{\mathcal{E}}(\mathcal{D})$$

from [50, Proposition 4.8.1.17] (where $\mathcal{D}$ and $\mathcal{E}$ are presentable ∞-categories) is a priori only natural in $\mathcal{D}$ and $\mathcal{E}$ up to (possibly non-coherent) homotopy. To enhance this equivalence to one between bifunctors of ∞-categories, one in addition needs to know that passing from left adjoint to right adjoint constitutes an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C})^{\mathrm{op}}$$

(where the left-hand side denotes the ∞-category of left adjoint functors and the right-hand side the ∞-category of right adjoint functors) that is natural (in the fully coherent sense) in both variables. Tracing through the proof of [49, Proposition 5.2.6.3], this in turn follows once we know that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Cat}_\infty \arrow[r, "F"] \arrow[dr, "\mathrm{PSh}(-)"'] & \mathrm{Pr}^{\mathrm{L}}_\infty \arrow[d, "\simeq"] \\ & (\mathrm{Pr}^{\mathrm{R}}_\infty)^{\mathrm{op}} \end{tikzcd}$$

in which F is determined by the universal property of presheaves [49, Theorem 5.1.5.6], i.e. the partial left adjoint of the inclusion $\mathrm{Pr}^{\mathrm{L}}_\infty \hookrightarrow \widehat{\mathrm{Cat}}_\infty$. A priori, there is only such a commutative diagram once one passes to homotopy categories. That there also exists a fully coherent version of this diagram has only quite recently been shown by Haugseng-Linskens-Hebestreit-Nuiten [34] and by Ramzi [66], using different methods.

Remark 1.4.2.4. By definition of $\mathsf{Cat}_{\mathcal{B}}$, there is a canonical equivalence

$$\pi_A^* \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}$$

for every $A \in \mathcal{B}$. In fact, this follows from the computation

$$\pi_A^* \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}(\mathcal{B}_{/(\pi_A)_!(-)}) \simeq \mathsf{Cat}((\mathcal{B}_{/A})_{/-}) \simeq \mathsf{Cat}_{\mathcal{B}_{/A}},$$

see the discussion in Section 1.1.4 and in Section 1.2.6.

Remark 1.4.2.5. By applying $- \otimes \mathsf{Grpd}_{\mathcal{B}}$ to the equivalence of ∞-categories $(-)^{\mathrm{op}} \colon \mathsf{Cat}_\infty \simeq \mathsf{Cat}_\infty$, one obtains an equivalence $(-)^{\mathrm{op}} \colon \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}}$. On local sections over $A \in \mathcal{B}$, this equivalence recovers the map that carries a $\mathcal{B}_{/A}$-category to its opposite (cf. Remark 1.2.6.7).

Remark 1.4.2.6. By working internal to $\widehat{\mathcal{B}}$, we may define the (very large) $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ of large $\mathcal{B}$-categories. By regarding $\mathsf{Cat}_{\mathcal{B}}$ as a very large $\mathcal{B}$-category, we furthermore obtain a fully faithful functor $i \colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$. In fact, by the discussion in Section 1.2.4, the inclusion $\mathsf{Cat}(\mathcal{B}_{/A}) \hookrightarrow \mathsf{Cat}(\widehat{\mathcal{B}}_{/A})$ defines an embedding of presheaves $\mathsf{Cat}(\mathcal{B}_{/-}) \hookrightarrow \mathsf{Cat}(\widehat{\mathcal{B}}_{/-})$ on $\mathcal{B}$. Since moreover restriction along the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ defines an equivalence

$$\mathrm{Sh}_{\mathsf{Cat}_\infty^{\mathrm{W}}}(\widehat{\mathcal{B}}) \simeq \mathrm{Sh}_{\mathsf{Cat}_\infty^{\mathrm{W}}}(\mathcal{B})$$

(see Remark 1.1.4.1), we obtain the desired fully faithful functor $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ in $\mathsf{Cat}(\mathcal{B}^{\mathrm{W}})$. Explicitly, an object $A \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ in context $A \in \widehat{\mathcal{B}}$ that corresponds to a $\widehat{\mathcal{B}}_{/A}$-category $\mathsf{C} \to A$ is contained in $\mathsf{Cat}_{\mathcal{B}}$ precisely if for every map $s \colon A' \to A$ with $A' \in \mathcal{B}$ the pullback $s^*\mathsf{C}$ is small.

2. Presheaves

One of the most fundamental constructions in higher category theory is that of the ∞-category of *presheaves* on an ∞-category. To a large extent, its significance stems from Yoneda's lemma, which is an indispensable tool for the development of the theory of ∞-categories. In this chapter, our goal is to study the $\mathcal{B}$-categorical analogue of presheaves, and to derive a $\mathcal{B}$-categorical version of Yoneda's lemma. To that end, recall from Section 1.4.1 that the universe $\mathrm{Grpd}_{\mathcal{B}}$ is to be regarded as the $\mathcal{B}$-categorical analogue of the ∞-category Ani of ∞-groupoids. It therefore makes sense to define:

Definition 2.0.0.1. For any $\mathcal{B}$-category C, the (large) $\mathcal{B}$-category of *presheaves* on $\mathcal{B}$ is defined as $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}})$. Its underlying ∞-category of global sections will be denoted by $\mathrm{PSh}_{\mathcal{B}}(\mathsf{C})$.

Remark 2.0.0.2. Even when C is a large $\mathcal{B}$-category, we will continue to write $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ for the large functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}})$. By contrast, the (very large) functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\widehat{\mathcal{B}}})$ will be denoted by $\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$.

Remark 2.0.0.3. By combining Remark 1.4.1.2 with Proposition 1.2.5.4, one has a canonical equivalence $\pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C})$ for every $\mathcal{B}$-category C and every $A \in \mathcal{B}$.

The $\mathcal{B}$-categorical version of Yoneda's lemma will rely on the interplay between $\mathrm{Grpd}_{\mathcal{B}}$-valued functors on a $\mathcal{B}$-category C and *left fibrations* $p\colon \mathsf{P} \to \mathsf{C}$, a result that is commonly referred to as the *straightening and unstraightening*. The collection of left fibrations forms the right class of a factorisation system in $\mathrm{Cat}(\mathcal{B})$ whose left complement is comprised of *initial functors*. We discuss this factorisation system in Section 2.1, and in Section 2.2 we establish the straightening equivalence between such left fibration and functors into the universe $\mathrm{Grpd}_{\mathcal{B}}$. Finally, we use this machinery to state and prove Yoneda's lemma in Section 2.3.

Remark 2.0.0.4. Our strategy for the proof of Yoneda's lemma is inspired by Cisinski's proof of Yoneda's lemma for ∞-categories in [18].

2.1. Left fibrations and initial maps

The notion of a *left fibration* between simplicial sets, which is originally due to Joyal and thoroughly studied in [49, § 2.1], is a higher categorical generalisation of the classical notion of *categories fibred in groupoids*. The idea is straightforward: a map $p\colon P \to C$ of simplicial sets is a left fibration precisely if the lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "d^1"] & P \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & C \end{tikzcd}$$

admits a solution that is unique *up to coherent homotopy* [49, Corollary 2.1.2.10]. If p presents a map of ∞-categories, this simply means that p is *internally left orthogonal* (in the cartesian closed ∞-category Cat_∞) to the inclusion $d^1\colon \Delta^0 \hookrightarrow \Delta^1$. Reinterpreted in this way, the notion of a left fibration can be immediately generalised to $\mathcal{B}$-categories, and in fact more generally to simplicial objects in $\mathcal{B}$.

In Section 2.1.1, we will study the class of left fibrations between simplicial objects in $\mathcal{B}$. In Section 2.1.2, we focus on a subcollection of left fibrations that will be of particular importance later on: that of *slice $\mathcal{B}$-categories*. In Section 2.1.3 we focus on the class of maps that are *left orthogonal* to the collection of left fibrations: these are classically called *initial maps*, and we adopt the same terminology for $\mathcal{B}$-categories. Finally, in Section 2.1.4 we study the notion of a *covariant equivalence* between simplicial objects in $\mathcal{B}$: given a fixed simplicial object C, the collection of left fibrations over C sits reflectively inside $(\mathcal{B}_\Delta)_{/C}$, and a covariant equivalence is a map $P \to Q$ over C which is inverted by the localisation functor. We will provide a characterisation of covariant equivalences. which will, among other things, lead to a BB-categorical variant of Quillen's Theorem A.

2.1.1. Left fibrations

In this section we discuss left fibrations between $\mathcal{B}$-categories (and more generally between simplicial objects in $\mathcal{B}$) and discuss some of their basic properties.

Definition 2.1.1.1. A map $P \to C$ between simplicial objects in $\mathcal{B}$ is a *left fibration* if it is internally right orthogonal to the map $d^1 : \Delta^0 \hookrightarrow \Delta^1$. Dually, p is a *right fibration* if it is internally right orthogonal to the map $d^0 : \Delta^0 \hookrightarrow \Delta^1$. We denote by LFib and RFib (or $\mathrm{LFib}_{\mathcal{B}}$ and $\mathrm{RFib}_{\mathcal{B}}$ when we want to emphasise the dependency on the base ∞-topos) the full subcategories of $\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta)$ spanned by the left and right fibrations, respectively.

In what follows, we will mostly restrict the discussion to left fibrations. By dualising, however, all statements carry over unchanged to right fibrations. In more precise terms, this dualisation is obtained by taking opposite simplicial objects (as defined at the end of Section 1.2.1): Since the functor $(-)^{\mathrm{op}}$ sends the inclusion $d^1 : \Delta^0 \hookrightarrow \Delta^1$ to the map $d^0 : \Delta^0 \hookrightarrow \Delta^1$, one finds that the autoequivalence $(-)^{\mathrm{op}} : \mathcal{B}_\Delta \simeq \mathcal{B}_\Delta$ sends right fibrations to left fibrations and vice versa.

Lemma 2.1.1.2. *The saturated class of maps in $\mathcal{B}_\Delta$ that is generated by the maps $d^1 : E \hookrightarrow \Delta^1 \otimes E$ for any simplicial object E in $\mathcal{B}$ coincides with the saturation of the set*

$$\{d^{\{0\}} : A \hookrightarrow \Delta^n \otimes A \mid A \in \mathcal{B},\ n \geq 0\}.$$

Proof. Let S be the saturation of the set of maps $d^1 : E \hookrightarrow \Delta^1 \otimes E$ for $E \in \mathcal{B}_\Delta$. Then for any $A \in \mathcal{B}$ and any $n \geq 0$ the map $d^0 : (\Delta^0 \times \Delta^n) \otimes A \hookrightarrow (\Delta^1 \otimes \Delta^n) \otimes A$ is contained in S as well. Let $\alpha\colon \Delta^{n+1} \hookrightarrow \Delta^1 \times \Delta^n$ be defined by $\alpha(0) = (0,0)$ and $\alpha(k) = (1, k-1)$ for $1 \leq k \leq n$, and let $\beta\colon \Delta^1 \times \Delta^n \to \Delta^{n+1}$ be defined by $\beta(0,k) = 0$ and $\beta(1,k) = k+1$ for any $0 \leq k \leq n$. One then obtains a commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[d, "d^{\{0\}}"] & \Delta^n \arrow[r] \arrow[d, "d^1\times\mathrm{id}"] & \Delta^0 \arrow[d, "d^{\{0\}}"] \\
\Delta^{n+1} \arrow[r, "\alpha"] & \Delta^1 \times \Delta^n \arrow[r, "\beta"] & \Delta^{n+1},
\end{tikzcd}$$

and as $\beta\alpha \simeq \mathrm{id}_{\Delta^{n+1}}$, the map $d^{\{0\}} : \Delta^0 \hookrightarrow \Delta^{n+1}$ is a retract of $d^1 \times \mathrm{id} : \Delta^n \to \Delta^1 \times \Delta^n$. By tensoring with A, this shows that the map $d^{\{0\}} : A \hookrightarrow \Delta^n \otimes A$ is contained in S for all $n \geq 1$.

Conversely, let S be the saturation of the set of maps $d^{\{0\}} : A \hookrightarrow \Delta^n \otimes A$ for $n \geq 0$ and $A \in \mathcal{B}$, and let E be a simplicial object in $\mathcal{B}$. We need to show that the map $d^1\colon E \hookrightarrow \Delta^1 \otimes E$ is contained in S. By the same argument as in Lemma 1.2.3.5, we may assume without loss of generality $E \simeq \Delta^n \otimes A$ for some $n \geq 1$ and some $A \in \mathcal{B}$. Now with respect to the usual decomposition $\Delta^1 \times \Delta^n \simeq \Delta^{n+1} \sqcup_{\Delta^n} \cdots \sqcup_{\Delta^n} \Delta^{n+1}$ of the product $\Delta^1 \times \Delta^n$ into $n+1$ copies of Δ^{n+1}, the map $d^{\{0\}}\colon \Delta^n \hookrightarrow \Delta^1 \times \Delta^n$ is given by the iterated pushout

$$d^{\{0\}} \sqcup_{d^{\{0\}}} \cdots \sqcup_{d^{\{0,\dots,n-1\}}} d^{\{0,\dots,n\}}\colon \Delta^0 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^{n-1}} \Delta^n \to \Delta^{n+1} \sqcup_{\Delta^n} \cdots \sqcup_{\Delta^n} \Delta^{n+1}$$

in $\mathrm{Fun}(\Delta^1, \mathrm{Ani}_\Delta)$. It is therefore enough to show that for every integer $n \geq 1$ and every $0 \leq i \leq n$ the map $d^{\{0,\dots,i\}}\colon \Delta^i \otimes A \hookrightarrow \Delta^n \otimes A$ is contained in S, which follows immediately from the assumption by using item (2) of Proposition 1.1.5.2. $\square$

Proposition 2.1.1.3. *A map $P \to C$ between simplicial objects in $\mathcal{B}$ is a left fibration if and only if for every $n \geq 1$ the commutative diagram*

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, "d_{\{0\}}"] & C_n \arrow[d, "d_{\{0\}}"] \\ P_0 \arrow[r] & C_0 \end{tikzcd}$$

is cartesian.

Proof. This follows immediately from Lemma 2.1.1.2. $\square$

Remark 2.1.1.4. Proposition 2.1.1.3 implies that base change along both algebraic and geometric morphisms of ∞-topoi preserves left fibrations. Moreover, if $A \in \mathcal{B}$ is an object, the forgetful functor $(\pi_A)_!\colon (\mathcal{B}_{/A})_\Delta \to \mathcal{B}_\Delta$ commutes with pullbacks and therefore also preserves left fibrations.

Lemma 2.1.1.5. *Let S be the set of maps in $\mathcal{B}_\Delta$ that is internally generated by $d^1\colon \Delta^0 \hookrightarrow \Delta^1$. Then S contains the two maps $E^1 \to 1$ and $I^2 \hookrightarrow \Delta^2$. Dually, the set S' that is internally generated by $d^0\colon \Delta^0 \hookrightarrow \Delta^1$ contains the two maps $E^1 \to 1$ and $I^2 \hookrightarrow \Delta^2$.*

Proof. We show the statement for the set S, the proof for the dual case is analogous. Since d^1 is a section of the unique map $\Delta^1 \to \Delta^0$, the latter is contained in S, hence $\Delta^3 \to E^1$ is contained in S as well since saturated classes of maps are stable under colimits in $\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta)$ and pushouts in $\mathcal{B}_\Delta$. Since by Lemma 2.1.1.2 the map

$d^{\{0\}}\colon \Delta^0 \hookrightarrow \Delta^3$ defines an element of S, we find that the composition $\Delta^0 \to E^1$ is contained in S. As this is a section of the map $E^1 \to 1$, we conclude that the latter map is contained in S as well.

The inclusion $\Delta^1 \hookrightarrow I^2$ of the first copy of Δ^1 in I^2 is a pushout of $d^1\colon \Delta^0 \hookrightarrow \Delta^1$ and therefore an element of S. By precomposing with d^1, we thus obtain a map $\Delta^0 \hookrightarrow I^2$ in S such that its postcomposition with the inclusion $I^2 \hookrightarrow \Delta^2$ recovers $d^{\{0\}}\colon \Delta^0 \hookrightarrow \Delta^2$. Since Lemma 2.1.1.2 shows that this map is an element of S as well, we conclude that the inclusion $I^2 \hookrightarrow \Delta^2$ must be contained in S too, which finishes the proof. $\square$

Proposition 2.1.1.6. *Let $p\colon P \to \mathsf{C}$ be a left fibration in $\mathcal{B}_\Delta$ such that C is a $\mathcal{B}$-category. Then P is a $\mathcal{B}$-category as well.*

Proof. By Lemma 2.1.1.5, the map p is internally right orthogonal to the two maps $E^1 \to 1$ and $I^2 \hookrightarrow \Delta^2$. Since C is internally local with respect to these maps, we conclude that P is internally local with respect to the two maps as well and therefore a $\mathcal{B}$-category, as claimed. $\square$

Remark 2.1.1.7. By Proposition 1.2.3.12, a functor $p\colon \mathsf{P} \to \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})$ is a left fibration precisely if it is internally right orthogonal to the map d^1 *in* $\mathrm{Cat}(\mathcal{B})$. Therefore, Proposition 2.1.1.6 implies that the pullback of the cartesian fibration $\mathrm{LFib} \to \mathcal{B}_\Delta$ along the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ is given by the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B}))$ that is spanned by the left fibrations between $\mathcal{B}$-categories. We will denote the resulting cartesian fibration over $\mathrm{Cat}(\mathcal{B})$ by LFib as well. Note, moreover, that also the localisation functor $\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B})) \to \mathrm{LFib}$ arises as the restriction of the localisation functor $\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta) \to \mathrm{LFib}$.

Next, we show that left fibrations between $\mathcal{B}$-categories are *conservative* (in the sense of Definition 1.3.3.1) and therefore in particular *fibred in $\mathcal{B}$-groupoids*. To see this, note that since $d^0\colon \Delta^0 \hookrightarrow \Delta^1$ is a section of the unique map $\Delta^1 \to \Delta^0$, one finds:

Lemma 2.1.1.8. *Let S be a saturated class of maps in $\mathcal{B}_\Delta$ that contains the maps $d^0\colon K \hookrightarrow \Delta^1 \otimes K$ for all $K \in \mathcal{B}_\Delta$. Then S contains the class of maps in $\mathcal{B}_\Delta$ that is internally generated by $\Delta^1 \to \Delta^0$.* $\square$

Lemma 2.1.1.8 now shows:

Proposition 2.1.1.9. *Both left and right fibrations between simplicial objects in $\mathcal{B}$ are internally right orthogonal to $\Delta^1 \to \Delta^0$. In particular, left and right fibrations between $\mathcal{B}$-categories are conservative.* □

By using Proposition 2.1.1.9 and Corollary 1.3.3.6, one furthermore concludes:

Corollary 2.1.1.10. *The fibre of a left or right fibration in* $\mathrm{Cat}(\mathcal{B})$ *over any object in the codomain in context $A \in \mathcal{B}$ is a $\mathcal{B}_{/A}$-groupoid.* □

Remark 2.1.1.11. By Proposition 2.1.1.9 and Proposition 2.1.1.3, a map $C \to A$ in $\mathcal{B}_\Delta$ in which A is contained in $\mathcal{B}$ is a left or right fibration precisely if C is contained in $\mathcal{B}$ as well. Therefore, both localisation functors $\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta) \to \mathrm{RFib}$ and $\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta) \to \mathrm{LFib}$ recover the functor $\mathrm{colim}_{\Delta^{\mathrm{op}}} : \mathcal{B}_\Delta \to \mathcal{B}$ upon taking the fibre over the final object $1 \in \mathcal{B}_\Delta$. By restriction, the localisation functors $\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B})) \to \mathrm{RFib}$ and $\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B})) \to \mathrm{LFib}$ thus both induce the groupoidification functor on the fibres over $1 \in \mathrm{Cat}(\mathcal{B})$.

We conclude this section by showing that equivalences between left or right fibrations can be detected fibre-wise:

Proposition 2.1.1.12. *A map $f\colon P \to Q$ between left fibrations over a simplicial object C in $\widehat{\mathcal{B}}$ is an equivalence if and only if for every object $A \in \mathcal{B}$ and every map $c\colon A \to C$ the induced map $c^*P \to c^*Q$ is an equivalence in $\widehat{\mathcal{B}}_{/A}$. In particular, a map between left fibrations of large $\mathcal{B}$-categories is an equivalence if and only if it induces an equivalence on the fibres over every object in the base $\mathcal{B}$-category.*

Proof. By item (2) of Proposition 1.1.5.2, the map f is a left fibration itself. Therefore f is an equivalence whenever the underlying map $f_0 : P_0 \to Q_0$ is one. The claim now follows from descent together with the fact that C_0 is canonically obtained as the colimit $\mathrm{colim}_{A \to C_0} A$. □

2.1.2. Slice $\mathcal{B}$-categories

In this section we will discuss one particularly important example of left fibrations between $\mathcal{B}$-categories - that of *slice* $\mathcal{B}$*-categories*.

Definition 2.1.2.1. Let $f\colon \mathsf{D} \to \mathsf{C}$ and $g\colon \mathsf{E} \to \mathsf{C}$ be two functors between $\mathcal{B}$-categories. The *comma $\mathcal{B}$-category* $\mathsf{D} \downarrow_{\mathsf{C}} \mathsf{E}$ is defined as the pullback

$$\begin{tikzcd} \mathsf{D} \downarrow_{\mathsf{C}} \mathsf{E} \arrow[r] \arrow[d] & \mathsf{C}^{\Delta^1} \arrow[d, "{(d_1,d_0)}"] \\ \mathsf{D} \times \mathsf{E} \arrow[r, "f \times g"] & \mathsf{C} \times \mathsf{C}. \end{tikzcd}$$

If g is given by an object $c\colon A \to \mathsf{C}$, we write $\mathsf{D}_{/c} = \mathsf{D} \downarrow_{\mathsf{C}} A$, and if in addition f is the identity on C we refer to this $\mathcal{B}$-category as the *slice $\mathcal{B}$-category* over c. Dually if f is given by an object $c\colon A \to \mathsf{C}$ we write $\mathsf{D}_{c/} = A \downarrow_{\mathsf{C}} \mathsf{D}$ and refer to this $\mathcal{B}$-category as the slice $\mathcal{B}$-category under c when furthermore g is the identity on C.

In the situation of Definition 2.1.2.1, the slice $\mathcal{B}$-category $\mathsf{C}_{c/}$ comes along with a canonical map to $A \times \mathsf{C}$ which we will denote by $(\pi_c)_! \colon \mathsf{C}_{c/} \to A \times \mathsf{C}$. Furthermore, note that the identity $\mathrm{id}_c \colon A \to \mathsf{C}^{\Delta^1}$ induces a map $A \to \mathsf{C}_{c/}$ over $(\mathrm{id}_A, c)\colon A \to A \times \mathsf{C}$ that we will denote by id_c as well.

Remark 2.1.2.2. Let C be a $\mathcal{B}$-category and let $c\colon A \to \mathsf{C}$ be an object in C, which can be equivalently regarded as an object $\bar{c}\colon 1 \to \pi_A^* \mathsf{C}$ (see Remark 1.2.7.1). Using Proposition 1.2.4.6 and Remark 1.2.5.6, the map $(\pi_c)_! \colon \mathsf{C}_{c/} \to A \times \mathsf{C}$ is equivalent to the image of the projection $(\pi_c)_! \colon (\pi_A^* \mathsf{C})_{c/} \to \pi_A^* \mathsf{C}$ along the forgetful functor $(\pi_A)_! \colon \mathrm{Cat}(\mathcal{B}_{/A}) \to \mathrm{Cat}(\mathcal{B})$. In other words, when viewed as a $\mathcal{B}_{/A}$-category, we can identify $\mathsf{C}_{c/}$ with the slice of $\bar{c}\colon 1 \to \pi_A^* \mathsf{C}$, which allows us to restrict our attention to slices under (or over) objects that are defined in *global* context.

Hereafter, our goal is to prove that the projection $(\pi_c)_! \colon \mathsf{C}_{c/} \to A \times \mathsf{C}$ is a left fibration for any $\mathcal{B}$-category C and any object c in C in context $A \in \mathcal{B}$. We will achieve this by identifying $(\pi_c)_!$ as a pullback of the *twisted arrow $\mathcal{B}$-category* on C. To that end, let $- \star - \colon \Delta \times \Delta \to \Delta$ be the ordinal sum bifunctor. We may then define:

Definition 2.1.2.3. Let $\epsilon\colon \Delta \to \Delta$ denote the functor $\langle n \rangle \mapsto \langle n \rangle^{\mathrm{op}} \star \langle n \rangle$. For any $\mathcal{B}$-category C, we define the *twisted arrow $\mathcal{B}$-category* $\mathrm{Tw}(\mathsf{C})$ to be the simplicial object given by the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\epsilon^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{\mathsf{C}} \mathcal{B}.$$

This defines a functor $\mathsf{Tw}\colon \mathsf{Cat}(\mathcal{B}) \to \mathcal{B}_\Delta$.

Note that the functor ϵ in Definition 2.1.2.3 comes along with two canonical natural transformations

$$(-)^{\mathrm{op}} \to \epsilon \leftarrow \mathrm{id}_\Delta$$

which induces a map of simplicial objects

$$\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$$

that is natural in C.

Proposition 2.1.2.4. *For any $\mathcal{B}$-category C, the simplicial object $\mathsf{Tw}(\mathsf{C})$ is a $\mathcal{B}$-category as well, and the map $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is a left fibration.*

Proof. We will begin by showing that for any $n \geq 1$ the square

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{C})_n \arrow[r] \arrow[d, "d_{\{0\}}"] & \mathsf{C}_n^{\mathrm{op}} \times \mathsf{C}_n^{\mathrm{op}} \arrow[d, "d_{\{0\}}"] \\
\mathsf{Tw}(\mathsf{C})_0 \arrow[r] & \mathsf{C}_0^{\mathrm{op}} \times \mathsf{C}_0
\end{tikzcd}$$

is a pullback diagram. Unwinding the definitions, this is equivalent to the diagram

$$\begin{tikzcd}
 & & \mathsf{C}_{2n+1} \arrow[dll, "d_{\{0,\dots,n\}}"'] \arrow[d, "d_{\{n,n+1\}}"] \arrow[drr, "d_{\{n+1,\dots,2n+1\}}"] & & \\
\mathsf{C}_n \arrow[dr, "d_{\{n\}}"'] & & \mathsf{C}_1 \arrow[dl, "d_{\{0\}}"] \arrow[dr, "d_{\{1\}}"'] & & \mathsf{C}_n \arrow[dl, "d_{\{0\}}"] \\
 & \mathsf{C}_0 & & \mathsf{C}_0 &
\end{tikzcd}$$

being a limit diagram, which follows easily from the Segal conditions. By Proposition 2.1.1.3, the map $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is therefore a left fibration. Since the codomain of this map defines a $\mathcal{B}$-category, Proposition 2.1.1.6 now implies that $\mathsf{Tw}(\mathsf{C})$ is a $\mathcal{B}$-category as well. $\square$

Remark 2.1.2.5. For every $\mathcal{B}$-category C, note that Remark 1.2.5.6 and the definition of the twisted arrow $\mathcal{B}$-category immediately imply that the twisted arrow construction is preserved by base change along geometric morphisms and along *étale* algebraic morphisms of ∞-topoi.

We proceed with our goal of exhibiting the slice projection $(\pi_c)_! \colon \mathsf{C}_{c/} \to A \times \mathsf{C}$ as a pullback of the left fibration $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$. To that end, note that the ordinal sum functor $\star \colon \Delta \times \Delta \to \Delta$ fits into the commutative square

$$\begin{tikzcd} \Delta \times \Delta \arrow[r, "\star"] \arrow[d, hook] & \Delta \arrow[d, hook] \\ \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty \arrow[r, "\diamond"] & \mathrm{Cat}_\infty \end{tikzcd}$$

in which $- \diamond -$ denotes the bifunctor that sends a pair $(\mathcal{C}, \mathcal{D})$ of ∞-categories to the pushout

$$\begin{tikzcd} (\mathcal{C} \times \mathcal{D}) \sqcup (\mathcal{C} \times \mathcal{D}) \arrow[r, "{(d^1, d^0)}"] \arrow[d, "\mathrm{pr}_0 \sqcup \mathrm{pr}_1"] & \mathcal{C} \times \mathcal{D} \times \Delta^1 \arrow[d] \\ \mathcal{C} \sqcup \mathcal{D} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \mathcal{C} \diamond \mathcal{D}. \end{tikzcd}$$

In fact, the inclusions $\langle m \rangle \hookrightarrow \langle m \rangle \star \langle n \rangle \hookleftarrow \langle n \rangle$ in Δ induce a map $\Delta^m \sqcup \Delta^n \to \Delta^{m \star n}$ that is natural in m and n, and we may also define a map $\Delta^n \times \Delta^m \times \Delta^1 \to \Delta^{n \star m}$ of 1-categories naturally in m and n by sending a triple (i, j, k) to i if $k = 0$ and to $m + j$ otherwise. This construction gives rise to a natural map $\Delta^m \diamond \Delta^n \to \Delta^{m \star n}$ that is an equivalence by [49, Proposition 4.2.1.2]. Combining this observation with Proposition 1.2.1.4, we therefore conclude that for any $\mathcal{B}$-category the underlying simplicial object of $\mathsf{Tw}(\mathsf{C})$ is obtained by applying the core functor to the simplicial object $\mathsf{C}^{(\Delta^\bullet)^{\mathrm{op}} \diamond \Delta^\bullet}$ in $\mathsf{Cat}(\mathcal{B})$.

Lemma 2.1.2.6. *For any integer $n \geq 0$, the canonical square*

$$\begin{tikzcd} (\Delta^n)^{\mathrm{op}} \sqcup \Delta^n \arrow[r] \arrow[d] & (\Delta^n)^{\mathrm{op}} \diamond \Delta^n \arrow[d] \\ \Delta^0 \sqcup \Delta^n \arrow[r] & \Delta^0 \diamond \Delta^n \end{tikzcd}$$

is a pushout in Cat_∞.

Proof. By definition of the bifunctor $\diamond$ and the pasting lemma for pushout squares, the commutative square in the statement of the lemma is a pushout if and only if the square

$$\begin{tikzcd} ((\Delta^n)^{\mathrm{op}} \times \Delta^n) \sqcup ((\Delta^n)^{\mathrm{op}} \times \Delta^n) \arrow[r, "{(d^1, d^0)}"] \arrow[d, "\mathrm{pr}_1 \sqcup \mathrm{pr}_1"] & (\Delta^n)^{\mathrm{op}} \times \Delta^n \times \Delta^1 \arrow[d, "\mathrm{pr}_1 \times \mathrm{id}_{\Delta^1}"] \\ \Delta^n \sqcup \Delta^n \arrow[r, "{(d^1, d^0)}"] & \Delta^n \times \Delta^1 \end{tikzcd}$$

is cocartesian. As the functor $- \times \Delta^n$ preserves colimits, we may assume $\Delta^n = \Delta^0$. Moreover, in light of the decomposition $\Delta^n \simeq \Delta^1 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^0} \Delta^1$ in Cat_∞, we may assume $(\Delta^n)^{\mathrm{op}} = (\Delta^1)^{\mathrm{op}}$. We now have to show that the commutative square

$$\begin{tikzcd}
(\Delta^1)^{\mathrm{op}} \sqcup (\Delta^1)^{\mathrm{op}} \arrow[r, "{(d^1,d^0)}"] \arrow[d] & (\Delta^1)^{\mathrm{op}} \times \Delta^1 \arrow[d] \\
\Delta^0 \sqcup \Delta^0 \arrow[r, "{(d^1,d^0)}"] & \Delta^1
\end{tikzcd}$$

is a pushout, which is easily shown by making use of the equivalence

$$(\Delta^1)^{\mathrm{op}} \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$$

and the fact that the diagram

$$\begin{tikzcd}
\Delta^1 \arrow[r, hook, "d^0"] \arrow[d] & \Delta^2 \arrow[d] \\
\Delta^0 \arrow[r, hook, "d^0"] & \Delta^1
\end{tikzcd}$$

is cocartesian in Cat_∞. $\square$

By making use of Lemma 2.1.2.6, one now obtains a cartesian square

$$\begin{tikzcd}
(\mathsf{C}^{\Delta^0 \diamond \Delta^\bullet})_0 \arrow[r] \arrow[d] & \mathsf{Tw}(\mathsf{C}) \arrow[d] \\
\mathsf{C}_0 \times \mathsf{C} \arrow[r] & \mathsf{C}^{\mathrm{op}} \times \mathsf{C}.
\end{tikzcd}$$

On the other hand, the defining pushout for $\Delta^0 \diamond \Delta^\bullet$ gives rise to a cartesian square

$$\begin{tikzcd}
(\mathsf{C}^{(\Delta^0)^{\mathrm{op}} \diamond \Delta^\bullet})_0 \arrow[r] \arrow[d] & \mathsf{C}^{\Delta^1} \arrow[d] \\
\mathsf{C}_0 \times \mathsf{C} \arrow[r] & \mathsf{C} \times \mathsf{C},
\end{tikzcd}$$

which in particular shows:

Proposition 2.1.2.7. *Let* C *be a* $\mathcal{B}$*-category. For any object* $c\colon A \to \mathsf{C}$*, the canonical map* $(\pi_c)_! \colon \mathsf{C}_{c/} \to A \times \mathsf{C}$ *fits into a cartesian square*

$$\begin{tikzcd} \mathsf{C}_{c/} \arrow[r] \arrow[d] & \mathsf{Tw}(\mathsf{C}) \arrow[d] \\ A \times \mathsf{C} \arrow[r, "c\times \mathrm{id}"] & \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$*. In particular,* $(\pi_c)_!$ *is a left fibration.* □

2.1.3. Initial functors

We will now focus on the left complement of the class of left fibrations. The results in this section are heavily inspired by Cisinski's book [18].

Definition 2.1.3.1. A map $J \to I$ between simplicial objects in $\mathcal{B}$ is said to be *initial* if it is left orthogonal to every left fibration in $\mathcal{B}_\Delta$. Dually, $J \to I$ is *final* if it is left orthogonal to every right fibration in $\mathcal{B}_\Delta$.

Remark 2.1.3.2. A map $J \to I$ between simplicial objects in $\mathcal{B}$ is initial if and only if its opposite $J^{\mathrm{op}} \to I^{\mathrm{op}}$ is final. Therefore all properties of initial maps carry over to final maps upon taking opposite simplicial objects. We will therefore restrict our attention to the case of initial maps.

Remark 2.1.3.3. By Remark 2.1.1.4, base change along every algebraic morphism preserves initial maps. Moreover, if $A \in \mathcal{B}$ is an arbitrary object, then the forgetful functor $(\pi_A)_! \colon (\mathcal{B}_{/A})_\Delta \to \mathcal{B}_\Delta$ preserves initial maps as well. It even *creates* initial maps since every map in $(\mathcal{B}_{/A})_\Delta$ arises as a pullback of a map that is in the image of $\pi_A^* \colon \mathcal{B}_\Delta \to (\mathcal{B}_{/A})_\Delta$.

Remark 2.1.3.4. A functor $\mathsf{J} \to \mathsf{I}$ between $\mathcal{B}$-categories is initial (final) in $\mathcal{B}_\Delta$ precisely if it is internally left orthogonal to left (right) fibrations in $\mathrm{Cat}(\mathcal{B})$. This is easily seen as a consequence of Proposition 2.1.1.6.

Example 2.1.3.5. By Lemma 2.1.1.8, any map between simplicial objects in $\mathcal{B}$ that is in the internal saturation of the map $\Delta^1 \to \Delta^0$ defines both an initial and a final map. In particular, all iterated localisations are both initial and final.

Definition 2.1.3.6. Let C be a $\mathcal{B}$-category. An object $c\colon A \to \mathsf{C}$ is said to be *initial* if the transpose map $1 \to \pi_A^* \mathsf{C}$ defines an initial functor in $\mathrm{Cat}(\mathcal{B}_{/A})$. Dually, c is *final* if the transpose map $1 \to \pi_A^* \mathsf{C}$ defines a final functor in $\mathrm{Cat}(\mathcal{B}_{/A})$.

Remark 2.1.3.7. In Corollary 2.1.3.16 we will see that initial and final objects satisfy the expected universal property, which in particular implies that an object in an ordinary ∞-category is initial or final in the sense of Definition 2.1.3.6 precisely if it is initial or final in the usual sense.

Warning 2.1.3.8. Note that an object $c\colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C being initial is different from the condition that c is initial when viewed as a functor in $\mathrm{Cat}(\mathcal{B})$. In fact, by Remark 2.1.3.3 the first condition is equivalent to $(c, \mathrm{id})\colon A \to A \times \mathsf{C}$ being initial as a functor in $\mathrm{Cat}(\mathcal{B})$, hence either of the two conditions implying the other would imply that the projection $A \times \mathsf{C} \to \mathsf{C}$ is initial as well, which is not true in general. By contrast, if C is the underlying $\mathcal{B}$-category of a $\mathcal{B}_{/A}$-category and if $c\colon A \to \mathsf{C}$ is a *section* of the structure map $\mathsf{C} \to A$, then c defines an initial object of the $\mathcal{B}_{/A}$-category C in *global* context $1_{\mathcal{B}_{/A}}$ if and only if the map $c\colon A \to \mathsf{C}$ is initial in $\mathrm{Cat}(\mathcal{B})$.

Any functor between $\mathcal{B}$-categories admits a unique factorisation into an initial map followed by a left fibration. In what follows, our goal is to describe this factorisation explicitly for the case where the domain is the final object $1 \in \mathcal{B}$, i.e. encodes an object in global context. To that end, recall that to any $\mathcal{B}$-category C and any object $c\colon 1 \to \mathsf{C}$ one can associate the slice $\mathcal{B}$-category $\mathsf{C}_{c/} \to \mathsf{C}$ such that the identity map on c defines a lift $\mathrm{id}_c\colon 1 \to \mathsf{C}_{c/}$ of $c\colon 1 \to \mathsf{C}$.

Proposition 2.1.3.9. *For any $\mathcal{B}$-category and any object c in C in context $1 \in \mathcal{B}$, the object $\mathrm{id}_c\colon 1 \to \mathsf{C}_{c/}$ is initial.*

Remark 2.1.3.10. By combining Remark 2.1.2.2 with Remark 2.1.3.3, Proposition 2.1.3.9 furthermore implies that if $c\colon A \to \mathsf{C}$ is an object in arbitrary context $A \in \mathcal{B}$, then the induced section $\mathrm{id}_c\colon A \to \mathsf{C}_{c/}$ over A is an initial map in $\mathrm{Cat}(\mathcal{B})$.

In order to show Proposition 2.1.3.9, we will need a convenient criterion how to detect initial objects. This will make use of the notion of *cocomma $\mathcal{B}$-categories*:

Definition 2.1.3.11. Let $f\colon \mathsf{C} \to \mathsf{D}$ and $g\colon \mathsf{C} \to \mathsf{E}$ be functors in $\mathrm{Cat}(\mathcal{B})$. The *cocomma* $\mathcal{B}$*-category* $\mathsf{D} \diamond_{\mathsf{C}} \mathsf{E}$ is the $\mathcal{B}$-category that is defined by the pushout square

$$\begin{tikzcd} \mathsf{C} \sqcup \mathsf{C} \arrow[r, "(d^1,d^0)"] \arrow[d, "f \sqcup g"] & \Delta^1 \otimes \mathsf{C} \arrow[d] \\ \mathsf{D} \sqcup \mathsf{E} \arrow[r] & \mathsf{D} \diamond_{\mathsf{C}} \mathsf{E}. \end{tikzcd}$$

If $A \in \mathcal{B}$ is an arbitrary object and $\mathsf{C} \to A$ is a map (i.e. if C is a $\mathcal{B}_{/A}$-category), we write $\mathsf{C}^{\triangleright} = \mathsf{C} \diamond_{\mathsf{C}} A$ and refer to this $\mathcal{B}$-category as the *right cone of* $\mathsf{C} \to A$. Dually, we write $\mathsf{C}^{\triangleleft} = A \diamond_{\mathsf{C}} \mathsf{C}$ and refer to this $\mathcal{B}$-category as the *left cone of* $\mathsf{C} \to A$.

Note that if if C is a $\mathcal{B}$-category and $c\colon 1 \to \mathsf{C}$ is an arbitrary object, then there is always a canonical map $\infty \to c$ in $\mathsf{C}^{\triangleleft}$, where $\infty\colon 1 \to \mathsf{C}^{\triangleleft}$ denotes the cone point. In fact, this map can be defined as the image of the map $\mathrm{id} \otimes c\colon \Delta^1 \to \Delta^1 \otimes \mathsf{C}$ along the map $\Delta^1 \otimes \mathsf{C} \to \mathsf{C}^{\triangleleft}$. We are now ready to state and prove the desired criterion for initiality:

Lemma 2.1.3.12. *Let* C *be a* $\mathcal{B}$*-category and let* $c\colon 1 \to \mathsf{C}$ *be an object. Then the following are equivalent:*

1. *the canonical inclusion* $\mathsf{C} \to \mathsf{C}^{\triangleleft}$ *admits a retraction* $r\colon \mathsf{C}^{\triangleleft} \to \mathsf{C}$ *that carries the canonical map* $\infty \to c$ *in* $\mathsf{C}^{\triangleleft}$ *to an equivalence in* C*;*
2. *there is a map* $\epsilon\colon \Delta^1 \otimes \mathsf{C} \to \mathsf{C}$ *such that*
   a) *the composite* $\epsilon d^1\colon \mathsf{C} \to \mathsf{C}$ *is equivalent to the constant map with value* c*, i.e. to the composite* $\mathsf{C} \to 1 \to \mathsf{C}$ *in which the second arrow is given by* c*;*
   b) *the composite* $\epsilon d^0\colon \mathsf{C} \to \mathsf{C}$ *is equivalent to the identity;*
   c) *the map* $\epsilon \circ (\mathrm{id} \otimes c)\colon \Delta^1 \to \mathsf{C}$ *is an equivalence.*

Moreover, if either of these conditions is satisfied, then c *is initial.*

Proof. If (1) holds, then one can define ϵ as the composition of r with the canonical map $\Delta^1 \otimes \mathsf{C} \to \mathsf{C}^{\triangleleft}$. The conditions on r then immediately imply that (2) holds. Conversely, if (2) holds, the universal property of pushouts gives rise to a map

$r\colon \mathsf{C}^{\triangleleft} \to \mathsf{C}$ which carries the map $\infty \to c$ to an equivalence. Lastly, if (1) holds, then the map r gives rise to a commutative diagram

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}"] \arrow[d, hook, "d^1"] & 1 \arrow[r, "\mathrm{id}"] \arrow[d] & 1 \arrow[d, "c"] \\
\Delta^1 \arrow[r] \arrow[rr, bend left, "\mathrm{id}_c"] & \mathsf{C}^{\triangleleft} \arrow[r, "r"] & \mathsf{C} \\
1 \arrow[r, "c"] \arrow[u, "d^0"] & \mathsf{C} \arrow[r, "\mathrm{id}"] \arrow[u] & \mathsf{C} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

in $\mathsf{Cat}(\mathcal{B})$. As a consequence, the map $c\colon 1 \to \mathsf{C}$ is seen to be a retract of the map $\Delta^1 \to \mathsf{C}^{\triangleleft}$, hence it suffices to show that the latter is initial. Since both d^1 and $\infty\colon 1 \to \mathsf{C}^{\triangleleft}$ are initial, this follows immediately from item (2) in Proposition 1.1.5.2. □

Proof of Proposition 2.1.3.9. We would like to apply Lemma 2.1.3.12 to the pair $(\mathsf{C}_{c/}, \mathrm{id}_c)$. To that end, let $d\colon \Delta^1 \times \Delta^1 \to \Delta^1$ be the projection onto the diagonal that is given by composing the equivalence $\Delta^1 \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$ with $s^0 \sqcup_{\Delta^1} s^0 : \Delta^2 \sqcup_{\Delta^1} \Delta^2 \to \Delta^1$. Then the map $d^* : \mathsf{C}^{\Delta^1} \to \mathsf{C}^{\Delta^1 \times \Delta^1}$ fits into the two commutative squares

$$\begin{tikzcd}
\mathsf{C}^{\Delta^1} \arrow[r, "d^*"] \arrow[d, "d_1"] & \mathsf{C}^{\Delta^1 \times \Delta^1} \arrow[d, "(\mathrm{id} \times d^1)^*"] \\
\mathsf{C} \arrow[r, "s_0"] & \mathsf{C}^{\Delta^1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C}^{\Delta^1} \arrow[r, "d^*"] \arrow[d, "d_1"] & \mathsf{C}^{\Delta^1 \times \Delta^1} \arrow[d, "(d^1 \times \mathrm{id})^*"] \\
\mathsf{C} \arrow[r, "s_0"] & \mathsf{C}^{\Delta^1}.
\end{tikzcd}$$

Transposing d^* along the adjunction $\Delta^1 \otimes - \dashv (-)^{\Delta^1}$ thus determines a map $e\colon \Delta^1 \otimes \mathsf{C}^{\Delta^1} \to \mathsf{C}^{\Delta^1}$ together with two commutative squares

$$\begin{tikzcd}
\Delta^1 \otimes \mathsf{C}^{\Delta^1} \arrow[r, "e"] \arrow[d, "\mathrm{id} \otimes d_1"] & \mathsf{C}^{\Delta^1} \arrow[d, "d_1"] \\
\Delta^1 \otimes \mathsf{C} \arrow[r, "s^0 \otimes \mathrm{id}"] & \mathsf{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C}^{\Delta^1} \arrow[r, "d^1 \otimes \mathrm{id}"] \arrow[d, "d_1"] & \Delta^1 \otimes \mathsf{C}^{\Delta^1} \arrow[d, "e"] \\
\mathsf{C} \arrow[r, "s_0"] & \mathsf{C}^{\Delta^1}.
\end{tikzcd}$$

By pasting the right square with the pullback diagram that defines the slice $\mathcal{B}$-category $\mathsf{C}_{c/}$, we obtain a map $h\colon (\mathsf{C}_{c/})^{\triangleleft} \to \mathsf{C}^{\Delta^1}$ that fits into the commutative diagram

$$\begin{tikzcd}
 & \mathsf{C}_{c/} \arrow[rr, "d^1 \otimes \mathrm{id}"] \arrow[dl] \arrow[dd] & & \Delta^1 \otimes \mathsf{C}_{c/} \arrow[rr, "\mathrm{id} \otimes \pi_{\mathsf{C}_{c/}}"] \arrow[dl] \arrow[dd] & & \Delta^1 \arrow[dl, "\mathrm{id} \otimes c"] \arrow[dd] \\
\mathsf{C}^{\Delta^1} \arrow[rr, "d^1 \otimes \mathrm{id}"] \arrow[dd, "d_1"] & & \Delta^1 \otimes \mathsf{C}^{\Delta^1} \arrow[rr, "\mathrm{id} \otimes d_1"] \arrow[dd, "e"] & & \Delta^1 \otimes \mathsf{C} \arrow[dd, "s^0 \otimes \mathrm{id}"] & \\
 & 1 \arrow[rr] \arrow[dl, "c"] & & (\mathsf{C}_{c/})^{\triangleleft} \arrow[rr] \arrow[dl, "h"] & & 1 \arrow[dl, "c"] \\
\mathsf{C} \arrow[rr, "s_0"] & & \mathsf{C}^{\Delta^1}. \arrow[rr, "d_1"] & & \mathsf{C}. &
\end{tikzcd}$$

The horizontal square on the bottom right in this diagram now gives rise to the desired map $r\colon (\mathsf{C}_{c/})^{\triangleleft} \to \mathsf{C}_{c/}$. By inspection of the above commutative diagram, it is clear that r sends the canonical map $\infty \to \mathrm{id}_c$ to the identity on $\mathrm{id}_c\colon 1 \to \mathsf{C}_{c/}$ and therefore to an equivalence. Lastly, the observation that the composition

$$\mathsf{C}^{\Delta^1} \xrightarrow{d^0 \otimes \mathrm{id}} \Delta^1 \otimes \mathsf{C}^{\Delta^1} \xrightarrow{e} \mathsf{C}^{\Delta^1}$$

recovers the identity on C^{Δ^1} implies that r is a retract of the map $\mathsf{C}_{c/} \to (\mathsf{C}_{c/})^{\triangleleft}$, which finishes the proof. □

Corollary 2.1.3.13. *Let* C *be a* $\mathcal{B}$*-category and let* $c\colon 1 \to \mathsf{C}$ *be an object in* C*. Then the composition* $(\pi_c)_! \,\mathrm{id}_c\colon 1 \to \mathsf{C}_{c/} \to \mathsf{C}$ *yields the unique factorisation of* c *into an initial map and a left fibration.*

Remark 2.1.3.14. In light of Remark 2.1.2.2 and the fact that the forgetful functor $(\pi_A)_!$ preserves initial maps, Corollary 2.1.3.13 more generally shows that if $c\colon A \to \mathsf{C}$ is an object in arbitrary context $A \in \mathcal{B}$, the factorisation of c into an initial map and a left fibration is given by the composition

$$\mathrm{pr}_1 (\pi_c)_! \,\mathrm{id}_c\colon A \to \mathsf{C}_{c/} \to \mathsf{C},$$

where $\mathrm{pr}_1\colon A \times \mathsf{C} \to \mathsf{C}$ denotes the projection.

Lemma 2.1.3.12 can furthermore be used to derive the following characterisation of initial objects in a $\mathcal{B}$-category:

Proposition 2.1.3.15. *Let* C *be a* $\mathcal{B}$*-category. For any object* $c\colon 1 \to \mathsf{C}$*, the following are equivalent:*

1. *c is an initial object;*

2. *the projection $(\pi_c)_! \colon \mathsf{C}_{c/} \to \mathsf{C}$ is an equivalence;*

3. *for any object $d \colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ the map $\operatorname{map}_{\mathsf{C}}(\pi_A^*(c), d) \to A$ is an equivalence in $\mathcal{B}$.*

Proof. If c is initial, Corollary 2.1.3.13 implies that the left fibration $(\pi_c)_! \colon \mathsf{C}_{c/} \to \mathsf{C}$ must be initial as well and therefore an equivalence. Conversely, if this map is an equivalence, Corollary 2.1.3.13 implies that c is initial. Lastly, since the map $(\pi_c)_! \colon \mathsf{C}_{c/} \to \mathsf{C}$ is a left fibration, Proposition 2.1.1.12 implies that this map is an equivalence whenever the induced map $(\mathsf{C}_{c/})|_d \to A$ is an equivalence for any object $d \colon A \to \mathsf{C}$. As this map recovers the morphism in (3), the claim follows. □

Corollary 2.1.3.16. *Let C be a $\mathcal{B}$-category and let c and d be objects in C in context $1 \in \mathcal{B}$ such that c is initial. Then there is a unique map $c \to d$ in C that is an equivalence if and only if d is initial as well.*

Proof. By Proposition 2.1.3.15, the map $\operatorname{map}_{\mathsf{C}}(c, d) \to 1$ is an equivalence. Therefore, there is a unique map $f \colon c \to d$ that corresponds to the unique section $1 \to \operatorname{map}_{\mathsf{C}}(c, d)$. If d is initial, then by the same argumentation there is a unique map $g \colon d \to c$, and by uniqueness this must be an inverse of f. Hence f is an equivalence. Conversely, if f is an equivalence, then c and d are equivalent as objects in C, which implies that the two maps $c, d \colon 1 \rightrightarrows \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B}_{/A})$ are equivalent, which shows that d must be initial. □

As a consequence of Corollary 2.1.3.16 (and its dual), initial (and final) objects in a $\mathcal{B}$-category C are unique. We will usually denote an initial object by $\varnothing_{\mathsf{C}} \colon 1 \to \mathsf{C}$ and a final object by $1_{\mathsf{C}} \colon 1 \to \mathsf{C}$.

2.1.4. Covariant equivalences

Recall from Section 1.1.5 that the inclusion $\mathrm{LFib} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta)$ admits a left adjoint L. We will denote by $L_{/C} \colon (\mathcal{B}_\Delta)_{/C} \to \mathrm{LFib}(C)$ the induced functor on the fibre over a simplicial object C in $\mathcal{B}$, i.e. the left adjoint of the inclusion $\mathrm{LFib}(C) \hookrightarrow (\mathcal{B}_\Delta)_{/C}$.

Definition 2.1.4.1. Let C be a simplicial object in $\mathcal{B}$ and let $f\colon P \to Q$ be a map in $(\mathcal{B}_\Delta)_{/C}$. Then f is said to be a *covariant equivalence* if $L_{/C}(f)$ is an equivalence in $\mathrm{LFib}(C)$.

Remark 2.1.4.2. In the context of Definition 2.1.4.1, the map $L_{/C}(f)$ is constructed by means of the unique commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\
L_{/C}(P) \arrow[r, "L_{/C}(f)"] \arrow[dr] & L_{/C}(Q) \arrow[d] \\
& C
\end{tikzcd}$$

in which the two vertical maps are initial and the two diagonal maps are left fibrations. In particular, if f is initial then f is a covariant equivalence over C. The converse implication is true whenever the map $Q \to C$ is already a left fibration.

The main goal of this section is to prove the following characterisation of covariant equivalences over C:

Proposition 2.1.4.3. *Let* C *be a* $\mathcal{B}$*-category and let*

$$\begin{tikzcd}
P \arrow[rr, "f"] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\
& \mathsf{C} &
\end{tikzcd}$$

be a commutative triangle in $\mathcal{B}_\Delta$*. Then the following are equivalent:*

1. *f is a covariant equivalence over* C*;*
2. *for any object $c\colon A \to \mathsf{C}$ the induced map $f_{/c}\colon P_{/c} \to Q_{/c}$ is a covariant equivalence over* $\mathsf{C}_{/c}$*;*
3. *for any object $c\colon A \to \mathsf{C}$ the induced map*

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}(f_{/c})\colon \operatorname*{colim}_{\Delta^{\mathrm{op}}}(P_{/c}) \to \operatorname*{colim}_{\Delta^{\mathrm{op}}}(Q_{/c})$$

is an equivalence in $\mathcal{B}$*.*

The proof of Proposition 2.1.4.3 is based on the concept of a *proper map*. Observe that for any map $p\colon P \to C$ in $\mathcal{B}_\Delta$ the commutative square

$$\begin{tikzcd} \mathrm{LFib}(C) \arrow[r, hook] \arrow[d, "p^*"] & (\mathcal{B}_\Delta)_{/C} \arrow[d, "p^*"] \\ \mathrm{LFib}(P) \arrow[r, hook] & (\mathcal{B}_\Delta)_{/P} \end{tikzcd}$$

gives rise to a left lax square

$$\begin{tikzcd} \mathrm{LFib}(C) \arrow[d, "p^*"] & (\mathcal{B}_\Delta)_{/C} \arrow[l, "L_{/C}"'] \arrow[d, "p^*"] \arrow[dl, phantom, "\Nwarrow"] \\ \mathrm{LFib}(P) & (\mathcal{B}_\Delta)_{/\mathrm{P}} \arrow[l, "L_{/D}"'] \end{tikzcd}$$

by means of the mate construction. As L does not preserve pullbacks, this square does not commute in general.

Definition 2.1.4.4. A map $p\colon P \to C$ in $\mathcal{B}_\Delta$ is said to be *proper* if for any cartesian square

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "q"] & P \arrow[d, "p"] \\ D \arrow[r] & C \end{tikzcd}$$

in $\mathcal{B}_\Delta$ the left lax square

$$\begin{tikzcd} \mathrm{LFib}(D) \arrow[d, "q^*"] & (\mathcal{B}_\Delta)_{/D} \arrow[l, "L_{/D}"'] \arrow[d, "q^*"] \arrow[dl, phantom, "\Nwarrow"] \\ \mathrm{LFib}(Q) & (\mathcal{B}_\Delta)_{/Q} \arrow[l, "L_{/Q}"'] \end{tikzcd}$$

commutes. Dually, a map $p\colon P \to C$ is *smooth* if $P^{\mathrm{op}} \to C^{\mathrm{op}}$ is proper.

Proposition 2.1.4.5. *A map $p\colon P \to C$ in $\mathcal{B}_\Delta$ is proper if and only if the pullback functor $p^*\colon (\mathcal{B}_\Delta)_{/C} \to (\mathcal{B}_\Delta)_{/P}$ preserves initial maps (where a map in $(\mathcal{B}_\Delta)_{/C}$ is said to be initial if its image in $\mathcal{B}_\Delta$ is initial, and similarly for $(\mathcal{B}_\Delta)_{/P}$).*

Proof. Unwinding the definitions, the left lax square from Definition 2.1.4.4 is commutative if and only if for any $f\colon E \to D$ the lower square in the commutative diagram

$$\begin{tikzcd}
q^*E \arrow[r] \arrow[d] \arrow[dd, bend right=60, "q^*f"'] & E \arrow[d] \arrow[dd, bend left=60, "f"] \\
L(q^*E) \arrow[r] \arrow[d] & L(E) \arrow[d] \\
Q \arrow[r, "q"] & D
\end{tikzcd}$$

is cartesian. Here the vertical maps are given by the factorisation of $E \to D$ and $q^*E \to Q$ into an initial map and a left fibration. In particular, if p^* preserves initial maps, then the map $L(q^*E) \to q^*L(E)$ is initial. But since this map must also be a left fibration, it is necessarily an equivalence. The converse direction follows from chasing an initial map through the commutative square that is provided in the definition of proper maps. $\square$

Remark 2.1.4.6. Proposition 2.1.4.5 in particular implies that proper maps are preserved by étale base change. In fact, if $A \in \mathcal{B}$ is an arbitrary object and if $p\colon P \to C$ is proper, then by Remark 2.1.3.3 the map $\pi_A^*(P) \to \pi_A^* C$ is proper as soon as its image along $(\pi_A)_!$ is proper. As the latter can be identified with $p \times \mathrm{id}\colon P \times A \to C \times A$, this follows immediately from the definition.

By definition, proper maps preserve covariant equivalences:

Proposition 2.1.4.7. *If $p\colon P \to C$ is a proper map between simplicial objects in $\mathcal{B}$, then the base change functor $p^*\colon (\mathcal{B}_\Delta)_{/C} \to (\mathcal{B}_\Delta)_{/P}$ carries covariant equivalences over C to covariant equivalences over P.* $\square$

Proposition 2.1.4.8. *For any two simplicial objects C and D in $\mathcal{B}$ the projection $C \times D \to C$ is proper.*

Proof. This follows immediately from the fact that initial maps are stable under products (since initial maps are internally left orthogonal to left fibrations). $\square$

The central ingredient towards the proof of Proposition 2.1.4.3 is the following proposition:

Proposition 2.1.4.9. *Any right fibration between simplicial objects in $\mathcal{B}$ is proper.*

Proof. Since right fibrations are stable under pullbacks, it suffices to show that the base change along a right fibration preserves initial maps. To that end, let S be the set of maps in $\mathcal{B}_\Delta$ whose base change along right fibrations results in an initial map. We claim that S is saturated. In fact, it is obvious that S is closed under composition and contains all equivalences, and the stability of S under pushouts and small colimits in $\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta)$ follows from the fact that RFib defines a sheaf on $\mathcal{B}_\Delta$ (as it is defined as the right orthogonality class of a factorisation system). As a consequence, it suffices to show that the initial map $d^1 : D \hookrightarrow \Delta^1 \otimes D$ is contained in S for any simplicial object D in $\mathcal{B}$.

Let therefore

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d] & P \arrow[d, "p"] \\ D \arrow[r, "d^1"] & \Delta^1 \otimes D \end{tikzcd}$$

be a cartesian square in $\mathcal{B}_\Delta$ such that p is a right fibration. Let $\tau\colon \Delta^1 \times \Delta^1 \to \Delta^1$ be the map that sends the final vertex $(1, 1)$ to 1 and any other vertex to 0. We then obtain a commutative diagram

$$\begin{tikzcd} & \Delta^1 \arrow[r, "s_0"] \arrow[d, "\mathrm{id} \times d^1"'] & \Delta^0 \arrow[d, "d^1"] \\ \Delta^1 \arrow[r, "d^0 \times \mathrm{id}"] & \Delta^1 \times \Delta^1 \arrow[r, "\tau"] & \Delta^1 \\ & \Delta^1 \arrow[u, "\mathrm{id} \times d^0"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

of ∞-categories such that the composition of the two horizontal arrows is the identity. Now let $(\Delta^1 \otimes Q) \sqcup_Q P$ be the pushout of f along the map $d^0 : Q \to \Delta^1 \otimes Q$, and observe that the induced map $(\Delta^1 \otimes Q) \sqcup_Q P \to \Delta^1 \otimes P$ is final (by making use of item (2) of Proposition 1.1.5.2). Therefore the lifting problem

$$\begin{tikzcd} (\Delta^1 \otimes Q) \sqcup_Q P \arrow[r, "(f \otimes s^0, \mathrm{id})"] \arrow[d] & P \arrow[d, "p"] \\ \Delta^1 \otimes P \arrow[r, "(\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes p)"'] \arrow[ur, dotted, "h"] & \Delta^1 \otimes D \end{tikzcd}$$

admits a unique solution h. Let $r\colon P \to Q$ be defined as the unique functor that makes the diagram

$$\begin{tikzcd}
 & P \arrow[rr, "d^1"] \arrow[dl, "r"'] \arrow[dd, "p"] & & \Delta^1 \otimes P \arrow[dl, "h"'] \arrow[dd, "\mathrm{id} \otimes p"] \\
Q \arrow[rr, "f"] \arrow[dd, "q"] & & P \arrow[dd, "p"] & \\
 & \Delta^1 \otimes D \arrow[rr, "d^1 \otimes \mathrm{id}"] \arrow[dl, "s^0"'] & & (\Delta^1 \times \Delta^1) \otimes D \arrow[dl, "\tau \otimes \mathrm{id}"] \\
D \arrow[rr, "d^1"] & & \Delta^1 \otimes D &
\end{tikzcd}$$

commute. By construction, this map satisfies $rf \simeq \mathrm{id}$ and moreover fits into the commutative diagram

$$\begin{tikzcd}
P \arrow[d, "d^1"] \arrow[dr, "fr"] & \\
\Delta^1 \otimes P \arrow[r, "h"] & P \\
P. \arrow[u, "d^0"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Hence the commutative diagram

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "f"] & \Delta^1 \otimes Q \sqcup_Q P \arrow[r, "{(s^0, r)}"] \arrow[d] & Q \arrow[d, "f"] \\
P \arrow[r, "d^1"] & \Delta^1 \otimes P \arrow[r, "s^0"] & P
\end{tikzcd}$$

exhibits f as a retract of the initial map $\Delta^1 \otimes Q \sqcup_Q P \to \Delta^1 \otimes P$ (in which the domain is the pushout of f along the inclusion $d^1\colon Q \to \Delta^1 \otimes Q$) and therefore as an initial map itself. □

Proof of Proposition 2.1.4.3. Suppose first that f is a covariant equivalence over C. Since the projection $(\pi_c)_!\colon \mathsf{C}_{/c} \to \mathsf{C} \times A$ is a right fibration and since by Proposition 2.1.4.9 any right fibration is proper, Proposition 2.1.4.7 implies that the map $f_{/c}$ must be a covariant equivalence over $\mathsf{C}_{/c}$.

Suppose now that $f_{/c}$ is a covariant equivalence over $\mathsf{C}_{/c}$, i.e. that $L_{/(\mathsf{C}_{/c})}(f)$ is an equivalence. Quite generally, note that for any simplicial object D in $\mathcal{B}$ the base change functor $\pi_D^*\colon \mathcal{B}_\Delta \to (\mathcal{B}_\Delta)_{/\mathrm{D}}$ admits a left adjoint $(\pi_D)_!$ that is given by the forgetful functor, which implies that the base change functor $\pi_D^* \colon \mathcal{B} \to \mathrm{LFib}(D)$ admits a left adjoint $(\pi_D)_!$ as well that is explicitly given by the composition

$$\mathrm{LFib}(D) \hookrightarrow (\mathcal{B}_\Delta)_{/D} \xrightarrow{(\pi_D)_!} \mathcal{B}_\Delta \xrightarrow{\mathrm{colim}_{\Delta^{\mathrm{op}}}} \mathcal{B},$$

cf. Remark 2.1.1.11. One consequently obtains a commutative square

$$\begin{tikzcd}
(\mathcal{B}_\Delta)_{/D} \arrow[r, "(\pi_D)_!"] \arrow[d, "L_{/D}"] & \mathcal{B}_\Delta \arrow[d, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] \\
\mathrm{LFib}(D) \arrow[r, "(\pi_D)_!"] & \mathcal{B}.
\end{tikzcd}$$

Applying this observation to $D = \mathsf{C}_{/c}$, one finds that the map $\mathrm{colim}_{\Delta^{\mathrm{op}}}(f_{/c})$ arises as the image of $L_{/(\mathsf{C}_{/c})}(f)$ along the functor $(\pi_{\mathsf{C}_{/c}})_!$ and is therefore an equivalence.

Lastly, assume that $\mathrm{colim}_{\Delta^{\mathrm{op}}}(f_{/c})$ is an equivalence in $\mathcal{B}$ for every object $c \colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$. We need to show that the map

$$L_{/\mathsf{C}}(f) \colon L_{/\mathsf{C}}(P) \to L_{/\mathsf{C}}(Q)$$

in $\mathrm{LFib}(\mathsf{C})$ is an equivalence. By Proposition 2.1.1.12, it suffices to show that the map $L_{/\mathsf{C}}(f)|_c \colon L_{/\mathsf{C}}(P)|_c \to L_{/\mathsf{C}}(Q)|_c$ that is induced on the fibres over $c \colon A \to \mathsf{C}$ is an equivalence in $\mathcal{B}$ for all objects c in C. By making use of the factorisation of c into the canonical final map $\mathrm{id}_c \colon A \to \mathsf{C}_{/c}$ followed by the right fibration $\mathrm{pr}_1(\pi_c)_! \colon \mathsf{C}_{/c} \to \mathsf{C}$, one obtains a pullback square

$$\begin{tikzcd}
L_{/\mathsf{C}}(P)|_c \arrow[r, "L_{/\mathsf{C}}(f)|_c"] \arrow[d] & L_{/\mathsf{C}}(Q)|_c \arrow[d] \\
L_{/\mathsf{C}}(P)_{/c} \arrow[r, "(L_{/\mathsf{C}}(f))_{/c}"] & L_{/\mathsf{C}}(Q)_{/c}
\end{tikzcd}$$

in which the vertical maps are final since they arise as pullbacks of the final map id_c along left fibrations and since the dual of Proposition 2.1.4.9 implies that left fibrations are smooth. Note that the fibres $L_{/\mathsf{C}}(P)|_c$ and $L_{/\mathsf{C}}(Q)|_c$ are contained in $\mathcal{B}$. Hence the functor $\mathrm{colim}_{\Delta^{\mathrm{op}}}$ sends the two vertical maps in the above square to equivalences in $\mathcal{B}$ while leaving the upper horizontal map unchanged. We conclude that $L_{/\mathsf{C}}(\mathrm{f})|_c$ is an equivalence whenever $\mathrm{colim}_{\Delta^{\mathrm{op}}}((L_{/\mathsf{C}}(f))_{/c})$ is one, and since the latter recovers the map $\operatorname{colim}_{\Delta^{\mathrm{op}}}(f_{/c})$ (again using properness of the right fibration $\mathrm{pr}_0(\pi_c)_! \colon \mathsf{C}_{/c} \to \mathsf{C}$), the result follows. ☐

Proposition 2.1.4.3 can be used to derive an internal version of Quillen's theorem A:

Corollary 2.1.4.10. *A functor $f \colon \mathsf{J} \to \mathsf{I}$ between $\mathcal{B}$-categories is initial if and only if for every object i in I in context $A \in \mathcal{B}$ the canonical map $(\mathsf{J}_{/i})^{\mathrm{gpd}} \to A$ is an equivalence.*

Proof. On account of Remark 2.1.4.2, the map f is initial if and only if it is a covariant equivalence over I. By Proposition 2.1.4.3, this is the case if and only if for every object $i \colon A \to \mathsf{I}$ the map $(\mathsf{J}_{/i})^{\mathrm{gpd}} \to (\mathsf{I}_{/i})^{\mathrm{gpd}}$ is an equivalence. By construction, there is a commutative diagram

$$\begin{tikzcd} (\mathsf{J}_{/i})^{\mathrm{gpd}} \arrow[rr] \arrow[dr] & & (\mathsf{I}_{/i})^{\mathrm{gpd}} \arrow[dl] \\ & A. & \end{tikzcd}$$

Therefore, the proof is finished once we show that the map $(\mathsf{I}_{/i})^{\mathrm{gpd}} \to A$ is an equivalence. But this follows from the observation that the map $\mathsf{I}_{/i} \to A$ is final as it is a retraction of the final section $\mathrm{id}_c \colon A \to \mathsf{I}_{/c}$. ☐

Remark 2.1.4.11. As a consequence of Corollary 2.1.4.10, the condition of a functor between $\mathcal{B}$-categories to be initial is *local*. More precisely, if $\bigsqcup_k A_k \twoheadrightarrow 1$ is a cover in $\mathcal{B}$ and if $f \colon \mathsf{J} \to \mathsf{I}$ is a functor between $\mathcal{B}$-categories, then f is initial if and only if $\pi_{A_k}^* f$ is initial for all k. In fact, Corollary 2.1.4.10 tells us that f being initial is equivalent to the map $(\mathsf{J}_{/i})^{\mathrm{gpd}} \to A$ being an equivalence for every object $i \colon A \to \mathsf{I}$. Using Remark 1.2.5.6, we obtain an equivalence $\pi_{A_k}^*(\mathsf{J}_{/i}) \simeq (\pi_{A_k}^*\mathsf{J})_{/\pi_{A_k}^* i}$ over A_k for all k. Since $\pi_{A_k}^*$ moreover commutes with the groupoidification functor (Proposition 1.2.4.4), we may thus identify the pullback of the map $(\mathsf{J}_{/i})^{\mathrm{gpd}} \to A$ along π_{A_k} with the map $(\pi_{A_k}^*\mathsf{J}_{/\pi_{A_k}^* i})^{\mathrm{gpd}} \to \pi_{A_k}^* A$ in $\mathcal{B}_{/A_k}$. The claim now follows from the fact that the algebraic morphism $\mathcal{B} \to \prod_k \mathcal{B}_{/A_k}$ is conservative since $\bigsqcup_k A_k \twoheadrightarrow 1$ is a cover in $\mathcal{B}$.

We end this section with yet another characterisation of covariant equivalences that will be useful later:

Proposition 2.1.4.12. *Let* C *be a* $\mathcal{B}$*-category and let*

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\ & \mathsf{C} & \end{tikzcd}$$

be a commutative triangle in $\mathcal{B}_\Delta$ *in which both* p *and* q *are smooth. Then* f *is a covariant equivalence over* C *if and only if for any object* c *in* C *in arbitrary context* $A \in \mathcal{B}$ *the induced map* $\operatorname{colim}_{\Delta^{\mathrm{op}}} f|_c \colon \operatorname{colim}_{\Delta^{\mathrm{op}}}(P|_c) \to \operatorname{colim}_{\Delta^{\mathrm{op}}}(Q|_c)$ *is an equivalence.*

Proof. f is a covariant equivalence over C if and only if $L_{/\mathsf{C}}(f)$ is an equivalence. Let

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "i"] & Q \arrow[d, "j"] \\ L_{/\mathsf{C}}(P) \arrow[r, "L_{/\mathsf{C}}(f)"] & L_{/\mathsf{C}}(Q) \end{tikzcd}$$

be the canonical square in which the two vertical maps are obtained from the adjunction unit and are therefore initial. Since $L_{/\mathsf{C}}(f)$ is a map in of left fibrations over C, Proposition 2.1.1.12 implies that this map is an equivalence if and only if the induced map $L_{/\mathsf{C}}(f)|_c$ on the fibres over c is one for every object $c \colon A \to \mathsf{C}$. It therefore suffices to show that in the induced commutative diagram

$$\begin{tikzcd} P|_c \arrow[r, "f|_c"] \arrow[d, "i|_c"] & Q|_c \arrow[d, "j|_c"] \\ L_{/\mathsf{C}}(P)|_c \arrow[r, "L_{/\mathsf{C}}(f)|_c"] & L_{/\mathsf{C}}(Q)|_c \end{tikzcd}$$

of the fibres over c the maps $i|_c$ and $j|_c$ are initial. We will show this for $i|_c$, the case of $j|_c$ is analogous.

Let $p' \colon L_{/\mathsf{C}}(P) \to \mathsf{C}$ be the structure map, and consider the commutative diagram

$$\begin{tikzcd}
 & P\vert_c \arrow[rr] \arrow[dl, "i\vert_c"'] \arrow[dd, "\mathrm{id}"] & & P_{/c} \arrow[rr] \arrow[dl, "i_{/c}"'] \arrow[dd, "\mathrm{id}"] & & P \arrow[dl, "i"'] \arrow[dd, "\mathrm{id}"] \\
L_{/\mathsf{C}}(P)\vert_c \arrow[rr] \arrow[dd, "p'\vert_c"'] & & L_{/\mathsf{C}}(P)_{/c} \arrow[rr] \arrow[dd, "p'_{/c}"'] & & L_{/\mathsf{C}}(P) \arrow[dd, "p'"'] & \\
 & P\vert_c \arrow[rr] \arrow[dl, "p\vert_c"'] & & P_{/c} \arrow[rr] \arrow[dl, "p_{/c}"'] & & P \arrow[dl, "p"'] \\
A \arrow[rr, "\mathrm{id}_c"] & & \mathsf{C}_{/c} \arrow[rr, "\mathrm{pr}_0(\pi_c)_!"] & & \mathsf{C}. &
\end{tikzcd}$$

The projection $\mathrm{pr}_0(\pi_c)_!$ is a right fibration, which implies that the two maps $P_{/c} \to P$ and $L_{/\mathsf{C}}(P)_{/c} \to L_{/\mathsf{C}}(P)$ must be right fibrations as well. Hence Proposition 2.1.4.9 implies that the map $i_{/c}$ must be initial. Moreover, since p and p' are smooth the maps $p_{/c}$ and $p'_{/c}$ must be smooth as well, which implies that the maps $P\vert_c \to P_{/c}$ and $L_{/\mathsf{C}}(P)\vert_c \to L_{/\mathsf{C}}(P)_{/c}$ must be final since id_c is final. We therefore obtain a pullback square

$$\begin{tikzcd}
P\vert_c \arrow[r] \arrow[d, "i\vert_c"] & P_{/c} \arrow[d, "i_{/c}"] \\
L_{/\mathsf{C}}(P)\vert_c \arrow[r] & L_{/\mathsf{C}}(P)_{/c}
\end{tikzcd}$$

in which the horizontal maps are final and the vertical map on the right is initial. Since the functor $\mathrm{colim}_{\Delta^{\mathrm{op}}}$ carries both final and initial maps to equivalences in $\mathcal{B}$ (cf. Remark 1.1.5.3 and Remark 2.1.1.11), the map $\mathrm{colim}_{\Delta^{\mathrm{op}}}(i\vert_c)$ must be an equivalence. But as $L_{/\mathsf{C}}(P)\vert_c$ is already contained in $\mathcal{B}$, the map $i\vert_c$ is equivalent to the composition of an initial map with an equivalence and therefore initial itself. □

2.2. Straightening of left fibrations

In [49, § 2.1.1], Lurie proves the *straightening equivalence* for left fibrations, which states that to every left fibration $p\colon P \to C$ of simplicial sets one can associated a functor $f\colon C \to \mathrm{Ani}$ which is unique up to coherent homotopy. The functor f is called the *straightening* of p. In this section, our goal is to obtain a $\mathcal{B}$-categorical version of this result. Recall that the analogue of Ani in the world of $\mathcal{B}$-categories is the universe for $\mathcal{B}$-groupoids $\mathrm{Grpd}_{\mathcal{B}}$. Therefore, a variant of straightening for $\mathcal{B}$-categories ought to associate to each left fibration $p\colon P \to C$ of simplicial objects in $\mathcal{B}$ a map $f\colon C \to \mathrm{Grpd}_{\mathcal{B}}$, and this assignment should induce an equivalence between a (suitably defined) $\mathcal{B}$-category of left fibrations over C and the functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(C, \mathrm{Grpd}_{\mathcal{B}})$. We accomplish this goal in Section 2.2.1. As a consequence, there is a *universal* left fibration, which is defined as the unique left fibration $\phi\colon (\mathrm{Grpd}_{\mathcal{B}})_* \to \mathrm{Grpd}_{\mathcal{B}}$ whose straightening is the equivalence on $\mathrm{Grpd}_{\mathcal{B}}$. In Section 2.2.2, we study this map in some detail.

2.2.1. The straightening equivalence

For any $\mathcal{B}$-category C, we will denote by $\mathsf{LFib}_{\mathsf{C}}$ the $\widehat{\mathrm{Cat}}_\infty$-valued presheaf on $\mathcal{B}$ that is given by the assignment $A \mapsto \mathrm{LFib}(\mathsf{C} \times A)$. This defines a functor $\mathsf{C} \mapsto \mathsf{LFib}_{\mathsf{C}}$ from $\mathrm{Cat}(\mathcal{B})$ into $\mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$.

Theorem 2.2.1.1. *For every $\mathcal{B}$-category C, the presheaf $\mathsf{LFib}_{\mathsf{C}}$ defines a large $\mathcal{B}$-category, and there is a canonical equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \simeq \mathsf{LFib}_{\mathsf{C}}$$

that is natural in C.

Theorem 2.2.1.1 can be easily deduced from the following more general statement:

Proposition 2.2.1.2. *There is an equivalence*

$$\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathrm{LFib}$$

of $\widehat{\mathrm{Cat}}_\infty$-valued sheaves on $\mathcal{B}_\Delta$.

Remark 2.2.1.3. In the situation of Proposition 2.2.1.2, it is clear that the presheaf $\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}})$ defines a sheaf on $\mathcal{B}_\Delta$, but the claim that LFib is a sheaf (without knowing that LFib is equivalent to $\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}})$) requires an argument. Recall from Proposition 1.4.1.6 that LFib is a sheaf if and only if the class of left fibrations in $\mathcal{B}_\Delta$ is *local*. In light of the characterisation of left fibrations from Proposition 2.1.1.3, this is a straightforward consequence of descent in $\mathcal{B}$.

Proof of Theorem 2.2.1.1. By making use of the embedding $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Cat}_\infty}(\mathcal{B})$, one sees that the functor $\underline{\mathrm{Fun}}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}})$ is equivalent to the bifunctor

$$\mathrm{Fun}_{\mathcal{B}}(- \times -, \mathrm{Grpd}_{\mathcal{B}}) \colon \mathcal{B}^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty.$$

Similarly, the functor $\mathrm{LFib}_{(-)}$ corresponds to the bifunctor

$$\mathrm{LFib}(- \times -) \colon \mathcal{B}^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

under this identification. Thus, the claim follows by restricting the equivalence

$$\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathrm{LFib}$$

along the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$. □

The remainder of this section is devoted to the proof of Proposition 2.2.1.2. Fix a small full subcategory $\mathcal{G} \subset \mathcal{B}$ as in Remark 1.2.1.3 The main ingredient will be to establish an equivalence

$$\mathrm{Fun}_{\widehat{\mathcal{B}}}(\Delta^\bullet \otimes -, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathrm{LFib}(\Delta^\bullet \otimes -)$$

of $\widehat{\mathrm{Cat}}_\infty$-valued presheaves on $\Delta \times \mathcal{G}$. This will require a few preparations. First, for any $n \geq 0$ an any $G \in \mathcal{G}$ let us denote by $\Delta^n \times G$ the presheaf on $\Delta \times \mathcal{G}$ that is represented by the pair $(\langle n \rangle, G)$. We thus obtain $L(\Delta^n \times G) \simeq \Delta^n \otimes G$ (where $L \colon \mathrm{PSh}(\Delta \times \mathcal{G}) \to \mathcal{B}_\Delta$ is the localisation functor).

Lemma 2.2.1.4. *Assigning to a left fibration $P \to \Delta^n \otimes G$ in $\mathcal{B}_\Delta$ its pullback along the adjunction unit $\Delta^n \times G \to \Delta^n \otimes G$ in $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$ defines an embedding*

$$\mathrm{LFib}(\Delta^\bullet \otimes -) \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^\bullet \times -}$$

of presheaves on $\Delta \times \mathcal{G}$.

Proof. Since the presheaf $\mathrm{LFib}(\Delta^\bullet \otimes -)$ embeds into the presheaf $(\mathcal{B}_\Delta)_{/\Delta^\bullet \otimes -}$, it suffices to prove that one can define an embedding

$$(\mathcal{B}_\Delta)_{/\Delta^\bullet \otimes -} \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^\bullet \times -}$$

in the desired way. Since the localisation functor $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G}) \to \mathcal{B}_\Delta$ is left exact, there is a functorial map

$$\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^\bullet \times -} \to (\mathcal{B}_\Delta)_{/\Delta^\bullet \otimes -}$$

that is given on the level of cartesian fibrations by the pullback of the natural map

$$\mathrm{Fun}(\Delta^1, \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})) \to \mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta) \times_{\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})} \mathcal{B}_\Delta$$

along the Yoneda embedding $\Delta \times \mathcal{G} \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$. On the fibre over $(\langle n \rangle, G)$, this functor is given by the map that is naturally induced by the localisation functor $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G}) \to \mathcal{B}_\Delta$ upon taking slice ∞-categories. Hence there are fibre-wise right adjoints that are given by composing the natural map

$$(\mathcal{B}_\Delta)_{/\Delta^n \otimes G} \to \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^n \otimes G}$$

that is induced by the inclusion $\mathcal{B}_\Delta \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$ with the pullback functor along the adjunction unit $\Delta^n \times G \to \Delta^n \otimes G$ in $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$. Note that each of these fibre-wise right adjoints is fully faithful since the localisation functor commutes with pullbacks. We conclude by observing that these fibre-wise right adjoints assemble to a map of $\widehat{\mathrm{Cat}}_\infty$-valued presheaves on $\Delta \times \mathcal{G}$. $\square$

By [25, Proposition 9.8] there is a functorial equivalence

$$\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^\bullet \times -} \simeq \mathrm{PSh}_{\mathrm{Ani}}((\Delta \times \mathcal{G})_{/\Delta^\bullet \times -})$$

where the right-hand side can furthermore be identified with $\mathrm{PSh}_{\mathrm{Ani}}(\Delta_{/\Delta^\bullet} \times \mathcal{G}_{/-})$. Combining this result with Lemma 2.2.1.4, we conclude that there is an embedding

$$\mathrm{LFib}(\Delta^\bullet \otimes -) \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta_{/\Delta^\bullet} \times \mathcal{G}_{/-})$$

that sends a left fibration $P \to \Delta^n \otimes G$ in $\mathcal{B}_\Delta$ to the presheaf that maps a pair (τ, s) (where $\tau \colon \langle k \rangle \to \langle n \rangle$ is a map in Δ and $s \colon H \to G$ is a map in $\mathcal{G}$) to the fibre

$$\begin{tikzcd} P_k(H)|_{(\tau,s)} \arrow[r] \arrow[d] & P_k(H) \arrow[d] \\ 1 \arrow[r, "{(\tau,s)}"] & \mathrm{const}_{\mathcal{B}}(\Delta^n_k) \times G(H). \end{tikzcd}$$

Note that this embedding factors through the inclusion

$$\mathrm{PSh}_{\mathcal{B}_{/-}}(\Delta_{/\Delta^\bullet}) \hookrightarrow \mathrm{PSh}_{\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{G}_{/-})}(\Delta_{/\Delta^\bullet}) \simeq \mathrm{PSh}_{\mathrm{Ani}}(\Delta_{/\Delta^\bullet} \times \mathcal{G}_{/-}),$$

which implies that we end up with a functorial embedding

$$\mathrm{LFib}(\Delta^\bullet \otimes -) \hookrightarrow \mathrm{PSh}_{\mathcal{B}_{/-}}(\Delta_{/\Delta^\bullet}).$$

Our next goal is to characterise the essential image of this embedding:

Lemma 2.2.1.5. *For any $G \in \mathcal{G}$ and any $n \geq 0$, a presheaf $F \in \mathrm{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n})$ is contained in the essential image of the inclusion $\mathrm{LFib}(\Delta^n \otimes G) \hookrightarrow \mathrm{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n})$ if and only if for any $k \geq 1$ and any map $\tau\colon \langle k \rangle \to \langle n \rangle$ in Δ, the inclusion $\delta^{\{0\}} : \langle 0 \rangle \to \langle k \rangle$ induces an equivalence $F(\tau) \simeq F(\tau\delta^{\{0\}})$.*

Proof. Let F be a $\mathcal{B}_{/G}$-valued presheaf on $\Delta_{/\Delta^n}$ and let $P \to \Delta^n \times G$ be the map in $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$ that corresponds to F in view of the inclusion

$$\mathrm{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n}) \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^n \times G}.$$

Then $P \to \Delta^n \times G$ is in the essential image of the inclusion

$$(\mathcal{B}_\Delta)_{/\Delta^n \otimes G} \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})_{/\Delta^n \times G}.$$

To see this, let $(L \dashv i)$ denotes the adjunction $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G}) \rightleftarrows \mathcal{B}_\Delta$. We need to show that the commutative square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & iL(P) \arrow[d] \\ \Delta^n \times G \arrow[r] & i(\Delta^n \otimes G) \end{tikzcd}$$

that is induced by the adjunction unit $\mathrm{id} \to iL$ is a pullback square in $\mathrm{PSh}_{\mathrm{Ani}}(\Delta \times \mathcal{G})$. It suffices to show this for each $k \in \Delta$ individually. In this case, the map $P_k \to \mathrm{const}(\Delta^n_k) \times G$ is given by the coproduct

$$\bigsqcup_{\tau\colon \langle k \rangle \to \langle n \rangle} F(\tau) \to \bigsqcup_{\tau\colon \langle k \rangle \to \langle n \rangle} G,$$

hence it suffices to show that for each map $\tau\colon \langle k\rangle \to \langle n\rangle$ in Δ the square

$$\begin{tikzcd}
F(\tau) \arrow[r] \arrow[d] & iL(F(\tau)) \arrow[d] \\
G \arrow[r] & iL(G)
\end{tikzcd}$$

is cartesian, which follows from $F(\tau)$ being contained in $\mathcal{B}$. Hence F is contained in $\mathrm{LFib}(\Delta^n \otimes G)$ precisely if the map $L(P) \to \Delta^n \otimes G$ is a left fibration.

Using that L is left exact as well as Proposition 2.1.1.3, one finds that map $L(P) \to \Delta^n \otimes G$ being a left fibration is equivalent to the square

$$\begin{tikzcd}
P_k \arrow[r, "d_{\{0\}}"] \arrow[d] & P_0 \arrow[d] \\
\Delta^n_k \times G \arrow[r, "d_{\{0\}}"] & \Delta^n_0 \times G
\end{tikzcd}$$

being a pullback diagram for all $k \geq 1$. On account of the commutative diagrams

$$\begin{tikzcd}
& F(\tau) \arrow[rr] \arrow[dl] \arrow[dd] & & F(\tau\delta^{\{0\}}) \arrow[dl] \arrow[dd] \\
P_k \arrow[rr, "d_{\{0\}}"] \arrow[dd] & & P_0 \arrow[dd] & \\
& G \arrow[rr, "\mathrm{id}"] \arrow[dl, "\tau"'] & & G \arrow[dl, "\tau\delta^{\{0\}}"] \\
\Delta^n_k \times G \arrow[rr, "d_{\{0\}}"] & & \Delta^n_0 \times G &
\end{tikzcd}$$

for all $\tau\colon \langle k\rangle \to \langle n\rangle$, in which the squares on the left and on the right are cartesian, this is seen to be equivalent to the map $\delta^{\{0\}}\colon \langle 0\rangle \to \langle k\rangle$ inducing an equivalence $F(\tau) \simeq F(\tau\delta^{\{0\}})$. $\square$

Recall that there is an equivalence

$$\mathrm{Fun}_{\mathcal{B}}(\Delta^\bullet \otimes -, \mathsf{Grpd}_{\mathcal{B}}) \simeq \mathrm{Fun}(\Delta^\bullet, \mathsf{Grpd}_{\mathcal{B}}(-)) \simeq \mathrm{Fun}(\Delta^\bullet, \mathcal{B}_{/-})$$

of $\widehat{\mathrm{Cat}}_\infty$-valued presheaves on $\Delta \times \mathcal{G}$. Let

$$\lambda\colon \Delta^n \to \left(\Delta_{/\Delta^n}\right)^{\mathrm{op}}$$

denote the functor that sends $k \leq n$ to the inclusion $d^{\{k,\dots,n\}}\colon \Delta^{n-k} \subset \Delta^n$. This functor admits a right adjoint

$$\epsilon\colon \left(\Delta_{/\Delta^n}\right)^{\mathrm{op}} \to \Delta^n$$

that sends $\tau\colon \langle k\rangle \to \langle n\rangle$ to $\tau(0)$. One easily checks that ϵ is natural in n. Moreover, since $\lambda\sigma$ is the identity functor on Δ^n, this adjunction exhibits Δ^n as a localisation of $\left(\Delta_{/\Delta^n}\right)^{\mathrm{op}}$. By precomposition, we therefore obtain a functorial embedding

$$\epsilon^*\colon \operatorname{Fun}(\Delta^\bullet, \mathcal{B}_{/-}) \hookrightarrow \operatorname{PSh}_{\mathcal{B}_{/-}}(\Delta_{/\Delta^\bullet})$$

that exhibits each ∞-category $\operatorname{Fun}(\Delta^n, \mathcal{B}_{/G})$ as a colocalisation of $\operatorname{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n})$, with the right adjoint given by λ^*.

Lemma 2.2.1.6. *For any pair $(\langle n\rangle, G) \in \Delta \times \mathcal{G}$, the essential image of the functor*

$$\epsilon^*\colon \operatorname{Fun}(\Delta^n, \mathcal{B}_{/G}) \hookrightarrow \operatorname{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n})$$

coincides with the essential image of the embedding

$$\operatorname{LFib}(\Delta^n \otimes G) \hookrightarrow \operatorname{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n}).$$

Proof. We first claim that for any $\sigma\colon \Delta^n \to \mathcal{B}_{/G}$ the associated presheaf

$$\sigma\epsilon\colon (\Delta_{/\Delta^n})^{\mathrm{op}} \to \mathcal{B}_{/G}$$

satisfies the condition of Lemma 2.2.1.5. In fact, if $\tau\colon \langle k\rangle \to \langle n\rangle$ is a map in Δ with $k \geq 1$, then the map in $\mathcal{B}_{/G}$ that is induced by the inclusion $\delta^{\{0\}}\colon \langle 0\rangle \to \langle k\rangle$ is simply the identity $\sigma(\tau(0)) \simeq \sigma(\tau(0))$, hence the claim follows.

To finish the proof, it now suffices to show that for any $F \in \operatorname{PSh}_{\mathcal{B}_{/G}}(\Delta_{/\Delta^n})$ that satisfies the condition of Lemma 2.2.1.5 the adjunction counit $\epsilon^*\lambda^* F \to F$ is an equivalence. Since this map is given by precomposition with the adjunction counit of $\lambda \dashv \epsilon$, the map $\epsilon^*\lambda^*F \to F$ is defined on each object $\tau\colon \langle k\rangle \to \langle n\rangle$ in $\Delta_{/\Delta^n}$ by applying F to the map $\langle k\rangle \to \langle n - \tau(0)\rangle$ over $\langle n\rangle$, where the structure map of the codomain into $\langle n\rangle$ is given by the inclusion $\delta^{\{\tau(0),\dots,n\}}$. Since precomposing this map with $\delta^{\{0\}}\colon \langle 0\rangle \to \langle k\rangle$ recovers the inclusion $\delta^{\{0\}}\colon \langle 0\rangle \to \langle n - \tau(0)\rangle$, the two out of three property of equivalences and the condition on F imply that F sends the map $\langle k\rangle \to \langle n - \tau(0)\rangle$ to an equivalence in $\mathcal{B}_{/G}$, as desired. ☐

Proof of Proposition 2.2.1.2. By Lemma 2.2.1.6, there is an equivalence

$$\operatorname{LFib}(\Delta^\bullet \otimes -) \simeq \operatorname{Fun}_{\mathcal{B}}(\Delta^\bullet \otimes -, \operatorname{Grpd}_{\mathcal{B}})$$

of functors $\Delta^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B}_{\Delta}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$. As both LFib and $\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}})$ are sheaves on $\mathcal{B}_{\Delta}$ (see Remark 2.2.1.3), Remark 1.2.1.3 implies that the above equivalences can be uniquely extended to an equivalence

$$\mathrm{LFib} \simeq \mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}}),$$

which finishes the proof. □

Remark 2.2.1.7. Let $A \in \mathcal{B}$ be an arbitrary object. By combining Theorem 2.2.1.1, Remark 1.4.1.2 and Proposition 1.2.5.4, one obtains an equivalence

$$\pi_A^* \,\mathsf{LFib}_{\mathsf{C}} \simeq \mathsf{LFib}_{\pi_A^* \mathsf{C}}$$

that is natural in C. By unwinding the constructions, this equivalence is explicitly given by the chain of equivalences

$$\begin{aligned} \mathrm{Fun}_{\mathcal{B}_{/A}}(-, \mathsf{LFib}_{\pi_A^*\mathsf{C}}) &\simeq \mathrm{LFib}(- \times_A \pi_A^* \mathsf{C}) \\ &\simeq \mathrm{LFib}((\pi_A)_!(-) \times \mathsf{C}) \\ &\simeq \mathrm{Fun}_{\mathcal{B}_{/A}}(-, \pi_A^* \,\mathsf{LFib}_{\mathsf{C}}) \end{aligned}$$

in which the middle equivalence is induced by the equivalence

$$(\pi_A)_!(- \times_A \pi_A^* \mathsf{C}) \simeq (\pi_A)_!(-) \times \mathsf{C},$$

together with the evident observation that the forgetful functor

$$(\pi_A)_! \colon \ \mathrm{Cat}(\mathcal{B}_{/A}) \to \mathrm{Cat}(\mathcal{B})$$

yields an equivalence $\mathrm{LFib}_{\mathcal{B}}((\pi_A)_!(-)) \simeq \mathrm{LFib}_{\mathcal{B}_{/A}}(-)$ of sheaves on $\mathrm{Cat}(\mathcal{B}_{/A})$.

Remark 2.2.1.8. The proof of Proposition 2.2.1.2 shows that the restriction of the equivalence $\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathrm{LFib}$ along the inclusion $\mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ recovers the equivalence

$$\mathrm{Fun}_{\mathcal{B}}(-, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathcal{B}_{/-}$$

of $\widehat{\mathrm{Cat}}_{\infty}$-valued sheaves on $\mathcal{B}$.

Let $p\colon P \to C$ be a map between simplicial objects in $\widehat{\mathcal{B}}$. We will say that p is *small* if for every map $D \to C$ in $\widehat{\mathcal{B}}_\Delta$ in which D is small (i.e. contained in $\mathcal{B}_\Delta$), the pullback $p^*D = P \times_C D$ is small as well. The collection of small maps defines a cartesian subfibration of the codomain fibration $\mathrm{Fun}(\Delta^1, \widehat{\mathcal{B}}_\Delta) \to \widehat{\mathcal{B}}_\Delta$. We therefore obtain a subpresheaf $\mathrm{LFib}^{\mathbf{U}}_{\widehat{\mathcal{B}}} \hookrightarrow \mathrm{LFib}_{\widehat{\mathcal{B}}}$ of the sheaf of left fibrations on $\widehat{\mathcal{B}}_\Delta$ that is spanned by the small left fibrations.

Proposition 2.2.1.9. *Let $p\colon P \to C$ be a left fibration between simplicial objects in $\widehat{\mathcal{B}}$. Then p is small if and only if for all maps $c\colon A \to C$ with $A \in \mathcal{B}$ the fibre $P|_c = P \times_C A$ is contained in $\mathcal{B}_{/A}$.*

Proof. The condition is clearly necessary. For the converse direction, it suffices to show that if $p\colon P \to C$ is a left fibration in $\widehat{\mathcal{B}}_\Delta$ such that C is small and such that the fibre $P|_c$ is small for every map $c\colon A \to C$ with $A \in \mathcal{B}$, the simplicial object P is small as well. To see this, note that since p is a left fibration and C is small, it suffices to show that P_0 is small. But this follows from the fact that P_0 arises as the fibre of p over the map $C_0 \to C$, see Corollary 1.3.3.5. □

By combining Remark 1.4.1.4 and Remark 2.2.1.8 with Proposition 2.2.1.9, we now obtain:

Corollary 2.2.1.10. *For every large $\mathcal{B}$-category* C*, the subpresheaf*

$$\mathrm{LFib}^{\mathbf{U}}(- \times \mathsf{C}) \hookrightarrow \mathrm{LFib}(- \times \mathsf{C})$$

defines a large $\mathcal{B}$-category $\mathsf{LFib}^{\mathbf{U}}_{\mathsf{C}}$*. Moreover, the restriction of the equivalence*

$$\mathsf{LFib}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})$$

along the fully faithful functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})$ *gives rise to an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \simeq \mathsf{LFib}^{\mathbf{U}}_{\mathsf{C}}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$. □

Remark 2.2.1.11. The equivalence $(-)^{\mathrm{op}} \colon \mathcal{B}_\Delta \simeq \mathcal{B}_\Delta$ from Section 1.2.1 induces a commutative square

$$\begin{tikzcd} \mathrm{RFib} \arrow[r, "(-)^{\mathrm{op}}"] \arrow[d] & \mathrm{LFib} \arrow[d] \\ \mathcal{B}_\Delta \arrow[r, "(-)^{\mathrm{op}}"] & \mathcal{B}_\Delta \end{tikzcd}$$

and therefore by Theorem 2.2.1.1 an equivalence

$$\mathrm{RFib} \simeq \mathrm{Fun}_{\mathcal{B}}((-)^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) = \underline{\mathsf{PSh}}_{\mathcal{B}}(-)$$

of $\widehat{\mathrm{Cat}}_\infty$-valued sheaves on $\mathcal{B}_\Delta$. Since the diagonal embedding $\mathcal{B} \hookrightarrow \mathcal{B}_\Delta$ commutes with taking opposite simplicial objects in $\mathcal{B}$, we thus obtain an equivalence

$$\mathsf{RFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$$

(where the large $\mathcal{B}$-category $\mathsf{RFib}_{\mathsf{C}}$ is given by the $\widehat{\mathrm{Cat}}_\infty$-valued sheaf $\mathrm{RFib}(\mathsf{C}\times -)$) that is natural in $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$. Similarly, one also obtains an equivalence

$$\mathsf{RFib}_{\mathsf{C}}^{\mathbf{U}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}})$$

for every *large* $\mathcal{B}$-category C.

2.2.2. The universal left fibration

By Corollary 2.2.1.10, the identity $\mathrm{id}_{\mathsf{Grpd}_{\mathcal{B}}} \colon \mathsf{Grpd}_{\mathcal{B}} \simeq \mathsf{Grpd}_{\mathcal{B}}$ determines a small left fibration $\phi \colon (\mathsf{Grpd}_{\mathcal{B}})_* \to \mathsf{Grpd}_{\mathcal{B}}$ of large $\mathcal{B}$-categories.

Definition 2.2.2.1. The left fibration $\phi \colon (\mathsf{Grpd}_{\mathcal{B}})_* \to \mathsf{Grpd}_{\mathcal{B}}$ is referred to as the the *universal left fibration*.

Remark 2.2.2.2. Yoneda's lemma for ∞-categories implies that the equivalence

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-, \mathsf{Grpd}_{\mathcal{B}}) \simeq (\mathrm{LFib}^{\mathbf{U}})^{\simeq}$$

that is induced by the equivalence in Corollary 2.2.1.10 on the underlying $\widehat{\mathrm{Ani}}$-valued sheaves is induced by assigning to each functor $f \colon \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$ the left fibration $\int f \to \mathsf{C}$ that is determined by the pullback square

$$\begin{tikzcd} \int f \arrow[r] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})_* \arrow[d, "\phi"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{Grpd}_{\mathcal{B}}\,. \end{tikzcd}$$

We say that the left fibration $\int f \to \mathsf{C}$ is *classified* by f. Conversely, given a small left fibration $p\colon \mathsf{P} \to \mathsf{C}$ of large $\mathcal{B}$-categories, the functor $\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$ that classifies p acts by carrying an object $c\colon A \to \mathsf{P}$ to the $\mathcal{B}_{/A}$-groupoid $\mathsf{P}|_c$ that is determined by the cartesian square

$$\begin{tikzcd} \mathsf{P}|_c \arrow[r] \arrow[d] & \mathsf{P} \arrow[d] \\ A \arrow[r, "c"] & \mathsf{C}. \end{tikzcd}$$

Remark 2.2.2.3. For any $A \in \mathcal{B}$ the functor π_A^* carries the universal left fibration in $\mathcal{B}$ to the universal left fibration in $\mathcal{B}_{/A}$. In fact, in view of Remark 2.2.1.7, this follows immediately from the observation that we have a commutative diagram

$$\begin{tikzcd} & 1_{\mathcal{B}_{/A}} \arrow[dl, "\pi_A^*(\mathrm{id}_{\mathsf{Grpd}_{\mathcal{B}}})"'] \arrow[dr, "\mathrm{id}_{\mathsf{Grpd}_{\mathcal{B}_{/A}}}"] & \\ \pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[rr, "\simeq"] & & \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{Grpd}_{\mathcal{B}_{/A}}, \mathsf{Grpd}_{\mathcal{B}_{/A}}) \end{tikzcd}$$

(see Remark 1.4.1.2 and Proposition 1.2.5.4).

The main goal of this section is to prove that the universal left fibration admits the following explicit description:

Proposition 2.2.2.4. *The global section* $1\colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$ *that is determined by the final object* $1_{\mathcal{B}} \in \mathcal{B}$ *defines a final object in the universe* $\mathsf{Grpd}_{\mathcal{B}}$*. Moreover, there is an equivalence* $(\mathsf{Grpd}_{\mathcal{B}})_{1/} \simeq (\mathsf{Grpd}_{\mathcal{B}})_*$ *that fits into the commutative diagram*

$$\begin{tikzcd} (\mathsf{Grpd}_{\mathcal{B}})_{1/} \arrow[r, "\simeq"] \arrow[dr, "(\pi_1)_!"'] & (\mathsf{Grpd}_{\mathcal{B}})_* \arrow[d, "\phi"] \\ & \mathsf{Grpd}_{\mathcal{B}}\,. \end{tikzcd}$$

The remainder of this section is devoted to the proof of Proposition 2.2.2.4. The core ingredient will be the following proposition:

Proposition 2.2.2.5. *Let* $g\colon \mathsf{D} \to \mathsf{C}$ *be a functor between large* $\mathcal{B}$*-categories and suppose that* g *admits a factorisation* $pi\colon \mathsf{D} \to \mathsf{P} \to \mathsf{C}$ *into a final map and a* small *right fibration. Let* $f\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ *be the associated functor. Consider the left fibration* π *that is defined by the cartesian square*

$$\begin{tikzcd} \mathsf{Z} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, (\mathsf{Grpd}_{\mathcal{B}})_*) \arrow[d, "\phi_*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "g^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}). \end{tikzcd}$$

Then there is an initial object $z\colon 1 \to \mathsf{Z}$ *whose image along* π *is* f*.*

Proof. Since $i\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{P}^{\mathrm{op}}$ is initial and $(\mathsf{Grpd}_{\mathcal{B}})_* \to \mathsf{Grpd}_{\mathcal{B}}$ is a left fibration, the pullback square in the statement of the lemma decomposes into two cartesian squares

$$\begin{tikzcd} \mathsf{Z} \arrow[r] \arrow[d, "\pi"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{P}^{\mathrm{op}}, (\mathsf{Grpd}_{\mathcal{B}})_*) \arrow[r, "i^*"] \arrow[d, "\phi_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, (\mathsf{Grpd}_{\mathcal{B}})_*) \arrow[d, "\phi_*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "p^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}) \arrow[r, "i^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}). \end{tikzcd}$$

By applying the Yoneda embedding $\mathsf{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathsf{PSh}_{\widehat{\mathrm{Ani}}}(\mathsf{Cat}(\widehat{\mathcal{B}}))$ to the left square, we obtain a pullback square

$$\begin{tikzcd} \mathrm{map}_{\mathsf{Cat}(\widehat{\mathcal{B}})}(-, \mathsf{Z}) \arrow[r] \arrow[d] & \mathrm{map}_{\mathsf{Cat}(\widehat{\mathcal{B}})}(- \times \mathsf{P}^{\mathrm{op}}, (\mathsf{Grpd}_{\mathcal{B}})_*) \arrow[d, "\phi_*"] \\ \mathrm{map}_{\mathsf{Cat}(\widehat{\mathcal{B}})}(- \times \mathsf{C}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "(\mathrm{id} \times p)^*"] & \mathrm{map}_{\mathsf{Cat}(\widehat{\mathcal{B}})}(- \times \mathsf{P}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) \end{tikzcd}$$

of $\widehat{\mathrm{Ani}}$-valued presheaves on $\mathsf{Cat}(\widehat{\mathcal{B}})$. By the formula for the mapping spaces in arrow ∞-categories (see for example [28, Proposition 2.3]), we thus obtain an equivalence

$$\mathrm{map}_{\mathsf{Cat}(\widehat{\mathcal{B}})}(-, \mathsf{Z}) \simeq \mathrm{map}_{\mathsf{LFib}^{\mathbf{U}}}(s_0(-) \times p^{\mathrm{op}}, \phi) \tag{$*$}$$

(in which $s_0\colon \mathsf{Cat}(\widehat{\mathcal{B}}) \to \mathsf{LFib}^{\mathbf{U}}$ is the functor that carries a $\mathcal{B}$-category C to the identity id_{C}). As a consequence, the cartesian square

$$\begin{tikzcd} \mathsf{P}^{\mathrm{op}} \arrow[r] \arrow[d, "p"] & (\mathsf{Grpd}_{\mathcal{B}})_* \arrow[d] \\ \mathsf{C}^{\mathrm{op}} \arrow[r, "f"] & \mathsf{Grpd}_{\mathcal{B}} \end{tikzcd}$$

gives rise to an object $z\colon 1 \to \mathsf{Z}$ whose image along π is f.

We still need to show that z is initial. To that end, note that by the equivalence $(*)$, the identity on Z corresponds to a commutative square

$$\begin{tikzcd}
\mathsf{Z}\times\mathsf{P}^{\mathrm{op}} \arrow[r] \arrow[d, "\mathrm{id}\times p"] & (\mathrm{Grpd}_{\mathcal{B}})_* \arrow[d] \\
\mathsf{Z}\times\mathsf{C}^{\mathrm{op}} \arrow[r, "k"] & \mathrm{Grpd}_{\mathcal{B}},
\end{tikzcd}$$

and therefore gives rise to a map $\mathsf{Z}\times\mathsf{P}^{\mathrm{op}} \to \int k$ of left fibrations over $\mathsf{Z}\times\mathsf{C}^{\mathrm{op}}$. Using Corollary 2.2.1.10, this map corresponds to a morphism $\alpha\colon \mathrm{pr}_1^*(f) \to k$ in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{Z}\times\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}})$, where $\mathrm{pr}_1\colon \mathsf{Z}\times\mathsf{C}^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$ denotes the projection. Let us consider the evident commutative square

$$\begin{tikzcd}
\mathrm{pr}_1^*(f) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathrm{pr}_1^*(f) \arrow[d, "\alpha"] \\
\mathrm{pr}_1^*(f) \arrow[r, "\alpha"] & k
\end{tikzcd}$$

as a map $\tau\colon \mathrm{id} \to \alpha$ in the $\mathcal{B}$-category $\underline{\mathrm{Fun}}_{\mathcal{B}}((\Delta^1\otimes\mathsf{Z})\times\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}})$. By again using Corollary 2.2.1.10, this map corresponds to a morphism $(\Delta^1\otimes\mathsf{Z})\times\mathsf{P}^{\mathrm{op}} \to \int\alpha$ of left fibrations over $(\Delta^1\otimes\mathsf{Z})\times\mathsf{C}^{\mathrm{op}}$ and therefore by the equivalence $(*)$ to a map $\epsilon\colon \Delta^1\otimes\mathsf{Z}\to\mathsf{Z}$. By functoriality, the map $\mathsf{Z}\to\mathsf{Z}$ that is induced by restricting ϵ along the inclusion $d^1\colon \mathsf{Z}\to\Delta^1\otimes\mathsf{Z}$ corresponds to the outer square in the commutative diagram

$$\begin{tikzcd}
\mathsf{Z}\times\mathsf{P}^{\mathrm{op}} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{id}\times p"] & \mathsf{P}^{\mathrm{op}} \arrow[r] \arrow[d, "p"] & (\mathrm{Grpd}_{\mathcal{B}})_* \arrow[d] \\
\mathsf{Z}\times\mathsf{C}^{\mathrm{op}} \arrow[r, "\mathrm{pr}_1"] & \mathsf{C}^{\mathrm{op}} \arrow[r, "f"] & \mathrm{Grpd}_{\mathcal{B}},
\end{tikzcd}$$

hence this map is equivalent to the constant functor $z\colon \mathsf{Z}\to 1\to\mathsf{Z}$. Precomposing the map $\Delta^1\otimes\mathsf{Z}\to\mathsf{Z}$ with the inclusion $d^0\colon \mathsf{Z}\to\Delta^1\otimes\mathsf{Z}$, on the other hand, produces the identity on Z. As moreover the restriction of the map $\Delta^1\otimes\mathsf{Z}\to\mathsf{Z}$ along $z\colon 1\to\mathsf{Z}$ recovers the identity on z, Lemma 2.1.3.12 implies that z is initial. $\square$

Proposition 2.2.2.5 has the following immediate consequence:

Corollary 2.2.2.6. *Let* $g\colon \mathsf{D} \to \mathsf{C}$ *be a functor between large* $\mathcal{B}$*-categories, and suppose that g admits a factorisation pi* : $\mathsf{D} \to \mathsf{P} \to \mathsf{C}$ *into a final map and a* small *right fibration. Let* $f\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ *be the associated functor. Then there is a cartesian square*

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{f/} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, (\mathsf{Grpd}_{\mathcal{B}})_{*}) \arrow[d, "\phi_*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "g^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}). \end{tikzcd}$$

Proof. By Proposition 2.2.2.5, the $\mathcal{B}$-category Z that is defined by the cartesian square

$$\begin{tikzcd} \mathsf{Z} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, (\mathsf{Grpd}_{\mathcal{B}})_{*}) \arrow[d, "\phi_*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "g^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}). \end{tikzcd}$$

admits an initial object $\varnothing_{\mathsf{Z}}\colon 1 \to \mathsf{Z}$ whose image in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is f. By Corollary 2.1.3.13 we obtain a commutative square

$$\begin{tikzcd} 1 \arrow[r, "\varnothing_{\mathsf{Z}}"] \arrow[d, "f"] & \mathsf{Z} \arrow[d] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{f/} \arrow[r, "(\pi_f)_!"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which the two maps starting in the upper left corner are initial and the two maps ending in the lower right corner are left fibrations. When regarded as a lifting problem, the above square thus admits a unique filler $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{f/} \to \mathsf{Z}$ that is both initial and a left fibration and therefore an equivalence. ☐

Proof of Proposition 2.2.2.4. Applying Corollary 2.2.2.6 to $\mathsf{D} \simeq \mathsf{C} \simeq 1$, we immediately conclude that the object $1\colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$ gives rise to a cartesian square

$$\begin{tikzcd} (\mathsf{Grpd}_{\mathcal{B}})_{1/} \arrow[r] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})_{*} \arrow[d] \\ \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\mathrm{id}"] & \mathsf{Grpd}_{\mathcal{B}}, \end{tikzcd}$$

which implies that the upper horizontal map must be an equivalence. Moreover, in light of Remark 1.2.6.8, evaluating the map $(\pi_1)_! \colon (\mathsf{Grpd}_{\mathcal{B}})_{/1} \to \mathsf{Grpd}_{\mathcal{B}}$ at $A \in \mathcal{B}$ recovers the map $(\pi_A)_! \colon (\mathcal{B}_{/A})_{/A} \to \mathcal{B}_{/A}$ and is therefore an equivalence. Hence Proposition 2.1.3.15 implies that $1 \colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$ is final. □

Remark 2.2.2.7. Proposition 2.2.2.4 shows, together with Remark 1.2.6.8, that the $\widehat{\mathrm{Cat}}_\infty$-valued sheaf that corresponds to $(\mathsf{Grpd}_{\mathcal{B}})_*$ is given by sending $A \in \mathcal{B}$ to the ∞-category of pointed objects in $\mathcal{B}_{/A}$.

Corollary 2.2.2.8. *Let* C *be a (large)* $\mathcal{B}$*-category and let* $p \colon \mathsf{P} \to \mathsf{C}$ *be a small left fibration that is classified by a functor* $f \colon \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$*. Then* $\Gamma(p)$ *is classified by the functor* $\Gamma \circ \Gamma(f) \colon \Gamma(\mathsf{C}) \to \mathcal{B} \to \mathrm{Ani}$.

Proof. By Proposition 2.2.2.4, there is a commutative diagram

$$\begin{tikzcd} \Gamma(\mathsf{P}) \arrow[r] \arrow[d, "\Gamma(p)"] & \mathcal{B}_{1/} \arrow[r] \arrow[d] & \mathrm{Ani}_{1/} \arrow[d] \\ \Gamma(\mathsf{C}) \arrow[r, "\Gamma(f)"] & \mathcal{B} \arrow[r, "\Gamma"] & \mathrm{Ani} \end{tikzcd}$$

in which both squares are cartesian and in which the left square arises from applying the global sections functor Γ to the cartesian square in $\mathrm{Cat}(\widehat{\mathcal{B}})$ that exhibits p as being classified by f. □

2.3. Yoneda's lemma

The goal of this section is to make use of the theory of left fibrations and their interplay with $\mathsf{Grpd}_{\mathcal{B}}$-valued functors via the straightening equivalence in order to prove a version of Yoneda's lemma for $\mathcal{B}$-categories. Recall that if $\mathcal{C}$ is an ∞-category, one has a functor $\mathrm{map}_{\mathcal{C}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ani}$ whose transpose yields the *Yoneda embedding* $h_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$, and Yoneda's lemma asserts that is a natural equivalence $\mathrm{map}_{\mathrm{PSh}(\mathcal{C})}(h_{\mathcal{C}}(c), F) \simeq F(c)$ for every $c \in \mathcal{C}$ and every $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ani}$. Thus, in order to even *state* Yoneda's lemma for $\mathcal{B}$-categories, we first need to construct a functor $\mathrm{map}_{\mathsf{C}}(-,-) \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$ for every $\mathcal{B}$-category C. Using the straightening equivalence for left fibrations, this is easy: we may simply straighten the left fibration $p \colon \mathrm{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ from Section 2.1.2.

However, in the case where the $\mathcal{B}$-category is *large* (such as $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$), we run into a problem: in this case, we can only straighten p if this is a *small* left fibration. This is precisely the condition that the $\mathcal{B}$-category in question is *locally small*. We study this notion in some detail in Section 2.3.1, before we move on to state and prove Yoneda's lemma in Section 2.3.2.

2.3.1. Locally small $\mathcal{B}$-categories

In higher category theory, a large ∞-category $\mathcal{C}$ is called locally small if $\mathrm{map}_{\mathcal{C}}(c, d)$ is small for every pair of objects $c, d \in \mathcal{C}$. In this section, we generalise this notion to $\mathcal{B}$-categories. Recall from the discussion at the end of Section 2.2.1 the definition of a *small* left fibration. We may now define:

Definition 2.3.1.1. A *locally small* $\mathcal{B}$-category is a large $\mathcal{B}$-category C for which the left fibration $\mathrm{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is small.

Remark 2.3.1.2. The base change of a locally small along every geometric morphism and every *étale* algebraic morphism of ∞-topoi is locally small as well. In fact, in light of Remark 2.1.2.5, it suffices to show that base change along these functors preserve small left fibrations. This is a straightforward consequence of the fact that each of these functors and their left adjoints all preserve the property of being small (see Section 1.2.4).

As a consequence of Proposition 2.1.2.4, the fibre of $\mathrm{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ over any pair of objects $(c, d)\colon A \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ in context $A \in \mathcal{B}$ is a large $\mathcal{B}_{/A}$-groupoid, hence the fibre can be computed as the fibre of the induced map of core $\mathcal{B}$-groupoids, which is simply the pullback

$$\begin{tikzcd} \mathrm{map}_{\mathsf{C}}(c, d) \arrow[r] \arrow[d] & \mathsf{C}_1 \arrow[d] \\ A \arrow[r, "(c,d)"] & \mathsf{C}_0 \times \mathsf{C}_0. \end{tikzcd}$$

Using Proposition 2.2.1.9, we may therefore deduce:

Proposition 2.3.1.3. *A $\mathcal{B}$-category C is locally small if and only if for any pair of objects c, d in C in context $A \in \mathcal{B}$ the (a priori large) mapping $\mathcal{B}_{/A}$-groupoid $\mathrm{map}_{\mathsf{C}}(c, d)$ is contained in $\mathcal{B}_{/A}$.* $\square$

Example 2.3.1.4. Using Proposition 2.3.1.3 together with Proposition 1.4.1.3, one finds that the universe $\mathrm{Grpd}_{\mathcal{B}}$ is locally small.

Proposition 2.3.1.5. *A locally small $\mathcal{B}$-category C is small if and only if $\mathsf{C}^{\simeq}$ is a small $\mathcal{B}$-groupoid.*

Proof. The condition is clearly necessary, so let us assume that C is locally small and that $\mathsf{C}^{\simeq}$ is a small $\mathcal{B}$-groupoid. By making use of the Segal conditions, we need only show that C_1 is contained in $\mathcal{B}$. Since $\mathsf{C}_0 \times \mathsf{C}_0$ is an object of $\mathcal{B}$, this follows from the observation that C_1 is recovered as the mapping $\mathcal{B}$-groupoid of the pair $(\mathrm{pr}_0, \mathrm{pr}_1)\colon \mathsf{C}_0 \times \mathsf{C}_0 \to \mathsf{C}_0 \times \mathsf{C}_0$. □

Lemma 2.3.1.6. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor such that C is small and D is locally small. Then the essential image E of f is small.*

Proof. Being a full subcategory of D, the $\mathcal{B}$-category E is locally small (using Proposition 1.3.2.7), hence Proposition 2.3.1.5 implies that E is small whenever $\mathsf{E}^{\simeq}$ is a small $\mathcal{B}$-groupoid. By Corollary 1.3.2.15, E_0 is the image of the map $f_0\colon \mathsf{C}_0 \to \mathsf{D}_0$, hence one finds that E_0 is the colimit of the Čech nerve of $\mathsf{C}_0 \to \mathsf{D}_0$, i.e. $\mathsf{E}_0 \simeq \mathrm{colim}_n\, \mathsf{C}_0 \times_{\mathsf{D}_0} \cdots \times_{\mathsf{D}_0} \mathsf{C}_0$. As Δ is a small 1-category, it suffices to show that for each $n \geq 0$ the $(n+1)$-fold fibre product $\mathsf{C}_0 \times_{\mathsf{D}_0} \cdots \times_{\mathsf{D}_0} \mathsf{C}_0 \in \widehat{\mathcal{B}}$ is contained in $\mathcal{B}$. We may identify this object as the pullback of the map $\mathsf{C}_0^{n+1} \to \mathsf{D}_0^{n+1}$ along the diagonal $\mathsf{D}_0 \to \mathsf{D}_0^{n+1}$. Since the map $\mathsf{D}_0 \to \mathsf{D}_n$ is a monomorphism in $\widehat{\mathcal{B}}$, we obtain a monomorphism

$$\mathsf{C}_0 \times_{\mathsf{D}_0} \cdots \times_{\mathsf{D}_0} \mathsf{C}_0 \hookrightarrow \mathrm{map}_{\mathsf{D}}(\mathrm{pr}_0^* f_0, \dots, \mathrm{pr}_n^* f_n)$$

where $\mathrm{pr}_i\colon \mathsf{C}_0^{n+1} \to \mathsf{C}_0$ denotes the ith projection. Since D is by assumption locally small, the codomain of this map is contained in $\mathcal{B}$, hence the result follows. □

Proposition 2.3.1.7. *For any small $\mathcal{B}$-category C and any locally small $\mathcal{B}$-category D, the (large) functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is locally small as well.*

Proof. Using Proposition 2.3.1.3, we need to show that for any pair of objects $f, g\colon A \rightrightarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ the mapping $\mathcal{B}$-groupoid $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})}(f, g)$ is small. Let E be the essential image of f when viewed as a functor $A \times \mathsf{C} \to \mathsf{D}$, and let E' be the essential image of g when viewed as a functor $A \times \mathsf{C} \to \mathsf{D}$. Then both E and E' are small $\mathcal{B}$-categories by Lemma 2.3.1.6. By the same argument, the essential image E'' of the functor $\mathsf{E} \sqcup \mathsf{E}' \to \mathsf{D}$ that is induced by the two inclusions is a small $\mathcal{B}$-category that embeds fully faithfully into D. By construction, both f and g are contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}'')$. As the latter is small, the (a priori large) mapping $\mathcal{B}$-groupoid $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})}(f, g) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}'')}(f, g)$ must be contained in $\mathcal{B}$. $\square$

2.3.2. Yoneda's lemma

If C is an arbitrary locally small $\mathcal{B}$-category, applying Theorem 2.2.1.1 to the left fibration $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ gives rise the *mapping $\mathcal{B}$-groupoid functor*

$$\operatorname{map}_{\mathsf{C}}(-,-)\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}.$$

By transposing across the adjunction $\mathsf{C}^{\mathrm{op}} \times - \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, -)$, this functor determines the *Yoneda embedding*

$$h\colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) = \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).$$

Remark 2.3.2.1. By combining Remark 2.1.2.5 and Remark 2.2.2.3, one obtains a commutative diagram

$$\begin{tikzcd} & \pi_A^* \mathsf{C}^{\mathrm{op}} \times \pi_A^* \mathsf{C} \arrow[dl, "\pi_A^* \operatorname{map}_{\mathsf{C}}(-,-)"'] \arrow[dr, "\operatorname{map}_{\pi_A^*\mathsf{C}}(-,-)"] & \\ \pi_A^* \mathsf{Grpd}_{\mathcal{B}} \arrow[rr, "\simeq"] & & \mathsf{Grpd}_{\mathcal{B}_{/A}} \end{tikzcd}$$

for every $A \in \mathcal{B}$ and every $\mathsf{C} \in \mathsf{Cat}(\mathcal{B})$. By combining this observation with Remark 1.2.5.5, one furthermore finds that there is a commutative diagram

$$\begin{tikzcd} \pi_A^* \mathsf{C} \arrow[r, "\pi_A^* h_{\mathsf{C}}"] \arrow[dr, "h_{\pi_A^* \mathsf{C}}"'] & \pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "\simeq"] \\ & \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}). \end{tikzcd}$$

Remark 2.3.2.2. Let C be a locally small $\mathcal{B}$-category, let $A \in \mathcal{B}$ be an arbitrary object and let c be an object in C in context A. Let us furthermore fix a map $f\colon d \to e$ in C in context A. Applying the mapping $\mathcal{B}$-groupoid functor $\mathrm{map}_{\mathsf{C}}(-,-)$ to the pair (id_c, f) of maps in C then results in a morphism $f_! \colon \mathrm{map}_{\mathsf{C}}(c,d) \to \mathrm{map}_{\mathsf{C}}(c,e)$ in $\mathcal{B}_{/A}$. Explicitly, this map is given by applying the chain of equivalences $\mathrm{LFib}^{\mathbf{U}}(\Delta^1 \otimes A) \simeq \mathrm{Fun}_{\mathcal{B}}(\Delta^1 \otimes A, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{B}_{/A})$ to the left fibration $\mathsf{P} \to \Delta^1 \otimes A$ that arises as the pullback

$$\begin{tikzcd} \mathsf{P} \arrow[r] \arrow[d] & \mathsf{C}_{c/} \arrow[d] \\ \Delta^1 \otimes A \arrow[r, "(\mathrm{pr}_1, f)"] & A \times \mathsf{C} \end{tikzcd}$$

in which $\mathrm{pr}_1 \colon \Delta^1 \otimes A \to A$ denotes the projection. By construction of the equivalence of ∞-categories $\mathrm{LFib}^{\mathbf{U}}(\Delta^1 \otimes A) \simeq \mathrm{Fun}(\Delta^1, \mathcal{B}_{/A})$, one now sees that the map $f_!$ fits into the commutative diagram

$$\begin{tikzcd} & \mathrm{map}_{\mathsf{C}}(c,d) \arrow[dl] \arrow[dd] \arrow[rrrr, bend left, "f_!"] & & Z \arrow[ll, "\simeq"'] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{map}_{\mathsf{C}}(c,e) \arrow[dl] \arrow[dd] \\ (\mathsf{C}_{c/})_0 \arrow[dd] & & (\mathsf{C}_{c/})_1 \arrow[ll, "d_1"'] \arrow[rr, "d_0"] \arrow[dd] & & (\mathsf{C}_{c/})_0 \arrow[dd] & \\ & A \arrow[dl, "d"] & & A \arrow[ll, "\mathrm{id}"'] \arrow[rr, "\mathrm{id}"] \arrow[dl, "f"] & & A \arrow[dl, "e"] \\ \mathsf{C}_0 & & \mathsf{C}_1 \arrow[ll, "d_1"'] \arrow[rr, "d_0"] & & \mathsf{C}_0. & \end{tikzcd}$$

Let $g\colon c \to d$ be an arbitrary map in C in context A, and let $\sigma\colon \Delta^2 \otimes A \to \mathsf{C}$ be the 2-morphism that is encoded by the commutative diagram

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[dr, "fg"'] & d \arrow[d, "f"] \\ & e. \end{tikzcd}$$

Let furthermore $\tau\colon \Delta^1 \otimes A \to \mathsf{C}$ be the 2-morphism that is determined by the commutative diagram

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}"] \arrow[dr, "fg"'] & c \arrow[d, "fg"] \\ & e. \end{tikzcd}$$

On account of the decomposition $\Delta^1 \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$, the pair (τ, σ) gives rise to a map $(\Delta^1 \times \Delta^1) \otimes A \to \mathsf{C}$ that by construction defines a section $A \to Z$. Furthermore, the composition $A \to Z \simeq \mathrm{map}_{\mathsf{C}}(c, d)$ recovers g and the composition $A \to Z \to \mathrm{map}_{\mathsf{C}}(c, e)$ recovers fg. Therefore, the map $f_!$ acts by sending a map $g\colon c \to d$ to the composition $fg\colon c \to d \to e$. By a dual argument, the map $f^*\colon \mathrm{map}_{\mathsf{C}}(e, c) \to \mathrm{map}_{\mathsf{C}}(d, c)$ that is determined by applying the mapping $\mathcal{B}$-groupoid functor to the pair (f, id_c) sends a map $g\colon e \to c$ to the composition $gf\colon d \to e \to c$.

If C is a $\mathcal{B}$-category, let us denote by $\mathrm{ev}\colon \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{Grpd}_{\mathcal{B}}$ the evaluation functor, i.e. the counit of the adjunction $\mathsf{C}^{\mathrm{op}} \times - \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, -)$.

Theorem 2.3.2.3 (Yoneda's lemma)**.** *For any $\mathcal{B}$-category C, there is a commutative diagram*

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "h\times\mathrm{id}"] \arrow[dr, "\mathrm{ev}"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-,-)"] \\
 & \mathsf{Grpd}_{\mathcal{B}}
\end{tikzcd}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$.

The proof of Theorem 2.3.2.3 employs a strategy that is similar to the one used by Cisinski in [18] for a proof of Yoneda's lemma for ∞-categories. We will need the following lemma:

Lemma 2.3.2.4. *Let*

$$\begin{tikzcd}
\mathsf{P} \arrow[r, "f"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\
\mathsf{C} \times \mathsf{E} \arrow[r, "\mathrm{id}\times g"] & \mathsf{C} \times \mathsf{D}
\end{tikzcd}$$

be a commutative diagram in $\mathrm{Cat}(\mathcal{B})$ *such that the maps p and q are left fibrations, and suppose that for any object $c\colon A \to \mathsf{C}$ the induced map $f|_c\colon \mathsf{P}|_c \to \mathsf{Q}|_c$ is initial. Then f is initial.*

Proof. By Remark 2.1.4.2 and the fact that q is a left fibration, it suffices to show that f is a covariant equivalence over $\mathsf{C} \times \mathsf{D}$. Using Proposition 2.1.4.3, it is moreover enough to show that for any object $d\colon A \to \mathsf{D}$ the induced map $f_{/d}$ (that is obtained by pulling back f along the right fibration $\mathsf{C}\times\mathsf{D}_{/d} \to \mathsf{C}\times\mathsf{D}$) is a covariant equivalence over C. In fact, if this is the case, then Proposition 2.1.4.3 implies that for any object $c\colon A \to \mathsf{C}$ the induced map $(f_{/d})_{/c}$ becomes an equivalence after applying the groupoidification functor. Now it is straightforward to see that this map is equivalently obtained by the pullback $f_{/(c,d)}$ of f along the right fibration $(\mathsf{C} \times \mathsf{D})_{/(c,d)} \to \mathsf{C} \times \mathsf{D}$ that is determined by the object $(c, d)\colon A \to \mathsf{C} \times \mathsf{D}$, hence another application of Proposition 2.1.4.3 implies that f is a covariant equivalence over $\mathsf{C} \times \mathsf{D}$.

Now since the projections $\mathsf{C} \times \mathsf{E}_{/d} \to \mathsf{C}$ and $\mathsf{C} \times \mathsf{D}_{/d} \to \mathsf{C}$ are smooth, the diagonal maps in the induced commutative diagram

$$\begin{tikzcd} \mathsf{P}_{/d} \arrow[rr, "f_{/d}"] \arrow[dr] & & \mathsf{Q}_{/d} \arrow[dl] \\ & \mathsf{C} & \end{tikzcd}$$

are smooth (as left fibrations are smooth by the dual of Proposition 2.1.4.9). As the induced map $\left(f_{/d}\right)|_c$ on the fibres over $c\colon A \to \mathsf{C}$ is a pullback of the initial functor $f|_c$ along a proper map, this functor must be initial as well, hence we may apply Proposition 2.1.4.12 to deduce that $f_{/d}$ is a covariant equivalence over C, as required. □

Remark 2.3.2.5. In the situation of Lemma 2.3.2.4, let $\pi_A^*(f)|_{\bar{c}}\colon \pi_A^*\mathsf{P}|_{\bar{c}} \to \pi_A^*\mathsf{Q}|_{\bar{c}}$ be the functor that arises as the fibre of the morphism $\pi_A^*(f)\colon \pi_A^*(\mathsf{P}) \to \pi_A^*(\mathsf{Q})$ over the object $\bar{c}\colon 1 \to \pi_A^*\mathsf{C}$ that corresponds to $c\colon A \to \mathsf{C}$ by transposition. Then $f|_c$ is obtained as the image of $\pi_A^*(f)|_{\bar{c}}$ along the forgetful functor $(\pi_A)_!$, hence Remark 2.1.3.3 implies that $f|_c$ is initial if and only if $\pi_A^*(f)|_{\bar{c}}$ is initial. Therefore, we deduce from Lemma 2.3.2.4 that f is an initial map if and only if for every object $A \in \mathcal{B}$ the fibre of $\pi_A^*(f)$ over every *global* object $c\colon 1 \to \pi_A^*\mathsf{C}$ is initial.

Proof of Theorem 2.3.2.3. Let $\int \mathrm{ev} \to \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the left fibration that classifies the evaluation functor ev. Using Theorem 2.2.1.1, it suffices to show that there is a cartesian square

$$\begin{tikzcd}
\int \mathrm{ev} \arrow[r] \arrow[d] & \mathsf{Tw}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \arrow[d] \\
\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "h\times\mathrm{id}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).
\end{tikzcd}$$

Note that by definition of the evaluation functor, there is a cartesian square

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{C}) \arrow[r, "f"] \arrow[d] & \int \mathrm{ev} \arrow[d] \\
\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "\mathrm{id}\times h"] & \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).
\end{tikzcd}$$

Moreover, using functoriality of the twisted arrow $\mathcal{B}$-category construction, we may construct a commutative diagram

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{C}) \arrow[r, "g"] \arrow[d] \arrow[rr, bend left, "\mathsf{Tw}(h)"] & \mathsf{P} \arrow[r] \arrow[d] & \mathsf{Tw}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \arrow[d] \\
\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "\mathrm{id}\times h"] \arrow[rr, bend right, "h\times h"'] & \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "h\times\mathrm{id}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

in which the right square is cartesian. As a consequence, one obtains a commutative square

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{C}) \arrow[r, "f"] \arrow[d, "g"] & \int \mathrm{ev} \arrow[d] \\
\mathsf{P} \arrow[r] & \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).
\end{tikzcd}$$

To complete the proof, it therefore suffices to produce a lift $\mathsf{P} \to \int \mathrm{ev}$ in the previous square and to show that this map is an equivalence. This is possible once we verify that the two maps f and g are initial.

In order to show that the map g is initial, note that we are in the situation of Lemma 2.3.2.4, which means that it suffices to show that for any object $c\colon A \to \mathsf{C}$

the induced functor $g|_c\colon \mathsf{Tw}(\mathsf{C})|_c \to \mathsf{P}|_c$ is initial. By construction of P, this map is equivalent to the functor $\mathsf{Tw}(h)|_c\colon \mathsf{Tw}(\mathsf{C})|_c \to \mathsf{Tw}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))|_{h(c)}$, and by using Proposition 2.1.2.7 this map can be identified with the functor

$$\mathsf{C}_{c/} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{h(c)/}.$$

This map is initial as it sends the initial section id_c to the initial section $\mathrm{id}_{h(c)}$.

In order to prove that $f\colon \mathsf{Tw}(\mathsf{C}) \to \int \mathrm{ev}$ is initial, we employ Lemma 2.3.2.4 once more to conclude that it will be sufficient to show that the map $f|_c$ in the induced cartesian square

$$\begin{tikzcd} \mathsf{C}_{c/} \arrow[r, "f|_c"] \arrow[d] & \int \mathrm{ev}|_c \arrow[d] \\ A \times \mathsf{C} \arrow[r, "\mathrm{id}\times h"] & A \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

is initial. Remark 2.3.2.5 implies that we may regard this square as a diagram in $\mathsf{Cat}(\widehat{\mathcal{B}}_{/A})$. By combining Remark 2.1.2.5, Remark 2.3.2.1 and Remark 1.2.5.5, we may thus assume without loss of generality $A \simeq 1$.

Now the key observation is that since the composition $\mathrm{ev} \circ (c \times \mathrm{id})$ can be identified with c^*, we may apply Proposition 2.2.2.5 to the factorisation $1 \to \mathsf{C}_{/c} \to \mathsf{C}$ of c into a final map followed by a right fibration (cf. Corollary 2.1.3.13) to conclude that there is a lift $z\colon 1 \to \int \mathrm{ev}|_c$ of $h(c)$ that defines an initial object in $\int \mathrm{ev}|_c$. Since the fibre of $\int \mathrm{ev}|_c$ over $h(c)$ is given by $\mathrm{map}_{\mathsf{C}}(c,c)$, the object z determines a map $w\colon 1 \to \mathrm{map}_{\mathsf{C}}(c,c)$ in C, and it suffices to show that this map is equivalent to id_c. To that end, note that the proof of Proposition 2.2.2.5 shows that in light of the equivalence

$$\mathrm{map}_{\mathsf{Cat}(\mathcal{B})}(1, \textstyle\int \mathrm{ev}|_c) \simeq \mathrm{map}_{\mathsf{LFib}}((\pi_c)_!^{\mathrm{op}}, \phi)$$

(where $(\pi_c)_!^{\mathrm{op}}\colon (\mathsf{C}_{/c})^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$ is the projection and ϕ is the universal left fibration), the initial object $z\colon 1 \to \int \mathrm{ev}|_c$ is determined by the outer square in the commutative diagram

$$\begin{tikzcd} (\mathsf{C}_{/c})^{\mathrm{op}} \arrow[r] \arrow[d, "(\pi_c)_!^{\mathrm{op}}"] & \mathsf{Tw}(\mathsf{C}) \arrow[r] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})_* \arrow[d, "\phi"] \\ \mathsf{C}^{\mathrm{op}} \arrow[r, "\mathrm{id}\times c"] & \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "\mathrm{map}_{\mathsf{C}}(-,-)"] & \mathsf{Grpd}_{\mathcal{B}} \end{tikzcd}$$

in which the left square is the pullback diagram from Proposition 2.1.2.7. As a consequence, the section $w\colon 1 \to \mathrm{map}_{\mathsf{C}}(c, c)$ is encoded by pasting the previous diagram with the lift

$$\begin{tikzcd} & (\mathsf{C}_{/c})^{\mathrm{op}} \arrow[d, "(\pi_c)_!^{\mathrm{op}}"] \\ 1 \arrow[ur, bend left, "\mathrm{id}_c"] \arrow[r, "c"] & \mathsf{C}^{\mathrm{op}}, \end{tikzcd}$$

which immediately implies the claim. □

Corollary 2.3.2.6. *For every* $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$, *the Yoneda embedding* $h\colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *is fully faithful.*

Proof. By Theorem 2.3.2.3 the canonical square

$$\begin{tikzcd} \mathsf{Tw}(\mathsf{C}) \arrow[r] \arrow[d] & \mathsf{Tw}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \arrow[d] \\ \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "h^{\mathrm{op}} \times h"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

that is obtained by functoriality of the twisted arrow construction is cartesian, which proves the claim upon applying the core $\mathcal{B}$-groupoid functor. □

Remark 2.3.2.7. Theorem 2.3.2.3 moreover implies that if $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is a presheaf, then the associated right fibration $\int F \to \mathsf{C} \times A$ fits into a pullback square

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[d] \\ \mathsf{C} \times A \arrow[r, hook, "h_{\mathsf{C}} \times \mathrm{id}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times A. \end{tikzcd}$$

In other words, there is an equivalence $\int F \simeq \mathsf{C}_{/F}$ of right fibrations over C in context A.

Definition 2.3.2.8. Let C be a $\mathcal{B}$-category. Then a presheaf $f\colon A \times \mathsf{C}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ is said to be *representable* by an object $c\colon A \to \mathsf{C}$ if there is an equivalence $f \simeq h_{\mathsf{C}}(c)$ (where h_{C} denotes the Yoneda embedding).

Remark 2.3.2.9. If $p\colon \mathsf{P} \to A \times \mathsf{C}$ is a right fibration between $\mathcal{B}$-categories (i.e. an object $A \to \mathsf{RFib}_{\mathsf{C}}$), we may say that p is *representable* if the associated presheaf $A \times \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ that classifies p is representable in the sense of Definition 2.3.2.8. Equivalently, this means that there is an object $c\colon A \to \mathsf{C}$ and an equivalence $\mathsf{C}_{/c} \simeq \mathsf{P}$ over $A \times \mathsf{C}$.

Remark 2.3.2.10. The property of being representable is *local* in $\mathcal{B}$: given any cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ and presheaf $f\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, then f is representable if and only if $s_i^*(f)$ is representable for each i. In fact, this is certainly a necessary condition, and it is also sufficient on account of the lifting problem

$$\begin{tikzcd}
\bigsqcup_i A_i \arrow[r, "(s_i^* f)"] \arrow[d, two heads, "(s_i)"'] & \mathsf{C} \arrow[d, hook, "h_{\mathsf{C}}"] \\
A \arrow[r, "f"'] \arrow[ur, dashed] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

having a unique solution.

Proposition 2.3.2.11. *A right fibration $p\colon \mathsf{P} \to \mathsf{C} \times A$ between $\mathcal{B}$-categories is representable by an object $c\colon A \to \mathsf{C}$ if and only if there is a final section $A \to \mathsf{P}$ over A.*

Proof. In light of Remark 2.1.3.3, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Now if p is representable by an object $c\colon 1 \to \mathsf{C}$ then there is an equivalence $\mathsf{P} \simeq \mathsf{C}_{/c}$ over C, hence Proposition 2.1.3.9 implies that there is a final object $1 \to \mathsf{P}$. Conversely, if there is such a final object $d\colon 1 \to \mathsf{P}$, the lifting problem

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_{p(d)}"] \arrow[d, "d"'] & \mathsf{C}_{/p(d)} \arrow[d, "(\pi_{p(d)})_!"] \\
\mathsf{P} \arrow[r, "p"'] \arrow[ur, dotted] & \mathsf{C}
\end{tikzcd}$$

admits a unique solution which is necessarily an equivalence since $\mathrm{id}_{p(d)}$ is final and p is a right fibration. □

We conclude this section with noting that Corollary 2.3.2.6 furthermore implies that equivalences of functors between $\mathcal{B}$-categories can be checked object-wise:

Proposition 2.3.2.12. *Let* C *and* D *be* $\mathcal{B}$*-categories and let* $\colon \Delta^1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ *be a morphism in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$*. Then* α *is an equivalence if and only if for all* $c \colon A \to \mathsf{C}$ *the map* $\alpha(c) \colon \Delta^1 \otimes A \to \mathsf{D}$ *is an equivalence in* D*.*

Proof. The condition is clearly necessary, so suppose that $\alpha(c)$ is an equivalence for every object $c \colon A \to \mathsf{C}$. By Corollary 2.3.2.6, the functor

$$h_* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}})$$

is fully faithful and therefore in particular conservative. It therefore suffices to show that the map $h_*\alpha$ is an equivalence. For any $(c, d) \colon 1 \to \mathsf{C} \times \mathsf{D}^{\mathrm{op}}$, the map $h_*\alpha(c, d)$ corresponds to the image of $\alpha(c)$ along $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \to \mathsf{Grpd}_{\mathcal{B}}$ in which the second arrow is given by evaluation at d. As a consequence, the map $h_*\alpha(c, d)$ must be an equivalence in $\mathsf{Grpd}_{\mathcal{B}}$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and using Remark 1.2.5.5 and Remark 2.3.2.1, the same is true when (c, d) is in arbitrary context. By replacing C with $\mathsf{C} \times \mathsf{D}^{\mathrm{op}}$, we may therefore assume without loss of generality $\mathsf{D} \simeq \mathsf{Grpd}_{\mathcal{B}}$. In this case, the desired result follows from Proposition 2.1.1.12. □

3. Colimits and cocompletion

Recall from Section 2.3.2 that to every $\mathcal{B}$-category C we can associate its *Yoneda embedding* $h_{\mathsf{C}} \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. One of the main goals in this chapter to establish the *universal property* of this map: it exhibits $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ as the *free cocompletion* of C. In order to make the this result precise, we first need to introduce a rather substantial portion of categorical tools in the setting of $\mathcal{B}$-categories.

We will begin in Section 3.1 by studying adjunction between $\mathcal{B}$-categories. The main interesting observation here is that, while one obtains a functioning theory of adjunctions which is completely parallel to that of adjunctions between ∞-categories, one can also describe adjunctions between $\mathcal{B}$-categories quite explicitly as section-wise adjunctions of the corresponding sheaves of ∞-categories on $\mathcal{B}$ that are compatible with the transition functors in a certain way.

In Section 3.2 and Section 3.3, we develop the theory of *limits and colimits* in a $\mathcal{B}$-category. In particular, we make precise what it means for a $\mathcal{B}$-category to be *cocomplete*. Perhaps surprisingly, the latter does not simply mean that a $\mathcal{B}$-category C admits colimits indexed by every indexing $\mathcal{B}$-category I, but also that the étale base change $\pi_A^*\mathsf{C}$ admits colimits indexed by every $\mathcal{B}_{/A}$-category I, for every $A \in \mathcal{B}$. In other words, the notion of cocompleteness is designed so that it is stable under étale base change, which is a prerequisite for every reasonably defined notion.

As a next step, we discuss *Kan extensions* of functors between $\mathcal{B}$-categories in Section 3.4. The main outcome of this section will be that (left) Kan extensions always exist provided that the target is sufficiently cocomplete, as it is the case in ordinary (higher) category theory.

At this point, we will have built sufficient machinery to be able to establish the universal property of presheaf $\mathcal{B}$-categories in Section 3.5. More generally, we will construct the *free* U*-cocompletion* of a small $\mathcal{B}$-category for any so called *internal class* U of $\mathcal{B}$-categories, which can simply be defined as a full subcategory of the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ of $\mathcal{B}$-categories. As an application, we will provide a general method to *decompose* colimits into simpler pieces. In particular, this will allow us to derive a very explicit description of the notion of cocompleteness.

3.1. Adjunctions

In this section we will study *adjunctions* between $\mathcal{B}$-categories. We begin in Section 3.1.1 by defining such adjunctions as ordinary adjunctions in the underlying bicategory of $\mathrm{Cat}(\mathcal{B})$. In Section 3.1.2 we compare our definition with *relative* adjunctions and prove a convenient section-wise criterion for when a functor admits a left or right adjoint. In Section 3.1.3 we discuss an alternative approach to adjunctions based on an equivalence of mapping $\mathcal{B}$-groupoids. Finally, we discuss the special case of *reflective subcategories* in Section 3.1.4.

3.1.1. Definitions and basic properties

Let C and D be $\mathcal{B}$-categories, let $f, g \colon \mathsf{C} \rightrightarrows \mathsf{D}$ be two functors and let $\alpha \colon f \to g$ be a morphism of functors, i.e. a map in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$. If $h \colon \mathsf{E} \to \mathsf{C}$ is any other functor, we denote by $\alpha h \colon fh \to gh$ the map $h^*(\alpha)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{E}, \mathsf{D})$. Dually, if $k \colon \mathsf{D} \to \mathsf{E}$ is an arbitrary functor, we denote by $k\alpha \colon kf \to kg$ the map $k_*(\alpha)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$. Having established the necessary notational conventions, we may now define:

Definition 3.1.1.1. Let C and D be $\mathcal{B}$-categories. An *adjunction* between C and D is a tuple (l, r, η, ϵ), where $l \colon \mathsf{C} \to \mathsf{D}$ and $r \colon \mathsf{D} \to \mathsf{C}$ are functors and where $\eta \colon \mathrm{id}_{\mathsf{D}} \to rl$ and $\epsilon \colon lr \to \mathrm{id}_{\mathsf{C}}$ are maps such that there are commutative triangles

$$\begin{tikzcd} l \arrow[r, "l\eta"] \arrow[dr, "\mathrm{id}"'] & lrl \arrow[d, "\epsilon l"] \\ & l \end{tikzcd} \qquad \begin{tikzcd} rlr \arrow[d, "r\epsilon"'] & r \arrow[l, "\eta r"'] \arrow[dl, "\mathrm{id}"] \\ r & \end{tikzcd}$$

in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ and in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{D}, \mathsf{C})$, respectively. We denote such an adjunction by $l \dashv r$, and we refer to η as the *unit* and to ϵ as the *counit* of the adjunction. We say that a pair $(l, r) \colon \mathsf{C} \rightleftarrows \mathsf{D}$ *defines an adjunction* if there exist transformations η and ϵ as above such that the tuple (l, r, η, ϵ) is an adjunction.

Analogous to how adjunctions between ∞-categories can be defined (see [42, §17]), Definition 3.1.1.1 is equivalent to that of an adjunction in the underlying homotopy bicategory of the $(\infty, 2)$-category $\mathrm{Cat}(\mathcal{B})$ (see Section 1.2.5). We may therefore make use of the usual bicategorical arguments to derive results for adjunctions in $\mathrm{Cat}(\mathcal{B})$. Hereafter, we list a few of these results, we refer the reader to [29, § I.6] and [73, § 2.1] for proofs.

Proposition 3.1.1.2. *If $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ and $(l' \dashv r')\colon \mathsf{D} \rightleftarrows \mathsf{E}$ are adjunctions between $\mathcal{B}$-categories, then the composite functors define an adjunction $(ll' \dashv r'r)\colon \mathsf{C} \rightleftarrows \mathsf{E}$.* □

Proposition 3.1.1.3. *Adjoints are unique if they exist, i.e if $(l \dashv r)$ and $(l \dashv r')$ are adjunctions between $\mathcal{B}$-categories, then $r \simeq r'$. Dually, if $(l \dashv r)$ and $(l' \dashv r)$ are adjunctions, then $l \simeq l'$.* □

Proposition 3.1.1.4. *In order for a pair $(l, r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ of functors between $\mathcal{B}$-categories to define an adjunction, it suffices to provide maps $\eta\colon \mathrm{id}_{\mathsf{D}} \to rl$ and $\epsilon\colon lr \to \mathrm{id}_{\mathsf{C}}$ such that the compositions $\epsilon l \circ l\eta$ and $r\epsilon \circ \eta r$ are equivalences.* □

Corollary 3.1.1.5. *If $f\colon \mathsf{C} \to \mathsf{D}$ is an equivalence between $\mathcal{B}$-categories, then the pair (f, f^{-1}) defines an adjunction.* □

Corollary 3.1.1.6. *For any adjunction $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ between $\mathcal{B}$-categories and any equivalence $f\colon D \simeq D'$, the induced pair $(lf^{-1}, fr)\colon \mathsf{C} \rightleftarrows \mathsf{D}'$ defines an adjunction as well.* □

If $\mathcal{A}$ and $\mathcal{B}$ are ∞-topoi and $f\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ is a functor, we will often need to know whether f carries an adjunction $l \dashv r$ in $\mathrm{Cat}(\mathcal{B})$ to an adjunction $f(l) \dashv f(r)$ in $\mathrm{Cat}(\mathcal{A})$. This is obviously the case whenever f is a functor of $(\infty, 2)$-categories. Since we do not wish to dive too deep into $(\infty, 2)$-categorical arguments, we will instead make use of the straightforward observation that f preserves adjunctions whenever there is a bifunctorial map

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))$$

that recovers the action of f on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor.

Lemma 3.1.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be ∞-topoi and let $f\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ be a functor that preserves finite products. Suppose furthermore that there is a map*

$$\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}},$$

where $\mathrm{const}_{\mathcal{B}} : \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathcal{B})$ *and* $\mathrm{const}_{\mathcal{A}} : \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathcal{A})$ *are the constant sheaf functors. Then f induces a bifunctorial map*

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))$$

that recovers the action of f on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor. Moreover, if f is fully faithful and if $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$ *restricts to an equivalence on the essential image of f, then this map is an equivalence.*

Proof. Since f preserves finite products, the map $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$ induces a map

$$- \otimes f(-) \to f(- \otimes -)$$

of bifunctors $\mathrm{Cat}_\infty \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$. This map gives rise to the first arrow in the composition

$$\begin{aligned}\mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(f(- \otimes -), f(-)) &\to \mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(- \otimes f(-), f(-)) \\ &\simeq \mathrm{map}_{\mathrm{Cat}_\infty}(-, \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))),\end{aligned}$$

and by precomposition with the morphism

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(- \otimes -, -) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(f(- \otimes -), f(-))$$

that is induced by f and Yoneda's lemma, we end up with the desired morphism of functors

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))$$

that recovers the morphism $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-,-) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(f(-), f(-))$ upon restriction to core ∞-groupoids. By construction, this map is an equivalence whenever f is fully faithful and $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$ is an equivalence. □

Remark 3.1.1.8. In the situation of Lemma 3.1.1.7, the construction in the proof shows that if C and D are $\mathcal{B}$-categories, the functor

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \to \mathrm{Fun}_{\mathcal{A}}(f(\mathsf{C}), f(\mathsf{D}))$$

that is induced by f and the morphism of functors $\phi\colon -\otimes f(-)\to f(-\otimes -)$ is given as the transpose of the composition

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C},\mathsf{D})\otimes f(\mathsf{C})\xrightarrow{\phi} f(\mathrm{Fun}_{\mathcal{B}}(\mathsf{C},\mathsf{D})\otimes\mathsf{C})\xrightarrow{f(\mathrm{ev})} f(\mathsf{D})$$

in which $\mathrm{ev}\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C},\mathsf{D})\otimes\mathsf{C}\to\mathsf{D}$ denotes the counit of the adjunction $-\otimes\mathsf{C}\dashv\mathrm{Fun}_{\mathcal{B}}(\mathsf{C},-)$.

Using Lemma 3.1.1.7, one now finds:

Corollary 3.1.1.9. *Let $f_*\colon\mathcal{B}\to\mathcal{A}$ be a geometric morphism of ∞-topoi. If a pair (l,r) of functors in* $\mathrm{Cat}(\mathcal{B})$ *defines an adjunction, then the pair $(f_*(l),f_*(r))$ defines an adjunction in* $\mathrm{Cat}(\mathcal{A})$*. Moreover, the converse is true whenever f_* is fully faithful.*

Dually, for any algebraic morphism $f^*\colon\mathcal{A}\to\mathcal{B}$ of ∞-topoi, if a pair (l,r) of functors in $\mathrm{Cat}(A)$ *defines an adjunction, then the pair $(f^*(l),f^*(r))$ defines an adjunction in* $\mathrm{Cat}(\mathcal{B})$*, and the converse is true whenever f^* is fully faithful.*

Proof. On account of the equivalence $\mathrm{const}_{\mathcal{B}}\simeq f^*\circ\mathrm{const}_{\mathcal{A}}$ as well as the morphism $\mathrm{const}_{\mathcal{A}}\to f_*\,\mathrm{const}_{\mathcal{B}}$ that is induced by the adjunction unit $\mathrm{id}_{\mathcal{A}}\to f_*f^*$, the claim follows immediately from Lemma 3.1.1.7. □

Corollary 3.1.1.10. *For any simplicial object $K\in\mathcal{B}_\Delta$, the endofunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(K,-)$ on* $\mathrm{Cat}(\mathcal{B})$ *preserves adjunctions in* $\mathrm{Cat}(\mathcal{B})$.

Proof. By bifunctoriality of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$, precomposition with the terminal map $K\to 1$ in $\mathcal{B}_\Delta$ gives rise to the diagonal functor $\mathrm{id}_{\mathrm{Cat}(\mathcal{B})}\to\underline{\mathsf{Fun}}_{\mathcal{B}}(K,-)$, and combining this map with the functor $\mathrm{const}_{\mathcal{B}}$ then defines a map

$$\mathrm{const}_{\mathcal{B}}(-)\to\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathrm{const}_{\mathcal{B}}(-)),$$

hence Lemma 3.1.1.7 applies. □

Remark 3.1.1.11. In the situation of Corollary 3.1.1.10, Remark 3.1.1.8 shows that for any two $\mathcal{B}$-categories C and D, the induced map

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C},\mathsf{D})\to\mathrm{Fun}_{\mathcal{B}}(\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{C}),\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{D}))$$

is the one that is determined by the composition

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C},\mathsf{D}) \otimes (\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{C}) \times K) \xrightarrow{\mathrm{id}\otimes \mathrm{ev}_K} \mathrm{Fun}_{\mathcal{B}}(\mathsf{C},\mathsf{D}) \otimes \mathsf{C} \xrightarrow{\mathrm{ev}_{\mathsf{C}}} \mathsf{D}$$

in light of the two adjunctions $- \times K \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(K,-)$ and $- \otimes \mathsf{C} \dashv \mathrm{Fun}_{\mathcal{B}}(\mathsf{C},-)$. Here ev_K and ev_{C}, respectively, denote the counits of these adjunctions.

Combining Corollary 3.1.1.9 with Corollary 3.1.1.10, one furthermore obtains:

Corollary 3.1.1.12. *For any simplicial object $K \in \mathcal{B}_\Delta$, the functor*

$$\mathrm{Fun}_{\mathcal{B}}(K,-)\colon\ \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$$

carries adjunctions in $\mathrm{Cat}(\mathcal{B})$ *to adjunctions in* Cat_∞. □

Similarly as above, if $\mathcal{A}$ and $\mathcal{B}$ are ∞-topoi and if $f\colon\ \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ is a functor such that there is a bifunctorial map

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))^{\mathrm{op}}$$

that recovers the action of f on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor, the functor f sends an adjunction $l \dashv r$ in $\mathrm{Cat}(\mathcal{B})$ to an adjunction $f(r) \dashv f(l)$ in $\mathrm{Cat}(\mathcal{A})$. One therefore finds:

Proposition 3.1.1.13. *The equivalence* $(-)^{\mathrm{op}}\colon\ \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$ *sends an adjunction $l \dashv r$ to an adjunction $r^{\mathrm{op}} \dashv l^{\mathrm{op}}$.*

Proof. This follows from the evident equivalence

$$(-)^{\mathrm{op}}\colon\ \mathrm{Fun}_{\mathcal{B}}(-,-) \simeq \mathrm{Fun}_{\mathcal{B}}((-)^{\mathrm{op}}, (-)^{\mathrm{op}})^{\mathrm{op}}$$

of bifunctors $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$, which shows that if $l \dashv r$ is an adjunction with unit η and counit ϵ, then the pair $(r^{\mathrm{op}}, l^{\mathrm{op}})$ defines an adjunction that is witnessed by the two maps $\epsilon^{\mathrm{op}}\colon\ \mathrm{id} \to l^{\mathrm{op}}r^{\mathrm{op}}$ and $\eta^{\mathrm{op}}\colon\ r^{\mathrm{op}}l^{\mathrm{op}} \to \mathrm{id}$ that correspond to ϵ and η via the above equivalence. □

The contravariant versions of the functors which are considered in Corollary 3.1.1.10 and Corollary 3.1.1.12 preserve adjunctions as well: If C is an arbitrary $\mathcal{B}$-category, functoriality of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,\mathsf{C})$ defines a map

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{E},\mathsf{D}) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{C}))$$

that is natural in E and D. The composition

$$\begin{aligned}\operatorname{map}_{\mathrm{Cat}(\mathcal{B})}(-\otimes \mathsf{E}, \mathsf{D}) &\to \operatorname{map}_{\mathrm{Cat}(\mathcal{B})}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(-\otimes \mathsf{E}, \mathsf{C}))\\ &\simeq \operatorname{map}_{\mathrm{Cat}(\mathcal{B})}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}) \times (-\otimes \mathsf{E}), \mathsf{C})\\ &\simeq \operatorname{map}_{\mathrm{Cat}(\mathcal{B})}((-\otimes \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C})) \times \mathsf{E}, \mathsf{C})\\ &\simeq \operatorname{map}_{\mathrm{Cat}(\mathcal{B})}(-\otimes \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}, \mathsf{C}))\end{aligned}$$

(in which each step is natural in D and E) and Yoneda's lemma now give rise to a map

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{E}, \mathsf{D}) \to \mathrm{Fun}_{\mathcal{B}}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}, \mathsf{C}))$$

that defines a morphism of functors $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_{\infty}$ and that recovers the action of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{C})$ on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor. One therefore finds:

Proposition 3.1.1.14. *For every $\mathcal{B}$-category* C*, applying the functors* $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{C})$ *and* $\mathrm{Fun}_{\mathcal{B}}(-, \mathsf{C})$ *to an adjunction $l \dashv r$ in* $\mathrm{Cat}(\mathcal{B})$ *yields an adjunction $r^* \dashv l^*$ in* $\mathrm{Cat}(\mathcal{B})$ *and in* Cat_{∞}*, respectively.* □

We end this section by showing that an adjunction between $\mathcal{B}$-categories induces an adjunction when passing to slice $\mathcal{B}$-categories:

Proposition 3.1.1.15. *Let* $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ *be an adjunction between $\mathcal{B}$-categories, and let $c\colon A \to \mathsf{C}$ be an arbitrary object. Then the induced map $r_{/c}\colon \mathsf{C}_{/c} \to \mathsf{D}_{/r(c)}$ of $\mathcal{B}_{/A}$-categories admits a left adjoint l_c that is explicitly given by the composition*

$$l_c\colon \mathsf{D}_{/r(c)} \xrightarrow{l_{/r(c)}} \mathsf{C}_{/lr(c)} \xrightarrow{(\epsilon c)_!} \mathsf{C}_{/c},$$

where $\epsilon c\colon lr(c) \to c$ is the counit of the adjunction $(l \dashv r)$.

Proof. Using Remark 2.1.2.2, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and the adjunction $l \dashv r$ by its image along π_A^* and can therefore assume without loss of generality that $A \simeq 1$. Let us fix an adjunction unit η and an adjunction counit ϵ. In light of the commutative diagram

$$
\begin{tikzcd}
 & & \mathsf{C}_{/c} \arrow[rr, "r_{/c}"] \arrow[ddl, bend left, "(\pi_c)_!"] & & \mathsf{D}_{/r(c)} \arrow[ddl, bend left, "(\pi_{r(c)})_!"] \\
\mathsf{D}_{/r(c)} \arrow[r, "l_{/r(c)}"] \arrow[d, "(\pi_{r(c)})_!"'] \arrow[urr, bend left, "l_c"] & \mathsf{C}_{/lr(c)} \arrow[ur, "(\epsilon c)_!"] \arrow[rr, "r_{/lr(c)}"] \arrow[d, "(\pi_{lr(c)})_!"'] & & \mathsf{D}_{/rlr(c)} \arrow[ur, "(\epsilon r(c))_!"] \arrow[d, "(\pi_{rlr(c)})_!"'] & \\
\mathsf{D} \arrow[r, "l"] & \mathsf{C} \arrow[rr, "r"] & & \mathsf{D} &
\end{tikzcd}
$$

we obtain an equivalence $rl(\pi_{r(c)})_! \simeq (\pi_{r(c)})_! r_{/c} l_c$, which in turn yields a commutative diagram

$$
\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_{r(c)}"] \arrow[d, hook, "d^0"] \arrow[rr, bend left, "\epsilon r(c)"] & \mathsf{D}_{/r(c)} \arrow[r, "r_{/c} l_c"] \arrow[d, "d^0"] & \mathsf{D}_{/r(c)} \arrow[d, "(\pi_{r(c)})_!"] \\
\Delta^1 \arrow[r, "\mathrm{id} \otimes \mathrm{id}_{r(c)}"] \arrow[rr, bend right, "r\eta c"'] & \Delta^1 \otimes \mathsf{D}_{/r(c)} \arrow[r, "\eta(\pi_{r(c)})_!"] \arrow[ur, dashed, "\eta_c"] & \mathsf{D}.
\end{tikzcd}
$$

Note that as d^0 is a final functor, the lift η_c exists. Moreover, since restricting η_c along $\mathrm{id} \otimes \mathrm{id}_{r(c)}$ produces a lift of the outer square in the above diagram, the uniqueness of such lifts and the triangle identities for the adjunction $l \dashv r$ imply that η_c carries the final object $\mathrm{id}_{r(c)}$ to the map in $\mathsf{D}_{/r(c)}$ that is encoded by the commutative triangle

$$
\begin{tikzcd}
r(c) \arrow[r, "r\eta(c)"] \arrow[dr, "\mathrm{id}_{r(c)}"'] & rlr(c) \arrow[d, "\epsilon r(c)"] \\
 & r(c).
\end{tikzcd}
$$

In particular, the functor $\mathsf{D}_{/r(c)} \xrightarrow{d^1} \Delta^1 \otimes \mathsf{D}_{/r(c)} \xrightarrow{\eta_c} \mathsf{D}_{/r(c)}$ preserves the final object. Since this functor by construction commutes with the projection $(\pi_{r(c)})_!$, it must therefore be equivalent to the identity on $\mathsf{D}_{/r(c)}$, so that η_c encodes a map $\mathrm{id}_{\mathsf{D}_{/r(c)}} \to r_{/c} l_c$.

Dually, we also have an equivalence $lr(\pi_c)_! \simeq (\pi_c)_! l_c r_{/c}$, so that the map $\epsilon(\pi_c)_! : \Delta^1 \otimes \mathsf{C}_{/c} \to \mathsf{C}$ encodes a morphism of functors $(\pi_c)_! l_c r_{/c} \to (\pi_c)_!$. By an analogous argument as above, we can now construct a lift $\epsilon_c\colon \Delta^1 \otimes \mathsf{C}_{/c} \to \mathsf{C}_{/c}$ of $\epsilon(\pi_c)_!$ along $(\pi_c)_!$ that encodes a morphism of functors $l_c r_{/c} \to \mathrm{id}_{\mathsf{C}_{/c}}$. To complete the proof, it now suffices to show that the two compositions

$$r_{/c} \xrightarrow{\eta_c r_{/c}} r_{/c} l_c r_{/c} \xrightarrow{r_{/c}\epsilon_c} r_{/c} \qquad l_c \xrightarrow{l_c\eta_c} l_c r_{/c} l_c \xrightarrow{\epsilon l_c} l_c$$

are equivalences, cf. Proposition 3.1.1.4. Using that $(\pi_{r(c)})_!$ and $(\pi_c)_!$ are right fibrations and therefore in particular conservative, it suffices to show that these two morphisms become equivalences after postcomposition with $(\pi_{r(c)})_!$ and $(\pi_c)_!$, respectively. Therefore, the claim follows from the triangle identities for η and ϵ, together with the observation that by construction we may identify $(\pi_{r(c)})_!\eta_c \simeq \eta(\pi_{r(c)})_!$ and $(\pi_c)_!\epsilon_c \simeq \epsilon(\pi_c)_!$. $\square$

3.1.2. Adjunctions via relative adjunctions of cartesian fibrations

Recall from the discussion in Section 1.2.6 that every pair $(l, r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ of functors between (large) $\mathcal{B}$-categories give rise to functors $(\int l, \int r)\colon \int \mathsf{C} \rightleftarrows \int \mathsf{D}$ between the associated cartesian fibrations over $\mathcal{B}$. In this section, our goal is to characterise those pairs $(\int l, \int r)$ that come from an adjunction $l \dashv r$.

Given any small ∞-category $\mathcal{C}$, there is a bifunctor

$$-\otimes -\colon \mathrm{Cat}_\infty \times \mathrm{Cart}(\mathcal{C}) \to \mathrm{Cart}(\mathcal{C})$$

that sends a pair $(\mathcal{X}, \mathcal{P} \to \mathcal{C})$ to the cartesian fibration $\mathcal{X} \times \mathcal{P} \to \mathcal{P} \to \mathcal{C}$ in which the first arrow is the natural projection. Explicitly, a morphism in $\mathcal{X} \times \mathcal{P}$ is cartesian precisely if its projection to $\mathcal{P}$ is cartesian in $\mathcal{P}$ and its projection to $\mathcal{X}$ is an equivalence. For an arbitrary fixed cartesian fibration $\mathcal{P} \to \mathcal{C}$, the functor $-\otimes \mathcal{P}\colon \mathrm{Cat}_\infty \to \mathrm{Cart}(\mathcal{C}) \hookrightarrow (\widehat{\mathrm{Cat}}_\infty)_{/\mathcal{C}}$ admits a right adjoint $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{P}, -)$ that sends a map $\mathcal{Q} \to \mathcal{C}$ to the ∞-category that is defined by the pullback square

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{C}}(\mathcal{P}, \mathcal{Q}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{P}, \mathcal{Q}) \arrow[d] \\ 1 \arrow[r] & \mathrm{Fun}(\mathcal{P}, \mathcal{C}) \end{tikzcd}$$

in which the vertical map on the right is given by postcomposition with $\mathcal{Q} \to \mathcal{C}$ and in which the lower horizontal arrow picks out the cartesian fibration $\mathcal{P} \to \mathcal{C}$

[49, Proposition 5.2.5.1]. If $\mathcal{Q} \to \mathcal{C}$ is a cartesian fibration, let us denote by $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P}, \mathcal{Q}) \hookrightarrow \mathrm{Fun}_{/\mathcal{C}}(\mathcal{P}, \mathcal{Q})$ the full subcategory that is spanned by those functors that preserve cartesian edges, and observe that this defines a functor

$$\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P}, -)\colon \ \mathrm{Cart}(\mathcal{C}) \to \mathrm{Cat}_\infty .$$

As the equivalence $\mathrm{map}_{/\mathcal{C}}(\mathcal{X} \otimes \mathcal{P}, \mathcal{Q}) \simeq \mathrm{map}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{X}, \mathrm{Fun}_{/\mathcal{C}}(\mathcal{P}, \mathcal{Q}))$ identifies functors $\mathcal{X} \otimes \mathcal{P} \to \mathcal{Q}$ that preserve cartesian arrows with functors $\mathcal{X} \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{P}, \mathcal{Q})$ that take values in the full subcategory $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P}, \mathcal{Q})$, one obtains an adjunction $(- \otimes \mathcal{P} \dashv \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P}, -))\colon \ \widehat{\mathrm{Cat}}_\infty \rightleftarrows \mathrm{Cart}(\mathcal{C})$. By making use of the bifunctoriality of $- \otimes -$, the assignment $\mathcal{P} \mapsto \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P}, -)$ gives rise to a bifunctor $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(-, -)$ in a unique way such that there is an equivalence

$$\mathrm{map}_{\mathrm{Cart}(\mathcal{C})}(- \otimes -, -) \simeq \mathrm{map}_{\mathrm{Cat}_\infty}(-, \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(-, -)).$$

Now observe that by functoriality of the unstraightening equivalence, there is an equivalence $\int(- \otimes -) \simeq - \otimes \int(-)$ of bifunctors $\mathrm{Cat}_\infty \times \mathrm{Cat}(\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})) \to \mathrm{Cart}(\mathcal{C})$ in which the tensoring on the left-hand side is given by the canonical tensoring in $\mathrm{Cat}(\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}))$ over Cat_∞, i.e. by the bifunctor $\mathrm{const}(-) \times -$. By the uniqueness of adjoints, one therefore finds:

Proposition 3.1.2.1. *For any small ∞-category $\mathcal{C}$, there is an equivalence*

$$\mathrm{Fun}_{\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})}(-, -) \simeq \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\textstyle\int(-), \int(-))$$

of bifunctors $\mathrm{Cat}(\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}))^{\mathrm{op}} \times \mathrm{Cat}(\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})) \to \mathrm{Cat}_\infty$ *that recovers the action of the equivalence* $\int\colon \ \mathrm{Cat}(\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})) \simeq \mathrm{Cart}(\mathcal{C})$ *on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor.* □

Recall the notion of a *relative adjunction* between cartesian fibrations as defined by Lurie in [50, § 7.3]:

Definition 3.1.2.2. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{P}$ and $\mathcal{Q}$ be cartesian fibrations over $\mathcal{C}$. A *relative adjunction* between $\mathcal{P}$ and $\mathcal{Q}$ is defined to be an adjunction $(l \dashv r)\colon \ \mathcal{Q} \rightleftarrows \mathcal{P}$ between the underlying ∞-categories such that both l and r define maps in $\mathrm{Cart}(\mathcal{C})$ and such that the structure map $p\colon \ \mathcal{P} \to \mathcal{C}$ sends the adjunction counit ϵ to the identity transformation on p and the structure map $q\colon \ \mathcal{Q} \to \mathcal{C}$ sends the adjunction unit η to the identity transformation on q.

By construction of the bifunctor $\mathrm{Fun}^{\mathrm{Cart}}_{\mathcal{C}}(-,-)$, it is immediate that a pair $(l,r)\colon \mathcal{Q} \rightleftarrows \mathcal{P}$ of maps in $\mathrm{Cart}(\mathcal{C})$ defines a relative adjunction if and only if there are maps $\eta\colon \mathrm{id}_{\mathcal{Q}} \to rl$ and $\epsilon\colon lr \to \mathrm{id}_{\mathcal{P}}$ in $\mathrm{Fun}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{Q},\mathcal{Q})$ and $\mathrm{Fun}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{P},\mathcal{P})$, respectively, that satisfy the triangle identities from Definition 3.1.1.1. Proposition 3.1.2.1 therefore implies:

Corollary 3.1.2.3. *For any small ∞-category $\mathcal{C}$, a pair $(l,r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ of functors between $\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$-categories defines an adjunction if and only if the associated pair $(\int l, \int r)$ defines a relative adjunction in* $\mathrm{Cart}(\mathcal{C})$. ☐

Observe that as by [49, Lemma 6.3.5.28] the inclusion $\widehat{\mathcal{B}} \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$ defines a geometric morphism of ∞-topoi (relative to the universe $\mathbf{V}$), Corollary 3.1.1.9 implies that the pair (l,r) defines an adjunction between large $\mathcal{B}$-categories if and only if it defines an adjunction in $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$. We may therefore conclude:

Corollary 3.1.2.4. *A pair $(l,r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ of functors between large $\mathcal{B}$-categories defines an adjunction if and only if the associated pair $(\int l, \int r)$ defines a relative adjunction in* $\mathrm{Cart}(\mathcal{B})$. ☐

The upshot of Corollary 3.1.2.4 is that we may make use of Lurie's results on relative adjunctions in order to formulate a useful criterion for when a functor between $\mathcal{B}$-categories admits a right and a left adjoint, respectively. For this we need to recall the *mate* construction:

Definition 3.1.2.5. For any right lax square in $\mathrm{Cat}(\mathcal{B})$ of the form

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "r_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\swArrow \phi"] & \mathsf{D}_1 \arrow[d, "g"] \\ \mathsf{C}_2 \arrow[r, "r_2"] & \mathsf{D}_2 \end{tikzcd}$$

such that both r_1 and r_2 admit left adjoints l_1 and l_2 exhibited by units $\eta_i\colon \mathrm{id} \to r_i l_i$ and counits $\epsilon_i\colon l_i r_i \to \mathrm{id}$, there is a left lax square

$$\begin{tikzcd} \mathsf{C}_1 \arrow[d, "f"'] & \mathsf{D}_1 \arrow[l, "l_1"'] \arrow[d, "g"] \arrow[dl, phantom, "\nwArrow \psi"] \\ \mathsf{C}_2 & \mathsf{D}_2 \arrow[l, "l_2"] \end{tikzcd}$$

in which ψ is defined as the composite map

$$l_2 g \xrightarrow{l_2 g \eta_1} l_2 g r_1 l_1 \xrightarrow{l_2 \phi l_1} l_2 r_2 f l_1 \xrightarrow{\epsilon_2 f l_1} f l_1.$$

Conversely, when starting with the latter left lax square, the original right lax square is recovered by means of the composition

$$g r_1 \xrightarrow{\eta_2 g r_1} r_2 l_2 g r_1 \xrightarrow{r_2 \psi r_1} r_2 f l_1 r_1 \xrightarrow{r_2 f \epsilon_1} r_2 f.$$

The left lax square determined by ψ is referred to as the *mate* of the right lax square determined by ψ, and vice versa.

Remark 3.1.2.6. In the 2-categorical context mates have been studied under the name *adjoint squares* by Gray in [29, §I.6], and under the name mate in [45, §2]. In the $(\infty, 2)$-categorical setting they have been studied by Haugseng, see the discussion following [33, Remark 4.5]. In the case where the starting 2-cell is invertible, which we will mostly use, they are also already considered in [50, Definition 4.7.4.13].

Remark 3.1.2.7. The mate construction is functorial in the following sense: Consider the composition of right lax squares

$$\begin{tikzcd}
\mathsf{C}_1 \arrow[r, "r_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\overset{\phi_1}{\swArrow}"] & \mathsf{D}_1 \arrow[d, "g_1"] \\
\mathsf{C}_2 \arrow[r, "r_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\overset{\phi_2}{\swArrow}"] & \mathsf{D}_2 \arrow[d, "g_2"] \\
\mathsf{C}_3 \arrow[r, "r_3"] & \mathsf{D}_3,
\end{tikzcd}$$

by which we simply mean the composition $(\phi_2 f_1) \circ (g_2 \phi_1)$. Then the mate of the composite square is given by the composition of left lax squares

$$\begin{tikzcd}
\mathsf{C}_1 \arrow[d, "f_1"'] \arrow[dr, phantom, "\overset{\psi_1}{\nwArrow}"] & \mathsf{D}_1 \arrow[l, "l_1"'] \arrow[d, "g_1"] \\
\mathsf{C}_2 \arrow[d, "f_2"'] \arrow[dr, phantom, "\overset{\psi_2}{\nwArrow}"] & \mathsf{D}_2 \arrow[l, "l_2"'] \arrow[d, "g_2"] \\
\mathsf{C}_3 & \mathsf{D}_3, \arrow[l, "l_3"']
\end{tikzcd}$$

in which ψ_1 denotes the mate of ϕ_1 and ψ_2 denotes the mate of ϕ_2. This is easily checked using the triangle identities for adjunctions and the interchange law in bicategories.

Similarly, one can show that the mate of the *horizontal* composition of right lax squares

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "r_1"] \arrow[d, "f"] & \mathsf{D}_1 \arrow[r, "r_1'"] \arrow[d, "g"] \arrow[dl, Rightarrow, "\phi_1"'] & \mathsf{E}_1 \arrow[d, "h"] \arrow[dl, Rightarrow, "\phi_2"'] \\ \mathsf{C}_2 \arrow[r, "r_2"'] & \mathsf{D}_2 \arrow[r, "r_2'"'] & \mathsf{E}_2 \end{tikzcd}$$

(i.e. the composite $r_2'\phi_1 \circ \phi_2 r_1$) is given by the horizontal composition of the associated mates.

Lemma 3.1.2.8. *Let $\mathcal{C}$ be an ∞-category and let $p\colon \mathcal{P} \to \mathcal{C}$ and $q\colon \mathcal{Q} \to \mathcal{C}$ be cartesian fibrations. A map $r\colon \mathcal{P} \to \mathcal{Q}$ in $\mathrm{Cart}(\mathcal{C})$ is a relative right adjoint if and only if*

1. *for all $c \in \mathcal{C}$ the functor $r|_c\colon \mathcal{P}|_c \to \mathcal{Q}|_c$ that is induced by r on the fibres over c admits a left adjoint $l_c\colon \mathcal{Q}|_c \to \mathcal{P}|_c$;*

2. *For every morphism $g\colon d \to c$ in $\mathcal{C}$, the mate of the commutative square*

$$\begin{tikzcd} \mathcal{P}|_c \arrow[r, "r|_c"] \arrow[d, "g^*"'] & \mathcal{Q}|_c \arrow[d, "g^*"] \arrow[dl, Rightarrow, "\simeq"'] \\ \mathcal{P}|_d \arrow[r, "r|_d"'] & \mathcal{Q}|_d \end{tikzcd}$$

commutes.

If this is the case, the relative left adjoint l of r recovers the map l_c on the fibres over $c \in \mathcal{C}$.

Dually, a map $l\colon \mathcal{Q} \to \mathcal{P}$ in $\mathrm{Cart}(\mathcal{C})$ is a relative left adjoint if and only if

1. *for all $c \in \mathcal{C}$ the functor $l|_c\colon \mathcal{Q}|_c \to \mathcal{P}|_c$ that is induced by l on the fibres over c admits a right adjoint $r_c\colon \mathcal{P}|_c \to \mathcal{Q}|_c$;*

2. *For every morphism $g\colon d \to c$ in $\mathcal{C}$, the mate of the commutative square*

$$\begin{tikzcd} \mathcal{P}|_c \arrow[d, "g^*"'] & \mathcal{Q}|_c \arrow[l, "l|_c"'] \arrow[d, "g^*"] \arrow[dl, phantom, "\simeq"] \\ \mathcal{P}|_d & \mathcal{Q}|_d \arrow[l, "l|_d"] \end{tikzcd}$$

commutes.

If this is the case, the relative right adjoint r of l recovers the map r_c on the fibres over $c \in \mathcal{C}$.

Proof. The second claim is the content of (the dual of) [50, Proposition 7.3.2.11]. The first claim, on the other hand, is a formal consequence of the second: in fact, in light of the straightening equivalence for cartesian fibrations, there is an equivalence $(-)^{\vee,\mathrm{op}}\colon \mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Cart}(\mathcal{C})$ that is determined by the equivalence

$$(-)^{\mathrm{op}}_*\colon \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{C}) \simeq \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{C})$$

given by postcomposition with the involution $(-)^{\mathrm{op}}\colon \widehat{\mathrm{Cat}}_\infty \simeq \widehat{\mathrm{Cat}}_\infty$. By combining Proposition 3.1.1.13 with Corollary 3.1.2.3, the equivalence $(-)^{\vee,\mathrm{op}}$ carries a relative left adjoint to a relative right adjoint, and it is evidently true that it translates the two conditions in the second statement to the two conditions in the first one. Since we already know that the second statement is verified, the first one therefore follows as well. $\square$

By combining Corollary 3.1.2.4 with Lemma 3.1.2.8, we conclude:

Proposition 3.1.2.9. *A functor $r\colon \mathsf{C} \to \mathsf{D}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$ is a right adjoint if and only if the following two conditions hold:*

1. *For any object $A \in \mathcal{B}$, the induced functor $r(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ is the right adjoint in an adjunction $(l_A, r(A), \eta_A, \epsilon_A)$.*

2. *For any morphism $s\colon B \to A$ in $\mathcal{B}$, the mate of the commutative square*

$$\begin{tikzcd} \mathsf{C}(A) \arrow[r, "r(A)"] \arrow[d, "s^*"'] & \mathsf{D}(A) \arrow[d, "s^*"] \arrow[dl, phantom, "\simeq"] \\ \mathsf{C}(B) \arrow[r, "r(B)"] & \mathsf{D}(B) \end{tikzcd}$$

commutes.

If this is the case, then the left adjoint l of r is given on objects $A \in \mathcal{B}$ by l_A and on morphisms $s\colon B \to A$ by the mate of the commutative square defined by $r(s)$.

Dually, a functor $l\colon \mathsf{D} \to \mathsf{C}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$ is a left adjoint if and only if the following two conditions hold:

1. *For any object $A \in \mathcal{B}$, the induced map $l(A)\colon \mathsf{D}(A) \to \mathsf{C}(A)$ is the left adjoint in an adjunction $(l(A), r_A, \eta_A, \epsilon_A)$.*
2. *For any morphism $s\colon B \to A$ in $\mathcal{B}$, the mate of the commutative square*

$$\begin{tikzcd} \mathsf{C}(A) \arrow[d, "s^*"'] & \mathsf{D}(A) \arrow[l, "l(A)"'] \arrow[d, "s^*"] \arrow[dl, phantom, "\Leftarrow\simeq"] \\ \mathsf{C}(B) & \mathsf{D}(B) \arrow[l, "l(B)"] \end{tikzcd}$$

commutes.

If this is the case, then a right adjoint r of l is given on objects $A \in \mathcal{B}$ by r_A and on morphisms $s\colon B \to A$ by the mate of the commutative square defined by $l(s)$. □

Remark 3.1.2.10. In the situation of Proposition 3.1.2.9, suppose that the functor $r\colon \mathsf{C} \to \mathsf{D}$ is fully faithful and suppose that condition (1) is satisfied. Since the mate of the commutative square in condition (2) is given by the composition

$$l_B g^* \xrightarrow{l_B g^* \eta_A} l_B g^* r(A) l_A \xrightarrow{\simeq} l_B r(B) g^* l_A \xrightarrow{\epsilon_B g^* l_A} g^* l_A$$

in which the map ϵ_B is an equivalence, the composition is an equivalence whenever the map $l_B g^* \eta_A$ is an equivalence. Since furthermore the map $l_A \eta_A$ is an equivalence as well, we may in this case replace condition (2) by the a priori weaker condition that there exists an *arbitrary* equivalence $l_B g^* \simeq g^* l_A$.

Combining Lemma 3.1.2.8 with Corollary 3.1.2.3 furthermore implies:

Corollary 3.1.2.11. *Let $r\colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories and choose a left exact localisation $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$, where $\mathcal{C}$ is some small ∞-category. Then r is a right adjoint if and only if the following two conditions hold:*

1. *For any object $c \in \mathcal{C}$, the induced functor $r(Lc)\colon \mathsf{C}(Lc) \to \mathsf{D}(Lc)$ is a right adjoint.*

2. *For any morphism $s\colon d \to c$ in $\mathcal{C}$, the mate of the commutative square*

$$\begin{tikzcd} \mathsf{C}(Lc) \arrow[r, "r(Lc)"] \arrow[d, "Ls^*"'] & \mathsf{D}(Lc) \arrow[d, "Ls^*"] \arrow[dl, phantom, "\overset{\simeq}{\Swarrow}"] \\ \mathsf{C}(Ld) \arrow[r, "r(Ld)"] & \mathsf{D}(Ld) \end{tikzcd}$$

commutes. □

Using the criterion from Proposition 3.1.2.9, we are now able to provide a large class of examples for adjunctions between $\mathcal{B}$-categories:

Example 3.1.2.12. In Construction 1.4.2.1, we defined a functor

$$- \otimes \mathrm{Grpd}_{\mathcal{B}}\colon \mathrm{Pr}^{\mathrm{R}}_{\infty} \to \mathrm{Cat}(\mathcal{B})$$

that carries a presentable ∞-category $\mathcal{C}$ to the sheaf of ∞-categories $\mathcal{C} \otimes \mathcal{B}_{/-}$ (where $- \otimes -$ is Lurie's tensor product of presentable ∞-categories). Therefore, if $g\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint functor between presentable ∞-categories, we get an induced functor

$$g \otimes \mathrm{Grpd}_{\mathcal{B}}\colon \mathcal{C} \otimes \mathrm{Grpd}_{\mathcal{B}} \to \mathcal{D} \otimes \mathrm{Grpd}_{\mathcal{B}}$$

of large $\mathcal{B}$-categories. We note that for any morphism $s\colon B \to A$ in $\mathcal{B}$ the mate of the commutative square

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{B}_{/A} \arrow[r, "g\otimes\mathcal{B}_{/A}"] \arrow[d, "\mathcal{C}\otimes s^*"] & \mathcal{D} \otimes \mathcal{B}_{/A} \arrow[d, "\mathcal{D}\otimes s^*"] \\ \mathcal{C} \otimes \mathcal{B}_{/B} \arrow[r, "g\otimes\mathcal{B}_{/B}"] & \mathcal{D} \otimes \mathcal{B}_{/B} \end{tikzcd}$$

may be identified with the square induced by passing to left adjoints in the commutative diagram

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{B}_{/A} \arrow[r, "g\otimes\mathcal{B}_{/A}"] & \mathcal{D} \otimes \mathcal{B}_{/A} \\ \mathcal{C} \otimes \mathcal{B}_{/B} \arrow[r, "g\otimes\mathcal{B}_{/B}"] \arrow[u, "\mathcal{C}\otimes s_*"] & \mathcal{D} \otimes \mathcal{B}_{/B} \arrow[u, "\mathcal{D}\otimes s_*"'] \end{tikzcd}$$

Thus it follows from Proposition 3.1.2.9 that $g \otimes \mathrm{Grpd}_{\mathcal{B}}$ is a right adjoint.

We conclude this section by applying the above example in two concrete cases. At first we note that by using Remark 1.2.6.8, the large $\mathcal{B}$-category $(\mathsf{Grpd}_{\mathcal{B}})_{\Delta} = \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ (where Δ is viewed as a constant $\mathcal{B}$-category) may naturally be identified with the large $\mathcal{B}$-category $\mathrm{Ani}_{\Delta} \otimes \mathsf{Grpd}_{\mathcal{B}}$. Therefore, by applying the functor $- \otimes \mathsf{Grpd}_{\mathcal{B}}$ from Construction 1.4.2.1 to the inclusion $\mathrm{Cat}_{\infty} \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\Delta)$, one obtains a canonical inclusion of large $\mathcal{B}$-categories

$$\iota\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta).$$

Now Example 3.1.2.12 shows:

Proposition 3.1.2.13. *The inclusion* $\iota\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ *admits a left adjoint.* □

Similarly, the inclusion $\mathrm{Ani} \hookrightarrow \mathrm{Cat}_{\infty}$ induces an inclusion $\mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$, so that Example 3.1.2.12 furthermore yields:

Proposition 3.1.2.14. *The inclusion* $\mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ *admits both a right adjoint* $(-)^{\simeq}$ *and a left adjoint* $(-)^{\mathrm{gpd}}$ *that recover the core $\mathcal{B}$-groupoid and the groupoidification functor on local sections.* □

3.1.3. Adjunctions in terms of mapping $\mathcal{B}$-groupoids

The notion of an adjunction between ∞-categories can be formalised in several ways. One way is the bicategorical approach that we have chosen in Definition 3.1.1.1, but an equivalent way to define an adjunction is by means of a triple (l, r, α) in which $(l, r)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ is a pair of functors and

$$\alpha\colon \operatorname{map}_{\mathcal{D}}(-, r(-)) \simeq \operatorname{map}_{\mathcal{C}}(l(-), -)$$

is an equivalence (see for Example [18, Theorem 6.1.23]). The aim of this section is to obtain an analogous characterisation for adjunctions between $\mathcal{B}$-categories. To that end, recall from Section 2.1.1 that there is a factorisation system in $\mathrm{Cat}(\mathcal{B})$ between initial functors and left fibrations. Recall, furthermore, that there is a functor $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}(\widehat{\mathcal{B}})$ that carries a $\mathcal{B}$-category C to the large $\mathcal{B}$-category $\mathsf{LFib}_{\mathsf{C}}$ of left fibrations over C and that carries a functor $f\colon \mathsf{C} \to \mathsf{D}$ to the pullback functor $f^*\colon \mathsf{LFib}_{\mathsf{C}} \to \mathsf{LFib}_{\mathsf{D}}$ that carries a left fibration $q\colon \mathsf{Q} \to A \times \mathsf{D}$ in context

$A \in \mathcal{B}$ to its pullback along $\mathrm{id} \times f\colon A \times \mathsf{C} \to A \times \mathsf{D}$. Now the key result from which we will derive our desired characterisation of adjunctions is the following statement:

Proposition 3.1.3.1. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories. Then the pullback functor*

$$f^*\colon \mathsf{LFib}_{\mathsf{D}} \to \mathsf{LFib}_{\mathsf{C}}$$

admits a left adjoint $f_!$ that is fully faithful whenever f is. If $p\colon \mathsf{P} \to A \times \mathsf{C}$ is an object in $\mathsf{LFib}_{\mathsf{C}}$, the left fibration $f_!(p)$ over $A \times \mathsf{D}$ is the unique functor that fits into a commutative diagram

$$\begin{tikzcd} \mathsf{P} \arrow[r, "i"] \arrow[d, "p"] & f_!\mathsf{P} \arrow[d, "f_!(p)"] \\ A \times \mathsf{C} \arrow[r, "\mathrm{id} \times f"] & A \times \mathsf{D} \end{tikzcd}$$

such that i is initial.

In order to prove Proposition 3.1.3.1, we need the following lemma:

Lemma 3.1.3.2. *If $f\colon \mathsf{C} \to \mathsf{D}$ and $g\colon \mathsf{D} \to \mathsf{E}$ are functors in $\mathrm{Cat}(\mathcal{B})$ such that g is fully faithful, then gf is initial if and only if both f and g are initial.*

Proof. As initial functors are closed under composition, gf is initial whenever both f and g are, so it suffices to show the converse direction. Since initial functors are the left complement in a factorisation system, they satisfy the left cancellability property, so that it suffices to show that f is initial given that gf is. We will make use of the $\mathcal{B}$-categorical version of Quillen's theorem A (Corollary 2.1.4.10). Let therefore $d\colon A \to \mathsf{D}$ be an object in context $A \in \mathcal{B}$. On account of the commutative diagram

$$\begin{tikzcd} \mathsf{C}_{/d} \arrow[r] \arrow[d] & \mathsf{D}_{/d} \arrow[r] \arrow[d] & \mathsf{E}_{/g(d)} \arrow[d] \\ \mathsf{C} \times A \arrow[r] & \mathsf{D} \times A \arrow[r] & \mathsf{E} \times A \end{tikzcd}$$

in which the left square is a pullback, it suffices to show that the right square is a pullback as well, which follows immediately from g being fully faithful. □

Proof of Proposition 3.1.3.1. We wish to apply Proposition 3.1.2.9. Fixing an object $A \in \mathcal{B}$, first note that the functor

$$f^* \colon \mathrm{LFib}(A \times \mathsf{D}) \to \mathrm{LFib}(A \times \mathsf{C})$$

that is given by pullback along $(\mathrm{id} \times f) \colon A \times \mathsf{C} \to A \times \mathsf{D}$ has a left adjoint $f_!$. In fact, on account of the commutative square

$$\begin{tikzcd}
\mathrm{LFib}(A \times \mathsf{D}) \arrow[r, "f^*"] \arrow[d, hook, "i"] & \mathrm{LFib}(A \times \mathsf{C}) \arrow[d, hook, "i"] \\
\mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{D}} \arrow[r, "f^*"] & \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}},
\end{tikzcd}$$

one may factor $f_!$ as the composition $L_{/A \times \mathsf{D}} \circ (\mathrm{id} \times f)_! \circ i$, where

$$L_{/A \times \mathsf{C}} \colon \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{D}} \to \mathrm{LFib}(A \times \mathsf{D})$$

denotes the localisation functor and where $(\mathrm{id} \times f)_!$ denotes the forgetful functor. By construction, this functor sends $p \colon \mathsf{P} \to A \times \mathsf{C}$ to the left fibration $f_!(p) \colon \mathsf{Q} \to A \times \mathsf{D}$ that arises from the factorisation of $(\mathrm{id} \times f)p \colon \mathsf{P} \to A \times \mathsf{D}$ into an initial map and a left fibration. Note that the counit of this adjunction is given by the canonical map $\mathsf{P} \to \mathsf{Q} \times_{\mathsf{C}} \mathsf{D}$. If f is fully faithful, Lemma 3.1.3.2 implies that this map is initial and therefore an equivalence since it is already a left fibration. As a consequence f being fully faithful implies that $f_!$ is fully faithful as well. Therefore, by using Proposition 3.1.2.9 the proof is complete once we show that for any map $s \colon B \to A$ in $\mathcal{B}$, the lax square

$$\begin{tikzcd}
\mathrm{LFib}(A \times \mathsf{D}) \arrow[d, "s^*"'] & \mathrm{LFib}(A \times \mathsf{C}) \arrow[l, "f_!"'] \arrow[d, "s^*"] \arrow[dl, phantom, "\Leftarrow \phi"] \\
\mathrm{LFib}(B \times \mathsf{D}) & \mathrm{LFib}(B \times \mathsf{C}) \arrow[l, "f_!"']
\end{tikzcd}$$

commutes. To see this, let $p \colon \mathsf{P} \to A \times \mathsf{C}$ be a left fibration, and consider the commutative diagram

$$\begin{tikzcd}
 & s^* f_! \mathsf{P} \arrow[rr] \arrow[dd, "s^* f_!(p)" near start] & & f_! \mathsf{P} \arrow[dd, "f_!(p)"] \\
s^* \mathsf{P} \arrow[ur, "s^* i"] \arrow[rr, crossing over] \arrow[dd, "s^* p"] & & \mathsf{P} \arrow[ur, "i"] & \\
 & B \times \mathsf{D} \arrow[rr, "s \times \mathrm{id}"' near start] & & A \times \mathsf{D} \\
B \times \mathsf{C} \arrow[ur, "\mathrm{id} \times f"'] \arrow[rr, "s \times \mathrm{id}"'] & & A \times \mathsf{C} \arrow[ur, "\mathrm{id} \times f"'] \arrow[from=uu, crossing over, "p" near start] &
\end{tikzcd}$$

in which $f_!(p)i \colon \mathsf{P} \to f_! \mathsf{P} \to A \times \mathsf{D}$ is the factorisation of $(\mathrm{id} \times f)p$ into an initial map and a left fibration. The map $\phi \colon f_! s^*(p) \to s^* f_!(p)$ is given by the unique lift in the commutative square

$$\begin{tikzcd}
s^* \mathsf{P} \arrow[r, "s^* i"] \arrow[d, "j"] & s^* f_! \mathsf{P} \arrow[d, "s^* f_!(p)"] \\
f_! s^* \mathsf{P} \arrow[r, "f_! s^* p"] \arrow[ur, dotted, "\phi"] & B \times \mathsf{D}
\end{tikzcd}$$

in which j is initial. To complete the proof, it therefore suffices to show that $s^* i$ is initial, which follows from the fact that the map $s \colon B \to A$ is a right fibration and therefore proper, cf. Section 2.1.4. $\square$

Corollary 3.1.3.3. *For any functor $f \colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories, the functor*

$$f^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$$

admits a left adjoint $f_!$ that fits into a commutative diagram

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[r, "f^{\mathrm{op}}"] \arrow[d, hook, "h_{\mathsf{C}^{\mathrm{op}}}"] & \mathsf{D}^{\mathrm{op}} \arrow[d, hook, "h_{\mathsf{D}^{\mathrm{op}}}"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "f_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}})
\end{tikzcd}$$

in which the two vertical arrows are given by the Yoneda embedding. Moreover, f is fully faithful if and only if $f_!$ is fully faithful.

Proof. The existence of the left adjoint $f_!$ follows immediately from Proposition 3.1.3.1 on account of the straightening/unstraightening equivalence for left fibrations (Theorem 2.2.1.1). To show that the composition

$$\mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}})$$

factors through the Yoneda embedding $\mathsf{D}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}})$, it suffices to show that for every representable left fibration $p\colon \mathsf{P} \to A \times \mathsf{C}$ the associated left fibration $f_!(p)\colon \mathsf{Q} \to A \times \mathsf{D}$ is representable as well. This follows immediately from the fact that there is an initial map $i\colon \mathsf{P} \to \mathsf{Q}$, which implies that Q admits an initial section $A \to \mathsf{Q}$ over A whenever P admits such a section (cf. Proposition 2.3.2.11). ☐

Proposition 3.1.3.4. *A pair of functors $(l, r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ between $\mathcal{B}$-categories defines an adjunction if and only if there is an equivalence of functors*

$$\alpha\colon \operatorname{map}_{\mathsf{D}}(l(-), -) \simeq \operatorname{map}_{\mathsf{C}}(-, r(-)).$$

Proof. Suppose that $l \dashv r$ is an adjunction in $\mathrm{Cat}(\mathcal{B})$. Then Proposition 3.1.1.14 gives rise to an adjunction $l^* \dashv r^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \rightleftarrows \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. On the other hand, Corollary 3.1.3.3 provides a left adjoint $r_!$ to r^*, hence the uniqueness of adjoints implies that there is an equivalence $\beta\colon r_! \simeq l^*$. We therefore conclude that there is an equivalence $\alpha\colon h_{\mathsf{C}} r \simeq l^* h_{\mathsf{D}}$, where h_{C} and h_{D} denotes the Yoneda embedding of C and D, respectively. By transposing α across the adjunction $- \times \mathsf{D}^{\mathrm{op}} \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, -)$, we thus end up with an equivalence

$$\alpha\colon \operatorname{map}_{\mathsf{D}}(l(-), -) \simeq \operatorname{map}_{\mathsf{C}}(-, r(-)),$$

as desired.

Conversely, suppose that the pair (l, r) comes along with an equivalence α as above. As functoriality of the twisted arrow construction (Definition 2.1.2.3) gives rise to a morphism of functors $\operatorname{map}_{\mathsf{C}}(-, -) \to \operatorname{map}_{\mathsf{D}}(l(-), l(-))$, one obtains a map

$$\operatorname{map}_{\mathsf{C}}(-, -) \to \operatorname{map}_{\mathsf{D}}(l(-), l(-)) \simeq \operatorname{map}_{\mathsf{C}}(-, rl(-)).$$

As the Yoneda embedding is fully faithful (Corollary 2.3.2.6), this map arises uniquely from a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to rl$. In fact, we may view the above map as a functor

$$\mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\Delta^1}$$

that sends an object $d\colon A \to \mathsf{C}$ to the map

$$\operatorname{map}_{\mathsf{C}}(-, d) \to \operatorname{map}_{\mathsf{D}}(l(-), l(d)) \simeq \operatorname{map}_{\mathsf{C}}(-, rl(d))$$

in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$. As the Yoneda embedding $\mathsf{C} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ is fully faithful, this map must arise from a map in C, hence the above functor factors through the fully faithful functor $\mathsf{C}^{\Delta^1} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})^{\Delta^1}$ that is induced by the Yoneda embedding. By a similar argument, one obtains a map $\epsilon\colon lr \to \mathrm{id}_{\mathsf{D}}$. We complete the proof by showing that η and ϵ satisfy the conditions of Proposition 3.1.1.4, i.e. that the maps $r\epsilon \circ \eta r$ and $\epsilon l \circ l\eta$ are equivalences. We show this for the first case, the second case follows from an analogous argument. Since equivalences of functors can be detected object-wise by Proposition 2.3.2.12, it suffices to show that for any object $d\colon A \to \mathsf{D}$ the map

$$r(d) \xrightarrow{\eta r d} rlr(d) \xrightarrow{r\epsilon d} r(d)$$

is an equivalence. Now bifunctoriality of the equivalence

$$\mathrm{map}_{\mathsf{D}}(l(-), -) \simeq \mathrm{map}_{\mathsf{C}}(-, r(-))$$

implies that there is a commutative diagram

$$\begin{tikzcd} r(d) \arrow[r, "\eta r d"] \arrow[d, "\mathrm{id}_{r(d)}"] & rlr(d) \arrow[d, "r\epsilon d"] \\ r(d) \arrow[r, "\mathrm{id}_{r(d)}"] & r(d) \end{tikzcd}$$

that arises from the transposed commutative diagram

$$\begin{tikzcd} lr(d) \arrow[r, "\mathrm{id}_{lr(d)}"] \arrow[d, "\mathrm{id}_{lr(d)}"] & lr(d) \arrow[d, "\epsilon d"] \\ lr(d) \arrow[r, "\epsilon d"] & d, \end{tikzcd}$$

which proves the claim. □

Recall from Definition 2.1.2.1 that if $r\colon \mathsf{D} \to \mathsf{C}$ is a functor between $\mathcal{B}$-categories and if $c\colon A \to \mathsf{D}$ is an arbitrary object, we denote by $\mathsf{D}_{c/} \to A \times \mathsf{D}$ the left fibration that arises as pullback of the slice projection $(\pi_c)_!\colon \mathsf{C}_{c/} \to A \times \mathsf{C}$ along $\mathrm{id} \times r\colon A \times \mathsf{D} \to A \times \mathsf{C}$ (cf. Section 2.1.2). Note that by functoriality of the straightening equivalence (Theorem 2.2.1.1), this left fibration is classified by the functor $\mathrm{map}_{\mathsf{C}}(c, r(-))\colon A \times \mathsf{D} \to \mathrm{Grpd}_{\mathcal{B}}$. We now obtain:

Corollary 3.1.3.5. *Let $r\colon \mathsf{D} \to \mathsf{C}$ be a functor between large $\mathcal{B}$-categories. Then r admits a left adjoint l if and only if for any object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ the copresheaf $\mathrm{map}_{\mathsf{C}}(c, r(-))$ (viewed as an object in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}})$ in context A) is representable by an object in D, in which case the representing object is given by $l(c)$ and the associated initial object in $\mathsf{D}_{c/}$ is given by the unit map $\eta c\colon c \to rl(c)$.*

Proof. By Proposition 3.1.3.4, the functor r admits a left adjoint if and only if there is a functor $l\colon \mathsf{C} \to \mathsf{D}$ and an equivalence

$$\alpha\colon \mathrm{map}_{\mathsf{D}}(l(-), -) \simeq \mathrm{map}_{\mathsf{C}}(-, r(-)).$$

Therefore, if r admits a left adjoint then $\mathrm{map}_{\mathsf{C}}(c, r(-))$ is representable by the object $l(c)\colon A \to \mathsf{D}$, and the explicit construction of the equivalence α in Proposition 3.1.3.4 shows that the equivalence

$$\mathsf{D}_{l(c)/} \simeq \mathsf{D}_{c/}$$

over $A \times \mathsf{D}$ that arises from α sends the initial section $\mathrm{id}_{l(c)}\colon A \to \mathsf{D}_{l(c)/}$ to the unit map $\eta c\colon c \to rl(c)$.

Conversely, if $\mathrm{map}_{\mathsf{C}}(c, r(-))$ is representable for every object c in C in context $A \in \mathcal{B}$, then the functor $hr\colon \mathsf{D} \to \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$ transposes to a functor

$$\mathsf{C}^{\mathrm{op}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})$$

that factors through the Yoneda embedding $\mathsf{D}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})$ and therefore defines a functor $l\colon \mathsf{C} \to \mathsf{D}$. By construction, this functor comes with an equivalence

$$\mathrm{map}_{\mathsf{D}}(l(-), -) \simeq \mathrm{map}_{\mathsf{C}}(-, r(-)),$$

hence the claim follows. □

Let C and D be $\mathcal{B}$-categories and let $\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C})$ be the full subcategory that is spanned by those functors $\pi_A^* \mathsf{D} \to \pi_A^* \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B}_{/A})$ (for every $A \in \mathcal{B}$) that admit a left adjoint. Dually, let $\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ denote the full subcategory spanned by those functors that admit a right adjoint.

Remark 3.1.3.6. If C and D are $\mathcal{B}$-categories and $A \in \mathcal{B}$ is an arbitrary object, the property of a functor $f\colon \pi_A^* \mathsf{C} \to \pi_A^* \mathsf{D}$ to be a right adjoint is *local* in $\mathcal{B}$.

In fact, by Corollary 3.1.3.5 this property is equivalent to the condition that for every object c in $\pi_A^*\mathsf{C}$ (in arbitrary context), the functor $\operatorname{map}_{\pi_A^*\mathsf{C}}(c, f(-))$ is representable. Hence the claim follows from the fact that the representability of such functors is a local condition (see Remark 2.3.2.10). In particular, this implies that every object in $\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ in context $A \in \mathcal{B}$ encodes a right adjoint functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ (cf. Remark 1.3.2.18), and one furthermore has a canonical equivalence

$$\pi_A^*\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{D},\mathsf{C}) \simeq \underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{D},\pi_A^*\mathsf{C})$$

for every $A \in \mathcal{B}$.

Corollary 3.1.3.7. *For any two $\mathcal{B}$-categories C and D, there is an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{D},\mathsf{C}) \simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}$$

that sends a functor between D and C to its left adjoint, and vice versa.

Proof. Postcomposition with the Yoneda embeddings h_{C} and h_{D} gives rise to fully faithful functors

$$\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{D},\mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{C}^{\mathrm{op}},\mathsf{Grpd}_{\mathcal{B}}) \hookleftarrow \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}.$$

Thus, to finish the proof, we only need to show that for any $A \in \mathcal{B}$, an object $f\colon A\times\mathsf{D}\times\mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{C}^{\mathrm{op}},\mathsf{Grpd}_{\mathcal{B}})$ is contained in the essential image of $\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ if and only if it is contained in the essential image of $\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}$. By Remark 3.1.3.6 and Remark 2.3.2.1 and Proposition 1.2.5.4, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$. By Corollary 3.1.3.5, the functor f is contained in $\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ if and only if $f(d,-)$ is representable for any object d in D and $f(-,c)$ is representable for any object c in C, which is in turn equivalent to f being contained in the essential image of $\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}$. Thus the claim follows. □

3.1.4. Reflective subcategories

In this brief section we discuss the special case of an adjunction where the right adjoint is fully faithful. Again this material is quite standard for ordinary ∞-categories, see for example [49, §5.2.7].

Definition 3.1.4.1. Let $i\colon \mathsf{C} \hookrightarrow \mathsf{D}$ be a fully faithful functor of $\mathcal{B}$-categories. Then C is said to be *reflective* in D if i admits a left adjoint. Dually, C is *coreflective* if i admits a right adjoint.

Proposition 3.1.4.2. *If $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ is an adjunction between $\mathcal{B}$-categories, then l is fully faithful if and only if the adjunction unit η is an equivalence, and r is fully faithful if and only if the adjunction counit ϵ is an equivalence.*

Proof. The functor l is fully faithful if and only if the map

$$\mathrm{map}_{\mathsf{C}}(-,-) \to \mathrm{map}_{\mathsf{D}}(l(-), l(-))$$

is an equivalence (by combining Proposition 1.3.2.7 with Proposition 2.3.2.12). By postcomposition with the equivalence

$$\mathrm{map}_{\mathsf{D}}(l(-), l(-)) \simeq \mathrm{map}_{\mathsf{C}}(-, rl(-))$$

that is provided by Proposition 3.1.3.4, this is in turn equivalent to the map

$$\mathrm{map}_{\mathsf{C}}(-,-) \to \mathrm{map}_{\mathsf{C}}(-, rl(-))$$

being an equivalence. But this map is obtained as the image of the adjunction unit η along the fully faithful functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$ that is induced by postcomposition with the Yoneda embedding $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. The claim thus follows from the observation that fully faithful functors are conservative (see Lemma 1.3.2.8). The dual statement about r and ϵ is proved by an analogous argument. □

By combining Proposition 3.1.4.2 with Proposition 3.1.1.4, one immediately deduces:

Corollary 3.1.4.3. *Let $i\colon \mathsf{D} \hookrightarrow \mathsf{C}$ be a fully faithful functor between $\mathcal{B}$-categories. Then D is reflective in C if and only if i admits a retraction $L\colon \mathsf{C} \to \mathsf{D}$ together with a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to iL$ such that both ηi and $L\eta$ are equivalences.* □

If $\mathsf{D} \hookrightarrow \mathsf{C}$ is a reflective subcategory, then the reflection functor $L\colon \mathsf{C} \to \mathsf{D}$ is a retraction and therefore in particular essentially surjective (cf. Corollary 1.3.2.15). Consequently, we may recover the subcategory D from the functor $iL\colon \mathsf{C} \to \mathsf{C}$

be means of its factorisation into an essentially surjective and a fully faithful functor. Conversely, given an arbitrary endofunctor $f\colon \mathsf{C} \to \mathsf{C}$, Corollary 3.1.4.3 shows that the essential image of f defines a reflective subcategory precisely if there is a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to f$ such that both ηf and $f\eta$ are equivalences. Let us record this observation for future use in the following proposition.

Proposition 3.1.4.4. *Let* C *be a* $\mathcal{B}$*-category, let* $f\colon \mathsf{C} \to \mathsf{C}$ *be a functor and let* $iL\colon \mathsf{C} \twoheadrightarrow \mathsf{D} \hookrightarrow \mathsf{C}$ *be its factorisation into an essentially surjective and a fully faithful functor. Then* $L \dashv i$ *precisely if there is a map* $\eta\colon \mathrm{id}_{\mathsf{C}} \to f$ *such that both* ηf *and* $f\eta$ *are equivalences.* □

Example 3.1.4.5. If $(\mathcal{L}, \mathcal{R})$ is a modality in $\mathcal{B}$ (see Example 1.4.1.9), then for any $A \in \mathcal{B}$ the full subcategory $\mathcal{R}_{/A} \hookrightarrow \mathcal{B}_{/A}$ is reflective: the associated reflection functor $L_{/A}\colon \mathcal{B}_{/A} \to \mathcal{R}_{/A}$ is induced by the unique factorisation of maps. Since $(\mathcal{L}, \mathcal{R})$ being a *modality* precisely means that for every map $s\colon B \to A$ in $\mathcal{B}$ the natural map $L_{/B}s^* \to s^*L_{/A}$ is an equivalence, we deduce from Proposition 3.1.2.9, that the right orthogonality class $\mathcal{R}$ of any modality $(\mathcal{L}, \mathcal{R})$ defines a reflective subcategory of $\mathrm{Grpd}_{\mathcal{B}}$. In Example 3.3.2.6 below, we will characterise those reflective subcategories of $\mathrm{Grpd}_{\mathcal{B}}$ that arise in such a way.

Reflective subcategories are examples of *localisations* in the sense of Section 1.3.3:

Proposition 3.1.4.6. *Let* $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ *be a reflective subcategory. Then* l *is the localisation of* C *at the subcategory* $\mathsf{S} = l^{-1}\mathsf{D}^{\simeq} \hookrightarrow \mathsf{C}$.

Proof. By construction of S, we obtain a commutative diagram

$$\begin{tikzcd}
\mathsf{S} \arrow[r] \arrow[d, hook] & \mathsf{S}^{\mathrm{gpd}} \arrow[r] \arrow[d] & \mathsf{D}^{\simeq} \arrow[d, hook] \\
\mathsf{C} \arrow[r, "L"] \arrow[rr, bend right, "l"'] & \mathsf{S}^{-1}\mathsf{C} \arrow[r, "g"] & \mathsf{D},
\end{tikzcd}$$

hence we only need to show that g is an equivalence. Let us define $h = Lr$. Then $gh \simeq lr \simeq \mathrm{id}$, hence h is a right inverse of g. We finish the proof by showing that h is a left inverse of g as well. Since $L^*\colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C}, \mathsf{S}^{-1}\mathsf{C}) \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{S}^{-1}\mathsf{C})$

is fully faithful by Proposition 1.3.3.20, it suffices to produce an equivalence $hgL \simeq L$. Let $\eta \colon \mathrm{id} \to rl$ be the adjunction unit. Since $l\eta$ is an equivalence, the map $l\eta c$ factors through the core $\mathsf{D}^{\simeq} \hookrightarrow \mathsf{D}$ for every object $c \colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$. By construction of S, this means that ηc is contained in S, hence $L\eta c$ is an equivalence. Since equivalences of functors can be detected object-wise (see Proposition 2.3.2.12), we conclude that $L\eta \colon L \to Lrl \simeq hgL$ is the desired equivalence. □

It will be useful to have a name for the class of localisations that arise from reflective subcategories:

Definition 3.1.4.7. If $\mathsf{S} \to \mathsf{C}$ is a functor between $\mathcal{B}$-categories, we say that the associated localisation $L \colon \mathsf{C} \to S^{-1}\mathsf{C}$ is *Bousfield* if L admits a fully faithful right adjoint $i \colon \mathsf{S}^{-1}\mathsf{C} \hookrightarrow \mathsf{C}$.

Remark 3.1.4.8. The extra condition on the right adjoint in Definition 3.1.4.7 to be fully faithful is superfluous: in fact, by Proposition 1.3.3.20 the functor $L^* \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(S^{-1}\mathsf{C}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is fully faithful and by Proposition 3.1.1.14 L^* is left adjoint to i^*. We therefore obtain an equivalence $L^* \simeq i_!$, hence Corollary 3.1.3.3 implies that i must be fully faithful as well.

3.2. Limits and colimits

In this section we study limits and colimits in a $\mathcal{B}$-category. We set up the general theory in Section 3.2.1–3.2.4. All in all our treatment is quite parallel to the one in ordinary higher category theory, see for example [42, §19] or [18, §6.2]. In Section 3.2.5 and Section 3.2.6 we discuss limits and colimits in the universe $\mathsf{Grpd}_{\mathcal{B}}$ and in the $\mathcal{B}$-category of $\mathcal{B}$-categories $\mathsf{Cat}_{\mathcal{B}}$. In Section 3.2.7 we show that initial and final functors can be characterised by their property of preserving limits and colimits. Finally, in Section 3.5.4 we explain how general internal limits and colimits can be decomposed into $\mathcal{B}$-groupoidal and ∞-categorical ones.

3.2.1. Definitions and first examples

Let C be a $\mathcal{B}$-category. Recall from Proposition 1.2.3.12 that for any simplicial object I in $\mathcal{B}$ the internal hom $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in $\mathcal{B}_{\Delta}$ is a $\mathcal{B}$-category. We refer to the objects of this $\mathcal{B}$-category as *I-indexed diagrams in* C. Note that this $\mathcal{B}$-category is equivalent to $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$, where I is the image of the simplicial object I along the localisation functor $\mathcal{B}_{\Delta} \to \mathrm{Cat}(\mathcal{B})$. Thus, in what follows we can always safely assume that I is a $\mathcal{B}$-category.

Now recall from Definition 2.1.2.1 that to any pair of maps $f\colon \mathsf{D} \to \mathsf{C}$ and $g\colon \mathsf{E} \to \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})$ we can associate the *comma $\mathcal{B}$-category*

$$\mathsf{D} \downarrow_{\mathsf{C}} \mathsf{E} = (\mathsf{D} \times \mathsf{E}) \times_{\mathsf{C} \times \mathsf{C}} \mathsf{C}^{\Delta^1}.$$

We may now define:

Definition 3.2.1.1. Let C be a $\mathcal{B}$-category and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ be an I-indexed diagram in C in context $A \in \mathcal{B}$, for some $I \in \mathcal{B}_{\Delta}$. The *$\mathcal{B}$-category of cones over d* is defined as the comma $\mathcal{B}$-category $\mathsf{C}_{/d} = \mathsf{C} \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})} A$ formed from $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ and the diagonal map $\mathrm{diag}\colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$. Dually, the *$\mathcal{B}$-category of cocones under d* is defined as the comma $\mathcal{B}$-category $\mathsf{C}_{d/} = A \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})} \mathsf{C}$.

In the situation of Definition 3.2.1.1, the $\mathcal{B}$-category of cones $\mathsf{C}_{/d}$ admits a structure map into $\mathsf{C} \times A$ that fits into the pullback square

$$\begin{tikzcd}
\mathsf{C}_{/d} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})_{/d} \arrow[d, "(\pi_d)_!"] \\
\mathsf{C} \times A \arrow[r, "\mathrm{diag} \times \mathrm{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \times A.
\end{tikzcd}$$

Since $(\pi_d)_!$ is a right fibration, so is the map $\mathsf{C}_{/d} \to \mathsf{C} \times A$. In other words, we may regard this map as an object in $\mathsf{RFib}_{\mathsf{C}}$ in context A. Dually, the map $\mathsf{C}_{d/} \to A \times \mathsf{C}$ is a left fibration and therefore defines an object in $\mathsf{LFib}_{\mathsf{C}}$ in context A. With respect to the straightening/unstraightening equivalence $\mathsf{RFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ from Theorem 2.2.1.1, the right fibration $\mathsf{C}_{/d} \to \mathsf{C} \times A$ corresponds to the presheaf $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(\mathrm{diag}(-), d)$ on C, and the left fibration $\mathsf{C}_{d/} \to A \times \mathsf{C}$ corresponds to the copresheaf $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(d, \mathrm{diag}(-))$ on C.

Remark 3.2.1.2. In the situation of Definition 3.2.1.1, let

$$\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$$

be the transpose of d. Since the forgetful functor $(\pi_A)_!\colon \mathrm{Cat}(\mathcal{B}_{/A}) \to \mathrm{Cat}(\mathcal{B})$ preserves pullbacks, we deduce from Remark 2.1.2.2 that the map $\mathsf{C}_{/d} \to \mathsf{C} \times A$ arises as the image of $(\pi_A^*\mathsf{C})_{/\bar{d}} \to \pi_A^*\mathsf{C}$ along $(\pi_A)_!$. In other words, when regarded as a $\mathcal{B}_{/A}$-category, we can identify $\mathsf{C}_{/d}$ with $(\pi_A^*\mathsf{C})_{/\bar{d}}$.

Remark 3.2.1.3. Let I be a simplicial object in $\mathcal{B}$ and let C be a $\mathcal{B}$-category. Recall from Definition 2.1.3.11 that the *right cone* $I^{\triangleright}$ is given by the pushout

$$\begin{tikzcd} I \sqcup I \arrow[r, "{(d^1,d^0)}"] \arrow[d, "\mathrm{id}\times\pi_I"] & \Delta^1 \otimes I \arrow[d] \\ I \sqcup 1 \arrow[r, "{(\iota,\infty)}"] & I^{\triangleright}. \end{tikzcd}$$

By applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,\mathsf{C})$ to this diagram, one obtains an equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright},\mathsf{C}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C}) \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})} \mathsf{C}$$

over $\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C}) \times \mathsf{C}$, in which the right-hand side denotes the comma $\mathcal{B}$-category that is formed from the cospan

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C}) \xrightarrow{\mathrm{id}} \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C}) \xleftarrow{\mathrm{diag}} \mathsf{C}.$$

By construction, if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})$ is an I-indexed diagram in C in context $A \in \mathcal{B}$, one obtains a pullback square

$$\begin{tikzcd} \mathsf{C}_{d/} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright},\mathsf{C}) \arrow[d, "{(\iota^*,\infty^*)}"] \\ A \times \mathsf{C} \arrow[r, "d\times\mathrm{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C}) \times \mathsf{C}. \end{tikzcd}$$

In other words, the pullback of $\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright},\mathsf{C})$ along $d \times \mathrm{id}$ recovers the $\mathcal{B}$-category of cocones under d. We may therefore regard any object $\bar{d}\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright},\mathsf{C})$ as a cocone $d \to \mathrm{diag}\, c$ under the diagram $d = \iota^*\bar{d}$ with $c = \infty^*\bar{d}$, where $\infty\colon 1 \to I^{\triangleright}$ denotes the cone point.

Dually, one defines the *left cone* $I^{\triangleleft}$ as the pushout

$$\begin{tikzcd} I \sqcup I \arrow[r, "{(d^1,d^0)}"] \arrow[d, "\pi_I\times\mathrm{id}"] & \Delta^1 \otimes I \arrow[d] \\ 1 \times I \arrow[r] & I^{\triangleleft} \end{tikzcd}$$

and therefore obtains an equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleleft}, \mathsf{C}) \simeq \mathsf{C} \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})} \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$$

over $\mathsf{C} \times \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$. Consequently, the pullback of $\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleleft}, \mathsf{C})$ along $\mathrm{id} \times d$ recovers the $\mathcal{B}$-category of cones $\mathsf{C}_{/d}$ over d.

Definition 3.2.1.4. Let C be a $\mathcal{B}$-category and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ be an I-indexed diagram in context A in C, for some $A \in \mathcal{B}$ and some $I \in \mathcal{B}_{\Delta}$. A *limit* cone of d is a map $\mathrm{diag}(\lim d) \to d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in context A that defines a final section $A \to \mathsf{C}_{/d}$ over A. Dually, a *colimit* cocone of d is a map $d \to \mathrm{diag}(\operatorname{colim} d)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in context A that defines an initial section $A \to \mathsf{C}_{d/}$ over A.

Remark 3.2.1.5. The above definition is a direct analogue of Joyal's original definition of limits and colimits in an ∞-category [41].

Remark 3.2.1.6. In the situation of Definition 3.2.1.4, Proposition 2.3.2.11 implies that an I-indexed diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a colimit cocone if and only if the presheaf $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(d, \mathrm{diag}(-))$ is representable, in which case the representing object is given by $\operatorname{colim} d$. In other words, if d admits a colimit cocone, one obtains an equivalence $\mathsf{C}_{\operatorname{colim} d/} \simeq \mathsf{C}_{d/}$ over $A \times \mathsf{C}$, and conversely if there is an object $c\colon A \to \mathsf{C}$ and an equivalence $\mathsf{C}_{c/} \simeq \mathsf{C}_{d/}$ over $A \times \mathsf{C}$ then the image of the object id_c in $\mathsf{C}_{c/}$ along this equivalence defines a colimit cocone of d. A similar observation can be made for limits. In particular, the colimit and limit of a diagram are unique up to equivalence if they exist.

Remark 3.2.1.7. The existence of limits and colimits is a *local* condition: in fact, by the same reasoning as in Remark 3.1.3.6, given any cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, a diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a limit in C if and only if the diagram $s_i^*(d)\colon A_i \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a limit in C for every i. Analogous observations can be made for colimits.

Remark 3.2.1.8. In light of Remark 3.2.1.2, a cone $\mathrm{diag}\, c \to d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in context A transposes to a cone $\mathrm{diag}\, \bar{c} \to \bar{d}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$ in context $1_{\mathcal{B}_{/A}}$ (where $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^* \mathsf{C}$ and $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$ are the transpose of c and d, respectively), and the former defines an initial section $A \to \mathsf{C}_{/d}$ over

A if and only if the latter defines an initial object $1_{\mathcal{B}_{/A}} \to (\pi_A^* \mathsf{C})_{/\bar{d}}$. In other words, we may compute the limit of $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ as the transpose of the limit of $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$. Analogous observations can be made for colimits.

Example 3.2.1.9. Let C be $\mathcal{B}$-category and let $c\colon A \to \mathsf{C}$ be an object, viewed as a 1-indexed diagram $c\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(1, \mathsf{C}) \simeq \mathsf{C}$. Then there are equivalences $\lim c \simeq c \simeq \operatorname{colim} c$, and the associated limit and colimit cocones are given by $\mathrm{id}_c\colon A \to \mathsf{C}_{/c}$ and $\mathrm{id}_c\colon A \to \mathsf{C}_{c/}$.

Example 3.2.1.10. For any $\mathcal{B}$-category C and any object $c\colon A \to \mathsf{C}$, the object c is initial if and only if it defines a colimit of the initial diagram $d\colon A \times \varnothing \simeq \varnothing \to \mathsf{C}$, and dually c is final if and only if it defines a limit of d. In fact, by Remark 3.2.1.8 we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$. In this case, since $\varnothing$ is initial in $\mathrm{Cat}(\mathcal{B})$, there is an equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\varnothing, \mathsf{C}) \simeq 1$, which implies that the left fibration $\mathsf{C}_{d/} \to \mathsf{C}$ is an equivalence. Consequently, an object $1 \to \mathsf{C}_{d/}$ is initial if and only if its image $1 \to \mathsf{C}$ is. The analogous statement about final objects and limits follows by dualisation.

Proposition 3.2.1.11. *Let C be a $\mathcal{B}$-category and let I be a simplicial object in $\mathcal{B}$. The following conditions are equivalent:*

1. *every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a colimit $\operatorname{colim} d$;*
2. *the diagonal $\mathrm{diag}\colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a left adjoint*

$$\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \to \mathsf{C}.$$

If either of these conditions are satisfied, the functor colim carries d to $\operatorname{colim} d$, and the adjunction unit $d \to \mathrm{diag}\operatorname{colim} d$ defines a colimit cocone of d. The dual statement for limits holds as well.

Proof. By the dual of Corollary 3.1.3.5, the functor diag admits a left adjoint if and only if for every $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ the functor $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})}(d, \mathrm{diag}(-))$ is representable by an object in C, in which case the left adjoint sends d to the representing object in C. By definition, this functor classifies the left fibration $\mathsf{C}_{d/} \to A \times \mathsf{C}$. Therefore, Remark 3.2.1.6 shows that diag admits a left adjoint if and only if every diagram d admits a colimit $\operatorname{colim} d\colon A \to \mathsf{C}$, in which case this is the representing object of the functor $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(d, \operatorname{diag}(-))$. Corollary 3.1.3.5 moreover shows that in this case the adjunction unit $d \to \operatorname{diag}\operatorname{colim} d$ defines an initial section $A \to \mathsf{C}_{d/}$. □

Example 3.2.1.12. Let C be a large $\mathcal{B}$-category and G be a $\mathcal{B}$-groupoid. By using Proposition 3.1.2.9, the following two conditions are equivalent:

1. C admits G-indexed colimits;
2. for every $A \in \mathcal{B}$ the functor $\pi_{\mathsf{G}}^* \colon \mathsf{C}(A) \to \mathsf{C}(\mathsf{G} \times A)$ admits a left adjoint $(\pi_{\mathsf{G}})_!$ such that for every map $s\colon B \to A$ in $\mathcal{B}$ the natural morphism $(\pi_{\mathsf{G}})_! s^* \to s^*(\pi_{\mathsf{G}})_!$ is an equivalence.

In particular, if C has G-indexed colimits, then the colimit of $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{C})$ can be identified with the image of $d \in \mathsf{C}(\mathsf{G} \times A)$ along the functor $(\pi_{\mathsf{G}})_!$.

Dually, the following two conditions are equivalent:

1. C admits G-indexed limits;
2. for every $A \in \mathcal{B}$ the functor $\pi_{\mathsf{G}}^* \colon \mathsf{C}(A) \to \mathsf{C}(\mathsf{G} \times A)$ admits a right adjoint $(\pi_{\mathsf{G}})_*$ such that for every map $s\colon B \to A$ in $\mathcal{B}$ the natural morphism $s^*(\pi_{\mathsf{G}})_* \to (\pi_{\mathsf{G}})_* s^*$ is an equivalence.

In particular, if C has G-indexed limits, then the limit of $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{C})$ can be identified with the image of $d \in \mathsf{C}(\mathsf{G} \times A)$ along the functor $(\pi_{\mathsf{G}})_*$.

Example 3.2.1.13. Let C be a large $\mathcal{B}$-category and let $\mathcal{J}$ be an ∞-category. By using Proposition 3.1.2.9, the following two conditions are equivalent:

1. C admits $\mathcal{J}$-indexed colimits;
2. for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits $\mathcal{J}$-indexed colimits, and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^* \colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves such colimits.

Dually, the following two conditions are equivalent:

1. C admits $\mathcal{J}$-indexed limits;

2. for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits $\mathcal{J}$-indexed limits, and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^* \colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves such limits.

Remark 3.2.1.14. Let $\mathcal{C}$ be a small ∞-category such that $\mathcal{B}$ is a left exact and accessible localisation of $\mathrm{PSh}(\mathcal{C})$. Let $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ be the localisation functor. Then Corollary 3.1.2.11 implies that in the situation of Example 3.2.1.12 and Example 3.2.1.13, it suffices to check the condition in (2) for the special case where $A = L(c)$, $B = L(d)$ and $s = L(t)$ for some objects $c, d \in \mathcal{C}$ and some map $t\colon d \to c$ in $\mathcal{C}$.

3.2.2. Preservation of limits and colimits

Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories and I be a simplicial object in $\mathcal{B}$. Let $f_* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ be the functor that is given by postcomposition with f. For any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$, the functor f_* gives rise to two evident commutative squares

$$\begin{tikzcd} \mathsf{C}_{/d} \arrow[r, "f_*"] \arrow[d] & \mathsf{D}_{/f_*d} \arrow[d] \\ \mathsf{C} \times A \arrow[r, "f\times\mathrm{id}"] & \mathsf{D} \times A \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}_{d/} \arrow[r, "f_*"] \arrow[d] & \mathsf{D}_{f_*d/} \arrow[d] \\ A \times \mathsf{C} \arrow[r, "\mathrm{id}\times f"] & A \times \mathsf{D} \end{tikzcd}$$

Definition 3.2.2.1. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories, let $d\colon I \times A \to \mathsf{C}$ be a diagram and suppose that d admits a limit in C. Then f is said to *preserve* this limit if the induced functor $f_* \colon \mathsf{C}_{/d} \to \mathsf{D}_{/f_*d}$ is final. Dually, if d admits a colimit in C, then f is said to preserve this colimit if the functor $f_* \colon \mathsf{C}_{d/} \to \mathsf{D}_{f_*d/}$ is initial.

Remark 3.2.2.2. In the situation of Definition 3.2.2.1, the condition that f_* is final is equivalent to the condition that this functor carries the limit cone $A \to \mathsf{C}_{/d}$ to a final section $A \to \mathsf{D}_{/f_*d}$ over A. In particular, the latter defines a limit cone over the diagram f_*d. The same observation can be made for the preservation of a colimit.

Remark 3.2.2.3. The property that a functor $f\colon \mathsf{C} \to \mathsf{D}$ preserves the limit (colimit) of a diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ is a *local* condition: if $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$

is a cover in $\mathcal{B}$. then f preserves the limit (colimit) of d if and only if for every i the limit (colimit) of the induced diagram $s_i^*(d)$ is preserved by f. This follows immediately from the fact that initiality (and therefore also finality) is a local condition (Remark 2.1.4.11).

Remark 3.2.2.4. When viewing the map f_* from Definition 3.2.2.1 as a functor of $\mathcal{B}_{/A}$-categories, Remark 3.2.1.2 implies that this map can be identified with the functor

$$(\pi_A^* f)_* \colon (\pi_A^* \mathsf{C})_{/\bar{d}} \to (\pi_A^* \mathsf{D})_{/(\pi_A^* f)_* \bar{d}}$$

(where $\bar{d} \colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$ denotes the transpose of d). In combination with Remark 3.2.1.8 and Remark 2.1.3.3, this implies that f preserves the limit of d if and only if $\pi_A^* f$ preserves the limit of $\bar{d}$. Analogous observations hold for colimits.

Lemma 3.2.2.5. *Let $(l \dashv r) \colon \mathsf{C} \rightleftarrows \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories, and let $f \colon c \to r(d)$ be a map in C in context $A \in \mathcal{B}$. Then f is an equivalence if and only if the transpose map $g \colon l(c) \to d$ defines a final section of $\mathsf{C}_{/d}$ over A.*

Proof. By Corollary 3.1.3.5, the counit $\epsilon d \colon lr(d) \to d$ defines a final section of $\mathsf{C}_{/d}$ over A, hence the dual of Corollary 2.1.3.16 implies that there is a map $g \to \epsilon d$ in $\mathsf{C}_{/d}$ that is an equivalence if and only if g is final. On account of the equivalence $\mathsf{C}_{/d} \simeq \mathsf{C}_{/r(d)}$, this map corresponds to a map $f \to \mathrm{id}_{r(d)}$ in $\mathsf{C}_{/r(d)}$. The result now follows from the straightforward observation that the latter is an equivalence if and only if f is an equivalence in C. $\square$

Proposition 3.2.2.6. *Let $f \colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories and let I be a simplicial object in $\mathcal{B}$ such that C and D admit all I-indexed limits, i.e the diagonal maps $\mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ and $\mathsf{D} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ admit right adjoints (cf. Proposition 3.2.1.11). Then f preserves all I-indexed limits precisely if the mate of the commutative square*

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \arrow[d, "f_*"] & \mathsf{C} \arrow[l, "\mathrm{diag}"'] \arrow[d, "f"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}) & \mathsf{D} \arrow[l, "\mathrm{diag}"']
\end{tikzcd}$$

commutes. The dual statement about colimits holds as well.

Proof. Suppose that f preserves all I-indexed limits. The mate of the commutative square in the statement of the proposition is encoded by a map $\phi\colon\ f\lim \to \lim f_*$ that is given by the composite

$$f\lim \xrightarrow{\eta f\lim} \lim \operatorname{diag} f\lim \xrightarrow{\simeq} \lim f_* \operatorname{diag}\lim \xrightarrow{\lim f_*\epsilon} \lim f_*$$

in which η and ϵ are the units and counits of the two adjunctions $\operatorname{diag} \dashv \lim$. By Proposition 2.3.2.12, this map is an equivalence if and only if for any diagram $d\colon\ A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ the associated map $\phi(d)\colon\ f(\lim d) \to \lim f_* d$ is an equivalence in D. Now since the transpose map $\operatorname{diag} f(\lim d) \to f_* d$ is given by postcomposing the equivalence $\operatorname{diag} f(\lim d) \simeq f_* \operatorname{diag}(\lim d)$ with the map $f_*\epsilon d$ and since Proposition 3.2.1.11 implies that ϵd is precisely the limit cone over d in D, the claim follows from Lemma 3.2.2.5. □

Remark 3.2.2.7. Let $f\colon\ \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories, let I be an arbitrary simplicial object in $\mathcal{B}$ and let $d\colon\ A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ be a diagram that has a limit in C. Suppose furthermore that $f_* d$ has a limit in D. Then the universal property of final objects (see Corollary 2.1.3.16) gives rise to a unique map

$$\begin{tikzcd}
\operatorname{diag} f(\lim d) \arrow[rr] \arrow[dr] & & \operatorname{diag}\lim f_* d \arrow[dl] \\
& f_* d &
\end{tikzcd}$$

in $\mathsf{D}_{/f_* d}$ that is an equivalence if and only if f preserves the limit of d. Since $\mathsf{D}_{/f_* d} \to \mathsf{D}$ is a right fibration and therefore in particular conservative, this is in turn equivalent to the map $f(\lim d) \to \lim f_* d$ being an equivalence in D. If both C and D admit I-indexed limits, this map is nothing but the mate transformation $f\lim \to \lim f_*$ from Proposition 3.2.2.6 evaluated at the object d.

Example 3.2.2.8. Let $f\colon\ \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories and let G be a $\mathcal{B}$-groupoid. Suppose that both C and D admit G-indexed colimits. By using Proposition 3.1.2.9 and Proposition 3.2.2.6, the following two conditions are equivalent:

1. f preserves G-indexed colimits;

2. for every $A \in \mathcal{B}$ the natural morphism $(\pi_{\mathsf{G}})_! f(\mathsf{G} \times A) \to f(A)(\pi_{\mathsf{G}})_!$ is an equivalence.

Dually, if C and D admit G-indexed limits, the following two conditions are equivalent:

1. f preserves G-indexed limits;

2. for every $A \in \mathcal{B}$ the natural morphism $f(A)(\pi_{\mathsf{G}})_* \to (\pi_{\mathsf{G}})_* f(A)$ is an equivalence.

Example 3.2.2.9. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories, let $\mathcal{I}$ be an ∞-category and suppose that both C and D admit $\mathcal{I}$-indexed colimits. By using Proposition 3.1.2.9 and Proposition 3.2.2.6, the following two conditions are equivalent:

1. f preserves $\mathcal{I}$-indexed colimits;

2. for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ preserves $\mathcal{I}$-indexed colimits.

Dually, if C and D admit $\mathcal{I}$-indexed limits, the following two conditions are equivalent:

1. f preserves $\mathcal{I}$-indexed limits;

2. for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ preserves $\mathcal{I}$-indexed limits.

Checking whether a functor between $\mathcal{B}$-categories preserves certain limits or colimits becomes simpler when the functor is fully faithful:

Proposition 3.2.2.10. *Let $f\colon \mathsf{C} \hookrightarrow \mathsf{D}$ be a fully faithful functor between $\mathcal{B}$-categories, let I be a simplicial object in $\mathcal{B}$ and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ be a diagram in C. Suppose that $f_*(d)$ admits a colimit in D such that $\operatorname{colim} f_* d$ is contained in C. Then $\operatorname{colim} f_* d$ already defines a colimit of d in C. The analogous statement for limits holds as well.*

Proof. Since f is fully faithful, the canonical square

$$\begin{tikzcd} \mathsf{C}_{d/} \arrow[r, hook, "f_*"] \arrow[d] & \mathsf{D}_{f_*d/} \arrow[d] \\ A \times \mathsf{C} \arrow[r, hook, "\mathrm{id} \times f"] & A \times \mathsf{D} \end{tikzcd}$$

is a pullback and f_* is fully faithful. Therefore, if $\operatorname{colim} f_*d \colon A \to \mathsf{D}_{f_*d/}$ is an initial section such that the underlying object $\operatorname{colim} f_*d$ in D is contained in C, then the entire colimit cocone is contained in the essential image of f_*, i.e. defines a section $A \to \mathsf{C}_{d/}$ over A. By Lemma 3.1.3.2, this section must be initial as well, hence the result follows. □

Corollary 3.2.2.11. *Let $f \colon \mathsf{C} \hookrightarrow \mathsf{D}$ be a fully faithful functor between $\mathcal{B}$-categories, and suppose that both C and D admit I-indexed colimits for some simplicial object I in $\mathcal{B}$. Then f preserves I-indexed colimits if and only if the restriction of* $\operatorname{colim} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}) \to \mathsf{D}$ *along the inclusion $f_* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ factors through the inclusion $f \colon \mathsf{C} \hookrightarrow \mathsf{D}$. The analogous statement for limits holds as well.* □

We conclude this section with a discussion of the preservation of (co)limits by adjoint functors. We will need the following lemma:

Lemma 3.2.2.12. *Let $(l \dashv r) \colon \mathsf{C} \rightleftarrows \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories and let $i \colon L \to K$ be a map between simplicial objects in $\mathcal{B}$. Then the two commutative squares*

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[r, "l_*"] \arrow[d, "i^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}) \arrow[d, "i^*"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C}) \arrow[r, "l_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{D}) \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[d, "i^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}) \arrow[l, "r_*"'] \arrow[d, "i^*"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C}) & \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{D}) \arrow[l, "r_*"'] \end{tikzcd}$$

that are obtained from the bifunctoriality of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ are related by the mate correspondence.

Proof. To prove the lemma, we may argue in the homotopy bicategory of the $(\infty, 2)$-category $\mathrm{Cat}(\mathcal{B})$. Then the claim follows from the fact that

$$i^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, -) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L, -)$$

determines a pseudo-natural transformation between 2-functors. See [45, Proposition 2.5] for an argument in the strict case. $\square$

Proposition 3.2.2.13. *Let* $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ *be an adjunction between* $\mathcal{B}$*-categories. Then* l *preserves all colimits that exist in* C*, and* r *preserves all limits that exist in* D.

Proof. We will show that the right adjoint $r\colon \mathsf{D} \to \mathsf{C}$ preserves all limits that exist in D, the dual statement about l and colimits follows by taking opposite $\mathcal{B}$-categories. Let therefore I be a simplicial object in $\mathcal{B}$ and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ be a diagram that has a limit in D. We need to show that the image of the final section $\operatorname{diag}\lim d \to d$ along $r_*\colon \mathsf{D}_{/d} \to \mathsf{C}_{/r_* d}$ is final. By Corollary 3.1.1.10, the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, -)$ sends the adjunction $l \dashv r$ to an adjunction

$$l_* \dashv r_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \rightleftarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}),$$

hence by using Proposition 3.1.3.4 one obtains a chain of equivalences

$$\begin{aligned} \operatorname{map}_{\mathsf{C}}(-, r(\lim d)) &\simeq \operatorname{map}_{\mathsf{D}}(l(-), \lim d) \\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(\operatorname{diag} l(-), d) \\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(l_* \operatorname{diag}(-), d) \\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(\operatorname{diag}(-), r_* d) \end{aligned}$$

of presheaves on C. We complete the proof by showing that this equivalence sends the identity $\mathrm{id}_{r(\lim d)}$ to the map $\operatorname{diag} r(\lim d) \simeq r_* \operatorname{diag}\lim d \to r_* d$ that arises as the image of the limit cone $\operatorname{diag}\lim d \to d$ under the functor r_*. By construction, the image of the identity $\mathrm{id}_{r(\lim d)}$ under this chain of equivalences is given by the composition

$$\begin{aligned} \operatorname{diag} r(\lim d) &\xrightarrow{\eta \operatorname{diag} r} r_* l_* \operatorname{diag} r(\lim d) \\ &\xrightarrow{\simeq} r_* \operatorname{diag} lr(\lim d) \\ &\xrightarrow{r_* \operatorname{diag} \epsilon} r_* \operatorname{diag}\lim d \to r_* d \end{aligned}$$

in which the right-most map is the image of the limit cone $\operatorname{diag}\lim d \to d$ under the functor r_*, the map η denotes the unit of the adjunction $l_* \dashv r_*$ and ϵ denotes the counit of the adjunction $l \dashv r$. As the composition of the first three maps is precisely the mate of the equivalence $l_* \operatorname{diag} \simeq \operatorname{diag} l$ and therefore recovers the equivalence $\operatorname{diag} r(\lim d) \simeq r_* \operatorname{diag}(\lim d)$ by Lemma 3.2.2.12, the result follows. □

Proposition 3.2.2.14. *Let* $(l \dashv r)\colon \mathsf{C} \rightleftarrows \mathsf{D}$ *be an adjunction in* $\mathrm{Cat}(\mathcal{B})$ *that exhibits* D *as a reflective subcategory of* C*, let* I *be a simplicial object in* $\mathcal{B}$ *and let* $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ *be a diagram in context* $A \in \mathcal{B}$ *such that* $r_* d$ *admits a colimit in* C*. Then* $l(\operatorname{colim} r_* d)$ *defines a colimit of* d *in* D*. Dually, if* $r_* d$ *admits a limit in* C*, then* $l(\lim r_* d)$ *defines a limit of* d *in* D*.*

Proof. Suppose first that $r_* d$ admits a colimit in C. Since r is fully faithful, we obtain a chain of equivalences

$$\begin{aligned}
\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(d, \operatorname{diag}(-)) &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(r_* d, \operatorname{diag} r(-)) \\
&\simeq \operatorname{map}_{\mathsf{C}}(\operatorname{colim} r_* d, r(-)) \\
&\simeq \operatorname{map}_{\mathsf{D}}(l(\operatorname{colim} r_* d), -),
\end{aligned}$$

which shows that the colimit of d in D exists and is explicitly given by $l(\operatorname{colim} r_* d)$.

Next, let us suppose that $r_* d$ admits a limit in C. By the triangle identities, the functor l sends the adjunction unit $\eta\colon \mathrm{id} \to rl$ to an equivalence. In particular, the map $\lim r_* d \to rl(\lim r_* d)$ is sent to an equivalence in D. Note that on account of the equivalence

$$\operatorname{map}_{\mathsf{C}}(-, \lim r_* d) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(\operatorname{diag} l(-), d),$$

the presheaf $\operatorname{map}_{\mathsf{C}}(-, \lim r_* d)$ sends any map in C that is inverted by l to an equivalence in $\mathrm{Grpd}_{\mathcal{B}}$. Applying this observation to $\eta\colon \lim r_* d \to rl(\lim r_* d)$, we obtain a retraction $\phi\colon rl(\lim r_* d) \to \lim r_* d$ of η that gives rise to a retract diagram

$$\begin{tikzcd}
\lim r_* d \arrow[r, "\eta"] \arrow[d, "\eta"] & rl(\lim r_* d) \arrow[r, "\phi"] \arrow[d, "\eta"] & \lim r_* d \arrow[d, "\eta"] \\
rl(\lim r_* d) \arrow[r, "rl\eta"] & rlrl(\lim r_* d) \arrow[r, "rl\phi"] & rl(\lim r_* d)
\end{tikzcd}$$

in which the two maps in the lower row are equivalences. By the triangle identities and the fact that since r is fully faithful the adjunction counit $\epsilon\colon lr \to \mathrm{id}$

is an equivalence (see Proposition 3.1.4.2), the vertical map in the middle must be an equivalence as well, hence we conclude that $\eta\colon \lim r_* d \to rl(\lim r_* d)$ too is an equivalence. Therefore, the computation

$$\begin{aligned}
\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(\operatorname{diag}(-), d) &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(\operatorname{diag} r(-), r_* d) \\
&\simeq \operatorname{map}_{\mathsf{C}}(r(-), \lim r_* d) \\
&\simeq \operatorname{map}_{\mathsf{C}}(r(-), rl(\lim r_* d)) \\
&\simeq \operatorname{map}_{\mathsf{D}}(lr(-), l(\lim r_* d)) \\
&\simeq \operatorname{map}_{\mathsf{D}}(-, l(\lim r_* d))
\end{aligned}$$

proves the claim. □

Remark 3.2.2.15. We adopted the strategy for the proof of the second claim in Proposition 3.2.2.14 from Denis-Charles Cisinski's proof of the analogous statement for ∞-categories, see [18, Proposition 6.2.17].

3.2.3. Limits and colimits in functor $\mathcal{B}$-categories

In this section, we discuss the familiar fact that limits and colimits in functor ∞-categories can be computed object-wise in the context of $\mathcal{B}$-categories.

Proposition 3.2.3.1. *Let I be a simplicial object in $\mathcal{B}$ and let C be a $\mathcal{B}$-category that admits all I-indexed limits. Then $\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{C})$ admits all I-indexed limits for any simplicial object K in $\mathcal{B}$, and the precomposition functor $i^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L,\mathsf{C})$ preserves I-indexed limits for any map $i\colon L \to K$ in $\mathcal{B}_\Delta$. The dual statement for colimits is true as well.*

Proof. We show the statement for limits. Note that we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{C}) \arrow[r, "\operatorname{diag}_*"] \arrow[d, "i^*"] \arrow[rr, bend left, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K,\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})) \arrow[r, "\simeq"] \arrow[d, "i^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\underline{\mathsf{Fun}}_{\mathcal{B}}(K,\mathsf{C})) \arrow[d, "(i^*)_*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(L,\mathsf{C}) \arrow[r, "\operatorname{diag}_*"] \arrow[rr, bend right, "\operatorname{diag}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(L,\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})) \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\underline{\mathsf{Fun}}_{\mathcal{B}}(L,\mathsf{C})).
\end{tikzcd}$$

Since Proposition 3.2.1.11 implies that the two functors labelled with diag_* have a right adjoint, so do the functors labelled with diag, so that both $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})$ and $\underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C})$ admit I-indexed limits. Moreover, the functoriality of the mate construction (cf. Remark 3.1.2.7) implies that in order to show that the functor $i^* : \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C})$ preserves I-indexed limits, we only need to show that the mate of the left square in the above diagram commutes, which is an immediate consequence of Lemma 3.2.2.12. □

Proposition 3.2.3.2. *Let I be a simplicial object in $\mathcal{B}$ and let C and D be $\mathcal{B}$-categories such that D admits I-indexed limits. Let $d\colon I \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ be a diagram in global context, and let $\mathrm{diag}\, F \to d$ be a cone over d, where $F\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is an arbitrary object. Then $\mathrm{diag}\, F \to d$ is a limit cone if and only if for every $A \in \mathcal{B}$ and every $c\colon A \to \mathsf{C}$ the induced map $\mathrm{diag}(\bar{F})(c) \to \bar{d}(c)$ is a limit cone in D. The dual statements for colimits holds as well.*

Proof. By means of the adjunction $\mathrm{diag} \dashv \lim$ and Lemma 3.2.2.5, the map $\mathrm{diag}\, F \to d$ defines a limit cone if and only if the transpose map $F \to \lim d$ is an equivalence in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$. Using that equivalences in functor $\mathcal{B}$-categories are detected object-wise (see Proposition 2.3.2.12, this is in turn the case precisely if for every $c\colon A \to \mathsf{C}$ the map $F(c) \to (\lim d)(c)$ is an equivalence in context A. Note that by Remark 3.2.1.7, this map transposes to the map $\pi_A^*(F)(\bar{c}) \to \lim \pi_A^*(d)(\bar{c})$ (where $\bar{c} : 1_{\mathcal{B}_{/A}} \to \pi_A^* \mathsf{C}$ is the transpose of c). Using Proposition 3.2.3.1, we can identify the latter with the map $\pi_A^*(F)(\bar{c}) \to \lim(\pi_A^*(d)(c))$, i.e. with the transpose of the morphism of diagrams $\mathrm{diag}\, \pi_A^*(F)(\bar{c}) \to \pi_A^*(d)(\bar{c})$. Hence, we conclude that $\mathrm{diag}\, F \to d$ is a limit cone if and only if $\mathrm{diag}\, \pi_A^*(F)(\bar{c}) \to \pi_A^*(d)(\bar{c})$ is one for each $c\colon A \to \mathsf{C}$. Now by Remark 3.2.1.2, the latter transposes to $\mathrm{diag}\, F(c) \to d(c)$, hence the claim follows from Remark 3.2.1.8. □

Proposition 3.2.3.3. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories, let I be a simplicial object in $\mathcal{B}$ and suppose that both C and D admits I-indexed limits and that f preserves such limits. Then for every simplicial object K in $\mathcal{B}$, the induced functor $f_* : \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D})$ preserves I-indexed limits as well. The dual statement for colimits holds too.*

Proof. Similarly as in the proof in Proposition 3.2.3.1, we need to show that the mate of the left square in the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[r, "\mathrm{diag}_*"] \arrow[d, "f_*"] \arrow[rr, bend left, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})) \arrow[r, "\simeq"] \arrow[d, "(f_*)_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})) \arrow[d, "(f_*)_*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}) \arrow[r, "\mathrm{diag}_*"] \arrow[rr, bend right, "\mathrm{diag}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})) \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}))
\end{tikzcd}$$

commutes, which follows from the observation that this mate is obtained by applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$ to the mate of the commutative square

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathrm{diag}"] \arrow[d, "f"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \arrow[d, "f_*"] \\
\mathsf{D} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}),
\end{tikzcd}$$

which by assumption is an equivalence. Hence the claim follows. □

3.2.4. Colimits in slice $\mathcal{B}$-categories

It is well-known that if $\mathcal{C}$ is an ∞-category and $c \in \mathcal{C}$ is an arbitrary object, the colimit of a diagram $d \colon \mathcal{I} \to \mathcal{C}_{/c}$ can be computed as the colimit of the underlying diagram $(\pi_c)_! d \colon \mathcal{I} \to \mathcal{C}$. In this section, we will establish the analogous statement for $\mathcal{B}$-categories.

Lemma 3.2.4.1. *Let* C *be a* $\mathcal{B}$*-category and let* $f \colon c \to d$ *be a map in* C *in context* $1 \in \mathcal{B}$ *such that* c *is an initial object in* C*. Then* f *defines an initial object in* $\mathsf{C}_{/d}$*.*

Proof. Let $g \colon c' \to d$ be an arbitrary map in C in context $1 \in \mathcal{B}$. We have an evident commutative square

$$\begin{tikzcd}
(\mathsf{C}_{/d})_{/g} \arrow[r, "\simeq"] \arrow[d, "(\pi_g)_!"] & \mathsf{C}_{/c'} \arrow[d, "(\pi_{c'})_!"] \\
\mathsf{C}_{/d} \arrow[r, "(\pi_d)_!"] & \mathsf{C}.
\end{tikzcd}$$

in which the upper horizontal map is an equivalence as it is a right fibration that preserves final objects. Moreover, since c is initial, the diagram

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & \mathsf{C}_{/d} \arrow[d, "(\pi_d)_!"] \\ 1 \arrow[r, "c"] & \mathsf{C} \end{tikzcd}$$

is a pullback. Consequently, we obtain an equivalence $\mathrm{map}_{/d}(f, g) \simeq \mathrm{map}_{\mathsf{C}}(c, c')$. Since c is initial, we conclude that $\mathrm{map}_{/d}(f, g) \simeq 1$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, the same conclusion holds for every map $g\colon c' \to \pi_A^* d$ in context A. Hence, we deduce from Proposition 2.1.3.15 that f is initial. $\square$

Lemma 3.2.4.2. *Let* C *be a* $\mathcal{B}$*-category and let* $f\colon c \to d$ *be a map in* C *in context* $1 \in \mathcal{B}$*. Then there is an equivalence* $(\mathsf{C}_{c/})_{/f} \simeq (\mathsf{C}_{/d})_{f/}$ *that commutes with the projections to* $\mathsf{C}_{/d}$ *and* $\mathsf{C}_{c/}$*.*

Proof. Note that the projection $(\pi_c)_!\colon \mathsf{C}_{c/} \to \mathsf{C}$ induces a left fibration

$$(\pi_c)_!\colon (\mathsf{C}_{c/})_{/f} \to \mathsf{C}_{/d}.$$

By considering the commutative square

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & c \arrow[d, "f"] \\ c \arrow[r, "f"] & d \end{tikzcd}$$

as an object $\phi\colon 1 \to (\mathsf{C}_{c/})_{/f}$, we obtain a commutative square

$$\begin{tikzcd} 1 \arrow[r, "\phi"] \arrow[d, "\mathrm{id}_f"] & (\mathsf{C}_{c/})_{/f} \arrow[d, "(\pi_c)_!"] \\ (\mathsf{C}_{/d})_{f/} \arrow[r, "(\pi_c)_!"] \arrow[ur, dotted] & \mathsf{C}_{/d}. \end{tikzcd}$$

As the left vertical map is initial, the dotted filler exists, hence the proof is complete once we show that ϕ is initial too. By construction, the right fibration $(\pi_f)_!\colon (\mathsf{C}_{c/})_{/f} \to \mathsf{C}_{c/}$ carries ϕ to an initial object. The desired result therefore follows from Lemma 3.2.4.1. $\square$

Proposition 3.2.4.3. *Let I and C be $\mathcal{B}$-categories and let $c\colon 1 \to \mathsf{C}$ be an object. Let $d\colon \mathsf{I} \to \mathsf{C}_{/c}$ be a diagram and suppose that the diagram $(\pi_c)_! d\colon \mathsf{I} \to \mathsf{C}$ admits a colimit in C. Then d admits a colimit in $\mathsf{C}_{/c}$, and $(\pi_c)_!$ preserves this colimit.*

Proof. On account of the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/\operatorname{diag}(c)}$, we may equivalently regard the diagram $d\colon \mathsf{K} \to \mathsf{C}_{/c}$ as an object $d' = (\pi_c)_! d \to \operatorname{diag}(c)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/\operatorname{diag}(c)}$, which can in turn be equivalently regarded as a cocone $\overline{d'}\colon 1 \to \mathsf{C}_{d'/}$. One therefore obtains a unique map

$$\begin{tikzcd} & d' \arrow[dl] \arrow[dr, "\overline{d'}"] & \\ \operatorname{diag}(\operatorname{colim} d') \arrow[rr] & & \operatorname{diag}(c) \end{tikzcd}$$

in $\mathsf{C}_{d'/}$ (by the universal property of initial objects, see Proposition 2.1.3.15) which can be regarded as an object in $(\mathsf{C}_{d'/})_{/\overline{d'}}$. Now Lemma 3.2.4.2 gives rise to an equivalence

$$(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{d'/})_{/\overline{d'}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c})_{d/}$$

over $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c})$ the pullback of which along the diagonal map determines an equivalence $(\mathsf{C}_{d'/})_{/\overline{d'}} \simeq (\mathsf{C}_{/c})_{d/}$ that fits into a commutative diagram

$$\begin{tikzcd} (\mathsf{C}_{d'/})_{/\overline{d'}} \arrow[rr, "\simeq"] \arrow[dr, "(\pi_{\overline{d'}})_!"'] & & (\mathsf{C}_{/c})_{d/} \arrow[dl, "(\pi_c)_!"] \\ & \mathsf{C}_{d/} & \end{tikzcd}$$

Consequently, the colimit cocone $d' \to \operatorname{colim} d'$ lifts along $(\pi_c)_!$ to a cocone under d. By Lemma 3.2.4.1, this lift defines an initial object and therefore a colimit cocone, hence the claim follows. □

3.2.5. Limits and colimits in the universe

Our goal of this section is to prove that the universe $\mathsf{Grpd}_{\mathcal{B}}$ for $\mathcal{B}$-groupoids admits small limits and colimits, and to give explicit constructions of those. We start with the case of colimits:

Proposition 3.2.5.1. *The universe* $\mathrm{Grpd}_{\mathcal{B}}$ *for small* $\mathcal{B}$*-groupoids admits small colimits. Moreover, if* I *is a* $\mathcal{B}$*-category and if* $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}})$ *is an* I*-indexed diagram in context* $A \in \mathcal{B}$*, then the colimit* $\operatorname{colim} d\colon A \to \mathsf{Grpd}_{\mathcal{B}}$ *is given by the* $\mathcal{B}_{/A}$*-groupoid* $(\int d)^{\mathrm{gpd}}$ *, where* $\int d \to A \times \mathsf{I}$ *denotes the left fibration that is classified by d.*

Proof. In light of Proposition 3.2.1.11, we need to show that the diagonal functor

$$\mathrm{diag}\colon \mathsf{Grpd}_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}})$$

has a left adjoint, which is a consequence of Corollary 3.1.3.3. The explicit description of this colimit furthermore follows from Proposition 3.1.3.1. □

Remark 3.2.5.2. For the special case $\mathcal{B} \simeq \mathrm{Ani}$, the explicit construction of colimits in Proposition 3.2.5.1 is given in [49, Corollary 3.3.4.6].

Remark 3.2.5.3. Let $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ be a left exact accessible localisation with left adjoint L, where $\mathcal{C}$ is a small ∞-category. Let I be a $\mathcal{B}$-category and let $d\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}})$ be a diagram classified by a left fibration $\mathsf{P} \to \mathsf{I}$. By Proposition 3.2.5.1 we have that $\operatorname{colim} d \simeq \mathsf{P}^{\mathrm{gpd}} \simeq \operatorname{colim}_{\Delta^{\mathrm{op}}} \mathsf{P}$. Therefore $\operatorname{colim} d$ is given by applying L to the presheaf

$$c \mapsto (\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathsf{P})(c) \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathsf{P}(c)) \simeq \mathsf{P}(c)^{\mathrm{gpd}}.$$

Since Corollary 2.2.2.8 implies that for every $c \in \mathcal{C}$ the left fibration $\mathsf{P}(c) \to \mathsf{I}(c)$ classifies the functor $\Gamma_{\mathcal{B}_{/L(c)}} \circ d(c)\colon \mathsf{I}(c) \to \mathcal{S}$, we conclude that $\operatorname{colim} d \in \mathcal{B}$ is given by applying L to the presheaf $c \mapsto \operatorname{colim}(\Gamma \circ d(c))$.

We will now proceed by showing that $\mathsf{Grpd}_{\mathcal{B}}$ also admits small limits. By Proposition 3.2.1.11, we need to show that for any $\mathcal{B}$-category I the diagonal functor

$$\mathrm{diag}\colon \mathsf{Grpd}_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}})$$

admits a right adjoint. To that end, recall that since $\mathrm{Cat}(\mathcal{B})$ is cartesian closed, the pullback functor $\pi_{\mathsf{I}}^*\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{I}}$ admits a right adjoint $(\pi_{\mathsf{I}})_*$ that is given by sending a functor $p\colon \mathsf{P} \to \mathsf{I}$ to the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{P})_{/\mathsf{I}}$ that is defined by the pullback square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{P})_{/\mathsf{I}} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{P}) \arrow[d, "p_*"] \\ 1 \arrow[r, "\mathrm{id}_{\mathsf{I}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{I}). \end{tikzcd}$$

If p is a left fibration, then so is p_*, hence $(\pi_{\mathsf{I}})_*$ sends p to a $\mathcal{B}$-groupoid in this case. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (where $A \in \mathcal{B}$ is an arbitrary object) and using Remark 2.2.1.7, this argument also shows that the pullback functor $\pi_{\mathsf{I}}^* : \mathcal{B}_{/A} \to \mathrm{LFib}(A \times \mathsf{I})$ admits a right adjoint $(\pi_{\mathsf{I}})_*$ for any $A \in \mathcal{B}$. Moreover, if $s \colon B \to A$ is a map in $\mathcal{B}$, the natural map $s^*(\pi_{\mathsf{I}})_* \to (\pi_{\mathsf{I}})_* s^*$ is an equivalence whenever the transpose map $s_!(\pi_{\mathsf{I}})^* \to (\pi_{\mathsf{I}})^* s_!$ is one, and as this latter condition is evidently satisfied, Proposition 3.1.2.9 and Theorem 2.2.1.1 now show:

Proposition 3.2.5.4. *The universe* $\mathrm{Grpd}_{\mathcal{B}}$ *for small* $\mathcal{B}$*-groupoids admits small limits. More precisely, if* I *is a* $\mathcal{B}$*-category and if* $d \colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathrm{Grpd}_{\mathcal{B}})$ *is an* I*-indexed diagram in context* $A \in \mathcal{B}$*, then the limit* $\lim d \colon A \to \mathrm{Grpd}_{\mathcal{B}}$ *is given by the* $\mathcal{B}_{/A}$*-groupoid* $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{I}, \int \bar{d})_{/\pi_A^*\mathsf{I}}$ *in* $\mathcal{B}_{/A}$*, where* $\int \bar{d} \to \pi_A^*\mathsf{I}$ *is the left fibration that is classified by the transpose* $\bar{d} \colon \pi_A^*\mathsf{I} \to \mathrm{Grpd}_{\mathcal{B}_{/A}}$ *of* d. □

Remark 3.2.5.5. For the special case $\mathcal{B} \simeq \mathrm{Ani}$, the explicit construction of limits in Proposition 3.2.5.4 is given in [49, Corollary 3.3.3.3].

If I is an arbitrary $\mathcal{B}$-category, the fact that right adjoint functors preserve limits (Proposition 3.2.2.13) combined with the fact that the final object $1_{\mathrm{Grpd}_{\mathcal{B}}}$ is the limit of the unique diagram $\varnothing \to \mathrm{Grpd}_{\mathcal{B}}$ (Example 3.2.1.10) show that $\mathrm{diag}(1_{\mathrm{Grpd}_{\mathcal{B}}}) \colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$ defines a final object in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$. We will denote this object by $1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})}$. Proposition 3.2.5.4 now implies:

Corollary 3.2.5.6. *For any* $\mathcal{B}$*-category* I*, the limit functor*

$$\lim_{\mathsf{I}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Grpd}_{\mathcal{B}}$$

is explicitly given by the representable functor $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})}, -)$.

Proof. Since Proposition 3.2.5.4 already implies the existence of $\lim_{\mathsf{I}}$, the claim follows from the equivalence

$$\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})}, -) \simeq \mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(1_{\mathrm{Grpd}_{\mathcal{B}}}, \lim_{\mathsf{I}}(-))$$

and the fact that $\mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(1_{\mathrm{Grpd}_{\mathcal{B}}}, -)$ is equivalent to the identity functor on $\mathrm{Grpd}_{\mathcal{B}}$, see Proposition 2.2.2.4. □

Recall from Remark 1.4.1.4 that there is an embedding $i\colon \mathrm{Grpd}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\widehat{\mathcal{B}}}$. For later use, we note:

Proposition 3.2.5.7. *The inclusion* $i\colon \mathrm{Grpd}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\widehat{\mathcal{B}}}$ *preserves small limits and colimits.*

Proof. We begin with the case of colimits. Using Corollary 3.2.2.11, it suffices to show that the restriction of the colimit functor $\mathrm{colim}\colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\widehat{\mathcal{B}}}) \to \mathrm{Grpd}_{\widehat{\mathcal{B}}}$ along the inclusion $i_*\colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\widehat{\mathcal{B}}})$ takes values in $\mathrm{Grpd}_{\mathcal{B}}$ for any $\mathcal{B}$-category I. Since Proposition 3.2.5.1 implies that the colimit of any diagram $d\colon A \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\widehat{\mathcal{B}}})$ is given by the (large) $\mathcal{B}_{/A}$-groupoid $(\int d)^{\mathrm{gpd}}$, the claim follows from Proposition 1.2.4.4, together with the fact that d taking values in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$ is tantamount to $\int d$ being a small $\mathcal{B}_{/A}$-category, cf. Corollary 2.2.1.10.

As for the case of limits, by Corollary 3.2.5.6 we need to verify that the functor

$$\mathrm{map}_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\widehat{\mathcal{B}}})}(1_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\widehat{\mathcal{B}}})}, i_*(-))\colon \underline{\mathrm{Fun}}_{\mathcal{B}}(I, \mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Grpd}_{\widehat{\mathcal{B}}}$$

takes values in $\mathrm{Grpd}_{\mathcal{B}}$. Since we have $1_{\mathrm{Grpd}_{\widehat{\mathcal{B}}}} \simeq i(1_{\mathrm{Grpd}_{\mathcal{B}}})$, we find that the final object $1_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\widehat{\mathcal{B}}})}$ can be identified with the image of $1_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\mathcal{B}})}$ along i_*, so that we obtain an equivalence

$$\mathrm{map}_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\widehat{\mathcal{B}}})}(1_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\widehat{\mathcal{B}}})}, i_*(-)) \simeq \mathrm{map}_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\mathcal{B}})}(1_{\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I},\mathrm{Grpd}_{\mathcal{B}})}, -)$$

as i_* is fully faithful. Hence the claim follows. □

We have now assembled the necessary results in order to prove the following:

Proposition 3.2.5.8. *For any $\mathcal{B}$-category* C*, the $\mathcal{B}$-category* $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ *of presheaves on* C *admits small limits and colimits. Moreover, for any $\mathcal{B}$-category* I *and any diagram* $d\colon A \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$*, a cone* $\mathrm{diag}\, c \to d$ *defines a limit of d if and only if the induced cone* $\mathrm{diag}\, h(c) \to h_* d$ *defines a limit in* $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$*. In particular, the Yoneda embedding h preserves small limits.*

Proof. The fact that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ admits small limits and colimits follows immediately from combining Proposition 3.2.3.1 with Proposition 3.2.5.4 and Proposition 3.2.5.1. Now if we fix an I-indexed diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in C and if $\operatorname{diag} c \to d$ is an arbitrary cone that is represented by a section $A \to \mathsf{C}_{/d}$ over A, we obtain a commutative diagram

$$\begin{tikzcd}
& \mathsf{C}_{/c} \arrow[rr, hook] \arrow[dl] \arrow[dd] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \arrow[dl] \arrow[dd] \\
\mathsf{C}_{/d} \arrow[rr, hook, crossing over] \arrow[dd] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_*d} & \\
& \mathsf{C} \times A \arrow[rr, hook, "h\times\mathrm{id}" near start] \arrow[dl, "\mathrm{id}"] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times A \arrow[dl, "\mathrm{id}"] \\
\mathsf{C} \times A \arrow[rr, hook, "h\times\mathrm{id}"] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times A \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in which the square in the front and the one in the back are cartesian as h is fully faithful. Therefore, the upper horizontal square must be cartesian as well. The cone $\operatorname{diag} c \to d$ defines a limit of d if and only if the map $\mathsf{C}_{/c} \to \mathsf{C}_{/d}$ is an equivalence. Likewise, the induced cone $\operatorname{diag} h(c) \to h_*d$ defines a limit of h_*d precisely if the map $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_*d}$ is an equivalence. To complete the proof, we therefore need to show that the first map is an equivalence if and only if the second map is one. As the upper square in the previous diagram is cartesian, the second condition implies the first. Conversely, the map $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_*d}$ corresponds via Theorem 2.3.2.3 to a map between presheaves on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ which are both representable by objects in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Therefore, there is a unique map $h(c) \to \lim h_*d$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that the induced map

$$\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-, h(c)) \to \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-, \lim h_*d)$$

recovers the morphism $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_*d}$ on the level of presheaves on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As Yoneda's lemma (Theorem 2.3.2.3) implies that restricting this map along $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ recovers the map $h(c) \to \lim h_*d$, the latter being an equivalence implies that the morphism $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_*d}$ is an equivalence as well, as desired. $\square$

Corollary 3.2.5.9. *For any $\mathcal{B}$-category* C *and any object* $c\colon A \to \mathsf{C}$ *in context* $A \in \mathcal{B}$*, the corepresentable functor* $\operatorname{map}_{\mathsf{C}}(c,-)\colon A \times \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$ *transposes to a functor* $\pi_A^*\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ *that preserves all limits that exist in* $\pi_A^*\mathsf{C}$*.*

Proof. By Remark 2.3.2.1, the transpose of $\operatorname{map}_{\mathsf{C}}(c,-)$ can be identified with $\operatorname{map}_{\pi_A^*\mathsf{C}}(\bar{c},-)$, where $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$ is the transpose of c. Therefore, by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume that $A \simeq 1$. On account of Yoneda's lemma, the functor $\operatorname{map}_{\mathsf{C}}(c,-)$ is equivalent to the composition c^*h, where h denotes the Yoneda embedding and $c^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{Grpd}_{\mathcal{B}}$ is the evaluation functor at c. By Proposition 3.2.5.8 and Proposition 3.2.3.1, both of these functors preserve limits, hence the claim follows. □

Our next goal is to show that $\mathsf{Grpd}_{\mathcal{B}}$ is *cartesian closed.* To that end, note that Proposition 3.2.5.4 in particular implies that $\mathsf{Grpd}_{\mathcal{B}}$ admits products. We denote the resulting product functor by $-\times-\colon \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$.

Proposition 3.2.5.10. *The universe* $\mathsf{Grpd}_{\mathcal{B}}$ *for small $\mathcal{B}$-groupoids is cartesian closed, in that there is an equivalence*

$$\operatorname{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-\times-,-) \simeq \operatorname{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-,\operatorname{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-,-))$$

of functors $\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$.

Proof. First, we claim that the transpose $\phi\colon \mathsf{Grpd}_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}},\mathsf{Grpd}_{\mathcal{B}})$ of the product bifunctor $-\times-\colon \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ takes values in $\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}},\mathsf{Grpd}_{\mathcal{B}})$. To see this, we need to show that the image of every $\mathcal{B}_{/A}$-groupoid G along ϕ defines a left adjoint functor of $\mathcal{B}_{/A}$-categories. Note that since π_A^* preserves adjunctions (Corollary 3.1.1.9) and the internal hom (Proposition 1.2.5.4), we may identify $\pi_A^*(-\times-)$ with the product bifunctor of $\pi_A^*\mathsf{Grpd}_{\mathcal{B}}$ and $\pi_A^*(\phi)$ with its transpose. Along with the equivalence $\pi_A^*\mathsf{Grpd}_{\mathcal{B}} \simeq \mathsf{Grpd}_{\mathcal{B}_{/A}}$ from Remark 1.4.1.2, this implies that $\phi(\mathsf{G})\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}},\mathsf{Grpd}_{\mathcal{B}})$ transposes to the product functor $\mathsf{G}\times-\colon \mathsf{Grpd}_{\mathcal{B}_{/A}} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$. Thus, by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume without loss of generality that $A \simeq 1$. In this case, Example 3.2.1.13 implies that the functor $\mathsf{G}\times-\colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ is given on local sections over $A \in \mathcal{B}$ by the ∞-categorical product functor

$$\mathcal{B}_{/A} \xrightarrow{\pi_A^*\mathsf{G}\times-} \mathcal{B}_{/A}$$

which admits a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\pi_A^* \mathrm{G}, -)$. If $s\colon B \to A$ is a map in $\mathcal{B}$, we deduce from Proposition 1.2.5.4 that the natural map

$$s^* \underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\pi_A^* \mathrm{G}, -) \to \underline{\mathrm{Hom}}_{\mathcal{B}_{/B}}(\pi_B^* \mathrm{G}, s^*(-))$$

is an equivalence, hence Proposition 3.1.2.9 shows that the functor

$$\mathrm{G} \times - \colon \ \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$$

admits a right adjoint, as desired.

As a consequence of what we have just shown and Corollary 3.1.3.7, we now obtain a bifunctor $f\colon \ \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ that fits into an equivalence

$$\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(- \times -, -) \simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-, f(-,-)).$$

We complete the proof by showing that f is equivalent to $\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-,-)$. Note that by Proposition 2.2.2.4 the functor $\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}, -)$ is equivalent to the identity on $\mathsf{Grpd}_{\mathcal{B}}$. Hence the chain of equivalences

$$\begin{aligned} f(-,-) &\simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}, f(-,-)) \\ &\simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}} \times -, -) \\ &\simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-,-) \end{aligned}$$

in which the second step follows from the equivalence $1_{\mathsf{Grpd}_{\mathcal{B}}} \times - \simeq \mathrm{id}_{\mathsf{Grpd}_{\mathcal{B}}}$ gives rise to the desired identification. $\square$

In Proposition 1.4.1.3, we claimed without proof that for any two object $P, Q \in \mathcal{B}_{/A}$, viewed as objects of $\mathsf{Grpd}_{\mathcal{B}}$ in context $A \in \mathcal{B}$, there is an equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, Q) \simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(P, Q)$$

of $\mathcal{B}_{/A}$-groupoids (where $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, Q)$ denotes the internal hom in $\mathcal{B}_{/A}$). We are finally in the position to prove this statement. In fact, we even show that there is a *functorial* equivalence:

Proposition 3.2.5.11. *The evaluation of the map of $\mathcal{B}$-categories* $\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(-,-)$ *at $A \in \mathcal{B}$ recovers the internal hom bifunctor* $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(-,-)\colon \ \mathcal{B}_{/A}^{\mathrm{op}} \times \mathcal{B}_{/A} \to \mathcal{B}_{/A}$.

Proof. By Remark 2.3.2.1 and Remark 1.4.1.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$, so that we can assume without loss of generality that $A \simeq 1$. Also, Corollary 2.2.2.8 implies that one may identify the bifunctor $\mathrm{map}_{\mathcal{B}}(-,-)\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Ani}$ with the composition

$$\mathcal{B}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{\Gamma_{\mathcal{B}}(\mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(-,-))} \mathcal{B} \xrightarrow{\Gamma_{\mathcal{B}}} \mathrm{Ani}\,.$$

Since applying $\Gamma_{\mathcal{B}}$ to the bifunctor $- \times -\colon\ \mathrm{Grpd}_{\mathcal{B}} \times \mathrm{Grpd}_{\mathcal{B}} \to \mathrm{Grpd}_{\mathcal{B}}$ recovers the ordinary product bifunctor on $\mathcal{B}$, Proposition 3.2.5.10 yields an equivalence

$$\mathrm{map}_{\mathcal{B}}(- \times -, -) \simeq \mathrm{map}_{\mathcal{B}}(-, \Gamma_{\mathcal{B}}(\mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(-,-))),$$

which finishes the proof. □

3.2.6. Limits and colimits of $\mathcal{B}$-categories

Recall that by the discussion in Section 1.4.2, the assignment $A \mapsto \mathrm{Cat}(\mathcal{B}_{/A})$ defines a sheaf of ∞-categories on $\mathcal{B}$ that we denote by $\mathrm{Cat}_{\mathcal{B}}$ and that we refer to as the $\mathcal{B}$-category of (small) $\mathcal{B}$-categories. By combining Proposition 3.2.2.14 with Proposition 3.1.2.13 and the fact that presheaf $\mathcal{B}$-categories admits small limits and colimits (Proposition 3.2.5.8), we find:

Proposition 3.2.6.1. *The $\mathcal{B}$-category $\mathrm{Cat}_{\mathcal{B}}$ admits small limits and colimits.* □

Remark 3.2.6.2. Similar to the case of diagrams in $\mathrm{Grpd}_{\mathcal{B}}$, one can give explicit formulas for limits and colimits of diagrams in $\mathrm{Cat}_{\mathcal{B}}$. However, these formulas rely on the theory of cartesian and cocartesian fibrations for $\mathcal{B}$-categories, which will be the subject of Chapter 4. The explicit formulas for limits and colimits in $\mathrm{Cat}_{\mathcal{B}}$ will be derived in Section 4.5.1.

Next, our goal is to show that $\mathrm{Cat}_{\mathcal{B}}$ is *cartesian closed*. Let

$$- \times -\colon\ \mathrm{Cat}_{\mathcal{B}} \times \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$$

be the product functor.

Proposition 3.2.6.3. *There is a functor* $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ *together with an equivalence*

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(- \times -, -) \simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(-, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)).$$

In other words, the $\mathcal{B}$*-category* $\mathsf{Cat}_{\mathcal{B}}$ *is cartesian closed.*

Proof. This is proved in exactly the same way as Proposition 3.2.5.10. Namely, by using Corollary 3.1.3.7, it is enough to show that the product bifunctor transposes to a functor $\mathsf{Cat}_{\mathcal{B}} \to \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{Cat}_{\mathcal{B}}, \mathsf{Cat}_{\mathcal{B}})$. Using the equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}$ from Remark 1.4.2.4, we may carry out the same reduction steps as in the proof of Proposition 3.2.5.10, so that it will be sufficient to prove that for every $\mathcal{B}$-category C the functor $\mathsf{C} \times - \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ has a right adjoint. To see this, note that this functor is given on local sections over $A \in \mathcal{B}$ by the ∞-categorical product functor

$$\mathrm{Cat}(\mathcal{B}_{/A}) \xrightarrow{\pi_A^* \mathsf{C} \times -} \mathrm{Cat}(\mathcal{B}_{/A}).$$

which admits a right adjoint $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}, -)$. Furthermore, if $s\colon B \to A$ is a map in $\mathcal{B}$, we deduce from Proposition 1.2.5.4 that the natural map

$$s^* \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}, -) \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/B}}(\pi_B^* \mathsf{C}, s^*(-))$$

is an equivalence. Hence, Proposition 3.1.2.9 shows that the functor

$$\mathsf{C} \times - \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$$

admits a right adjoint, as desired. □

Remark 3.2.6.4. By making use of Corollary 2.2.2.8 and the fact that the product bifunctor $-\times-$ on $\mathsf{Cat}_{\mathcal{B}}$ recovers the ∞-categorical product bifunctor on $\mathrm{Cat}(\mathcal{B}_{/A})$ upon taking local sections over $A \in \mathcal{B}$, the equivalence

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(- \times -, -) \simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(-, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-))$$

from Proposition 3.2.6.3 implies that $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ recovers the internal hom of $\mathrm{Cat}(\mathcal{B}_{/A})$ when being evaluated at $A \in \mathcal{B}$, which justifies our choice of notation.

Corollary 3.2.6.5. *The bifunctor* $\operatorname{map}_{\mathsf{Cat}_{\mathcal{B}}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}}\times\mathsf{Cat}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ *is equivalent to the composition of the bifunctor* $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}}\times\mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ *with the core* $\mathcal{B}$*-groupoid functor* $(-)^{\simeq}\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$.

Proof. On account of Proposition 3.1.2.14 and the fact that $\operatorname{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}},-)$ is equivalent to the identity on $\mathsf{Grpd}_{\mathcal{B}}$ (see Proposition 2.2.2.4), we obtain equivalences

$$\begin{aligned}\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)^{\simeq} &\simeq \operatorname{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}},\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)^{\simeq})\\ &\simeq \operatorname{map}_{\mathsf{Cat}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}},\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-))\\ &\simeq \operatorname{map}_{\mathsf{Cat}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}\times -,-)\\ &\simeq \operatorname{map}_{\mathsf{Cat}_{\mathcal{B}}}(-,-)\end{aligned}$$

in which the last equivalence follows from the equivalence $1_{\mathsf{Grpd}_{\mathcal{B}}}\times - \simeq \mathrm{id}_{\mathsf{Cat}_{\mathcal{B}}}$. □

3.2.7. A characterisation of initial and final functors

In this section, we show that initial and final functors (see Section 2.1.1) can be characterised as those functors along which restriction of diagrams does not change their limits and colimits, respectively. For the case $\mathcal{B}\simeq \mathrm{Ani}$, this characterisation is proved in [49, Proposition 4.1.1.8] or [18, Theorem 6.4.5]. For the general case, note that precomposition with a functor $i\colon \mathsf{J}\to\mathsf{I}$ of $\mathcal{B}$-categories defines a functor $i^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\mathsf{C})$ that induces a functor $i^*\colon \mathsf{C}_{d/}\to\mathsf{C}_{i^*d/}$ over $A\times\mathsf{C}$ for every I-indexed diagram $d\colon A\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ in C.

Proposition 3.2.7.1. *For any functor* $i\colon \mathsf{J}\to\mathsf{I}$ *between* $\mathcal{B}$*-categories, the following are equivalent:*

1. *i is final;*

2. *for every large $\mathcal{B}$-category C and every diagram $d\colon A\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ in context $A\in\mathcal{B}$, the functor $i^*\colon \mathsf{C}_{d/}\to\mathsf{C}_{i^*d/}$ is an equivalence;*

3. *For every large $\mathcal{B}$-category C and every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in context $A \in \mathcal{B}$ that admits a colimit* $\operatorname{colim} d$*, the image of the colimit cocone* $d \to \operatorname{diag}\operatorname{colim} d$ *along the functor $i^* : \mathsf{C}_{d/} \to \mathsf{C}_{i^*d/}$ defines a colimit cocone of $i^* d$.*

4. *The mate of the commutative square*

$$\begin{tikzcd} \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\operatorname{diag}"] \arrow[d, "\operatorname{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "i^*"] \\ \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \end{tikzcd}$$

commutes.

The dual characterisation of initial functors holds as well.

Proof. Suppose that i is final, and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ be an arbitrary diagram. By making use of Remark 3.2.1.8 and Remark 2.1.3.3, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. To show (2), note that on account of Proposition 2.1.1.12, it suffices to show that the induced map $i^*|_c$ on the fibres over every $c\colon A \to \mathsf{C}$ is an equivalence. By the same argument as above, we may again assume $A \simeq 1$. Now the commutative diagram

$$\begin{tikzcd} 1 \arrow[rr, "c"] \arrow[d, "d"] & & \mathsf{C} \arrow[d, "d \times \operatorname{diag}"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \arrow[rr, "\operatorname{id} \times \operatorname{diag}(c)"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \end{tikzcd}$$

shows that the fibre of the left fibration $\mathsf{C}_{d/} \to \mathsf{C}$ over c is equivalent to the fibre of the right fibration $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ (that is given by postcomposition with $(\pi_c)_! : \mathsf{C}_{/c} \to \mathsf{C}$) over $d\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$. Similarly, the fibre of $\mathsf{C}_{i^*d/} \to \mathsf{C}$ over c is equivalent to the fibre of the right fibration $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C}_{/c}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C})$

over i^*d such that the map $i^*|_c$ fits into the commutative diagram

$$\begin{tikzcd}
& \mathsf{C}_{i^*d/}|_c \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C}_{/c}) \arrow[dd] \\
\mathsf{C}_{d/}|_c \arrow[ur, "i^*|_c"] \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c}) \arrow[ur, "i^*"] \arrow[dd] & \\
& 1 \arrow[rr, "i^*d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C}) \\
1 \arrow[ur, "\mathrm{id}"] \arrow[rr, "d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \arrow[ur, "i^*"] &
\end{tikzcd}$$

in which the two squares in the front and in the back are cartesian. Since i is final, the right square must be cartesian as well, hence $i^*|_c$ is an equivalence, so that (2) holds. Condition (3) follows immediately from (2). For the special case $\mathsf{C} = \mathsf{Grpd}_{\mathcal{B}}$, the same argument as in the proof of Proposition 3.2.2.6 shows that condition (3) is equivalent to the condition that the map $\operatorname{colim}_{\mathsf{J}} i^* \to \operatorname{colim}_{\mathsf{I}}$ must be an equivalence, hence condition (3) implies condition (4). Lastly, suppose that the map $\operatorname{colim}_{\mathsf{J}} i^* \to \operatorname{colim}_{\mathsf{I}}$ is an equivalence, and let us show that i is final. It will be enough to show that i is internally left orthogonal to the universal right fibration $(\mathsf{Grpd}_{\mathcal{B}})_*^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}$ as every right fibration between (small) $\mathcal{B}$-categories arises as a pullback of this functor. By Proposition 3.2.5.4, the universe $\mathsf{Grpd}_{\mathcal{B}}$ admits small limits, hence if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})$ is an arbitrary diagram both $(\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{d/}$ and $(\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{i^*d/}$ admits an initial section. By assumption, the functor $i^* : (\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{d/} \to (\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{i^*d/}$ sends the colimit cocone $d \to \operatorname{diag} \operatorname{colim} d$ to an initial section of $(\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{i^*d/}$, which implies that the functor $i^* : (\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{d/} \to (\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{i^*d/}$ must be initial as well. But this map is already a left fibration since it can be regarded as a map between left fibrations over $\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}$, hence we conclude that this functor must be an equivalence. Similarly as above and by making use of the equivalence $(\mathsf{Grpd}_{\mathcal{B}})_* \simeq (\mathsf{Grpd}_{\mathcal{B}})_{1_{\mathsf{Grpd}_{\mathcal{B}}}/}$ over

$\mathsf{Grpd}_{\mathcal{B}}$ from Proposition 2.2.2.4, one obtains a commutative diagram

$$\begin{tikzcd}[column sep=small]
 & (\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{i^*d/|\pi_A^*(1_{\mathsf{Grpd}_{\mathcal{B}}})} \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, (\mathsf{Grpd}_{\mathcal{B}})_*^{\mathrm{op}}) \arrow[dd] \\
(\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})_{d/|\pi_A^*(1_{\mathsf{Grpd}_{\mathcal{B}}})} \arrow[ur, "i^*|_{\pi_A^*(1_{\mathsf{Grpd}_{\mathcal{B}}})}"] \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\mathsf{Grpd}_{\mathcal{B}})_*^{\mathrm{op}}) \arrow[ur, "i^*"] \arrow[dd] & \\
 & A \arrow[rr, "i^*d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}) \\
A \arrow[ur, "\mathrm{id}"] \arrow[rr, "d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}) \arrow[ur, "i^*"] &
\end{tikzcd}$$

in which the squares in the front, in the back and on the left are cartesian. As the maps

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\mathsf{Grpd}_{\mathcal{B}})_*^{\mathrm{op}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})$$

and

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, (\mathsf{Grpd}_{\mathcal{B}})_*^{\mathrm{op}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}})$$

are right fibrations, the vertical square on the right is cartesian already when its underlying square of core $\mathcal{B}$-groupoids is. We therefore deduce that this square must be a pullback as well, which means that i is final. $\square$

Remark 3.2.7.2. Let C be a large $\mathcal{B}$-category, let $i\colon \mathsf{J} \to \mathsf{I}$ be a functor between $\mathcal{B}$-categories and let us fix an I-indexed diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$. Suppose that both d and i^*d admit a colimit in C. Then the universal property of initial objects (see Corollary 2.1.3.16) gives rise to a unique map

$$\begin{tikzcd}
 & i^*d \arrow[dl] \arrow[dr] & \\
\operatorname{diag}\operatorname{colim} i^*d \arrow[rr] & & \operatorname{diag}\operatorname{colim} d
\end{tikzcd}$$

in $\mathsf{C}_{i^*d/}$ that is an equivalence if and only if the cocone $i^*d \to \operatorname{diag}\operatorname{colim} d$ (which is the image of the colimit cocone $d \to \operatorname{diag}\operatorname{colim} d$ along i^*) is a colimit cocone. Proposition 3.2.7.1 now implies that this map is always an equivalence when i is final, and conversely i must be final whenever this map is an equivalence for every $\mathcal{B}$-category C and every diagram d that has a colimit in C (in fact, Proposition 3.2.7.1 shows that it suffices to consider $\mathsf{C} = \mathsf{Grpd}_{\mathcal{B}}$).

3.3. Cocompleteness

This section is dedicated to a more global study of (co)limits in a $\mathcal{B}$-category. More precisely, if U is an *internal class* of $\mathcal{B}$-categories (i.e. a full subcategory of $\mathsf{Cat}_{\mathcal{B}}$, see Definition 3.3.1.1), we define and study what it means for a $\mathcal{B}$-category C to be U-*(co)complete* and for a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories to be U-*(co)continuous*. For the maximal case, i.e. where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$, this will yield the correct internal analogue of the usual notion of cocompleteness and cocontinuity in (higher) category theory. One should note that this will be a strictly stronger notion than to simply admit all internal colimits that are indexed by small $\mathcal{B}$-categories, cf. Example 3.5.4.8 below. We begin in Section 3.3.1 by defining the notion of an internal class U of $\mathcal{B}$-categories, which is the internal analogue of a collection of ∞-categories. In Section 3.3.2, we give the definition of U-cocompleteness and U-cocontinuity with respect to such an internal class and we recast some of the results from Section 3.2 in this language. In Section 3.3.3, we define the large $\mathcal{B}$-category of U-cocomplete $\mathcal{B}$-categories and $\mathcal{B}$-categories of U-cocontinuous functors between U-cocomplete $\mathcal{B}$-categories.

3.3.1. Internal classes

In this section we introduce the correct $\mathcal{B}$-categorical analogue of *classes* of ∞-categories:

Definition 3.3.1.1. An *internal class* of $\mathcal{B}$-categories is a full subcategory U of $\mathsf{Cat}_{\mathcal{B}}$.

Remark 3.3.1.2. The reason why we define an internal class as a full subcategory $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ rather than just a full subcategory $\mathcal{U} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ in the usual ∞-categorical sense is that when using internal classes as indexing classes for colimits, only the former notion leads to a theory of cocompleteness that is *local* in $\mathcal{B}$, whereas the latter does not. For example, it is not reasonable to call a $\mathcal{B}$-category *cocomplete* even when it admits I-indexed colimits for every $\mathcal{B}$-category I, because it could still happen that there is a $\mathcal{B}_{/A}$-category J (for some $A \in \mathcal{B}$) such that $\pi_A^*\mathsf{C}$ does not have all J-indexed colimits (see Example 3.5.4.8 below). Instead, on should ask that C admits all colimits indexed by the maximal internal class $\mathsf{Cat}_{\mathcal{B}}$ (Example 3.3.1.3), which precisely amounts to asking that every small diagram $\mathsf{I} \to \pi_A^* \mathsf{C}$ of $\mathcal{B}_{/A}$-categories admits a colimit for every $A \in \mathcal{B}$. In this way, the notion of cocompleteness is forced to be local.

Example 3.3.1.3. By Remark 1.4.2.6, the (large) $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ may be regarded as an internal class of large $\mathcal{B}$-categories, so as a full subcategory of the (very large) $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}$. Consequently, every internal class of (small) $\mathcal{B}$-categories can also be regarded as an internal class of large $\mathcal{B}$-categories.

Example 3.3.1.4. Let $\mathcal{K} \subset \mathrm{Cat}_\infty$ be a class of ∞-categories. Note that on account of the adjunction $\mathrm{const} \dashv \Gamma \colon \widehat{\mathrm{Cat}}_\infty \rightleftarrows \mathrm{Cat}(\widehat{\mathcal{B}})$, the transpose of the functor

$$\mathrm{const}\,|_{\mathcal{K}} \colon \mathcal{K} \hookrightarrow \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathcal{B}) \simeq \Gamma(\mathsf{Cat}_{\mathcal{B}})$$

defines a map $\mathrm{const}(\mathcal{K}) \to \mathsf{Cat}_{\mathcal{B}}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$. The essential image of this functor thus defines an internal class of $\mathcal{B}$-categories that we denote by $\mathsf{LConst}_{\mathcal{K}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ and that we refer to as the internal class of *locally $\mathcal{K}$-constant $\mathcal{B}$-categories.* By construction, this is the full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the $\mathcal{K}$-constant $\mathcal{B}$-categories, i.e. by those objects $1 \to \mathsf{Cat}_{\mathcal{B}}$ that correspond to categories of the form $\mathrm{const}(K)$ for some $K \in \mathcal{K}$. Thus, a $\mathcal{B}_{/A}$-category C defines an object in LConst in context $A \in \mathcal{B}$ precisely if there is a cover $(s_i)_{i \in I} \colon \bigsqcup_{i \in I} A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that for every i there is an equivalence $s_i^* \mathsf{C} \simeq \mathrm{const}_{\mathcal{B}_{/A_i}}(K_i)$ for some $K_i \in \mathcal{K}$. In the maximal case $\mathcal{K} = \mathrm{Cat}_\infty$, we simply write LConst for the associated internal class of *locally constant* $\mathcal{B}$-categories.

Example 3.3.1.5. Recall from the discussion in Section 1.4.1 that every local class S of morphisms in $\mathcal{B}$ corresponds to a *subuniverse* $\mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ (i.e. full subcategory of $\mathsf{Grpd}_{\mathcal{B}}$). On account of the inclusion $\mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ from Proposition 3.1.2.14, we may thus regard Grpd_S as an internal class of $\mathcal{B}$-categories.

3.3.2. U-cocomplete $\mathcal{B}$-categories

In this section we define and study the condition on a $\mathcal{B}$-category to admit colimits indexed by objects in an internal class U of $\mathcal{B}$-categories (see Definition 3.3.1.1).

Definition 3.3.2.1. Let U be an internal class of $\mathcal{B}$-categories. A $\mathcal{B}$-category C is said to be U*-cocomplete* if $\pi_A^* \mathsf{C}$ admits I-indexed colimits for every object $\mathsf{I} \in \mathsf{U}(A)$

and every $A \in \mathcal{B}$. Similarly, if $f\colon \mathsf{C} \to \mathsf{D}$ is a functor between $\mathcal{B}$-categories that are both U-cocomplete, we say that f is U-*cocontinuous* if $\pi_A^* f$ preserves I-indexed colimits for any $A \in \mathcal{B}$ and any $\mathsf{I} \in \mathsf{U}(A)$. We simply say that a (large) $\mathcal{B}$-category C is *cocomplete* if it is $\mathsf{Cat}_{\mathcal{B}}$-cocomplete (when viewing $\mathsf{Cat}_{\mathcal{B}}$ as an internal class of $\widehat{\mathcal{B}}$-categories), and we call a functor between cocomplete (large) $\mathcal{B}$-categories *cocontinuous* if it is $\mathsf{Cat}_{\mathcal{B}}$-cocontinuous.

Dually, we say that a $\mathcal{B}$-category C is U-*complete* if $\pi_A^* \mathsf{C}$ admits I-indexed limits for every object $\mathsf{I} \in \mathsf{U}(A)$ and every $A \in \mathcal{B}$. If $f\colon \mathsf{C} \to \mathsf{D}$ is a functor between $\mathcal{B}$-categories that are both U-complete, we say that f is U-*continuous* if $\pi_A^* f$ preserves I-indexed limits for any $A \in \mathcal{B}$ and any $\mathsf{I} \in \mathsf{U}(A)$. We simply say that a (large) $\mathcal{B}$-category C is *complete* if it is $\mathsf{Cat}_{\mathcal{B}}$-complete, and we call a functor between complete (large) $\mathcal{B}$-categories *continuous* if it is $\mathsf{Cat}_{\mathcal{B}}$-continuous.

Remark 3.3.2.2. If U is an internal class of $\mathcal{B}$-categories, let $\mathrm{op}(\mathsf{U})$ be the internal class that arises as the image of U along the equivalence $(-)^{\mathrm{op}}\colon\ \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}}$ from Remark 1.4.2.5. Then a $\mathcal{B}$-category C is U-complete if and only if C^{op} is $\mathrm{op}(\mathsf{U})$-cocomplete, and a functor f is U-continuous if and only if f^{op} is $\mathrm{op}(\mathsf{U})$-cocontinuous. Hence, we may dualise statements about $\mathrm{op}(\mathsf{U})$-cocompleteness and $\mathrm{op}(\mathsf{U})$-cocontinuity to obtain the corresponding dual statements about U-completeness and U-continuity.

Remark 3.3.2.3. Since both the existence of (co)limits and the preservation of such (co)limits are local conditions (Remark 3.2.1.7 and Remark 3.2.2.3), one finds that if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, a $\mathcal{B}$-category C is U-(co)complete if and only if $\pi_{A_i}^* \mathsf{C}$ is $\pi_{A_i}^* \mathsf{U}$-(co)complete, and a functor $f\colon \mathsf{C} \to \mathsf{D}$ between U-(co)complete $\mathcal{B}$-categories is U-(co)continuous if and only if $\pi_{A_i}^*(f)$ is $\pi_{A_i}^* \mathsf{U}$-(co)continuous.

Remark 3.3.2.4. Let U be an internal class of $\mathcal{B}$-categories that is spanned by a collection of objects $(\mathsf{I}_i \in \mathsf{Cat}_{\mathcal{B}}(A_i))_{i \in I}$ in $\mathsf{Cat}_{\mathcal{B}}$ (in the sense of Definition 1.3.2.17). Then Remark 3.2.1.7 implies that a $\mathcal{B}$-category C is U-cocomplete whenever $\pi_{A_i}^* \mathsf{C}$ has I_i-indexed colimits for all $i \in I$. Moreover, Remark 3.2.2.3 implies that a functor $f\colon \mathsf{C} \to \mathsf{D}$ between U-cocomplete $\mathcal{B}$-categories is U-cocontinuous whenever $\pi_{A_i}^* f$ preserves I_i-indexed colimits for all $i \in I$.

For the examples of internal classes U that we introduced in Section 3.3.1, the associated notion of U-cocompleteness and U-cocontinuity admits a quite explicit description. We begin with the case $\mathsf{U} = \mathsf{Grpd}_S$, where S is a local class of maps in $\mathcal{B}$ (see Example 3.3.1.5). By combining Example 3.2.1.12 with Example 3.2.2.8, we find:

Proposition 3.3.2.5. *Let S be a local class of maps in $\mathcal{B}$ and let Grpd_S be the associated internal class of (large) $\mathcal{B}$-categories. Then a large $\mathcal{B}$-category C is Grpd_S-cocomplete if and only if the following two conditions are satisfied:*

1. *for every map $p\colon P \to A$ in S, the functor $p^* \colon \mathsf{C}(A) \to \mathsf{C}(P)$ admits a left adjoint $p_!$;*
2. *for every pullback square*
$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$
in $\mathcal{B}$ in which p and q are contained in S, the natural map $q_! t^* \to s^* p_!$ is an equivalence.

Furthermore, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between (large) Grpd_S-cocomplete $\mathcal{B}$-categories is Grpd_S-cocontinuous precisely if for every map $p\colon P \to A$ in S the natural map $p_! f(P) \to f(A) p_!$ is an equivalence. □

Example 3.3.2.6. If S is a local class in $\mathcal{B}$, the associated subuniverse

$$\mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$$

is closed under Grpd_S-colimits (i.e. Grpd_S is Grpd_S-cocomplete and the inclusion of Grpd_S into $\mathsf{Grpd}_{\mathcal{B}}$ is Grpd_S-cocontinuous) if and only if S is stable under composition. Moreover, recall from Example 3.1.4.5 that every modality $(\mathcal{L}, \mathcal{R})$ in $\mathcal{B}$ (i.e. a factorisation system in which $\mathcal{L}$ is stable under base change in $\mathcal{B}$) determines a reflective subcategory $\mathsf{Grpd}_{\mathcal{R}}$ of $\mathsf{Grpd}_{\mathcal{B}}$. Conversely, if $\mathsf{Grpd}_{\mathcal{R}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ is an arbitrary reflective subcategory, then [81, Theorem 4.8] shows that the associated local class $\mathcal{R}$ in $\mathcal{B}$ arises from a modality as in Example 3.1.4.5 precisely if $\mathcal{R}$ is stable under composition. Hence modalities in $\mathcal{B}$ correspond precisely to those reflective subuniverses that are closed under self-indexed colimits in $\mathsf{Grpd}_{\mathcal{B}}$.

Next, let $\mathcal{K} \subset \mathrm{Cat}_\infty$ be a class of ∞-categories, and let $\mathsf{LConst}_{\mathcal{K}}$ be the associated internal class (see Example 3.3.1.4). Using Remark 3.3.2.4, Example 3.2.1.13 and Example 3.2.2.9 now imply:

Proposition 3.3.2.7. *If $\mathcal{K}$ is a class of ∞-categories, a (large) $\mathcal{B}$-category C is $\mathsf{LConst}_{\mathcal{K}}$-cocomplete if and only if for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits colimits indexed by every object in $\mathcal{K}$ and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^* : \mathsf{C}(A) \to \mathsf{C}(B)$ preserves such colimits. Furthermore, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathsf{LConst}_{\mathcal{K}}$-cocomplete $\mathcal{B}$-categories is $\mathsf{LConst}_{\mathcal{K}}$-cocontinuous if and only if for all $A \in \mathcal{B}$ the functor $f(A)$ preserves all colimits that are indexed by objects in $\mathcal{K}$.* $\square$

In Construction 1.4.2.1, we defined a functor $- \otimes \mathsf{Grpd}_{\mathcal{B}} : \mathrm{Pr}^{\mathrm{R}}_\infty \to \mathsf{Cat}(\widehat{\mathcal{B}})$. Its explicit formula and Proposition 3.3.2.7 now yield:

Corollary 3.3.2.8. *For every class of ∞-categories $\mathcal{K}$ there is an equivalence*

$$\mathsf{Cat}_{\mathcal{B}}^{\mathsf{LConst}_{\mathcal{K}}\text{-cc}} \simeq \mathsf{Cat}_\infty^{\mathcal{K}\text{-cc}} \otimes \mathsf{Grpd}_{\mathcal{B}}$$

with respect to which the inclusion $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{LConst}_{\mathcal{K}}\text{-cc}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is obtained by applying $- \otimes \mathsf{Grpd}_{\mathcal{B}}$ to the inclusion $\mathsf{Cat}_\infty^{\mathcal{K}\text{-cc}} \hookrightarrow \mathsf{Cat}_\infty$. $\square$

Remark 3.3.2.9. We may also combine Proposition 3.3.2.5 and Proposition 3.3.2.7 in the following way: if S is a local class of maps in $\mathcal{B}$ and $\mathcal{K}$ a class of ∞-categories, we may consider the internal class $\langle S, \mathcal{K}\rangle$ generated by Grpd_S and $\mathsf{LConst}_{\mathcal{K}}$ (i.e. the essential image of the functor $\mathsf{Grpd}_S \sqcup \mathsf{LConst}_{\mathcal{K}} \to \mathsf{Cat}_{\mathcal{B}}$). Then Remark 3.3.2.4 shows that a $\mathcal{B}$-category C is $\langle S, \mathcal{K}\rangle$-cocomplete if and only if

1. for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits colimits indexed by objects in $\mathcal{K}$, and for every map $s\colon B \to A$ in $\mathcal{B}$ the transition functor $s^* : \mathsf{C}(A) \to \mathsf{C}(B)$ preserve these colimits;

2. for every map $p\colon P \to A$ in S the functor p^* admits a left adjoint $p_!$ that is compatible with base change in the sense of Proposition 3.3.2.5.

The goal for the remainder of this section is to recast some of the results that we obtained in Section 3.2 in the language of U-cocompleteness and U-cocontinuity.

We begin with the preservation of limits and colimits by adjoint functors. Since by Corollary 3.1.1.9 the functor π_A^* carries adjunctions in $\mathcal{B}$ to adjunctions in $\mathcal{B}_{/A}$ for every $A \in \mathcal{B}$, we immediately obtain from Proposition 3.2.2.13:

Proposition 3.3.2.10. *A left adjoint functor between* U*-cocomplete categories is* U*-cocontinuous, while a right adjoint between* U*-complete categories is* U*-continuous.* □

Similarly, Proposition 3.2.2.14 shows:

Proposition 3.3.2.11. *Suppose that* U *is an internal class of* $\mathcal{B}$*-categories and let* D *be a* U*-cocomplete* $\mathcal{B}$*-category. Then any reflective and any coreflective subcategory of* D *is* U*-cocomplete as well.* □

Using Proposition 1.2.5.4, we furthermore deduce from Proposition 3.2.3.1 and Proposition 3.2.3.3:

Proposition 3.3.2.12. *Suppose that* $f\colon \mathsf{C} \to \mathsf{D}$ *is a* U*-cocontinuous functor between* U*-cocomplete* $\mathcal{B}$*-categories, and let* K *be a simplicial object in* $\mathcal{B}$*. Then the postcomposition functor* $f_* : \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D})$ *is a* U*-cocontinuous functor between* U*-cocomplete* $\mathcal{B}$*-categories. Moreover, for all* $i : L \to K$ *in* $\mathcal{B}_{\Delta}$*, the map* $i^* : \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C})$ *is* U*-cocontinuous as well.* □

By combining Proposition 3.2.4.3 with Remark 1.2.5.6, we obtain:

Proposition 3.3.2.13. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category. For every object* $c : 1 \to \mathsf{C}$*, the slice* $\mathcal{B}$*-category* $\mathsf{C}_{/c}$ *is* U*-cocomplete, and the forgetful functor* $(\pi_c)_!$ *is* U*-cocontinuous.* □

Example 3.3.2.14. The universe $\mathrm{Grpd}_{\mathcal{B}}$ for small $\mathcal{B}$-groupoids is complete and cocomplete since $\mathrm{Grpd}_{\mathcal{B}}$ admits small limits and colimits (Proposition 3.2.5.1 and Proposition 3.2.5.4) and since for any $A \in \mathcal{B}$ there is a natural equivalence $\pi_A^* \mathrm{Grpd}_{\mathcal{B}} \simeq \mathrm{Grpd}_{\mathcal{B}_{/A}}$ (Remark 1.4.1.2). By the same argument and Proposition 3.2.5.7, the inclusion $i : \mathrm{Grpd}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\widehat{\mathcal{B}}}$ is continuous and cocontinuous.

Proposition 3.3.2.15. *For any* $\mathcal{B}$*-category* C*, the presheaf* $\mathcal{B}$*-category* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *is complete and cocomplete. If* C *is furthermore* U*-complete for some internal class* U*, the Yoneda embedding* $h_{\mathsf{C}} : \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *is* U*-continuous, and for every*

$c\colon A \to \mathsf{C}$ the corepresentable copresheaf $\mathrm{map}_{\mathsf{C}}(c,-)\colon A\times\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$ *transposes to a $\pi_A^*\mathsf{U}$-continuous functor $\pi_A^*\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$.*

Proof. The first claim is an immediate consequence of Example 3.3.2.14 and Proposition 3.3.2.12. Regarding the second claim, we have to see that the embedding $\pi_A^* h\colon \pi_A^*\mathsf{C} \to \pi_A^*\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ preserves all limits indexed by the objects in $\mathsf{U}(A)$. By Remark 2.3.2.1, we may identify $\pi_A^* h_{\mathsf{C}}$ with $h_{\pi_A^*\mathsf{C}}$, so that we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Now the claim follows from Proposition 3.2.5.8. Lastly, the third claim is a direct consequence of Corollary 3.2.5.9. □

Example 3.3.2.16. By combining Proposition 3.3.2.15 with Proposition 3.3.2.11 and Proposition 3.1.2.13, one finds that the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ is complete and cocomplete.

3.3.3. The large $\mathcal{B}$-category of U-cocomplete $\mathcal{B}$-categories

Recall that we showed in Proposition 1.3.1.12 that in order to define a (non-full) subcategory of a $\mathcal{B}$-category C, it suffices to specify a subobject of its object of morphisms C_1. With this in mind, we define (cf. Definition 1.3.1.14):

Definition 3.3.3.1. For any internal class U of $\mathcal{B}$-categories, we define the large $\mathcal{B}$-category of U-cocomplete $\mathcal{B}$-categories $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ as the subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the $\pi_A^*\mathsf{U}$-cocontinuous functors between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. We write $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}$ for the underlying ∞-category of global sections. In the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$ (viewed as an internal class of large $\mathcal{B}$-categories), we denote the resulting very large $\mathcal{B}$-category by $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc}}$ and its underlying ∞-category of global sections by $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}}$.

Remark 3.3.3.2. The subobject of $(\mathsf{Cat}_{\mathcal{B}})_1$ spanned by the $\pi_A^*\mathsf{U}$-cocontinuous functors between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}$-categories is stable under equivalences and composition in the sense of Proposition 1.3.1.17. As moreover U-cocompleteness and U-cocontinuity are local conditions (Remark 3.3.2.3), we conclude that an object $A \to \mathsf{Cat}_{\mathcal{B}}$ is contained in $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ if and only if the associated $\mathcal{B}_{/A}$-category is $\pi_A^*\mathsf{U}$-complete, and a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}_{/A}$-categories defines a morphism in $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ in context $A \in \mathcal{B}$ precisely if it is a $\pi_A^*\mathsf{U}$-cocontinuous functor between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories. In particular, if C and D are

$\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories, a functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is contained in the image of the monomorphism

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}}(\mathsf{C}, \mathsf{D}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\mathsf{C}, \mathsf{D})$$

if and only if it is $\pi_A^*\mathsf{U}$-cocontinuous. As a further consequence of the above, we obtain a canonical equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}\text{-cc}}$ for every $A \in \mathcal{B}$.

Definition 3.3.3.3. Let U be an internal class of $\mathcal{B}$-categories. If C and D are U-cocomplete $\mathcal{B}$-categories, we will denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{D})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ that is spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ in context $A \in \mathcal{B}$ such that the corresponding functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is $\pi_A^*\mathsf{U}$-cocontinuous. We will denote by $\mathrm{Fun}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{D})$ the underlying ∞-category of global sections. In the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$, we will denote the associated large $\mathcal{B}$-category by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{C}, \mathsf{D})$ and its underlying ∞-category of global sections by $\mathrm{Fun}_{\mathcal{B}}^{\mathsf{c}}\, c(\mathsf{C}, \mathsf{D})$.

Remark 3.3.3.4. In the situation of Definition 3.3.3.3, note that by combining Remark 3.2.6.4 and Corollary 3.2.6.5 with Remark 3.3.3.2, we obtain an equivalence

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}}(\mathsf{C}, \mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{D})^{\simeq}.$$

As a consequence, Remark 3.3.3.2 implies that an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{D})$ if and only if the associated functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is $\pi_A^*\mathsf{U}$-cocontinuous, and we obtain a canonical equivalence

$$\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}\text{-cc}}(\pi_A^*\mathsf{C}, \pi_A^*\mathsf{D})$$

for every $A \in \mathcal{B}$.

The notion of U-cocompleteness and U-cocontinuity allows for some flexibility in the choice of internal class U. For example, Proposition 3.2.7.1 implies that whenever I is a $\mathcal{B}$-category that is contained in U and $f\colon \mathsf{I} \to \mathsf{J}$ is a final functor, adjoining the $\mathcal{B}$-category J to U does not affect whether a $\mathcal{B}$-category is U-cocomplete or not. As it will be convenient later to impose certain stability conditions on an internal class, we define:

Definition 3.3.3.5. A *colimit class* in $\mathcal{B}$ is an internal class U of $\mathcal{B}$-categories that contains the final $\mathcal{B}$-category 1 and that is stable under final functors, i.e. satisfies the property that whenever $\mathsf{I} \to \mathsf{J}$ is a final functor in $\mathcal{B}_{/A}$ for some $A \in \mathcal{B}$, then $\mathsf{I} \in \mathsf{U}(A)$ implies that $\mathsf{J} \in \mathsf{U}(A)$.

For every internal class U of $\mathcal{B}$-categories one can construct a colimit class $\mathsf{U}^{\mathrm{colim}}$ that is uniquely specified by the condition that $\mathsf{U}^{\mathrm{colim}}$ is the minimal colimit class that contains U. Explicitly, this class is spanned by those $\mathcal{B}_{/A}$-categories J that admit a final functor from either an object in $\mathsf{U}(A)$ or the final $\mathcal{B}_{/A}$-category $1 \in \mathrm{Cat}(\mathcal{B}_{/A})$. Thus, a $\mathcal{B}_{/A}$-category I is contained in $\mathsf{U}^{\mathrm{colim}}(A)$ if and only if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that for each i the $\mathcal{B}_{/A_i}$-category $s_i^*\mathsf{I}$ admits a final functor from either an object in $\mathsf{U}(A_i)$ or the final object $1 \in \mathrm{Cat}(\mathcal{B}_{/A_i})$. By combining Proposition 3.2.7.1 with Remark 3.3.2.4, we deduce that a $\mathcal{B}$-category C is U-cocomplete if and only if it is $\mathsf{U}^{\mathrm{colim}}$-cocomplete, and similarly a functor $f\colon \mathsf{C} \to \mathsf{D}$ is U-cocontinuous if and only if it is $\mathsf{U}^{\mathrm{colim}}$-cocontinuous. Together with the evident observation that the above description of the objects in $\mathsf{U}^{\mathrm{colim}}$ is local in $\mathcal{B}$ (so that one obtains an equivalence $\pi_A^*(\mathsf{U}^{\mathrm{colim}}) \simeq (\pi_A^*\mathsf{U})^{\mathrm{colim}}$ for all $A \in \mathcal{B}$), this implies that one has $\mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \simeq \mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}^{\mathrm{colim}}\text{-cc}}$. Thus, for the sake of discussing colimits, we may therefore always assume that an internal class is a colimit class.

3.4. Kan extensions

The goal of this section is to develop the theory of Kan extensions of functors between $\mathcal{B}$-categories. The main theorem about the existence of Kan extensions will be discussed in Section 3.4.3, but its proof requires a few preliminary steps. We begin in Section 3.4.1 by discussing the *co-Yoneda lemma*, which states that every presheaf can be obtained as the colimit of its Grothendieck construction. Secondly, Section 3.4.2 contains a discussion of what we call U-*small* presheaves, those that can be obtained as U-colimits of representables. Lastly, after having established our main existence theorem for Kan extensions, we apply this result to obtain a characterisation of colimit cocones in Section 3.4.4.

3.4.1. The co-Yoneda lemma

Recall from Remark 2.3.2.7 that if C is a $\mathcal{B}$-category and if $F\colon \mathsf{C}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ is a presheaf on C, we may identify the pullback $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ of the right fibration $(\pi_F)_!\colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ along the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ with the right fibration $\int F \to \mathsf{C}$ that is classified by F. Let us denote by

$h_{/F}\colon \mathsf{C}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ the induced embedding. Since $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ admits a final object $\mathrm{id}_F\colon 1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$, Proposition 3.2.7.1 implies that the functor $(\pi_F)_!$ admits a colimit that is given by F itself (cf. Example 3.2.1.9). Using Remark 3.2.7.2, the functor $h_{/F}$ therefore induces a canonical map

$$\operatorname{colim} hp \simeq \operatorname{colim}(\pi_F)_! h_{/F} \to \operatorname{colim}(\pi_F)_! \simeq F$$

of presheaves on C. Our goal in this section is to prove that this map is an equivalence:

Proposition 3.4.1.1. *Let* C *be a* $\mathcal{B}$*-category, let* $F\colon 1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *be a presheaf on* C *and let* $p\colon C_{/F} \to \mathsf{C}$ *be the associated right fibration. Then the map* $\operatorname{colim} hp \to F$ *is an equivalence.*

Remark 3.4.1.2. The analogue of Proposition 3.4.1.1 for ∞-categories can be found in [49, Lemma 5.1.5.3].

The proof of Proposition 3.4.1.1 requires a few preparations. We begin with the following special case:

Proposition 3.4.1.3. *For any* $\mathcal{B}$*-category* C*, the colimit of the Yoneda embedding is given by the final object* $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$*.*

Proof. Using Proposition 3.2.5.7 in conjunction with Proposition 3.2.2.10, it suffices to show that the colimit of $\hat{h}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})$ is given by $1_{\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})}$. On account of the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{pr}_0^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})) \arrow[d, "\simeq"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{Grpd}_{\widehat{\mathcal{B}}}) \arrow[r, "(\mathrm{id} \times h)^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})
\end{tikzcd}$$

and Corollary 3.1.3.3, the colimit of $\hat{h}$ in $\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$ can be identified with the presheaf $(\mathrm{pr}_0)_!(\mathrm{id} \times h)_!(i \operatorname{map}_{\mathsf{C}})$, where $i\colon \mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\widehat{\mathcal{B}}}$ denotes the inclusion. On the other hand, Yoneda's lemma provides a commutative square

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{C}) \arrow[r, "j"] \arrow[d] & \int \mathrm{ev} \arrow[d] \\
\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "\mathrm{id} \times h"] & \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

in which j is initial, which together with Proposition 3.1.3.1 implies that the functor $(\mathrm{id} \times h)_!(i \operatorname{map}_{\mathsf{C}})$ is given by $i \circ \mathrm{ev}$. On account of the commutative diagram

$$\begin{tikzcd} & \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \arrow[dl, "\mathrm{pr}_0^*"'] \arrow[dr, "\mathrm{diag}"] & \\ \mathrm{Fun}(\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{Grpd}_{\widehat{\mathcal{B}}}) \arrow[rr, "\simeq"] & & \mathrm{Fun}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})) \end{tikzcd}$$

in which the lower equivalence carries $i \circ \mathrm{ev}$ to $i_* : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$, we conclude that the colimit of $\hat{h}$ is equivalent to the colimit of i_*. Since $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ is a final object, the result thus follows from Proposition 3.2.7.1, together with Example 3.2.1.9. □

Lemma 3.4.1.4. *Let* C *be a* $\mathcal{B}$*-category and let* $F \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ *be a presheaf on* C*. Then there is a canonical equivalence* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ *that fits into a commutative diagram*

$$\begin{tikzcd} \mathsf{C}_{/F} \arrow[d, hook, "h_{(\mathsf{C}_{/F})}"] \arrow[dr, "(h_{\mathsf{C}})_{/F}"] & \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \arrow[r, "\simeq"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \end{tikzcd}$$

Proof. Let $p \colon \mathsf{C}_{/F} \to \mathsf{C}$ be the projection, and let $p_! : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the left adjoint of the precomposition functor p^*. By Corollary 3.1.3.3, there is an equivalence $p_! h_{(\mathsf{C}_{/F})} \simeq h_{\mathsf{C}} p$, so it suffices to show that $p_!$ factors through $(\pi_F)_! : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ via an equivalence. By construction of $p_!$, this functor sends the final object $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ to F, hence we obtain a lifting problem

$$\begin{tikzcd} 1 \arrow[r, "F"] \arrow[d, "1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[d, "(\pi_F)_!"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \arrow[r, "p_!"] \arrow[ur, dashed] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which F and $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ define final maps and $(\pi_F)_!$ is a right fibration. On account of the factorisation system between final maps and right fibrations, the dashed arrow exists and has to be final as well. To complete the proof, it therefore suffices to show that it is also a right fibration, which follows once we verify that

$p_!$ is a right fibration. By Proposition 3.1.3.1, this map evaluates at any $A \in \mathcal{B}$ to the the functor $\mathrm{RFib}(A \times \mathsf{C}_{/F}) \to \mathrm{RFib}(A \times \mathsf{C})$ that is given by restricting the right fibration $\mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}_{/F}} \to \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}}$ of ∞-categories. Since the canonical square

$$\begin{tikzcd} \mathrm{RFib}(A \times \mathsf{C}_{/F}) \arrow[r, hook] \arrow[d] & \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}_{/F}} \arrow[d] \\ \mathrm{RFib}(A \times \mathsf{C}) \arrow[r, hook] & \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}} \end{tikzcd}$$

is a pullback, it thus follows that $p_!$ is section-wise a right fibration and must therefore be a right fibration itself. □

Proof of Proposition 3.4.1.1. The map $\operatorname{colim} hp \to F$ is determined by the cocone under the functor $hp \simeq (\pi_F)_! h_{/F}$ that arises as the image of the colimit cocone $(\pi_F)_! \to \mathrm{diag}(F)$ along

$$(h_{/F})^* \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{(\pi_F)_!/} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{hp/}.$$

Using the equivalence $\phi \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})$ from Lemma 3.4.1.4, we now obtain a commutative square

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})_{\phi/} \arrow[r, "(h_{/F})^*"] \arrow[d, "(p_!)_*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})_{h_{\mathsf{C}_{/F}}/} \arrow[d, "(p_!)_*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{(\pi_F)_!/} \arrow[r, "(h_{/F})^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{hp/}. \end{tikzcd}$$

As $p_!$ is a left adjoint and therefore preserves colimits, we may thus replace C by $\mathsf{C}_{/F}$ and can therefore assume without loss of generality $F = 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$, in which case the desired result follows from Proposition 3.4.1.3. □

3.4.2. U-small presheaves

In this section we study those subcategories of the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ of presheaves on a $\mathcal{B}$-category C that are spanned by U-colimits of representable presheaves for an arbitrary internal class U of $\mathcal{B}$-categories.

Definition 3.4.2.1. Let C be a $\mathcal{B}$-category and let U be an internal class of $\mathcal{B}$-categories. We say that a presheaf $F \colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context $A \in \mathcal{B}$ is U-*small* if $\mathsf{C}_{/F}$ is contained in $\mathsf{U}^{\mathrm{colim}}(A)$ (see the discussion after Definition 3.3.3.5). We denote by $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by the U-small presheaves, and we denote by $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ the underlying ∞-category of global sections.

Remark 3.4.2.2. Since for every $A \in \mathcal{B}$ we have an equivalence

$$\pi_A^*(\mathsf{U}^{\mathrm{colim}}) \simeq (\pi_A^* \mathsf{U})^{\mathrm{colim}}$$

(see the discussion following Definition 3.3.3.5) and on account of Remark 2.1.2.2, it follows that a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is U-small if and only if its transpose $\hat{F}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C})$ is $\pi_A^*\mathsf{U}$-small.

Remark 3.4.2.3. The property of a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ to be U-small is local in $\mathcal{B}$. That is, for every cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, the presheaf F is U-small if and only if $s_i^*(F)$ is U-small. This follows immediately from the fact that since $\mathsf{U}^{\mathrm{colim}}$ is itself a $\mathcal{B}$-category, the property to be contained in $\mathsf{U}^{\mathrm{colim}}(A)$ can be checked locally. As a consequence, a presheaf F is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}(\mathsf{C})$ if and only if F is U-small. From this observation and Remark 3.4.2.2, it furthermore follows that there is a natural equivalence

$$\underline{\mathsf{Small}}^{\pi_A^*\mathsf{U}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}) \simeq \pi_A^* \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}).$$

for every $A \in \mathcal{B}$.

Remark 3.4.2.4. For the special case where $\mathcal{B} \simeq \mathsf{Ani}$ and where U is the class of κ-filtered ∞-categories for some regular cardinal κ, the ∞-category of U-small presheaves on a small ∞-category is precisely its ind-completion by κ-filtered colimits in the sense of [49, § 5.3.5]. In general, however, the $\mathcal{B}$-category $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ need not be a free cocompletion, see Section 3.5.1 below.

Example 3.4.2.5. For any internal class U of $\mathcal{B}$-categories and for any $\mathcal{B}$-category C, the presheaf represented by an object c in C in context $A \in \mathcal{B}$ is U-small: in fact, by Remark 3.4.2.2 and Remark 2.3.2.1, we may assume (by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$) that $A \simeq 1$. In this case, the canonical section $\mathrm{id}_c\colon 1 \to \mathsf{C}_{/c}$ provides a final map from an object contained in $\mathsf{U}^{\mathrm{colim}}(1)$, which implies that $\mathsf{C}_{/c}$ defines an object of $\mathsf{U}^{\mathrm{colim}}$ as well. Hence, by furthermore using Proposition 1.3.2.14, the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ thus factors through the inclusion $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Proposition 3.4.2.6. *For any $\mathcal{B}$-category C and any internal class U of small $\mathcal{B}$-categories, the $\mathcal{B}$-category $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is closed under U-colimits of representables in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. More precisely, for any object $A \to \mathsf{U}$ in context $A \in \mathcal{B}$ that corresponds to a $\mathcal{B}_{/A}$-category I, the functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \to \pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ restricts to a functor*

$$\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C}) \to \pi_A^* \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}).$$

Proof. By using Remark 2.3.2.1 and Remark 3.4.2.3, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$, so that it will be enough to show that for any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in context $A \in \mathcal{B}$ the colimit $\operatorname{colim} hd\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is a U-small presheaf on C. By the same argument and Remark 3.2.1.8, we can again reduce to $A \simeq 1$ by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$. Let $pi\colon \mathsf{I} \to \mathsf{P} \to \mathsf{C}$ be the factorisation of d into a final functor and a right fibration. By Proposition 3.2.7.1 we find $\operatorname{colim} hd \simeq \operatorname{colim} hp$, hence Proposition 3.4.1.1 implies $\mathsf{P} \simeq \mathsf{C}_{/\operatorname{colim} hd}$. Since i is a final functor into P from the $\mathcal{B}$-category $\mathsf{I} \in \mathsf{U}(1)$, this shows that $\operatorname{colim} hd$ is U-small. □

We finish this section by showing that for any $\mathcal{B}$-category C, the functor $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ that is induced by the Yoneda embedding has a left adjoint whenever C is U-cocomplete.

Proposition 3.4.2.7. *Let U be an internal class of $\mathcal{B}$-categories. If C is a U-cocomplete $\mathcal{B}$-category, the functor $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ that is induced by the Yoneda embedding admits a left adjoint $L\colon \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{C}$.*

Proof. As C being U-cocomplete is equivalent to C being $\mathsf{U}^{\mathrm{colim}}$-cocomplete, we may assume without loss of generality that U is already a colimit class. Let F be an object in $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ in context $A \in \mathcal{B}$. On account of Corollary 3.1.3.5, it suffices to show that the copresheaf $\operatorname{map}_{\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}(F, h(-))$ is corepresentable by an object in C. Using Remark 2.3.2.1 and Remark 3.4.2.3, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume without loss of generality that F is a U-small presheaf in context $1 \in \mathcal{B}$. In this case, we have $\mathsf{C}_{/F} \in \mathsf{U}(1)$, where $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ is the right fibration that is classified by F. Now Proposition 3.4.1.1

and Proposition 3.2.7.1 give rise to an equivalence $F \simeq \operatorname{colim} hp$. Thus, one obtains a chain of equivalences

$$\begin{aligned}\operatorname{map}_{\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}(F, h(-)) &\simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\operatorname{colim} hp, h(-)) \\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))}(hp, \operatorname{diag} h(-)) \\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathsf{C})}(p, \operatorname{diag}(-)) \\ &\simeq \operatorname{map}_{\mathsf{C}}(\operatorname{colim} p, -),\end{aligned}$$

which shows that the presheaf $\operatorname{map}_{\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}(F, h(-))$ is represented by the object $L(F) = \operatorname{colim} p$. □

3.4.3. The functor of left Kan extension

Throughout this section, let C, D and E be $\mathcal{B}$-categories and let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor.

Definition 3.4.3.1. A *left Kan extension* of a functor $F\colon \mathsf{C} \to \mathsf{E}$ along f is a functor $f_!F\colon \mathsf{D} \to \mathsf{E}$ together with an equivalence

$$\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})}(f_!F, -) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})}(F, f^*(-)).$$

Dually, a *right Kan extension* of F along f is a functor $f_*F\colon \mathsf{D} \to \mathsf{E}$ together with an equivalence

$$\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})}(-, f_*F) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})}(f^*(-), F).$$

Remark 3.4.3.2. In the situation of Definition 3.4.3.1, if $A \in \mathcal{B}$ is an arbitrary object, one easily deduces from Proposition 1.2.5.4 and Remark 2.3.2.1 that the functor $\pi_A^*(f_!F)$ is a left Kan extension of π_A^*F along π_A^*f.

Remark 3.4.3.3. As usual, the theory of right Kan extensions can be formally obtained from the theory of left Kan extensions by taking opposite $\mathcal{B}$-categories. We will therefore only discuss the case of left Kan extensions.

Remark 3.4.3.4. The theory of Kan extensions for the special case $\mathcal{B} \simeq \mathrm{Ani}$ is discussed in [42, § 22], [49, § 4.3], or [18, § 6.4].

The main goal of this section is to prove the following theorem about the existence of left Kan extensions:

Theorem 3.4.3.5. *Let* U *be an internal class of* $\mathcal{B}$*-categories such that for every object* $d\colon A \to \mathsf{D}$ *in context* $A \in \mathcal{B}$ *the* $\mathcal{B}_{/A}$*-category* $\mathsf{C}_{/d}$ *is contained in* $\mathsf{U}^{\mathrm{colim}}(A)$*. Then, whenever* E *is* U*-cocomplete, the functor* $f^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})$ *has a left adjoint* $f_!$ *which is fully faithful whenever* f *is fully faithful.*

Proof. To begin with, by replacing U with $\mathsf{U}^{\mathrm{colim}}$, we may assume without loss of generality that U is a colimit class and therefore that $\mathsf{C}_{/d}$ is contained in U for every object d in D.

By Corollary 3.1.3.3, the functor

$$(f\times \mathrm{id})^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{E}^{\mathrm{op}},\mathsf{Grpd}_{\mathcal{B}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{E}^{\mathrm{op}},\mathsf{Grpd}_{\mathcal{B}})$$

admits a left adjoint $(f\times\mathrm{id})_!$. We now claim that the composition

$$\begin{aligned}\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}) &\xrightarrow{h_*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}))\\ &\xrightarrow{\simeq} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{E}^{\mathrm{op}},\mathsf{Grpd}_{\mathcal{B}})\\ &\xrightarrow{(f\times\mathrm{id})_!} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{E}^{\mathrm{op}},\mathsf{Grpd}_{\mathcal{B}})\\ &\xrightarrow{\simeq} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}))\end{aligned}$$

takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{E}))$. To see this, let $F\colon A\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})$ be an object in context $A\in\mathcal{B}$. Using Proposition 1.2.5.4 together with Remark 2.3.2.1 and Remark 3.4.2.3 as well as the fact that as π_A^* preserves adjunctions (Corollary 3.1.1.9) we may identify $\pi_A^*(f\times\mathrm{id})_!$ with $(\pi_A^*(f)\times\mathrm{id})_!$, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore to reduce to the case where $A\simeq 1$. Let $p\colon \mathsf{P}\to\mathsf{C}\times\mathsf{E}^{\mathrm{op}}$ be the left fibration that is classified by the transpose of hF, and let

$$\mathsf{P}\xrightarrow{i}\mathsf{Q}\xrightarrow{q}\mathsf{D}\times\mathsf{E}^{\mathrm{op}}$$

be the factorisation of $(f\times\mathrm{id})p$ into an initial functor and a left fibration. Then $q\colon \mathsf{Q}\to\mathsf{D}\times\mathsf{E}^{\mathrm{op}}$ classifies $(f\times\mathrm{id})_!(hF)$, hence we need to show that for any object $d\colon A\to\mathsf{D}$ in context $A\in\mathcal{B}$ the fibre $\mathsf{Q}|_d\to A\times\mathsf{E}^{\mathrm{op}}$ is classified by a

U-small presheaf on E. By the same argument as above, we may again assume that $A \simeq 1$. Consider the commutative diagram

$$\begin{tikzcd}
 & & & & \mathsf{Q}|_d \arrow[dl] \arrow[dd] \arrow[dr, "s"] & \\
 & \mathsf{P}_{/d} \arrow[rr, "i_{/d}"] \arrow[dl] \arrow[dd] & & \mathsf{Q}_{/d} \arrow[rr, "j"] \arrow[dl] \arrow[dd] & & \mathsf{R} \arrow[dd, "r"] \\
\mathsf{P} \arrow[rr, "i"] \arrow[dd, "p"] & & \mathsf{Q} \arrow[dd, "q"] & & \mathsf{E}^{\mathrm{op}} \arrow[dl] \arrow[dr, "\mathrm{id}"] & \\
 & \mathsf{C}_{/d} \times \mathsf{E}^{\mathrm{op}} \arrow[rr] \arrow[dl] & & \mathsf{D}_{/d} \times \mathsf{E}^{\mathrm{op}} \arrow[rr] \arrow[dl] & & \mathsf{E}^{\mathrm{op}} \\
\mathsf{C} \times \mathsf{E}^{\mathrm{op}} \arrow[rr, "f \times \mathrm{id}"] & & \mathsf{D} \times \mathsf{E}^{\mathrm{op}} & & &
\end{tikzcd}$$

in which R is uniquely determined by the condition that j be initial and r be a left fibration. Since $i_{/d}$ is the pullback of i along a right fibration and since right fibrations are proper by Proposition 2.1.4.9, this map is initial. As a consequence, the composition $ji_{/d}$ is initial as well, which implies that the left fibration r is classified by the colimit of the composition $\mathsf{C}_{/d} \to \mathsf{C} \to \mathsf{E} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E})$. By Proposition 3.4.2.6 and the condition on $\mathsf{C}_{/d}$ to be contained in $\mathsf{U}(1)$, the left fibration r is classified by a U-small presheaf. To prove our claim, we therefore need only show that the map $s\colon \mathsf{Q}|_d \to \mathsf{R}$ is an equivalence. As this is a map of right fibrations over E^{op}, we may work fibre-wise (Proposition 2.1.1.12). If $e\colon A \to \mathsf{E}^{\mathrm{op}}$ is an object in context $A \in \mathcal{B}$, we obtain an induced commutative triangle

$$\begin{tikzcd}
 & (\mathsf{Q}|_d)|_e \arrow[dl] \arrow[dr, "s|_e"] & \\
(\mathsf{Q}_{/d})|_e \arrow[rr, "j|_e"] & & \mathsf{R}|_e
\end{tikzcd}$$

over A. Since the projections $\mathsf{Q}_{/d} \to \mathsf{E}^{\mathrm{op}}$ and $\mathsf{R} \to \mathsf{E}^{\mathrm{op}}$ are left fibrations and therefore smooth (by the dual of Proposition 2.1.4.9) and since initial functors are a fortiori covariant equivalences (see Section 2.1.4), we deduce from Proposition 2.1.4.12 that $j|_e$ exhibits $\mathsf{R}|_e$ as the groupoidification of $(\mathsf{Q}_{/d})|_e$. Moreover, the map $(\mathsf{Q}|_d)|_e \to (\mathsf{Q}_{/d})|_e$ is a pullback of the final map $A \to \mathsf{D}_{/d} \times A$ along a smooth map and therefore final as well. Since final functors induce equivalences on groupoidifications, we thus conclude that $s|_e$ must be an equivalence, as desired.

By making use of the discussion thus far, we may now define $f_!$ as the composition of the two horizontal arrows in the top row of the commutative diagram

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \arrow[r] \arrow[d, hook, "h"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{E})) \arrow[r, "L_*"] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}) \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{E}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "(f\times \mathrm{id})_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D} \times \mathsf{E}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) & \end{tikzcd}$$

in which L denotes the left adjoint to the Yoneda embedding that is supplied by Proposition 3.4.2.7. It is now clear from the construction of $f_!$ that this functor defines a left adjoint of f^*.

Lastly, suppose that f is fully faithful. We show that in this case the adjunction counit $\mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})} \to f^* f_!$ is an equivalence. Since equivalences are computed object-wise (see Proposition 2.3.2.12), we only have to show that for every object F in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ the induced map $F \to f^* f_! F$ is an equivalence. Since π_A^* preserves adjunctions and the internal hom (Corollary 3.1.1.9 and Proposition 1.2.5.4), we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that F is in context $1 \in \mathcal{B}$. By construction of the adjunction $f_! \dashv f^*$, the unit $F \to f^* f_! F$ is determined by the composition

$$\begin{aligned} h_*(F) &\xrightarrow{\eta_1 h_*(F)} (f \times \mathrm{id})^*(f \times \mathrm{id})_! h_*(F) \\ &\xrightarrow{(f\times\mathrm{id})^*\eta_2(f\times\mathrm{id})_! h_*(F)} (f \times \mathrm{id})^* h_* L_*(f \times \mathrm{id})_! h_*(F) \end{aligned}$$

in which η_1 is the unit of the adjunction $(f\times\mathrm{id})_! \dashv (f\times\mathrm{id})^*$ and η_2 is the unit of the adjunction $L_* \dashv h_*$. By Corollary 3.1.3.3, the first map is an equivalence, hence it suffices to show that the second one is an equivalence as well. Again, it suffices to show this object-wise. Let therefore c be an object of C, as above without loss of generality in context $1 \in \mathcal{B}$. By the above argument, the object $(f \times \mathrm{id})_! h_*(F)(c)$ is given by the colimit of the diagram $hF(\pi_c)_! \colon \mathsf{C}_{/c} \to \mathsf{C} \to \mathsf{E} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E})$. By making use of the final section $\mathrm{id}_c \colon 1 \to \mathsf{C}_{/c}$, this presheaf is therefore representable by $F(c)$, which implies the claim. $\square$

Remark 3.4.3.6. In the situation of Theorem 3.4.3.5, the construction of $f_!$ shows that if $F\colon \mathsf{D} \to \mathsf{E}$ is a functor, the counit $f_! f^* F \to F$ is given by the composition

$$L_*(f \times \mathrm{id})_!(f \times \mathrm{id})^* h_*(F) \xrightarrow{L_*\epsilon_1 h_* F} L_* h_*(F) \xrightarrow{\epsilon_2} F$$

where ϵ_1 is the counit of the adjunction $(f \times \mathrm{id})_! \dashv (f \times \mathrm{id})^*$ and ϵ_2 is the counit of the adjunction $L_* \dashv h_*$. Since the latter is an equivalence, the functor F arises as the left Kan extension of f^*F precisely if the map $L_*\epsilon_1 h_*(F)$ is an equivalence. Let $q\colon \mathsf{Q} \to \mathsf{D} \times \mathsf{E}^{\mathrm{op}}$ be the left fibration that is classified by $h_*(F)$ and let $p\colon \mathsf{P} \to \mathsf{C} \times \mathsf{E}^{\mathrm{op}}$ be the pullback of q along $f \times \mathrm{id}$. Let furthermore $q'\colon \mathsf{Q}' \to \mathsf{C} \times \mathsf{E}^{\mathrm{op}}$ be the functor that arises from factoring $(f \times \mathrm{id})p$ into an initial map and a left fibration. On the level of left fibrations over $\mathsf{D} \times \mathsf{E}^{\mathrm{op}}$, the map $\epsilon_1 h_*(F)$ is then given by the map g that arises as the unique lift in the commutative diagram

$$\begin{tikzcd} \mathsf{P} \arrow[r, "i"] \arrow[d, "i^*"'] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{Q}' \arrow[r, "q"] \arrow[ur, dotted, "g"] & \mathsf{D} \times \mathsf{E}^{\mathrm{op}}. \end{tikzcd}$$

Then the condition that $L_*\epsilon_1 h_* F$ is an equivalence corresponds to the condition that for any object $d\colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ the map $g|_d\colon \mathsf{Q}'|_d \to \mathsf{Q}|_d$, viewed as a map over $\pi_A^*\mathsf{E}^{\mathrm{op}}$, induces an equivalence $\mathrm{colim}(q'|_d^{\mathrm{op}}) \simeq \mathrm{colim}(q|_d^{\mathrm{op}})$ in $\pi_A^*\mathsf{E}$. Note that by a similar argument as in the proof of Theorem 3.4.3.5, the map $g|_d$ fits into a commutative square

$$\begin{tikzcd} \mathsf{Q}'_{/d} \arrow[r, "g_{/d}"] \arrow[d, "j'"] & \mathsf{Q}_{/d} \arrow[d, "j"] \\ \mathsf{Q}'|_d \arrow[r, "g|_d"] & \mathsf{Q}|_d \end{tikzcd}$$

in which j' and j are initial. As a consequence, the map $g|_d$ is determined by the factorisation of the map $ji_{/d}$ in the commutative diagram

$$\begin{tikzcd} \mathsf{P}_{/d} \arrow[r, "i_{/d}"] \arrow[d] & \mathsf{Q}_{/d} \arrow[r, "j"] \arrow[d] & \mathsf{Q}|_d \arrow[d] \\ \mathsf{C}_{/d} \times \mathsf{E}^{\mathrm{op}} \times A \arrow[r, "f_{/d} \times \mathrm{id}"] & \mathsf{D}_{/d} \times \mathsf{E}^{\mathrm{op}} \times A \arrow[r] & \mathsf{E}^{\mathrm{op}} \times A \end{tikzcd}$$

into an initial map and a right fibration. This argument shows that the map $g|_d$ classifies the canonical map

$$\mathrm{colim}\, hF(\pi_d)_! f_{/d} \to \mathrm{colim}\, hF(\pi_d)_!$$

of presheaves on $\pi_A^* \mathsf{E}$ that is induced by the functor $f_{/d} \colon \mathsf{C}_{/d} \to \mathsf{D}_{/d}$. Since L is a left inverse of h that preserves colimits, we thus conclude that F is a left Kan extension of its restriction $f^* F$ precisely if the map $f_{/d} \colon \mathsf{C}_{/d} \to \mathsf{D}_{/d}$ induces an equivalence

$$\operatorname{colim} F(\pi_d)_! f_{/d} \simeq \operatorname{colim} F(\pi_d)_! \simeq F(d)$$

in $\pi_A^* \mathsf{E}$ for every object $d \colon A \to \mathsf{D}$.

Recall from Section 2.3.1 that a large $\mathcal{B}$-category D is *locally* small if the left fibration $\mathsf{Tw}(\mathsf{D}) \to \mathsf{D}^{\mathrm{op}} \times \mathsf{D}$ is small. Theorem 3.4.3.5 now implies:

Corollary 3.4.3.7. *If $f \colon \mathsf{C} \to \mathsf{D}$ is a functor of $\mathcal{B}$-categories such that C is small and D is locally small (but not necessarily small). If E is a cocomplete large $\mathcal{B}$-category, the functor of left Kan extension $f_!$ always exists.*

Proof. By Theorem 3.4.3.5, it suffices to show that for any object $d \colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/d}$ is small, which follows immediately from the observation that the right fibration $\mathsf{C}_{/d} \to \mathsf{C} \times A$ a pullback of the small fibration $\mathsf{Tw}(\mathsf{D}) \to \mathsf{D}^{\mathrm{op}} \times \mathsf{D}$ and therefore small itself. □

3.4.4. Application: colimit cocones

In this section, we apply the theory of Kan extensions to obtain a characterisation of *colimit cocones*. Recall from Definition 2.1.3.11 that if I is a $\mathcal{B}$-category, the associated right cone $\mathsf{I}^{\triangleright}$ comes equipped with two functors $\iota \colon \mathsf{I} \to \mathsf{I}^{\triangleright}$ and $\infty \colon 1 \to \mathsf{I}^{\triangleright}$. Moreover, recall from Remark 3.2.1.3 that for every $\mathcal{B}$-category C we may identify $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$ with the $\mathcal{B}$-category of *cocones* over I-indexed diagrams in C. Our goal is to prove:

Proposition 3.4.4.1. *Let I and C be $\mathcal{B}$-categories and suppose that C admits I-indexed colimits. Then the functor of left Kan extension*

$$\iota_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$$

along $\iota \colon \mathsf{I} \to \mathsf{I}^{\triangleright}$ exists and is fully faithful, and its essential image coincides with the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$ that is spanned by the colimit cocones.

The proof of Proposition 3.4.4.1 relies on the following two general facts:

Lemma 3.4.4.2. *Suppose that*

$$\begin{tikzcd} \mathsf{P} \arrow[r, "g"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

is a cartesian square in $\mathrm{Cat}(\mathcal{B})$ *such that* q *admits a fully faithful left adjoint. Then* p *admits a fully faithful left adjoint as well.*

Proof. By assumption q has a section $l_1 \colon \mathsf{D} \to \mathsf{Q}$ which pulls back along f to form a section $l_0 \colon \mathsf{C} \hookrightarrow \mathsf{P}$ of p. Moreover, the adjunction counit $\epsilon_1 \colon \Delta^1 \otimes \mathsf{Q} \to \mathsf{Q}$ fits into a commutative diagram

$$\begin{tikzcd} \Delta^1 \otimes \mathsf{D} \arrow[r, "\mathrm{id} \otimes l_1"] \arrow[d, "s^0"] & \Delta^1 \otimes \mathsf{Q} \arrow[r, "s^0"] \arrow[d, "\epsilon_1"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{D} \arrow[r, "l_1"] & \mathsf{Q} \arrow[r, "q"] & \mathsf{D}, \end{tikzcd}$$

hence pullback along f defines a map $\epsilon_0 \colon \Delta^1 \otimes \mathsf{P} \to \mathsf{P}$ that fits into a commutative square

$$\begin{tikzcd} \Delta^1 \otimes \mathsf{C} \arrow[r, "\mathrm{id} \otimes l_0"] \arrow[d, "s^0"] & \Delta^1 \otimes \mathsf{P} \arrow[r, "s^0"] \arrow[d, "\epsilon_0"] & \mathsf{P} \arrow[d, "p"] \\ \mathsf{C} \arrow[r, "l_0"] & \mathsf{P} \arrow[r, "p"] & \mathsf{C}, \end{tikzcd}$$

By construction, the map $\epsilon_0 d^0$ is equivalent to the identity on P, and the map $\epsilon_1 d^1$ recovers the functor $l_1 p$. The previous commutative diagram now precisely expresses that both $p\epsilon_0$ and $\epsilon_0 l_0$ are equivalence, hence the desired result follows from Corollary 3.1.4.3. □

Lemma 3.4.4.3. *Fully faithful functors in* $\mathrm{Cat}(\mathcal{B})$ *are stable under pushout.*

Proof. If

$$\begin{tikzcd} \mathsf{C} \arrow[r, "h"] \arrow[d, hook, "f"] & \mathsf{E} \arrow[d, "g"] \\ \mathsf{D} \arrow[r, "k"] & \mathsf{F} \end{tikzcd}$$

is a pushout square in $\mathrm{Cat}(\mathcal{B})$ in which f is fully faithful, applying the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)$ results in a pullback square

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{F}) \arrow[r, "g^*"] \arrow[d, "k^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}) \arrow[d, "h^*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r, "f^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which f^* admits a fully faithful left adjoint $f_!$. By Lemma 3.4.4.2, this implies that g^* admits a fully faithful left adjoint as well, hence that the functor of left Kan extension $g_!$ is fully faithful. This in turn implies that g must be fully faithful too, see Corollary 3.1.3.3. $\square$

Proof of Proposition 3.4.4.1. Let U be the smallest colimit class in $\mathcal{B}$ that contains I. Then C is U-cocomplete (by Remark 3.3.2.4). Hence the existence of $\iota_!$ follows from Theorem 3.4.3.5 once we show that for every object $j\colon A \to \mathsf{I}^{\triangleright}$ the $\mathcal{B}_{/A}$-category $\mathsf{I}_{/j}$ is contained in $\mathsf{U}(A)$. By definition of the right cone, we have a cover $\mathsf{I}_0 \sqcup 1 \twoheadrightarrow (\mathsf{I}^{\triangleright})_0$ which induces a cover $A_0 \sqcup A_1 \twoheadrightarrow A$ by taking the pullback along $j\colon A \to (\mathsf{I}^{\triangleright})_0$. Let $j_0 : A_0 \to \mathsf{I}^{\triangleright}$ and $j_1 : A_1 \to \mathsf{I}^{\triangleright}$ be the induced objects. Since j_0 factors through the inclusion $\iota_0 : \mathsf{I}_0 \hookrightarrow (\mathsf{I}^{\triangleright})_0$ and since ι is fully faithful by Lemma 3.4.4.3, we obtain an equivalence $\mathsf{I}_{/j_0} \simeq \mathsf{I}_{/j_0'}$ over A_0, where j_0' is the unique object in I such that $\iota(j_0') \simeq j_0$. Since j_1 factors through the inclusion of the cone point $\infty\colon 1 \to \mathsf{I}^{\triangleright}$ which defines a final object in $\mathsf{I}^{\triangleright}$, we furthermore obtain an equivalence $\mathsf{I}_{/j_1} \simeq \pi_{A_i}^*\mathsf{I}$. Therefore the $\mathcal{B}_{/A}$-category $\mathsf{I}_{/j}$ is *locally* contained in U and therefore contained in U itself, for U defines a sheaf on $\mathcal{B}$. We therefore deduce that the functor of left Kan extension $\iota_!$ exists. Since Lemma 3.4.4.3 implies that ι is fully faithful, Corollary 3.1.3.3 furthermore shows that $\iota_!$ is fully faithful as well.

We finish the proof by identifying the essential image of $\iota_!$. By combining Remark 3.2.1.3 with Lemma 3.4.4.2, if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ is a diagram, the object $\iota_!(d)$ defines a fully faithful left adjoint $A \to \mathsf{C}_{d/}$ to the projection $\mathsf{C}_{d/} \to A$. By Example 3.2.1.10, this precisely means that $\iota_!(d)$ is an initial section over A and is therefore a colimit cocone. Conversely, if $\bar{d} : A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$ is a cocone under $d = \iota^*\bar{d}$, the map $\epsilon\bar{d} : \iota_! d \to \bar{d}$ defines a map in $\mathsf{C}_{d/}$. By the above argument, the domain of this map is a colimit cocone, hence if $\bar{d}$ defines a colimit cocone in $\mathsf{C}_{d/}$

as well, the map $\bar{\epsilon d}$ must necessarily be an equivalence since *any* map between two initial objects in a $\mathcal{B}_{/A}$-category is an equivalence (see Corollary 2.1.3.16). $\square$

3.5. Cocompletion

The main goal of this section is to construct and study the *free cocompletion* by U-colimits of an arbitrary $\mathcal{B}$-category, for any internal class U of $\mathcal{B}$-categories. In Section 3.5.1 we give the construction of this $\mathcal{B}$-category and prove its universal property. In Section 3.5.2 we discuss a criterion to detect free cocompletions. We finish this section by studying the free U-cocompletion of the point in Section 3.5.3.

3.5.1. The free U-cocompletion

Let C be a $\mathcal{B}$-category and let U be an internal class of $\mathcal{B}$-categories. The goal of this section is to construct the free U-cocompletion of C, i.e. the initial U-cocomplete $\mathcal{B}$-category that is equipped with a functor from C.

We begin our discussion with the maximal case $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$:

Theorem 3.5.1.1. *For every $\mathcal{B}$-category C and every cocomplete large $\mathcal{B}$-category E, the functor of left Kan extension $(h_{\mathsf{C}})_!$ along $h_{\mathsf{C}} \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ induces an equivalence*

$$(h_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}).$$

In other words, the Yoneda embedding $h_{\mathsf{C}} \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ exhibits the $\mathcal{B}$-category of presheaves on C as the free cocompletion of C.

Remark 3.5.1.2. The analogue of Theorem 3.5.1.1 for ∞-categories is the content of [49, Theorem 5.1.5.6] or [18, Theorem 6.3.13].

The proof of Theorem 3.5.1.1 relies on the following lemma:

Lemma 3.5.1.3. *Let $f \colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories and assume that C is small. Then the left Kan extension $(h_{\mathsf{C}})_!(h_{\mathsf{D}} f) \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{D})$ of $h_{\mathsf{D}} F$ along h_{C} is equivalent to the composition*

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \xhookrightarrow{i_*} \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \xrightarrow{f_!} \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{D}),$$

where $i\colon \mathrm{Grpd}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\widehat{\mathcal{B}}}$ *denotes the inclusion.*

Proof. Since $(h_{\mathsf{C}})_!$ is fully faithful and since the restriction of $f_! i_*$ along h_{C} recovers the functor $h_{\mathsf{D}} f$, it suffices to show that $f_! i_*$ is a left Kan extension along its restriction. By Remark 3.4.3.6, this is the case precisely if for any presheaf F on C the inclusion $h_{/F}\colon \mathsf{C}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ induces an equivalence

$$\operatorname{colim} f_!(\pi_F)_! h_{/F} \simeq \operatorname{colim} f_!(\pi_F)_! \simeq f_!(F).$$

Since $f_! i_*$ commutes with small colimits (Proposition 3.2.5.7) and as $\mathrm{PSh}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})$ admits small colimits (Proposition 3.3.2.12), it suffices to show that the map

$$\operatorname{colim}(\pi_F)_! h_{/F} \to F$$

is an equivalence in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, which follows from Proposition 3.4.1.1. $\square$

Proof of Theorem 3.5.1.1. Let us first show that for any object $f\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ in context $A \in \mathcal{B}$ the object $(h_{\mathsf{C}})_!(f)$ is contained in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$. By making use of Proposition 1.2.5.4 and Remark 3.3.3.4, Remark 2.3.2.1 and Remark 3.4.3.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Hence, we only need to show that $h_!(f)$ is cocontinuous. By again making use of Remark 3.4.3.2 and Remark 2.3.2.1, it is furthermore enough to show that $h_!(f)$ preserves I-indexed colimits for every small $\mathcal{B}$-category I. By Lemma 3.5.1.3 and the explicit construction of $h_!$ in Theorem 3.4.3.5, the functor $h_!(f)$ is equivalent to the composition

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \xhookrightarrow{i_*} \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \xrightarrow{f_!} \underline{\mathsf{Small}}^{\mathsf{Cat}_{\mathcal{B}}}_{\widehat{\mathcal{B}}}(\mathsf{E}) \xrightarrow{L} \mathsf{E}$$

in which L is left adjoint to the Yoneda embedding h_{E}. Since all three functors preserve small colimits, the claim follows.

By what we have just shown, $h_!$ takes values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$ and therefore determines an inclusion $h_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$. To show that this functor is essentially surjective as well, we need only show that any object $g\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$ in context $A \in \mathcal{B}$ whose associated functor in $\mathrm{Cat}(\widehat{\mathcal{B}}_{/A})$ is cocontinuous is a left Kan extension of its restriction along h. By the same reduction argument as above, we may again assume $A \simeq 1$. By using

Remark 3.4.3.6, the functor g is a Kan extension of gh precisely if for any presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ the functor $h_{/F}\colon \mathsf{C}_{/F} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ induces an equivalence

$$\operatorname{colim} g(\pi_F)_! h_{/F} \simeq g(F)$$

in E. Since Proposition 3.4.1.1 implies that the canonical map $\operatorname{colim}(\pi_F)_! h_{/F} \to F$ is an equivalence in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ and since g is cocontinuous, this is immediate. $\square$

Remark 3.5.1.4. In the situation of Theorem 3.5.1.1, suppose that E is in addition locally small. If $f\colon \mathsf{C} \to \mathsf{E}$ is an arbitrary functor, its left Kan extension $h_!(f)$ is not only cocontinuous, but even admits a right adjoint. In fact, by the explicit construction of $h_!(f)$ in the proof of Theorem 3.5.1.1, we may compute

$$\begin{aligned} \operatorname{map}_{\mathsf{E}}(h_!(f)(-), -) &\simeq \operatorname{map}_{\mathsf{E}}(Lf_! i_*(-), -) \\ &\simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})}(i_*(-), f^* h_{\mathsf{E}}(-)) \end{aligned}$$

and since E is locally small, the functor $f^* h_{\mathsf{E}}$ takes values in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, hence the claim follows. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and using Remark 3.4.3.2 and Remark 2.3.2.1, the same argument works for arbitrary objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$, hence we conclude that the functor $h_!$ takes values in $\underline{\mathsf{Fun}}^{\mathsf{L}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$ and therefore gives rise to an equivalence

$$\underline{\mathsf{Fun}}^{\mathsf{L}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathsf{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}).$$

This is a special (and in a certain sense universal) case of the adjoint functor theorem for presentable $\mathcal{B}$-categories. We will treat the general case in Section 5.4.3.

Our next goal is to generalise Theorem 3.5.1.1 to an arbitrary internal class U of $\mathcal{B}$-categories. For this, we need to make the following general observation:

Lemma 3.5.1.5. *Let* E *be a* $\mathcal{B}$*-category, let* $\mathsf{C} \hookrightarrow \mathsf{E}$ *be a full subcategory and let* $\mathsf{U} \subset \mathsf{V}$ *be two internal classes of* $\mathcal{B}$*-categories. Suppose that* E *is* V*-cocomplete. Then there exists a full subcategory* $\mathsf{D} \hookrightarrow \mathsf{E}$ *that is closed under* U*-colimits (i.e. that is* U*-cocomplete and the inclusion into* E *is* U*-cocontinuous), contains* C *and is the smallest full subcategory of* E *with these properties, in that whenever* $\mathsf{D}' \hookrightarrow \mathsf{E}$ *has the same properties there is an inclusion* $\mathsf{D} \hookrightarrow \mathsf{D}'$ *over* E.

Proof. By Proposition 1.4.1.7, the poset $\mathrm{Sub}_{\mathrm{full}}(\mathsf{E})$ of full subcategories in E is presentable and therefore admits all limits. To complete the proof, we therefore only need to show that the collection of full subcategories of E that contain C and that are closed under U-colimits in E is closed under limits in $\mathrm{Sub}_{\mathrm{full}}(\mathsf{E})$. Clearly, if $(\mathsf{D}_i)_{i\in I}$ is a collection of full subcategories in E that each contain C, then so does their meet $\mathsf{D} = \bigwedge_i \mathsf{D}_i$. Similarly, suppose that each $\mathcal{B}$-category D_i is closed under U-colimits in E, and let $A \in \mathcal{B}$ be an arbitrary context. Since π_A^* commutes with limits and carries fully faithful functors to fully faithful functors, we may assume without loss of generality that $A \simeq 1$. We thus only need to show that the meet of the D_i is closed under I-indexed colimits in E for any $\mathsf{I} \in \mathsf{U}(1)$. Let $d \colon B \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{D})$ be a diagram in context $B \in \mathcal{B}$. Since by assumption the object $\operatorname{colim} d$ is contained in D_i for every $i \in I$ and thus defines an object in D, the result follows. □

In light of Lemma 3.5.1.5, we may now define:

Definition 3.5.1.6. For any $\mathcal{B}$-category C and any internal class U of small $\mathcal{B}$-categories, we define the large $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ as the smallest full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that contains C and is closed under U-colimits. We will denote by $\mathrm{PSh}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ the underlying ∞-category of global sections.

Remark 3.5.1.7. Suppose that U is a *small* internal class of $\mathcal{B}$-categories and C is a $\mathcal{B}$-category. Then $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is small as well. To see this, let us first fix a small full subcategory of generators $\mathcal{G} \subset \mathcal{B}$ (i.e. a full subcategory such that every object in $\mathcal{B}$ admits a small cover by objects in $\mathcal{G}$). Since U is small, there exists a small regular cardinal κ such that for every $\mathcal{B}$-category I in U in context $G \in \mathcal{G}$ the object $\mathsf{I}_0 \in \mathcal{B}_{/G}$ is κ-compact. We construct a diagram $\mathsf{E}^{\bullet} \colon \kappa \to \mathrm{Sub}_{\mathrm{full}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ by transfinite recursion as follows: set $\mathsf{E}^0 = \mathsf{C}$ and $\mathsf{E}^{\lambda} = \bigvee_{\tau<\lambda} \mathsf{E}^{\tau}$ for any limit ordinal $\lambda < \kappa$, where the right-hand side denotes the join operation in the poset $\mathrm{Sub}_{\mathrm{full}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. For $\lambda < \kappa$, we furthermore define $\mathsf{E}^{\lambda+1}$ to be the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by E^{λ} together with those objects that arise as the colimit of a diagram of the form $d \colon \mathsf{I} \to \pi_G^* \mathsf{E}^{\lambda}$ for $G \in \mathcal{G}$ and $\mathsf{I} \in \mathsf{U}(G)$. Let us set $\mathsf{E} = \bigvee_{\tau<\kappa} \mathsf{E}^{\tau}$. Since κ is small and E^{τ} is a small large $\mathcal{B}$-category for every $\tau < \kappa$, the large $\mathcal{B}$-category E is small as well. We claim that E is U-cocomplete. In fact, it suffices to show that for every $G \in \mathcal{G}$ and every diagram $d \colon \mathsf{I} \to \pi_G^* \mathsf{E}$ the object $\operatorname{colim} d$ is contained in $\pi_G^* \mathsf{E}$ as well. Since I_0 is κ-compact in $\mathcal{B}_{/G}$ and since κ is κ-filtered as it is regular, the map $d_0 \colon \mathsf{I}_0 \to \mathsf{E}_0 = \bigvee_{\tau<\kappa} \mathsf{E}_0^\tau$ factors through E_0^τ for some $\tau < \kappa$. As a consequence, the colimit $\operatorname{colim} d$ is contained in $\mathsf{E}^{\tau+1}$ and therefore a fortiori in E, as claimed. Now since E is U-cocomplete and contains C, it must also contain $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$, which is therefore small.

In the situation of Definition 3.5.1.6, Proposition 3.4.1.1 implies that there are inclusions

$$\mathsf{C} \hookrightarrow \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).$$

In general, the middle inclusion is not an equivalence, as the following example shows.

Example 3.5.1.8. Let $\mathcal{B} = \mathrm{Ani}$ be ∞-topos of spaces, let $\mathsf{C} = (\Delta^1)^{\mathrm{op}}$ and let U be the smallest colimit class that contains Λ_0^2. An ∞-category is thus U-cocomplete precisely if it admits pushouts. An object in $\mathrm{Fun}(\Delta^1, \mathrm{Ani})$ is representable when viewed as a presheaf on $(\Delta^1)^{\mathrm{op}}$ precisely if it is one of the two maps $0 \to 1$ and $1 \to 1$. Hence $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Ani})$ that is spanned by the maps $n \to 1$ for natural numbers $n \leq 2$. But this ∞-category is not closed under pushouts in $\mathrm{Fun}(\Delta^1, \mathrm{Ani})$: for example, the map $S^1 \to 1$ is a pushout of objects in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ which is not contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ itself.

Our first goal is to verify that the construction from Definition 3.5.1.6 is stable under étale base change:

Proposition 3.5.1.9. *For any $\mathcal{B}$-category C, any internal class U of $\mathcal{B}$-categories and any object $A \in \mathcal{B}$, there is a natural equivalence*

$$\pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}^{\pi_A^* \mathsf{U}}(\pi_A^* \mathsf{C}).$$

The proof of Proposition 3.5.1.9 relies on the following two lemmas:

Lemma 3.5.1.10. *Let $A \in \mathcal{B}$ be an arbitrary object, let U be an internal class of $\mathcal{B}$-categories and let $f \colon \mathsf{C} \to \mathsf{D}$ be a $\pi_A^* \mathsf{U}$-cocontinuous functor of $\pi_A^* \mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-category. Then $(\pi_A)_*(f)$ is a U-cocontinuous functor of U-cocomplete $\mathcal{B}$-categories.*

Proof. Let $B \in \mathcal{B}$ be an arbitrary object. We need to show that for every $\mathsf{I} \in \mathsf{U}(B)$ the $\mathcal{B}_{/B}$-categories $\pi_B^*(\pi_A)_*\mathsf{C}$ and $\pi_B^*(\pi_A)_*\mathsf{D}$ admit I-indexed colimits and that $\pi_B^*(\pi_A)_*(f)$ preserves these. Note that if $\mathrm{pr}_0 \colon A \times B \to A$ and $\mathrm{pr}_1 \colon A \times B \to B$ are the two projections, the natural map $\pi_B^*(\pi_A)_* \to (\mathrm{pr}_1)_* \mathrm{pr}_0^*$ is an equivalence, owing to the transpose map $(\mathrm{pr}_0)_! \mathrm{pr}_1^* \to \pi_A^*(\pi_B)_!$ being one. Thus, we may identify $\pi_B^*(\pi_A)_*(f)$ with $(\mathrm{pr}_1)_* \mathrm{pr}_0^*(f)$. Now since f is a $\pi_A^*\mathsf{U}$-cocontinuous functor between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories, it follows that $\mathrm{pr}_0^*(f)$ is a $\pi_{A\times B}^*\mathsf{U}$-cocontinuous functor between $\pi_{A\times B}^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A\times B}$-categories (Remark 3.3.2.3). Therefore, by passing to $\mathcal{B}_{/B}$, we can assume that $B \simeq 1$. In other words, we only need to show that for every $\mathsf{I} \in \mathsf{U}(1)$ the two horizontal maps in the commutative square

$$\begin{tikzcd}
(\pi_A)_*\mathsf{C} \arrow[r, "\mathrm{diag}"] \arrow[d, "(\pi_A)_*(f)"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\pi_A)_*\mathsf{C}) \arrow[d, "(\pi_A)_*(f)_*"] \\
(\pi_A)_*\mathsf{C} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\pi_A)_*\mathsf{C})
\end{tikzcd}$$

have left adjoints and that the associated mate transformation is an equivalence. This is a consequence of the equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(-, (\pi_A)_*(-)) \simeq (\pi_A)_*\underline{\mathsf{Fun}}_{\mathcal{B}}(\pi_A^*(-), -),$$

which follows by adjunction from the evident equivalence

$$\pi_A^*(- \times -) \simeq \pi_A^*(-) \times_A \pi_A^*(-),$$

and the fact that by Corollary 3.1.1.9 the geometric morphism $(\pi_A)_*$ preserves adjunctions. □

Lemma 3.5.1.11. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let*

$$\begin{tikzcd}
\mathsf{Q} \arrow[r, "g"] \arrow[d, "q"] & \mathsf{P} \arrow[d, "p"] \\
\mathsf{D} \arrow[r, "f"] & \mathsf{C}
\end{tikzcd}$$

be a pullback square in $\mathrm{Cat}(\mathcal{B})$ *in which* D, C *and* P *are* U*-cocomplete and both f and p are* U*-cocontinuous. Then* Q *is* U*-cocomplete and both q and g are* U*-cocontinuous.*

Proof. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ for $A \in \mathcal{B}$ if necessary and using Remark 3.2.1.8 and Remark 3.2.2.4, it will suffice to prove that any diagram $d\colon \mathsf{K} \to \mathsf{Q}$ with $\mathsf{K} \in \mathsf{U}(1)$ admits a colimit in Q and that furthermore q preserves this colimit. The pullback square in the statement of the lemma induces a commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[rr] \arrow[dl, "\bar{d}"'] \arrow[dd] & & 1 \arrow[dl, "\overline{gd}"'] \arrow[dd] \\
\mathsf{Q}_{d/} \arrow[rr, "g_*"] \arrow[dd, "q_*"'] & & \mathsf{P}_{gd/} \arrow[dd, "p_*"] & \\
 & 1 \arrow[rr] \arrow[dl, "\overline{qd}"'] & & 1 \arrow[dl, "\overline{pgd}"] \\
\mathsf{D}_{qd/} \arrow[rr, "f_*"] & & \mathsf{C}_{pgd/} &
\end{tikzcd}$$

in which the front square is a pullback, the three cocones $\overline{qd}$, $\overline{gd}$ and $\overline{pgd}$ are colimit cocones and the cocone $\bar{d}$ is determined by the universal property of pullbacks. To finish the proof, it suffices to show that $\bar{d}$ is a colimit cocone, i.e. initial. Given any $\bar{d}' \colon 1 \to \mathsf{Q}_{d/}$, we obtain a pullback square

$$\begin{tikzcd}
\operatorname{map}_{\mathsf{Q}_{d/}}(\bar{d}, \bar{d}') \arrow[r] \arrow[d] & \operatorname{map}_{\mathsf{P}_{gd/}}(\overline{gd}, g_*\bar{d}') \arrow[d] \\
\operatorname{map}_{\mathsf{D}_{fd/}}(\overline{qd}, q_*\bar{d}') \arrow[r] & \operatorname{map}_{\mathsf{C}_{pgd/}}(\overline{pgd}, p_*g_*\bar{d}')
\end{tikzcd}$$

in $\mathcal{B}$. Since $\overline{qd}$, $\overline{gd}$ and $\overline{pgd}$ are initial, it follows that the cospan in the lower right corner is constant on the final object $1 \in \mathcal{B}$, hence $\operatorname{map}_{\mathsf{Q}_{d/}}(\bar{d}, \bar{d}') \simeq 1$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and $\bar{d}$ with $\pi_A^*(\bar{d})$, the same is true for any object $\bar{d}' \colon A \to \mathsf{Q}_{d/}$. As a consequence, $\bar{d}$ must be initial. □

Proof of Proposition 3.5.1.9. It follows from Remark 2.3.2.1 that there is a commutative diagram

$$\begin{tikzcd}
\pi_A^*\mathsf{C} \arrow[d, hook, "\pi_A^* h"] \arrow[ddr, hook, bend left, "h_{\pi_A^*\mathsf{C}}"] & \\
\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \arrow[d, hook] \arrow[dr, hook] & \\
\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "\simeq"] & \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}),
\end{tikzcd}$$

and it is clear that $\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is closed under $\pi_A^*\mathsf{U}$-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$. It therefore suffices to show that if $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$ is a full subcategory that contains $\pi_A^*\mathsf{C}$ and that is likewise closed under $\pi_A^*\mathsf{U}$-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$, this subcategory must contain $\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. Consider the commutative diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, hook] \arrow[d, "\eta_A"] & \mathsf{D}' \arrow[r, hook] \arrow[d] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "\eta_A"] \\
(\pi_A)_*\pi_A^*\mathsf{C} \arrow[r, hook] & (\pi_A)_*\mathsf{D} \arrow[r, hook] & (\pi_A)_*\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

in which η_A denotes the adjunction unit of $\pi_A^* \dashv (\pi_A)_*$ and in which D' is defined by the condition that the right square is a pullback. Note that the triangle identities for the adjunction $\pi_A^* \dashv (\pi_A)_*$ imply that D contains $\pi_A^*\mathsf{D}'$. The proof is therefore finished once we show that D' is closed under U-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. To prove this claim, note that we may identify $(\pi_A)_*\pi_A^* \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(A, -)$. With respect to this identification, the unit η_A corresponds to precomposition with the unique map $\pi_A \colon A \to 1$. Thus, Proposition 3.3.2.12 implies that η_A is a U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories. Also, Lemma 3.5.1.10 implies that the inclusion $(\pi_A)_*\mathsf{D} \hookrightarrow (\pi_A)_*\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is closed under U-colimits. Therefore, the result follows from Lemma 3.5.1.11. □

Theorem 3.5.1.12. *Let* C *be a* $\mathcal{B}$*-category, let* U *be an internal class of* $\mathcal{B}$*-categories and let* E *be a* U*-cocomplete large* $\mathcal{B}$*-category. Then the functor of left Kan extension along* $h_{\mathsf{C}} \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ *exists and determines an equivalence*

$$(h_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{E}).$$

In other words, the $\mathcal{B}$*-category* $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ *is the free* U*-cocompletion of* C.

Proof. Let us define $\mathsf{E}' = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}, \mathsf{Grpd}_{\widehat{\mathcal{B}}})^{\mathrm{op}}$. Then Proposition 3.3.2.15 implies that the inclusion $h_{\mathsf{E}}^{\mathrm{op}} \colon \mathsf{E} \hookrightarrow \mathsf{E}'$ is U-cocontinuous. Let $j \colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the inclusion. By Theorem 3.4.3.5, the functors of left Kan extension along h_{C} and j exist and define inclusions

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}') \xhookrightarrow{(h_{\mathsf{C}})_!} \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{E}') \xhookrightarrow{j_!} \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}'),$$

and by Theorem 3.5.1.1 the essential image of the composition is spanned by those objects in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}')$ which define cocontinuous functors. Since j is by construction U-cocontinuous, the restriction functor

$$j^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}') \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{E}')$$

restricts to a functor

$$j^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}') \to \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{E}').$$

Consequently, we deduce that the left Kan extension functor

$$(h_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}') \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{E}')$$

factors through an inclusion

$$(h_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}') \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{E}').$$

We claim that this functor is essentially surjective and therefore an equivalence. On account of Remark 3.3.3.4 and Remark 3.4.3.2 as well as Proposition 3.5.1.9, it suffices to show (by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$) that any U-cocontinuous functor $f \colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \to \mathsf{E}'$ is a left Kan extension along its restriction to C. Let $\epsilon \colon (h_{\mathsf{C}})_! h_{\mathsf{C}}^* f \to f$ be the adjunction counit, and let D be the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ that is spanned by those objects F in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ (in arbitrary context) for which ϵF is an equivalence. We need to show that $\mathsf{D} = \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. By construction, we have $\mathsf{C} \hookrightarrow \mathsf{D}$, so that it suffices to show that D is closed under U-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. Note that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is precisely the pullback of $s_0 \colon \mathsf{E}' \hookrightarrow (\mathsf{E}')^{\Delta^1}$ along $\epsilon \colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \to (\mathsf{E}')^{\Delta^1}$. Since Proposition 3.3.2.12 implies that s_0 is cocontinuous and since ϵ is U-cocontinuous (this is a special case of Lemma 5.5.1.3 below), we deduce from Lemma 3.5.1.11 that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is indeed closed under U-colimits.

To finish the proof, we still need to show that the equivalence

$$c(h_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}') \simeq \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}')$$

restricts to the desired equivalence

$$(h_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}).$$

As clearly h_{C}^* restricts in the desired way, it suffices to show that $(h_{\mathsf{C}})_!$ restricts as well. By the same reduction steps as above, this follows once we show that for every functor $f \colon \mathsf{C} \to \mathsf{E}$, the left Kan extension $(h_{\mathsf{C}})_! f \colon \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{E}'$ factors through E. Consider the commutative diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, hook, dotted] \arrow[dr, hook, "h"'] & \mathsf{D} \arrow[r] \arrow[d, hook] & \mathsf{E} \arrow[d, hook] \\
& \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "(h_{\mathsf{C}})_! f"] & \mathsf{E}'
\end{tikzcd}$$

in which the square is a pullback. Since both $\hat{f}$ and $\mathsf{E} \hookrightarrow \mathsf{E}'$ are U-cocontinuous, it follows from Lemma 3.5.1.11 that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is closed under U-colimits and must therefore be an equivalence. As a consequence, the functor $(h_{\mathsf{C}})_! f$ factors through E, as needed. □

Corollary 3.5.1.13. *Let* C *be a* $\mathcal{B}$*-category and let* $\mathsf{U} \subset \mathsf{V}$ *be internal classes such that* $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *is* V*-cocomplete. Then the inclusion* $i \colon \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{V}}_{\mathcal{B}}(\mathsf{C})$ *admits a left adjoint. In particular, if* C *itself is* V*-cocomplete, the inclusion* $h_{\mathsf{C}} \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{V}}_{\mathcal{B}}(\mathsf{C})$ *admits a left adjoint.*

Proof. By choosing $\mathsf{U} = \varnothing$ (i.e. the initial object in $\mathrm{Cat}(\mathcal{B})$), the second claim is an immediate consequence of the first. To prove the first statement, let $j \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ be the inclusion. Then Theorem 3.5.1.12 allows us to construct a candidate for the left adjoint $L \colon \underline{\mathsf{PSh}}^{\mathsf{V}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ of i as the left Kan extension of j along ij. By construction, L is V-cocontinuous. As i is U-cocontinuous and since we have equivalences $j^*(Li) \simeq (ij)^*(L) \simeq j$, Theorem 3.5.1.12 moreover gives rise to an equivalence $Li \simeq \mathrm{id}_{\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}$. Similarly, since $j^*(i) \simeq ij$, one obtains an equivalence $i \simeq j_!(ij)$. Therefore, transposing the identity on ij across the adjunction $(ij)_! \dashv (ij)^*$ gives rise to a map $\eta\colon \mathrm{id}_{\underline{\mathrm{PSh}}^{\mathsf{V}}_{\mathcal{B}}(\mathsf{C})} \to iL$ such that ηi is an equivalence, being a map between U-cocontinuous functors that restricts to an equivalence along j. By making use of Corollary 3.1.4.3, we conclude that L is a left adjoint once we verify that $L\eta$ is an equivalence as well. As both domain and codomain of this map are V-cocontinuous functors, this is the case already if its restriction along ij is an equivalence, which follows from the construction of η. ☐

Corollary 3.5.1.14. *Let* U *be a* small *internal class of* $\mathcal{B}$*-categories. Then the inclusion* $\mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \hookrightarrow \mathrm{Cat}_{\mathcal{B}}$ *admits a left adjoint that carries a* $\mathcal{B}$*-category* C *to its free* U*-cocompletion. Moreover, the adjunction unit is given by the Yoneda embedding* $\mathsf{C} \hookrightarrow \underline{\mathrm{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$*.*

Proof. By Remark 3.5.1.7, the free U-cocompletion $\underline{\mathrm{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is indeed a small $\mathcal{B}$-category. Therefore, the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathrm{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is a well-defined map in $\mathrm{Cat}_{\mathcal{B}}$. By Corollary 3.1.3.5, it suffices to show that the composition

$$\phi\colon \operatorname{map}_{\mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}}(\underline{\mathrm{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}), -) \hookrightarrow \operatorname{map}_{\mathrm{Cat}_{\mathcal{B}}}(\underline{\mathrm{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}), -) \to \operatorname{map}_{\mathrm{Cat}_{\mathcal{B}}}(\mathsf{C}, -)$$

is an equivalence of functors $\mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \to \mathrm{Grpd}_{\mathcal{B}}$. Using that equivalences of functors are detected object-wise (Proposition 2.3.2.12), this follows once we show that the evaluation of this map at any object $A \to \mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ yields an equivalence of $\mathcal{B}_{/A}$-groupoids. By combining Remark 3.3.3.2 and Remark 2.3.2.1 with Proposition 3.5.1.9, we may pass to $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. In this case, the result follows from Theorem 3.5.1.12 in light of the observation that by Remark 3.3.3.4, the evaluation of ϕ at a U-cocomplete $\mathcal{B}$-category E is precisely the restriction of the equivalence from Theorem 3.5.1.12 to core $\mathcal{B}$-groupoids. ☐

3.5.2. Detecting cocompletions

In this section we give a characterisation when a functor $f\colon \mathsf{C} \to \mathsf{D}$ exhibits D as the free U-cocompletion of C. To achieve this, we need the notion of U-*cocontinuous objects*, which is in a certain way an internal analogue of the notion of a κ-compact object in an ∞-category:

Definition 3.5.2.1. Let D be a U-cocomplete $\mathcal{B}$-category. We define the full subcategory $\mathsf{D}^{\mathsf{U}\text{-cc}} \hookrightarrow \mathsf{D}$ of U-*cocontinuous objects* as the pullback

$$\begin{tikzcd} \mathsf{D}^{\mathsf{U}\text{-cc}} \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}} \arrow[d, hook] \\ \mathsf{D} \arrow[r, hook, "h_{\mathsf{D}^{\mathrm{op}}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}}. \end{tikzcd}$$

Remark 3.5.2.2. In the situation of Definition 3.5.2.1, we may combine Remark 3.3.3.4 and Remark 2.3.2.1 to deduce that there is a canonical equivalence $\pi_A^*(\mathsf{D}^{\mathsf{U}\text{-cc}}) \simeq (\pi_A^*\mathsf{D})^{\pi_A^*\mathsf{U}\text{-cc}}$ of full subcategories of $\pi_A^*\mathsf{D}$, for every $A \in \mathcal{B}$. Consequently, an object $d\colon A \to \mathsf{D}$ is contained in $\mathsf{D}^{\mathsf{U}\text{-cc}}$ if and only if its transpose $\bar{d}\colon 1 \to \pi_A^*\mathsf{D}$ is $\pi_A^*\mathsf{U}$-cocontinuous.

We now come to the main result of this section. If $\mathsf{C} \hookrightarrow \mathsf{D}$ is a full subcategory of a U-cocomplete $\mathcal{B}$-category D, we will say that D is *generated* by C under U-colimits if D is the smallest full subcategory of itself that is closed under U-colimits and that contains C. We now obtain the following recognition principle of free cocompletions, whose proof we adopted from its ∞-categorical analogue [49, Proposition 5.1.6.10]:

Proposition 3.5.2.3. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories such that D is U-cocomplete, and let $\hat{f}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \to \mathsf{D}$ be its unique U-cocontinuous extension. Then the following are equivalent:*

1. *$\hat{f}$ is an equivalence;*
2. *f is fully faithful, takes values in $\mathsf{D}^{\mathsf{U}\text{-cc}}$, and generates D under U-colimits.*

Proof. We first note that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})^{\mathsf{U}\text{-cc}}$ contains C. Indeed, Yoneda's lemma implies that the composition

$$\mathsf{C} \xhookrightarrow{h_{\mathsf{C}}} \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \xhookrightarrow{h_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})^{\mathrm{op}}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}}$$

can be identified with the opposite of the transpose of the evaluation map. Together with Proposition 3.5.1.9 and Remark 1.2.5.5 and Remark 2.3.2.1, this implies that the image of $c\colon A \to \mathsf{C}$ along this composition transposes to the functor

$$\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}}(\pi_A^*\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}) \xrightarrow{\mathrm{ev}_c} \mathsf{Grpd}_{\mathcal{B}_{/A}}$$

which is $\pi_A^*\mathsf{U}$-cocontinuous by Proposition 3.3.2.12. Therefore, (1) implies (2).

Conversely, suppose that condition (2) is satisfied. We first prove that $\hat{f}$ is fully faithful. To that end, if $c\colon A \to \mathsf{C}$ is an arbitrary object, we claim that the morphism

$$\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})}(c, -) \to \mathrm{map}_{\mathsf{D}}(\hat{f}(c), \hat{f}(-))$$

is an equivalence. By combining Remark 3.4.3.2 and Remark 2.3.2.1 with Proposition 3.5.1.9, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$. In this case, the fact that C is contained in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})^{\mathsf{U}\text{-cc}}$ and condition (2) imply that both domain and codomain of the morphism are U-cocontinuous functors. Using Lemma 5.5.1.3 below, this implies that the above map is itself U-cocontinuous when viewed as a functor $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \to \mathsf{Grpd}_{\mathcal{B}}^{\Delta^1}$. Since the restriction of this map to C is an equivalence , the universal property of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ thus implies that the entire morphism must be an equivalence. Thus, by what we just have shown, if $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is an arbitrary object, the natural transformation

$$\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})}(-, F) \to \mathrm{map}_{\mathsf{D}}(\hat{f}(-), \hat{f}(F))$$

restricts to an equivalence on C. As this map transposes to a morphism of $\pi_A^*\mathsf{U}$-cocontinuous functors (using Proposition 3.3.2.15 and the fact that $\hat{f}$ is U-cocontinuous), the same argument as above shows that the entire natural transformation is in fact an equivalence and therefore that $\hat{f}$ is fully faithful, as desired. As therefore $\hat{f}$ exhibits $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ as a full subcategory of D that is closed under U-colimits and that contains C, the assumption that D is generated by C under U-colimits implies that $\hat{f}$ is an equivalence. □

3.5.3. Cocompletion of the point

Let U be an internal class of $\mathcal{B}$-categories. Our goal in this section is to study the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$. To that end, let us denote by $\mathrm{gpd}(\mathsf{U}) \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ the essential image of U along the groupoidification functor $(-)^{\mathrm{gpd}}\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ from Proposition 3.1.2.14.

Definition 3.5.3.1. We call an internal class U *closed under groupoidification*, if for any $A \in \mathcal{B}$ and $\mathsf{I} \in \mathsf{U}(A)$ the groupoidification $\mathsf{I}^{\mathrm{gpd}}$ is also contained in U.

For any internal class U we can form its *closure under groupoidification*, denoted $\overline{\mathsf{U}}^{\mathrm{gpd}}$, that is defined as the internal class spanned by U and $\mathrm{gpd}(\mathsf{U})$.

Remark 3.5.3.2. Since for any $\mathcal{B}$-category I, the morphism $\mathsf{I} \to \mathsf{I}^{\mathrm{gpd}}$ is final, it follows that any colimit class (in the sense of Definition 3.3.3.5) is closed under groupoidification. Furthermore, for any internal class U, we have inclusions $\mathsf{U} \subseteq \overline{\mathsf{U}}^{\mathrm{gpd}} \subseteq \mathsf{U}^{\mathrm{colim}}$. In particular the discussion after Definition 3.3.3.5 shows that a $\mathcal{B}$-category is U-cocomplete if and only if it is $\overline{\mathsf{U}}^{\mathrm{gpd}}$-cocomplete. The same statement holds for U-cocontinuity.

Remark 3.5.3.3. If U is closed under groupodification, the adjunction $(-)^{\mathrm{gpd}} \dashv \iota$ restricts to an adjunction

$$((-)^{\mathrm{gpd}} \dashv i) : \mathsf{U} \rightleftarrows \mathrm{gpd}(\mathsf{U}).$$

Proposition 3.5.3.4. *For any internal class* U *of* $\mathcal{B}$*-categories there is a canonical inclusion* $\mathrm{gpd}(\mathsf{U}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ *which is an equivalence whenever* $\overline{\mathsf{U}}^{\mathrm{gpd}}$ *is closed under* U*-colimits in* $\mathsf{Cat}_{\mathcal{B}}$*.*

Proof. By construction, the canonical map $\mathsf{U} \hookrightarrow \overline{\mathsf{U}}^{\mathrm{gpd}}$ induces an equivalence of subuniverses $\mathrm{gpd}(\mathsf{U}) \simeq \mathrm{gpd}(\overline{\mathsf{U}}^{\mathrm{gpd}})$. Therefore we may assume that U is closed under groupoidification. For any $\mathsf{I} \in \mathsf{U}(1)$, its groupoidification $\mathsf{I}^{\mathrm{gpd}}$ is the colimit of the functor $\mathsf{I} \to 1 \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ (see Proposition 3.2.5.1) and therefore by definition contained in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$. Note that by using Remark 1.4.1.2 and Remark 1.4.2.4 as well as Corollary 3.1.1.9, for every $A \in \mathcal{B}$ the functor π_A^* carries the adjunction $(-)^{\mathrm{gpd}} \dashv \iota \colon \mathsf{Cat}_{\mathcal{B}} \rightleftarrows \mathsf{Grpd}_{\mathcal{B}}$ to the adjunction $(-)^{\mathrm{gpd}} \dashv \iota \colon \mathsf{Cat}_{\mathcal{B}_{/A}} \rightleftarrows \mathsf{Grpd}_{\mathcal{B}_{/A}}$. Together with Proposition 3.5.1.9, this observation and the above argument also yields that for every $\mathsf{I} \in \mathsf{U}(A)$ the groupoidification $\mathsf{I}^{\mathrm{gpd}}$ defines an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$. Thus, the groupoidification functor $(-)^{\mathrm{gpd}} \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ restricts to a functor $\mathsf{U} \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ and therefore gives rise to the desired inclusion $\mathrm{gpd}(\mathsf{U}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$. Now by definition of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$, this inclusion is an equivalence if and only if $\mathrm{gpd}(\mathsf{U})$ is closed under U-colimits in $\mathsf{Grpd}_{\mathcal{B}}$. But if the subcategory $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is closed under U-colimits in $\mathsf{Cat}_{\mathcal{B}}$ it follows by Remark 3.5.3.3 that we have $\mathrm{gpd}(\mathsf{U}) = \mathsf{U} \cap \mathsf{Grpd}_{\mathcal{B}}$, hence the claim follows from Lemma 3.5.1.11. $\square$

Example 3.5.3.5. Let S be a local class of maps in $\mathcal{B}$ and let $\mathrm{Grpd}_S \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ be the associated full subcategory of $\mathrm{Grpd}_{\mathcal{B}}$. Then Grpd_S is clearly closed under groupoidification. Recall that Grpd_S is closed under Grpd_S-colimits in $\mathrm{Grpd}_{\mathcal{B}}$ precisely if the local class S is stable under composition (see Example 3.3.2.6). Therefore, whenever S is stable under composition, Proposition 3.5.3.4 provides an equivalence of subuniverses $\mathrm{Grpd}_S \simeq \underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathrm{Grpd}_S}(1)$.

If S is not closed under composition, the free cocompletion $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathrm{Grpd}_S}(1)$ still admits an explicit description. Namely, an object $c \colon A \to \mathrm{Grpd}_{\mathcal{B}}$ in context $A \in \mathcal{B}$ defines an object of $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathrm{Grpd}_S}(1)$ if and only if it is locally a composition of two morphisms in S. To be more precise, c is in $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathrm{Grpd}_S}(1)$ if and only if there is a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that every $s_i^* c \in \mathrm{Grpd}_{\mathcal{B}}(A_i) = \mathcal{B}_{/A_i}$ can be written as a composition $g_i f_i$ of two morphisms $g_i : P_i \to Q_i$ and $f_i : Q_i \to A_i$ that are in S. This description holds since the full subcategory spanned by these objects is clearly closed under Grpd_S-indexed colimits and it is easy to see that it is the smallest full subcategory of $\mathrm{Grpd}_{\mathcal{B}}$ with this property.

Example 3.5.3.6. The following observation is due to Bastiaan Cnossen: Suppose that we have $\mathcal{B} = \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ with pullbacks, and let S be some class of morphisms in $\mathcal{C}$ that is closed under pullbacks in $\mathcal{C}$. It generates a local class in $\mathcal{B} = \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C})$ that we denote by W. As in Remark 3.3.2.9, we obtain an internal class $\mathsf{U}_S = \langle W, \mathrm{Cat}_\infty \rangle$, so that we may now consider the free U_S-cocompletion $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathsf{U}_S}(1)$ of the point. It may be explicitly described as the presheaf on $\mathcal{C}$ given by

$$\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathsf{U}_S}(1) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad c \mapsto \mathrm{PSh}_{\mathrm{Ani}}(S_{/c})$$

where $S_{/c}$ denotes the full subcategory of $\mathcal{C}_{/c}$ spanned by the morphisms in S. In particular, it agrees with the $\mathrm{PSh}(\mathcal{C})$-category underlying the *initial cocomplete pullback formalism* described in [21, § 4]. One can use this observation to give an alternative proof of [21, corollary 4.9]. In fact, one can prove something more general since the proof in [21] relies on $\mathcal{C}$ being a 1-category, which is not necessary in our framework.

We conclude this section by showing that any U-cocomplete large $\mathcal{B}$-category E is *tensored* over $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ in the following sense:

Definition 3.5.3.7. A large $\mathcal{B}$-category E is *tensored* over $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)$ if there is a functor $- \otimes - \colon \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1) \times \mathsf{E} \to \mathsf{E}$ together with an equivalence

$$\mathrm{map}_{\mathsf{E}}(- \otimes -, -) \simeq \mathrm{map}_{\mathsf{Grpd}_{\widehat{\mathcal{B}}}}(-, \mathrm{map}_{\mathsf{E}}(-,-)).$$

Proposition 3.5.3.8. *If* E *is a* U*-cocomplete large* $\mathcal{B}$*-category, then* E *is tensored over* $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)$.

Proof. Since E is U-cocomplete, Proposition 3.2.3.1 implies that the functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{E})$ is U-cocomplete as well. As a consequence, we may apply Theorem 3.5.1.12 to extend the identity $\mathrm{id}_{\mathsf{E}} \colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{E})$ in a unique way to a U-cocontinuous functor $f \colon \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{E})$. We define the desired bifunctor $- \otimes -$ as the transpose of f. To see that it has the desired property, note that $\mathrm{map}_{\mathsf{E}}(- \otimes -, -)$ is the transpose of the composition

$$\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}^{\mathrm{op}},\mathsf{E}^{\mathrm{op}}) \xhookrightarrow{(h_{\mathsf{E}^{\mathrm{op}}})_*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}^{\mathrm{op}} \times \mathsf{E}, \mathsf{Grpd}_{\widehat{\mathcal{B}}}),$$

whereas the functor $\mathrm{map}_{\mathsf{Grpd}_{\widehat{\mathcal{B}}}}(-, \mathrm{map}_{\mathsf{E}}(-,-))$ transposes to the functor

$$\begin{aligned}\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)^{\mathrm{op}} \overset{i}{\hookrightarrow} \mathsf{Grpd}^{\mathrm{op}}_{\widehat{\mathcal{B}}} &\\ &\xhookrightarrow{h_{\mathsf{Grpd}^{\mathrm{op}}_{\widehat{\mathcal{B}}}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\widehat{\mathcal{B}}}, \mathsf{Grpd}_{\widehat{\mathcal{B}}}) \\ &\xrightarrow{\mathrm{map}^*_{\mathsf{E}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}^{\mathrm{op}} \times \mathsf{E}, \mathsf{Grpd}_{\widehat{\mathcal{B}}}).\end{aligned}$$

As the opposite of either of these functors is U-cocontinuous, Theorem 3.5.1.12 implies that they are both uniquely determined by their value at the point $1 \colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$. Since $\mathrm{map}_{\mathsf{Grpd}_{\widehat{\mathcal{B}}}}(1,-)$ is equivalent to the identity functor, we find that both of these functors send $1 \colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$ to $\mathrm{map}_{\mathsf{E}}$ and that they are therefore equivalent, as required. □

Remark 3.5.3.9. By dualising Proposition 3.5.3.8, one obtains that a U-complete large $\mathcal{B}$-category E is *powered* over $\underline{\mathsf{PSh}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(1)$: since $\underline{\mathsf{PSh}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(1)^{\mathrm{op}}$ is the free U-completion of the final $\mathcal{B}$-category $1 \in \mathsf{Cat}(\mathcal{B})$, there is a functor

$$(-)^{(-)} \colon \underline{\mathsf{PSh}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(1)^{\mathrm{op}} \times \mathsf{E} \to \mathsf{E}$$

that fits into an equivalence

$$\mathrm{map}_{\mathsf{E}}(-, (-)^{(-)}) \simeq \mathrm{map}_{\mathsf{Grpd}_{\widehat{\mathcal{B}}}}(-, \mathrm{map}_{\mathsf{E}}(-,-)).$$

3.5.4. Application: decomposition of colimits

In [49, § 4.2], Lurie provides techniques for computing colimits in an ∞-category by means of decomposing diagrams into more manageable pieces. For example, he proves that an ∞-category has small colimits if and only if it has small coproducts and pushouts. In this section, we aim for similar results in the context of $\mathcal{B}$-category theory. More precisely, our main goal is to show:

Proposition 3.5.4.1. *Let* U *be an internal class, let* $d\colon \mathsf{I} \to \mathsf{U}$ *be a diagram such that* $\mathsf{I} \in \mathsf{U}(1)$*, and let* $\mathsf{K} = \operatorname{colim} d$*. Then every* U*-cocomplete* $\mathcal{B}$*-category admits* K*-indexed colimits, and every* U*-cocontinuous functor between* U*-cocomplete* $\mathcal{B}$*-categories preserves* K*-indexed colimits.*

The proof of Proposition 3.5.4.1 requires some preparations and will be given at the end of this section. For now, we will focus on some consequences of this result. Our most important corollary is the following:

Corollary 3.5.4.2. *A large* $\mathcal{B}$*-category* C *is cocomplete if and only if it is both* $\mathsf{Grpd}_{\mathcal{B}}$*- and* LConst*-cocomplete, and a functor of cocomplete large* $\mathcal{B}$*-categories is cocontinuous if and only if it is both* $\mathsf{Grpd}_{\mathcal{B}}$*- and* LConst*-cocontinuous.*

Proof. The condition is clearly necessary, so it suffices to show that it is sufficient. Therefore, let C be a $\mathsf{Grpd}_{\mathcal{B}}$- and LConst-cocomplete $\mathcal{B}$-category. In order to show that C is cocomplete, the fact that for every $A \in \mathcal{B}$ we can identify $\pi_A^* \mathsf{LConst}$ with the internal class of locally constant $\mathcal{B}_{/A}$-category as well as Remark 1.4.1.2 imply that it will be sufficient to prove that C admits I-indexed colimits for every $\mathcal{B}$-category I. To that end, let $\mathsf{U} = \mathsf{Grpd}_{\mathcal{B}} \cup \mathsf{LConst} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ be the internal class generated by $\mathsf{Grpd}_{\mathcal{B}}$ and LConst, i.e. the essential image of the induced functor $\mathsf{Grpd}_{\mathcal{B}} \sqcup \mathsf{LConst} \to \mathsf{Cat}_{\mathcal{B}}$. By Remark 3.3.2.4, we find that C is U-cocomplete. Now by Remark 1.2.1.3, we can find a diagram $d\colon \mathcal{J} \to \mathrm{Cat}(\mathcal{B})$ such that $\operatorname{colim} d \simeq \mathsf{I}$ and such that $d(i) \simeq \Delta^n \otimes \mathsf{G}$ for some $n \geq 0$ and some $\mathsf{G} \in \mathcal{B}$. Since $d^{(}\{n\}) \colon \mathsf{G} \to \Delta^n \otimes \mathsf{G}$ is a final functor, the diagram d thus takes values in $\mathsf{U}^{\mathrm{colim}}(1) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ (where $\mathsf{U}^{\mathrm{colim}}$ is the *colimit class* generated by U in the sense of Definition 3.3.3.5. By the discussion following Definition 3.3.3.5, the $\mathcal{B}$-category is $\mathsf{U}^{\mathrm{colim}}$-cocomplete. Hence, the observation that (the constant $\mathcal{B}$-category associated with) $\mathcal{J}$ defines an object of $\mathsf{U}^{\mathrm{colim}}(1)$ itself and Proposition 3.5.4.1

now imply that C admits I-indexed colimits, as desired. The claim that every $\mathrm{Grpd}_{\mathcal{B}}$- and LConst-cocontinuous functor is already cocontinuous is shown analogously. $\square$

Remark 3.5.4.3. By the results in [49, § 4.2], an ∞-category admits small colimits if and only if it admits small coproducts and pushouts, and a functor preserves small colimits if and only if it preserves small coproducts and pushouts. Since coproducts are a particular case of ∞-groupoidal colimits, Corollary 3.5.4.2 even implies that a $\mathcal{B}$-category C is cocomplete if and only if it is $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete and admits pushouts. Likewise, a functor of cocomplete $\mathcal{B}$-categories is cocontinuous precisely if it is $\mathrm{Grpd}_{\mathcal{B}}$-cocontinuous and preserves pushouts.

By combining Corollary 3.5.4.2 with Proposition 3.3.2.5 and 3.3.2.7, we now arrive at the following explicit description of cocompleteness and cocontinuity:

Corollary 3.5.4.4. *A $\mathcal{B}$-category C is cocomplete if and only if the following conditions are satisfied:*

1. *For every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ is cocomplete and for any $s\colon B \to A$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves colimits.*
2. *For every map $p\colon P \to A$ in $\mathcal{B}$ the functor p^* has a left adjoint $p_!$ such that for every pullback square*
$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$
the natural map $q_! t^* \to s^* p_!$ is an equivalence.

Furthermore a functor $f\colon \mathsf{C} \to \mathsf{D}$ of cocomplete $\mathcal{B}$-categories is cocontinuous if and only if for every $A \in \mathcal{B}$ the functor $f(A)$ preserves colimits, and for every map $p\colon P \to A$ in $\mathcal{B}$ the natural map $p_! f(P) \to f(A) p_!$ is an equivalence. $\square$

Remark 3.5.4.5. Let $\mathcal{C}$ be a small ∞-category such that $\mathcal{B}$ is a left exact and accessible localisation of $\mathrm{PSh}(\mathcal{C})$, and let $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ be the localisation functor. Then in order to see that a $\mathcal{B}$-category C is cocomplete, it suffices to check the conditions of Corollary 3.5.4.4 for objects in $\mathcal{C}$: Indeed, as the existence of colimits is a local condition (Remark 3.2.1.7), one may assume without loss of generality that the object A appearing in condition (1) and (2) of Proposition 3.3.2.5 is of the form $L(a)$ for some $a \in \mathcal{C}$. By furthermore using Remark 3.2.1.14, one can also assume that $B = L(b)$ and $s = L(s')$ for some $d \in \mathcal{C}$ and some map $s' : b \to a$ in $\mathcal{C}$. Finally, provided that C is LConst-cocomplete, Corollary 3.5.4.2 allows us to further assume that $P = L(p)$ and $u = L(u')$ for some $p \in \mathcal{C}$ and some map $u' : p \to a$ in $\mathcal{C}$. Together with Corollary 3.5.4.2, these observations imply that C is cocomplete if and only if

1. for every $a \in \mathcal{C}$ the ∞-category $\mathsf{C}(L(a))$ has small colimits, and for every $t : b \to a$ in $\mathcal{C}$ the functor $L(t)^* : \mathsf{C}(L(a)) \to \mathsf{C}(L(b))$ preserves small colimits;

2. for every pullback square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "v"] & p \arrow[d, "u"] \\ b \arrow[r, "s"] & a \end{tikzcd}$$

in $\mathrm{PSh}(\mathcal{C})$ in which both $s : b \to a$ and $u : p \to a$ are maps in $\mathcal{C}$, the two functors $L(u)^* : \mathsf{C}(L(a)) \to \mathsf{C}(L(p))$ and $L(v)^* : \mathsf{C}(L(d)) \to \mathsf{C}(L(Q))$ admits left adjoints $L(u)_!$ and $L(v)_!$ such that the natural map $L(v)_! L(t)^* \to L(s)^* L(u)_!$ is an equivalence.

Example 3.5.4.6. Let $\mathcal{C}$ be a presentable ∞-category. Then Corollary 3.5.4.4 and its dual show that the $\mathcal{B}$-category of Construction 1.4.2.1 is both complete and cocomplete. In fact, $\mathcal{C} \otimes \mathsf{Grpd}_{\mathcal{B}}$ will give rise to an example of a *presentable $\mathcal{B}$-category* in the sense of Section 5.4.

Example 3.5.4.7. Let $\mathcal{X}$ be an ∞-topos and let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism. We may consider the limit-preserving functor

$$\mathcal{X}_{/f^*(-)} : \mathcal{B}^{\mathrm{op}} \xrightarrow{(f^*)^{\mathrm{op}}} \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathcal{X}_{/-}} \widehat{\mathrm{Cat}}_\infty$$

which defines a large $\mathcal{B}$-category X. Clearly X is LConst-cocomplete. Furthermore, for every pullback square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

in $\mathcal{B}$, the lax square

$$\begin{tikzcd} \mathcal{X}_{/f^*(Q)} \arrow[d, "f^*(q)_!"] \arrow[dr, phantom, "\Searrow"] & \mathcal{X}_{/f^*(P)} \arrow[l, "f^*(t)^*"'] \arrow[d, "f^*(p)_!"] \\ \mathcal{X}_{/f^*(B)} & \mathcal{X}_{/f^*(A)} \arrow[l, "f^*(s)^*"] \end{tikzcd}$$

commutes since f^* preserves pullbacks. Thus it follows from Corollary 3.5.4.4 that X is cocomplete. Dually, one shows that X is also complete. In fact, X will be an example of a $\mathcal{B}$-*topos* in the sense of Chapter 6

Example 3.5.4.8. We can finally explain why the notion of being cocomplete is strictly stronger than simply admitting small colimits. For a concrete counterexample, consider be the category of (topological) manifolds Man. There is a functor

$$\underline{\mathrm{Sh}}_{\mathcal{B}} = \mathrm{Sh}(-)\colon \mathrm{Man} \to \mathrm{Pr}^{\mathrm{L}}$$

that takes a manifold M to the ∞-category of sheaves of spaces on M. This defines a limit-preserving functor

$$\underline{\mathrm{Sh}}_{\mathcal{B}}\colon \mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$$

via Kan extension and thus a $\mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})$-category that is in particular LConst-cocomplete. Furthermore, $\underline{\mathrm{Sh}}_{\mathcal{B}}$ has colimits indexed by arbitrary $\mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})$-groupoids: as we show in Corollary 3.5.4.2, it suffices to see this for representable $\mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})$-groupoids. By Corollary 3.1.2.11, we have to check that for any two manifolds M and N the functor

$$\pi_M^*\colon \mathrm{Sh}(N) \to \mathrm{Sh}(M \times N)$$

admits a left adjoint and for any map $\alpha\colon N' \to N$ the mate of the commutative square

$$\begin{tikzcd} \mathrm{Sh}(N) \arrow[r, "\pi_M^*"] \arrow[d, "\alpha^*"] & \mathrm{Sh}(M \times N) \arrow[d, "\alpha_X^*"] \\ \mathrm{Sh}(N') \arrow[r, "\pi_M^*"] & \mathrm{Sh}(M \times N') \end{tikzcd}$$

is an equivalence. Since the projections $M \times N \to N$ and $M \times N' \to N'$ are topological submersions, the left adjoint exists and the mate is an equivalence by the smooth base change isomorphism, see [82, Lemma 3.25]. Therefore $\underline{\mathrm{Sh}}_{\mathcal{B}}$ admits all colimits indexed by (small) $\mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})$-categories. However, if $\underline{\mathrm{Sh}}_{\mathcal{B}}$ was cocomplete, it would follow that for *any* continuous map $f \colon M \to N$ of manifolds, the pullback functor

$$f^* \colon \mathrm{Sh}(N) \to \mathrm{Sh}(M)$$

has a left adjoint. This is certainly not the case. For example, if Y is a point, the pullback f^* is simply the stalk functor at the point determined by f, and in general stalk functors do not preserve infinite products. However, if we let Sub denote the local class in $\mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})$ that is generated by the topological submersions in Man, the above arguments show that the $\mathrm{PSh}_{\mathrm{Ani}}(\mathrm{Man})$-category $\underline{\mathrm{Sh}}_{\mathcal{B}}$ is in fact $\langle \mathrm{Sub}, \mathrm{Cat}_\infty \rangle$-cocomplete (see Remark 3.3.2.9).

We now turn to the proof of Proposition 3.5.4.1. Our strategy will be to to take the colimit of a K-indexed diagram in the free cocompletion of C (i.e. in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$) and to show that this colimit can be reflected back into C. We therefore need to study such K-indexed colimits in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ first.

Lemma 3.5.4.9. *For every $\mathcal{B}$-category* C*, the large $\mathcal{B}$-category* $\mathsf{RFib}_{\mathsf{C}}$ *is a reflective subcategory of* $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$.

Proof. To begin with, we note that the sheaf associated with $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ is given by $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times -}$. In fact, the latter defines a $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$-category, and there is a right fibration of $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$-categories $p \colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times -} \to \mathrm{Cat}(\mathcal{B})_{/-}$ that is section-wise given by postcomposition with the projection onto the second factor. By Proposition 1.2.4.6, the codomain can be identified with (the underlying $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$-category of) the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$. Since $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times -}$ has a final object (in the $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$-categorical sense, which is easily deduced from Example 3.2.1.10 and Example 3.2.1.13) that is carried to C along the right fibration p, we thus obtain an equivalence $(\mathrm{Cat}_{\mathcal{B}})_{/\mathsf{C}} \simeq \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times -}$ of $\mathrm{PSh}_{\widehat{\mathrm{Ani}}}(\mathcal{B})$-categories. Since the domain is a (large) $\mathcal{B}$-category, so is the codomain, and this equivalence defines an identification of (large) $\mathcal{B}$-category. Now using this identification, we find that the inclusion $\mathrm{RFib}(\mathsf{C} \times -) \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times -}$ determines a fully faithful functor $i\colon \mathsf{RFib}_{\mathsf{C}} \hookrightarrow (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ such that $i(A)$ admits a left adjoint L_A for every $A \in \mathcal{B}$. Moreover, if $s\colon B \to A$ is an arbitrary map in $\mathcal{B}$, the fact that s is smooth implies that the natural map $L_B s^* \to s^* L_A$ is an equivalence (see the discussion in Section 2.1.4), hence the claim follows. □

Proposition 3.5.4.10. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* C *be a small* $\mathcal{B}$*-category. Then, for any diagram* $d\colon \mathsf{I} \to \mathsf{U}$ *with colimit* K *in* $\mathsf{Cat}_{\mathcal{B}}$ *and any diagram* $p\colon \mathsf{K} \to \mathsf{C}$ *with colimit* F *in* $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$*, there is a diagram* $d'\colon \mathsf{I} \to \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *such that* $F \simeq \operatorname{colim} d'$*.*

Proof. The cocone $d \to \mathrm{diag}(\mathsf{C})$ implies that we may equivalently regard d as a diagram $d\colon \mathsf{I} \to \mathsf{U}_{/\mathsf{C}} \hookrightarrow (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$. By Proposition 3.2.4.3, the colimit of this diagram is $p\colon \mathsf{K} \to \mathsf{C}$. Since $F \simeq \operatorname{colim} p$, there is a final functor $\mathsf{K} \to \mathsf{C}_{/F}$ over C, hence Lemma 3.5.4.9 implies that the localisation functor $L\colon (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}}$ carries $p\colon \mathsf{K} \to \mathsf{C}$ to the right fibration $\mathsf{C}_{/F} \to \mathsf{C}$. In other words, the presheaf F arises as the colimit of the diagram $d' = Ld\colon \mathsf{I} \to \mathsf{U}_{/\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. It now suffices to observe that by construction of L, this functor takes values in $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. □

Proof of Proposition 3.5.4.1. Suppose that $f\colon \mathsf{C} \to \mathsf{D}$ is a U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories. Consider the commutative diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, hook, "h_{\mathsf{C}}"] & \mathsf{D} \arrow[d, hook, "h_{\mathsf{D}}"] \\ \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "\hat{f}"] & \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

that arises from applying the universal property of $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ to the composition $\mathsf{C} \to \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{D})$. As C and D are U-cocomplete, the vertical inclusions admit left adjoints L_{C} and L_{D} (see Corollary 3.5.1.13). Now if $p\colon \mathsf{K} \to \mathsf{C}$ is a diagram, Proposition 3.5.4.10 implies that there is a diagram $p' \colon \mathsf{I} \to \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ such that $\operatorname{colim} p'$ is equivalent to the colimit of $h_{\mathsf{C}} p$. In particular, the colimit of $h_{\mathsf{C}} p$ is contained in $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Consequently, $\operatorname{colim} L_{\mathsf{C}} p'$ defines a colimit of p (Proposition 3.3.2.10). By replacing C with D, this argument also shows that every diagram $\mathsf{K} \to \mathsf{D}$ admits a colimit in D. Moreover, as f and $\hat{f}$ are U-cocontinuous, the universal property of $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ implies that the canonical map $L_{\mathsf{D}} \hat{f} \to f L_{\mathsf{C}}$ is an equivalence. Consequently, as $L_{\mathsf{D}} \hat{f}$ preserves the colimit of $h_{\mathsf{C}} p$, so does $f L_{\mathsf{C}}$. As the colimit cocone of p is the image of the colimit cocone of $h_{\mathsf{D}} p$ along L_{C}, we conclude that f preserves the colimit of p. Now by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the above argumentation, one concludes that both C and D admit K-indexed colimits and that f commutes with such colimits. $\square$

4. Cocartesian fibrations and the straightening equivalence

One of the fundamental results in higher category theory is Lurie's *straightening theorem* [49], which provides an equivalence $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \simeq \mathrm{Cocart}(\mathcal{C})$ between the ∞-category of Cat_∞-valued functors on an ∞-category $\mathcal{C}$ and that of *cocartesian fibrations* over $\mathcal{C}$. This theorem generalises Grothendieck's classical result on the equivalence between pseudo-functors from a 1-category into the 2-category of 1-categories and Grothendieck opfibrations over that 1-category. As it is notoriously challenging to directly construct a functor $\mathcal{C} \to \mathrm{Cat}_\infty$ due to the infinite tower of coherence conditions, the straightening theorem provides an indispensable tool for the study of such functors.

The main goal of this chapter is to provide a $\mathcal{B}$-categorical analogue of Lurie's result: we will define the notion of a *cocartesian fibration* between $\mathcal{B}$-categories, and we will establish the associated *straightening equivalence* which assigns to every such cocartesian fibration $p \colon \mathsf{P} \to \mathsf{C}$ its *straightening* $\mathrm{St}_{\mathsf{C}}(p) \colon \mathsf{C} \to \mathsf{Cat}_{\mathcal{B}}$.

Our strategy for the proof of this result is to build upon the straightening equivalence $\mathsf{LFib}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$ for left fibrations that we already established in Section 2.2.1, a strategy that has previously been outlined by Vladimir Hinich [36] as a proof of the straightening equivalence for cocartesian fibrations of ∞-categories. Since the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ embeds into $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$, we may regard a functor $\mathsf{C} \to \mathsf{Cat}_{\mathcal{B}}$ as a simplicial object in $\mathrm{Fun}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$, or equivalently in $\mathsf{LFib}_{\mathsf{C}}$. We will show that cocartesian fibrations over C are *powered* over Δ, i.e. that there is a functor $(-)^{\Delta^\bullet} \colon \Delta^{\mathrm{op}} \times \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{Cocart}_{\mathsf{C}}$, where $\mathsf{Cocart}_{\mathsf{C}}$ is the (suitably defined) $\mathcal{B}$-category of cocartesian fibrations with codomain C. By combining this fact with the observation that there is an inclusion $\mathsf{LFib}_{\mathsf{C}} \hookrightarrow \mathsf{Cocart}_{\mathsf{C}}$ which admits a right adjoint $(-)_\sharp$, we can now define the straightening functor

St_{C} via the composition

$$\mathsf{Cocart}_{\mathsf{C}} \xrightarrow{(-)^{\Delta^\bullet}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{Cocart}_{\mathsf{C}}) \xrightarrow{(-)_\sharp} \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{LFib}_{\mathsf{C}}).$$

Conversely, the $\mathcal{B}$-category of cocartesian fibrations over C is also *tensored* over Δ in the form of a functor $\Delta^\bullet \otimes - \colon \Delta \times \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{Cocart}_{\mathsf{C}}$. By making use of the universal property of presheaf $\mathcal{B}$-categories that was established in Section 3.5.1, we may therefore define the *unstraightening* functor

$$\mathrm{Un}_{\mathsf{C}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\,\mathsf{Grpd}_{\mathcal{B}})) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta \times \mathsf{C}^{\mathrm{op}}) \to \mathsf{Cocart}_{\mathsf{C}}$$

as the left Kan extension of the functor $\Delta^\bullet \otimes \mathsf{C}_{-/} \colon \Delta^{\mathrm{op}} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{Cocart}_{\mathsf{C}}$ along the Yoneda embedding $h_{\Delta \times \mathsf{C}^{\mathrm{op}}}$. By construction, the unstraightening functor is left adjoint to the straightening functor. We will complete our argument by showing that this adjunction is natural in C in the appropriate sense, so that we can reduce to the case $\mathsf{C} = 1$, in which case the desired result follows trivially.

We begin in Section 4.1 by defining the notion of a cocartesian fibrations between $\mathcal{B}$-categories via an internal analogue of what is sometimes known as the *Chevalley criterion* in (higher) category theory (see for example [73, Theorem 5.2.8]). Moreover, we study the concept of cocartesian morphisms in this context and show that a cocartesian fibration can be characterised by the existence of a sufficient amount of such cocartesian morphisms in the domain.

In Section 4.2, we establish an internal analogue of Lurie's *marked model structure* for cocartesian fibrations. We define the marked simplex category Δ_+ and study the ∞-topos $\mathcal{B}^+_\Delta$ of marked simplicial objects in $\mathcal{B}$, i.e. of $\mathcal{B}$-valued presheaves on Δ_+ . The benefit of passing to marked simplicial objects is that cocartesian fibrations are determined by a factorisation system in $\mathcal{B}^+_\Delta$, which enables us to make use of the many desirable properties of factorisation systems in an ∞-topos to deepen our study of cocartesian fibrations. This already comes in handy when we define and study the $\mathcal{B}$-category $\mathsf{Cocart}_{\mathsf{C}}$ of cocartesian fibrations over a fixed $\mathcal{B}$-category C in Section 4.3.

In Section 4.4, we set up and discuss the straightening and unstraightening functors, and we prove the main result of this chapter, the fact that they form an adjoint equivalence. We complement this result with a study of the *universal* cocartesian fibration, which helps us understand how the straightening of a cocartesian fibration of $\mathcal{B}$-categories is related to the straightening of the underlying cocartesian fibration of ∞-categories that is obtained by passing to global sections. Lastly, we investigate the special case of cocartesian fibrations over the interval Δ^1 and how these can be used to characterise adjunctions between $\mathcal{B}$-categories.

We conclude this chapter by briefly mentioning two applications of the straightening equivalence in Section 4.5. The first application gives a formula for the limit and colimit of $\mathrm{Cat}_{\mathcal{B}}$-valued diagrams in terms of the associated cocartesian fibrations. In the second application, we use our knowledge of cocartesian fibrations over the interval to establish that passing from a right adjoint functor to its left adjoint (and vice versa) constitutes an equivalence between the $\mathcal{B}$-category of $\mathcal{B}$-categories with right adjoint functors and that of $\mathcal{B}$-categories with left adjoint functors.

4.1. Cocartesian fibrations

In this section, we define and study cocartesian fibrations between $\mathcal{B}$-categories. We introduce the notion in Section 4.1.1, where we also discuss a sheaf-theoretic characterisation. In Section 4.1.2, we study cocartesian morphisms and show that cocartesian fibrations are precisely those functors with respect to which their domain has sufficiently many cocartesian morphisms.

4.1.1. Definition and section-wise characterisation

If $p\colon \mathsf{P} \to \mathsf{C}$ is a functor of $\mathcal{B}$-categories, we obtain a functor $\mathrm{res}_p\colon \mathsf{P}^{\Delta^1} \to \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C}$ that makes the diagram

$$\begin{tikzcd}
\mathsf{P}^{\Delta^1} \arrow[r, "\mathrm{res}_p"] \arrow[rr, bend left, "p_*"] \arrow[d] & \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \arrow[r] \arrow[d] & \mathsf{C}^{\Delta^1} \arrow[d] \\
\mathsf{P} \times \mathsf{P} \arrow[r, "\mathrm{id} \times p"] & \mathsf{P} \times \mathsf{C} \arrow[r, "p \times \mathrm{id}"] & \mathsf{C} \times \mathsf{C}
\end{tikzcd}$$

commute. Here the right square is a pullback by definition of the comma $\mathcal{B}$-category (see Definition 2.1.2.1).

Definition 4.1.1.1. A functor $p\colon \mathsf{P} \to \mathsf{C}$ between $\mathcal{B}$-categories is said to be a *cocartesian fibration* if the functor $\mathrm{res}_p\colon \mathsf{P}^{\Delta^1} \to \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C}$ admits a fully faithful left adjoint

$$\mathrm{lift}_p\colon \;\colon \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \hookrightarrow \mathsf{P}^{\Delta^1}.$$

If $p\colon \mathsf{P} \to \mathsf{C}$ and $q\colon \mathsf{Q} \to \mathsf{D}$ are cocartesian fibrations, a *cocartesian functor* between p and q is a commutative square

$$\begin{tikzcd} \mathsf{P} \arrow[r, "g"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

such that the mate of the induced commutative square

$$\begin{tikzcd} \mathsf{P}^{\Delta^1} \arrow[r, "g_*"] \arrow[d, "\mathrm{res}_p"] & \mathsf{Q}^{\Delta^1} \arrow[d, "\mathrm{res}_q"] \\ \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \arrow[r, "f_*"] & \mathsf{Q} \downarrow_{\mathsf{D}} \mathsf{D} \end{tikzcd}$$

commutes as well.

Remark 4.1.1.2. Recall from Remark 3.1.3.6 that the condition of a functor to be a right adjoint is *local* in $\mathcal{B}$. Since similarly the condition of a functor to be fully faithful is local as well, we conclude that for every cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$, a functor $p\colon \mathsf{P} \to \mathsf{C}$ is a cocartesian fibration if and only if $\pi_{A_i}^*(p)$ is a cocartesian fibration of $\mathcal{B}_{/A_i}$-categories for all i. A similar observation can be made for cocartesian functors.

Remark 4.1.1.3. In the case $\mathcal{B} \simeq \mathrm{Ani}$, our definition recovers the notion of cocartesian fibrations and cocartesian functors in Cat_∞ in the sense of [49], cf. Proposition 4.1.2.7 below.

Remark 4.1.1.4. There is an evident dual notion of *cartesian* fibrations, namely those maps $p\colon \mathsf{P} \to \mathsf{C}$ for which the restriction functor $\mathrm{res}_p\colon \mathsf{P}^{\Delta^1} \to \mathsf{C} \downarrow_{\mathsf{C}} \mathsf{P}$ admits a fully faithful *right* adjoint. One defines cartesian functors between such cartesian fibrations in the obvious way. By Proposition 3.1.1.13, a functor p is a cartesian fibration if and only if p^{op} is a cocartesian fibration, and a map between cartesian fibrations defines a cartesian functor if and only if its opposite defines a cocartesian functor. In what follows, we will therefore restrict our attention to cocartesian fibrations, as every statement about those can be dualised in the appropriate sense to be turned into a statement about cartesian fibrations.

Remark 4.1.1.5. In the situation of Definition 4.1.1.1, Remark 3.1.2.10 shows that the square

$$\begin{tikzcd} \mathsf{P} \arrow[r, "g"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

defines a cocartesian functor already when there is an *arbitrary* equivalence $\mathrm{lift}_q f_* \simeq g_* \mathrm{lift}_p$.

Remark 4.1.1.6. Suppose that $p\colon \mathsf{P} \to \mathsf{C}$ is a cocartesian fibration. Since the projection $\mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \to \mathsf{P}$ is the pullback of $d_1 : \mathsf{C}^{\Delta^1} \to \mathsf{C}$ along p and since d_1 admits a fully faithful left adjoint s_0, the projection $\mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \to \mathsf{P}$ also admits a fully faithful left adjoint (Lemma 3.4.4.2), which we will denote by s_0 as well. By the uniqueness of adjoints, we thus obtain a commutative diagram

$$\begin{tikzcd} \mathsf{P} \arrow[d, hook, "s_0"] \arrow[dr, hook, "s_0"] & \\ \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \arrow[r, hook, "\mathrm{lift}_p"] & \mathsf{P}^{\Delta^1}. \end{tikzcd}$$

Proposition 4.1.1.7. *A functor $p\colon \mathsf{P} \to \mathsf{C}$ in $\mathcal{B}$ is a cocartesian fibration if and only if*

1. *for every $A \in \mathcal{B}$ the functor $p(A)$ is a cocartesian fibration of ∞-categories;*
2. *for every $s\colon B \to A$ in $\mathcal{B}$ the commutative square*

$$\begin{tikzcd} \mathsf{P}(A) \arrow[r, "s^*"] \arrow[d, "p(A)"] & \mathsf{P}(B) \arrow[d, "p(B)"] \\ \mathsf{C}(A) \arrow[r, "s^*"] & \mathsf{C}(B) \end{tikzcd}$$

defines a cocartesian functor.

Furthermore, if $p\colon \mathsf{P} \to \mathsf{C}$ *and* $q\colon \mathsf{Q} \to \mathsf{D}$ *are cocartesian fibrations, a commutative square*

$$\begin{tikzcd} \mathsf{P} \arrow[r, "g"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

defines a cocartesian functor precisely if it does so section-wise.

Proof. Since the local sections functor $\mathrm{Fun}_{\mathcal{B}}(A, -)$ commutes with limits and the powering functor and preserves full faithfulness, this statement is an immediate consequence of the sheaf-theoretic characterisation of right adjoint functors (Proposition 3.1.2.9), together with Remark 3.1.2.10 and the fact that full faithfulness can be detected section-wise as well (Proposition 1.3.2.7). □

Proposition 4.1.1.8. *Suppose that*

$$\begin{tikzcd} \mathsf{P} \arrow[r, "g"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

is a pullback square in $\mathrm{Cat}(\mathcal{B})$ *such that* q *is a cocartesian fibration. Then* p *is a cocartesian fibration, and the square itself defines a cocartesian functor.*

Proof. The pullback square gives rise to a commutative square

$$\begin{tikzcd} \mathsf{P}^{\Delta^1} \arrow[r, "g_*"] \arrow[d, "\mathrm{res}_p"] & \mathsf{Q}^{\Delta^1} \arrow[d, "\mathrm{res}_q"] \\ P \downarrow_{\mathsf{C}} \mathsf{C} \arrow[r, "f_*"] & \mathsf{P} \downarrow_{\mathsf{D}} \mathsf{D} \end{tikzcd}$$

that is easily seen to be a pullback too. Thus the claim follows from Lemma 3.4.4.2. □

We denote by $\mathrm{Cocart} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B}))$ the subcategory that is spanned by the cocartesian fibrations and cocartesian squares. By Proposition 4.1.1.8, this defines a cartesian subfibration of $d_0\colon \mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B})) \to \mathrm{Cat}(\mathcal{B})$. For $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$, we denote by $\mathrm{Cocart}(\mathsf{C})$ the fibre of Cocart over C. Clearly we have $\mathrm{Cocart}(A) \simeq \mathrm{Cat}(\mathcal{B}_{/A})$ for any $A \in \mathcal{B}$ since for any $\mathcal{B}_{/A}$-category P and for $\pi_{\mathsf{P}} \colon \mathsf{P} \to A$ the structure map, the restriction functor $\mathrm{res}_{\pi_{\mathsf{P}}}$ is an equivalence. In other words, the restriction of the presheaf Cocart along the inclusion $\mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ recovers the sheaf $\mathrm{Cat}(\mathcal{B}_{/-})$.

Proposition 4.1.1.9. *For every $\mathcal{B}$-category C and every simplicial object K in $\mathcal{B}$, the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$ restricts to a functor $\mathrm{Cocart}(\mathsf{C}) \to \mathrm{Cocart}(\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}))$.*

Proof. This follows from the observation that $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$ commutes with limits and powering, carries adjunctions to adjunctions (see Corollary 3.1.1.10) and preserves the property of functors to be fully faithful. $\square$

4.1.2. Cocartesian morphisms

Let $p \colon \mathsf{P} \to \mathsf{C}$ be a cocartesian fibration. Then $\mathrm{lift}_p \colon \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C} \hookrightarrow \mathsf{P}^{\Delta^1}$ determines a subobject of maps in P. Our goal in this section is to study these maps.

Definition 4.1.2.1. Let $p \colon \mathsf{P} \to \mathsf{C}$ be a functor between $\mathcal{B}$-categories. A map $f \colon x \to y$ in P (in context $A \in \mathcal{B}$) is said to be *p-cocartesian* if the commutative square

$$\begin{tikzcd} \mathsf{P}_{y/} \arrow[r, "f^*"] \arrow[d, "p"] & \mathsf{P}_{x/} \arrow[d, "p"] \\ \mathsf{C}_{p(y)/} \arrow[r, "p(f)^*"] & \mathsf{C}_{p(x)/} \end{tikzcd}$$

is a pullback square in $\mathrm{Cat}(\mathcal{B}_{/A})$.

Remark 4.1.2.2. The commutative square in Definition 4.1.2.1 formally arises from evaluating the morphism of bifunctors $\mathrm{map}_{\mathsf{P}}(-, -) \to \mathrm{map}_{\mathsf{C}}(p(-), p(-))$ (which is itself constructed by using functoriality of the twisted arrow construction) at $f \colon \Delta^1 \otimes A \to \mathsf{P}^{\mathrm{op}}$ and by using the straightening equivalence for left fibrations (Theorem 2.2.1.1). This produces a commutative square

$$\begin{tikzcd} \mathsf{P}_{y/} \arrow[r, "f^*"] \arrow[d, "p"] & \mathsf{P}_{x/} \arrow[d, "p"] \\ \mathsf{C}_{p(y)/} \times_{\mathsf{C}} \mathsf{P} \arrow[r, "p(f)^*"] & \mathsf{C}_{p(x)/} \times_{\mathsf{C}} \mathsf{P}, \end{tikzcd}$$

that recovers the square from Definition 4.1.2.1 upon pasting with the pullback square

$$\begin{tikzcd} \mathsf{C}_{p(y)/} \times_{\mathsf{C}} \mathsf{P} \arrow[r, "p(f)^*"] \arrow[d] & \mathsf{C}_{p(x)/} \times_{\mathsf{C}} \mathsf{P} \arrow[d] \\ \mathsf{C}_{p(y)/} \arrow[r, "p(f)^*"] & \mathsf{C}_{p(x)/}. \end{tikzcd}$$

Remark 4.1.2.3. The notion of a map to be p-cocartesian is *local* in $\mathcal{B}$: if $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ is a cover in $\mathcal{B}$, then $f\colon x \to y$ in context A is p-cocartesian if and only if each $s_i^*(f)$ is. This follows immediately from the observation that the functor $\mathrm{Cat}(\mathcal{B}_{/A}) \to \prod_i \mathrm{Cat}(\mathcal{B}_{/A_i})$ is conservative. As a consequence, there is a subobject $E \hookrightarrow \mathsf{P}_1$ that is determined by the condition that a map $f\colon x \to y$ in P in context A is contained in E if and only if it is p-cocartesian.

Remark 4.1.2.4. In the situation of Definition 4.1.2.1, the pasting lemma for pullback squares implies that the subobject $E \hookrightarrow \mathsf{P}_1$ that is determined by the cocartesian morphisms in P is closed under composition and equivalences in the sense of Proposition 1.3.1.17. Furthermore, the pasting lemma shows that if $f\colon x \to y$ is a cocartesian map and $g\colon y \to z$ is an arbitrary map, then gf is cocartesian if and only if g is cocartesian.

Let $p\colon \mathsf{P} \to \mathsf{C}$ be a functor in $\mathrm{Cat}(\mathcal{B})$ and let $f\colon x \to y$ be a map in P in context $A \in \mathcal{B}$ with image $\alpha\colon c \to d$ in C. Let $\mathsf{P}^{\Delta^2}|_f$ and $\mathsf{P}^{\Lambda^2_0}|_f$ be the fibres of $d_{\{0,1\}}\colon \mathsf{P}^{\Delta^2} \to \mathsf{P}^{\Delta^1}$ and $d_{\{0,1\}}\colon \mathsf{P}^{\Lambda^2_0} \to \mathsf{P}^{\Delta^1}$ over $f\colon A \to \mathsf{P}^{\Delta^1}$. Define $\mathsf{C}^{\Delta^2}|_\alpha$ and $\mathsf{C}^{\Lambda^2_0}|_\alpha$ likewise. One then obtains:

Proposition 4.1.2.5. *Let $p\colon \mathsf{P} \to \mathsf{C}$ be a functor in $\mathrm{Cat}(\mathcal{B})$ and let $f\colon x \to y$ be a map in P in context $A \in \mathcal{B}$. Let α be the image of f along p. Then f is cocartesian if and only if the commutative square*

$$\begin{tikzcd} \mathsf{P}^{\Delta^2}|_f \arrow[r] \arrow[d] & \mathsf{P}^{\Lambda^2_0}|_f \arrow[d] \\ \mathsf{C}^{\Delta^2}|_\alpha \arrow[r] & \mathsf{C}^{\Lambda^2_0}|_\alpha \end{tikzcd}$$

is cartesian.

Proof. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and f with its transpose, we may assume without loss of generality that $A \simeq 1$. Moreover, note that there is a commutative diagram

$$\begin{tikzcd}
\mathsf{P}^{\Delta^2}|_f \arrow[rr] \arrow[dr, "d_{\{2\}}"'] & & \mathsf{P}^{\Lambda^2_0}|_f \arrow[dl, "d_{\{2\}}"] \\
& \mathsf{P}. &
\end{tikzcd}$$

in which the diagonal maps are left fibrations. Hence the map $\mathsf{P}^{\Delta^2}|_f \to \mathsf{P}^{\Lambda^2_0}|_f$ is a left fibration as well. As the same is true for the map $\mathsf{C}^{\Delta^2}|_\alpha \to \mathsf{C}^{\Lambda^2_0}|_\alpha$, the square in the statement of the proposition is a pullback if and only if it is carried to a pullback square by the core $\mathcal{B}$-groupoid functor. Let τ be the tautological object in P in context P_0, i.e. the object that is determined by the identity $\mathsf{P}_0 \simeq \mathsf{P}_0$. We then obtain a commutative diagram

$$\begin{tikzcd}[column sep=small]
& \mathrm{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* y, \tau) \arrow[dl] \arrow[dd] & & (\mathsf{P}^{\Delta^2}|_f)_0 \arrow[ll] \arrow[rr, dashed] \arrow[dl] \arrow[dd] & & (\mathsf{P}^{\Lambda^2_0}|_f)_0 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* x, \tau) \arrow[dl] \arrow[dd] \\
\mathsf{P}_1 \arrow[dd, "{(d_1,d_0)}"] & & \mathsf{P}_2 \arrow[ll, "d_{\{1,2\}}"'] \arrow[rr] \arrow[dd, "{(d_{\{0,1\}},d_{\{2\}})}"] & & (\mathsf{P}^{\Lambda^2_0})_0 \arrow[rr, "d_{\{0,2\}}"] \arrow[dd, "{(d_{\{0,1\}},d_{\{2\}})}"] & & \mathsf{P}_1 \arrow[dd, "{(d_1,d_0)}"] & \\
& \mathsf{P}_0 \arrow[dl, "y\times\mathrm{id}"] & & \mathsf{P}_0 \arrow[ll, "\mathrm{id}"'] \arrow[rr, "\mathrm{id}"] \arrow[dl, "f\times\mathrm{id}"] & & \mathsf{P}_0 \arrow[rr, "\mathrm{id}"] \arrow[dl, "f\times\mathrm{id}"] & & \mathsf{P}_0 \arrow[dl, "x\times\mathrm{id}"] \\
\mathsf{P}_0 \times \mathsf{P}_0 & & \mathsf{P}_1 \times \mathsf{P}_0 \arrow[ll, "d_0\times\mathrm{id}"'] \arrow[rr, "\mathrm{id}"] & & \mathsf{P}_1 \times \mathsf{P}_0 \arrow[rr, "d_1\times\mathrm{id}"] & & \mathsf{P}_0 \times \mathsf{P}_0 &
\end{tikzcd}$$

in which the dotted arrow is the map that is induced by the functor $\mathsf{P}^{\Delta^2}|_f \to \mathsf{P}^{\Lambda^2_0}|_f$ upon applying the core $\mathcal{B}$-groupoid functor. Note that both the front left and the front right square is cartesian, hence both $(\mathsf{P}^{\Delta^2}|_f)_0 \to \mathrm{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* y, \tau)$ and $(\mathsf{P}^{\Lambda^2_0}|_f)_0 \to \mathrm{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* x, \tau)$ must be an equivalence. Now by using the argument in Remark 2.3.2.2, the composition

$$\mathrm{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* y, z) \xrightarrow{\simeq} (\mathsf{P}^{\Delta^2}|_f)_0 \to (\mathsf{P}^{\Lambda^2_0}|_f)_0 \xrightarrow{\simeq} \mathrm{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* x, z)$$

recovers the map f^*. We now note that the above construction is natural in P, in that we may identify the commutative square that arises from applying $(-)^{\simeq}$ to the square in the statement of the proposition with the commutative diagram

$$\begin{tikzcd} \operatorname{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* y, \tau) \arrow[r, "(\pi_{\mathsf{P}_0}^* f)^*"] \arrow[d] & \operatorname{map}_{\mathsf{P}}(\pi_{\mathsf{P}_0}^* x, \tau) \arrow[d] \\ \operatorname{map}_{\mathsf{C}}(\pi_{\mathsf{P}_0}^* d, p(\tau)) \arrow[r, "(\pi_{\mathsf{P}_0}^* \alpha)^*"] & \operatorname{map}_{\mathsf{C}}(\pi_{\mathsf{P}_0}^* c, p(\tau)) \end{tikzcd}$$

that is obtained by evaluating the square from Definition 4.1.2.1 at τ. As τ is the tautological object, we conclude that this diagram is a pullback if and only if f is cocartesian, as desired. □

Let C be a $\mathcal{B}$-category. Observe that the evaluation functor $\mathrm{ev}\colon \Delta^1 \otimes \mathsf{C}^{\Delta^1} \to \mathsf{C}$ can be regarded as a morphism $\phi\colon d_1 \to d_0$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\Delta^1}, \mathsf{C})$. By postcomposition with the Yoneda embedding, one thus obtains a map

$$\phi_*\colon \operatorname{map}_{\mathsf{C}}(-, d_1(-)) \to \operatorname{map}_{\mathsf{C}}(-, d_0(-)).$$

Dually, one obtains a map

$$\phi^*\colon \operatorname{map}_{\mathsf{C}}(d_0(-), -) \to \operatorname{map}_{\mathsf{C}}(d_1(-), -).$$

Lemma 4.1.2.6. *For any $\mathcal{B}$-category C, there is a cartesian square*

$$\begin{tikzcd} \operatorname{map}_{\mathsf{C}^{\Delta^1}}(-, -) \arrow[r] \arrow[d] & \operatorname{map}_{\mathsf{C}}(d_0(-), d_0(-)) \arrow[d, "\phi^*"] \\ \operatorname{map}_{\mathsf{C}}(d_1(-), d_1(-)) \arrow[r, "\phi_*"] & \operatorname{map}_{\mathsf{C}}(d_1(-), d_0(-)) \end{tikzcd}$$

in which the left vertical and the upper horizontal map are given by the action of the functors $d_1, d_0\colon \mathsf{C}^{\Delta^1} \rightrightarrows \mathsf{C}$ on mapping $\mathcal{B}$-groupoids.

Proof. Let $\epsilon\colon s_0 d_1 \to \mathrm{id}$ be the counit of the adjunction $s_0 \dashv d_1$ and $\eta\colon \mathrm{id} \to s_0 d_0$ be the unit of the adjunction $d_0 \dashv s_0$. Then $\phi\colon \Delta^1 \otimes \mathsf{C}^{\Delta^1} \to \mathsf{C}$ can be recovered both by postcomposing η with d_1 and ϵ with d_0. We may therefore construct a commutative square as in the statement of the lemma as the unique square that makes the diagram

$$\begin{tikzcd}
& \mathrm{map}_{\mathsf{C}^{\Delta^1}}(-,-) \arrow[rr] \arrow[dl, "\simeq"'] \arrow[dd] & & \mathrm{map}_{\mathsf{C}}(d_0(-),d_0(-)) \arrow[dl, "\simeq"'] \arrow[dd, "\phi^*"] \\
\mathrm{map}_{\mathsf{C}^{\Delta^1}}(-,-) \arrow[rr, "\eta_*"] \arrow[dd, "\epsilon^*"] & & \mathrm{map}_{\mathsf{C}^{\Delta^1}}(-,s_0d_0(-)) \arrow[dd, "\epsilon^*", near start] & \\
& \mathrm{map}_{\mathsf{C}}(d_1(-),d_1(-)) \arrow[rr, "\phi_*", near start] \arrow[dl, "\simeq"'] & & \mathrm{map}_{\mathsf{C}}(d_1(-),d_0(-)) \arrow[dl, "\simeq"'] \\
\mathrm{map}_{\mathsf{C}^{\Delta^1}}(s_0d_1(-),-) \arrow[rr, "\eta_*"] & & \mathrm{map}_{\mathsf{C}^{\Delta^1}}(s_0d_1(-),s_0d_0(-)) &
\end{tikzcd}$$

commute. We still need to show that this square is cartesian, for which it suffices to show that it becomes a pullback after being evaluated at an arbitrary pair of maps $f\colon c \to d$ and $g\colon c' \to d'$ in C in context $A \in \mathcal{B}$, see Proposition 3.2.3.2. This in turn allows us to argue section-wise in $\mathcal{B}$, which by using Corollary 2.2.2.8 lets us further reduce the statement to its analogue for ∞-categories. This appears (in a more general form) as [28, Proposition 2.3]. □

Proposition 4.1.2.7. *A functor $p\colon \mathsf{P} \to \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})$ is cocartesian if and only if for every object x in P in context $A \in \mathcal{B}$ and every map $\alpha\colon c \simeq p(x) \to d$ in C, there exists a cocartesian lift of α, i.e. a cocartesian morphism $f\colon x \to y$ such that $p(f) \simeq \alpha$.*

Proof. The datum of an object $x\colon A \to \mathsf{P}$ and a map $\alpha\colon c \simeq p(x) \to d$ in C is tantamount to an object $w\colon A \to \mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C}$. In light of this observation, the datum of a lift $f\colon x \to y$ of α is equivalent to a lift of w along res_p. Given such a lift, the definition of comma $\mathcal{B}$-categories and Lemma 4.1.2.6 provide a commutative diagram

$$\begin{tikzcd}
& & \mathrm{map}_{\mathsf{P}^{\Delta^1}}(f,-) \arrow[rr] \arrow[dll, dashed, "\rho"'] \arrow[dl] \arrow[dd] & & \mathrm{map}_{\mathsf{P}}(y,d_0(-)) \arrow[dl] \arrow[dd, "f^*"] \\
\mathrm{map}_{\mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C}}(w,\mathrm{res}_p(-)) \arrow[r] \arrow[dd] & \mathrm{map}_{\mathsf{C}^{\Delta^1}}(\alpha,p_*(-)) \arrow[rr] \arrow[dd] & & \mathrm{map}_{\mathsf{C}}(d,pd_0(-)) \arrow[dd, "\alpha^*", near start] & \\
& & \mathrm{map}_{\mathsf{P}}(x,d_1(-)) \arrow[rr, "\phi_*", near start] \arrow[dll, "\mathrm{id}"'] \arrow[dl] & & \mathrm{map}_{\mathsf{P}}(x,d_0(-)) \arrow[dl] \\
\mathrm{map}_{\mathsf{P}}(x,d_1(-)) \arrow[r] & \mathrm{map}_{\mathsf{C}}(c,pd_1(-)) \arrow[rr, "p\phi_*"] & & \mathrm{map}_{\mathsf{C}}(c,pd_0(-)) &
\end{tikzcd}$$

in which the two squares in the front and the one in the back are cartesian and in which the dotted arrow ρ is given by the composition

$$\mathrm{map}_{\mathsf{P}^{\Delta^1}}(f,-) \to \mathrm{map}_{\mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C}}(\mathrm{res}_p(f),\mathrm{res}_p(-)) \xrightarrow{\eta^*} \mathrm{map}_{\mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C}}(w,\mathrm{res}_p(-))$$

in which $\eta\colon w \simeq \mathrm{res}_p(f)$ is the specified equivalence that exhibits f as a lift of w. Now by Corollary 3.1.3.5, p is cocartesian precisely if every object w in $\mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C}$ admits a lift f along res_p such that the induced map ρ is an equivalence. The proof is thus finished once we show that ρ is an equivalence if and only if f is a cocartesian morphism. If f is a cocartesian morphism, then the right square in the above diagram is a pullback, which clearly implies that ρ is a pullback of the identity on $\mathrm{map}_{\mathsf{P}}(x,d_1(-))$ and therefore an equivalence as well. Conversely, if ρ is an equivalence, one obtains a pullback square

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{P}^{\Delta^1}}(f,-) \arrow[r] \arrow[d] & \mathrm{map}_{\mathsf{C}}(d,pd_0(-)) \arrow[d, "\alpha^*"] \\
\mathrm{map}_{\mathsf{P}}(x,d_1(-)) \arrow[r] & \mathrm{map}_{\mathsf{C}}(c,pd_0(-))
\end{tikzcd}$$

that recovers the square from Definition 4.1.2.1 upon precomposition with the map $s_0\colon \mathsf{P}\hookrightarrow \mathsf{P}^{\Delta^1}$. □

Remark 4.1.2.8. The proof of Proposition 4.1.2.7 shows that if $p\colon \mathsf{P}\to\mathsf{C}$ is a cocartesian fibration, a map $f\colon x\to y$ in P in context $A\in\mathcal{B}$ is contained in the subobject $(\mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C})_0\hookrightarrow \mathsf{P}_1$ if and only if it is a cocartesian morphism. In particular, the map f is cocartesian with respect to p if and only if it is cocartesian with respect to $p(A)$ when viewed as a map in the ∞-category $\mathsf{P}(A)$.

Remark 4.1.2.9. The proof of Proposition 4.1.2.7 shows that if $x\colon A\to\mathsf{P}$ is an arbitrary object, a map $\alpha\colon c\simeq p(x)\to d$ admits a cocartesian lift $f\colon x\to y$ if and only if the copresheaf $\mathrm{map}_{\mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C}}(w,\mathrm{res}_p(-))$ is corepresentable, where $w\colon A\to \mathsf{P}\downarrow_{\mathsf{C}}\mathsf{C}$ is the object that corresponds to the datum $(x,\alpha\colon c\simeq p(x)\to d)$. By making use of Remark 2.3.2.10, we thus conclude that α admits a cocartesian lift $f\colon x\to y$ if and only if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^*\alpha$ admits a cocartesian lift $f_i\colon s_i^*x\to y_i$ for each i.

Corollary 4.1.2.10. *Let $p\colon \mathsf{P} \to \mathsf{C}$ and $q\colon \mathsf{Q} \to \mathsf{C}$ be cocartesian fibrations and let $h\colon \mathsf{P} \to \mathsf{Q}$ be a cocartesian functor over C. Then h is an equivalence precisely if it is a fibre-wise equivalence, i.e. if for every object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ the induced map $h|_c\colon \mathsf{P}|_c \to \mathsf{Q}|_c$ between the fibres is an equivalence.*

Proof. Suppose that h is a fibre-wise equivalence. Then h is certainly essentially surjective, so it suffices to show that it is fully faithful as well. To that end, let $x, y\colon A \rightrightarrows \mathsf{P}$ be two objects in P. We wish to show that the induced map

$$\operatorname{map}_{\mathsf{P}}(x,y) \to \operatorname{map}_{\mathsf{Q}}(h(x),h(y)) \tag{$*$}$$

is an equivalence in $\mathcal{B}_{/A}$. Let $c = p(x)$ and $d = p(y)$. Then the above map lies over $\operatorname{map}_{\mathsf{C}}(c,d)$, which in particular implies that it arises as a retract of its pullback along the projection $\operatorname{map}_{\mathsf{C}}(c,d) \times_A \operatorname{map}_{\mathsf{C}}(c,d) \to \operatorname{map}_{\mathsf{C}}(c,d)$. Hence, by using Remark 2.3.2.1, we may assume (upon replacing $\mathcal{B}_{/A}$ with $\mathcal{B}_{/\operatorname{map}_{\mathsf{C}}(c,d)}$) that (1) there exists a map $\alpha\colon c \to d$ in context A and that (2) we only need to check that the fibre of ($*$) over α is an equivalence. Using Proposition 4.1.2.7, we may choose a cocartesian lift $f\colon x \to z$ of α in P. By Remark 4.1.2.8 and the assumption that h is a cocartesian functor, the map $h(f)$ must be cocartesian as well. Therefore, we obtain a pullback square

$$\begin{tikzcd}
\operatorname{map}_{\mathsf{P}}(z,y) \arrow[r] \arrow[d, "f^*"] & \operatorname{map}_{\mathsf{Q}}(h(z),h(y)) \arrow[d, "h(f)^*"] \\
\operatorname{map}_{\mathsf{P}}(x,y) \arrow[r] & \operatorname{map}_{\mathsf{Q}}(h(x),h(y))
\end{tikzcd}$$

in which the upper horizontal map lies over $\operatorname{map}_{\mathsf{C}}(d,d)$, such that its fibre over id_d coincides with the fibre of $\operatorname{map}_{\mathsf{P}}(x,y) \to \operatorname{map}_{\mathsf{Q}}(h(x),h(y))$ over the object $\alpha\colon A \to \operatorname{map}_{\mathsf{C}}(c,d)$. We may therefore assume without loss of generality $c = d$ and $\alpha = \mathrm{id}_d$. Thus the claim follows from the assumption that $h|_d$ is fully faithful. □

Remark 4.1.2.11. If $p\colon \mathsf{P} \to \mathsf{C}$ is a functor between $\mathcal{B}$-categories, we can define a map $f\colon y \to x$ in P to be *cartesian* if it is cocartesian when viewed as a map $x \to y$ in P^{op}. Explicitly, this amounts to the condition that the commutative square

$$\begin{tikzcd} \mathsf{P}_{/y} \arrow[r, "f_!"] \arrow[d, "p"] & \mathsf{P}_{/x} \arrow[d, "p"] \\ \mathsf{C}_{/d} \arrow[r, "\alpha_!"] & \mathsf{C}_{/c} \end{tikzcd}$$

(where $\alpha\colon d \to c$ is the image of f along p) is a pullback. The results in this section can therefore be dualised to cartesian fibrations, with cartesian maps in place of cocartesian maps.

4.2. The marked model for cocartesian fibrations

As opposed to left fibrations between $\mathcal{B}$-categories, cocartesian fibration do not arise as the right complement of a factorisation system in $\mathrm{Cat}(\mathcal{B})$. In order to rectify this, one must treat cocartesian maps as extra data. This naturally leads us to the study of *marked simplicial objects* in $\mathcal{B}$, which are an internal (and higher-categorical) analogue of marked simplicial sets as studied in [49, § 3.1]. In Section 4.2.1 we introduce the ∞-topos of marked simplicial objects in $\mathcal{B}$ and study its basic properties. In Section 4.2.2 and Section 4.2.3, we study the factorisation system in the ∞-topos of marked simplicial objects that gives rise to the desired model for cocartesian fibrations. In Section 4.2.4, we discuss how left fibrations can be recovered in the marked model. Finally, Section 4.2.5 features a discussion of the notion of marked *proper* and marked *smooth* maps, which will be important for our study of the $\mathcal{B}$-category of cocartesian fibrations over a fixed base $\mathcal{B}$-category.

4.2.1. Marked simplicial objects

Recall from Appendix A.1 the definition of the marked simplex (∞-)category Δ_+. It comes equipped with a fully faithful functor $\iota\colon \Delta \hookrightarrow \Delta_+$ that admits both a left adjoint $\flat$ and a right adjoint $\sharp$. Moreover, there is a single object in Δ_+ that is not contained in the essential image of ι; we will denote this object by $+$. We may now define:

Definition 4.2.1.1. A *marked simplicial object* in $\mathcal{B}$ is a functor $\Delta_+^{\mathrm{op}} \to \mathcal{B}$. We denote by $\mathcal{B}_\Delta^+ = \mathrm{Fun}(\Delta_+^{\mathrm{op}}, \mathcal{B})$ the ∞-topos of marked simplicial objects in $\mathcal{B}$.

Remark 4.2.1.2. Analogous to the case of simplicial objects in $\mathcal{B}$, postcomposition with the global sections functor induces a geometric morphism

$$\Gamma\colon \mathcal{B}^+_\Delta \to \mathrm{Ani}^+_\Delta,$$

which in particular implies that the ∞-topos $\mathcal{B}^+_\Delta$ is tensored and powered over Ani^+_Δ. For $\langle n\rangle \in \Delta$, we will denote by Δ^n_+ the object in Ani^+_Δ that is represented by $\iota\langle n\rangle$, and we will denote by Δ^+_+ the marked simplicial ∞-groupoid that is represented by $+$. As usual, we will implicitly identify such marked simplicial ∞-groupoids with the associated constant marked simplicial objects in $\mathcal{B}$. Note that analogously as in the case of simplicial objects in $\mathcal{B}$, the identity functor on $\mathcal{B}^+_\Delta$ is equivalent to the composition $\mathrm{ev}_0 \circ (-)^{\Delta^\bullet_+}$. In other words, for every marked simplicial object P in $\mathcal{B}$ there is an equivalence $P_\bullet \simeq (P^{\Delta^\bullet_+})_0$ which is natural in P.

By precomposition, the restriction functor $(-)|_\Delta = \iota^*\colon \mathcal{B}^+_\Delta \to \mathcal{B}_\Delta$ admits both a left adjoint $(-)^\flat$ and a right adjoint $(-)^\sharp$, both of which are fully faithful. We denote by $(-)_\sharp$ the right adjoint of $(-)^\sharp$ that is given by right Kan extension along $\sharp$. There is also a further left adjoint $(-)_\flat$ of $(-)^\flat$ given by left Kan extension along $\flat$, but we will not need this functor. Note that applying the unit of the adjunction $(-)|_\Delta \dashv (-)^\sharp$ to $(-)^\flat$ gives rise to a canonical morphism $(-)^\flat \to (-)^\sharp$. Explicitly, this map is given by precomposition with $\flat\epsilon\colon \sharp \simeq \flat\iota\sharp \to \flat$, where ϵ is the counit of the adjunction $\iota \dashv \sharp$.

Remark 4.2.1.3. Since the map $\flat\epsilon\colon \sharp \to \flat$ evaluates to the identity on $\langle 0\rangle$, the natural morphism $(-)^\flat \to (-)^\sharp$ is an equivalence when restricted to $\mathcal{B} \hookrightarrow \mathcal{B}_\Delta$.

Observe that the fact that $(-)^\flat$ is left adjoint to $(-)|_\Delta$ and therefore equivalent to the functor of left Kan extension $\iota_!$ implies that there is a canonical equivalence $\Delta^\bullet_+ \simeq (\Delta^\bullet)^\flat$ of functors $\Delta \to \mathcal{B}^+_\Delta$. We will also need to identify the marked simplicial object Δ^+_+. To that end, observe that there is an equivalence $(\Delta^+_+)|_\Delta \simeq \Delta^1$ and therefore a canonical morphism $\Delta^+_+ \to (\Delta^1)^\sharp$ in $\mathcal{B}^+_\Delta$.

Lemma 4.2.1.4. *The map $\Delta^+_+ \to (\Delta^1)^\sharp$ is an equivalence.*

Proof. We can assume $\mathcal{B} \simeq \mathrm{Ani}$. By construction, the restriction functor $(-)|_\Delta$ carries the map $\Delta^+_+ \to (\Delta^1)^\sharp$ to an equivalence. As equivalences in Ani^+_Δ are detected object-wise, it therefore suffices to show that the evaluation of this map at $+ \in \Delta_+$ is an equivalence as well. On account of Yoneda's lemma, this amounts to showing that the morphism

$$\mathrm{map}_{\Delta_+}(+,+) \to \mathrm{map}_{\Delta}(\langle 1\rangle, \langle 1\rangle)$$

that is induced by the action of the functor $\sharp$ on mapping ∞-groupoids is an equivalence. In light of the explicit computation of $\mathrm{map}_{\Delta_+}(+,+)$ in Appendix A.1, this is immediate. $\square$

Remark 4.2.1.5. The canonical map $\Delta^1_+ \to \Delta^+_+$ gives rise to a commutative diagram

$$\begin{tikzcd}
(\Delta^1_+)|^\flat_\Delta \arrow[r] \arrow[d] & \Delta^1_+ \arrow[r] \arrow[d] & (\Delta^1_+)|^\sharp_\Delta \arrow[d] \\
(\Delta^+_+)|^\flat_\Delta \arrow[r] & \Delta^+_+ \arrow[r] & (\Delta^+_+)|^\sharp_\Delta
\end{tikzcd}$$

in which the two horizontal maps on the left are given by the counit of the adjunction $(-)^\flat \dashv (-)|_\Delta$ and the ones on the right are given by the unit of the adjunction $(-)|_\Delta \dashv (-)^\sharp$. As Δ^1_+ is in the essential image of $(-)^\flat$, the upper left horizontal map is an equivalence, and by Lemma 4.2.1.4 the lower right horizontal map is an equivalence too. Hence the morphism $(\Delta^1)^\flat \to (\Delta^1)^\sharp$ recovers the canonical map $\Delta^1_+ \to \Delta^+_+$ upon identifying $(\Delta^1)^\flat \simeq \Delta^1_+$ and $\Delta^+_+ \simeq (\Delta^1)^\sharp$.

4.2.2. Marked left anodyne morphisms

The goal of this section is to construct a saturated class of maps in $\mathcal{B}^+_\Delta$ whose right complement ought to model cocartesian fibrations. Our approach is in large parts an adaptation of Lurie's construction of the cocartesian model structure in [49, § 3.1], but as we work internally there will be some deviations. In particular, the generators that we list in Definition 4.2.2.1 are slightly different from the class of marked anodyne morphisms as defined in [49, Definition 3.1.1.1].

Definition 4.2.2.1. A map in $\mathcal{B}^+_\Delta$ is said to be *marked left anodyne* if it is contained in the internal saturation of the following collection of maps:

1. $(I^2)^\flat \hookrightarrow (\Delta^2)^\flat$;

2. $(E^1)^\flat \to 1$;
3. $(\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^1)^\sharp \to (\Delta^1)^\sharp$;
4. $d^1 : 1 \hookrightarrow (\Delta^1)^\sharp$.

For practical purposes, we will need a slightly smaller set of generators for the collection of marked left anodyne maps. In what follows, we shall adopt Jay Shah's notation in [77] and let ${}_\natural(\Delta^n)^\flat = (\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^n)^\flat$ denote the pushout of $(\Delta^1)^\flat \to (\Delta^1)^\sharp$ along $d^{\{0,1\}} : (\Delta^1)^\flat \hookrightarrow (\Delta^n)^\flat$ for every $n \geq 2$. We will use the same notation for any subobject of Δ^n that contains the edge $\{0, 1\}$.

Proposition 4.2.2.2. *A map in $\mathcal{B}_\Delta^+$ is marked left anodyne if and only if it is contained in the saturation of the following collection of maps:*

1. $(I^2 \otimes K)^\flat \hookrightarrow (\Delta^2 \otimes K)^\flat$ *for all* $K \in \mathcal{B}_\Delta$*;*
2. $(E^1 \otimes K)^\flat \to K^\flat$ *for all* $K \in \mathcal{B}_\Delta$*;*
3. $(\Delta^1 \otimes A)^\sharp \sqcup_{(\Delta^1 \otimes A)^\flat} (\Delta^1 \otimes A)^\sharp \to (\Delta^1 \otimes A)^\sharp$ *for all* $A \in \mathcal{B}$*;*
4. ${}_\natural(\Lambda^2_0)^\flat \otimes A \hookrightarrow {}_\natural(\Delta^2)^\flat \otimes A$ *for all* $A \in \mathcal{B}$*;*
5. $d^1 : A^\sharp \hookrightarrow (\Delta^1 \otimes A)^\sharp$ *for all* $A \in \mathcal{B}$*;*
6. $(I^2 \otimes A)^\sharp \hookrightarrow (\Delta^2 \otimes A)^\sharp$ *for all* $A \in \mathcal{B}$.

We will spread out the proof of Proposition 4.2.2.2 over Lemma 4.2.2.4 and Lemma 4.2.2.6. Both of them will make repeated use of the following basic observation:

Lemma 4.2.2.3. *Let*

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"] & M \arrow[d, "g"] \\ L \arrow[r] & N \end{tikzcd}$$

be a commutative square in $\mathcal{B}_\Delta^+$ such that $f|_\Delta$ and $g|_\Delta$ are equivalences. Then the square is a pushout if and only if it becomes a pushout after evaluation at $+ \in \Delta_+$. In particular, if $C \to D$ is a map in $\mathcal{B}_\Delta$, the map $C^\sharp \sqcup_{C^\flat} D^\flat \to D^\sharp$ is an equivalence if and only if $C_1 \sqcup_{C_0} D_0 \to D_1$ is an equivalence. An analogous result holds for pullbacks.

Proof. The square is a pushout if and only if its evaluation at each object in Δ_+ is a pushout in $\mathcal{B}$. Since all but the object $+ \in \Delta_+$ are contained in the essential image of the inclusion $\Delta \hookrightarrow \Delta_+$ and since the functor $(-)|_\Delta$ by assumption carries the vertical maps to equivalences, the first claim follows. As for the second claim, it suffices to observe that the map $C_1 \sqcup_{C_0} D_0 \to D_1$ is precisely the evaluation of $C^\sharp \sqcup_{C^\flat} D^\flat \to D^\sharp$ at $+ \in \Delta_+$. □

Lemma 4.2.2.4. *The internal saturation of the maps in (1)–(3) in Definition 4.2.2.1 is equal to the saturation of the following maps:*

1. $(I^2 \otimes K)^\flat \hookrightarrow (\Delta^2 \otimes K)^\flat$ *for all* $K \in \mathcal{B}_\Delta$*;*

2. $(E^1 \otimes K)^\flat \to K^\flat$ *for all* $K \in \mathcal{B}_\Delta$*;*

3. $(\Delta^1 \otimes A)^\sharp \sqcup_{(\Delta^1 \otimes A)^\flat} (\Delta^1 \otimes A)^\sharp \to (\Delta^1 \otimes A)^\sharp$ *for all* $A \in \mathcal{B}$.

Proof. Let S be the saturation of the maps in (1)–(3) in the lemma. As the internal saturation of the maps in (1)–(3) in Definition 4.2.2.1 clearly contains S, it suffices to prove the converse direction. We need to show that for every marked simplicial object K, the map $f \otimes \mathrm{id}_K$ is contained in S, where f is one of the maps in (1)–(3) in Definition 4.2.2.1. As every marked simplicial object can be obtained as a small colimit of objects of the form $\Delta^n_+ \otimes A$, where $A \in \mathcal{B}$ and either $n \geq 0$ or $n = +$, we only need to show this for $K \in \{(\Delta^n)^\flat \mid n \geq 0\} \cup \{(\Delta^1)^\sharp\}$. There are therefore six cases:

1. For $n \geq 0$, the map $(I^2 \times \Delta^n)^\flat \hookrightarrow (\Delta^2 \times \Delta^n)^\flat$ is by definition contained in S.

2. In order to show that the map $(I^2)^\flat \times (\Delta^1)^\sharp \hookrightarrow (\Delta^2)^\flat \times (\Delta^1)^\sharp$ is contained in S, it suffices to show that the map
$$((\Delta^2)^\flat \times (\Delta^1)^\sharp) \sqcup_{(I^2)^\flat \times (\Delta^1)^\flat} ((I^2)^\flat \times (\Delta^1)^\sharp) \to (\Delta^2)^\flat \times (\Delta^1)^\sharp$$
is contained in S. Using Lemma 4.2.2.3, one easily verifies that this map is an equivalence.

3. The maps $(E^1 \times \Delta^n)^\flat \to (\Delta^n)^\flat$ are by definition contained in S.

4. Consider the commutative diagram

$$\begin{tikzcd}
& (\Delta^1 \sqcup \Delta^1)^\flat \arrow[rr] \arrow[dd] & & (\Delta^1 \sqcup \Delta^1)^\sharp \arrow[dd] \\
(\Delta^1 \sqcup \Delta^1)^\flat \arrow[ur, "\mathrm{id}"] \arrow[rr] \arrow[dd] & & (\Delta^1 \sqcup \Delta^1)^\sharp \arrow[ur, "\mathrm{id}"] \arrow[dd] & \\
& (\Delta^1)^\flat \arrow[rr] & & (\Delta^1)^\flat \sqcup_{(\Delta^1 \sqcup \Delta^1)^\flat} (\Delta^1 \sqcup \Delta^1)^\sharp \\
(E^1 \times \Delta^1)^\flat \arrow[ur] \arrow[rr] & & (E^1)^\flat \times (\Delta^1)^\sharp \arrow[ur, "\phi"'] &
\end{tikzcd}$$

in which the two vertical maps in the front square are induced by the inclusion of the two points of E^1. Since Lemma 4.2.2.3 implies that the front square in this diagram is a pushout, the map ϕ is obtained as a pushout of maps that are contained in S and must therefore be in S too. Hence, to show that $(E^1)^\flat \times (\Delta^1)^\sharp \to (\Delta^1)^\sharp$ is contained in S, it suffices to show that $(\Delta^1)^\flat \sqcup_{(\Delta^1 \sqcup \Delta^1)^\flat} (\Delta^1 \sqcup \Delta^1)^\sharp \to (\Delta^1)^\sharp$ is in S, which follows from the observation that this is precisely the map in (3) in the case where $A \simeq 1$.

5. Let us set $L = (\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^1)^\sharp$. We have a commutative diagram

$$\begin{tikzcd}
& \coprod_{i \in \langle n \rangle} (\Delta^1)^\flat \sqcup \coprod_{i \in \langle n \rangle} (\Delta^1)^\flat \arrow[rr] \arrow[dl] \arrow[dd] & & \coprod_{i \in \langle n \rangle} (\Delta^1)^\sharp \sqcup \coprod_{i \in \langle n \rangle} (\Delta^1)^\sharp \arrow[dl] \arrow[dd] \\
(\Delta^1 \times \Delta^n)^\flat \sqcup (\Delta^1 \times \Delta^n)^\flat \arrow[rr] \arrow[dd] & & ((\Delta^1)^\sharp \times (\Delta^n)^\flat) \sqcup ((\Delta^1)^\sharp \times (\Delta^n)^\flat) \arrow[dd] & \\
& \coprod_{i \in \langle n \rangle} (\Delta^1)^\flat \arrow[rr] \arrow[dl] & & \coprod_{i \in \langle n \rangle} L \arrow[dl] \\
(\Delta^1 \times \Delta^n)^\flat \arrow[rr] & & L \times (\Delta^n)^\flat &
\end{tikzcd}$$

in which both the front and the back square are pushouts. By making use of Lemma 4.2.2.3, one moreover easily verifies that the top square is a pushout too, which implies that the bottom square is one as well. As a consequence, we obtain a commutative diagram

$$\begin{tikzcd} \bigsqcup_{i\in\langle n\rangle}(\Delta^1)^\flat \arrow[r] \arrow[d] & \bigsqcup_{i\in\langle n\rangle} L \arrow[r] \arrow[d] & \bigsqcup_{i\in\langle n\rangle}(\Delta^1)^\sharp \arrow[d] \\ (\Delta^1 \times \Delta^n)^\flat \arrow[r] & L \times (\Delta^n)^\flat \arrow[r] & (\Delta^1)^\sharp \times (\Delta^n)^\flat \end{tikzcd}$$

in which the left square is cocartesian. Since the map $K \to (\Delta^1)^\sharp$ is contained in S, we conclude that the map $K \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$ is an element of S whenever the right square is a pushout diagram. This follows from the observation that the outer square of this diagram is cocartesian, which is easily verified using Lemma 4.2.2.3.

6. Let again $K = (\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^1)^\sharp$ and consider the commutative diagram
7. Let again $L = (\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^1)^\sharp$ and consider the commutative diagram

$$\begin{tikzcd} (\Delta^1 \sqcup \Delta^1)^\flat \arrow[r] \arrow[d] & L \times (\Delta^1)^\flat \arrow[r] \arrow[d] & (\Delta^1 \times \Delta^1)^\flat \arrow[d] \\ (\Delta^1 \sqcup \Delta^1)^\sharp \arrow[r] & L \times (\Delta^1)^\sharp \arrow[r] & (\Delta^1)^\flat \times (\Delta^1)^\sharp \end{tikzcd}$$

in which the two horizontal maps on the left are induced by the inclusion of the two points of L_0. Using Lemma 4.2.2.3, one finds that the composite square is cocartesian, and the fact that the two horizontal maps on the left induce an equivalence when evaluated at $+ \in \Delta_+$ similarly implies that the left square is a pushout too. We thus conclude that the right square is cocartesian. As the upper right horizontal morphism is contained in S, this shows that the map $L \times (\Delta^1)^\sharp \to (\Delta^1)^\flat \times (\Delta^1)^\sharp$ is in S as well. □

Remark 4.2.2.5. Note that the internal saturation of the maps in (1)–(3) in Definition 4.2.2.1 also contains the map $K^\sharp \sqcup_{K^\flat} K^\sharp \to K^\sharp$ for every simplicial object K. In fact, this follows from the observation that this map arises as a retract of the morphism

$$(\Delta^1 \otimes K)^\sharp \sqcup_{(\Delta^1 \otimes K)^\flat} (\Delta^1 \otimes K)^\sharp \to (\Delta^1 \otimes K)^\sharp.$$

Lemma 4.2.2.6. *Let S be a saturated class of maps in $\mathcal{B}^+_\Delta$ that contains the internal saturation of the maps in (1)–(3) in Definition 4.2.2.1. Then S contains the internal saturation of $d^1 \colon 1 \hookrightarrow (\Delta^1)^\sharp$ if and only if it contains the following maps:*

1. ${}_\natural(\Lambda^2_0)^\flat \otimes A \hookrightarrow {}_\natural(\Delta^2)^\flat \otimes A$ *for all* $A \in \mathcal{B}$;
2. $d^1 \colon A^\sharp \hookrightarrow (\Delta^1 \otimes A)^\sharp$ *for all* $A \in \mathcal{B}$;
3. $(I^2 \otimes A)^\sharp \hookrightarrow (\Delta^2 \otimes A)^\sharp$ *for all* $A \in \mathcal{B}$.

Proof. Suppose first that S contains the internal saturation of $d^1 \colon 1 \hookrightarrow (\Delta^1)^\sharp$. There are now three cases to consider:

1. Let $K \to L$ be the unique map in $\mathcal{B}^+_\Delta$ that fits into the diagram

$$\begin{tikzcd}
& (\Lambda^2_0)^\flat \arrow[rr, hook] \arrow[dl] \arrow[dd, hook, "d^1"] & & (\Delta^2)^\flat \arrow[dl] \arrow[dd, hook] \arrow[ddrr, bend left, "d^1"] & & \\
{}_\natural(\Lambda^2_0)^\flat \arrow[rr, hook] \arrow[dd, hook, "d^1"] & & {}_\natural(\Delta^2)^\flat \arrow[dd, hook] \arrow[ddrr, bend left, "d^1"] & & & \\
& (\Delta^1 \times \Lambda^2_0)^\flat \arrow[rr, hook] \arrow[dl] & & K \arrow[rr, hook] \arrow[dl] & & (\Delta^1 \times \Delta^2)^\flat \arrow[dl] \\
(\Delta^1)^\sharp \times {}_\natural(\Lambda^2_0)^\flat \arrow[rr, hook] & & L \arrow[rr, hook] & & (\Delta^1)^\sharp \times {}_\natural(\Delta^2)^\flat &
\end{tikzcd}$$

   such that both the front and the back square is a pushout. Then the inclusion $L \hookrightarrow (\Delta^1)^\sharp \times {}_\natural(\Delta^2)^\flat$ is contained in S. We claim that the inclusion ${}_\natural(\Lambda^2_0)^\flat \hookrightarrow {}_\natural(\Delta^2)^\flat$ is a retract of this map. To see this, first note that the two squares on the bottom of the above diagram are cocartesian by Lemma 4.2.2.3. Now let $r \colon \Delta^1 \times \Delta^2 \to \Delta^2$ be the map given by $r(0,1) = 0$ and $r(k,l) = l$ else. The $r^\flat$ restricts to a map $(r')^\flat \colon K \to (\Lambda^2_0)^\flat$. We obtain a commutative diagram

$$\begin{tikzcd}
& (\Lambda^2_0)^\flat \arrow[rr, "d^0"] \arrow[dl] \arrow[dd] & & K \arrow[rr, "(r')^\flat"] \arrow[dl] \arrow[dd] & & (\Lambda^2_0)^\flat \arrow[dl] \arrow[dd] \\
(\Delta^2)^\flat \arrow[rr, "d^0"] \arrow[dd, dashed] & & (\Delta^1 \times \Delta^2)^\flat \arrow[rr, "r^\flat"] \arrow[dd, dashed] & & (\Delta^2)^\flat \arrow[dd, dashed] & \\
& {}_\natural(\Lambda^2_0)^\flat \arrow[rr, "d^0"] \arrow[dl, dashed] & & L \arrow[rr] \arrow[dl, dashed] & & {}_\natural(\Lambda^2_0)^\flat \arrow[dl, dashed] \\
{}_\natural(\Delta^2)^\flat \arrow[rr, dashed] & & (\Delta^1)^\sharp \times {}_\natural(\Delta^2)^\flat \arrow[rr, dashed] & & {}_\natural(\Delta^2)^\flat &
\end{tikzcd}$$

in which the upper row is a retract diagram. Since the lower row is obtained as a pushout of the upper row, the claim follows. As a consequence, the maps in (1) can be realised as retracts of maps in S, which shows that they too must be contained in S.

2. The maps $d^1 : A^\sharp \hookrightarrow (\Delta^1 \otimes A)^\sharp$ are by definition contained in S.

3. Note that the map $d^{\{0,1\}} : (\Delta^1 \otimes A)^\sharp \hookrightarrow (I^2 \otimes A)^\sharp$ is a pushout of the inclusion $d^1 : A^\sharp \hookrightarrow (\Delta^1 \otimes A)^\sharp$ and therefore contained in S. Hence, to show that the maps in (3) are in S, it suffices to prove that the map $d^{\{0,1\}} : (\Delta^1 \otimes A)^\sharp \hookrightarrow (\Delta^2 \otimes A)^\sharp$ is an element of S. This in turn follows from the observation that this map is a retract of the morphism $d^1 : (\Delta^1 \otimes A)^\sharp \hookrightarrow (\Delta^1 \otimes (\Delta^1 \otimes A))^\sharp$.

We now show the converse inclusion. As in the proof of Lemma 4.2.2.4, we only need to show that the map $d^1 : K \hookrightarrow (\Delta^1)^\sharp \otimes K$ is contained in S for every

$$K \in \{(\Delta^n \otimes A)^\flat \mid n \geq 0,\ A \in \mathcal{B}\} \cup \{(\Delta^1 \otimes A)^\sharp \mid A \in \mathcal{B}\}.$$

As $d^1 : A^\sharp \hookrightarrow (\Delta^1 \otimes A)^\sharp$ is contained in S, we can replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and therefore always assume $A \simeq 1$. Moreover, since S by assumption contains the internal saturation of $(I^2)^\flat \hookrightarrow (\Delta^2)^\flat$, the map $(I^n)^\flat \otimes K \hookrightarrow (\Delta^n)^\flat \otimes K$ is in S too, for every integer $n \geq 2$ (see Lemma 1.2.3.5). Thus, if $f \in S$ is an arbitrary map such that $\mathrm{id}_{(\Delta^1)^\flat} \otimes f$ is contained in S, the map $\mathrm{id}_{(\Delta^n)^\flat} \otimes f$ must be in S too for every integer $n \geq 0$. In total, these considerations allow us to assume $K \in \{(\Delta^1)^\flat, (\Delta^1)^\sharp\}$. There are therefore two cases:

1. To show that $d^1 : (\Delta^1)^\flat \hookrightarrow (\Delta^1)^\sharp \times (\Delta^1)^\flat$ is contained in S, first note that the codomain of this map is given by the pushout

$$\begin{tikzcd} (\Delta^1 \sqcup \Delta^1)^\flat \arrow[r] \arrow[d, "(d^1\times\mathrm{id}, d^0\times\mathrm{id})"] & (\Delta^1 \sqcup \Delta^1)^\sharp \arrow[d] \\ (\Delta^1 \times \Delta^1)^\flat \arrow[r] & (\Delta^1)^\flat \times (\Delta^1)^\sharp. \end{tikzcd}$$

Therefore, by using the decomposition $\Delta^1 \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$, we obtain an equivalence of marked simplicial objects $(\Delta^1)^\flat \times (\Delta^1)^\sharp \simeq H \sqcup_{(\Delta^1)^\flat} K$, where

H and K are defined as the pushouts

$$\begin{tikzcd} (\Delta^1)^\flat \arrow[r] \arrow[d, hook, "d^{\{1,2\}}"] & (\Delta^1)^\sharp \arrow[d] \\ (\Delta^2)^\flat \arrow[r] & H \end{tikzcd} \qquad \begin{tikzcd} (\Delta^1)^\flat \arrow[r] \arrow[d, hook, "d^{\{0,1\}}"] & (\Delta^1)^\sharp \arrow[d] \\ (\Delta^2)^\flat \arrow[r] & K. \end{tikzcd}$$

With respect to this identification, the inclusion $d^1 : (\Delta^1)^\flat \hookrightarrow (\Delta^1)^\sharp \times (\Delta^1)^\flat$ is obtained by the composition

$$(\Delta^1)^\flat \xhookrightarrow{d^{\{0,1\}}} H \hookrightarrow H \sqcup_{(\Delta^1)^\flat} K.$$

It therefore suffices to show that $d^{\{0,1\}} : (\Delta^1)^\flat \hookrightarrow H$ and $d^{\{0,2\}} : (\Delta^1)^\flat \hookrightarrow K$ are contained in S. We begin with the first map. Observe that this morphism is equivalent to the composition

$$(\Delta^1)^\flat \xhookrightarrow{d^{\{0,1\}}} (\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (I^2)^\flat \hookrightarrow (\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^2)^\flat.$$

Here the right map is of the form (1) in Definition 4.2.2.1 and therefore included in S. The left map, on the other hand, is obtained as a pushout of $d^1 : (\Delta^0)^\sharp \hookrightarrow (\Delta^1)^\sharp$, hence contained in S too. In order to show that $d^{\{0,2\}} : (\Delta^1)^\flat \hookrightarrow K$ defines an element of S, it suffices to observe that this map can be obtained as the composition

$$(\Delta^1)^\flat \xhookrightarrow{d^{\{0,2\}"}} {}_\natural(\Lambda^2_0)^\flat \hookrightarrow {}_\natural(\Delta^2)^\flat$$

in which the right map is of the form (1) and therefore in S and in which the left map is a pushout of $d^1 : (\Delta^0)^\sharp \hookrightarrow (\Delta^1)^\sharp$, so contained in S as well.

2. Finally, we show that the map $d^1 : (\Delta^1)^\sharp \hookrightarrow (\Delta^1 \times \Delta^1)^\sharp$ is contained in S. On account of the commutative diagram

$$\begin{tikzcd} & (\Delta^1)^\sharp \arrow[rr, "d^{\{0,2\}}"] \arrow[dd, "d^{\{0,2\}}"] & & (\Delta^2)^\sharp \arrow[dd] \\ (\Delta^0)^\sharp \arrow[ur, "d^1"] \arrow[rr, "d^1"] \arrow[dd, "\mathrm{id}"] & & (\Delta^1)^\sharp \arrow[ur, "d^{\{0,1\}}"] \arrow[dd, "\mathrm{id}"] & \\ & (\Delta^2)^\sharp \arrow[rr, hook] & & (\Delta^1 \times \Delta^1)^\sharp \\ (\Delta^0)^\sharp \arrow[ur, "d^{\{0\}}"] \arrow[rr, "d^1"] & & (\Delta^1)^\sharp \arrow[ur] & \end{tikzcd}$$

in which both the front and the back square is a pushout and in which the map $d^1\colon (\Delta^0)^\sharp \hookrightarrow (\Delta^1)^\sharp$ is contained in S, it will be sufficient to show that the two maps $d^{\{0\}}\colon (\Delta^0)^\sharp \hookrightarrow (\Delta^2)^\sharp$ and $d^{\{0,1\}}\colon (\Delta^1)^\sharp \hookrightarrow (\Delta^2)^\sharp$ are contained in S as well. As the first of these two maps can be factored into $d^1\colon (\Delta^0)^\sharp \hookrightarrow (\Delta^1)^\sharp$ followed by $d^{\{0,1\}}\colon (\Delta^1)^\sharp \hookrightarrow (\Delta^2)^\sharp$, we only need to prove this for the second map. By in turn factoring this morphism as

$$(\Delta^1)^\sharp \xhookrightarrow{d^{\{0,1\}}} (I^2)^\sharp \hookrightarrow (\Delta^2)^\sharp,$$

this is a consequence of the observation that the map $(\Delta^1)^\sharp \hookrightarrow (I^2)^\sharp$ is obtained as a pushout of $d^1\colon (\Delta^0)^\sharp \hookrightarrow (\Delta^1)^\sharp$. □

Proof of Proposition 4.2.2.2. Combine Lemma 4.2.2.4 and Lemma 4.2.2.6. □

4.2.3. Marked cocartesian fibrations

In this section we turn to studying the right complement of the class of marked left anodyne morphisms in $\mathcal{B}^+_\Delta$.

Definition 4.2.3.1. A map in $\mathcal{B}^+_\Delta$ is a *marked cocartesian fibration* if it is right orthogonal to the class of marked left anodyne maps. We write

$$\mathrm{Cocart}^+ \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{B}^+_\Delta)$$

for the full cartesian subfibration over $\mathcal{B}^+_\Delta$ that is spanned by the marked cocartesian fibrations.

The following proposition shows that marked cocartesian fibrations faithfully generalise cocartesian fibrations of $\mathcal{B}$-categories. The analogous result for cocartesian fibrations of ∞-categories appears as (the dual of) [49, Proposition 3.1.1.6].

Proposition 4.2.3.2. *For any $\mathcal{B}$-category C, a map $p\colon P \to \mathsf{C}^\sharp$ is a marked cocartesian fibration if and only if $P|_\Delta$ is a $\mathcal{B}$-category, the map $p|_\Delta$ is a cocartesian fibration in* $\mathrm{Cat}(\mathcal{B})$*, and the map $P_+ \to P_1$ is a monomorphism that identifies P_+ with the subobject of cocartesian morphisms of $p|_\Delta$.*

Proof. The map p being right orthogonal to the maps of the form (1) and (2) in Proposition 4.2.2.2 is equivalent to $P|_{\Delta}$ being a $\mathcal{B}$-category. Moreover, p is right orthogonal to the maps in (3) in Proposition 4.2.2.2 if and only if $P_+ \to P_1$ is a monomorphism in $\mathcal{B}$.

Suppose now that p is right orthogonal to the morphisms that are listed in (1)–(3) in Proposition 4.2.2.2, and let us denote by $\mathsf{P} = P|_{\Delta}$ the underlying $\mathcal{B}$-category of P. The condition that p is right orthogonal to the maps in (4) in Proposition 4.2.2.2 is now equivalent to the commutative diagram

$$\begin{tikzcd}
P_+ \times_{\mathsf{P}_1} \mathsf{P}_2 \arrow[r] \arrow[d] & \mathsf{C}_1 \times_{\mathsf{C}_1} \mathsf{C}_2 \arrow[d] \\
P_+ \times_{\mathsf{P}_1} (\mathsf{P}^{\Lambda^2_0})_0 \arrow[r] & \mathsf{C}_1 \times_{\mathsf{C}_1} (\mathsf{C}^{\Lambda^2_0})_0
\end{tikzcd}$$

being a pullback square. By employing Proposition 4.1.2.5, this is equivalent to the condition that the inclusion $P_+ \hookrightarrow \mathsf{P}_1$ defines a cocartesian morphism in $\mathsf{P} = P|_{\Delta}$. Therefore, if p is in addition right orthogonal to the maps in (5) in Proposition 4.2.2.2, we conclude from Proposition 4.1.2.7 that $p|_{\Delta}$ must be a cocartesian fibration. Furthermore, under these conditions *every* cocartesian map factors through $P_+ \hookrightarrow \mathsf{P}_1$. To see this, suppose that $f\colon x \to y$ is a cocartesian morphism in P in context $A \in \mathcal{B}$, and let $\alpha\colon c \to d$ be the image of f along $p|_{\Delta}$. Using the maps in (5) in Proposition 4.2.2.2, there exists a marked lift of α, i.e. a map $g\colon x \to z$ in P that is contained in $P_+ \hookrightarrow \mathsf{P}_1$ and that is sent to α by $p|_{\Delta}$. Since g is marked and therefore cocartesian, Proposition 4.1.2.5 implies that one can find a map $h\colon z \to y$ in P that is sent to id_d by $p|_{\Delta}$ such that $hg \simeq f$. This implies that h must be cocartesian as well and therefore an equivalence. Thus f is marked too, i.e contained in the image of $P_+ \hookrightarrow \mathsf{P}_1$.

So far, we have shown that if p is a marked cocartesian fibration, then the simplicial object $P|_{\Delta}$ is a $\mathcal{B}$-category and $p|_{\Delta}$ is a cocartesian fibration such that $P_+ \to P_1$ is a monomorphism that identifies P_+ with the subobject of cocartesian maps in $P|_{\Delta}$. Conversely, if the map p satisfies these conditions, the above argumentation shows that the proof is complete once we show that p is right orthogonal to the maps in (5) and (6) in Proposition 4.2.2.2. Since orthogonality to the maps in (5) precisely means that the map $P_+ \to P_0 \times_{C_0} C_1$ is an equivalence, this is immediate by the assumption that the inclusion $P_+ \hookrightarrow P_1$ identifies P_+

with the subobject of cocartesian maps in $P|_{\Delta}$, cf. Remark 4.1.2.8. Orthogonality to the maps in (6), on the other hand, translates into the condition that the map $(P_{\sharp})_2 \to P_+ \times_{P_0} P_+$ is an equivalence. To show this, first note that by Lemma 4.2.2.3 the commutative square

$$\begin{tikzcd} (\Delta^1 \sqcup I^2)^{\flat} \arrow[r] \arrow[d] & (\Delta^1 \sqcup I^2)^{\sharp} \arrow[d] \\ (\Delta^2)^{\flat} \arrow[r] & (\Delta^2)^{\sharp} \end{tikzcd}$$

is a pushout. Here the map $\Delta^1 \sqcup I^2 \to \Delta^2$ is given by $d^{\{0,2\}}$ on the first summand and by the canonical inclusion on the second one. One therefore obtains a pullback square

$$\begin{tikzcd} (P_{\sharp})_2 \arrow[r] \arrow[d, hook] & P_+ \arrow[d, hook] \\ P_+ \times_{P_0} P_+ \arrow[r] & P_1 \end{tikzcd}$$

in which the lower horizontal map is given by the composition

$$P_+ \times_{P_0} P_+ \hookrightarrow P_1 \times_{P_0} P_1 \simeq P_2 \xrightarrow{d_{\{0,2\}}} P_1.$$

Since cocartesian maps in $P|_{\Delta}$ are closed under composition (see Remark 4.1.2.4), we thus conclude that the lower horizontal map factors through the inclusion $P_+ \hookrightarrow P_1$. This shows that the map $(P_{\sharp})_2 \hookrightarrow P_+ \times_{P_0} P_+$ is an equivalence, as desired. □

As a consequence of Proposition 4.2.3.2, we obtain a commutative square

$$\begin{tikzcd} \mathrm{Cocart}^+ \times_{\mathcal{B}^+_{\Delta}} \mathrm{Cat}(\mathcal{B}) \arrow[r, "(-)|_{\Delta}"] \arrow[d, hook] & \mathrm{Cocart} \arrow[d, hook] \\ \mathrm{Fun}(\Delta^1, \mathcal{B}^+_{\Delta}) \arrow[r, "(-)|_{\Delta}"] & \mathrm{Fun}(\Delta^1, \mathcal{B}_{\Delta}) \end{tikzcd}$$

Out next goal is to show:

Proposition 4.2.3.3. *The functor* $(-)|_{\Delta} \colon \mathrm{Cocart}^+ \times_{\mathcal{B}^+_{\Delta}} \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cocart}$ *is an equivalence.*

The proof of Proposition 4.2.3.3 will need the following lemma:

Lemma 4.2.3.4. *Given two presheaves $\sigma, \tau \in \mathrm{PSh}_{\mathcal{B}}(\Delta^2)$ such that $\tau(1) \to \tau(0)$ is a monomorphism in $\mathcal{B}$, the map*

$$\mathrm{map}_{\mathrm{PSh}_{\mathcal{B}}(\Delta^2)}(\sigma, \tau) \to \mathrm{map}_{\mathrm{PSh}_{\mathcal{B}}(\Delta^1)}(d^*_{\{0,2\}}\sigma, d^*_{\{0,2\}}\tau)$$

is a monomorphism in Ani *whose image consists of those maps $d^*_{\{0,2\}}\sigma \to d^*_{\{0,2\}}\tau$ for which the composition $\sigma(1) \to \sigma(0) \to \tau(0)$ takes values in $\tau(1) \hookrightarrow \tau(0)$.*

Proof. By making use of the adjunction $s^*_{\{0,1\}} \dashv d^*_{\{0,2\}}$, the map is equivalently given by postcomposition with the adjunction unit $\eta \colon \tau \to s^*_{\{0,1\}} d^*_{\{0,2\}}\tau$ which is explicitly given by the commutative diagram

$$\begin{tikzcd}
 & & & \tau(0) \arrow[ddr, "\mathrm{id}"] & \\
 & & \tau(1) \arrow[ur, hook] \arrow[ddr] & & \\
 & & \tau(2) \arrow[uur] \arrow[rr] & & \tau(0) \\
\tau(2) \arrow[uurr] \arrow[urr, "\mathrm{id}"'] \arrow[rrr] & & & \tau(0). \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Since $\tau(1) \hookrightarrow \tau(0)$ is by assumption a monomorphism, the entire map η must be a monomorphism too. As a consequence, postcomposition with η defines a monomorphism in Ani, and it is clear from the description of η that the image of this map is of the desired form. $\square$

Proof of Proposition 4.2.3.3. We first show that the functor is fully faithful. To that end, let us fix two objects $p \colon P \to \mathrm{C}^\sharp$ and $q \colon Q \to \mathrm{D}^\sharp$ in $\mathrm{Cocart}^+ \times_{\mathcal{B}^+_\Delta} \mathrm{Cat}(\mathcal{B})$. We then obtain a pullback square

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Cocart}^+}(p, q) \arrow[r] \arrow[d] & \mathrm{map}_{\mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta)}(p|_\Delta, q|_\Delta) \arrow[d] \\
\mathrm{map}_{\mathrm{Fun}(\Delta^1, \mathrm{PSh}_{\mathcal{B}}(\Delta^2))}(\nu^* p, \nu^* q) \arrow[r] & \mathrm{map}_{\mathrm{Fun}(\Delta^1, \mathrm{PSh}_{\mathcal{B}}(\Delta^1))}(\sigma_0^* p|_\Delta, \sigma_0^* q|_\Delta).
\end{tikzcd}$$

By Lemma 4.2.3.4, the lower horizontal map is a monomorphism, hence the upper horizontal map is one as well. The lemma furthermore implies that a map

$p|_{\Delta} \to q|_{\Delta}$ is contained in the image of the upper horizontal map if and only if the composition $P_+ \hookrightarrow P_1 \to Q_1$ takes values in Q_+, which by Proposition 4.2.3.2 is the case precisely when the map $p|_{\Delta} \to q|_{\Delta}$ is a cocartesian functor. Hence the restriction functor $(-)|_{\Delta}$ induces an equivalence

$$\mathrm{map}_{\mathrm{Cocart}^+}(p, q) \simeq \mathrm{map}_{\mathrm{Cocart}}(p|_{\Delta}, q|_{\Delta})$$

and is thus fully faithful.

We complete the proof by showing that the functor is essentially surjective. If $p\colon \mathsf{P} \to \mathsf{C}$ is a cocartesian fibration in $\mathrm{Cat}(\mathcal{B})$ and if $E \hookrightarrow \mathsf{P}_1$ denotes the subobject that is spanned by the cocartesian maps, Remark 4.1.2.4 implies that the map $s_0\colon \mathsf{P}_0 \to \mathsf{P}_1$ factors through E and therefore determines a map $(\Delta^2)^{\mathrm{op}} \to \mathcal{B}$ whose restriction along $d_{\{0,2\}}\colon \Delta^1 \hookrightarrow \Delta^2$ recovers $s_0\colon \mathsf{P}_0 \to \mathsf{P}_1$. In light of the equivalence $\mathcal{B}_{\Delta}^+ \simeq \mathcal{B}_{\Delta} \times_{\mathrm{PSh}_{\mathcal{B}}(\Delta^1)} \mathrm{PSh}_{\mathcal{B}}(\Delta^2)$, we therefore obtain a marked simplicial object $\mathsf{P}^{\natural}$ with $\mathsf{P}^{\natural}_+ = E$ such that $\mathsf{P}^{\natural}|_{\Delta} \simeq \mathsf{P}$. By construction, the object $\mathsf{P}^{\natural}$ comes equipped with a map $p^{\natural}\colon \mathsf{P}^{\natural} \to \mathsf{C}^{\sharp}$. By Proposition 4.2.3.2, we now conclude that $p^{\natural}$ defines the desired object of Cocart^+ that satisfies $p^{\natural}|_{\Delta} \simeq p$. □

Corollary 4.2.3.5. *There is a pullback square*

$$\begin{tikzcd} \mathrm{Cocart} \arrow[r, hook, "(-)^{\natural}"] \arrow[d] & \mathrm{Cocart}^+ \arrow[d] \\ \mathrm{Cat}(\mathcal{B}) \arrow[r, hook, "(-)^{\sharp}"] & \mathcal{B}_{\Delta}^+ \end{tikzcd}$$

of ∞-categories. □

Example 4.2.3.6. If G is a $\mathcal{B}$-groupoid, every functor of $\mathcal{B}$-categories $p\colon \mathsf{P} \to \mathsf{G}$ is a cocartesian fibration, and the subobject of cocartesian maps in P is given by $s_0\colon \mathsf{P}_0 \hookrightarrow \mathsf{P}_1$. Hence Proposition 4.2.3.2 shows that the associated map $p^{\flat}\colon \mathsf{P}^{\flat} \to \mathsf{G}^{\flat} \simeq \mathsf{G}^{\sharp}$ is a marked cocartesian fibration and can therefore be identified with $p^{\natural}\colon \mathsf{P}^{\natural} \to \mathsf{G}^{\sharp}$.

Remark 4.2.3.7. There is an evident way to dually define a factorisation system of *marked right anodyne maps* and *marked cartesian fibrations* in $\mathcal{B}_{\Delta}^+$. Since the equivalence $\mathrm{op}\colon \Delta \simeq \Delta$ can be uniquely extended to an equivalence $\mathrm{op}\colon \Delta_+ \simeq \Delta_+$ upon specifying that op carries the factorisation $\langle 1 \rangle \to + \to \langle 0 \rangle$ to itself, we may simply define a map f in $\mathcal{B}^+_\Delta$ to be marked right anodyne if f^{op} is marked left anodyne. Explicitly, the class of marked right anodyne maps is the internal saturation of the maps in (1)–(3) in Definition 4.2.2.1 together with the map

4′) $d^0 \colon 1 \hookrightarrow (\Delta^1)^\sharp$.

A map f in $\mathcal{B}^+_\Delta$ is then a marked cartesian fibration if it is right orthogonal to the class of marked right anodyne maps, or equivalently if f^{op} is a marked cocartesian fibration. We denote by Cart^+ the associated cartesian fibration over $\mathcal{B}^+_\Delta$. Note that by similarly replacing the maps in (4) and (5) in Proposition 4.2.2.2, one obtains an analogous collection of generators for the dual case. In particular, Proposition 4.2.3.2 carries over to the case of cartesian fibrations, which implies that we also have a pullback square

$$\begin{tikzcd} \mathrm{Cart} \arrow[r, hook, "(-)^\natural"] \arrow[d] & \mathrm{Cart}^+ \arrow[d] \\ \mathrm{Cat}(\mathcal{B}) \arrow[r, hook, "(-)^\sharp"] & \mathcal{B}^+_\Delta. \end{tikzcd}$$

4.2.4. Marked left fibrations

In this section we discuss the marked analogue of the class of left fibrations between simplicial objects in $\mathcal{B}$. We will use this notion to show that left fibrations form a coreflective subcategory of cocartesian fibrations.

Definition 4.2.4.1. A map in $\mathcal{B}^+_\Delta$ is *marked initial* if it is contained in the internal saturation of the two maps $d^1 \colon 1 \hookrightarrow (\Delta^1)^\flat$ and $d^1 \colon 1 \hookrightarrow (\Delta^1)^\sharp$.

Remark 4.2.4.2. On account of the commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "d^1"] \arrow[dr, "d^1"'] & (\Delta^1)^\flat \arrow[d] \\ & (\Delta^1)^\sharp, \end{tikzcd}$$

the class of marked initial maps in $\mathcal{B}^+_\Delta$ is equivalent to the internal saturation of the two maps $d^1 \colon 1 \hookrightarrow (\Delta^1)^\flat$ and $(\Delta^1)^\flat \hookrightarrow (\Delta^1)^\sharp$.

Proposition 4.2.4.3. *Every marked left anodyne map is marked initial.*

Proof. We only need to show that the maps in (1)–(3) in Definition 4.2.2.1 are marked initial. By Lemma 4.2.2.4, we may equivalently show this for the maps in (1)–(3) in Lemma 4.2.2.4. The case of the first two maps is an immediate consequence of Lemma 2.1.1.5. As for the map $(\Delta^1)^\sharp \sqcup_{(\Delta^1)^\flat} (\Delta^1)^\sharp \to (\Delta^1)^\sharp$, this follows from the fact that $(\Delta^1)^\flat \hookrightarrow (\Delta^1)^\sharp$ is marked initial. □

Definition 4.2.4.4. A map $p\colon P \to C$ in $\mathcal{B}_\Delta^+$ is called a *marked left fibration* if it is internally right orthogonal to both $d^1 : 1 \hookrightarrow (\Delta^1)^\flat$ and $d^1 : 1 \hookrightarrow (\Delta^1)^\sharp$. We write $\mathrm{LFib}^+ \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta^+)$ for the full cartesian subfibration over $\mathcal{B}_\Delta^+$ that is spanned by the marked left fibrations.

As a consequence of Proposition 4.2.2.2, one has:

Proposition 4.2.4.5. *Every marked left fibration is marked cocartesian.* □

Lemma 4.2.4.6. *Let S be the saturation of the maps $d^1 : K^\flat \hookrightarrow (\Delta^1 \otimes K)^\flat$ for every $K \in \mathcal{B}_\Delta$ and $(\Delta^1 \otimes A)^\flat \to (\Delta^1 \otimes A)^\sharp$ for all $A \in \mathcal{B}$. Then S contains every marked initial map.*

Proof. We begin by showing that S contains the internal saturation of the inclusion $d^1 : 1 \hookrightarrow (\Delta^1)^\flat$. To that end, note that S is stable under taking products with any object $A \in \mathcal{B}$. Therefore, it suffices to show that S contains the map $d^1 : (\Delta^1)^\sharp \hookrightarrow (\Delta^1)^\flat \times (\Delta^1)^\sharp$. The pushout square

$$\begin{tikzcd}
(\Delta^1 \sqcup \Delta^1)^\flat \arrow[r] \arrow[d, "{(d^1,d^0)}"] & (\Delta^1 \sqcup \Delta^1)^\sharp \arrow[d] \\
(\Delta^1 \times \Delta^1)^\flat \arrow[r] & (\Delta^1)^\flat \times (\Delta^1)^\sharp
\end{tikzcd}$$

implies that $(\Delta^1 \times \Delta^1)^\flat \hookrightarrow (\Delta^1)^\flat \times (\Delta^1)^\sharp$ is in S. On account of the commutative square

$$\begin{tikzcd}
(\Delta^1)^\flat \arrow[r] \arrow[d, "d^1"] & (\Delta^1)^\sharp \arrow[d, "d^1"] \\
(\Delta^1 \times \Delta^1)^\flat \arrow[r] & (\Delta^1)^\flat \times (\Delta^1)^\sharp,
\end{tikzcd}$$

we therefore conclude that the right vertical map must be contained in S as well.

We still need to show that S also contains the internal saturation of the inclusion $d^1\colon 1 \hookrightarrow (\Delta^1)^\sharp$. By Lemma 4.2.2.3, the commutative square

$$\begin{tikzcd} \mathrm{sk}_1(\Delta^n)^\flat \arrow[r] \arrow[d, hook] & \mathrm{sk}_1(\Delta^n)^\sharp \arrow[d, hook] \\ (\Delta^n)^\flat \arrow[r] & (\Delta^n)^\sharp \end{tikzcd}$$

(where $\mathrm{sk}_1(\Delta^n)$ is the 1-skeleton of Δ^n, cf. Section 1.3.1) is a pushout for every $n \geq 2$. Hence S contains the map $(\Delta^n \otimes A)^\flat \to (\Delta^n \otimes A)^\sharp$ for all $n \geq 0$ and all $A \in \mathcal{B}$ and therefore also the maps $d^1\colon A^\sharp \hookrightarrow (\Delta^n \otimes A)^\sharp$. Using Lemma 2.1.1.2, we conclude that for every $K \in \mathcal{B}_\Delta$ the map $K^\sharp \hookrightarrow (\Delta^1 \otimes K)^\sharp$ is an element of S. To finish the proof, we now only need to verify that the morphism $d^1\colon (\Delta^n)^\flat \hookrightarrow (\Delta^1)^\sharp \times (\Delta^n)^\flat$ is in S too. To that end, Lemma 2.1.1.5 implies that the maps $(I^n)^\flat \hookrightarrow (\Delta^n)^\flat$ are contained in S and that we can therefore assume $n \in \{0, 1\}$. By Remark 4.2.4.2, we can further reduce this to $n = 1$. Now the commutative square

$$\begin{tikzcd} 1 \arrow[r, "d^1"] \arrow[d, "d^1"] & (\Delta^1)^\sharp \arrow[d, "\mathrm{id} \times d^1"] \\ (\Delta^1)^\flat \arrow[r, "d^1 \times \mathrm{id}"] & (\Delta^1)^\sharp \times (\Delta^1)^\flat \end{tikzcd}$$

and the first part of the proof show that the lower horizontal map is contained in S, as desired. □

Proposition 4.2.4.7. *A map $p\colon P \to C$ of marked simplicial objects in $\mathcal{B}$ is a marked left fibration if and only if*

1. *the map $p|_\Delta\colon P|_\Delta \to C|_\Delta$ is a left fibration of simplicial objects in $\mathcal{B}$;*
2. *the commutative square*

$$\begin{tikzcd} P_+ \arrow[r] \arrow[d, "p_+"] & P_1 \arrow[d, "p_1"] \\ C_+ \arrow[r] & C_1 \end{tikzcd}$$

is a pullback.

Proof. Condition (1) is equivalent to p being right orthogonal to the inclusion $d^1\colon K^\flat \hookrightarrow (\Delta^1 \times K)^\flat$ for all $K \in \mathcal{B}_\Delta$, whereas condition (2) is equivalent to p being right orthogonal to the map $(\Delta^1 \otimes A)^\flat \to (\Delta^1 \otimes A)^\sharp$ for all $A \in \mathcal{B}$. Hence the result follows from Lemma 4.2.4.6. □

Corollary 4.2.4.8. *Let C be a simplicial object in $\mathcal{B}$. Then a map $p\colon P \to C^\sharp$ is a marked left fibration if and only if the map $P_+ \to P_1$ is an equivalence and $p|_\Delta\colon P|_\Delta \to C$ is a left fibration.*

Proof. Since the map $C_+^\sharp \to C_1^\sharp$ is an equivalence, this follows immediately from Proposition 4.2.4.7. □

Remark 4.2.4.9. There is a dual version of Corollary 4.2.4.8 with $(-)^\flat$ in place of $(-)^\sharp$: a map $p\colon P \to C^\flat$ is a marked left fibration if and only if $p|_\Delta$ is a left fibration and $P_0 \to P_+$ is an equivalence. To see this, note that $P_0 \to P_+$ is an equivalence precisely if p is local with respect to $s^0\colon (\Delta^1 \otimes A)^\sharp \to A^\sharp$ for all $A \in \mathcal{B}$. As the latter map is a retraction of $d^1\colon A^\sharp \to (\Delta^1 \otimes A)^\sharp$, this condition is equivalent to P being local with respect to $d^1\colon A \to (\Delta^1 \otimes A)^\sharp$. Since $C^\flat$ is local with respect to this map as well, we conclude that $P_0 \to P_+$ is an equivalence if and only if p is right orthogonal to $d^1\colon A \to (\Delta^1 \otimes A)^\sharp$. By applying Lemma 4.2.4.6, the claim now follows.

Recall from Section 2.1.1 that the collection of left fibrations in $\mathcal{B}_\Delta$ determines a cartesian fibration $\mathrm{LFib} \to \mathcal{B}_\Delta$. Corollary 4.2.4.8 now implies:

Corollary 4.2.4.10. *The commutative square*

$$\begin{tikzcd} \mathrm{LFib} \arrow[r, hook, "(-)^\sharp"] \arrow[d] & \mathrm{LFib}^+ \arrow[d] \\ \mathcal{B}_\Delta \arrow[r, hook, "(-)^\sharp"] & \mathcal{B}_\Delta^+ \end{tikzcd}$$

is a pullback diagram of ∞-categories. □

As marked cocartesian fibrations are internally right orthogonal to the inclusion $d^1\colon 1 \hookrightarrow (\Delta^1)^\sharp$, the adjunction $(-)^\sharp \dashv (-)_\sharp\colon \mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta^+) \rightleftarrows \mathrm{Fun}(\Delta^1, \mathcal{B}_\Delta)$ restricts to an adjunction

$$(-)^\sharp \dashv (-)_\sharp\colon \mathrm{Cocart}^+ \rightleftarrows \mathrm{LFib}\,.$$

Upon restriction to the full subcategory $\mathsf{Cocart} = \mathsf{Cocart}^+ \times_{\mathcal{B}^+_\Delta} \mathcal{B}_\Delta \hookrightarrow \mathsf{Cocart}^+$, this yields:

Proposition 4.2.4.11. *The inclusion* $\mathsf{LFib} \hookrightarrow \mathsf{Cocart}$ *admits a relative right adjoint* $(-)_\sharp$ *over* $\mathcal{B}_\Delta$. □

Remark 4.2.4.12. By Corollary 4.2.3.5, the cartesian fibration $\mathsf{Cocart} \to \mathsf{Cat}(\mathcal{B})$ as defined in Section 4.1.1 arises as the pullback of $\mathsf{Cocart} \to \mathcal{B}_\Delta$ along the inclusion $\mathsf{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$. Therefore, our choice of using the same notation for both fibrations should not lead to confusion.

Remark 4.2.4.13. As usual, one can dualise the notion of marked initial maps and marked left fibrations in the evident way to obtain marked final maps and marked right fibrations. All statements about marked left fibrations carry over to analogous statements about marked right fibrations. In particular, upon defining $\mathsf{Cart} = \mathsf{Cart}^+ \times_{\mathcal{B}^+_\Delta} \mathcal{B}_\Delta$, one obtains an inclusion $\mathsf{RFib} \hookrightarrow \mathsf{Cart}$ that admits a relative right adjoint $(-)_\sharp \colon \mathsf{Cart} \to \mathsf{RFib}$ over $\mathcal{B}_\Delta$ as well.

Remark 4.2.4.14. If $p \colon \mathsf{P} \to \mathsf{C}$ is a cocartesian fibration, Proposition 4.2.3.2 implies that the adjunction counit $\mathsf{P}_\sharp \to \mathsf{P}$ is a monomorphism that identifies $\mathsf{P}_\sharp$ with the subcategory of P that is spanned by the subobject $(\mathsf{P} \downarrow_{\mathsf{C}} \mathsf{C})_0 \hookrightarrow \mathsf{P}_1$ of cocartesian maps.

4.2.5. Proper maps of marked simplicial objects

Recall from Section 2.1.4 that a map $p \colon P \to C$ between simplicial objects in $\mathcal{B}$ is *proper* if for every base change $q \colon Q \to D$ of p along some map $f \colon D \to C$ the lax square

$$\begin{tikzcd} \mathsf{LFib}(D) \arrow[d, "p^*"] & (\mathcal{B}_\Delta)_{/D} \arrow[l, "L_{/D}"'] \arrow[d, "q^*"] \arrow[dl, phantom, "\Leftarrow"] \\ \mathsf{LFib}(Q) & (\mathcal{B}_\Delta)_{/Q} \arrow[l, "L_{/Q}"'] \end{tikzcd}$$

commutes. In this section we will discuss the analogous notion of proper maps between *marked* simplicial objects.

Definition 4.2.5.1. A map $p\colon P \to C$ in $\mathcal{B}^+_\Delta$ is *marked proper* if for every cartesian square

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "q"] & P \arrow[d, "p"] \\ D \arrow[r] & C \end{tikzcd}$$

in $\mathcal{B}^+_\Delta$ the left lax square

$$\begin{tikzcd} \mathrm{Cocart}^+(D) \arrow[d, "q^*"] & (\mathcal{B}^+_\Delta)_{/D} \arrow[l, "L_{/D}"'] \arrow[d, "q^*"] \arrow[dl, phantom, "\Leftarrow"] \\ \mathrm{Cocart}^+(Q) & (\mathcal{B}^+_\Delta)_{/Q} \arrow[l, "L_{/Q}"'] \end{tikzcd}$$

(where $L_{/D}$ and $L_{/Q}$ are the localisation functors) commutes.

Note that by the same argument as in the proof of Proposition 2.1.4.5, a map $p\colon P \to C$ in $\mathcal{B}^+_\Delta$ is marked proper if and only if $p^*\colon (\mathcal{B}^+_\Delta)_{/C} \to (\mathcal{B}^+_\Delta)_{/P}$ preserves marked left anodyne morphisms.

Proposition 4.2.5.2. *For every $C, D \in \mathcal{B}^+_\Delta$, the projection $C \times D \to D$ is marked proper.*

Proof. It suffices to show that the terminal map $\pi_C\colon C \to 1$ is marked proper, which follows immediately from the fact that marked left anodyne morphisms are *internally* saturated and therefore preserved by π_C^*. □

Remark 4.2.5.3. Given any $A \in \mathcal{B}$, the forgetful functor $(\pi_A)_!\colon (\mathcal{B}_{/A})^+_\Delta \to \mathcal{B}^+_\Delta$ preserves marked proper maps. In fact, since $(\pi_A)_!$ commutes with pullbacks, this follows from the straightforward observation that this functor also preserves the property of a map to be marked left anodyne. As a consequence, Proposition 4.2.5.2 also implies that the projection $C \times_A D \to D$ is marked proper for all $C, D \in (\mathcal{B}_{/A})^+_\Delta$.

In Proposition 2.1.4.9, we showed that every right fibration between simplicial objects in $\mathcal{B}$ is proper. Our next goal is to generalise this result to marked simplicial objects. We begin with the following lemma, the proof of which we learned from Denis-Charles Cisinski [18, Proposition 5.3.5].

Lemma 4.2.5.4. *Let* C *be a* $\mathcal{B}$*-category and let*

$$\begin{tikzcd} Q \arrow[r, hook, "j"] \arrow[d, "q"] & \mathsf{P} \arrow[d, "p"] \\ I^2 \otimes \mathsf{C} \arrow[r, hook, "i"] & \Delta^2 \otimes \mathsf{C} \end{tikzcd}$$

be a pullback square in $\mathcal{B}_\Delta$ *in which* p *is a right fibration. Then* j *is contained in the internal saturation of* $I^2 \hookrightarrow \Delta^2$ *and* $E^1 \to 1$.

Proof. Let $L\colon \mathcal{B}_\Delta \to \mathrm{Cat}(\mathcal{B})$ be the localisation functor. Since P is a $\mathcal{B}$-category, we obtain a factorisation $Q \to L(Q) \to \mathsf{P}$ of j, and our task is to show that the second map is an equivalence. Note that as i is an equivalence on level 0, so is j. As a consequence, the map $L(Q) \to \mathsf{P}$ is essentially surjective. Let us show that it is fully faithful too. Consider the commutative square

$$\begin{tikzcd} \mathrm{RFib}(Q) \arrow[r, "j_!"] \arrow[d, "q_!"] & \mathrm{RFib}(\mathsf{P}) \arrow[d, "p_!"] \\ \mathrm{RFib}(I^2 \otimes \mathsf{C}) \arrow[r, "i_!"] & \mathrm{RFib}(\Delta^2 \otimes \mathsf{C}) \end{tikzcd}$$

in which each arrow is the left adjoint of the corresponding pullback functor. We claim that $j_!$ is an equivalence. To see this, note that applying the functor $p_!$ to the adjunction counit $j_! j^* \to \mathrm{id}$ recovers the adjunction counit of $i_! \dashv i^*$. Since Theorem 2.2.1.1 implies that i^* is an equivalence and as $p_!$ is conservative since p is a right fibration, we thus find that $j_! j^* \to \mathrm{id}$ is an equivalence. As a consequence, j^* is fully faithful. But $i_!$ being an equivalence and both $p_!$ and $q_!$ being conservative also implies that $j_!$ is conservative. As a result, $j_!$ must be an equivalence. Upon applying the functor $- \times A$ to the original pullback square for any $A \in \mathcal{B}$, the above argumentation also shows that the functor $g_!\colon \mathrm{RFib}(Q \times A) \to \mathrm{RFib}(\mathsf{P} \times A)$ must be an equivalence. Together with Theorem 2.2.1.1, this shows that restriction along $L(Q) \to \mathsf{P}$ induces an equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(L(Q))$ of $\mathcal{B}$-categories. In light of Corollary 3.1.3.3, this implies that $L(Q) \to \mathsf{P}$ is fully faithful, as desired. $\square$

Proposition 4.2.5.5. *Every marked right fibration is marked proper.*

Proof. To begin with, note that as in Remark 2.2.1.3, the explicit description of marked right fibrations from (the dual of) Proposition 4.2.4.7 and descent in $\mathcal{B}$ immediately imply that marked right fibrations form a local class in $\mathcal{B}^+_\Delta$. Consequently, it suffices to prove that whenever there is a pullback square

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "q"] & P \arrow[d, "p"] \\ D \arrow[r, "f"] & C \end{tikzcd}$$

in which f is one of the maps in Proposition 4.2.2.2 and p is a marked right fibration, the map g is marked left anodyne. We will first go through the maps listed in Lemma 4.2.2.4:

1. We begin with the case where f is the inclusion $(I^2 \otimes K)^\flat \hookrightarrow (\Delta^2 \otimes K)^\flat$. As every simplicial object in $\mathcal{B}$ is a colimit of $\mathcal{B}$-categories, we can assume that K is a $\mathcal{B}$-category. By Remark 4.2.4.9, the map $P|^\flat_\Delta \to P$ is an equivalence. As a consequence, to show that g marked left anodyne, it suffices to show that $g|_\Delta$ is contained in the internal saturation of $I^2 \hookrightarrow \Delta^2$ and $E^1 \to 1$, which is a consequence of Lemma 4.2.5.4.

2. The case where f is the map $(E^1 \otimes K)^\flat \to K^\flat$ follows immediately from the fact that marked left anodyne maps are stable under products.

3. Finally, we prove the case where f is the map

$$(\Delta^1 \otimes A)^\sharp \sqcup_{(\Delta^1 \otimes A)^\flat} (\Delta^1 \otimes A)^\sharp \to (\Delta^1 \otimes A)^\sharp.$$

   By Corollary 4.2.4.8, the morphism $P \to P|^\sharp_\Delta$ is an equivalence. Let us use the notation $P' = P|_\Delta$ and $p' = p|_\Delta$. Then $p \simeq (p')^\sharp$. By Lemma 4.2.2.3 and the fact that right fibrations are conservative, the map

$$(P')^\flat \to (P')^\sharp \times_{(\Delta^1 \otimes A)^\sharp} (\Delta^1 \otimes A)^\flat$$

   is an equivalence. Therefore, the map $g\colon Q \to P$ is equivalent to

$$(P')^\sharp \sqcup_{(P')^\flat} (P')^\sharp \to (P')^\sharp.$$

   By Remark 4.2.2.5, this map is marked left anodyne.

By making use of Lemma 4.2.2.6, it now suffices to prove the case where f is of the form $d^1 \colon K \hookrightarrow (\Delta^1)^\sharp \otimes K$ for an arbitrary $K \in \mathcal{B}^+_\Delta$. This is done in the same way as in the proof of Proposition 2.1.4.9. Namely, the map $g \colon Q \to P$ can be shown to arise as a retract of the marked left anodyne map $((\Delta^1)^\sharp \otimes Q) \sqcup_Q P \to (\Delta^1)^\sharp \otimes P$ and is therefore marked left anodyne itself. □

Remark 4.2.5.6. In the situation of Proposition 4.2.5.5, note that the argument in the last paragraph of its proof also works when p is only a marked cartesian fibration, as this argument only requires p to be internally right orthogonal to $d^0 \colon 1 \hookrightarrow (\Delta^1)^\sharp$. We will need this observation later for the proof of Theorem 4.4.3.1.

Remark 4.2.5.7. One can dualise the discussion in this section to *marked smooth maps*: a map f in $\mathcal{B}^+_\Delta$ is said to be marked smooth if f^{op} is marked proper. Then Proposition 4.2.5.5 dualises to the statement that marked *left* fibrations are smooth.

4.3. The $\mathcal{B}$-category of cocartesian fibrations

The goal of this chapter is to construct and study the $\mathcal{B}$-category of cocartesian fibrations over a $\mathcal{B}$-category C. It will be useful to first adopt a slightly more global perspective, i.e. to allow C to vary. Therefore, we begin in Section 4.3.1 by studying the $\mathcal{B}_\Delta$-category $\mathrm{Grpd}^+_{\mathcal{B}_\Delta}$ of marked objects, which we use in Section 4.3.2 to obtain the $\mathcal{B}$-category of cocartesian fibrations over a fixed base $\mathcal{B}$-category and to show that it is tensored and powered over $\mathrm{Cat}_{\mathcal{B}}$. Lastly, Section 4.3.3 contains a discussion of the existence of limits and colimits in this $\mathcal{B}$-category.

4.3.1. The $\mathcal{B}_\Delta$-category of marked objects

Observe that the inclusion $(-)^\sharp \colon \mathcal{B}_\Delta \hookrightarrow \mathcal{B}^+_\Delta$ can be regarded as an algebraic morphism of ∞-topoi whose right adjoint is given by $(-)_\sharp \colon \mathcal{B}^+_\Delta \to \mathcal{B}_\Delta$. Similarly, the diagonal embedding $\mathcal{B} \hookrightarrow \mathcal{B}_\Delta$ is an algebraic morphism whose right adjoint is the functor $(-)_0$ of evaluation at 0. To avoid confusion, we will denote the extension of the latter to the level of categories by $(-)|_{\mathcal{B}} \colon \mathrm{Cat}(\mathcal{B}_\Delta) \to \mathrm{Cat}(\mathcal{B})$

and its left adjoint by $(-)|^{\mathcal{B}_\Delta}$. We now obtain adjunctions

$$\mathsf{Cat}(\mathcal{B}_\Delta^+) \underset{(-)_\sharp}{\overset{(-)^\sharp}{\leftrightarrows}} \mathsf{Cat}(\mathcal{B}_\Delta) \underset{(-)|_{\mathcal{B}}}{\overset{(-)|^{\mathcal{B}_\Delta}}{\leftrightarrows}} \mathsf{Cat}(\mathcal{B}).$$

Note that on the level of Cat_∞-valued sheaves, the two colocalisation functors $(-)_\sharp$ and $(-)|_{\mathcal{B}}$ are given by precomposition with the inclusions $(-)^\sharp\colon \mathcal{B}_\Delta \hookrightarrow \mathcal{B}_\Delta^+$ and $\mathcal{B} \hookrightarrow \mathcal{B}_\Delta$, respectively.

Warning 4.3.1.1. Be aware that there are now *two distinct* inclusion of $\mathsf{Cat}(\mathcal{B})$ into $\mathsf{Cat}(\mathcal{B}_\Delta)$: the first is given by $(-)^{\mathcal{B}_\Delta}$, and the second is given by the composition

$$\mathsf{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta \hookrightarrow \mathsf{Cat}(\mathcal{B}_\Delta)$$

in which the second map is the diagonal embedding. The latter inclusion identifies $\mathsf{Cat}(\mathcal{B})$ with a class of $\mathcal{B}_\Delta$-groupoids, whereas this is not the case for the former map.

Definition 4.3.1.2. We define the *$\mathcal{B}_\Delta$-category of marked objects* to be the (large) $\mathcal{B}_\Delta$-category $\mathsf{Grpd}^+_{\mathcal{B}_\Delta} = (\mathsf{Grpd}_{\mathcal{B}_\Delta^+})_\sharp$.

Remark 4.3.1.3. Explicitly, the $\mathcal{B}_\Delta$-category $\mathsf{Grpd}^+_{\mathcal{B}_\Delta}$ of marked objects can be identified with the sheaf $(\mathcal{B}_\Delta^+)_{/(-)^\sharp}$.

Recall from Remark 1.2.6.8 that there is a pullback square

$$\begin{tikzcd} \int \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \arrow[r, hook] \arrow[d] & \mathsf{Fun}(\Delta^1, \mathcal{B}_\Delta) \arrow[d] \\ \mathcal{B} \arrow[r, hook] & \mathcal{B}_\Delta, \end{tikzcd}$$

where $\int \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \to \mathcal{B}$ is the cartesian fibration that corresponds to the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$. In other words, we may identify $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \simeq (\mathsf{Grpd}_{\mathcal{B}_\Delta})|_{\mathcal{B}}$. Similarly, the inclusion $(-)^\flat\colon \mathcal{B}_\Delta \hookrightarrow \mathcal{B}_\Delta^+$ determines an embedding

$$\mathsf{Grpd}_{\mathcal{B}_\Delta} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}_\Delta^+}|_\Delta$$

in $\mathsf{Cat}(\mathcal{B}_\Delta)$ and therefore in particular an embedding

$$\mathsf{Grpd}_{\mathcal{B}_\Delta}|_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}_\Delta^+}|_\Delta|_{\mathcal{B}}.$$

By making use of the natural equivalence $(-)|_{\Delta}|_{\mathcal{B}} \simeq (-)_{\sharp}|_{\mathcal{B}}$ from Remark 4.2.1.3, we therefore end up with a functor

$$(-)^{\flat} : \underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_{\Delta}} \to \mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}$$

of $\mathcal{B}_{\Delta}$-categories that carries a simplicial object $P \to A$ in $\mathcal{B}_{/A}$ to the marked simplicial object $P^{\flat} \to A^{\flat} \simeq A^{\sharp}$ in $\mathcal{B}_{/A}$.

To proceed, recall that by Proposition 3.2.5.10, the universe $\mathrm{Grpd}_{\mathcal{B}^{+}_{\Delta}}$ is *cartesian closed*. Using Corollary 3.1.1.9 together with Remark 1.2.5.6 and Remark 2.1.2.5, one deduces that the property of being cartesian closed is preserved by base change along geometric morphisms of ∞-topoi. Consequently, the $\mathcal{B}_{\Delta}$-category $\mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}$ is cartesian closed as well. We will denote by $\underline{\mathrm{Hom}}_{\mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}}(-,-)$ the internal hom. Combining this structure with the functor $(-)^{\flat}$ from above, we may now define bifunctors

$$-\otimes - = (-)^{\flat} \times - : \underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_{\Delta}} \times \mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}} \to \mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}$$

and

$$(-)^{(-)} = \underline{\mathrm{Hom}}_{\mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}}((-)^{\flat}, -) : (\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_{\Delta}})^{\mathrm{op}} \times \mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}} \to \mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}.$$

By construction, these two functors exhibit $\mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}$ as being *tensored* and *powered* over $\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_{\Delta}}$:

Proposition 4.3.1.4. *The two bifunctors $-\otimes-$ and $(-)^{(-)}$ fit into an equivalence*

$$\mathrm{map}_{\mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}}(-\otimes -, -) \simeq \mathrm{map}_{\mathrm{Grpd}^{+}_{\mathcal{B}_{\Delta}}}(-, (-)^{(-)})$$

of mapping bifunctors. □

Remark 4.3.1.5. Let $A \in \mathcal{B}$ be an an arbitrary object. Note that by postcomposition, the adjunction $(\pi_A)_! \dashv \pi_A^* : \mathcal{B} \rightleftarrows \mathcal{B}_{/A}$ induces adjunctions

$$(\pi_A)_! \dashv \pi_A^* : \mathcal{B}_{\Delta} \rightleftarrows (\mathcal{B}_{/A})_{\Delta}$$

and

$$(\pi_A)_! \dashv \pi_A^* : \mathcal{B}^{+}_{\Delta} \rightleftarrows (\mathcal{B}_{/A})^{+}_{\Delta}$$

that give rise to a diagram

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{B}_{\Delta}^{+}) \arrow[r, shift left, hook', "(-)^{\sharp}"'] \arrow[d, shift right, "\pi_A^*"'] & \mathrm{Cat}(\mathcal{B}_{\Delta}) \arrow[l, shift left, "(-)_{\sharp}"] \arrow[r, shift left, "(-)|_{\mathcal{B}}"'] \arrow[d, shift right, "\pi_A^*"'] & \mathrm{Cat}(\mathcal{B}) \arrow[l, shift left, hook', "(-)|^{\mathcal{B}_{\Delta}}"'] \arrow[d, shift right, "\pi_A^*"'] \\
\mathrm{Cat}((\mathcal{B}_{/A})_{\Delta}^{+}) \arrow[r, shift left, hook', "(-)^{\sharp}"'] \arrow[u, shift right, "(\pi_A)_!"'] & \mathrm{Cat}((\mathcal{B}_{/A})_{\Delta}) \arrow[l, shift left, "(-)_{\sharp}"] \arrow[r, shift left, "(-)|_{\mathcal{B}_{/A}}"'] \arrow[u, shift right, "(\pi_A)_!"'] & \mathrm{Cat}(\mathcal{B}_{/A}) \arrow[l, shift left, hook', "(-)|^{(\mathcal{B}_{/A})_{\Delta}}"'] \arrow[u, shift right, "(\pi_A)_!"']
\end{tikzcd}$$

that commutes in every direction. Consequently, Remark 1.4.1.2 implies that one obtains a commutative square

$$\begin{tikzcd}
\pi_A^*\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_{\Delta}} \arrow[r, "\pi_A^*(-)^{\flat}"] \arrow[d, "\simeq"] & \pi_A^*\,\mathrm{Grpd}_{\mathcal{B}_{\Delta}}^{+} \arrow[d, "\simeq"] \\
\underline{\mathrm{PSh}}_{\mathcal{B}_{/A}}(\Delta)|^{(\mathcal{B}_{/A})_{\Delta}} \arrow[r, "(-)^{\flat}"] & \mathrm{Grpd}_{(\mathcal{B}_{/A})_{\Delta}}^{+}.
\end{tikzcd}$$

Moreover, by Corollary 3.1.1.9, π_A^* carries the product bifunctor of $\mathrm{Grpd}_{\mathcal{B}_{\Delta}}^{+}$ to the one of $\mathrm{Grpd}_{(\mathcal{B}_{/A})_{\Delta}}^{+}$. Together with Remark 2.1.2.5, this implies that π_A^* carries the tensoring and powering bifunctors of $\mathrm{Grpd}_{\mathcal{B}_{\Delta}}^{+}$ over $\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_{\Delta}}$ to the tensoring and powering bifunctors of $\mathrm{Grpd}_{(\mathcal{B}_{/A})_{\Delta}}^{+}$ over $\underline{\mathrm{PSh}}_{\mathcal{B}_{/A}}(\Delta)|^{(\mathcal{B}_{/A})_{\Delta}}$.

4.3.2. $\mathcal{B}$-categories of cocartesian fibrations

In this section we make use of the preparations made in Section 4.3.1 to define the $\mathcal{B}$-category of cocartesian fibrations over a fixed $\mathcal{B}$-category C and to show that it is both tensored and powered over $\mathrm{Cat}_{\mathcal{B}}$. To that end, recall from Warning 4.3.1.1, that we may regard C as a $\mathcal{B}_{\Delta}$-*groupoid* via combining the inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_{\Delta}$ with the diagonal embedding $\mathcal{B}_{\Delta} \hookrightarrow \mathrm{Cat}(\mathcal{B}_{\Delta})$. When regarded as such, consider the associated large $\mathcal{B}$-category $\underline{\mathrm{Fun}}_{\mathcal{B}_{\Delta}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}_{\Delta}}^{+})|_{\mathcal{B}}$, that is explicitly given by the $\widehat{\mathrm{Cat}}_{\infty}$-valued sheaf $(\mathcal{B}_{\Delta}^{+})_{/-\times \mathsf{C}^{\sharp}}$. We may now define:

Definition 4.3.2.1. For every $\mathcal{B}$-category C, the large $\mathcal{B}$-category $\mathrm{Cocart}_{\mathsf{C}}$ of cocartesian fibrations over C is the full subcategory of $\underline{\mathrm{Fun}}_{\mathcal{B}_{\Delta}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}_{\Delta}}^{+})|_{\mathcal{B}}$ that is spanned by the marked cocartesian fibrations $P \to A \times \mathsf{C}^{\sharp}$ for each $A \in \mathcal{B}$.

Remark 4.3.2.2. Leb $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ be a cover in $\mathcal{B}$ and $p\colon P \to A \times \mathsf{C}^\sharp$ be a map in $\mathcal{B}^+_\Delta$. Then p is a marked cocartesian fibration if and only if each $s_i^*(p)\colon s_i^*(P) \to A_i \times \mathsf{C}^\sharp$ is one. In fact, this is certainly a necessary condition, so it suffices to verify that it is sufficient as well. To that end, let $p'j\colon P \to P' \to A\times\mathsf{C}^\sharp$ be the unique factorisation of p into a marked left anodyne and a marked cocartesian fibration. We need to show that j is an equivalence. Since $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ is a cover, this is the case already when each $s_i^*(j)$ is an equivalence. But as the maps $A_i \times \mathsf{C}^\sharp \to A \times \mathsf{C}^\sharp$ are marked right fibrations and therefore in particular marked proper (Proposition 4.2.5.5), each $s_i^*(j)$ is both a marked cocartesian fibration and marked left anodyne and must therefore be an equivalence. Hence the claim follows. As a consequence, we conclude that an object in $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})_{\mathcal{B}}$ in context $A \in \mathcal{B}$ is contained in $\mathsf{Cocart}_{\mathsf{C}}$ if and only if it encodes a marked cocartesian fibration over $A \times \mathsf{C}^\sharp$. Furthermore, this observation implies that the $\widehat{\mathrm{Cat}}_\infty$-valued sheaf associated with $\mathsf{Cocart}_{\mathsf{C}}$ is given by by $\mathrm{Cocart}^+(- \times \mathsf{C}^\sharp)$ and therefore by $\mathrm{Cocart}(- \times \mathsf{C})$, using Corollary 4.2.3.5.

Remark 4.3.2.3. Fix an object $A \in \mathcal{B}$. Since the terminal map $\pi_A\colon A \to 1$ is a marked cocartesian fibration, a map $p\colon P \to C$ in $(\mathcal{B}_{/A})^+_\Delta$ is marked cocartesian if and only if $(\pi_A)_!(p)$ is marked cocartesian in $\mathcal{B}^+_\Delta$. Therefore, the equivalence

$$\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}} \simeq \underline{\mathsf{Fun}}_{(\mathcal{B}_{/A})_\Delta}(\pi_A^*\mathsf{C}, \mathsf{Grpd}^+_{(\mathcal{B}_{/A})_\Delta})|_{\mathcal{B}_{/A}}$$

(see Remark 4.3.1.5 and Proposition 1.2.5.4) restricts to an equivalence of large $\mathcal{B}$-categories $\pi_A^* \mathsf{Cocart}_{\mathsf{C}} \simeq \mathsf{Cocart}_{\pi_A^*\mathsf{C}}$.

Our next goal is to show that for every $\mathcal{B}$-category C, the large $\mathcal{B}$-category $\mathsf{Cocart}_{\mathsf{C}}$ is both tensored and powered over $\mathsf{Cat}_{\mathcal{B}}$. To that end, note it follows from Remark 2.1.2.5 that by applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, -)|_{\mathcal{B}}$ to the two maps from Proposition 4.3.1.4 and by precomposing the result (in the first variable) with the map

$$\mathrm{diag}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(1, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_\Delta})|_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)|^{\mathcal{B}_\Delta})|_{\mathcal{B}},$$

one ends up with bifunctors

$$-\otimes-\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \times \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$$

and

$$(-)^{(-)}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)^{\mathrm{op}} \times \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$$

that are natural in C and that fit into an equivalence

$$\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}}(- \otimes -, -) \simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}}(-, (-)^{(-)}).$$

In other words, they exhibit $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ as being both tensored and powered over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$.

Remark 4.3.2.4. By combining Remark 4.3.1.5 with Proposition 1.2.5.4, one finds that for every object $A \in \mathcal{B}$ and every $\mathcal{B}$-category C, the associated base change functor π_A^* carries the tensoring and powering of $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ to the tensoring and powering of $\underline{\mathsf{Fun}}_{(\mathcal{B}_{/A})_\Delta}(\pi_A^*\mathsf{C}, \mathsf{Grpd}^+_{(\mathcal{B}_{/A})_\Delta})|_{\mathcal{B}}$ over $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\Delta)$.

Remark 4.3.2.5. Using that the global sections functor $\Gamma_{\mathcal{B}}\colon \mathsf{Cat}(\widehat{\mathcal{B}}) \to \widehat{\mathsf{Cat}}_\infty$ carries the product bifunctor of a (large) $\mathcal{B}$-category to that of its underlying ∞-category (Example 3.2.1.13), we deduce that on global sections, the tensoring of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ recovers the composition

$$\begin{aligned} \mathcal{B}_\Delta \times (\mathcal{B}^+_\Delta)_{/\mathsf{C}^\sharp} &\xrightarrow{(-)^\flat \times \mathrm{id}} \mathcal{B}^+_\Delta \times (\mathcal{B}^+_\Delta)_{/\mathsf{C}^\sharp} \\ &\xrightarrow{\pi^*_{\mathsf{C}^\sharp} \times \mathrm{id}} (\mathcal{B}^+_\Delta)_{/\mathsf{C}^\sharp} \times (\mathcal{B}^+_\Delta)_{/\mathsf{C}^\sharp} \\ &\xrightarrow{- \times -} (\mathcal{B}^+_\Delta)_{/\mathsf{C}^\sharp} \times (\mathcal{B}^+_\Delta)_{/\mathsf{C}^\sharp}. \end{aligned}$$

Dually, this means (using Corollary 2.2.2.8 and Remark 2.1.2.5) that on global sections, the powering of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ is given by the functor that carries $K \in \mathcal{B}_\Delta$ and a map $P \to \mathsf{C}^\sharp$ in $\mathcal{B}^+_\Delta$ to the pullback

$$\begin{tikzcd} P^K \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(K^\flat, P) \arrow[d] \\ \mathsf{C}^\sharp \arrow[r, "\mathrm{diag}"] & \underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(K^\flat, \mathsf{C}^\sharp). \end{tikzcd}$$

Finally, by combining Remark 4.3.2.3, Remark 4.3.2.4 and Remark 4.3.2.5, it is clear that the tensoring and powering of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ restricts to a tensoring and powering of $\mathsf{Cocart}_{\mathsf{C}}$ over $\mathsf{Cat}_{\mathcal{B}}$, so that we obtain:

Corollary 4.3.2.6. *For any* $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$*, the large* $\mathcal{B}$*-category* $\mathsf{Cocart}_{\mathsf{C}}$ *is both tensored and powered over the* $\mathcal{B}$*-category* $\mathsf{Cat}_{\mathcal{B}}$*, and both tensoring and powering is natural in* C*.* □

Remark 4.3.2.7. Using Remark 4.3.2.3 and Remark 4.3.2.4, one deduces that for every $A \in \mathcal{B}$, the base change functor π_A^* carries the tensoring and powering of $\mathsf{Cocart}_{\mathsf{C}}$ over $\mathsf{Cat}_{\mathcal{B}}$ to the tensoring and powering of $\mathsf{Cocart}_{\pi_A^*\mathsf{C}}$ over $\mathsf{Cat}_{\mathcal{B}_{/A}}$.

Recall from Proposition 4.2.4.11 that the inclusion $\mathrm{LFib} \hookrightarrow \mathrm{Cocart}$ admits a relative right adjoint $(-)_\sharp$ over $\mathcal{B}_\Delta$. In the language established in this section, this result can be rephrased as follows (cf. Corollary 3.1.2.4):

Proposition 4.3.2.8. *For every* $\mathcal{B}$*-category* C*, the inclusion* $\mathsf{LFib}_{\mathsf{C}} \hookrightarrow \mathsf{Cocart}_{\mathsf{C}}$ *admits a right adjoint* $(-)_\sharp$*. Moreover, this right adjoint is natural in* C*.* □

Remark 4.3.2.9. Dually, we may define the large $\mathcal{B}$-category $\mathsf{Cart}_{\mathsf{C}}$ of *cartesian* fibrations over $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$ as the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ that is spanned by the marked cartesian fibrations over $A \times \mathsf{C}^\sharp$ for every $A \in \mathcal{B}$. Using this definition, it is immediate that all of the results discussed in this section dualise, so that we also obtain a tensoring and powering of $\mathsf{Cart}_{\mathsf{C}}$ over $\mathsf{Cat}_{\mathcal{B}}$ that is natural in C, and that we moreover have a coreflection of the inclusion $\mathsf{RFib}_{\mathsf{C}} \hookrightarrow \mathsf{Cart}_{\mathsf{C}}$ that is natural in C as well.

4.3.3. Limits and colimits of cocartesian fibrations

In this section we will study limits and colimits in the $\mathcal{B}$-categories of cocartesian fibrations that we defined in the previous section and the behaviour of these under base change. We begin with the following observation:

Proposition 4.3.3.1. *For every* $\mathcal{B}$*-category* C*, the large* $\mathcal{B}$*-category* $\mathsf{Cocart}_{\mathsf{C}}$ *is a reflective subcategory of* $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$.

Proof. In light of Proposition 3.1.2.9, this is a straightforward consequence of the fact that for every $A \in \mathcal{B}$ the inclusion $\mathrm{Cocart}^+(A \times \mathsf{C}^\sharp) \hookrightarrow (\mathcal{B}^+_\Delta)_{/A\times\mathsf{C}^\sharp}$

admits a left adjoint and that for every map $s\colon B \to A$ in $\mathcal{B}$ the induced map $s \times \mathrm{id}\colon B \times \mathsf{C}^\sharp \to A \times \mathsf{C}^\sharp$ is marked proper. □

To proceed, note that using the characterisation of (co)completeness in Corollary 3.5.4.4, it is immediate that the large $\mathcal{B}_\Delta$-category $\mathsf{Grpd}^+_{\mathcal{B}_\Delta}$ is complete and cocomplete, hence so is $\underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$ by the same result and Proposition 3.3.2.12. Together with Proposition 3.3.2.11, this shows:

Proposition 4.3.3.2. *For every $\mathsf{C} \in \mathsf{Cat}(\mathcal{B})$, the large $\mathcal{B}$-category $\mathsf{Cocart}_{\mathsf{C}}$ is complete and cocomplete.* □

Furthermore, we find:

Proposition 4.3.3.3. *For every functor $f\colon \mathsf{C} \to \mathsf{D}$ in $\mathsf{Cat}(\mathcal{B})$, the associated pullback functor $f^*\colon \mathsf{Cocart}_{\mathsf{D}} \to \mathsf{Cocart}_{\mathsf{C}}$ admits a left adjoint $f_!$. In particular, f^* is continuous. If the map $f^\sharp\colon \mathsf{C}^\sharp \to \mathsf{D}^\sharp$ is moreover marked proper, then f^* also admits a right adjoint f_*, so that f^* is also continuous in this case.*

Proof. Since $\mathsf{Grpd}^+_{\mathcal{B}_\Delta}$ is complete and cocomplete, the functor

$$f^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{D}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta}) \to \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})$$

admits both a left adjoint $f_!$ and a right adjoint f_* (Corollary 3.4.3.7 and its dual). As base change along geometric morphisms preserves adjunctions (Corollary 3.1.1.9), this implies that the functor

$$f^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{D}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$$

also admits a left adjoint $f_!$ and a right adjoint f_*. Together with Proposition 4.3.3.1, this immediately implies that $f^*\colon \mathsf{Cocart}_{\mathsf{D}} \to \mathsf{Cocart}_{\mathsf{C}}$ has a left adjoint $f_!$. If $f^\sharp$ is furthermore marked proper, then the right adjoint

$$f_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_\Delta}(\mathsf{D}, \mathsf{Grpd}^+_{\mathcal{B}_\Delta})|_{\mathcal{B}}$$

restricts to a right adjoint $f_*\colon \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{Cocart}_{\mathsf{D}}$: in fact, by making use of Remark 4.3.2.3, this follows once we show that for every marked cocartesian fibration $p\colon P \to \mathsf{C}^\sharp$, the induced map $f_*(p)\colon f_*P \to \mathsf{D}^\sharp$ is a marked cocartesian fibration as well. Now it follows straightforwardly from the definitions that $f_*(p)$

is a cocartesian fibration if and only if it is local (when viewed as an object in $(\mathcal{B}^+_\Delta)_{/\mathsf{D}^\sharp}$) with respect to those maps in $(\mathcal{B}^+_\Delta)_{/\mathsf{D}^\sharp}$ whose underlying map in $\mathcal{B}^+_\Delta$ is marked left anodyne. Consequently, the claim follows from the fact that f^* preserves marked left anodyne maps since $f^\sharp$ is marked proper. □

Remark 4.3.3.4. If C is a $\mathcal{B}$-category, the functor

$$(\pi_{\mathsf{C}})_! \colon \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{Cocart}_1 \simeq \mathsf{Cat}_{\mathcal{B}}$$

is explicitly given by sending a cocartesian fibration $p \colon \mathsf{P} \to A \times \mathsf{C}$ to the $\mathcal{B}_{/A}$-category $\mathsf{P}_\sharp^{-1}\mathsf{P}$, i.e. to the pushout

$$\begin{tikzcd} \mathsf{P}_\sharp \arrow[r] \arrow[d, hook] & \mathsf{P}_\sharp^{\mathrm{gpd}} \arrow[d] \\ \mathsf{P} \arrow[r] & \mathsf{P}_\sharp^{-1}\mathsf{P} \end{tikzcd}$$

in $\mathsf{Cat}(\mathcal{B}_{/A})$. To see this, Remark 4.3.2.3 implies that we may assume without loss of generality $A \simeq 1$. Consider the pushout square

$$\begin{tikzcd} (\mathsf{P}_\sharp^\natural)^\sharp \arrow[r] \arrow[d, hook] & ((\mathsf{P}_\sharp^\natural)^{\mathrm{gpd}})^\sharp \arrow[d] \\ \mathsf{P}^\natural \arrow[r] & Z \end{tikzcd}$$

in $\mathcal{B}^+_\Delta$. Note that the span in the upper left corner of the first square is obtained by applying the functor $(-)|_\Delta$ to the span in the upper left corner of the second square. We claim that $Z|_\Delta^\flat \to Z$ is an equivalence. In fact, since the object $Z_+ \in \mathcal{B}$ is computed as the pushout

$$\begin{tikzcd} P_+ \arrow[r] \arrow[d, "\mathrm{id}"] & (\mathsf{P}_\sharp^\natural)^{\mathrm{gpd}} \arrow[d] \\ P_+ \arrow[r] & Z_+, \end{tikzcd}$$

the map $(\mathsf{P}_\sharp^\natural)^{\mathrm{gpd}} \to Z_+$ must be an equivalence. But since by the same argument the map $(\mathsf{P}_\sharp^\natural)^{\mathrm{gpd}} \to Z_0$ is an equivalence as well, the claim holds. Now by construction, the map $\mathsf{P}^\natural \to Z$ is contained in the internal saturation of $s^0 \colon (\Delta^1)^\sharp \to 1$

and is therefore in particular marked left anodyne. Moreover, if we denote by $\mathsf{Z} \in \mathsf{Cat}(\mathcal{B})$ the image of $Z|_\Delta$ along the localisation functor $L\colon \mathcal{B}_\Delta \to \mathsf{Cat}(\mathcal{B})$, the associated localisation map $Z|_\Delta \to \mathsf{Z}$ induces a marked left anodyne map $Z \to \mathsf{Z}^\flat$. In total, we therefore obtain a marked left anodyne map $\mathsf{P}^\natural \to \mathsf{Z}^\flat$, which implies that the image of p along $(\pi_{\mathsf{C}})_!$ is given by Z. As the $\mathcal{B}$-category $\mathsf{P}_\sharp^{-1}\mathsf{P}$ is precisely computed by applying the functor $L(-)|_\Delta$ to the pushout square that defines Z, we obtain the desired equivalence $\mathsf{P}_\sharp^{-1}\mathsf{P} \simeq \mathsf{Z}$.

Remark 4.3.3.5. If C is an arbitrary $\mathcal{B}$-category, Proposition 4.3.3.3 and the fact that $\pi_{\mathsf{C}^\sharp}\colon \mathsf{C}^\sharp \to 1$ is marked proper imply that the functor

$$(\pi_{\mathsf{C}})^*\colon \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cocart}_1 \to \mathsf{Cocart}_{\mathsf{C}}$$

admits a right adjoint $(\pi_{\mathsf{C}})_*$. On global sections, this functor is given by restricting the geometric morphism

$$(\pi_{\mathsf{C}^\sharp})_*\colon (\mathcal{B}_\Delta^+)_{/\mathsf{C}^\sharp} \to \mathcal{B}_\Delta^+$$

to marked cocartesian fibrations. Recall that this map is equivalently given by the functor $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{C}^\sharp, -)_{/\mathsf{C}^\sharp}$ that sends a map $p\colon P \to \mathsf{C}^\sharp$ to the pullback

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{C}^\sharp, P)_{/\mathsf{C}^\sharp} \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{C}^\sharp, P) \arrow[d, "p_*"] \\
1 \arrow[r, "\mathrm{id}_{\mathsf{C}^\sharp}"] & \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{C}^\sharp, \mathsf{C}^\sharp).
\end{tikzcd}$$

We thus conclude that the functor $(\pi_{\mathsf{C}})_*$ carries a cocartesian fibration $p\colon \mathsf{P} \to \mathsf{C}$ to the associated $\mathcal{B}$-category $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{C}^\sharp, \mathsf{P}^\natural)_{/\mathsf{C}^\sharp}|_\Delta$ of *cocartesian sections* of p.

Remark 4.3.3.6. By combining Proposition 4.3.3.3 with Proposition 4.2.5.5, one deduces in particular that the base change functor $p^*\colon \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{Cocart}_{\mathsf{P}}$ along any right fibration $p\colon \mathsf{P} \to \mathsf{C}$ in $\mathsf{Cat}(\mathcal{B})$ admits a right adjoint p_*.

Remark 4.3.3.7. Note that by taking opposite $\mathcal{B}$-categories, one obtains an equivalence $(-)^{\mathrm{op}}\colon \mathsf{Cart}_{\mathsf{C}} \simeq \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}}$ that is natural in $\mathsf{C} \in \mathsf{Cat}(\mathcal{B})$. Therefore, the results that have been established in this section can also be dualised to cartesian fibrations.

4.4. Straightening and unstraightening

The main goal of this section is to construct an equivalence

$$\mathsf{Cocart}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$$

that is natural in $\mathsf{C} \in \mathsf{Cat}(\mathcal{B})$. For ∞-categories, such an equivalence has first been established by Lurie in [49], who referred to the functor from cocartesian fibrations to Cat_{∞}-valued functors as *straightening* and to its inverse as *unstraightening*. We will make use of the same terminology here, although our constructions will be substantially different from those in Lurie's approach. We construct a straightening functor in Section 2.2 and its left adjoint in Section 4.4.2. In Section 4.4.3, we prove that this adjunction defines an equivalence of $\mathcal{B}$-categories. As a consequence, one obtains a *universal* cocartesian fibration over $\mathsf{Cat}_{\mathcal{B}}$ which is studied in Section 4.4.4. We close this section by giving an explicit description of the straightening functor in the special case where the base $\mathcal{B}$-category is the interval Δ^1 in Section 4.4.5.

4.4.1. The straightening functor

Recall from Proposition 3.1.2.13 that $\mathsf{Cat}_{\mathcal{B}}$ is a reflective subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$. Let us denote by $\Delta^{\bullet} \colon \Delta \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ the Yoneda embedding. For any $n \geq 0$, the presheaf represented by $\langle n \rangle \colon 1 \to \Delta$ is given by $\Delta^n \in \mathcal{B}_{\Delta}$ (see Lemma 4.4.4.6 below) and therefore by a $\mathcal{B}$-category. Since Δ is a constant $\mathcal{B}$-category and therefore generated by the collection of global objects $\langle n \rangle \colon 1 \to \Delta$, this shows that the Yoneda embedding defines a functor $\Delta^{\bullet} \colon \Delta \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$. Therefore, given any $\mathcal{B}$-category C, Corollary 4.3.2.6 implies that $\mathsf{Cocart}_{\mathsf{C}}$ is both tensored and powered over Δ, and that both bifunctors are natural in C. We now define the *straightening* functor St_{C} as the composition

$$\begin{aligned} \mathsf{St}_{\mathsf{C}} \colon \ \mathsf{Cocart}_{\mathsf{C}} &\to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{Cocart}_{\mathsf{C}}) \\ &\to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{LFib}_{\mathsf{C}}) \\ &\simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)) \end{aligned}$$

in which the first map is the transpose of the powering bifunctor, the second map is given by postcomposition with the coreflection $(-)_{\sharp}$ from Proposition 4.3.2.8

and the last map is determined by the straightening equivalence for left fibrations (Theorem 2.2.1.1). Note that as powering, the coreflection $(-)_\sharp$ and the straightening equivalence for left fibrations all are natural in C, we deduce that St_{C} is natural in C as well.

Remark 4.4.1.1. In the case $\mathcal{B} = \mathrm{Ani}$, the above definition of the straightening functor has previously appeared in lecture notes by Hinich [36].

Remark 4.4.1.2. For the special case $\mathsf{C} = 1$, the straightening functor is given by the inclusion

$$\mathsf{Cocart}_1 \simeq \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta).$$

In fact, since in this case the tensoring is by construction simply given by the product bifunctor $- \times - \colon \mathsf{Cat}_{\mathcal{B}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$, the powering functor is the internal hom of $\mathsf{Cat}_{\mathcal{B}}$ from Proposition 3.2.6.3. Since the coreflection $(-)_\sharp \colon \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{LFib}_{\mathsf{C}}$ reduces to the core $\mathcal{B}$-groupoid functor when $\mathsf{C} = 1$, Corollary 3.2.6.5 implies that the straightening functor $\mathrm{St}_1 \colon \mathsf{Cat}_{\mathcal{B}} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ is transpose to

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\Delta^\bullet, -) \colon \Delta^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}\,.$$

and is therefore given by the inclusion $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ on account of Yoneda's lemma.

Remark 4.4.1.3. For every $A \in \mathcal{B}$, Remark 4.3.2.7 and Remark 2.2.1.7 together with Proposition 1.2.5.4 (and Remark 1.2.5.5) imply that the étale base change $\pi_A^* \mathrm{St}_{\mathsf{C}}$ can be identified with $\mathrm{St}_{\pi_A^* \mathsf{C}}$.

Proposition 4.4.1.4. *For every $\mathcal{B}$-category C, the straightening functor St_{C} takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$.*

Proof. Let $p \colon \mathsf{P} \to A \times \mathsf{C}$ be a cocartesian fibration in context $A \in \mathcal{B}$. We need to show that the functor $\mathrm{St}_{\mathsf{C}}(p) \colon A \times \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ takes values in $\mathsf{Cat}_{\mathcal{B}}$. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume without loss of generality $A \simeq 1$, cf. Remark 4.4.1.3. We can argue object-wise in C, i.e. it suffices to show that for every object $c \colon A \to \mathsf{C}$ the simplicial object $\mathrm{St}_{\mathsf{C}}(p)(c) \in (\mathcal{B}_{/A})_\Delta$ is a $\mathcal{B}_{/A}$-category. Again, we can assume $A \simeq 1$. In light of the naturality of the straightening functor, this argument implies that we may reduce to the case $\mathsf{C} \simeq 1$. In this case, Remark 4.4.1.2 shows that the straightening functor is simply the inclusion $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$, hence the claim follows. □

We conclude this section with the observation that for every $\mathcal{B}$-category C, restricting the straightening functor St_{C} along the inclusion $\mathsf{LFib}_{\mathsf{C}} \hookrightarrow \mathsf{Cocart}_{\mathsf{C}}$ recovers the equivalence $\mathsf{LFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}})$ from Theorem 2.2.1.1. More precisely, one has:

Proposition 4.4.1.5. *There is a commutative square*

$$\begin{tikzcd}
\mathsf{LFib}_{\mathsf{C}} \arrow[r, "\simeq"] \arrow[d, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}) \arrow[d, hook, "\iota_*"] \\
\mathsf{Cocart}_{\mathsf{C}} \arrow[r, "\mathrm{St}_{\mathsf{C}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))
\end{tikzcd}$$

in which $\iota\colon \mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ *denotes the diagonal embedding.*

Proof. We need to show that for every left fibration $p\colon \mathsf{P} \to A \times \mathsf{C}$ in context $A \in \mathcal{B}$ the straightening $\mathrm{St}_{\mathsf{C}}(p)\colon A \times \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ factors through $\iota\colon \mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$. As in the proof of Proposition 4.4.1.4, the fact that the straightening functor is natural in C and the fact that we may work objectwise in C allows us to reduce to the case where $\mathsf{C} \simeq 1 \simeq A$, in which case the result also immediately follows from Remark 4.4.1.2. □

Remark 4.4.1.6. Dually, one can construct a straightening functor

$$\mathrm{St}_{\mathsf{C}}\colon \mathsf{Cart}_{\mathsf{C}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{RFib}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))$$

for every $\mathcal{B}$-category C, using Remark 4.3.2.9. Note that by the explicit construction of the tensoring bifunctors, the equivalence $(-)^{\mathrm{op}}\colon \mathsf{Cart}_{\mathsf{C}} \simeq \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}}$ from Remark 4.3.2.9 fits into a commutative square

$$\begin{tikzcd}
\Delta \times \mathsf{Cart}_{\mathsf{C}} \arrow[r, "\Delta^{\bullet} \otimes -"] \arrow[d, "\mathrm{op} \times (-)^{\mathrm{op}}"] & \mathsf{Cart}_{\mathsf{C}} \arrow[d, "(-)^{\mathrm{op}}"] \\
\Delta \times \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}} \arrow[r, "\Delta^{\bullet} \otimes -"] & \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}} .
\end{tikzcd}$$

By making use of the adjunction between tensoring and powering, this square in turn induces a commutative diagram

$$\begin{tikzcd} \mathsf{Cart}_{\mathsf{C}} \arrow[r, "(-)^{\Delta^\bullet}"] \arrow[d, "(-)^{\mathrm{op}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{Cart}_{\mathsf{C}}) \arrow[d, "(-)^{\mathrm{op}}_*"] \\ \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}} \arrow[r, "(-)^{\Delta^\bullet, \mathrm{op}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}}). \end{tikzcd}$$

Since the coreflection of (co)cartesian fibrations into right (left) fibrations evidently commutes with taking opposite $\mathcal{B}$-categories, we conclude that there is a commutative square

$$\begin{tikzcd} \mathsf{Cart}_{\mathsf{C}} \arrow[r, "\mathsf{St}_{\mathsf{C}}"] \arrow[d, "(-)^{\mathrm{op}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{B}}) \arrow[d, "(-)^{\mathrm{op}}_*"] \\ \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}} \arrow[r, "\mathsf{St}_{\mathsf{C}^{\mathrm{op}}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{B}}). \end{tikzcd}$$

4.4.2. The unstraightening functor

In this section we construct a left adjoint to the straightening functor St_{C} for every $\mathcal{B}$-category C. To that end, given $\mathsf{C} \in \mathsf{Cat}(\mathcal{B})$, note that restricting the tensoring bifunctor $- \otimes - \colon \mathsf{Cat}_{\mathcal{B}} \times \mathsf{Cocart}_{\mathsf{C}} \to \mathsf{Cocart}_{\mathsf{C}}$ along the inclusion

$$\Delta^\bullet \times h_{\mathsf{C}^{\mathrm{op}}} \colon \Delta \times \mathsf{C}^{\mathrm{op}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}} \times \mathsf{LFib}_{\mathsf{C}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}} \times \mathsf{Cocart}_{\mathsf{C}}$$

that is induced by the Yoneda embedding on either factor gives rise to a functor

$$\Delta^\bullet \otimes h_{\mathsf{C}^{\mathrm{op}}}(-) \colon \Delta \times \mathsf{C}^{\mathrm{op}} \to \mathsf{Cocart}_{\mathsf{C}}\,.$$

In light of Proposition 4.3.3.2 and the universal property of presheaf $\mathcal{B}$-categories (Theorem 3.5.1.1), we may now define:

Definition 4.4.2.1. For every $\mathcal{B}$-category C, the *unstraightening* functor

$$\mathsf{Un}_{\mathsf{C}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)) \to \mathsf{Cocart}_{\mathsf{C}}$$

is defined as the left Kan extension of $\Delta^\bullet \otimes h_{\mathsf{C}^{\mathrm{op}}}(-) \colon \Delta \times \mathsf{C}^{\mathrm{op}} \to \mathsf{Cocart}_{\mathsf{C}}$ along the Yoneda embedding

$$h_{\Delta \times \mathsf{C}^{\mathrm{op}}} \colon \Delta \times \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta \times \mathsf{C}^{\mathrm{op}}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)).$$

Our next goal is to show that the unstraightening functor Un_{C} is left adjoint to the straightening functor St_{C}. To that end, recall from Section 2.3.1 that a large $\mathcal{B}$-category C is locally small if the left fibration $\mathrm{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is small.

Proposition 4.4.2.2. *For every* $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$*, the large* $\mathcal{B}$*-category* $\mathrm{Cocart}_{\mathsf{C}}$ *is locally small.*

Proof. The large $\mathcal{B}_\Delta^+$-category $\mathrm{Grpd}_{\mathcal{B}_\Delta^+}$ is locally small (see Example 2.3.1.4), hence so is $\mathrm{Grpd}_{\mathcal{B}_\Delta}^+$ by Remark 2.3.1.2 By the same remark and Proposition 2.3.1.7, this implies that the $\mathcal{B}$-category $\underline{\mathrm{Fun}}_{\mathcal{B}_\Delta}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}_\Delta}^+)|_{\mathcal{B}}$ is locally small as well. Being a full subcategory of the latter, this implies that $\mathrm{Cocart}_{\mathsf{C}}$ must be locally small, as desired. □

As a result of Proposition 4.4.2.2, we deduce from Remark 3.5.1.4 that the unstraightening functor Un_{C} admits a right adjoint r. The computation

$$\begin{aligned} r &\simeq \mathrm{map}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta \times \mathsf{C}^{\mathrm{op}})}(h_{\Delta \times \mathsf{C}^{\mathrm{op}}}(-,-), r(-)) \\ &\simeq \mathrm{map}_{\mathrm{Cocart}_{\mathsf{C}}}(\Delta^\bullet \otimes h_{\mathsf{C}^{\mathrm{op}}}(-), -) \\ &\simeq \mathrm{map}_{\mathrm{Cocart}_{\mathsf{C}}}(h_{\mathsf{C}^{\mathrm{op}}}(-), (-)^{\Delta^\bullet}) \\ &\simeq \mathrm{map}_{\mathrm{LFib}_{\mathsf{C}}}(h_{\mathsf{C}^{\mathrm{op}}}(-), (-)^{\Delta^\bullet}_{\sharp}) \\ &\simeq (-)^{\Delta^\bullet}_{\sharp} \end{aligned}$$

now shows:

Proposition 4.4.2.3. *The unstraightening functor* Un_{C} *is left adjoint to the straightening functor* St_{C}*.* □

As a direct consequence of Proposition 4.4.1.4 and Proposition 4.4.2.3, one obtains:

Corollary 4.4.2.4. *The straightening and unstraightening functors restrict to an adjunction*

$$(\mathrm{Un}_{\mathsf{C}} \dashv \mathrm{St}_{\mathsf{C}}) \colon \mathrm{Cocart}_{\mathsf{C}} \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathrm{Cat}_{\mathcal{B}})$$

for every $\mathcal{B}$*-category* C*.* □

Remark 4.4.2.5. Since adjoints are unique and by Remark 4.4.1.3, we find that for any $A \in \mathcal{B}$, the base change $\pi_A^* \operatorname{Un}_{\mathsf{C}}$ can be identified with $\operatorname{Un}_{\pi_A^* \mathsf{C}}$.

In general, we have no explicit way to compute the unstraightening $\operatorname{Un}_{\mathsf{C}}(f)$ of a functor $f\colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ unless f is contained in the image of the Yoneda embedding $h_{\Delta \times \mathsf{C}^{\mathrm{op}}}$, in which case the unstraightening is simply given by the tensoring in $\mathsf{Cocart}_{\mathsf{C}}$. We conclude this section by explaining how this description extends to a slightly larger class of functors.

Lemma 4.4.2.6. *Let* C *and* D *be* $\mathcal{B}$*-categories. Then there is a commutative square*

$$\begin{tikzcd}
\mathsf{C} \times \mathsf{D} \arrow[rr, "h_{\mathsf{C}\times\mathsf{D}}"] \arrow[d, "h_{\mathsf{C}} \times h_{\mathsf{D}}"] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}) \\
\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r, "\mathrm{pr}_0^* \times \mathrm{pr}_1^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}) \arrow[ur, "- \times -"'] &
\end{tikzcd}$$

in which $\mathrm{pr}_0 \colon \mathsf{C} \times \mathsf{D} \to \mathsf{C}$ *and* $\mathrm{pr}_1 \colon \mathsf{C} \times \mathsf{D} \to \mathsf{D}$ *are the two projections.*

Proof. Since $\mathsf{Tw}(-)$ commutes with products, we have a pullback square

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{C} \times \mathsf{D}) \arrow[r] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})_{1/} \times (\mathsf{Grpd}_{\mathcal{B}})_{1/} \arrow[d] \\
(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}) \times (\mathsf{D}^{\mathrm{op}} \times \mathsf{D}) \arrow[r, "\mathrm{map}_{\mathsf{C}} \times \mathrm{map}_{\mathsf{D}}"] & \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}} .
\end{tikzcd}$$

In light of the equivalence

$$(\mathsf{Grpd}_{\mathcal{B}})_{1/} \times (\mathsf{Grpd}_{\mathcal{B}})_{1/} \simeq (\mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}})_{(1,1)/},$$

the left fibration on the right is corepresented by $(1,1) \colon 1 \to \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}}$. As the latter is a final object, we conclude (using Corollary 3.2.5.6) that the copresheaf that classifies this left fibration is the product functor

$$- \times - \colon \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}} .$$

We therefore obtain a commutative square

$$\begin{tikzcd}
(\mathsf{C} \times \mathsf{D})^{\mathrm{op}} \times (\mathsf{C} \times \mathsf{D}) \arrow[r, "\mathrm{map}_{\mathsf{C}\times\mathsf{D}}"] \arrow[d, "\simeq"] & \mathsf{Grpd}_{\mathcal{B}} \\
(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}) \times (\mathsf{D}^{\mathrm{op}} \times \mathsf{D}) \arrow[r, "\mathrm{map}_{\mathsf{C}} \times \mathrm{map}_{\mathsf{D}}"] & \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}} \arrow[u, "- \times -"']
\end{tikzcd}$$

which translates into a commutative square

$$\begin{tikzcd} \mathsf{C} \times \mathsf{D} \arrow[r, "h_{\mathsf{C}\times\mathsf{D}}"] \arrow[d, "h_{\mathsf{C}}\times h_{\mathsf{D}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}) \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r] & \underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{C} \times \mathsf{D})^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}}) \arrow[u, "(-\times-)_*"'] \end{tikzcd}$$

in which the lower horizontal map is the transpose of

$$\mathrm{ev}_{\mathsf{C}} \times \mathrm{ev}_{\mathsf{D}} \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \times \mathsf{D}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}}.$$

It now suffices to observe that with respect to the equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{C} \times \mathsf{D})^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}} \times \mathsf{Grpd}_{\mathcal{B}}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}),$$

the lower horizontal map in this diagram corresponds to $\mathrm{pr}_0^* \times \mathrm{pr}_1^*$ and the right vertical map corresponds to the product functor on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D})$. □

Lemma 4.4.2.7. *Let* C *be a* $\mathcal{B}$*-category and let* $f \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ *be a presheaf on* C*. Then the product functor* $f \times - \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *has a right adjoint.*

Proof. Let $p \colon \mathsf{P} \to \mathsf{C}$ be the right fibration that is classified by f. Then $f \times -$ corresponds to the product functor $\mathsf{P} \times - \colon \mathsf{RFib}_{\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}}$. On local sections over $A \in \mathcal{B}$, this functor is given by the composition

$$\mathrm{RFib}(\pi_A^* \mathsf{C}) \xrightarrow{p^*} \mathrm{RFib}(\pi_A^* \mathsf{P}) \xrightarrow{p_!} \mathrm{RFib}(\pi_A^* \mathsf{C}).$$

By the theory of Kan extensions (Section 3.4.3) and the fact that $\mathsf{Grpd}_{\mathcal{B}}$ is complete, the functor $p^* \colon \mathsf{RFib}_{\mathsf{C}} \to \mathsf{RFib}_{\mathsf{P}}$ has a right adjoint p_*, which implies that $\mathsf{P} \times -$ section-wise admits a right adjoint that is given by the composition $p_* p^*$. Now if $s \colon B \to A$ is a map in $\mathcal{B}$, the mate transformation $s^* p_* p^* \to p_* p^* s^*$ is given by the composition

$$s^* p_* p^* \to p_* s^* p^* \simeq p_* p^* s^*$$

in which the first map is induced by the mate transformation $s^* p_* \to p_* s^*$. Since p_* is an *internal* right adjoint of p^*, this map must be an equivalence. Using Proposition 3.1.2.9, the claim follows. □

Lemma 4.4.2.6 implies that there is a commutative triangle

$$\begin{tikzcd} \Delta \times \mathsf{C}^{\mathrm{op}} \arrow[r, "h_{\Delta\times \mathsf{C}^{\mathrm{op}}}"] \arrow[d, "\mathrm{id}_\Delta \times h_{\mathsf{C}^{\mathrm{op}}}"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta \times \mathsf{C}^{\mathrm{op}}) \\ \Delta \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}). \arrow[ur, "\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-)"'] & \end{tikzcd}$$

We are now able to compute the unstraightening of all of those functors in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))$ that lie in the image of $\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-)$:

Proposition 4.4.2.8. *There is a commutative square*

$$\begin{tikzcd} \Delta \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}) \arrow[r, "\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-)"] \arrow[d, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)) \arrow[d, "\mathrm{Un}_{\mathsf{C}}"] \\ \Delta \times \mathsf{LFib}_{\mathsf{C}} \arrow[r, "\Delta^\bullet \otimes -"] & \mathsf{Cocart}_{\mathsf{C}}\,. \end{tikzcd}$$

Proof. By construction of the unstraightening functor, the square commutes when restricted along the inclusion $\mathrm{id}_\Delta \times h_{\mathsf{C}^{\mathrm{op}}} \colon \Delta \times \mathsf{C}^{\mathrm{op}} \hookrightarrow \Delta \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}})$. By making use of the universal property of presheaf $\mathcal{B}$-categories (Theorem 3.5.1.1) and the fact that Un_{C} is a left adjoint functor and therefore cocontinuous (Proposition 3.3.2.10), it is enough to show that for every integer $n \geq 0$ both $\Delta^n \otimes -$ and $\mathrm{pr}_0^*(\Delta^n) \times \mathrm{pr}_1^*(-)$ are cocontinuous as well. For the first functor, this follows from the observation that it has a right adjoint given by $(-)^{\Delta^n}_\sharp$. Regarding the second functor, since pr_1^* is cocontinuous, it suffices to show that $\mathrm{pr}_0^*(\Delta^n) \times -$ is cocontinuous as well, which follows from Lemma 4.4.2.7. □

4.4.3. The straightening equivalence

We are finally ready to state and prove the main theorem of this chapter:

Theorem 4.4.3.1. *For every $\mathcal{B}$-category C, the straightening functor*

$$\mathrm{St}_{\mathsf{C}} \colon \mathsf{Cocart}_{\mathsf{C}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$$

is an equivalence of large $\mathcal{B}$-categories that is natural in $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$.

To prove Theorem 4.4.3.1, we will show that both the unit η_{C} and the counit ϵ_{C} of the adjunction $\mathrm{Un}_{\mathsf{C}} \dashv \mathrm{St}_{\mathsf{C}}$ from Corollary 4.4.2.4 is an equivalence. By Corollary 4.1.2.10 and the fact that equivalences in functor $\mathcal{B}$-categories are detected object-wise (Proposition 2.3.2.12), it will be enough to show that for every object $c \colon A \to \mathsf{C}$ the maps $\eta_{\mathsf{C}}(c)$ and $\epsilon_{\mathsf{C}}(c)$ are equivalences. Using Remark 4.4.1.3 and Remark 4.4.2.5, we may assume that $A \simeq 1$. Since furthermore Remark 4.4.1.2 implies that St_1 is an equivalence, it will suffice to construct equivalences $c^*\eta_{\mathsf{C}} \simeq \eta_1 c^*$ and $c^*\epsilon_{\mathsf{C}} \simeq \epsilon_1 c^*$. In other words, we need to show that the map $\mathrm{Un}_1 c^* \to c^* \mathrm{Un}_{\mathsf{C}}$ that arises as the mate of the commutative square

$$\begin{tikzcd} \mathsf{Cocart}_{\mathsf{C}} \arrow[r, "\mathrm{St}_{\mathsf{C}}"] \arrow[d, "c^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}}) \arrow[d, "c^*"] \\ \mathsf{Cocart}_1 \arrow[r, "\mathrm{St}_1"] & \mathsf{Cat}_{\mathcal{B}} \end{tikzcd}$$

is an equivalence. This will require a few preparatory steps.

Lemma 4.4.3.2. *There is a commutative square*

$$\begin{tikzcd} \Delta \times \mathsf{LFib}_{\mathsf{C}} \arrow[r, "\Delta^\bullet \times \int^{-1}"] \arrow[d, "\Delta^\bullet \otimes -"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}) \arrow[d, "\mathrm{pr}_0^*(-) \times \mathrm{pr}_1^*(-)"] \\ \mathsf{Cocart}_{\mathsf{C}} \arrow[r, "\mathrm{St}_{\mathsf{C}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)) \end{tikzcd}$$

where $\int \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}) \simeq \mathsf{LFib}_{\mathsf{C}}$ *denotes the equivalence from Theorem 2.2.1.1.*

Proof. Being a right adjoint, the straightening functor commutes with products. Thus the claim follows from Proposition 4.4.1.5 as well as from combining Remark 4.4.1.2 with the naturality of straightening. □

The argument in the proof of the lemma below was communicated to the author by Maxime Ramzi:

Lemma 4.4.3.3. *Let* $(l \dashv r) \colon \mathsf{C} \rightleftarrows \mathsf{D}$ *be an adjunction between (not necessarily small)* $\mathcal{B}$*-categories in which* D *is cocomplete and locally small. Let* $\mathsf{E}^0 \hookrightarrow \mathsf{E}$ *be a full inclusion of* $\mathcal{B}$*-categories where* E^0 *is small and* E *is locally small, and let* $f \colon \mathsf{E} \to \mathsf{D}$ *be a functor such that* f *is the left Kan extension of* fi *along* i *and the* *identity on* D *is the left Kan extension of* fi *along itself. Suppose furthermore that there is an arbitrary equivalence* $\phi\colon f \simeq rlf$*. Then the adjunction unit* η *induces an equivalence* $\eta f\colon f \simeq rlf$*.*

Proof. Note that by Corollary 3.4.3.7 the assumptions on E^0, E and D make sure that the functors of left Kan extension exist. Using the triangle identities, one can construct a commutative diagram

$$\begin{tikzcd} f \arrow[r, "\eta f"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "\phi"] & rlf \arrow[r, dashed] \arrow[d, "rl\phi"] & f \arrow[d, "\phi"] \\ rlf \arrow[r, "rl\eta f"] \arrow[rr, bend right, "\mathrm{id}"'] & rlrlf \arrow[r, "r\epsilon lf"] & rlf \end{tikzcd}$$

in which ϵ denotes the adjunction counit. Therefore, the map $s = \phi^{-1}\eta f\colon f \to f$ admits a retraction r that is obtained by composing ϕ with the dashed arrow in the above diagram. We complete the proof by showing that s is an equivalence. Since by assumption f is the left Kan extension of fi along i and since i is fully faithful, the functor of left Kan extension $i_!$ being fully faithful (see Theorem 3.4.3.5) implies that it suffices to show that $i^*(s)$ is an equivalence. Since furthermore the identity on D is the left Kan extension of fi along itself, the two maps $i^*(s)$ and $i^*(r)$ induce maps $s'\colon \mathrm{id}_{\mathsf{D}} \to \mathrm{id}_{\mathsf{D}}$ and $r'\colon \mathrm{id}_{\mathsf{D}} \to \mathrm{id}_{\mathsf{D}}$ such that $r's' \simeq \mathrm{id}$ and such that $i^*f^*(s') \simeq i^*(s)$. It therefore suffices to show that s' is an equivalence. Given $d\colon A \to \mathsf{D}$, naturality of s' implies that there is a commutative square

$$\begin{tikzcd} d \arrow[r, "s'(d)"] \arrow[d, "r'(d)"] & d \arrow[d, "r'(d)"] \\ d \arrow[r, "s'(d)"] & d, \end{tikzcd}$$

hence $r'(d)$ is both a left and right inverse of $s'(d)$, which implies that $s'(d)$ is an equivalence. As d was chosen arbitrarily, the result follows. □

Proposition 4.4.3.4. *Let* $f\colon \mathsf{D} \to \mathsf{C}$ *be a functor of* $\mathcal{B}$*-categories with respect to which the pullback functor* $f^*\colon \mathrm{Cocart}_{\mathsf{C}} \to \mathrm{Cocart}_{\mathsf{D}}$ *is cocontinuous. Then the* *mate of the commutative square*

$$\begin{tikzcd} \mathsf{Cocart}_{\mathsf{C}} \arrow[r, "\mathrm{St}_{\mathsf{C}}"] \arrow[d, "f^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)) \arrow[d, "f^*"] \\ \mathsf{Cocart}_{\mathsf{D}} \arrow[r, "\mathrm{St}_{\mathsf{D}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)) \end{tikzcd}$$

is an equivalence.

Proof. By Proposition 4.3.3.2 and Proposition 3.3.2.12 and the assumption on the functor f, the mate transformation $\phi\colon \mathrm{Un}_{\mathsf{D}} f^* \to f^* \mathrm{Un}_{\mathsf{C}}$ is a map of cocontinuous functors between cocomplete large $\mathcal{B}$-categories. Using the universal property of presheaf $\mathcal{B}$-categories, the map ϕ is therefore an equivalence whenever its restriction along the Yoneda embedding

$$h_{\Delta\times\mathsf{C}^{\mathrm{op}}}\colon \Delta\times\mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))$$

is one. By making use of the commutative triangle

$$\begin{tikzcd} \Delta\times\mathsf{C}^{\mathrm{op}} \arrow[r, "h_{\Delta\times\mathsf{C}^{\mathrm{op}}}"] \arrow[d, "\mathrm{id}_\Delta\times h_{\mathsf{C}^{\mathrm{op}}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta\times\mathsf{C}^{\mathrm{op}}) \\ \Delta\times\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}), \arrow[ur, "\mathrm{pr}_0^*\Delta^\bullet\times\mathrm{pr}_1^*(-)"'] & \end{tikzcd}$$

from Lemma 4.4.2.6, we might as well show that $\phi(\mathrm{pr}_0^*\Delta^\bullet\times\mathrm{pr}_1^*(-))$ is an equivalence.

To that end, let us first show that the functor $\mathrm{pr}_0^*\Delta^\bullet\times\mathrm{pr}_1^*(-)$ is the left Kan extension of $h_{\Delta\times\mathsf{C}^{\mathrm{op}}}$ along $\mathrm{id}_\Delta\times h_{\mathsf{C}^{\mathrm{op}}}$. Note that Lemma 4.4.2.7 and the universal property of presheaf $\mathcal{B}$-categories imply that the the associated functor $\Delta\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)))$ factors through the inclusion

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))) \arrow[d, hook, "(h_{\mathsf{C}^{\mathrm{op}}})_!"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))). \end{tikzcd}$$

Consequently, $\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-)$ is in the essential image of the inclusion

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta \times \mathsf{C}^{\mathrm{op}}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))) \arrow[d, hook, "(\mathrm{id} \times h_{\mathsf{C}^{\mathrm{op}}})_!"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta))), \end{tikzcd}$$

as claimed.

By combining this observation with Proposition 4.4.2.8 and Lemma 4.4.3.2, we are in the situation of Lemma 4.4.3.3 and may therefore conclude that the map $\eta_{\mathsf{C}}(\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-))$ is an equivalence, where η_{C} denotes the unit of $\mathrm{Un}_{\mathsf{C}} \dashv \mathrm{St}_{\mathsf{C}}$. As a consequence, since ϕ is explicitly given by the composition

$$\mathrm{Un}_{\mathsf{D}} f^* \xrightarrow{\mathrm{Un}_{\mathsf{C}} f^* \eta_{\mathsf{C}}} \mathrm{Un}_{\mathsf{D}} f^* \mathrm{St}_{\mathsf{C}} \mathrm{Un}_{\mathsf{C}} \xrightarrow{\simeq} \mathrm{Un}_{\mathsf{D}} \mathrm{St}_{\mathsf{D}} f^* \mathrm{Un}_{\mathsf{C}} \xrightarrow{\epsilon_{\mathsf{D}} f^* \mathrm{Un}_{\mathsf{C}}} f^* \mathrm{Un}_{\mathsf{C}},$$

the proof is finished once we show that also the counit $\epsilon_{\mathsf{D}} f^* \mathrm{Un}_{\mathsf{C}}(\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-))$ is an equivalence. But as in light of Proposition 4.4.2.8 and the naturality of tensoring there is an equivalence

$$f^* \mathrm{Un}_{\mathsf{C}}(\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^*(-)) \simeq \mathrm{Un}_{\mathsf{D}}(\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^* f^*(-)),$$

the triangle identities for the adjunction $\mathrm{Un}_{\mathsf{D}} \dashv \mathrm{St}_{\mathsf{C}}$ imply that this follows once we prove that the map $\eta_{\mathsf{D}}(\mathrm{pr}_0^* \Delta^\bullet \times \mathrm{pr}_1^* f^*(-))$ is an equivalence, which has already been shown above. □

Proof of Theorem 4.4.3.1. Let $c \colon 1 \to \mathsf{C}$ be an arbitrary global object. As discussed in the beginning of this section, we only have to show that the natural map $\phi \colon \mathrm{Un}_1 c^* \to c^* \mathrm{Un}_{\mathsf{C}}$ is an equivalence. In light of the factorisation of c into the composition $(\pi_c)_! \mathrm{id}_c \colon 1 \to \mathsf{C}_{/c} \to \mathsf{C}$ of a final map and a right fibration (see Corollary 2.1.3.13), the map ϕ arises as the mate of the composite square in the commutative diagram

$$\begin{tikzcd} \mathsf{Cocart}_{\mathsf{C}} \arrow[r, "\mathrm{St}_{\mathsf{C}}"] \arrow[d, "(\pi_c)_!^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}}) \arrow[d, "(\pi_c)_!^*"] \\ \mathsf{Cocart}_{\mathsf{C}_{/c}} \arrow[r, "\mathrm{St}_{\mathsf{C}_{/c}}"] \arrow[d, "\mathrm{id}_c^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/c}, \mathsf{Cat}_{\mathcal{B}}) \arrow[d, "\mathrm{id}_c^*"] \\ \mathsf{Cocart}_1 \arrow[r, "\mathrm{St}_1"] & \mathsf{Cat}_{\mathcal{B}}\,. \end{tikzcd}$$

Using the functoriality of mates, it therefore suffices to show that the mate of each individual square in the diagram commutes. Using Proposition 4.4.3.4, this follows once we show that the two vertical maps on the left-hand side of the above diagram are cocontinuous. As for $(\pi_c)_!^*$, this is a consequence of Proposition 4.3.3.3, so it suffices to consider the map id_c^*. Let $\pi_{\mathsf{C}_{/c}}\colon \mathsf{C}_{/c} \to 1$ be the projection. In light of Proposition 4.3.3.3, we obtain a map

$$\phi\colon\ \mathrm{id}_c^* \to \mathrm{id}_c^*\, \pi_{\mathsf{C}_{/c}}^*(\pi_{\mathsf{C}_{/c}})_! \simeq (\pi_{\mathsf{C}_{/c}})_!$$

in which the map on the left-hand side is induced by the unit of the adjunction $(\pi_{\mathsf{C}_{/c}})_! \dashv \pi_{\mathsf{C}_{/c}}^*$ and the equivalence on the right-hand side follows from the fact that id_c is a section of $\pi_{\mathsf{C}_{/c}}$. Since $(\pi_{\mathsf{C}_{/c}})_!$ is a left adjoint and therefore cocontinuous (Proposition 3.3.2.10), it thus suffices to verify that ϕ is an equivalence. Explicitly, if $p\colon\ \mathsf{P} \to A \times \mathsf{C}$ is a cocartesian fibration, the map $\phi(p)$ is constructed via the commutative diagram

$$\begin{tikzcd}
(\mathsf{P}|_c)^\flat \arrow[r, "i"] \arrow[d] \arrow[rr, bend left, "\phi(p)^\flat"] & \mathsf{P}^\natural \arrow[r, "j"] \arrow[d, "p^\natural"] & (\pi_{\mathsf{C}_{/c}})_!(\mathsf{P})^\flat \arrow[d] \\
A \arrow[r, "c^\sharp"] & A \times \mathsf{P}^\sharp \arrow[r, "\mathrm{pr}_0"] & A
\end{tikzcd}$$

in $\mathcal{B}_\Delta^+$ in which the left square is cartesian and the right square is defined by the condition that j is marked left anodyne. Since c is final, the map $c^\sharp$ is contained in the internal saturation of $d^0 : \Delta^0 \hookrightarrow (\Delta^1)^\sharp$. Using the dual of Remark 4.2.5.6, we thus conclude that i is marked right anodyne. Note that since $\mathrm{Cocart}(A) \simeq \mathrm{Cart}(A)$ as full subcategories of $(\mathcal{B}_\Delta^+)_{/A}$, the map j is simultaneously the reflection map into $\mathrm{Cocart}(A)$ and $\mathrm{Cart}(A)$ and must therefore be marked right anodyne as well. We therefore conclude that also $\phi(p)^\flat$ is a marked right anodyne map. Being a morphism between marked cartesian fibrations over A, this is necessarily also a marked cartesian fibration. Hence $\phi(p)$ is an equivalence. □

Remark 4.4.3.5. Theorem 4.4.3.1 in particular implies that the $\widehat{\mathrm{Cat}}_\infty$-presheaf Cocart on $\mathrm{Cat}(\mathcal{B})$ is a sheaf. In fact, using the naturality of straightening and the fact that $\underline{\mathrm{Fun}}_{\mathcal{B}}(-, \mathrm{Cat}_{\mathcal{B}})$ is a sheaf, we find that $\mathrm{Cocart}_{(-)}\colon\ \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}(\widehat{\mathcal{B}})$

is a sheaf as well. Since postcomposing the latter with the global sections functor recovers the presheaf Cocart, the claim follows.

Remark 4.4.3.6. By combining Theorem 4.4.3.1 with Remark 4.4.1.6, one also obtains that the straightening functor $\mathrm{St}_{\mathsf{C}}\colon \mathsf{Cart}_{\mathsf{C}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{B}})$ is an equivalence.

4.4.4. The universal cocartesian fibration

Let C be a large $\mathcal{B}$-category. Recall that a functor $p\colon \mathsf{P} \to \mathsf{C}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$ is said to be *small* if for every (small) $\mathcal{B}$-category D and every functor $\mathsf{D} \to \mathsf{C}$ the pullback $\mathsf{P} \times_{\mathsf{C}} \mathsf{D}$ is small as well. The collection of small cocartesian fibrations defines a subpresheaf $\mathrm{Cocart}^{\mathbf{U}} \hookrightarrow \mathrm{Cocart}$ on $\mathrm{Cat}(\widehat{\mathcal{B}})$.

Lemma 4.4.4.1. *Let $p\colon \mathsf{P} \to \mathsf{C}$ be a cocartesian fibration between large $\mathcal{B}$-categories. Then p is small if and only if for all objects $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ the fibre $\mathsf{P}|_c$ is a small $\mathcal{B}$-category.*

Proof. The condition is clearly necessary. Conversely, it suffices to show that if C is small and if $\mathsf{P}|_c$ is small for all objects $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$, then $\mathcal{B}$-category P is small as well. By letting c be the tautological object $\mathsf{C}_0 \to \mathsf{C}$, one finds that P_0 is small. It therefore suffices to show that P is locally small, see Proposition 2.3.1.5. Using Proposition 2.3.1.3, we need to show that for any two objects $x, y\colon A \rightrightarrows \mathsf{P}$ in context $A \in \mathcal{B}$ the (large) mapping $\mathcal{B}$-groupoid $\mathrm{map}_{\mathsf{P}}(x, y)$ is contained in $\mathcal{B}$. Let $c = p(x)$ and $d = p(y)$. Note that the morphism $\mathrm{map}_{\mathsf{P}}(x, y) \to \mathrm{map}_{\mathsf{C}}(c, d)$ can be identified with the fibre of

$$\mathrm{map}_{\mathsf{P}}(x, y) \times_A \mathrm{map}_{\mathsf{C}}(c, d) \to \mathrm{map}_{\mathsf{C}}(c, d) \times_A \mathrm{map}_{\mathsf{C}}(c, d)$$

over the diagonal $\mathrm{map}_{\mathsf{C}}(c, d) \to \mathrm{map}_{\mathsf{C}}(c, d) \times_A \mathrm{map}_{\mathsf{C}}(c, d)$. Therefore, by replacing A with $\mathrm{map}_{\mathsf{C}}(c, d)$, we may assume that there exists a map $\alpha\colon c \to d$ in context A and that we only have to show that the fibre of $\mathrm{map}_{\mathsf{P}}(x, y) \to \mathrm{map}_{\mathsf{C}}(c, d)$ over α is contained in $\mathcal{B}$. Let $f\colon x \to z$ be a cocartesian lift of α. We then obtain a cartesian square

$$\begin{tikzcd} \mathrm{map}_{\mathsf{P}}(z,y) \arrow[r, "f^*"] \arrow[d] & \mathrm{map}_{\mathsf{P}}(x,y) \arrow[d] \\ \mathrm{map}_{\mathsf{C}}(d,d) \arrow[r, "\alpha^*"] & \mathrm{map}_{\mathsf{C}}(c,d) \end{tikzcd}$$

such that the fibre of the left vertical map over the identity $\mathrm{id}_d \colon A \to \mathrm{map}_{\mathsf{C}}(d,d)$ recovers the fibre of the right vertical morphism over α. Hence this fibre recovers the mapping $\mathcal{B}$-groupoid $\mathrm{map}_{\mathsf{P}|_d}(z,y)$ and is therefore small. □

Recall that there is an inclusion $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ of very large $\mathcal{B}$-categories that identifies $\mathsf{Cat}_{\mathcal{B}}$ with the full subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is spanned by the small functors $\mathsf{D} \to A$ over all $A \in \mathcal{B}$ (Remark 1.4.2.6). As a consequence of Lemma 4.4.4.1, we now obtain:

Proposition 4.4.4.2. *For every large $\mathcal{B}$-category C, the subpresheaf $\mathrm{Cocart}^{\mathbf{U}}(-\times\mathsf{C})$ of the sheaf $\mathrm{Cocart}(-\times\mathsf{C})$ that is spanned by the small cocartesian fibrations over C is a sheaf on $\mathcal{B}$ and hence defines an (a priori very large) $\mathcal{B}$-category $\mathsf{Cocart}^{\mathbf{U}}_{\mathsf{C}}$. Moreover, restricting the straightening functor St_{C} to $\mathsf{Cocart}^{\mathbf{U}}_{\mathsf{C}}$ determines an equivalence $\mathsf{Cocart}^{\mathbf{U}}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$.* □

As a consequence of Proposition 4.4.4.2, we may now define:

Definition 4.4.4.3. The *universal* cocartesian fibration $\phi_{\mathcal{B}} \colon (\mathsf{Cat}_{\mathcal{B}})_{1/\!/} \to \mathsf{Cat}_{\mathcal{B}}$ is the map that arises as the unstraightening of the identity $\mathrm{id} \colon \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}}$.

Remark 4.4.4.4. Given $A \in \mathcal{B}$, Remark 4.4.1.3 implies that the base change $\pi_A^*(\phi_{\mathcal{B}})$ of the universal cocartesian fibration $\phi_{\mathcal{B}} \colon (\mathsf{Cat}_{\mathcal{B}})_{1/\!/} \to \mathsf{Cat}_{\mathcal{B}}$ in $\mathcal{B}$ is equivalent to the universal cocartesian fibration $\phi_{\mathcal{B}_{/A}} \colon (\mathsf{Cat}_{\mathcal{B}_{/A}})_{1/\!/} \to \mathsf{Cat}_{\mathcal{B}_{/A}}$.

Remark 4.4.4.5. On account of the naturality of straightening, if $\mathsf{C} \to \mathsf{Cat}_{\mathcal{B}}$ is a functor in $\mathrm{Cat}(\widehat{\mathcal{B}})$, the associated small cocartesian fibration $\mathrm{Un}_{\mathsf{C}}(f) \to \mathsf{C}$ fits into a unique pullback square

$$\begin{tikzcd} \mathrm{Un}_{\mathsf{C}}(f) \arrow[r] \arrow[d] & (\mathsf{Cat}_{\mathcal{B}})_{1/\!/} \arrow[d, "\phi_{\mathcal{B}}"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{Cat}_{\mathcal{B}}\,. \end{tikzcd}$$

The goal for the remainder of this section is to relate the cocartesian fibration $\Gamma(\phi_{\mathcal{B}})$ that is obtained by taking global sections of the universal cocartesian fibration in $\mathcal{B}$ with the universal cocartesian fibration in Ani. This will require quite a few preparations first. We begin with the following lemma:

Lemma 4.4.4.6. *For any ∞-category $\mathcal{C}$, the transposition of the Yoneda embedding $\mathcal{C} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathcal{C})$ in $\mathrm{Cat}(\mathcal{B})$ across the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ yields the composition*

$$\mathcal{C} \xhookrightarrow{h_{\mathcal{C}}} \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) \xrightarrow{(\mathrm{const}_{\mathcal{B}})_*} \mathrm{PSh}_{\mathcal{B}}(\mathcal{C}).$$

Proof. Transposing the Yoneda embedding $\mathcal{C} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathcal{C})$ across $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ yields the composition

$$\mathcal{C} \xrightarrow{\eta} \Gamma(\mathcal{C}) \hookrightarrow \mathrm{PSh}_{\mathcal{B}}(\mathcal{C})$$

in which η is the adjunction unit of $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ and the right map is given by taking global sections of the Yoneda embedding in $\mathrm{Cat}(\mathcal{B})$. By in turn transposing the above map across the adjunction $\mathcal{C}^{\mathrm{op}} \times - \dashv \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, -)$ in $\widehat{\mathrm{Cat}}_\infty$, one ends up with the functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\eta} \Gamma(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \xrightarrow{\Gamma(\mathrm{map}_{\mathcal{C}})} \mathcal{B}.$$

On the other hand, the transpose of the composition $\mathcal{C} \hookrightarrow \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) \to \mathrm{PSh}_{\mathcal{B}}(\mathcal{C})$ yields

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{map}_{\mathcal{C}}} \mathrm{Ani} \xrightarrow{\mathrm{const}_{\mathcal{B}}} \mathcal{B},$$

so it suffices to show that these two functors are equivalent. By Corollary 2.2.2.8 the functor $\mathrm{map}_{\Gamma\mathcal{C}}$ is equivalent to the composition

$$\Gamma \circ \Gamma(\mathrm{map}_{\mathcal{C}}) \colon \Gamma\mathcal{C}^{\mathrm{op}} \times \Gamma\mathcal{C} \to \mathcal{B} \to \mathrm{Ani},$$

hence the morphism $\mathrm{map}_{\mathcal{C}} \to \mathrm{map}_{\Gamma\mathcal{C}} \circ \eta$ that is induced by the action of η on mapping ∞-groupoids determines a morphism $\mathrm{map}_{\mathcal{C}} \to \Gamma \circ \Gamma(\mathrm{map}_{\mathcal{C}}) \circ \eta$ which in turn transposes to a map

$$\mathrm{const}_{\mathcal{B}} \circ \mathrm{map}_{\mathcal{C}} \to \Gamma(\mathrm{map}_{\mathcal{C}}) \circ \eta.$$

By the triangle identities, this is an equivalence. □

Lemma 4.4.4.7. *For any ∞-category $\mathcal{C}$, there is a commutative square*

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}, \mathrm{Ani}) \arrow[r, "(\mathrm{const}_{\mathcal{B}})_*"] \arrow[d, "\simeq"] & \mathrm{Fun}(\mathcal{C}, \mathcal{B}) \arrow[d, "\simeq"] \\ \mathrm{LFib}_{\mathrm{Ani}}(\mathcal{C}) \arrow[r, "\mathrm{const}_{\mathcal{B}}"] & \mathrm{LFib}_{\mathcal{B}}(\mathcal{C}) \end{tikzcd}$$

in which the two vertical equivalences are given by the straightening equivalence for left fibrations in Ani *and in* $\mathcal{B}$*, respectively.*

Proof. By using that both $\mathrm{Fun}(-, \mathrm{Ani})$ and $\mathrm{Fun}(-, \mathcal{B})$ are sheaves of ∞-categories on Cat_∞, it suffices to show that we have a commutative square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^\bullet, \mathrm{Ani}) \arrow[r, "(\mathrm{const}_{\mathcal{B}})_*"] \arrow[d, "\simeq"] & \mathrm{Fun}(\Delta^\bullet, \mathcal{B}) \arrow[d, "\simeq"] \\ \mathrm{LFib}_{\mathrm{Ani}}(\Delta^\bullet) \arrow[r, "\mathrm{const}_{\mathcal{B}}"] & \mathrm{LFib}_{\mathcal{B}}(\Delta^\bullet) \end{tikzcd}$$

of functors $\Delta^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Recall from Section 2.2.1 that the straightening equivalence for left fibrations fits into a commutative diagram

$$\begin{tikzcd} \mathrm{LFib}_{\mathcal{B}}(\Delta^\bullet) \arrow[r, hook] \arrow[d, "\simeq"] & (\mathcal{B}_\Delta)_{/\Delta^\bullet} \arrow[d, "\simeq"] \\ \mathrm{Fun}(\Delta^\bullet, \mathcal{B}) \arrow[r, hook, "\epsilon^*"] & \mathrm{PSh}_{\mathcal{B}}(\Delta_{/\Delta^\bullet}) \end{tikzcd}$$

in which $\epsilon\colon (\Delta_{/\Delta^\bullet})^{\mathrm{op}} \to \Delta^\bullet$ carries a map $\tau\colon \langle k\rangle \to \langle n\rangle$ to $\tau(0) \in \langle n\rangle$. It is now straightforward to verify that the two horizontal maps and the equivalence on the right are natural in $\mathcal{B} \in \mathrm{Top}^{\mathrm{L}}$, hence the claim follows. □

Lemma 4.4.4.8. *For any* $\mathcal{B}$*-category* C *there exists a commutative square*

$$\begin{tikzcd} \mathrm{Cocart}(\mathsf{C}) \arrow[r, hook, "\Gamma(\mathrm{St}_{\mathsf{C}})"] \arrow[d, "\Gamma"] & \mathrm{LFib}(\mathsf{C})_\Delta \arrow[d, "\Gamma"] \\ \mathrm{Cocart}(\Gamma(\mathsf{C})) \arrow[r, hook, "\mathrm{St}_{\Gamma\mathsf{C}}"] & \mathrm{LFib}(\Gamma\mathsf{C})_\Delta. \end{tikzcd}$$

Proof. By construction of the straightening functor and by making use of Remark 4.3.2.5 and Lemma 4.4.4.6, the functor $\Gamma(\mathrm{St}_{\mathsf{C}})$ fits into the diagram

$$\begin{tikzcd} \Gamma(\mathrm{St}_{\mathsf{C}})(-)_{\bullet} \arrow[r, hook] \arrow[dr] & (-)^{\Delta^{\bullet}} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\bullet}, -) \arrow[d] \\ & \mathsf{C} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\bullet}, \mathsf{C}) \end{tikzcd}$$

in which the square is a pullback and the upper left horizontal map is given by the inclusion of the underlying left fibration. As Γ commutes with limits, cotensoring by ∞-categories and taking the underlying left fibration of a cocartesian fibration, the claim follows. $\square$

For our next result, note that in light of the straightening equivalence for left fibrations, the global sections functor $\Gamma\colon \mathrm{LFib}_{\mathcal{B}}(\mathsf{C}) \to \mathrm{LFib}_{\mathrm{Ani}}(\Gamma\mathsf{C})$ determines a map

$$\Gamma\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathrm{Fun}(\Gamma(\mathsf{C}), \mathrm{Ani})$$

for every $\mathcal{B}$-category C.

Lemma 4.4.4.9. *The functor* $\Gamma\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{Cat}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathrm{Fun}(\mathrm{Cat}(\mathcal{B}), \mathrm{Ani})$ *carries the simplicial object in* $\mathrm{Fun}_{\mathcal{B}}(\mathsf{Cat}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}})$ *that is given by* $\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\Delta^{\bullet}, -)$ *to the simplicial object* $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\Delta^{\bullet}, -)$ *in* $\mathrm{Fun}(\mathrm{Cat}(\mathcal{B}), \mathrm{Ani})$.

Proof. Let $i\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ denote the inclusion. Note that we can identify the simplicial object $\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\Delta^{\bullet}, -)$ with the image of the simplicial object $\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)}(\Delta^{\bullet}, -)$ in $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta), \mathsf{Grpd}_{\mathcal{B}})$ along the functor

$$i^*\colon \mathrm{Fun}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta), \mathsf{Grpd}_{\mathcal{B}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{Cat}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}}).$$

Analogously, the simplicial object $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\Delta^{\bullet}, -)$ is the image of $\mathrm{map}_{\mathcal{B}_{\Delta}}(\Delta^{\bullet}, -)$ along $\Gamma(i)^*$. As the global sections functor $\Gamma\colon \mathrm{LFib}_{\mathcal{B}}(\mathsf{C}) \to \mathrm{LFib}_{\mathrm{Ani}}(\Gamma\mathsf{C})$ is natural in C, we may thus replace $\mathsf{Cat}_{\mathcal{B}}$ by $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ and $\mathrm{Cat}(\mathcal{B})$ by $\mathcal{B}_{\Delta}$. Now the functor of left Kan extension

$$(h_{\Delta})_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta), \mathsf{Grpd}_{\mathcal{B}})$$

induces an inclusion $\mathrm{LFib}_{\mathcal{B}}(\Delta) \hookrightarrow \mathrm{LFib}_{\mathcal{B}}(\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta))$ that acts by sending a left fibration $P \to \Delta$ to the left fibration $Q \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)$ that arises from the factorisation of $P \to \Delta \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)$ into an initial map and a left fibration (see Corollary 3.1.3.3). In particular, one obtains an initial map $P \to Q$. In the case that P is corepresented by a global object in Δ, the left fibration $Q \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)$ is corepresented by its image along h_Δ. Under these conditions, the global sections functor Γ carries the initial map $P \to Q$ to an initial map in $\widehat{\mathrm{Cat}}_\infty$. As a consequence, the lax square

$$\begin{tikzcd} \mathrm{LFib}_{\mathcal{B}}(\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta)) \arrow[r, "\Gamma"] & \mathrm{LFib}_{\mathrm{Ani}}(\mathcal{B}_\Delta) \\ \mathrm{LFib}_{\mathcal{B}}(\Delta) \arrow[r, "\Gamma"] \arrow[u, hook, "(h_\Delta)_!"] & \mathrm{LFib}_{\mathrm{Ani}}(\Gamma\Delta) \arrow[u, hook, "(\Gamma h_\Delta)_!"'] \end{tikzcd}$$

commutes after restricting along the inclusion $\Gamma(h_{\Delta^{\mathrm{op}}}) \colon \Gamma\Delta^{\mathrm{op}} \hookrightarrow \mathrm{LFib}_{\mathcal{B}}(\Delta)$. Now by virtue of Lemma 4.4.4.6, the restriction of Γh_Δ along the adjunction unit $\eta_\Delta \colon \Delta \to \Gamma\Delta$ is equivalent to the composition

$$\Delta \xhookrightarrow{h_\Delta} \mathrm{Ani}_\Delta \xrightarrow{\mathrm{const}_{\mathcal{B}}} \mathcal{B}_\Delta.$$

Hence, the equivalence $\mathrm{map}_{\mathcal{B}_\Delta}(\Delta^\bullet, -) \simeq \mathrm{map}_{\mathrm{Ani}_\Delta}(\Delta^\bullet, \Gamma(-))$ implies that the simplicial object $\mathrm{map}_{\mathcal{B}_\Delta}(\Delta^\bullet, -)$ arises as the image of $\eta_\Delta^{\mathrm{op}} \in (\Gamma\Delta^{\mathrm{op}})_\Delta$ along the inclusion

$$(\Gamma h_\Delta)_! \circ h_{\Gamma\Delta^{\mathrm{op}}} \colon (\Gamma\Delta^{\mathrm{op}})_\Delta \hookrightarrow \mathrm{Fun}(\mathcal{B}_\Delta, \mathrm{Ani})_\Delta.$$

To complete the proof, it therefore suffices to construct a commutative diagram

$$\begin{tikzcd} \mathrm{LFib}_{\mathcal{B}}(\Delta) \arrow[r, "\Gamma"] & \mathrm{LFib}_{\mathrm{Ani}}(\Gamma\Delta) \\ \Delta^{\mathrm{op}}. \arrow[u, "\Gamma(h_{\Delta^{\mathrm{op}}})\circ\eta_\Delta^{\mathrm{op}}"] \arrow[ur, "h_{\Gamma\Delta^{\mathrm{op}}}\circ\eta_\Delta^{\mathrm{op}}"'] & \end{tikzcd}$$

Again by Lemma 4.4.4.6 and by moreover making use of Lemma 4.4.4.7, the vertical map is equivalent to the composition

$$\Delta^{\mathrm{op}} \xhookrightarrow{h_{\Delta^{\mathrm{op}}}} \mathrm{LFib}_{\mathrm{Ani}}(\Delta) \xrightarrow{\mathrm{const}_{\mathcal{B}}} \mathrm{LFib}_{\mathcal{B}}(\Delta).$$

Note that the adjunction unit η induces a map $\mathrm{id}_{\mathrm{LFib}_{\mathrm{Ani}}(\Delta)} \to \eta_\Delta^* \circ \Gamma \circ \mathrm{const}_{\mathcal{B}}$ in which the codomain denotes the composition

$$\mathrm{LFib}_{\mathrm{Ani}}(\Delta) \xrightarrow{\mathrm{const}_{\mathcal{B}}} \mathrm{LFib}_{\mathcal{B}}(\Delta) \xrightarrow{\Gamma} \mathrm{LFib}_{\mathrm{Ani}}(\Gamma\Delta) \xrightarrow{\eta_\Delta^*} \mathrm{LFib}_{\mathrm{Ani}}(\Delta).$$

By transposition, we therefore end up with a map $\phi\colon (\eta_\Delta)_! \to \Gamma \circ \mathrm{const}_{\mathcal{B}}$. On account of the equivalence $(\eta_\Delta)_! \circ h_{\Delta^{\mathrm{op}}} \simeq h_{\Gamma\Delta^{\mathrm{op}}} \circ \eta$, it now suffices to show that $\phi h_{\Delta^{\mathrm{op}}}$ is an equivalence. But if $n \geq 0$ is an arbitrary integer, the map $\eta\colon \Delta_{\langle n\rangle/} \to \Gamma\Delta_{\langle n\rangle/}$ is already initial, which implies the claim. □

Proposition 4.4.4.10. *There is a cartesian square*

$$\begin{tikzcd} \Gamma(\mathrm{Cat}_{\mathcal{B}})_{1/\!/} \arrow[r] \arrow[d, "\Gamma(\phi_{\mathcal{B}})"] & (\mathrm{Cat}_\infty)_{1/\!/} \arrow[d, "\phi_{\mathrm{Ani}}"] \\ \Gamma(\mathrm{Cat}_{\mathcal{B}}) \arrow[r, "\Gamma"] & \mathrm{Cat}_\infty \end{tikzcd}$$

of ∞*-categories.*

Proof. Identifying $\mathrm{Fun}_{\mathcal{B}}(\mathrm{Cat}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})$ with $\mathrm{LFib}(\mathrm{Cat}_{\mathcal{B}})$ and $\mathrm{Fun}(\mathrm{Cat}(\mathcal{B}), \mathrm{Ani})$ with $\mathrm{LFib}(\mathrm{Cat}(\mathcal{B}))$, Lemma 4.4.4.8 gives rise to a commutative square

$$\begin{tikzcd} \mathrm{Cocart}(\mathrm{Cat}_{\mathcal{B}}) \arrow[r, hook, "\Gamma(\mathrm{St}_{\mathrm{Cat}_{\mathcal{B}}})"] \arrow[d, "\Gamma"] & \mathrm{Fun}_{\mathcal{B}}(\mathrm{Cat}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})_\Delta \arrow[d] \\ \mathrm{Cocart}(\mathrm{Cat}(\mathcal{B})) \arrow[r, hook, "\mathrm{St}_{\mathrm{Cat}(\mathcal{B})}"] & \mathrm{Fun}(\mathrm{Cat}(\mathcal{B}), \mathrm{Ani})_\Delta. \end{tikzcd}$$

The functor $\Gamma(\mathrm{St}_{\mathrm{Cat}_{\mathcal{B}}})$ carries the universal cocartesian fibration to the simplicial object $\mathrm{map}_{\mathrm{Cat}_{\mathcal{B}}}(\Delta^\bullet, -)$. By Lemma 4.4.4.9, the right vertical map in the above diagram sends this simplicial object to $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\Delta^\bullet, -)$, which is equivalent to $\mathrm{map}_{\mathrm{Cat}_\infty}(\Delta^\bullet, \Gamma(-))$ by virtue of the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$. Using the naturality of straightening (in Cat_∞), we conclude that $\Gamma(\phi_{\mathcal{B}})$ is the pullback of ϕ_{Ani} along the global sections functor Γ, as claimed. □

Corollary 4.4.4.11. *Let* $f\colon \mathsf{C} \to \mathrm{Cat}_{\mathcal{B}}$ *be a functor and let* $\mathsf{P} \to \mathsf{C}$ *be the cocartesian fibration of* $\mathcal{B}$*-categories that is classified by* f*. Then the cocartesian fibration* $\Gamma\mathsf{P} \to \Gamma\mathsf{C}$ *is classified by* $\Gamma \circ \Gamma(f)\colon \Gamma\mathsf{C} \to \mathrm{Cat}_\infty$. □

4.4.5. Straightening over the interval

Let $p\colon \mathsf{M} \to \Delta^1$ be a cocartesian fibration in $\mathsf{Cat}(\mathcal{B})$, and let $\mathsf{M}|_0$ and $\mathsf{M}|_1$ be its fibres over $d^1 : \Delta^0 \hookrightarrow \Delta^1$ and $d^0 : \Delta^0 \hookrightarrow \Delta^1$, respectively. Our goal in this section is to understand the functor $f\colon \mathsf{M}|_0 \to \mathsf{M}|_1$ that arises from applying the straightening functor St_{Δ^1} to p. Note that the inclusion $d^1 : \mathsf{M}|_0^\flat \hookrightarrow (\Delta^1)^\sharp \otimes \mathsf{M}|_1^\flat$ being marked left anodyne implies that there exists a unique map $h : (\Delta^1)^\sharp \otimes \mathsf{M}|_0^\flat \to \mathsf{M}^\natural$ that makes the diagram

$$\begin{tikzcd} \mathsf{M}|_0^\flat \arrow[r, hook] \arrow[d, hook, "d_1"] & \mathsf{M}^\natural \arrow[d, "p^\natural"] \\ (\Delta^1)^\sharp \otimes \mathsf{M}|_0^\flat \arrow[r, "\mathrm{pr}_0"] \arrow[ur, dashed, "h"] & (\Delta^1)^\sharp \end{tikzcd}$$

commute. Upon applying the restriction functor $(-)|_\Delta$ to this diagram, we therefore end up with a morphism $h : \Delta^1 \otimes \mathsf{M}|_0 \to \mathsf{M}$ in $\mathsf{Cocart}(\Delta^1)$ whose fibre over $d^1 : \Delta^0 \hookrightarrow \Delta^1$ recovers the identity on $\mathsf{M}|_0$. Note that the cocartesian fibration $\mathrm{pr}_0 : \Delta^1 \otimes \mathsf{M}|_0 \to \Delta^1$ is the pullback of $\mathsf{M}|_0 \to 1$ along $s^0 : \Delta^1 \to 1$ and therefore corresponds via straightening to the identity on $\mathsf{M}|_0$. Consequently, applying the functor St_{Δ^1} to h results in a commutative square

$$\begin{tikzcd} \mathsf{M}|_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathsf{M}|_0 \arrow[d, "f"] \\ \mathsf{M}|_0 \arrow[r, "g"] & \mathsf{M}|_1 \end{tikzcd}$$

in $\mathsf{Cat}(\mathcal{B})$, which of course implies $f \simeq g$. In other words, we may recover f as the fibre of h over the final object $d^0 : \Delta^0 \hookrightarrow \Delta^1$. To proceed, we first need the following characterisation of cocartesian fibrations over Δ^1:

Proposition 4.4.5.1. *A functor $p\colon \mathsf{M} \to \Delta^1$ in $\mathsf{Cat}(\mathcal{B})$ is a cocartesian fibration if and only if the inclusion $i_1 : \mathsf{M}|_1 \hookrightarrow \mathsf{M}$ of the fibre of p over $d^0 : \Delta^0 \hookrightarrow \Delta^1$ admits a left adjoint L_1, in which case the adjunction unit $\eta : m \to i_1 L_1(m)$ is a cocartesian map for every object $m : A \to \mathsf{M}$ in context $A \in \mathcal{B}$.*

The proof of Proposition 4.4.5.1 will make repeated use of the following observation:

Lemma 4.4.5.2. *Let $\mathcal{C}$ be a poset. Then a functor $p\colon \mathsf{P} \to \mathcal{C}$ in $\mathrm{Cat}(\mathcal{B})$ is a cocartesian fibration if and only if for every $c < d$ in $\mathcal{C}$ and every object $x\colon A \to \mathsf{P}|_c$ there exists a morphism $f\colon x \to y$ in P such that $p(y) \simeq \pi_A^*(d)$ and such that the map*

$$f^*\colon \operatorname{map}_{\mathsf{P}}(y, i_{\geq d}(-)) \to \operatorname{map}_{\mathsf{P}}(x, i_{\geq d}(-))$$

is an equivalence, where $i_{\geq d}\colon \mathsf{P}_{\geq d} = \mathsf{P} \times_{\mathcal{C}} \mathcal{C}_{d/} \hookrightarrow \mathsf{P}$ denotes the pullback of the inclusion $(\pi_d)_!\colon \mathcal{C}_{d/} \hookrightarrow \mathcal{C}$ along p. If this is the case, the map f is a cocartesian morphism.

Proof. For any $c \leq d$ in the poset $\mathcal{C}$, we shall denote by $(c \leq d)\colon 1 \to \mathcal{C}_1$ the associated morphism in the constant $\mathcal{B}$-category. Note that $\mathcal{C}$ being constant implies that $\mathcal{C}_1$ admits a cover $\bigsqcup_{c\leq d} 1 \twoheadrightarrow \mathcal{C}_1$, which implies that for every map $f\colon A \to \mathcal{C}$ there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $s_i^*(f) \simeq \pi_{A_i}^*(c \leq d)$ for some $(c \leq d)$ in the poset $\mathcal{C}$. By combining this observation with Proposition 4.1.2.7 and Remark 4.1.2.9, we thus conclude that p is cocartesian if and only if for every $c \leq d$ and every object $x\colon A \to \mathsf{P}|_c$ there exists a cocartesian lift $f\colon x \to y$ of $\pi_A^*(c \leq d)$. Since this is always possible when $c \simeq d$, we may assume $c < d$. Note that since $\mathcal{C}$ is a poset, the map $(d_1, d_0)\colon \mathcal{C}_1 \to \mathcal{C}_0 \times \mathcal{C}_0$ is a monomorphism in $\mathcal{B}$. Therefore, a map $f\colon x \to y$ is a lift of $\pi_A^*(c < d)$ if and only if $p(y) \simeq \pi_A^*(d)$. Now in order to finish the proof, we only need to show that the map f is cocartesian if and only if the morphism

$$f^*\colon \operatorname{map}_{\mathsf{P}}(y, i_{\geq d}(-)) \to \operatorname{map}_{\mathsf{P}}(x, i_{\geq d}(-))$$

is an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we can assume that $A \simeq 1$. By definition, f being cocartesian means that the commutative square

$$\begin{tikzcd}
\operatorname{map}_{\mathsf{P}}(y,-) \arrow[r, "f^*"] \arrow[d] & \operatorname{map}_{\mathsf{P}}(x,-) \arrow[d] \\
\operatorname{map}_{\mathcal{C}}(d, p(-)) \arrow[r, "(c<d)^*"] & \operatorname{map}_{\mathcal{C}}(c, p(-))
\end{tikzcd}$$

is a pullback. Let $\mathcal{C}_{\not\geq d}$ be the full subposet of $\mathcal{C}$ that is spanned by the objects in $\mathcal{C}$ that do not admit a map from d, and let us set $\mathsf{P}_{\not\geq d} = \mathsf{P} \times_{\mathcal{C}} \mathcal{C}_{\not\geq d}$. Then $\mathcal{C}_0$ decomposes into a coproduct $(\mathcal{C}_{d/})_0 \sqcup (\mathcal{C}_{\not\geq d})_0$, which in turn induces a decomposition $\mathsf{P}_0 \simeq (\mathsf{P}_{\geq d})_0 \sqcup (\mathsf{P}_{\not\geq d})_0$. As a consequence, the above square is cartesian if and only if its restriction along both $i_{\geq d} : \mathsf{P}_{\geq d} \hookrightarrow \mathsf{P}$ and $i_{\not\geq d} : \mathsf{P}_{\not\geq d} \hookrightarrow \mathsf{P}$ is cartesian. By construction of $\mathcal{C}_{\not\geq d}$, the restriction of $\mathrm{map}_{\mathcal{C}}(d, -)$ along the inclusion $\mathcal{C}_{\not\geq d} \hookrightarrow \mathcal{C}$ yields the initial object. Consequently, restricting the above square along $i_{\not\geq d}$ trivially gives rise to a pullback diagram. On the other hand, the restriction of $(c < d)^*$ along the inclusion $\mathcal{C}_{d/} \hookrightarrow \mathcal{C}$ produces an equivalence, which shows that the above square being cartesian is equivalent to the condition that $f^* : \mathrm{map}_{\mathsf{P}}(y, i_{\geq d}(-)) \to \mathrm{map}_{\mathsf{P}}(x, i_{\geq d}(-))$ is an equivalence. $\square$

Proof of Proposition 4.4.5.1. Let us first assume that the inclusion $i_1 : \mathsf{M}|_1 \hookrightarrow \mathsf{M}$ admits a left adjoint L_1. Let $m : A \to \mathsf{M}$ be an arbitrary object and let us denote by $\eta : m \to iL(m)$ the adjunction unit. Then the map

$$\mathrm{map}_{\mathsf{M}}(i_1 L_1(m), i_1(-)) \xrightarrow{\eta^*} \mathrm{map}_{\mathsf{M}}(m, i_1(-))$$

is an equivalence. Hence Lemma 4.4.5.2 implies p is a cocartesian fibration and that η is a cocartesian morphism.

Conversely, suppose that p is a cocartesian fibration. Given $m : A \to \mathsf{M}$ and $c = p(m)$, the fact that 1 is a final object in Δ^1 gives rise to a unique map $\alpha : c \to 1$ in context A. Let $f : m \to m'$ be a cocartesian lift of α. By construction, m' is contained in the essential image of i_1. We would like to show that the map

$$\eta^* : \mathrm{map}_{\mathsf{M}}(m', i_1(-)) \to \mathrm{map}_{\mathsf{M}}(m, i_1(-))$$

is an equivalence. But the map

$$\alpha^* : \mathrm{map}_{\Delta^1}(p(m'), pi_1(-)) \to \mathrm{map}_{\Delta^1}(p(m), pi_1(-))$$

is an equivalence, hence η^* is one as well on account of η being cocartesian. This shows that i_1 admits a left adjoint that is given by sending m to m'. $\square$

Let $\chi : i_0 \to i_1 f$ be the morphism of functors $\mathsf{M}|_0 \to \mathsf{M}$ that is encoded by the map $h : \Delta^1 \otimes \mathsf{M}|_0 \to \mathsf{M}$. Note that as for every object $m : A \to \mathsf{M}|_0$ the associated map $(\pi_A^*(0 < 1), \mathrm{id}_m)$ in $\Delta^1 \otimes \mathsf{M}|_0$ is cocartesian, the map $\chi(m) : i_0(m) \to i_1 f(m)$ in M is cocartesian too. By Proposition 4.4.5.1, the left adjoint $L_1 : \mathsf{M} \to \mathsf{M}|_1$ to i_1 thus carries $\chi(m)$ to an equivalence in $\mathsf{M}|_1$. In other words, the map $L_1 \chi$ is an equivalence of functors. But since $L_1 i_1 \simeq \mathrm{id}_{\mathsf{M}|_1}$ via the counit of the adjunction $L_1 \dashv i_1$, we conclude:

Proposition 4.4.5.3. *The functor $f\colon \mathsf{M}|_0 \to \mathsf{M}|_1$ that classifies the cocartesian fibration $p\colon \mathsf{M} \to \Delta^1$ is equivalent to the composition $L_1 i_0\colon \mathsf{M}|_0 \hookrightarrow \mathsf{M} \to \mathsf{M}|_1$.* □

Remark 4.4.5.4. Proposition 4.4.5.3 in particular shows that for any object $m\colon A \to \mathsf{M}|_0$ in context $A \in \mathcal{B}$, the object $f(m)\colon A \to \mathsf{M}|_1$ is given by the codomain of the unique cocartesian lift $f\colon m \to m'$ of the map $\pi_A^*(0 < 1)$ in Δ^1. More generally, if $F\colon \mathsf{C} \to \mathsf{Cat}_{\mathcal{B}}$ is an arbitrary functor and if $f\colon c \to d$ is a map in C in context $A \in \mathcal{B}$, the straightforward observation that a lift $h\colon x \to y$ in $\mathrm{Un}_{\mathsf{C}}(F)$ of f is cocartesian if and only if it defines a cocartesian lift of $0 < 1$ in the pullback $\mathrm{Un}_{\pi_A^*\mathsf{C}}(\pi_A^* F) \times_{\pi_A^*\mathsf{C}} \Delta^1 \simeq \mathrm{Un}_{\Delta^1}(F(f))$ implies that the object $y\colon A \to \mathrm{Un}_{\mathsf{C}}(F)|_d \simeq F(d)$ recovers the image of x along $F(f)$.

Corollary 4.4.5.5. *A cocartesian fibration $p\colon \mathsf{M} \to \Delta^1$ is cartesian if and only if the functor $f\colon \mathsf{M}|_0 \to \mathsf{M}|_1$ admits a right adjoint $g\colon \mathsf{M}|_1 \to \mathsf{M}|_0$. If this is the case, then g^{op} is classified by the cocartesian fibration $p^{\mathrm{op}}\colon \mathsf{M}^{\mathrm{op}} \to (\Delta^1)^{\mathrm{op}} \simeq \Delta^1$.*

Proof. The dual of Proposition 4.4.5.1 implies that p is a cartesian fibration if and only if the inclusion $i_0\colon \mathsf{M}|_0 \hookrightarrow \mathsf{M}$ admits a right adjoint $R_0\colon \mathsf{M} \to \mathsf{M}|_0$. Hence Proposition 4.4.5.3 both shows that the functor $g = R_0 i_1$ is right adjoint to $f \simeq L_1 i_0$ and that g^{op} arises as the straightening of p^{op}. Conversely, suppose that f has a right adjoint g. For any object $m\colon A \to \mathsf{M}|_1$, we obtain a map $h\colon i_0 g(m) \to i_1(m)$ that is defined via the composition

$$i_0 g(m) \xrightarrow{\eta} i_1 L_1 i_0 g(m) \xrightarrow{\simeq} i_1 f g(m) \xrightarrow{\epsilon} i_1(m)$$

where η is the unit of the adjunction $L_1 \dashv i_1$ and ϵ is the counit of the adjunction $f \dashv g$. We claim the map

$$h_*\colon \mathrm{map}_{\mathsf{M}}(i_0(-), i_0 g(m)) \to \mathrm{map}_{\mathsf{M}}(i_0(-), i_1(m))$$

is an equivalence. Unwinding the definitions, the composition of h_* with the equivalence

$$\mathrm{map}_{\mathsf{M}|_0}(-, g(m)) \xrightarrow{\simeq} \mathrm{map}_{\mathsf{M}}(i_0(-), i_0 g(m))$$

turns out to be equivalent to the composition

$$\begin{aligned}\operatorname{map}_{\mathsf{M}|_0}(-, g(m)) &\xrightarrow{\simeq} \operatorname{map}_{\mathsf{M}|_1}(f(-), m)\\ &\xrightarrow{\simeq} \operatorname{map}_{\mathsf{M}|_1}(L_1 i_0(-), m)\\ &\xrightarrow{\simeq} \operatorname{map}_{\mathsf{M}}(i_0(-), i_1(m))\end{aligned}$$

in which the two outer equivalences are determined by the two adjunctions $f \dashv g$ and $L_1 \dashv i_1$. Consequently, h_* is an equivalence, hence the dual version of Lemma 4.4.5.2 implies that p is a cartesian fibration. $\square$

Remark 4.4.5.6. In large parts, our treatment of cocartesian fibrations over the interval is an adaptation of the discussion in [52, § 02FJ] to $\mathcal{B}$-categories.

4.5. Applications

In this section, we briefly mention two application of the straightening equivalence for (co)cartesian fibrations from Theorem 4.4.3.1. In Section 4.5.1 we give formulas for the limit and colimit of a diagram in $\mathrm{Cat}_{\mathcal{B}}$, and in Section 4.5.2 we discuss how passing from a left adjoint functor between $\mathcal{B}$-categories to its right adjoint can be turned into a functor.

4.5.1. Limits and colimits of $\mathcal{B}$-categories

The straightening equivalence allows us to derive formulas for the limit and the colimit of a diagram of the form $d\colon \mathsf{J} \to \mathrm{Cat}_{\mathcal{B}}$. As the colimit functor $\operatorname{colim}_{\mathsf{J}}$ is left adjoint to the diagonal functor, one can compute $\operatorname{colim} d$ by evaluating the left adjoint of the pullback map $\pi_{\mathsf{J}}^*\colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cocart}_{\mathsf{J}}$ at $\mathrm{Un}_{\mathsf{J}}(d)$. Together with Remark 4.3.3.4, this shows:

Proposition 4.5.1.1. *Let $d\colon \mathsf{J} \to \mathrm{Cat}_{\mathcal{B}}$ be a small diagram, and let $p\colon \mathsf{P} \to \mathsf{J}$ be the unstraightening of d. Then the $\mathcal{B}$-category* $\operatorname{colim} d$ *is equivalent to the localisation $\mathsf{P}_\sharp^{-1}\mathsf{P}$ of P at the subcategory $\mathsf{P}_\sharp \hookrightarrow \mathsf{P}$ that is spanned by the cocartesian maps.* $\square$

Dually, the limit functor $\lim_{\mathsf{J}}$ is right adjoint to the diagonal functor and therefore corresponds via unstraightening to the right adjoint of $\pi_{\mathsf{J}}^*\colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cocart}_{\mathsf{J}}$. Using Remark 4.3.3.5, this shows:

Proposition 4.5.1.2. *Let $d\colon \mathsf{J} \to \mathrm{Cat}_{\mathcal{B}}$ be a small diagram, and let $p\colon \mathsf{P} \to \mathsf{J}$ be the unstraightening of d. Then* $\lim d$ *is equivalent to the $\mathcal{B}$-category* $(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\sharp}, \mathsf{P}^{\natural})_{/\mathsf{J}^{\sharp}})|_{\Delta}$ *of cocartesian sections of p.* □

4.5.2. Functoriality of passing between right and left adjoints

Let $R \hookrightarrow (\mathrm{Cat}_{\mathcal{B}})_1$ be the subobject that is spanned by the right adjoint functors (in arbitrary context), and let $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{R}}$ be the subcategory of $\mathrm{Cat}_{\mathcal{B}}$ that is determined by R (see Section 1.3.1). Note that since the condition for a functor between $\mathcal{B}$-categories to be a right adjoint is local (Remark 3.1.3.6), a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}_{/A}$-categories defines an object in R if and only if f is a right adjoint. Moreover, R is closed under equivalences and composition in the sense of Proposition 1.3.1.17, hence the inclusion $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{R}} \hookrightarrow \mathrm{Cat}_{\mathcal{B}}$ induces an equivalence $(\mathrm{Cat}_{\mathcal{B}}^{\mathrm{R}})_1 \simeq R$. In particular, a functor between $\mathcal{B}_{/A}$-categories defines a map in $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{R}}$ if and only if it is a right adjoint. We define the subcategory $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{L}} \hookrightarrow \mathrm{Cat}_{\mathcal{B}}$ that is spanned by the *left* adjoints in an analogous fashion. Note that the equivalence $(-)^{\mathrm{op}}\colon\ \mathrm{Cat}_{\mathcal{B}} \simeq \mathrm{Cat}_{\mathcal{B}}$ restricts to an equivalence $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{R}} \simeq \mathrm{Cat}_{\mathcal{B}}^{\mathrm{L}}$. Our goal in this section is to prove:

Proposition 4.5.2.1. *There is an equivalence* $(\mathrm{Cat}_{\mathcal{B}}^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Cat}_{\mathcal{B}}^{\mathrm{L}}$ *that carries a right adjoint functor to its left adjoint.*

The proof of Proposition 4.5.2.1 requires the following lemma, whose analogue for cocartesian fibrations of ∞-categories appears as [52, Proposition 02FP].

Lemma 4.5.2.2. *A cocartesian fibration $p\colon\ \mathsf{P} \to \mathsf{C}$ in* $\mathrm{Cat}(\mathcal{B})$ *is a cartesian fibration if and only if for every morphism $f\colon\ \Delta^1 \otimes A \to \mathsf{C}$ the functor $\mathsf{P}|_f \to \Delta^1 \otimes A$ is a cartesian fibration.*

Proof. The condition is clearly necessary, so it suffices to prove the converse. Note that by Remark 4.4.3.5 and its dual, both Cocart and Cart are *sheaves* on $\mathrm{Cat}(\mathcal{B})$. Consequently, Remark 1.2.1.3 allows us to reduce to the case where $\mathsf{C} \simeq \Delta^n \otimes A$. Using Remark 4.3.2.3 (and its dual), we can furthermore reduce to the case $A \simeq 1$. By the dual of Lemma 4.4.5.2, we need to show that for any $k < l$ in Δ^n and any object $x\colon\ A \to \mathsf{P}|_l$, there exists a map $f\colon\ y \to x$ in P such that

$p(y) \simeq k$ and such that the map

$$f_*\colon \operatorname{map}_{\mathsf{P}}(i_{\leq k}(-), y) \to \operatorname{map}_{\mathsf{P}}(i_{\leq k}(-), x)$$

is an equivalence. By assumption, the pullback $\mathsf{P}|_{k<l} \to \Delta^1$ of p along the morphism $(k < l)\colon \Delta^1 \hookrightarrow \Delta^n$ is a cartesian fibration. We can therefore choose a map $f\colon y \to x$ that defines a cartesian morphism in $\mathsf{P}|_{k<l}$. It will be sufficient to show that for every object $z\colon A \to \mathsf{P}_{\leq k}$, the morphism

$$f_*\colon \operatorname{map}_{\mathsf{P}}(i_{\leq k}(z), y) \to \operatorname{map}_{\mathsf{P}}(i_{\leq k}(z), x)$$

is an equivalence in $\mathcal{B}_{/A}$. As z is locally contained in $\mathsf{P}|_j$ for some $j \leq k$, we can furthermore assume that z is already contained in $\mathsf{P}|_j$, i.e. that $p(z) \simeq j$ holds. Let $g\colon z \to w$ be a cocartesian morphism in P such that $p(w) \simeq k$. We then obtain a commutative square

$$\begin{tikzcd}
\operatorname{map}_{\mathsf{P}}(i_{\leq k}(w), y) \arrow[r, "f_*"] \arrow[d, "g^*"] & \operatorname{map}_{\mathsf{P}}(i_{\leq k}(w), x) \arrow[d, "g^*"] \\
\operatorname{map}_{\mathsf{P}}(i_{\leq k}(z), y) \arrow[r, "f_*"] & \operatorname{map}_{\mathsf{P}}(i_{\leq k}(z), x).
\end{tikzcd}$$

By Lemma 4.4.5.2, the two vertical maps are equivalences. On the other hand, since w defines an object in $\mathsf{P}|_{k<l}$, the dual of Lemma 4.4.5.2 shows that the upper horizontal map is an equivalence on account of f being a cartesian morphism in $\mathsf{P}|_{k<l}$. Hence the claim follows. □

Proof of Proposition 4.5.2.1. By making use of the straightening equivalence and Proposition 1.2.1.4, there is a chain of equivalences

$$\mathsf{Cat}_{\mathcal{B}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\bullet}, \mathsf{Cat}_{\mathcal{B}})^{\simeq} \simeq (\mathsf{Cocart}_{\Delta^{\bullet}})^{\simeq}$$

of simplicial objects in $\widehat{\mathcal{B}}$. Moreover, since $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is a subcategory, a functor $\Delta^n \to \mathsf{Cat}_{\mathcal{B}}$ factors through $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}}$ if and only if its restriction along any $f\colon \Delta^1 \otimes A \to \Delta^n$ factors through $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}}$ (see Proposition 1.3.1.9). By combining this observation with Corollary 4.4.5.5, one concludes that a cocartesian fibration $p\colon \mathsf{P} \to \Delta^n$ arises as the unstraightening of a functor $\Delta^n \to \mathsf{Cat}^{\mathrm{R}}_{\mathcal{B}}$ if and only if for every map $f\colon \Delta^1 \otimes A \to \Delta^n$ the functor $\mathsf{P}|_f \to \Delta^1 \otimes A$ is also a cartesian fibration. By Lemma 4.5.2.2, this is in turn equivalent to p being a cartesian fibration itself. Together with Remark 4.4.1.3 and its dual, the equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^n, \mathsf{Cat}_{\mathcal{B}})^{\simeq} \simeq (\mathsf{Cocart}_{\Delta^n})^{\simeq}$$

identifies the subobject $\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^n, \mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}})^{\simeq} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^n, \mathsf{Cat}_{\mathcal{B}})^{\simeq}$ with the subobject $(\mathsf{Cocart}^{\mathsf{Cart}}_{\Delta^n})^{\simeq} \hookrightarrow (\mathsf{Cocart}_{\Delta^n})^{\simeq}$ that is spanned by the cartesian and cocartesian fibrations. Since taking opposite $\mathcal{B}$-categories determines an equivalence $(\mathsf{Cocart}^{\mathsf{Cart}}_{\Delta^n})^{\simeq} \simeq (\mathsf{Cocart}^{\mathsf{Cart}}_{(\Delta^n)^{\mathrm{op}}})^{\simeq}$ that is natural in n, we obtain equivalences

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^\bullet, \mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}})^{\simeq} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}((\Delta^\bullet)^{\mathrm{op}}, \mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}})^{\simeq} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^\bullet, (\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}})^{\simeq}$$

of simplicial objects in $\widehat{\mathcal{B}}$ and thus an equivalence $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}} \simeq (\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}}$. The result now follows by composing this map with $(-)^{\mathrm{op}} \colon (\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \simeq (\mathsf{Cat}^{\mathrm{R}}_{\mathcal{B}})^{\mathrm{op}}$. $\square$

5. Accessible and presentable $\mathcal{B}$-categories

The notion of presentable ∞-categories has gained a central role within the theory of higher categories. This is due to the many favourable properties of presentable ∞-categories, such as the presence of adjoint functor theorems and the existence of a well-behaved and explicit tensor product. Furthermore, almost all cocomplete ∞-categories that arise in practice are in fact presentable, which allows for wide applications of these general results. Therefore, it will be crucial to have a notion of presentability in the world of $\mathcal{B}$-categories at our disposal. The main goal of this chapter is to develop this theory.

Recall that one can define a presentable ∞-category as one which is *accessible*, i.e. of the form $\mathrm{Ind}^{\kappa}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ and some regular cardinal κ, and which is furthermore *cocomplete*. In order to make sense of such a definition within $\mathcal{B}$-category theory, we therefore first need a $\mathcal{B}$-categorical notion of accessibility. Since we already have a functional theory of free cocompletions at our disposal, the only part that is still missing is an internal notion of *κ-filteredness*. Therefore, we begin this chapter in Section 5.1 with a discussion of *U-filtered $\mathcal{B}$-categories* with respect to an arbitrary internal class U of $\mathcal{B}$-categories. However, not every internal class U will lead to a well-functioning theory: in $\mathcal{B}$-category theory, it is no longer true that every $\mathcal{B}$-category can be written as a U-filtered colimit of U-small $\mathcal{B}$-categories (where U-small simply means being contained in U). Having such a property, which is crucial for the development of accessibility in the world of $\mathcal{B}$-categories, is a genuine condition on an internal class U. In Section 5.2, we discuss how one can construct an ample amount of internal classes U which satisfy this condition. Building upon these rather technical preparations, we then define the concept of U-accessibility in Section 5.3 and prove a few basic results that we will need for our discussion of presentable $\mathcal{B}$-categories. For example, we give a characterisation of U-accessible $\mathcal{B}$-categories by making use of the notion of U-*compactness*. In Section 5.4, we introduce and study presentable $\mathcal{B}$-categories. Aside from discussing multiple characterisations of these $\mathcal{B}$-categories, we prove an adjoint functor theorem and discuss limits and colimits of presentable $\mathcal{B}$-categories. Lastly, in Section 5.5 we discuss tensor products of $\mathcal{B}$-categories and in particular a symmetric monoidal structure on the ∞-category of presentable $\mathcal{B}$-categories. We use this structure to realise $\mathcal{B}$-modules in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ as presentable $\mathcal{B}$-categories.

5.1. Filtered $\mathcal{B}$-categories

Classically, if κ is a (regular) cardinal, a 1-category $\mathcal{J}$ is said to be *κ-filtered* if the colimit functor $\mathrm{colim}_{\mathcal{J}}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{Set}) \to \mathrm{Set}$ commutes with κ-small limits. In [49], Lurie generalised this concept to the notion of a *κ-filtered ∞-category* $\mathcal{J}$, which is an ∞-category for which $\mathrm{colim}_{\mathcal{J}}\colon \mathrm{Fun}(\mathcal{J}, \mathrm{Ani}) \to \mathrm{Ani}$ preserves κ-small limits. The main goal of this section is to discuss an analogous concept for $\mathcal{B}$-categories.

Following ideas introduced in 1-category theory by Adámek-Borceux-Lack-Rosický [1] and later generalised to ∞-categories by Charles Rezk [71], we will introduce the notion of a U-*filtered $\mathcal{B}$-category*, where U is an arbitrary internal class, i.e. a full subcategory of the large $\mathcal{B}$-category $\mathrm{Cat}_{\mathcal{B}}$ of $\mathcal{B}$-categories (cf. Section 1.4.2). The main definitions and basic properties of such U-filtered $\mathcal{B}$-categories are discussed in Section 5.1.1. In Section 5.1.2, we introduce a slightly weaker notion, that of a *weakly* U-filtered $\mathcal{B}$-category. Classically, a κ-filtered (∞-)category can be equivalently described as an (∞-)category in which every κ-small diagram has a cocone. The notion of weak U-filteredness is a generalisation of this condition. However, as the terminology suggests, this notion is a priori weaker than that of U-filteredness. Following [1], we will call an internal class U a *doctrine* if both conditions happen to be equivalent.

In Section 5.1.3 and Section 5.1.4, we will study two other important properties of internal classes: *regularity* and the *decomposition property*. Recall that a cardinal κ is said to be regular if κ-small sets are closed under κ-small sums. The notion of regularity for internal classes aims at capturing this property in the world of $\mathcal{B}$-categories. The decomposition property, on the other hand, is the condition that every $\mathcal{B}$-category can be obtained as a U-filtered colimit of objects in U. Hence, this notion can be viewed as an analogue to the fact that every (∞-)category is a κ-filtered colimit of κ-small (∞-)categories. We will make use of this property when we discuss the notion of U-compactness in Section 5.1.5.

5.1.1. U-filtered $\mathcal{B}$-categories

In this section, we introduce and study the notion of U-filteredness in the world of $\mathcal{B}$-categories, where U is an arbitrary internal class. We begin with the following definition, which is an evident generalisation of the classical concept of a κ-filtered (∞-)category:

Definition 5.1.1.1. For any internal class U of $\mathcal{B}$-categories, a $\mathcal{B}$-category J is said to be U-*filtered* if the colimit functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ is U-continuous. We define the internal class $\mathsf{Filt}_{\mathsf{U}}$ of U-filtered categories as the full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by those $\mathcal{B}_{/A}$-categories J that are $\pi_A^*\mathsf{U}$-filtered, for every $A \in \mathcal{B}$.

Remark 5.1.1.2. In the situation of Definition 5.1.1.1, the fact that U-continuity is a local condition (Remark 3.3.2.3) implies that *every* object $A \to \mathsf{Filt}_{\mathsf{U}}$ is $\pi_A^*\mathsf{U}$-filtered (which a priori has no reason to be true). In particular, the sheaf associated with $\mathsf{Filt}_{\mathsf{U}}$ is given on local sections over $A \in \mathcal{B}$ by the full subcategory of $\mathsf{Cat}(\mathcal{B}_{/A})$ that is spanned by the $\pi_A^*\mathsf{U}$-filtered categories. For any $A \in \mathcal{B}$, we therefore obtain a canonical equivalence $\pi_A^* \mathsf{Filt}_{\mathsf{U}} \simeq \mathsf{Filt}_{\pi_A^*\mathsf{U}}$.

Remark 5.1.1.3. Clearly, if $\mathsf{U} \hookrightarrow \mathsf{V}$ is an inclusion of internal classes, every V-filtered $\mathcal{B}$-category is in particular U-filtered. Therefore, one obtains an inclusion $\mathsf{Filt}_{\mathsf{V}} \hookrightarrow \mathsf{Filt}_{\mathsf{U}}$.

Remark 5.1.1.4. If I and J are $\mathcal{B}$-categories, note that the horizontal mate of the commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I} \times \mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "\operatorname{colim}_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[l, "\operatorname{diag}_*"'] \arrow[d, "\operatorname{colim}"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) & \mathsf{Grpd}_{\mathcal{B}} \arrow[l, "\operatorname{diag}"]
\end{tikzcd}$$

(with respect to the two adjunctions $\mathrm{diag} \dashv \lim$ and $\mathrm{diag}_* \dashv \lim_*$) is equivalent to the horizontal mate of the commutative square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "\mathrm{diag}_*"] \arrow[d, "\lim"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I} \times \mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "\lim_*"] \\ \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}). \end{tikzcd}$$

(with respect to the two adjunctions $\mathrm{colim} \dashv \mathrm{diag}$ and $\mathrm{colim}_* \dashv \mathrm{diag}_*$). As a consequence, the functor $\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ commutes with I-indexed limits if and only if the functor $\lim\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ commutes with J-indexed colimits. Thus, if U is an internal class of $\mathcal{B}$-categories, a $\mathcal{B}$-category J is U-filtered precisely if for all $A \in \mathcal{B}$ and all $\mathsf{I} \in \mathsf{U}(A)$ the limit functor $\lim\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}_{/A}}) \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ commutes with $\pi_A^*\mathsf{J}$-indexed colimits.

Recall from Section 2.1.2 the definition of the *right cone* $\mathsf{J}^{\triangleright}$ of a $\mathcal{B}$-category J. It comes with an inclusion $\iota\colon \mathsf{J} \hookrightarrow \mathsf{J}^{\triangleright}$ such that for every $\mathcal{B}$-category C that admits J-indexed colimits, the functor of left Kan extension $\iota_!$ exists and carries an I-indexed diagram in C to its colimit cocone (Proposition 3.4.4.1). We now obtain:

Proposition 5.1.1.5. *A $\mathcal{B}$-category J is U-filtered with respect to some internal class U if and only if the inclusion $\iota_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \mathsf{Grpd}_{\mathcal{B}})$ is U-continuous.*

Proof. By definition, the functor $\iota_!$ is U-continuous if and only if for all $A \in \mathcal{B}$ the functor $\pi_A^*(\iota_!) \simeq (\pi_A^*\iota)_!$ (cf. Proposition 1.2.5.4 and Corollary 3.1.1.9) preserves limits of I-indexed diagrams for all $\mathsf{I} \in \mathsf{U}(A)$. Furthermore, J is U-filtered if and only if the colimit functor $\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}_{/A}}) \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ commutes with I-indexed limits for all $\mathsf{I} \in \mathsf{U}(A)$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, it therefore suffices to show that for any $\mathsf{I} \in \mathsf{U}(1)$, the functor $\iota_!$ commutes with I-indexed limits if and only if $\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ preserves I-limits. Note that by combining Proposition 3.2.7.1 with the fact that the cone point $\infty\colon 1 \to \mathsf{I}^{\triangleright}$ is final, one finds that the colimit functor $\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ is given by evaluation at ∞. As a consequence, Proposition 3.2.3.1 implies that the colimit functor colim : $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Grpd}_{\mathcal{B}}$ preserves I-indexed limits. Owing to the commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathrm{Grpd}_{\mathcal{B}}) \arrow[d, hook, "\iota_!"] \arrow[dr, "\mathrm{colim}_{\mathsf{J}}"] & \\ \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \mathrm{Grpd}_{\mathcal{B}}) \arrow[r, "\mathrm{colim}_{\mathsf{J}^{\triangleright}}"'] & \mathrm{Grpd}_{\mathcal{B}}, \end{tikzcd}$$

the functoriality of mates thus implies that $i_!$ preserving I-indexed limits implies that $\mathrm{colim}_{\mathsf{J}}$ commutes with I-indexed limits as well. The converse direction, on the other hand, follows from combining the functoriality of the mate construction with the straightforward observation that

$$(\iota^*, \infty^*) \colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \mathrm{Grpd}_{\mathcal{B}}) \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathrm{Grpd}_{\mathcal{B}}) \times \mathrm{Grpd}_{\mathcal{B}}$$

is a conservative functor. □

By an analogous argument as in the proof of Proposition 5.1.1.5 and by furthermore using Remark 5.1.1.4, one obtains:

Proposition 5.1.1.6. *A category J in $\mathcal{B}$ is U-filtered with respect to some internal class U if and only if for all $A \in \mathcal{B}$ and all $\mathsf{I} \in \mathsf{U}(A)$ the functor of right Kan extension*

$$\iota_* \colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}_{/A}}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathrm{Grpd}_{\mathcal{B}_{/A}})$$

preserves $\pi_A^*\mathsf{J}$-indexed colimits. □

Remark 5.1.1.7. By Proposition 5.1.1.6 and Proposition 3.2.7.1, if $\mathsf{J} \to \mathsf{K}$ is a final functor such that J is U-filtered, then K must be U-filtered as well. Since the final $\mathcal{B}$-category 1 is trivially U-filtered for every choice of internal class U, this means that $\mathrm{Filt}_{\mathsf{U}}$ is a *colimit class* in the sense of Definition 3.3.3.5.

For later use, let us note the following closure property of U-filtered $\mathcal{B}$-categories:

Proposition 5.1.1.8. *The internal class $\mathrm{Filt}_{\mathsf{U}}$ is closed under $\mathrm{Filt}_{\mathsf{U}}$-colimits in $\mathrm{Cat}_{\mathcal{B}}$.*

Proof. By Remark 5.1.1.2, it suffices to show that if J is a U-filtered $\mathcal{B}$-category and $d\colon \mathsf{J} \to \mathsf{Filt}_{\mathsf{U}}$ is a diagram, its colimit K in $\mathrm{Cat}_{\mathcal{B}}$ is also U-filtered. Given any $\mathsf{I} \in \mathsf{U}(1)$, Proposition 3.5.4.1 shows that the functor of right Kan extension $\iota_* : \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathsf{Grpd}_{\mathcal{B}})$ commutes with K-indexed colimits. As for any $A \in \mathcal{B}$ the $\mathcal{B}_{/A}$-category $\pi_A^*\mathsf{K}$ is the colimit of $\pi_A^* d$, the same argument also shows that for all $\mathsf{I} \in \mathsf{U}(A)$ the functor

$$\iota_* : \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathsf{Grpd}_{\mathcal{B}_{/A}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}_{/A}})$$

commutes with $\pi_A^*\mathsf{K}$-indexed colimits. Hence Proposition 5.1.1.6 implies that K is U-filtered. $\square$

5.1.2. Weakly U-filtered $\mathcal{B}$-categories

Recall from Remark 3.3.2.2 that if U is an internal class, we denote by $\mathrm{op}(\mathsf{U})$ the internal class that arises as the image of U along $(-)^{\mathrm{op}} : \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}}$. In practice, we will often require that every $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category is U-filtered. However, this is not true for every internal class U, not even in the case $\mathcal{B} = \mathrm{Ani}$ [71, §6]. In this section, we will therefore study a slightly weaker notion than that of a filtered U-category, which will encompass the class of $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-categories. We adopted the idea of weak U-filteredness from Charles Rezk [71], who in turn generalised ideas from [1] to ∞-categories.

Definition 5.1.2.1. If U is an internal class of $\mathcal{B}$-categories, a $\mathcal{B}$-category J is *weakly* U*-filtered* if for every $A \in \mathcal{B}$ and every $\mathsf{I} \in \mathsf{U}(A)$ the diagonal functor $\pi_A^*\mathsf{J} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\mathrm{op}}, \pi_A^*\mathsf{J})$ is final. We define the internal class $\mathsf{wFilt}_{\mathsf{U}}$ as the full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the weakly $\pi_A^*\mathsf{U}$-filtered $\mathcal{B}_{/A}$-categories, for every $A \in \mathcal{B}$.

Remark 5.1.2.2. In the situation of Definition 5.1.2.1, as the condition of a functor of $\mathcal{B}$-categories being final is *local* in $\mathcal{B}$ by Remark 2.1.3.3, every object $A \to \mathsf{wFilt}_{\mathsf{U}}$ is weakly $\pi_A^*\mathsf{U}$-filtered. In particular, there is a canonical equivalence $\pi_A^* \mathsf{wFilt}_{\mathsf{U}} \simeq \mathsf{wFilt}_{\pi_A^*\mathsf{U}}$ for all $A \in \mathcal{B}$.

Remark 5.1.2.3. If $\mathsf{U} \hookrightarrow \mathsf{V}$ is an inclusion of internal classes, every weakly V-filtered $\mathcal{B}$-category is in particular weakly U-filtered. One thus obtains an inclusion $\mathsf{wFilt}_{\mathsf{V}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$.

Example 5.1.2.4. By Quillen's theorem A for $\mathcal{B}$-categories (Corollary 2.1.4.10), every functor that admits a left adjoint is final. Consequently, every $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category I is in particular weakly U-filtered.

Proposition 5.1.2.5. *A $\mathcal{B}$-category J is weakly U-filtered if and only if for every $A \in \mathcal{B}$ and every $\mathsf{I} \in \mathsf{U}(A)$ the colimit functor*

$$\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{J}, \mathsf{Grpd}_{\mathcal{B}_{/A}}) \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$$

preserves I-indexed limits of corepresentables.

Proof. To begin with, note that since the colimit of a diagram $f\colon \mathsf{J} \to \mathsf{Grpd}_{\mathcal{B}}$ is given by the groupoidification of the associated left fibration $\mathsf{J}_{f/}$ (Proposition 3.2.5.1), the colimit of every corepresentable is given by the final object in $\mathsf{Grpd}_{\mathcal{B}}$. In other words, there is a commutative square

$$\begin{tikzcd}
\mathsf{J}^{\mathrm{op}} \arrow[r, hook, "h_{\mathsf{J}^{\mathrm{op}}}"] \arrow[d, "\pi_{\mathsf{J}^{\mathrm{op}}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "\mathrm{colim}"] \\
1 \arrow[r, hook, "1_{\mathsf{Grpd}_{\mathcal{B}}}"] & \mathsf{Grpd}_{\mathcal{B}}.
\end{tikzcd}$$

As a result, for any diagram $d\colon \mathsf{I} \to \mathsf{J}^{\mathrm{op}}$, the presheaf $\mathrm{colim}\, h_{\mathsf{J}^{\mathrm{op}}} d$ is equivalent to the constant functor $1_{\mathsf{Grpd}_{\mathcal{B}}} \pi_{\mathsf{J}^{\mathrm{op}}} : \mathsf{J}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$. As the inclusion $1_{\mathsf{Grpd}_{\mathcal{B}}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ admits a left adjoint (Example 3.2.1.10) and is therefore continuous (Proposition 3.3.2.10), we conclude that the limit $\lim(\mathrm{colim}\, h_{\mathsf{J}^{\mathrm{op}}} d)$ is given by the final object in $\mathsf{Grpd}_{\mathcal{B}}$. Hence the canonical map

$$\mathrm{colim}(\lim h_{\mathsf{J}^{\mathrm{op}}} d) \to \lim(\mathrm{colim}\, h_{\mathsf{J}^{\mathrm{op}}} d)$$

is an equivalence if and only if the domain of this map is the final object as well. On account of the chain of equivalences

$$\begin{aligned}
\lim h_{\mathsf{J}^{\mathrm{op}}} d &\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}})}(h_{\mathsf{J}^{\mathrm{op}}}(-), \lim h_{\mathsf{J}^{\mathrm{op}}} d) \\
&\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}))}(\mathrm{diag}\, h_{\mathsf{J}^{\mathrm{op}}}(-), h_{\mathsf{J}^{\mathrm{op}}} d) \\
&\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{J}^{\mathrm{op}})}(\mathrm{diag}(-), d),
\end{aligned}$$

the functor $\lim h_{\mathsf{J}^{\mathrm{op}}} d$ classifies the left fibration $\mathsf{J}_{d^{\mathrm{op}}/} \to \mathsf{J}$. Hence $\operatorname{colim}(\lim h_{\mathsf{J}^{\mathrm{op}}} d)$ is the final object if and only if $(\mathsf{J}_{d^{\mathrm{op}}/})^{\mathrm{gpd}} \simeq 1$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, the same argumentation goes through for any $\mathsf{I} \in \mathsf{U}(A)$ and any diagram $d\colon \mathsf{I} \to \pi_A^* \mathsf{J}^{\mathrm{op}}$. By Quillen's theorem A for $\mathcal{B}$-categories (Corollary 2.1.4.10), the result thus follows. □

Corollary 5.1.2.6. *For every internal class* U *of* $\mathcal{B}$*-categories, any* U*-filtered* $\mathcal{B}$*-category is weakly* U*-filtered. In other words, there is an inclusion* $\mathsf{Filt}_{\mathsf{U}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$ *of internal classes.* □

Following the terminology introduced in [1], we may now make the following definition:

Definition 5.1.2.7. An internal class U of $\mathcal{B}$-categories is said to be *sound* if the inclusion $\mathsf{wFilt}_{\mathsf{U}} \hookrightarrow \mathsf{Filt}_{\mathsf{U}}$ is an equivalence. It is called *weakly sound* if for every $A \in \mathcal{B}$, every $\mathrm{op}(\pi_A^* \mathsf{U})$-cocomplete $\mathcal{B}_{/A}$-category is $\pi_A^* \mathsf{U}$-filtered.

Remark 5.1.2.8. On account of Remark 5.1.1.2 and Remark 5.1.2.2, the étale base change of a (weakly) sound internal class is (weakly) sound as well.

We finish this section with another characterisation of weakly U-filtered $\mathcal{B}$-categories that will be useful later. Recall from Definition 3.4.2.1 that if C is an arbitrary $\mathcal{B}$-category and V is an arbitrary internal class of $\mathcal{B}$-categories, we denote by $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{V}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ the full subcategory that is spanned by those objects $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ for which the domain of the associated right fibration $\mathsf{C}_{/F}$ is contained in $\mathsf{V}^{\mathrm{colim}}(A)$ (where $\mathsf{V}^{\mathrm{colim}}$ is the smallest colimit class containing V, see Definition 3.3.3.5). We now obtain:

Proposition 5.1.2.9. *A* $\mathcal{B}$*-category* J *is weakly* U*-filtered if and only if the inclusion*

$$\mathsf{J} \hookrightarrow \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$$

induced by the Yoneda embedding is final.

Proof. By Quillen's theorem A for $\mathcal{B}$-categories (Corollary 2.1.4.10), we find that the inclusion $\mathsf{J} \hookrightarrow \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ is final if and only if for every $A \in \mathcal{B}$ and every $\mathrm{op}(\mathsf{U})$-small presheaf $F\colon A \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ the groupoidification of the

$\mathcal{B}_{/A}$-category $\mathsf{J}_{F/}$ is the final object in $\mathcal{B}_{/A}$. By the same reasoning, J is weakly U-filtered if and only if for every $\mathsf{I} \in \mathsf{U}(A)$ and every diagram $d \colon \mathsf{I}^{\mathrm{op}} \to \pi_A^* \mathsf{J}$ the groupoidification of the $\mathcal{B}_{/A}$-category $\pi_A^* \mathsf{J}_{d/}$ is final in $\mathcal{B}_{/A}$. Hence it suffices to show that for every such diagram d, there is an object $F \colon A \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ such that $\pi_A^* \mathsf{J}_{d/} \simeq \mathsf{J}_{F/}$, and vice versa. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and by using Remark 3.4.2.3, we may assume that $A \simeq 1$. Now by Proposition 3.4.2.6, the colimit of $h_{\mathsf{J}} d \colon \mathsf{I} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})$ is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ and therefore defines a U-small presheaf F. By construction, we have an equivalence $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{J})_{F/} \simeq \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{J})_{h_{\mathsf{J}} d/}$ whose pullback along the Yoneda embedding determines an equivalence $\mathsf{J}_{d/} \simeq \mathsf{J}_{F/}$. Hence, if $\mathsf{J}_{F/}^{\mathrm{gpd}}$ is final, so is $\mathsf{J}_{d/}^{\mathrm{gpd}}$. Conversely, if we are given an arbitrary U-small presheaf F, the fact that $\mathsf{J}_{F/}^{\mathrm{gpd}}$ being final is *local* in $\mathcal{B}$ implies (by definition of what it means for a presheaf to be U-small) that we may safely assume that there is a diagram $d \colon \mathsf{I}^{\mathrm{op}} \to \mathsf{J}$ with $\mathsf{I} \in \mathsf{U}(1)$ such that $F \simeq \operatorname{colim} h_{\mathsf{J}} d$. By the same argument as above, we thus conclude that if $\mathsf{J}_{d/}^{\mathrm{gpd}}$ is final, so is $\mathsf{J}_{F/}$, which finishes the proof. □

5.1.3. Regular classes

Recall that a cardinal κ is said to be *regular* if it is infinite and if any κ-small union of κ-small sets is still κ-small. In this section, we will study an analogue of this condition in the context of internal classes of $\mathcal{B}$-categories. To that end, recall from the discussion in Section 4.4.1 that the Yoneda embedding $\Delta \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ (where Δ is implicitly regarded as a constant $\mathcal{B}$-category) factors through the embedding $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$, so that we may regard Δ as an internal class of $\mathcal{B}$-categories. We may now define:

Definition 5.1.3.1. An internal class U is said to be *right regular* if U contains Δ and if U is closed under U-colimits in $\mathsf{Cat}_{\mathcal{B}}$. We define the *right regularisation* $\mathsf{U}_{\to}^{\mathrm{reg}}$ of U to be the smallest right regular class that contains U.

Dually, U is called *left regular* if it contains Δ and is closed under $\mathrm{op}(\mathsf{U})$-colimits in $\mathsf{Cat}_{\mathcal{B}}$, and we define the *left regularisation* $\mathsf{U}_{\leftarrow}^{\mathrm{reg}}$ as the smallest left regular class that contains U.

Finally, we say that U is *regular* if it is both left and right regular, and we define the *regularisation* $\mathsf{U}^{\mathrm{reg}}$ of U as the smallest regular class that contains U.

Remark 5.1.3.2. An internal class U of $\mathcal{B}$-categories is left regular if and only if $\mathrm{op}(\mathsf{U})$ is right regular, and there is an evident equivalence $\mathrm{op}(\mathsf{U}^{\mathrm{reg}}_{\leftarrow}) \simeq \mathrm{op}(\mathsf{U})^{\mathrm{reg}}_{\rightarrow}$ of internal classes. In particular, if we have an equivalence $\mathsf{U} \simeq \mathrm{op}(\mathsf{U})$ of internal classes, then the notions of left and right regularity collapse to the notion of regularity, and the left/right regularisation of U is already its regularisation (cf. Corollary 5.1.3.5 below).

Remark 5.1.3.3. By the same argument as in the proof of Proposition 3.5.1.9, there is an equivalence $\pi_A^*(\mathsf{U}^{\mathrm{reg}}_{\rightarrow}) \simeq (\pi_A^*\mathsf{U})^{\mathrm{reg}}_{\rightarrow}$ for any internal class U and any $A \in \mathcal{B}$. In particular, the étale base change of a right regular class is still right regular. Similar observations can be made for the (left) regularisation of U.

Proposition 5.1.3.4. *For every internal class* U *of* $\mathcal{B}$*-categories, a* $\mathcal{B}$*-category is* U*-cocomplete if and only if it is* $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$*-cocomplete, and a functor between* $\mathcal{B}$*-categories is* U*-cocontinuous if and only if it is* $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$*-cocontinuous.*

Dually, a $\mathcal{B}$*-category is* U*-complete if and only if it is* $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$*-complete, and a functor between* $\mathcal{B}$*-categories is* U*-complete if and only if it is* $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$*-continuous.*

Proof. We only prove the first statement, the second one follows by dualisation. So let C be a U-cocomplete $\mathcal{B}$-category, and let V be the largest internal class of $\mathcal{B}$-categories subject to the condition that C is V-cocomplete. Clearly V contains Δ since every $\mathcal{B}$-category is Δ-cocomplete (cf. Remark 3.3.2.4). Moreover, Proposition 3.5.4.1 implies that for any $\mathsf{I} \in \mathsf{V}(1)$ and any diagram $d\colon \mathsf{I} \to \mathsf{V}$ with colimit K, the $\mathcal{B}$-category C admits K-indexed colimits, which implies that $\mathsf{K} \in \mathsf{V}(1)$ by maximality of V. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the same argument, one concludes that V is closed under V-colimits in $\mathsf{Cat}_{\mathcal{B}}$ and must therefore contain $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$. An analogous argument also shows that every U-cocontinuous functor is $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$-cocontinuous. $\square$

Corollary 5.1.3.5. *The right (left) regularisation of an internal class* U *is the smallest internal class that contains* U *and* Δ *and that is closed under* U*-colimits* ($\mathrm{op}(\mathsf{U})$*-colimits) in* $\mathsf{Cat}_{\mathcal{B}}$.

Proof. This is an immediate consequence of the observation that by Proposition 5.1.3.4, an internal class V of $\mathcal{B}$-categories is closed under U-colimits ($\mathrm{op}(\mathsf{U})$-colimits) in $\mathsf{Cat}_{\mathcal{B}}$ if and only if it is closed under $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$-colimits ($\mathrm{op}(\mathsf{U}^{\mathrm{reg}}_{\leftarrow})$-colimits) in $\mathsf{Cat}_{\mathcal{B}}$. $\square$

Proposition 5.1.3.6. *For every internal class* U*, the inclusion* $\mathsf{Filt}_{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}} \hookrightarrow \mathsf{Filt}_{\mathsf{U}}$ *is an equivalence.*

Proof. In light of Remark 5.1.3.3 and Remark 5.1.1.2, it suffices to show that every U-filtered $\mathcal{B}$-category J is already $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-filtered. This amounts to showing that the functor

$$\operatorname*{colim}_{\mathsf{J}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$$

is $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-continuous. By Proposition 5.1.3.4, this is immediate. □

Corollary 5.1.3.7. *The left regularisation of a (weakly) sound internal class is also (weakly) sound.*

Proof. Suppose that U is sound, i.e. that $\mathsf{Filt}_{\mathsf{U}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$ is an equivalence. Since $\mathsf{U} \hookrightarrow \mathsf{U}^{\mathrm{reg}}_{\leftarrow}$ implies that we have an inclusion $\mathsf{wFilt}_{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$, Proposition 5.1.3.6 implies that the inclusion $\mathsf{Filt}_{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}^{\mathrm{reg}}}$ is also an equivalence, hence $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$ is sound. The case where U is weakly sound follows from a similar argument. □

For the study of accessibility and presentability of $\mathcal{B}$-categories, we will generally need to restrict our attention to those internal classes of $\mathcal{B}$-categories that are themselves *small* $\mathcal{B}$-categories. It will therefore be useful to give such internal classes a dedicated name. Again following [1], we thus define:

Definition 5.1.3.8. An internal class U of $\mathcal{B}$-categories is a *doctrine* if U is a small $\mathcal{B}$-category.

Proposition 5.1.3.9. *The (left/right) regularisation of a doctrine is still a doctrine.*

Proof. It suffices to show that any doctrine U is contained in a regular doctrine V. We will explicitly construct such a doctrine in Section 5.2.2 below, cf. Remark 5.2.2.22. □

5.1.4. The decomposition property

It is well-known that for every regular cardinal κ, any ∞-category can be written as a κ-filtered colimit of κ-small ∞-categories. In order to obtain a well-behaved notion of accessibility internal to $\mathcal{B}$, it will be crucial to have an analogue of this property for $\mathcal{B}$-categories. This leads us to the following definition:

Definition 5.1.4.1. An internal class U of $\mathcal{B}$-categories is said to have the *decomposition property* if for every $A \in \mathcal{B}$ and every $\mathcal{B}_{/A}$-category C, there is a $\pi_A^*\mathsf{U}$-filtered $\mathcal{B}_{/A}$-category J and a diagram $d\colon \mathsf{J} \to \pi_A^*\mathsf{U}$ with colimit C.

Remark 5.1.4.2. In the situation of Definition 5.1.4.1, by applying the decomposition property to $\mathsf{D} = \mathsf{C}^{\mathrm{op}}$, one deduces that C can also be obtained as a $\pi_A^*\mathsf{U}$-filtered colimit of a diagram in $\mathrm{op}(\pi_A^*\mathsf{U})$.

The main goal of this section is to show:

Proposition 5.1.4.3. *Every left regular and weakly sound internal class* U *has the decomposition property.*

Before we can prove Proposition 5.1.4.3, we need the following lemma:

Lemma 5.1.4.4. *Let* C *be a small* $\mathcal{B}$*-category and let* $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *be a full subcategory that contains* C*. Then any presheaf* $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ *is the colimit of the diagram* $\mathsf{D}_{/F} \to \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Proof. By Proposition 3.2.4.3, it suffices to show that the final object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ is the colimit of the inclusion $\mathsf{D}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$. In light of the inclusions

$$\mathsf{C}_{/F} \hookrightarrow \mathsf{D}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$$

and by using the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})$ from Lemma 3.4.1.4, we may thus assume that F is the final object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Moreover, as the inclusion $\mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\widehat{\mathcal{B}}}$ is cocontinuous (Example 3.3.2.14) we may enlarge our universe and thus assume without loss of generality that D is *small*. Now let $i\colon \mathsf{C} \hookrightarrow \mathsf{D}$ and $j\colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the inclusions. Since the identity on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is the left Kan extension of the Yoneda embedding h along itself (Theorem 3.5.1.1), we obtain equivalences $j \simeq j^* j_! i_!(h) \simeq i_!(h)$, where the functor $i_!$ exists since D is small (Corollary 3.4.3.7). Therefore, the identity on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is also the left Kan extension of j along itself. The claim now follows from the explicit description of the left Kan extension in Remark 3.4.3.6, which implies that we have equivalences $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})} \simeq j_!(j)(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}) \simeq \operatorname{colim} j$. □

Proof of Proposition 5.1.4.3. By Remark 5.1.3.3 and Remark 5.1.2.8, it suffices to show that every $\mathcal{B}$-category C is a U-filtered colimit of a diagram in U. As U

by regularity contains Δ and since the localisation functor $\underline{\mathrm{PSh}}_{\mathcal{B}}(\Delta) \to \mathrm{Cat}_{\mathcal{B}}$ is cocontinuous, we deduce from Lemma 5.1.4.4 that C arises as the colimit of the diagram $\mathsf{U}_{/\mathsf{C}} \to \mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$. We therefore only need to show that $\mathsf{U}_{/\mathsf{C}}$ is U-filtered. Using that U is weakly sound, it will suffice to show that $\mathsf{U}_{/\mathsf{C}}$ is $\mathrm{op}(\mathsf{U})$-cocomplete. By Proposition 3.3.2.13, the $\mathcal{B}$-category $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ is cocomplete and the projection $(\pi_{\mathsf{C}})_!$ is cocontinuous. As the inclusion $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is closed under $\mathrm{op}(\mathsf{U})$-colimits, the desired result follows from Lemma 3.5.1.11. □

Corollary 5.1.4.5. *Let* U *be a weakly sound internal class of* $\mathcal{B}$*-categories. Then a (large)* $\mathcal{B}$*-category* C *is cocomplete if and only if* C *is both* $\mathrm{op}(\mathsf{U})$*- and* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete. Similarly, a functor* $f\colon \mathsf{C} \to \mathsf{D}$ *between cocomplete (large)* $\mathcal{B}$*-categories is cocontinuous if and only if it is both* $\mathrm{op}(\mathsf{U})$*- and* $\mathsf{Filt}_{\mathsf{U}}$*-cocontinuous.*

Proof. We prove the first statement, the second one follows by a similar argument. Since the claim is clearly necessary, it suffices to prove the converse. So let us assume that C is both $\mathrm{op}(\mathsf{U})$- and $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. By Proposition 5.1.3.4 and Proposition 5.1.3.6, we may assume without loss of generality that U is left regular. Proposition 5.1.4.3 now implies that U has the decomposition property. By definition and in light of Remark 5.1.4.2, this means that $(\mathrm{op}(\mathsf{U}) \cup \mathsf{Filt}_{\mathsf{U}})^{\mathrm{reg}}_{\to} = \mathsf{Cat}_{\mathcal{B}}$. Appealing once more to Proposition 5.1.3.4, the claim follows. □

5.1.5. U-compact objects

Recall from Section 3.5.2 that if V is an internal class and if C is a V-cocomplete $\mathcal{B}$-category, we say that an object $c\colon A \to \mathsf{C}$ is V-*cocontinuous* if the functor $\mathrm{map}_{\mathsf{C}}(c, -)\colon \pi_A^*\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ is $\pi_A^*\mathsf{V}$-cocontinuous. In this section, we specialise this concept to the case where $\mathsf{V} = \mathsf{Filt}_{\mathsf{U}}$ for some internal class U. This leads us to the notion of a U-*compact object*, which is the internal analogue of the concept of a κ-compact object in an ∞-category, where κ is a cardinal.

Definition 5.1.5.1. Let U be an internal class of $\mathcal{B}$-categories, and let C be a $\mathsf{Filt}_{\mathsf{U}}$-cocomplete $\mathcal{B}$-category. An object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ is said to be U-*compact* if it is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous, i.e. if the functor

$$\mathrm{map}_{\mathsf{C}}(c, -)\colon \pi_A^*\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$$

is $\mathsf{Filt}_{\pi_A^*\mathsf{U}}$-cocontinuous. We denote by $\mathsf{C}^{\mathsf{U}\text{-cpt}}$ the full subcategory of C that is spanned by the U-compact objects.

Remark 5.1.5.2. In the situation of Definition 5.1.5.1, an object $c\colon A \to \mathsf{C}$ is contained in $\mathsf{C}^{\mathsf{U}\text{-cpt}}$ if and only if it is U-compact. This is a direct consequence of Remark 3.5.2.2. Together with Remark 5.1.1.2, this implies that if $A \in \mathcal{B}$ is an arbitrary object in $\mathcal{B}$, there is a natural equivalence $\pi_A^*(\mathsf{C}^{\mathsf{U}\text{-cpt}}) \simeq (\pi_A^*\mathsf{C})^{\pi_A^*\mathsf{U}\text{-cpt}}$.

Lemma 5.1.5.3. *Let* U *be an internal class of* $\mathcal{B}$*-categories, and let* C *be a* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete* $\mathcal{B}$*-category. Then the full subcategory*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}\text{-cc}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$$

of $\mathsf{Filt}_{\mathsf{U}}$*-cocontinuous functors is closed under* U*-limits.*

Proof. Using Remark 3.3.3.4 and Remark 5.1.1.2, it will suffice to show that whenever I is a $\mathcal{B}$-category that is contained in $\mathsf{U}(1)$ and $d\colon \mathsf{I} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})^{\mathsf{Filt}_{\mathsf{U}}}$ is a diagram, then the limit $\lim d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous. We may compute $\lim d$ as the composition

$$\mathsf{C} \xrightarrow{d'} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{\lim_{\mathsf{I}}} \mathsf{Grpd}_{\mathcal{B}},$$

where d' is the transpose of d. Since $\lim_{\mathsf{I}}$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous by Remark 5.1.1.4, it thus suffices to show that d' is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous as well. This can be extracted as a special case of Lemma 5.5.1.3 below. □

Proposition 5.1.5.4. *Let* U *be an internal class and let* C *be an* $\mathrm{op}(\mathsf{U})$*- and* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete* $\mathcal{B}$*-category. Then the subcategory* $\mathsf{C}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{C}$ *is closed under* $\mathrm{op}(\mathsf{U})$*-colimits in* C*.*

Proof. By using Remark 5.1.5.2, it suffices to show that whenever I is a $\mathcal{B}$-category that is contained in $\mathsf{U}(1)$ and $d\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{C}^{\mathsf{U}\text{-cpt}}$ is a diagram, the colimit $\operatorname{colim} d$ in C is U-compact. As we have noted in Proposition 3.3.2.15, the Yoneda embedding $h_{\mathsf{C}^{\mathrm{op}}}\colon \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$ is $\mathrm{op}(\mathsf{U})$-continuous, so that we can identify $\operatorname{map}_{\mathsf{C}}(\operatorname{colim} d, -)$ with the limit of the diagram $h_{\mathsf{C}^{\mathrm{op}}} d^{\mathrm{op}}$. The desired result now follows from Lemma 5.1.5.3. □

Definition 5.1.5.5. If $\mathsf{C} \hookrightarrow \mathsf{D}$ is a fully faithful functor of $\mathcal{B}$-categories, the $\mathcal{B}$-category $\mathrm{Ret}_{\mathsf{D}}(\mathsf{C})$ of *retracts* of C in D is the full subcategory of D that is spanned by those objects $d\colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ for which there is an object $c\colon A \to \mathsf{C}$ and a commutative diagram

$$\begin{tikzcd} & c \arrow[dr] & \\ d \arrow[ur] \arrow[rr, "\mathrm{id}"] & & d. \end{tikzcd}$$

Remark 5.1.5.6. In the situation of Definition 5.1.5.5, there are inclusions $\mathsf{C} \hookrightarrow \mathrm{Ret}_{\mathsf{D}}(\mathsf{C}) \hookrightarrow \mathsf{D}$. Furthermore, an object $d\colon A \to \mathsf{D}$ is contained in $\mathrm{Ret}_{\mathsf{C}}(\mathsf{D})$ precisely if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $s_i^*(d)\colon A_i \to \mathsf{D}$ is a retract of an object $c\colon A_i \to \mathsf{C}$. Therefore, if C is small and D is locally small (in the sense of Definition 2.3.1.1), then $\mathrm{Ret}_{\mathsf{D}}(\mathsf{C})$ is small as well: in fact, by Proposition 2.3.1.5, this follows once we verify that $\mathrm{Ret}_{\mathsf{D}}(\mathsf{C})_0$ is small. Since the latter admits a small cover

$$\bigsqcup_{G\in\mathcal{G}} \bigsqcup_{d\in\mathrm{Ret}_{\mathsf{D}(G)}(\mathsf{C}(G))} G \twoheadrightarrow \mathrm{Ret}_{\mathsf{D}}(\mathsf{C})_0$$

where $\mathcal{G} \subset \mathcal{B}$ is a small generating subcategory and where $\mathrm{Ret}_{\mathsf{D}(G)}(\mathsf{C}(G))$ denotes the full subcategory of $\mathsf{D}(G)$ that is spanned by the retracts of $\mathsf{C}(G)$, which is clearly a small ∞-category, this is immediate.

Lemma 5.1.5.7. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category. Then the full subcategory*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$$

of U*-cocontinuous functors is closed under retracts.*

Proof. By Remark 3.3.3.4, it will suffice to show that whenever a copresheaf $F\colon \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$ is a retract of a U-cocontinuous functor $G\colon \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}$, then F is U-cocontinuous as well. Let $R = \Delta^2 \sqcup_{\Delta^1} \Delta^0$ be the walking retract diagram, i.e. the quotient of Δ^2 that is obtained by collapsing $d^1\colon \Delta^1 \hookrightarrow \Delta^2$ to a point. Then the datum of retract $F \to G \to F$ is tantamount to a map $r\colon \mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}^R$. Since the retract of an equivalence is an equivalence as well, the functor $d_{\{1\}}\colon \mathsf{Grpd}_{\mathcal{B}}^R \to \mathsf{Grpd}_{\mathcal{B}}$ that is obtained by evaluation at $\{1\} \in \Delta^2 \twoheadrightarrow R$

must be conservative. By combining this observation with the fact that $d_{\{1\}}$ is cocontinuous, the equivalence $d_{\{1\}}r \simeq G$ and the functoriality of mates, we conclude that the map $\operatorname{colim}_{\mathsf{I}} r_* \to r \operatorname{colim}_{\mathsf{I}}$ is an equivalence for every $\mathsf{I} \in \mathsf{U}(1)$. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the same argument, we thus find that r is U-cocontinuous. As we recover F by postcomposing r with the cocontinuous functor $d_{\{0\}} \colon \mathsf{Grpd}_{\mathcal{B}}^{R} \to \mathsf{Grpd}_{\mathcal{B}}$, the claim follows. □

Proposition 5.1.5.8. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* C *be a* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete* $\mathcal{B}$*-category. Then* $\mathsf{C}^{\mathsf{U}\text{-cpt}}$ *is closed under retracts in* C*, in the sense that the inclusion*

$$\mathsf{C}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{Ret}_{\mathsf{C}}(\mathsf{C}^{\mathsf{U}\text{-cpt}})$$

is an equivalence.

Proof. It suffices to show that the retract of a U-compact object in C is U-compact as well, which immediately follows from Lemma 5.1.5.7. □

We conclude this section with a characterisation of U-compact objects in presheaf $\mathcal{B}$-categories. This will require the following lemma:

Lemma 5.1.5.9. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* $\mathsf{C} \hookrightarrow \mathsf{D}$ *be a full inclusion of* $\mathcal{B}$*-categories such that* D *is* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete. Let* J *be a* U*-filtered* $\mathcal{B}$*-category, let* $d \colon \mathsf{J} \to \mathsf{C} \hookrightarrow \mathsf{D}$ *be a diagram and suppose that* $F = \operatorname{colim} d$ *is a* U*-compact object in* D*. Then* F *is contained in* $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$*.*

Proof. The object F being U-compact implies that the canonical map

$$\phi \colon \operatorname{colim} \operatorname{map}_{\mathsf{D}}(F, d(-)) \to \operatorname{map}_{\mathsf{D}}(F, F)$$

must be an equivalence. Thus the identity on F gives rise to a global section

$$\mathrm{id}_F \colon 1 \to \operatorname{colim} \operatorname{map}_{\mathsf{D}}(F, d(-)).$$

Let $p \colon \mathsf{P} \to \mathsf{J}$ be the left fibration classified by the copresheaf $\operatorname{map}_{\mathsf{D}}(F, d(-))$. Since the map $\mathsf{P} \to \mathsf{P}^{\mathrm{gpd}} \simeq \operatorname{colim} \operatorname{map}_{\mathsf{D}}(F, d(-))$ is essentially surjective (by Lemma 1.3.2.8), the map $\mathsf{P}_0 \to \mathsf{P}^{\mathrm{gpd}}$ is a cover in $\mathcal{B}$ (Corollary 1.3.2.15), so that we can find a cover $s \colon A \twoheadrightarrow 1$ in $\mathcal{B}$ and a local section $x \colon A \to \mathsf{P}$ such that the composite with $\mathsf{P} \to \mathsf{P}^{\mathrm{gpd}}$ recovers $\pi_A^* \mathrm{id}_F$. Let $j = p(x)$. Then x defines an object $f\colon A \to \mathsf{P}|_j \simeq \mathrm{map}_{\mathsf{D}}(\pi_A^* F, d(j))$ that is carried to $\pi_A^* \mathrm{id}_F$ by the canonical morphism $\mathrm{map}_{\mathsf{D}}(\pi_A^* F, d(j)) \to \mathrm{map}_{\mathsf{D}}(\pi_A^* F, \pi_A^* D)$. In other words, composing $f\colon \pi_A^* F \to d(j)$ with the map $d(j) \to \pi_A^* F$ into the colimit yields $\pi_A^* F$. As this precisely means that $\pi_A^* F$ is a retract of $d(j)$, the claim follows. □

Proposition 5.1.5.10. *Let* U *be an internal class of* $\mathcal{B}$*-categories that has the decomposition property, and let* C *be a* $\mathcal{B}$*-category. Then there is an equivalence*

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}} \simeq \mathsf{Ret}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}))$$

of full subcategories in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*. In particular,* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ *is small.*

Proof. Yoneda's lemma implies that every representable presheaf is U-compact. By combining this observation with Proposition 5.1.5.8 and Proposition 5.1.5.4, one thus obtains an inclusion

$$\mathsf{Ret}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}.$$

As for the converse inclusion, suppose that $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is a U-compact presheaf. By Remark 5.1.4.2, there exists a U-filtered $\mathcal{B}$-category J and a diagram $d\colon \mathsf{J} \to \mathrm{op}(\mathsf{U})$ such that $\mathsf{C}_{/F} \simeq \mathrm{colim}\, d$ in $\mathrm{Cat}_{\mathcal{B}}$. Proposition 3.5.4.10 then shows that F is the colimit of a J-indexed diagram in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})$. As F is U-compact and J is U-filtered, Lemma 5.1.5.9 shows that F is locally a retract of an object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})$. By Remark 5.1.5.2 and Remark 3.4.2.3, if $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is an arbitrary object, we can replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and carry out the same argument as above, which shows that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is contained in $\mathsf{Ret}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}))$. □

Corollary 5.1.5.11. *Let* U *be an internal class of* $\mathcal{B}$*-categories that has the decomposition property, and let* J *be a weakly* U*-filtered* $\mathcal{B}$*-category. Then the inclusion* $\mathsf{J} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ *is final.*

Proof. Proposition 5.1.5.10 shows that any U-compact presheaf $F\colon \mathsf{J}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ arises as a retract of some object $G\colon 1 \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ after passing to a suitable cover of $1 \in \mathcal{B}$. Thus, the right fibration $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})_{/F} \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ is *locally* a retract of a representable right fibration, so that we must have

$(\underline{\mathrm{Small}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(\mathsf{J})_{/F})^{\mathrm{gpd}} \simeq 1$ as the latter property can be checked locally in $\mathcal{B}$. By Remark 5.1.5.2 and Remark 3.4.2.3, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ to arrive at the same conclusion for any object $F\colon A \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$. Using Quillen's theorem A (Corollary 2.1.4.10), this shows that the inclusion $\underline{\mathrm{Small}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(\mathsf{J}) \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is final. Hence the claim follows from Proposition 5.1.2.9. $\square$

Corollary 5.1.5.12. *A left regular class* U *is sound if and only if it is weakly sound.*

Proof. Using Remark 5.1.1.2 and Remark 5.1.2.8, it suffices to show that whenever U is weakly sound, every weakly U-filtered $\mathcal{B}$-category J is U-filtered. Since Proposition 5.1.4.3 implies that U has the decomposition property, Corollary 5.1.5.11 shows that the inclusion $\mathsf{J} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is final. By Proposition 5.1.5.4, the $\mathcal{B}$-category $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is $\mathrm{op}(\mathsf{U})$-cocomplete and therefore U-filtered since U is by assumption weakly sound. Now if $\mathsf{I} \in \mathsf{U}(1)$ is chosen arbitrarily, the fact that $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$ is cocomplete allows us to extend any diagram $d\colon \mathsf{J} \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$ to a diagram $d'\colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}} \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$, using the universal property of presheaf $\mathcal{B}$-categories. As $\mathsf{J} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is final, the limit functor $\lim_{\mathsf{I}}\colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Grpd}_{\mathcal{B}}$ preserves the colimit of d if and only if it preserves the colimit of d' (see Proposition 3.2.7.1), which is indeed the case as $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is U-filtered. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and carrying out the same argument (which is possible by Remark 5.1.5.2), this already implies that J must be U-filtered, as desired. $\square$

5.2. Cardinality in internal higher category theory

In our treatment of accessibility and presentability for $\mathcal{B}$-categories later in this thesis, we will rely on the existence of an ample amount of doctrines that satisfy the decomposition property. Therefore, it will be crucial to know that there are sufficiently many *(left) regular and sound doctrines* in any ∞-topos $\mathcal{B}$. The main objective of this section is to construct such internal classes. More precisely, our approach is to first construct what we call the *canonical bifiltration* of the $\mathcal{B}$-category $\mathrm{Cat}_{\mathcal{B}}$, i.e. a 2-dimensional filtration by internal classes which can be regarded as a way to order $\mathcal{B}$-categories by *size*. The first dimension of this bifiltration is parametrised by *cardinals*, and the second one by the poset of *local classes* in $\mathcal{B}$. The canonical bifiltration will be *exhaustive*, so that every $\mathcal{B}$-category can be assigned an upper bound in size, and it will be exclusively comprised of regular doctrines. We carry out the construction of this bifiltration in Section 5.2.1. In Section 5.2.2, we discuss how one can extract a particularly well-behaved subfiltration from the canonical bifiltration that is still exhaustive and in which each member is *sound.* The latter will be parametrised by a class of cardinals that satisfy a property which depends on the ∞-topos $\mathcal{B}$ and that we refer to as $\mathcal{B}$*-regularity.* Finally, we discuss a particular member of the canonical bifiltration in Section 5.2.3, that of *finite* $\mathcal{B}$-categories.

5.2.1. The canonical bifiltration of the $\mathcal{B}$-category of $\mathcal{B}$-categories

Recall from Example 3.3.1.4 that if $\mathcal{K}$ is an arbitrary class of ∞-categories (i.e. a full subcategory of Cat_∞), we denote by $\mathsf{LConst}_{\mathcal{K}}$ the essential image of the canonical functor $\mathcal{K} \hookrightarrow \mathrm{Cat}_\infty \to \mathsf{Cat}_{\mathcal{B}}$. If S is a local class of morphisms in $\mathcal{B}$, we denote by $\langle \mathcal{K}, S\rangle$ the internal class of $\mathcal{B}$-categories that is generated by $\mathsf{LConst}_{\mathcal{K}}$ and Grpd_S.

Definition 5.2.1.1. Let $\mathcal{K} \subset \mathrm{Cat}_\infty$ be a class of ∞-categories and let S be a local class of morphisms in $\mathcal{B}$. We define the internal class $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$ of $\langle \mathcal{K}, S\rangle$*-small* $\mathcal{B}$*-categories* as the left regularisation of $\langle \mathcal{K}, S\rangle$. We denote its underlying ∞-category of global sections by $\mathrm{Cat}(\mathcal{B})^{\langle \mathcal{K}, S\rangle}$.

Remark 5.2.1.2. In the situation of Definition 5.2.1.1, let us denote by $\pi_A^* S$ the class of those maps in $\mathcal{B}_{/A}$ whose underlying map in $\mathcal{B}$ is contained in S. Since $(\pi_A)_!$ preserves small colimits and covers, this is still a local class, and one has a natural equivalence $\pi_A^*(\mathsf{Grpd}_S) \simeq \mathsf{Grpd}_{\pi_A^* S}$ of subuniverses. With this understood, Remark 5.1.3.3 gives rise to a canonical equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^{\langle \mathcal{K}, \pi_A^* S\rangle}$ for every $A \in \mathcal{B}$.

By combining Proposition 5.1.3.4 with Remark 3.3.2.9, one finds:

Proposition 5.2.1.3. *A (large)* $\mathcal{B}$*-category* C *is* $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$*-complete precisely if*

1. *The* ∞*-category* $\mathsf{C}(A)$ *admits limits indexed by objects in* $\mathcal{K}$*, and for every map* $s\colon B \to A$ *the transition functor* $s^* : \mathsf{C}(A) \to \mathsf{C}(B)$ *preserves these limits;*

2. *For every map $p\colon P \to A$ in S, the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(P)$ admits a right adjoint s_*, and for every cartesian square*

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

in $\mathcal{B}$ in which p (and therefore q) are contained in S, the natural map $s^* p_* \to q_* t^*$ is an equivalence.

Moreover, a functor $f\colon \mathsf{C} \to \mathsf{D}$ of $\mathrm{Cat}_{\mathcal{B}}^{\langle \mathcal{K},S\rangle}$-complete $\mathcal{B}$-categories is $\mathrm{Cat}_{\mathcal{B}}^{\langle \mathcal{K},S\rangle}$-continuous precisely if for all $A \in \mathcal{B}$ the functor $f(A)$ preserves limits indexed by objects in $\mathcal{K}$, and for all maps $p\colon P \to A$ in S the natural morphism $f(A)p_* \to p_* f(P)$ is an equivalence.

The dual statements about cocompleteness and cocontinuity (both understood with respect to the right regular class $\mathrm{op}(\mathrm{Cat}_{\mathcal{B}}^{\langle \mathcal{K},S\rangle})$) hold as well. □

In the situation of Definition 5.2.1.1, note that whenever $\mathcal{K}$ is a doctrine (i.e. a small ∞-category) and S is bounded (i.e. the subuniverse Grpd_S that corresponds to S is small), Proposition 5.1.3.9 implies that $\mathrm{Cat}_{\mathcal{B}}^{\langle \mathcal{K},S\rangle}$ is a doctrine. Therefore, assigning to a pair $(\mathcal{K}, S)$ the regular class $\mathrm{Cat}_{\mathcal{B}}^{\langle \mathcal{K},S\rangle}$ defines a map of posets

$$\mathrm{Sub}_{\mathrm{full}}^{\mathrm{small}}(\mathrm{Cat}_\infty) \times \mathrm{Sub}_{\mathrm{full}}^{\mathrm{small}}(\mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Sub}_{\mathrm{full}}^{\mathrm{small}}(\mathrm{Cat}_{\mathcal{B}})$$

that we refer to as the *canonical bifiltration of* $\mathrm{Cat}_{\mathcal{B}}$.

Remark 5.2.1.4. The canonical bifiltration is *exhaustive*. In fact, if C is an arbitrary $\mathcal{B}_{/A}$-category, we may find a diagram $d\colon \mathcal{J} \to \mathrm{Cat}(\mathcal{B}_{/A})$ with colimit C such that for all $j \in \mathcal{J}$ one has $d(j) \simeq \Delta^n \otimes B$ for some $n \geq 0$ and some $B \in \mathcal{B}_{/A}$. Note that $\Delta^n \otimes B$ can be identified with the B-indexed colimit of the constant diagram in $\mathrm{Cat}_{\mathcal{B}_{/A}}$ with value Δ^n. Therefore, by choosing $\mathcal{K}$ to be the doctrine of ∞-categories spanned by the single object $\mathcal{J} \in \mathrm{Cat}_\infty$ and choosing S to be the bounded local class that is generated by the maps $B \to A$, we find that C is $\langle \mathcal{K}, S\rangle$-small.

Example 5.2.1.5. Since every $\mathcal{B}$-category is a small colimit of objects of the form $\Delta^n \otimes A$ with $n \geq 0$ and $A \in \mathcal{B}$ (see Remark 1.2.1.3), we deduce that the regularisation of $\langle \mathrm{Cat}_\infty, \mathrm{all}\rangle$ is $\mathrm{Cat}_{\mathcal{B}}$ (where the local class all is the class of all morphisms in $\mathcal{B}$).

5.2.2. κ-small $\mathcal{B}$-categories

Let κ be a cardinal. Recall from [49, § 6.1.6] that a map $p\colon P \to A$ in $\mathcal{B}$ is said to be *relatively κ-compact* if for every κ-compact $B \in \mathcal{B}$ and every map $s\colon B \to A$, the pullback s^*P is κ-compact as well. We denote by κ-cpt the local class of morphisms in $\mathcal{B}$ that is generated by the relatively κ-compact morphisms, and we let $\mathrm{Grpd}_{\mathcal{B}}^{\kappa}$ be the associated subuniverse. Explicitly, a map $p\colon P \to A$ is contained in κ-cpt precisely if there is a cover $(s_i)_i\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $s_i^* p$ is relatively κ-compact for each i.

Let us denote by $\mathrm{Cat}_{\infty}^{\kappa}$ the doctrine of κ-small ∞-categories. We may now define:

Definition 5.2.2.1. A $\mathcal{B}$-category is said to be *κ-small* if it is $\langle \mathrm{Cat}_{\infty}^{\kappa}, \kappa\text{-cpt}\rangle$-small. We will use the notation $\mathrm{Cat}_{\mathcal{B}}^{\kappa} = \mathrm{Cat}_{\mathcal{B}}^{\langle \mathrm{Cat}_{\infty}^{\kappa}, \kappa\text{-cpt}\rangle}$ to denote the internal class of κ-small $\mathcal{B}$-categories, and we denote its underlying ∞-category of global sections by $\mathrm{Cat}(\mathcal{B})^{\kappa}$.

Remark 5.2.2.2. Note that for general $A \in \mathcal{B}$ there is no reason to expect an equivalence $\pi_A^*(\mathrm{Grpd}_{\mathcal{B}}^{\kappa}) \simeq \mathrm{Grpd}_{\mathcal{B}_{/A}}^{\kappa}$. Therefore, we can also not expect to have an equivalence $\pi_A^* \mathrm{Cat}_{\mathcal{B}}^{\kappa} \simeq \mathrm{Cat}_{\mathcal{B}_{/A}}^{\kappa}$. The situation improves, however, when A is assumed to be κ-compact. In this case, the observation that an object in $\mathcal{B}_{/A}$ is κ-compact if and only if its underlying object in $\mathcal{B}$ is κ-compact implies that a map in $\mathcal{B}_{/A}$ is relatively κ-compact if and only its underlying map in $\mathcal{B}$ is relatively κ-compact, so that we obtain an equivalence $\pi_A^*(\mathrm{Grpd}_{\mathcal{B}}^{\kappa}) \simeq \mathrm{Grpd}_{\mathcal{B}_{/A}}^{\kappa}$. By using Remark 5.2.1.2, this equivalence in turn induces an equivalence $\pi_A^* \mathrm{Cat}_{\mathcal{B}}^{\kappa} \simeq \mathrm{Cat}_{\mathcal{B}_{/A}}^{\kappa}$.

The internal class $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$ is not very well-behaved for arbitrary cardinals κ. Therefore, we will restrict our attention to a certain class of cardinals that are in a sense *adapted* to the ∞-topos $\mathcal{B}$.

Definition 5.2.2.3. We say that cardinal κ is *$\mathcal{B}$-regular* if

1. κ is regular and uncountable;
2. $\mathcal{B}$ is κ-accessible;
3. the full subcategory $\mathcal{B}^{\kappa\text{-cpt}} \hookrightarrow \mathcal{B}$ of κ-compact objects in $\mathcal{B}$ is closed under finite limits and subobjects in $\mathcal{B}$.

Remark 5.2.2.4. Every uncountable regular cardinal κ is Ani-regular. In fact, condition (2) is immediate, and $1 \in \mathrm{Ani}$ is certainly κ-compact. Moreover, if $P = A \times_C B$ is a pullback of κ-compact ∞-groupoids, descent implies $P \simeq \mathrm{colim}_{a \in A} P|_a$. Since κ-compact ∞-groupoids are precisely those which are κ-small [49, Corollary 5.4.1.5] and since κ-compact objects in Ani are stable under κ-small colimits, it suffices to show that $P|_a$ is κ-compact. We may therefore reduce to the case where $A \simeq 1$. By the same reasoning, we can assume $B \simeq 1$ as well. But then P can be identified with a mapping ∞-groupoid of C, which is κ-small by again making use of [49, Corollary 5.4.1.5]. Finally, the identification of κ-compact ∞-groupoids with κ-small ∞-groupoids also shows that these are stable under subobjects. Hence condition (3) is satisfied as well.

Remark 5.2.2.5. Note that there is an ample amount of $\mathcal{B}$-regular cardinals, in the sense that if κ' is an arbitrary cardinal one can always find a larger cardinal $\kappa \geq \kappa'$ that is $\mathcal{B}$-regular. Indeed, by enlarging κ' if necessary one can always arrange for $\mathcal{B}$ to be κ'-accessible. Then for any (uncountable) $\kappa \gg \kappa'$ (in the sense of [49, Definition A.2.6.3]) for which $\mathcal{B}^{\kappa'\text{-cpt}}$ is κ-small, an object in $\mathcal{B}$ is κ-compact if and only if the underlying presheaf on $\mathcal{B}^{\kappa'\text{-cpt}}$ takes values in the full subcategory $\mathrm{Ani}^{\kappa\text{-cpt}} \hookrightarrow \mathrm{Ani}$ of κ-compact ∞-groupoids [49, Lemma 5.4.7.5]. In combination with Remark 5.2.2.4, this shows that κ is $\mathcal{B}$-regular. In particular, this argument shows that we can always find a $\mathcal{B}$-regular κ such that $\kappa \gg \kappa'$.

Remark 5.2.2.6. If κ is a $\mathcal{B}$-regular cardinal, then κ is also $\mathcal{B}_{/A}$-regular for every *κ-compact* object $A \in \mathcal{B}$. In fact, since an object in $\mathcal{B}_{/A}$ is κ-compact if and only if its image along $(\pi_A)_!$ is κ-compact, every object in $\mathcal{B}_{/A}$ is a κ-filtered colimit of κ-compact objects, which shows that (2) is satisfied. Condition (3) follows from A being κ-compact, together with the fact that $(\pi_A)_!$ preserves pullbacks (and consequently also subobjects).

Remark 5.2.2.7. For every $\mathcal{B}$-regular cardinal κ, the ∞-topos $\mathcal{B}$ admits a presentation by the full subcategory $\mathcal{B}^{\kappa\text{-cpt}} \subset \mathcal{B}$ of κ-compact objects, in the sense that its Yoneda extension $\mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}) \to \mathcal{B}$ is a left exact and accessible Bousfield localisation [49, Proposition 6.1.5.2]. Moreover, the inclusion $\mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}})$ commutes with κ-filtered colimits. In particular, the global sections functor $\Gamma\colon \mathcal{B} \to \mathrm{Ani}$ commutes with κ-filtered colimits, so that $\mathrm{const}_{\mathcal{B}}$ restricts to a functor $\mathrm{Ani}^{\kappa\text{-cpt}} \to \mathcal{B}^{\kappa\text{-cpt}}$.

The first main result in this section will be the following characterisation of κ-small $\mathcal{B}$-categories when κ is $\mathcal{B}$-regular:

Proposition 5.2.2.8. *Let κ be a $\mathcal{B}$-regular cardinal, and let C be a $\mathcal{B}$-category. Then the following are equivalent:*

1. *C is κ-small;*
2. *C is a κ-compact object in $\mathrm{Cat}(\mathcal{B})$;*
3. *C is contained in the smallest full subcategory of $\mathrm{Cat}(\mathcal{B})$ that is spanned by objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and that is closed under κ-small colimits;*
4. *C is a κ-compact object in $\mathcal{B}_{\Delta}$;*
5. *C_0 and C_1 are κ-compact objects in $\mathcal{B}$.*

Remark 5.2.2.9. On account of Remark 5.2.2.2, Proposition 5.2.2.8 implies that for every κ-compact object $A \in \mathcal{B}$, we can identify $\mathrm{Cat}(\mathcal{B}_{/A})^{\kappa}$ with the full subcategory of κ-compact objects in $\mathrm{Cat}(\mathcal{B}_{/A})$.

The proof of Proposition 5.2.2.8 requires a few preparations. We begin by establishing that the class of relatively κ-compact maps in $\mathcal{B}$ is already local.

Lemma 5.2.2.10. *Let κ be a $\mathcal{B}$-regular cardinal and let I be a small set. For every $i \in I$, let $P_i \to A_i$ be a relatively κ-compact map in $\mathcal{B}$. Then $\bigsqcup_i P_i \to \bigsqcup_i A_i$ is relatively κ-compact.*

Proof. Let G be κ-compact, and let $s \colon G \to \bigsqcup_i A_i$ be a map. Write $I = \mathrm{colim}_j I_j$ as a κ-filtered union of its κ-small subsets, so that one obtains equivalences $\bigsqcup_i A_i \simeq \mathrm{colim}_j \bigsqcup_{i \in I_j} A_i$ and $\bigsqcup_i P_i \simeq \mathrm{colim}_j \bigsqcup_{i \in I_j} P_i$. As G is κ-compact, there is some j such that s factors through the inclusion $\bigsqcup_{i \in I_j} A_i \hookrightarrow \bigsqcup_i A_i$. By descent, we obtain a pullback diagram

$$\begin{tikzcd} \bigsqcup_{i\in I_j} P_i \arrow[r] \arrow[d] & \bigsqcup_i P_i \arrow[d] \\ \bigsqcup_{i\in I_j} A_i \arrow[r] & \bigsqcup_i A_i, \end{tikzcd}$$

which implies that the pullback of $\bigsqcup_i P_i \to \bigsqcup_i A_i$ to G is equivalent to the pullback of $\bigsqcup_{i\in I_j} P_i \to \bigsqcup_{i\in I_j} A_i$ to G. By again using descent, this pullback can be identified with the coproduct $\bigsqcup_{i\in I_j} P_i \times_{A_i} G_i$, where $G_i = G \times_{\bigsqcup_i A_i} A_i$. As G_i is a subobject of G and therefore κ-compact, the fibre product $P_i \times_{A_i} G_i$ is κ-compact as well. Since I_j is κ-small, we conclude that also $\bigsqcup_{i\in I_j} P_i \times_{A_i} G_i$ is κ-compact, as desired. □

Proposition 5.2.2.11. *Let κ be a $\mathcal{B}$-regular cardinal. Then every object in* $\mathrm{Grpd}_{\mathcal{B}}^{\kappa}$ *is already relatively κ-compact. In other words, the class of relatively κ-compact maps in $\mathcal{B}$ is local.*

Proof. Let $P \to A$ be an object in $\mathrm{Grpd}_{\mathcal{B}}^{\kappa}(A)$. By definition, there is a cover $(s_i)\colon \bigsqcup_{i\in I} A_i \twoheadrightarrow A$ such that $s_i^* P \to A_i$ is relatively κ-compact. By Lemma 5.2.2.10, the map $\bigsqcup_i P_i \to \bigsqcup_i A_i$ is relatively κ-compact. The result therefore follows once we show that relatively κ-compact maps are stable under Δ^{op}-indexed colimits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$. By [49, Lemma 6.1.6.6] they are stable under pushouts, so we only need to consider the case of small coproducts, which again follows from Lemma 5.2.2.10. □

Remark 5.2.2.12. In [49, Proposition 6.1.6.7], Lurie shows that the class of relatively κ-compact maps in $\mathcal{B}$ is local already when $\mathcal{B}$ is κ-accessible and $\mathcal{B}^{\kappa\text{-cpt}}$ is stable under finite limits in $\mathcal{B}$. However, we failed to understand how Lurie derives this result without the additional assumption that $\mathcal{B}^{\kappa\text{-cpt}}$ is also stable under subobjects in $\mathcal{B}$. Therefore, we decided to reiterate Lurie's proof with this added assumption.

Next, we need to establish that every $\mathcal{B}$-regular cardinal is also $\mathcal{B}_\Delta$-regular. This will be a consequence of the following characterisation of κ-compact simplicial objects in $\mathcal{B}$:

Proposition 5.2.2.13. *If κ is a $\mathcal{B}$-regular cardinal, then the ∞-topos $\mathcal{B}_\Delta$ is κ-accessible, and if C is a simplicial object in $\mathcal{B}$, the following are equivalent:*

1. *C is κ-compact;*
2. *$C_n \in \mathcal{B}^{\kappa\text{-cpt}}$ for all $n \geq 0$;*

3. *C is contained in the smallest subcategory of $\mathcal{B}_\Delta$ that is spanned by objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and that is closed under κ-small colimits;*

Proof. Remark 5.2.2.7 implies that the inclusion $\mathcal{B}_\Delta \hookrightarrow \mathrm{PSh}(\Delta \times \mathcal{B}^{\kappa\text{-cpt}})$ commutes with κ-filtered colimits, which immediately implies that $\mathcal{B}_\Delta$ is κ-accessible. Moreover, since Δ is a κ-small ∞-category, every simplicial object in $\mathcal{B}$ that is level-wise κ-compact is also κ-compact in $\mathcal{B}_\Delta$ [49, Proposition 5.3.4.13], hence (2) implies (1). If C satisfies (3), the fact that for every $k \geq 0$ the functor $(-)_k$ commutes with small colimits implies that C_k is contained in the smallest full subcategory of $\mathcal{B}$ that contains all objects of the form $\Delta^n_k \times G$ for $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and $n \geq 0$ and that is closed under κ-small colimits. Since Δ^n_k is a finite set, this implies that C_k is κ-compact, hence (2) follows. Finally, suppose that C is κ-compact. We may write C as a small colimit of objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and therefore by [49, Corollary 4.2.3.10] as a κ-filtered colimit $C \simeq \mathrm{colim}_i C^i$ where each C^i is a κ-small colimits of objects of the form $\Delta^n \otimes G$. As C is κ-compact, there is some i_0 such that the identity on C factors through $C^{i_0} \to \mathrm{C}$. In other words, C is a retract of C^{i_0}. As retracts are countable and therefore a fortiori κ-small colimits, (3) follows. □

Corollary 5.2.2.14. *If κ is a $\mathcal{B}$-regular cardinal, then κ is $\mathcal{B}_\Delta$-regular as well. Moreover, a map in $\mathcal{B}_\Delta$ is relatively κ-compact if and only if it is level-wise given by a relatively κ-compact morphism in $\mathcal{B}$.* □

Lemma 5.2.2.15. *If κ is a $\mathcal{B}$-regular cardinal and if C is a κ-compact simplicial object in $\mathcal{B}$, then C^K is κ-compact for every ω-compact simplicial ∞-groupoid K.*

Proof. As κ-compact objects in $\mathcal{B}_\Delta$ are stable under retracts and as every ω-compact simplicial ∞-groupoid is a retract of a finite colimit of n-simplices, we may assume without loss of generality that K is a finite colimit of n-simplices. Therefore C^K is a finite limit of objects of the form C^{Δ^n}, so that Corollary 5.2.2.14 implies that we may reduce to the case $K = \Delta^n$. Now on account of the identity $(C^{\Delta^n})_k \simeq (C^{\Delta^n \times \Delta^k})_0$ and by using the fact that $\Delta^n \times \Delta^k$ is again ω-compact, we can identify $(C^{\Delta^n})_k$ as a finite limit of objects of the form $(C^{\Delta^l})_0 \simeq C_l$, which shows that $(C^{\Delta^n})_k$ is κ-compact. By Proposition 5.2.2.13, one concludes that C^{Δ^n} is κ-compact in $\mathcal{B}_\Delta$. □

Lemma 5.2.2.16. *For every ω-compact simplicial ∞-groupoid K, the functor*

$$(-)^K : \mathcal{B}_\Delta \to \mathcal{B}_\Delta$$

commutes with filtered colimits.

Proof. As every ω-compact simplicial ∞-groupoid K is a retract of a finite colimit of n-simplices, we may assume without loss of generality $K = \Delta^n$. As it suffices to show that$(-)_k^{\Delta^n}$ commutes with filtered colimits for all $k \geq 0$, the same argumentation as in the proof of Lemma 5.2.2.15 shows that we may reduce to showing that $(-)_0^{\Delta^n}$ commutes with filtered colimits. On account of the equivalence $(-)_0^{\Delta^n} \simeq (-)_n$, this is immediate. $\square$

Lemma 5.2.2.17. *Let κ be a $\mathcal{B}$-regular cardinal, let C be a κ-compact simplicial object in $\mathcal{B}_\Delta$ and let $C \to L(C)$ be the unit of the adjunction $(L \dashv i) : \operatorname{Cat}(\mathcal{B}) \rightleftarrows \mathcal{B}_\Delta$. Then $L(C)$ is κ-compact as well.*

Proof. We will make use of the ∞-categorical version of the small object argument as developed in [5, § 2.3]. For the convenience of the reader, we briefly explain the setup, at least in the special case that is relevant for this proof. Suppose that S is a finite set of maps in Ani_Δ such that for every map $s : K \to L$ in S the functors $(-)^K$ and $(-)^L$ commute with filtered colimits in $\mathcal{B}_\Delta$. Let $(\mathcal{L}, \mathcal{R})$ be the factorisation system in $\mathcal{B}_\Delta$ that is internally generated by the set S. To any object $C \in \mathcal{B}_\Delta$, we can now assign a sequence

$$\mathbb{N} \to \mathcal{B}_\Delta, \quad k \mapsto C(k)$$

by setting $C(0) = C$ and by recursively defining a map $C(k) \to C(k+1)$ via the pushout

$$\begin{tikzcd} \bigsqcup_{s\colon K \to L} L \otimes C(k)^L \sqcup_{K \otimes C(k)^L} K \otimes C(k)^K \arrow[r] \arrow[d] & C(k) \arrow[d] \\ \bigsqcup_{s\colon K \to L} L \otimes C(k)^K \arrow[r] & C(k+1) \end{tikzcd}$$

in which the coproduct ranges over all maps $s : K \to L$ in S. Then [5, Theorem 2.3.4] shows that the object $\operatorname{colim}_k C(k)$ is internally local with respect to the maps in S, i.e. contained in $\mathcal{R}_{/1}$, and that furthermore the map $C \to \operatorname{colim}_k C(k)$ is contained in $\mathcal{L}$, so that it is equivalent to the unit of the adjunction $\mathcal{R}_{/1} \rightleftarrows \mathcal{B}_\Delta$ evaluated at $C \in \mathcal{B}_\Delta$.

Now if we let S be the set $\{E^1 \to 1, I^2 \hookrightarrow \Delta^2\}$, Lemma 5.2.2.16 shows that we are in the above situation. Consequently, if C is a κ-compact object in $\mathcal{B}_\Delta$, the $\mathcal{B}$-category $L(C)$ can be computed as a countable colimit of the objects $C(k)$ as constructed above. Hence it suffices to show that each $C(k)$ is κ-compact, which easily follows from κ being $\mathcal{B}_\Delta$-regular (Corollary 5.2.2.14) and Lemma 5.2.2.15. □

Proof of Proposition 5.2.2.8. We first show that (2)–(5) are equivalent. By combining Proposition 5.2.2.13 with the Segal conditions, one finds that (4) and (5) are equivalent. Moreover, since $\operatorname{Cat}(\mathcal{B})$ is an ω-accessible localisation of $\mathcal{B}_\Delta$, the localisation functor preserves κ-compact objects, which shows that (4) implies (2). Suppose now that C is a κ-compact object in $\operatorname{Cat}(\mathcal{B})$. As in the proof of Proposition 5.2.2.13, we can find a κ-filtered ∞-category $\mathcal{J}$ such that $\mathsf{C} \simeq \operatorname{colim}_{j\in\mathcal{J}} \mathsf{C}^j$ where each C^j is a κ-small colimit of objects of the form $\Delta^n \otimes G$, where $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$. Hence C is a retract of some C^j, so that (3) holds. Lastly, since we can compute any small colimit in $\operatorname{Cat}(\mathcal{B})$ by first taking the colimit of the underlying diagram in $\mathcal{B}_\Delta$ and then applying the reflector $L\colon \mathcal{B}_\Delta \to \operatorname{Cat}(\mathcal{B})$, Lemma 5.2.2.17 implies that every κ-small colimit in $\operatorname{Cat}(\mathcal{B})$ of objects of the form $\Delta^n \otimes G$ with $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ is also κ-compact in $\mathcal{B}_\Delta$. Thus (3) implies (4).

Finally, since $\mathsf{Cat}^\kappa_{\mathcal{B}}$ is closed under both $\mathsf{LConst}_{\mathrm{Cat}^\kappa_\infty}$- and $\mathsf{Grpd}^\kappa_{\mathcal{B}}$-colimits and since $\Delta^n \otimes G$ can be regarded as the colimit of the constant G-indexed diagram with value Δ^n, it is clear that (3) implies (1). To show the converse, let V be the internal class that is spanned by those $\mathcal{B}_{/A}$-categories (for $A \in \mathcal{B}^{\kappa\text{-cpt}}$) that satisfy the $\mathcal{B}_{/A}$-categorical analogue of (3). Note that if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover by κ-compact objects and if D is a $\mathcal{B}$-category such that $\pi^*_{A_i}\mathsf{D}$ satisfies the $\mathcal{B}_{/A_i}$-categorical analogue of condition (3), then D satisfies (3): in fact, since we have already established that (3) and (4) are equivalent, this is a consequence of Proposition 5.2.2.11 and Corollary 5.2.2.14. As a consequence, for every κ-compact object $A \in \mathcal{B}$, we can identify $\mathsf{V}(A)$ with the class of $\mathcal{B}_{/A}$-categories that satisfy the $\mathcal{B}_{/A}$-categorical version of condition (3). As V clearly contains both $\mathsf{LConst}_{\mathrm{Cat}^\kappa_\infty}$ and $\mathsf{Grpd}^\kappa_{\mathcal{B}}$, the proof will be complete once we show that V is closed under both $\mathsf{LConst}_{\mathrm{Cat}^\kappa_\infty}$-

and $\mathrm{Grpd}_{\mathcal{B}}^{\kappa}$-colimits. By our description of $\mathsf{V}(A)$ for every κ-compact $A \in \mathcal{B}$ and the fact that limits can be computed locally (Remark 3.2.1.7), this is clear for the first case. To show the second case, we need to verify that for every relatively κ-compact map $p\colon P \to A$, the functor $p_!\colon \mathrm{Cat}(\mathcal{B}_{/P}) \to \mathrm{Cat}(\mathcal{B}_{/A})$ restricts to a map $\mathsf{V}(P) \to \mathsf{V}(A)$. Using again that the class of relatively κ-compact maps is local, it is enough to consider the case where A (and therefore also P) is κ-compact. To show the claim, we may again use the explicit description of $\mathsf{V}(A)$ and $\mathsf{V}(P)$ to deduce that it suffices to verify that $p_!$ carries κ-small colimits of objects in $\mathrm{Cat}(\mathcal{B}_{/P})$ of the form $\Delta^n \otimes Q$ (with $Q \to P$ relatively κ-compact) to κ-small colimits of objects in $\mathrm{Cat}(\mathcal{B}_{/A})$ of the form $\Delta^n \otimes Q$ (with $Q \to A$ κ-compact). Since $p_!$ preserves small colimits and acts by postcomposition with p, this follows from the fact that relatively κ-compact maps are closed under composition. □

Corollary 5.2.2.18. *For every $\mathcal{B}$-regular cardinal κ, the internal class $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$ is a doctrine.*

Proof. As $\mathcal{B}$ is generated by $\mathcal{B}^{\kappa}$, Remark 5.2.2.2 implies that we only need to show that the collection of κ-small $\mathcal{B}$-categories is small, which is an immediate consequence of (2) in Proposition 5.2.2.8 □

By construction, the internal class $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$ is regular for every cardinal κ. We conclude this section by proving that whenever κ is $\mathcal{B}$-regular, the doctrine $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$ is sound.

Lemma 5.2.2.19. *Let κ be a $\mathcal{B}$-regular cardinal, and let $\mathcal{J}$ be a κ-filtered ∞-category. Then $\mathcal{J}$ is $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-filtered when viewed as a constant $\mathcal{B}$-category.*

Proof. By Proposition 5.1.1.5, we need to show that the inclusion

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{J}, \mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{J}^{\triangleright}, \mathsf{Grpd}_{\mathcal{B}})$$

is $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-continuous. Since $\mathcal{J}$ is κ-filtered, the inclusion section-wise preserves κ-small limits. It therefore suffices to show that it is $\mathrm{Grpd}_{\mathcal{B}}^{\kappa}$-continuous. This amounts to showing that for every $A \in \mathcal{B}$ and every $G \in \mathrm{Grpd}_{\mathcal{B}}^{\kappa}(A)$ the geometric morphism $\mathcal{B}_{/G} \to \mathcal{B}_{/A}$ commutes with $\mathcal{J}$-indexed colimits. As the preservation of colimits is a local condition (Remark 3.2.2.3) and as $\mathcal{B}$ is generated by the

κ-compact objects in $\mathcal{B}$, we may assume that A is κ-compact. In light of Remark 5.2.2.6, we may thus replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore reduce to the case $A \simeq 1$. As κ is $\mathcal{B}$-regular, the collection of κ-compact objects in $\mathcal{B}$ is stable under finite limits. Therefore, for every $H \in \mathcal{B}^{\kappa\text{-cpt}}$ the functor $\operatorname{map}_{\mathcal{B}}(G \times H, -)$ preserves $\mathcal{J}$-filtered colimits. By Yoneda's lemma, this implies that the functor $\underline{\operatorname{Hom}}_{\mathcal{B}}(G, -)$ also preserves $\mathcal{J}$-filtered colimits. On account of the pullback square

$$\begin{tikzcd}
(\pi_G)_* \arrow[r] \arrow[d] & \underline{\operatorname{Hom}}_{\mathcal{B}}(G, (\pi_G)_!(-)) \arrow[d] \\
\operatorname{diag}(1) \arrow[r, "\operatorname{id}_G"] & \operatorname{diag}(\underline{\operatorname{Hom}}_{\mathcal{B}}(G, G))
\end{tikzcd}$$

in $\operatorname{Fun}(\mathcal{B}_{/G}, \mathcal{B})$ and the fact that the cospan in the lower right corner consists of functors which preserve $\mathcal{J}$-indexed colimits, the claim follows from the fact that $\mathcal{J}$-indexed colimits commute with finite limits. $\square$

Lemma 5.2.2.20. *Let κ be a $\mathcal{B}$-regular cardinal, and let J be a $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-cocomplete $\mathcal{B}$-category. Then the canonical functor $\Gamma\mathsf{J} \to \mathsf{J}$ that is obtained from the counit of the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ is final.*

Proof. For every $G \in \mathcal{B}^{\kappa\text{-cpt}}$, the functor $\mathsf{J}(1) \to \mathsf{J}(G)$ admits a left adjoint and is therefore in particular final. In other words, if $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}})$ denotes the inclusion, then the functor $\epsilon\colon \Gamma_{\mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}})} i\mathsf{J} \to i\mathsf{J}$ is section-wise final. But since the local sections functor $\mathrm{ev}_G\colon \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}) \to \mathrm{Ani}$ defines an algebraic morphism of ∞-topoi and since every algebraic morphism preserves both final functors and right fibrations, applying ev_G to any factorisation of ϵ in $\mathrm{Cat}(\mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}))$ into a final functor and a right fibration yields a factorisation of $\epsilon(G)$ into a final functor and a right fibration in Cat_∞. Consequently, the map ϵ must already be final. As we recover the map $\Gamma\mathsf{J} \to \mathsf{J}$ by applying the algebraic morphism $L\colon \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}) \to \mathcal{B}$ to ϵ, the claim follows. $\square$

Proposition 5.2.2.21. *If κ is a $\mathcal{B}$-regular cardinal, then $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$ is sound.*

Proof. On account of Corollary 5.1.5.12, it suffices to show that $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$ is weakly sound. Together with the fact that $\mathcal{B}$ is generated by its κ-compact objects and Remark 5.2.2.2, it is therefore enough to prove that every $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-cocomplete

$\mathcal{B}$-category J is $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-filtered. By Lemma 5.2.2.20 and Remark 5.1.1.7, we can furthermore assume that J is the constant $\mathcal{B}$-category associated with an ∞-category that admits κ-small colimits and that is therefore κ-filtered [49, Proposition 5.3.3.3]. As a consequence, the result follows from Lemma 5.2.2.19. □

Remark 5.2.2.22. As a consequence of Proposition 5.2.2.21, if C is an arbitrary $\mathcal{B}$-category, there is always a regular and sound doctrine U such that $\mathsf{C} \in \mathsf{U}(1)$. In fact, we only need to choose a $\mathcal{B}$-regular cardinal κ such that C is κ-compact (and therefore κ-small by Proposition 5.2.2.8) and set $\mathsf{U} = \mathsf{Cat}^{\kappa}_{\mathcal{B}}$. More generally, if V is a doctrine, we can find a $\mathcal{B}$-regular cardinal κ such that V_0 is κ-compact and such that the tautological object $\tau\colon \mathsf{V}_0 \to \mathsf{V}$ corresponds to a κ-small $\mathcal{B}_{/\mathsf{V}_0}$-category. As every object of V (in arbitrary context $A \in \mathcal{B}$) arises as a pullback of τ, this implies that V is contained in $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$.

Corollary 5.2.2.23. *For every* $\mathcal{B}$*-regular cardinal* κ*, there is an equivalence*

$$\mathsf{Grpd}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}} \simeq \mathsf{Grpd}^{\kappa}_{\mathcal{B}}$$

of full subcategories in $\mathsf{Grpd}_{\mathcal{B}}$.

Proof. Since $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$ is a sound doctrine by Proposition 5.2.2.21, it has the decomposition property. We may therefore apply Proposition 5.1.5.10 to deduce an equivalence

$$\mathsf{Grpd}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}} \simeq \mathsf{Ret}_{\mathsf{Grpd}_{\mathcal{B}}}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}(1)).$$

Now if $\mathsf{G}\colon 1 \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}(1)$ is an arbitrary object, there is a cover $\bigsqcup_i A_i \twoheadrightarrow 1$ of $\mathcal{B}$ (without loss of generality by κ-compact objects) and for each i a κ-small $\mathcal{B}_{/A_i}$-category J such that $\pi_{A_i}^* \mathsf{G} \simeq \mathsf{J}^{\mathrm{gpd}}$. Since κ is by definition uncountable, we thus find that $\pi_{A_i}^* \mathsf{G}$ arises as a κ-small colimit of κ-compact objects in $\mathcal{B}_{/A_i}$ (using the characterisation of κ-small $\mathcal{B}_{/A_i}$-categories in Proposition 5.2.2.8) and is therefore itself κ-compact. Using Proposition 5.2.2.11, this implies that G is κ-compact itself. By Remark 3.4.2.3, the same argument can be carried out for every object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}(1)$ in context $A \in \mathcal{B}^{\kappa\text{-cpt}}$, and since the collection of κ-compact objects in $\mathcal{B}$ generate $\mathcal{B}$ under small colimits, this implies that we have an inclusion

$$\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}(1) \hookrightarrow \mathsf{Grpd}^{\kappa}_{\mathcal{B}}\,.$$

Using again that κ is uncountable, the collection of κ-compact objects in $\mathcal{B}$ is closed under retracts, and as the same is true for the class of κ-compact objects in $\mathcal{B}_{/A}$ for every $A \in \mathcal{B}^{\kappa\text{-cpt}}$, we find that $\mathsf{Grpd}^{\kappa}_{\mathcal{B}}$ is closed under retracts in $\mathsf{Grpd}_{\mathcal{B}}$, so that we obtain an inclusion

$$\mathsf{Grpd}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}} \hookrightarrow \mathsf{Grpd}^{\kappa}_{\mathcal{B}}.$$

Conversely, it is clear that whenever G is a κ-small $\mathcal{B}$-groupoid, the associated object $\mathsf{G}\colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$ is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}(1)$. Again, the same is true for every κ-small $\mathcal{B}_{/A}$-groupoid whenever A is κ-compact. Hence we obtain an inclusion

$$\mathsf{Grpd}^{\kappa}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}},$$

which finishes the proof. □

5.2.3. Finite $\mathcal{B}$-categories

In this section we will discuss another important example of a regular and sound doctrine. Recall that a quasicategory $\mathcal{C}$ is called *finite* if there is a finite simplicial set and a Joyal equivalence $K \to \mathcal{C}$. This is equivalent to $\mathcal{C}$ being contained in the smallest subcategory of Cat_∞ that contains $\varnothing$, Δ^0 and Δ^1 and is closed under pushouts (see [83, Proposition 2.4]). We denote the associated doctrine of ∞-categories by $\mathrm{Fin}_{\mathrm{Ani}}$. Let us denote by eq the local class of equivalences in $\mathcal{B}$. We may now define:

Definition 5.2.3.1. A $\mathcal{B}$-category is said to be *finite* if it is $(\mathrm{Fin}_{\mathrm{Ani}}, \mathrm{eq})$-small, and we shall denote by $\mathsf{Fin}_{\mathcal{B}} = \mathsf{Cat}_{\mathcal{B}}^{\langle \mathrm{Fin}_{\mathrm{Ani}}, \mathrm{eq}\rangle}$ the associated regular doctrine of finite $\mathcal{B}$-categories. We will denote by $\mathrm{Fin}(\mathcal{B})$ the underlying ∞-category of global sections. We say that a $\mathcal{B}$-category I is *filtered* if it is $\mathsf{Fin}_{\mathcal{B}}$-filtered. We will say that a $\mathcal{B}$-category *has finite (co)limits* if it is $\mathsf{Fin}_{\mathcal{B}}$-(co)complete, and a functor *preserves finite (co)limits* if it is $\mathsf{Fin}_{\mathcal{B}}$-(co)continuous. Dually, we say that a $\mathcal{B}$-category *has filtered colimits* if it is Filt-cocomplete, and a functor *preserves filtered colimits* if it is Filt-cocontinuous. If C is a $\mathcal{B}$-category that has filtered colimits, an object $c\colon A \to \mathsf{C}$ is said to be *compact* if it is $\mathsf{Filt}_{\mathsf{Fin}_{\mathcal{B}}}$-compact, and we denote the full subcategory of compact objects in C by $\mathsf{C}^{\mathrm{cpt}}$.

Remark 5.2.3.2. By Remark 5.2.1.2 and the evident fact that $\pi_A^* \,\mathrm{eq} = \mathrm{eq}$ as local classes in $\mathcal{B}_{/A}$, there is a canonical equivalence $\pi_A^* \,\mathsf{Fin}_{\mathcal{B}} \simeq \mathsf{Fin}_{\mathcal{B}_{/A}}$ for all $A \in \mathcal{B}$.

Remark 5.2.3.3. Every filtered $\mathcal{B}$-category J satisfies $\mathsf{J}^{\mathrm{gpd}} \simeq 1$. In fact, by Corollary 5.1.2.6 it is weakly filtered, thus in particular the unique functor $I \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\varnothing, \mathsf{I}) \simeq 1$ is final.

Proposition 5.2.3.4. *There is an equivalence* $\mathsf{Fin}_{\mathcal{B}} \simeq \mathsf{LConst}_{\mathrm{Fin}_{\mathrm{Ani}}}$ *of internal classes. In other words, a finite $\mathcal{B}$-category is simply a locally constant sheaf of finite ∞-categories.*

Proof. Since $\mathrm{Grpd}_{\mathrm{eq}} \simeq 1_{\mathrm{Grpd}_{\mathcal{B}}}$ as full subcategories, we can describe $\mathsf{Fin}_{\mathcal{B}}$ as the regularisation of $\mathsf{LConst}_{\mathrm{Fin}_{\mathrm{Ani}}}$. But since Cat_∞ is compactly generated, we may apply Corollary A.2.0.4 and conclude that $\mathsf{LConst}_{\mathrm{Fin}_{\mathrm{Ani}}}$ is already closed under $\mathsf{LConst}_{\mathrm{Fin}_{\mathrm{Ani}}}$-colimits in $\mathrm{Cat}_{\mathcal{B}}$. Hence the claim follows. □

By Proposition 5.2.1.3, finite limits and preservation of finite limits can be checked section-wise:

Proposition 5.2.3.5. *Let* C *be a* $\mathcal{B}$*-category. Then*

1. C *has finite limits if and only if* $\mathsf{C}(A)$ *has finite limits for every* $A \in \mathcal{B}$ *and for every* $s\colon B \to A$ *the functor* $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ *preserves finite limits.*

2. *A functor* $f\colon \mathsf{C} \to \mathsf{D}$ *between* $\mathcal{B}$*-categories that have finite limits preserves such limits if and only if* $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ *preserves finite limits for every* $A \in \mathcal{B}$.

The dual statements about finite colimits hold as well. □

One can construct an ample amount of filtered $\mathcal{B}$-categories from *presheaves* of filtered ∞-categories:

Proposition 5.2.3.6. *Say that* $\mathcal{B}$ *is given as a left exact accessible localisation* $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ *where* $\mathcal{C}$ *is a small* ∞*-category. Let* J *be any* $\mathrm{PSh}(\mathcal{C})$*-category such* $\mathsf{J}(c)$ *is filtered for every* $c \in \mathcal{C}$*. Then* $L\mathsf{J}$ *is a filtered* $\mathcal{B}$*-category.*

Proof. Let $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ be the inclusion. Since L is left exact, it induces a functor of $\mathrm{PSh}(\mathcal{C})$-categories $L\colon\ \mathsf{Grpd}_{\mathrm{PSh}(\mathcal{C})} \to i\,\mathsf{Grpd}_{\mathcal{B}}$ that for every $A \in \mathrm{PSh}(\mathcal{C})$ is given by

$$L_{/A}\colon\ \mathrm{PSh}(\mathcal{C})_{/A} \to \mathcal{B}_{/LA}.$$

By Proposition 5.2.3.5, the functor L thus preserves finite limits. Furthermore, it readily follows from Proposition 3.1.2.9 that L admits a right adjoint i that is fully faithful. Therefore, we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{Grpd}_{\mathrm{PSh}(\mathcal{C})}) \arrow[r, "\mathrm{colim}_{\mathsf{J}}"] \arrow[d, "L_*"] & \mathsf{Grpd}_{\mathrm{PSh}(\mathcal{C})} \arrow[d, "L"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, i\,\mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "\mathrm{colim}_{\mathsf{J}}"'] & i\,\mathsf{Grpd}_{\mathcal{B}}\,.
\end{tikzcd}$$

Since there is an equivalence $i\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, i\,\mathsf{Grpd}_{\mathcal{B}}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(L\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}})$ that is natural in J, the lower colimit functor in the above diagram can be identified with the functor

$$i\mathrm{colim}_{L\mathsf{J}}\colon\ i\underline{\mathsf{Fun}}_{\mathcal{B}}(L\mathsf{J}, \mathsf{Grpd}_{\mathcal{B}}) \to i\,\mathsf{Grpd}_{\mathcal{B}}\,.$$

Using that i is fully faithful, we get that this map is equivalent to the composition $L\,\mathrm{colim}_{\mathsf{J}}\, i_*$. Therefore, it suffices to show that the upper colimit functor in the above diagram preserves finite limits. To see this, since $\mathrm{PSh}(\mathcal{C})_{/c} \simeq \mathrm{PSh}(\mathcal{C}_{/c})$ for every $c \in \mathcal{C}$ and since $\mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ generates $\mathrm{PSh}(\mathcal{C})$ under small colimits, it suffices to show that the functor $(-)^{\mathrm{gpd}}\colon\ \mathrm{LFib}(\mathsf{J}) \to \mathrm{PSh}(\mathcal{C})$ commutes with finite limits, cf. Proposition 5.2.3.5 and Proposition 3.2.5.1. Since for every $c \in \mathcal{C}$ the evaluation functor $\mathrm{ev}_c\colon\ \mathrm{PSh}(\mathcal{C}) \to \mathrm{Ani}$ commutes with small colimits, the lax square

$$\begin{tikzcd}
\mathrm{LFib}_{\mathrm{PSh}(\mathcal{C})}(\mathsf{J}) \arrow[r, "(-)^{\mathrm{gpd}}"] \arrow[d, "\mathrm{ev}_c"] & \mathcal{B} \arrow[d, "\mathrm{ev}_c"] \\
\mathrm{LFib}_{\mathrm{Ani}}(\mathsf{J}(c)) \arrow[r, "(-)^{\mathrm{gpd}}"] & \mathrm{Ani}
\end{tikzcd}$$

is commutative. By assumption and the fact that ev_c preserves limits, the functor $(-)^{\mathrm{gpd}} \circ \mathrm{ev}_c$ commutes with finite limits, hence so does $\mathrm{ev}_c \circ (-)^{\mathrm{gpd}}$. The claim now follows from the fact that $(\mathrm{ev}_c)_{c\in\mathcal{C}}\colon\ \mathrm{PSh}(\mathcal{C}) \to \prod_{c\in\mathcal{C}} \mathrm{Ani}$ is a conservative functor. $\square$

This leads to the main result of this section:

Proposition 5.2.3.7. *The doctrine* $\mathsf{Fin}_{\mathcal{B}}$ *is sound.*

Proof. Since $\mathsf{Fin}_{\mathcal{B}}$ is by definition regular, Corollary 5.1.5.12 implies that suffices it to show that $\mathsf{Fin}_{\mathcal{B}}$ is weakly sound. Using Remark 5.2.3.2, we only need to show that every $\mathcal{B}$-category J that has finite colimits is already filtered. But since J in particular admits finite *constant* colimits, it is section-wise filtered, hence the result follows from Proposition 5.2.3.6. $\square$

As a result of Proposition 5.2.3.7, we can now classify the compact objects of $\mathsf{Grpd}_{\mathcal{B}}$. To that end, Let us denote by $\mathsf{LConst}_{\mathrm{Ani}^{\mathrm{cpt}}}$ the full subcategory of $\mathsf{Grpd}_{\mathcal{B}}$ that arises as the essential image of the map $\mathrm{Ani}^{\mathrm{cpt}} \to \mathsf{Grpd}_{\mathcal{B}}$ (which is defined as the transpose of $\mathrm{const}_{\mathcal{B}} \colon \mathrm{Ani}^{\mathrm{cpt}} \to \mathcal{B}$). We now obtain:

Corollary 5.2.3.8. *There is an equivalence*

$$\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{cpt}} \simeq \mathsf{LConst}_{\mathrm{Ani}^{\mathrm{cpt}}}$$

of full subcategories in $\mathsf{Grpd}_{\mathcal{B}}$.

Proof. Since $\mathsf{Fin}_{\mathcal{B}}$ is a sound doctrine by Proposition 5.2.3.7, it has the decomposition property. We may therefore apply Proposition 5.1.5.10 to deduce an equivalence

$$\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{cpt}} \simeq \mathsf{Ret}_{\mathsf{Grpd}_{\mathcal{B}}}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Fin}_{\mathcal{B}}}(1)).$$

Hence, if $\mathsf{G} \colon A \to \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{cpt}}$ is an arbitrary object, there is a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^* \mathsf{G}$ is a retract of an object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Fin}_{\mathcal{B}}}(1)$ in context A_i, for every i. By further refining this cover, we can furthermore assume that for each i there is a finite $\mathcal{B}_{/A_i}$-category J_i such that $\pi_{A_i}^* \mathsf{G}$ is a retract of $\mathsf{J}_i^{\mathrm{gpd}}$. Hence $s_i^* \mathsf{G}$ is a retract of an object in $\mathsf{LConst}_{\mathrm{Ani}^{\mathrm{cpt}}}$ in context A_i, so that Corollary A.2.0.4 implies that $s_i^* \mathsf{G}$ is itself contained in $\mathsf{LConst}_{\mathrm{Ani}^{\mathrm{cpt}}}$, which necessarily implies that G is contained in $\mathsf{LConst}_{\mathrm{Ani}^{\mathrm{cpt}}}$. Conversely, if G is an object of $\mathsf{LConst}_{\mathrm{Ani}^{\mathrm{cpt}}}$ in context $A \in \mathcal{B}$, we can find a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^* \mathsf{G}$ is a constant $\mathcal{B}_{/A_i}$-groupoid coming from a compact ∞-groupoid, which in turn implies that $s_i^* \mathsf{G}$ is a retract of a constant $\mathcal{B}_{/A_i}$-groupoid coming from a *finite* ∞-groupoid. As this implies that $s_i^* \mathsf{G}$ is a retract of an object in $\mathsf{Small}^{\mathsf{Fin}_{\mathcal{B}}}(1)$ in context A_i, we conclude that $s_i^* \mathsf{G}$ must be contained in $\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{cpt}}$, so that G is contained in $\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{cpt}}$ as well. □

The goal for the remainder of this section is to discuss a more explicit description of filtered $\mathcal{B}$-categories in the case where $\mathcal{B}$ is *hypercomplete*. To that end, recall that the filtered ∞-categories can be characterised as those ∞-categories $\mathcal{C}$ for which every map $\mathcal{K} \to \mathcal{C}$ from a finite ∞-category $\mathcal{K}$ can be extended to a map from the cone $\mathcal{K}^{\triangleright} \to \mathcal{C}$. In other words, the ∞-category $\mathcal{C}$ is filtered if and only if for any finite ∞-category $\mathcal{K}$ the functor $j^* \colon \mathrm{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$ induced by restricting along the inclusion $j \colon \mathcal{K} \hookrightarrow \mathcal{K}^{\triangleright}$ is essentially surjective. This characterisation admits an immediate internal analogue:

Definition 5.2.3.9. A $\mathcal{B}$-category J is called *quasi-filtered* if for every finite ∞-category $\mathcal{K}$ the functor $j^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}^{\triangleright}, \mathsf{J}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})$ is essentially surjective.

As the terminology suggests, every filtered $\mathcal{B}$-category is quasi-filtered. To prove this, we require the following lemma, which gives a very explicit description of the notion of quasi-filteredness:

Lemma 5.2.3.10. *Let J be a $\mathcal{B}$-category. Then J is quasi-filtered if and only if for any $A \in \mathcal{B}$ and any diagram $\mathcal{K} \to \mathsf{J}(A)$ where $\mathcal{K}$ is a finite ∞-category there exists a cover $(s_i)_i \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that for every i we can find a map $\mathcal{K}^{\triangleright} \to \mathsf{J}(A_i)$ making the diagram*

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d, hook, "j"] & \mathsf{J}(A) \arrow[d, "s_i^*"] \\ \mathcal{K}^{\triangleright} \arrow[r] & \mathsf{J}(A_i) \end{tikzcd}$$

commute.

Proof. Let us first assume that J is quasi-filtered. Choose a diagram $\mathcal{K} \to \mathsf{J}(A)$ that corresponds to a map $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})$, and let us form the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "s"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}^{\triangleright}, \mathsf{J})^{\simeq} \arrow[d, "(j^*)^{\simeq}"] \\ A \arrow[r] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}. \end{tikzcd}$$

Since j^* is essentially surjective, $(j^*)^{\simeq}$ is a cover (Corollary 1.3.2.15), hence so is the map s. Thus $s\colon P \twoheadrightarrow A$ gives the desired cover. For the converse, we may pick the diagram $\mathcal{K} \to \mathsf{J}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq})$ that is determined by the identity $\mathrm{id}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}$. By assumption we may now find a cover $(s_i)_i\colon \bigsqcup_i A_i \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}$ such that the diagram

$$\begin{tikzcd}
\bigsqcup_i A_i \arrow[r] \arrow[d, two heads] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}^{\triangleright}, \mathsf{J})^{\simeq} \arrow[d, "j^*"'] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq} \arrow[r, "\mathrm{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}
\end{tikzcd}$$

commutes. Thus j^* is also a cover, as desired. □

Proposition 5.2.3.11. *Every filtered $\mathcal{B}$-category is quasi-filtered.*

Proof. Suppose that J is a filtered $\mathcal{B}$-category. In light of Lemma 5.2.3.10, it suffices to show that for every finite ∞-category $\mathcal{K}$, every diagram $d\colon \mathcal{K} \to \mathsf{J}$ locally extends to a map $\mathcal{K}^{\triangleright} \to \mathsf{J}$. Note that J being filtered implies that $\mathsf{J}_{d/}^{\mathrm{gpd}} \simeq 1$. Therefore there is a cover $A \twoheadrightarrow 1$ in $\mathcal{B}$ such that $\mathsf{J}_{d/}(A)$ is non-empty. Unwinding the definitions, this exactly provides the desired local extension of d. □

In [9, Éxpose V, Definition 8.11], Deligne chose (a 1-categorical analogue of) Definition 5.2.3.9 to *define* filtered 1-categories internal to a 1-topos, so one might be inclined to surmise that the notions of filteredness and quasi-filteredness coincide. In light of Proposition 5.2.3.11, the second is always implied by the first, and the converse is in fact true in the case where $\mathcal{B} \simeq \mathrm{Ani}$ (see [49, Proposition 5.4.1.22]). For general ∞-topoi, however, this is no longer the case, the obstruction being the presence of non-trivial ∞-connected objects:

Proposition 5.2.3.12. *Let* $\mathrm{G} \in \mathcal{B}$ *be an ∞-connective object. Then* G *is a quasi-filtered $\mathcal{B}$-category.*

Proof. It is well-known (see [62]) that G is ∞-connective if and only if for an arbitrary finite ∞-category $\mathcal{K}$, the diagonal map $\mathrm{G} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathrm{G})$ is a cover. This clearly implies the claim. □

As any filtered $\mathcal{B}$-groupoid is necessarily equivalent to the final object, Proposition 5.2.3.12 shows that any non-trivial ∞-connective object gives rise to a $\mathcal{B}$-category that is quasi-filtered but not filtered. In the remainder of this section we will show that is essentially the only obstruction. More precisely we will show that if $\mathcal{B}$ is hypercomplete, then any quasi-filtered $\mathcal{B}$-category is filtered.

Lemma 5.2.3.13. *Let* C *be a quasi-filtered* $\mathcal{B}$*-category and assume that* $\mathcal{B}$ *is hypercomplete. Then* $\mathsf{C}^{\mathrm{gpd}} \simeq 1$.

Proof. Since $\mathcal{B}$ is by assumption hypercomplete, it will be sufficient to verify that the diagonal map

$$\mathsf{C}^{\mathrm{gpd}} \to \mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(K, \mathsf{C}^{\mathrm{gpd}})$$

is a cover for any finite ∞-groupoid K, as in this case $\mathsf{C}^{\mathrm{gpd}}$ is ∞-connective (see [62]). So it is enough to see that for every $A \in \mathcal{B}$, every map $f\colon K \to \mathsf{C}^{\mathrm{gpd}}(A)$ from a finite ∞-groupoid K locally factors through the point. Replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$ we may assume that $A = 1$, so that f corresponds to a map $g\colon K \to \mathsf{C}^{\mathrm{gpd}}$. Now recall that since the doctrine of finite $\mathcal{B}$-categories is sound and regular, we can find a filtered $\mathcal{B}$-category J and a diagram $d\colon \mathsf{J} \to \mathsf{Fin}_{\mathcal{B}}$ with colimit C. Since $(-)^{\mathrm{gpd}}$ is cocontinuous and K is a compact object of $\mathrm{Grpd}_{\mathcal{B}}$ by Corollary 5.2.3.8, we obtain an equivalence

$$\mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(K, \mathsf{C}^{\mathrm{gpd}}) \simeq (\mathsf{C}_{\mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(K, d(-)^{\mathrm{gpd}})/})^{\mathrm{gpd}}.$$

Therefore, we may find a cover $\bigsqcup_k A_k \twoheadrightarrow 1$ and objects $j_k\colon A_k \to \mathsf{J}$ for each k such that $\pi_{A_k}^* g$ factors through the canonical map $d(j_k)^{\mathrm{gpd}} \to \pi_{A_k}^* \mathsf{C}^{\mathrm{gpd}}$. Since $d(j_k)$ is a finite $\mathcal{B}_{/A}$-category we may pass to a further cover and can therefore assume that $d(j_k)$ is the constant $\mathcal{B}_{/A_k}$-category associated to a finite ∞-category. Therefore, the assumption that C is quasi-filtered implies that locally the map $d(j_k)^{\mathrm{gpd}} \to \pi_{A_k}^* \mathsf{C}^{\mathrm{gpd}}$ factors through the final object, hence the claim follows. □

Proposition 5.2.3.14. *Suppose that* $\mathcal{B}$ *is hypercomplete. Then any quasi-filtered* $\mathcal{B}$*-category is filtered.*

Proof. Let C be quasi-filtered. Since $\mathsf{Fin}_{\mathcal{B}}$ is sound, we only have to verify that for any finite $\mathcal{B}$-category K the diagonal functor $\mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{C})$ is final. Since being final is a local property, we may assume that K is the constant $\mathcal{B}$-category attached to some finite ∞-category $\mathcal{K}$ (see Proposition 5.2.3.4). Now for any diagram $d\colon \mathcal{K} \to \mathsf{C}$, we will show that the slice $\mathcal{B}$-category $\mathsf{C}_{d/}$ is again quasi-filtered. To see this, let $\mathcal{K}'$ be a finite ∞-category and consider an arbitrary map $f\colon \mathcal{K}' \to \mathsf{C}_{d/}(A)$ for some $A \in \mathcal{B}$. Passing from $\mathcal{B}$ to $\mathcal{B}_{/A}$ and using that $\pi_A^*(\mathsf{C}_{d/}) \simeq (\pi_A^*\mathsf{C})_{\pi_A^* d/}$ we may assume that $A \simeq 1$. Since $\mathcal{K}$ is constant, the global sections of $\mathsf{C}_{/d}$ recover the slice ∞-category $(\Gamma\mathsf{C})_{d/}$. Therefore f is given by a map $f'\colon \mathcal{K} \diamond \mathcal{K}' \to \mathsf{C}(A)$ out of the join such that the restriction along $\mathcal{K} \hookrightarrow \mathcal{K} \diamond \mathcal{K}'$ recovers d. But since finite ∞-categories are stable under the join construction, we may find a covering $(s_i)_i\colon \bigsqcup A_i \twoheadrightarrow 1$ and for every i an extension $(\mathcal{K} \diamond \mathcal{K}')^{\triangleright} \to \mathsf{C}(A_i)$ of $\pi_{A_i}^* f'$. But these precisely correspond to maps $(\mathcal{K}')^{\triangleright} \to \mathsf{C}(A_i)$ extending $s_i^* \circ f\colon \mathsf{C}(1) \to \mathsf{C}(A_i)$, which shows that $\mathsf{C}_{d/}$ is quasi-filtered and that therefore $\mathsf{C}_{d/}^{\mathrm{gpd}} \simeq 1$ by Lemma 5.2.3.13. Repeating the above argument with $\mathcal{B}_{/A}$ instead of $\mathcal{B}$ we get that the same holds for a diagram d in any context A, so the claim follows from Quillen's Theorem A (Corollary 2.1.4.10). □

5.3. Accessible $\mathcal{B}$-categories

In the classical 1-categorical literature, a *κ-accessible* 1-category is one that can be obtained as the free cocompletion of a small 1-category under κ-filtered colimits [46, 53]. In [49, § 5.4], Lurie generalises this concept to ∞-categories. In this section we will introduce and study an analogous notion for $\mathcal{B}$-categories, that of a U-*accessible* $\mathcal{B}$-category for any sound doctrine U. As with our discussion of U-filteredness, we draw much of our inspiration from ideas in [1] and [71]. Our exposition is tailored to the study of presentable $\mathcal{B}$-categories in Section 5.4, so we will not provide an exhaustive treatment of accessibility for $\mathcal{B}$-categories, but rather set up only the basic machinery that we will need for our discussion of presentability later on. We begin in Section 5.3.1 by giving the definition of a U-accessible $\mathcal{B}$-category and proving some basic results that will be useful later. In Section 5.3.2, we discuss accessible functors. In Section 5.3.3, we give a characterisation of U-accessible $\mathcal{B}$-categories as those that are generated by U-compact objects under U-filtered colimits. Finally, we discuss the notion of

U-*flatness* in Section 5.3.4.

5.3.1. Accessibility

If U is an arbitrary internal class of $\mathcal{B}$-categories and if C is a $\mathcal{B}$-category, we will use the notation $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) = \underline{\mathsf{PSh}}^{\mathsf{Filt}_{\mathsf{U}}}_{\mathcal{B}}(\mathsf{C})$ to denote the free $\mathsf{Filt}_{\mathsf{U}}$-cocompletion of U. We write $\mathsf{Ind}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ for the underlying ∞-category of global sections. If $\mathsf{U} = \mathsf{Fin}_{\mathcal{B}}$, we will simply write $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C})$ for the free $\mathsf{Filt}_{\mathsf{Fin}_{\mathcal{B}}}$-cocompletion and $\mathsf{Ind}_{\mathcal{B}}(\mathsf{C})$ for its underlying ∞-category of global sections. We may now define:

Definition 5.3.1.1. Let U be a sound doctrine. A large $\mathcal{B}$-category D is U-*accessible* if there is a $\mathcal{B}$-category C and an equivalence $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. A large $\mathcal{B}$-category is called *accessible* if it is U-accessible for some sound doctrine U.

Remark 5.3.1.2. By combining Remark 5.1.1.2 with Proposition 3.5.1.9, we find that for every $A \in \mathcal{B}$ there is a canonical identification $\pi_A^*\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}^{\pi_A^*\mathsf{U}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$ for every $\mathcal{B}$-category C and every sound doctrine U.

Remark 5.3.1.3. In light of Proposition 5.1.3.6 one has $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}^{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}}_{\mathcal{B}}(\mathsf{C})$ for every $\mathcal{B}$-category C and every internal class U. In particular, a large $\mathcal{B}$-category D is U-accessible if and only if it is $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-accessible. When arguing about accessible $\mathcal{B}$-categories, we can therefore always assume that U is in addition *left regular* (cf. Corollary 5.1.3.7).

Suppose that D is a U-accessible $\mathcal{B}$-category, i.e. that we have $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ for some $\mathcal{B}$-category C. Recall from Section 3.5.1 that there is an inclusion $\underline{\mathsf{Small}}^{\mathsf{Filt}_{\mathsf{U}}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. The following proposition shows that this inclusion is in fact an equivalence.

Proposition 5.3.1.4. *For any internal class* U *of* $\mathcal{B}$*-categories and any* $\mathcal{B}$*-category* C*, the fully faithful functor* $\underline{\mathsf{Small}}^{\mathsf{Filt}_{\mathsf{U}}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *is an equivalence. In other words, the Yoneda embedding* $\mathsf{C} \hookrightarrow \underline{\mathsf{Small}}^{\mathsf{Filt}_{\mathsf{U}}}_{\mathcal{B}}(\mathsf{C})$ *exhibits* $\underline{\mathsf{Small}}^{\mathsf{Filt}_{\mathsf{U}}}_{\mathcal{B}}(\mathsf{C})$ *as the free* $\mathsf{Filt}_{\mathsf{U}}$*-cocompletion of* C.

Proof. It will be enough to show that $\underline{\mathsf{Small}}^{\mathsf{Filt}_{\mathsf{U}}}_{\mathcal{B}}(\mathsf{C})$ is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. By combining Remark 5.1.1.2 and Remark 3.4.2.3, this follows once we prove that for any U-filtered $\mathcal{B}$-category J, the colimit of any diagram $d\colon \mathsf{J} \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$. Let us set $F = \operatorname{colim} d$ and let $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ be the associated right fibration. We need to show that $\mathsf{C}_{/F}$ is U-filtered. On account of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \mathsf{RFib}_{\mathsf{C}}$ and in light of Lemma 3.5.4.9, we may regard d as a diagram

$$d\colon \mathsf{J} \to \mathsf{RFib}_{\mathsf{C}} \hookrightarrow (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$$

that takes values in the full subcategory $(\mathsf{Filt}_{\mathsf{U}})_{/\mathsf{C}}$ (as $\mathsf{Filt}_{\mathsf{U}}$ is a colimit class by Remark 5.1.1.7). Let $\mathsf{K} \to \mathsf{C}$ be the colimit of d in $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$. As the right fibration $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ is the image of $\mathsf{K} \to \mathsf{C}$ along the localisation functor $L\colon (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}}$ (see Proposition 3.3.2.10), there is a final map $\mathsf{K} \to \mathsf{C}_{/F}$ over C. It therefore suffices to show that K is U-filtered. Now Proposition 3.2.4.3 implies that K is the colimit of the diagram $(\pi_{\mathsf{C}})_! d\colon \mathsf{J} \to (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}} \to \mathsf{Cat}_{\mathcal{B}}$. By construction, this diagram takes values in $\mathsf{Filt}_{\mathsf{U}}$. Therefore, the result follows from Proposition 5.1.1.8. □

Remark 5.3.1.5. In light of Proposition 5.3.1.4, if C is a $\mathcal{B}$-category and if U is a sound doctrine, Remark 5.1.1.7 implies that a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context $A \in \mathcal{B}$ is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ if and only if the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/F}$ is $\pi_A^* \mathsf{U}$-filtered.

For later use, let us record that our notion of accessibility is stable under the formation of slice $\mathcal{B}$-categories:

Proposition 5.3.1.6. *Let* U *be a sound doctrine and let* D *be a* U*-accessible* $\mathcal{B}$*-category. Then* $\mathsf{D}_{/d}$ *is* $\pi_A^*\mathsf{U}$*-accessible, for any choice of object* $d\colon A \to \mathsf{D}$.

Proof. Using Remark 5.3.1.2, we may assume that $A \simeq 1$. Choose a $\mathcal{B}$-category C such that $\mathsf{D} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. Let $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ be the presheaf that corresponds to d under this equivalence. We then obtain a commutative diagram

$$\begin{tikzcd} \mathsf{C}_{/F} \arrow[r, hook] \arrow[d, "p"] & \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[r, hook] \arrow[d, "(\pi_F)_!"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[d, "(\pi_F)_!"] \\ \mathsf{C} \arrow[r, hook, "h_{\mathsf{C}}"] & \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which both squares are cartesian. By Lemma 3.4.1.4, the vertical map on the right can be identified with $p_! \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that the upper row in the above diagram recovers the Yoneda embedding $h_{\mathsf{C}_{/F}}$. With respect to this identification, a presheaf on $\mathsf{C}_{/F}$ is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C})_{/F}$ precisely if the domain of the associated right fibration is U-filtered. We therefore obtain an equivalence $\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}_{/F})$, hence the result follows. □

5.3.2. Accessible functors

It will be convenient to also have a notion of accessibility for functors between accessible $\mathcal{B}$-categories at our disposal:

Definition 5.3.2.1. Let U be a sound doctrine. A functor $f\colon \mathsf{C} \to \mathsf{D}$ of large $\mathcal{B}$-categories is U-*accessible* if C and D are $\mathsf{Filt}_{\mathsf{U}}$-cocomplete and f is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous. We will call f *accessible* if it is U-accessible for some sound doctrine U. We denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{C}, \mathsf{D})$ the full subcategory spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ such that the corresponding $\mathcal{B}_{/A}$-functor $\pi_A^* \mathsf{C} \to \pi_A^* \mathsf{D}$ is accessible. We will denote by $\mathsf{Fun}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{C}, \mathsf{D})$ the underlying ∞-category of global sections.

Remark 5.3.2.2. Let $f\colon \mathsf{C} \to \mathsf{D}$ be U-accessible for some sound doctrine U. By Remark 5.2.2.22 we may find a $\mathcal{B}$-regular cardinal κ such that $\mathsf{U} \subset \mathsf{Cat}_{\mathcal{B}}^{\kappa}$. It follows that a functor is accessible if and only if it is $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$-accessible for some $\mathcal{B}$-regular cardinal κ.

Remark 5.3.2.3. Let $f\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{C}, \mathsf{D})$ be an arbitrary object. By definition, this means that there is a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that the functors $s_i^* f\colon \pi_{A_i}^* \mathsf{C} \to \pi_{A_i}^* \mathsf{D}$ are accessible for all $i \in I$. By Remark 5.3.2.2, we may find a $\mathcal{B}_{/A}$-regular cardinal κ such that all A_i are κ-compact (in $\mathcal{B}_{/A}$) and $s_i^* f$ is $\mathsf{Cat}_{\mathcal{B}/A_i}^{\kappa}$-accessible for every κ. Hence Remark 5.1.1.2 and Remark 5.2.2.2 together with Remark 3.3.2.3 imply that f is $\mathsf{Filt}_{\mathsf{Cat}_{\mathcal{B}/A}^{\kappa}}$-cocontinuous, so in particular accessible. Thus, an object $f\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{C}, \mathsf{D})$ if and only if f defines an accessible functor between $\mathcal{B}_{/A}$-categories. In particular, one obtains a canonical equivalence

$$\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{C}, \mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\mathrm{acc}}(\pi_A^* \mathsf{C}, \pi_A^* \mathsf{D})$$

for every $A \in \mathcal{B}$.

Somewhat surprisingly, provided that both domain and codomain have a sufficient amount of colimits, accessibility of a functor between $\mathcal{B}$-categories is an entirely section-wise concept:

Proposition 5.3.2.4. *Let κ be a $\mathcal{B}$-regular cardinal and let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between cocomplete $\mathcal{B}$-categories that is section-wise κ-accessible. Then the functor f is $\mathsf{Filt}_{\mathrm{Cat}_{\mathcal{B}}^{\kappa}}$-accessible.*

Proof. As κ is $\mathcal{B}$-regular, Remark 5.2.2.2 and Remark 5.1.1.2 imply that it suffices to show that f preserves the colimit of every diagram $d\colon \mathsf{J} \to \mathsf{C}$ with J a $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-filtered $\mathcal{B}$-category. As C is cocomplete, there exists an extension

$$d' : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathrm{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}} \to \mathsf{C}$$

of d. By Corollary 5.1.5.11, the inclusion $\mathsf{J} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathrm{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}}$ is final, hence we may replace J by $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathrm{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}}$ and d by d' and can thus assume that J is $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-cocomplete (see Proposition 5.1.5.4). Using Lemma 5.2.2.20 and Remark 5.1.1.7, we can further reduce to the case where J is the constant $\mathcal{B}$-category that is associated with an ∞-category with κ-small colimits. As by [49, Proposition 5.3.3.3] every such ∞-category is κ-filtered, the result follows. □

Corollary 5.3.2.5. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor of cocomplete large $\mathcal{B}$-categories. Then f is accessible if and only if f is section-wise accessible.*

Proof. By Remark 5.3.2.2, we can assume that f is $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-accessible for some $\mathcal{B}$-regular cardinal κ. Then it follows from Lemma 5.2.2.19 that f commutes with colimits indexed by constant $\mathcal{B}$-categories attached to κ-filtered ∞-categories. In other words, $f(A)$ commutes with κ-filtered colimits for every $A \in \mathcal{B}$ and is thus section-wise accessible. For the converse, we pick a small full subcategory $\mathcal{G} \hookrightarrow \mathcal{B}$ that generates $\mathcal{B}$ under small colimits. Then we may find a $\mathcal{B}$-regular cardinal κ such that $f(G)$ is κ-accessible for every $G \in \mathcal{G}$. Since the preservation of colimits is a local condition (Remark 3.2.2.3) and since every object $A \in \mathcal{B}$ admits a cover by objects in $\mathcal{G}$, we conclude that $f(A)$ preserves κ-filtered colimits for all $A \in \mathcal{B}$. Therefore f is accessible by Proposition 5.3.2.4. □

5.3.3. U-compact objects in accessible $\mathcal{B}$-categories

In [49, Proposition 5.4.2.2], Lurie characterises κ-accessible ∞-categories as those that are generated by a small collection of κ-compact objects under κ-filtered colimits. In this section, our goal is to obtain an analogue of this statement for accessible $\mathcal{B}$-categories. We begin with the following characterisation of the U-compact objects in a U-accessible $\mathcal{B}$-category:

Proposition 5.3.3.1. *Let* U *be an internal class of* $\mathcal{B}$*-categories, let* C *be a* $\mathcal{B}$*-category and let* $\mathsf{D} = \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$*. Then there is an equivalence* $\mathsf{D}^{\mathsf{U}\text{-cpt}} \simeq \mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$ *of full subcategories in* D*. In particular,* $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ *is small.*

Proof. In light of Remark 5.1.5.6, the second claim follows immediately from the first. Now by Yoneda's lemma and the fact that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits, every representable presheaf on C defines a U-compact object in D. In other words, one obtains an inclusion $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$. By combining this observation with Proposition 5.1.5.8, one obtains an inclusion $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C}) \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$. Conversely, let $F\colon A \to \mathsf{D}^{\mathsf{U}\text{-cpt}}$ be an arbitrary object. We need to show that F is contained in $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (which is made possible by Remark 5.1.5.2 and Remark 5.3.1.2), we can assume $A \simeq 1$. The desired result thus follows from Lemma 5.1.5.9. ☐

We can now state and prove our characterisation of U-accessible $\mathcal{B}$-categories. To that end, if D is a $\mathsf{Filt}_{\mathsf{U}}$-cocomplete $\mathcal{B}$-category and $\mathsf{C} \hookrightarrow \mathsf{D}$ is a full subcategory, recall that we say that D is *generated* under $\mathsf{Filt}_{\mathsf{U}}$-colimits by C if D is the smallest full subcategory of itself that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits and contains C. We now obtain:

Proposition 5.3.3.2. *Let* U *be a sound doctrine and let* D *be a large* $\mathcal{B}$*-category. Then the following are equivalent:*

1. D *is* U*-accessible;*
2. D *is locally small and* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete, the (a priori large)* $\mathcal{B}$*-category* $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ *is small and generates* D *under* $\mathsf{Filt}_{\mathsf{U}}$*-colimits;*
3. D *is* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete, and there is a small full subcategory* $\mathsf{C} \hookrightarrow \mathsf{D}$ *such that* $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$ *and such that* C *generates* D *under* $\mathsf{Filt}_{\mathsf{U}}$*-colimits.*

Proof. If D is U-accessible, there is a small $\mathcal{B}$-category C and an equivalence $\mathsf{D} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. In particular, D is locally small and $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. Furthermore, Proposition 5.3.3.1 implies that $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small. Since D is generated by C under $\mathsf{Filt}_{\mathsf{U}}$-colimits and therefore by $\mathsf{D}^{\mathsf{U}\text{-cpt}}$, we conclude that (1) implies (2). Moreover, (2) trivially implies (3), and the fact that (3) implies (1) immediately follows from Proposition 3.5.2.3. □

Recall from Definition 3.1.4.7 that a localisation $L\colon \mathsf{D} \to \mathsf{E}$ is a *Bousfield localisation* if L admits a (necessarily fully faithful) right adjoint i. Proposition 5.3.3.2 now implies:

Corollary 5.3.3.3. *Let* U *be a sound doctrine and let* D *be a* U*-accessible* $\mathcal{B}$*-category. Suppose that* E *is a Bousfield localisation of* D *such that the inclusion* $i\colon \mathsf{E} \hookrightarrow \mathsf{D}$ *is* $\mathsf{Filt}_{\mathsf{U}}$*-cocontinuous. Then* E *is* U*-accessible as well.*

Proof. Let $\mathsf{C} \hookrightarrow \mathsf{E}$ be the image of $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ along the localisation functor $L\colon \mathsf{D} \to \mathsf{E}$. As E is locally small and $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small by Proposition 5.3.3.2, the $\mathcal{B}$-category C is small as well (Lemma 2.3.1.6). In light of the adjunction $L \dashv i$, the assumption that i is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous implies that L preserves U-compact objects. In other words, we have $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$. By Proposition 5.3.3.2, the large $\mathcal{B}$-category D is generated by $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ under $\mathsf{Filt}_{\mathsf{U}}$-colimits, i.e. D is the smallest full subcategory of itself that contains $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ and that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits. Let $\mathsf{E}' \hookrightarrow \mathsf{E}$ be the smallest full subcategory that contains C and that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits, and let us consider the commutative diagram

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{U}\text{-cpt}} \arrow[r, hook] \arrow[d, two heads] & \mathsf{D}' \arrow[r, hook] \arrow[d] & \mathsf{D} \arrow[d, two heads, "L"] \\
\mathsf{C} \arrow[r, hook] & \mathsf{E}' \arrow[r, hook] & \mathsf{E}
\end{tikzcd}$$

in which the right square is a pullback. Since L is cocontinuous, the inclusion $\mathsf{D}' \hookrightarrow \mathsf{D}$ is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits (using Lemma 3.5.1.11) and must therefore be an equivalence. As the inclusion i is a section of L, this implies that the inclusion $\mathsf{E}' \hookrightarrow \mathsf{E}$ is an equivalence as well. By using Proposition 5.3.3.2, we thus conclude that D is U-accessible. □

Definition 5.3.3.4. A Bousfield localisation $L\colon \mathsf{D} \to \mathsf{E}$ of a U-accessible $\mathcal{B}$-category D is said to be U-*accessible* if the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$ is U-accessible. More generally, a Bousfield localisation $L\colon \mathsf{D} \to \mathsf{E}$ is accessible if there is a sound doctrine U such that D is U-accessible and the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous.

Remark 5.3.3.5. Proposition 5.3.3.2 also shows that accessibility is a *local* condition: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$ and if U is a sound doctrine, then a large $\mathcal{B}$-category D being U-accessible is equivalent to each $\pi_{A_i}^* \mathsf{D}$ being $\pi_{A_i}^* \mathsf{U}$-accessible. In fact, Remark 5.3.1.2 shows that we have an equivalence

$$\pi_A^* \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}_{\mathcal{B}_{/A}}^{\pi_A^* \mathsf{U}}(\pi_A^* \mathsf{C})$$

for every $\mathcal{B}$-category C and every $A \in \mathcal{B}$, hence the condition is necessary. To show that it is sufficient, first recall that since $\mathsf{Filt}_{\mathsf{U}}$-cocompleteness is a local condition (Remark 3.3.2.3), we deduce that D must be $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. Moreover, if $\mathsf{E} \hookrightarrow \mathsf{D}$ is the smallest full subcategory that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits in D and that contains $\mathsf{D}^{\mathsf{U}\text{-cpt}}$, the fact that $\pi_{A_i}^* \mathsf{E}$ is closed under $\mathsf{Filt}_{\pi_{A_i}^* \mathsf{U}}$-colimits and Remark 5.1.5.2 imply that the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$ is locally an equivalence and therefore already an equivalence. To show that D is U-accessible, Proposition 5.3.3.2 thus implies that it suffices to verify that also the condition of a large $\mathcal{B}$-category to be (locally) small is local in $\mathcal{B}$, which is clear from the definitions.

Proposition 5.3.3.2 can furthermore be used to show that presheaf $\mathcal{B}$-categories are U-accessible for *every* choice of a sound doctrine U:

Proposition 5.3.3.6. *For every $\mathcal{B}$-category C and every sound doctrine U, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is U-accessible.*

Proof. In light of Remark 5.3.1.3, we can assume that U is a left regular doctrine. By Proposition 5.1.5.10, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is small. Using Proposition 5.3.3.2, it therefore suffices to show that every object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ can be obtained as a U-filtered colimit of U-compact objects. If $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is an arbitrary presheaf, Lemma 5.1.4.4 shows that F is the colimit of the diagram

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}^{\mathsf{U}\text{-cpt}} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).$$

By Lemma 3.5.1.11, the $\mathcal{B}$-category $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}_{/F}$ is op(U)-cocomplete and therefore in particular U-filtered (Example 5.1.2.4). Hence the presheaf F is contained in $\underline{\mathrm{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}})$. Finally, upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (which is made possible by Remark 5.1.5.2), the same conclusion holds for objects in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context A, which finishes the proof. □

5.3.4. Flatness

Recall from Section 3.5.1 that if C is a $\mathcal{B}$-category, the functor of left Kan extension along the Yoneda embedding $h_{\mathsf{C}^{\mathrm{op}}} \colon \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})$ induces an equivalence

$$(h_{\mathsf{C}^{\mathrm{op}}})_! \colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathrm{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}}), \mathrm{Grpd}_{\mathcal{B}})$$

where the right-hand side denotes the large $\mathcal{B}$-category of cocontinuous functors between $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})$ and $\mathrm{Grpd}_{\mathcal{B}}$.

Definition 5.3.4.1. Let C be a $\mathcal{B}$-category and let U be an arbitrary internal class of $\mathcal{B}$-categories. A presheaf $F \colon A \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ is said to be U-*flat* if the functor

$$\underline{\mathrm{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}, \mathrm{Grpd}_{\mathcal{B}_{/A}}) \to \mathrm{Grpd}_{\mathcal{B}_{/A}}$$

that is encoded by $(h_{\mathsf{C}^{\mathrm{op}}})_!(F)$ is $\pi_A^*\mathsf{U}$-continuous. We denote the full subcategory of $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by the U-flat presheaves by $\underline{\mathrm{Flat}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$, and we denote its underlying ∞-category of global sections by $\mathrm{Flat}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$.

Remark 5.3.4.2. In the situation of Definition 5.3.4.1, the fact that U-continuity is a local condition (Remark 3.3.2.3) together with Remark 3.4.3.2 and Remark 2.3.2.1 implies that the presheaf F is U-flat if and only if for every cover $(s_i) \colon \bigsqcup A_i \twoheadrightarrow A$ in $\mathcal{B}$ the presheaf $s_i^* F$ is U-flat. In particular, *every* object in $\underline{\mathrm{Flat}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is U-flat, and there is a canonical equivalence $\pi_A^* \underline{\mathrm{Flat}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathrm{Flat}}^{\pi_A^*\mathsf{U}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{U})$ for every $A \in \mathcal{B}$.

Lemma 5.3.4.3. *Let* C *be a* $\mathcal{B}$*-category and let* $F \colon \mathsf{C}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ *be a presheaf. Then the Yoneda extension* $(h_{\mathsf{C}^{\mathrm{op}}})_!(F)$ *is equivalent to the composition*

$$\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}}) \xrightarrow{p^*} \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathrm{Grpd}_{\mathcal{B}}) \xrightarrow{\mathrm{colim}} \mathrm{Grpd}_{\mathcal{B}},$$

where $p \colon \mathsf{C}_{/F} \to \mathsf{C}$ *is the right fibration that is classified by* F*.*

Proof. As both p^* and colim are cocontinuous functors, the universal property of presheaf $\mathcal{B}$-categories implies that it suffices to find an equivalence colim $p^* h_{\mathsf{C}^{\mathrm{op}}} \simeq F$. Let us denote by $h_{/F} \colon \mathsf{C}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ the functor that is induced by the Yoneda embedding h_{C} by taking slice $\mathcal{B}$-categories. Now there is a commutative diagram

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[r, "h_{\mathsf{C}}^{\mathrm{op}}"] \arrow[d, "h_{\mathsf{C}^{\mathrm{op}}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \arrow[d, "h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "(h_{\mathsf{C}})_!"] \arrow[d, "p^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "(\pi_F)_!^*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "(h_{/F})_!"'] \arrow[dr, bend right, "\mathrm{colim}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "\mathrm{id}_F^*"] \\
 & \mathsf{Grpd}_{\mathcal{B}}
\end{tikzcd}$$

in which the commutativity of the lower square follows from the straightening equivalence for right fibrations (Theorem 2.2.1.1) together with p being a right fibration and therefore *proper* in the sense of Section 2.1.4, see Proposition 2.1.4.9. In light of Yoneda's lemma, it is now immediate that the composition of the upper horizontal map with the right column in the above diagram recovers F, as desired. □

Recall from Example 3.2.1.10 that if C is a $\mathcal{B}$-category that admits a final object $1_{\mathsf{C}} \colon 1 \to \mathsf{C}$, then this object is the limit of the unique diagram $\varnothing \to \mathsf{C}$. In other words, the map $1_{\mathsf{C}} \colon 1 \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\varnothing, \mathsf{C}) \to \mathsf{C}$ is right adjoint to the unique functor $\pi_{\mathsf{C}} \colon \mathsf{C} \to 1$. We will denote by $\pi \colon \mathrm{id}_{\mathsf{C}} \to 1_{\mathsf{C}}\pi_{\mathsf{C}}$ the associated adjunction unit.

Lemma 5.3.4.4. *If $p \colon \mathsf{P} \to \mathsf{C}$ is a right fibration of $\mathcal{B}$-categories, the commutative square*

$$\begin{tikzcd}
\mathrm{id}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \arrow[r, "\eta"] \arrow[d, "\pi"] & p^* p_! \arrow[d, "p^* p_! \pi"] \\
1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}\pi_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \arrow[r, "\eta 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}\pi_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}"] & p^* p_! 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}\pi_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}
\end{tikzcd}$$

is a pullback square in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}), \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}))$.

Proof. By Proposition 3.2.3.2, it suffices to show that for every $A \in \mathcal{B}$ and every object $F\colon A \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})$ the induced diagram

$$\begin{tikzcd}
F \arrow[r, "\eta F"] \arrow[d, "\pi"] & p^* p_!(F) \arrow[d, "p^* p_! \pi"] \\
\pi_A^*(1_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})}) \arrow[r, "\eta \pi_A^*(1_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})})"] & p^* p_!(\pi_A^*(1_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})}))
\end{tikzcd}$$

is a pullback. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we can assume $A \simeq 1$. In light of the straightening equivalence for right fibrations, this diagram corresponds to the commutative square

$$\begin{tikzcd}
\mathsf{P}_{/F} \arrow[r] \arrow[d] & \mathsf{P}_{/F} \times_{\mathsf{C}} \mathsf{P} \arrow[d] \\
\mathsf{P} \arrow[r] & \mathsf{P} \times_{\mathsf{C}} \mathsf{P}
\end{tikzcd}$$

of right fibrations over P. As this square is evidently a pullback, the claim follows. □

Lemma 5.3.4.5. *Let* I *be a* $\mathcal{B}$*-category and let*

$$\begin{tikzcd}
d \arrow[r, "\phi"] \arrow[d] & h \arrow[d] \\
\mathrm{diag}(\mathsf{G}) \arrow[r, "\mathrm{diag}(s)"] & \mathrm{diag}(\mathsf{H})
\end{tikzcd}$$

be a pullback square in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})$*, where* $s\colon \mathsf{G} \to \mathsf{H}$ *is an arbitrary map of* $\mathcal{B}$*-groupoids. Then the commutative square*

$$\begin{tikzcd}
\mathrm{colim}(d) \arrow[r, "\mathrm{colim}(\phi)"] \arrow[d] & \mathrm{colim}(h) \arrow[d] \\
\mathsf{G} \arrow[r, "s"] & \mathsf{H}
\end{tikzcd}$$

that is obtained by transposing the first square across the adjunction $\mathrm{colim} \dashv \mathrm{diag}$ *is a pullback square as well.*

Proof. In light of the Grothendieck construction, the above pullback square corresponds to a pullback square

$$\begin{tikzcd} \mathsf{I}_{/d} \arrow[r] \arrow[d] & \mathsf{I}_{/h} \arrow[d] \\ \mathsf{G} \arrow[r, "s"] & \mathsf{H} \end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$. By Proposition 3.2.5.1, we need to show that the groupoidification functor carries this diagram to a pullback square in $\mathcal{B}$. As s is a right fibration and therefore proper (Proposition 2.1.4.9) this is immediate. □

Proposition 5.3.4.6. *Let* U *be a sound internal class and let* C *be a* $\mathcal{B}$*-category. Then there is an equivalence* $\underline{\mathsf{Flat}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *of full subcategories in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Proof. In light of Remark 5.3.1.2 and Remark 5.3.4.2, it will be enough to show that a presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ defines an object of $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ if and only if it is U-flat. So suppose first that F is contained in $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Then $\mathsf{C}_{/F}$ is U-filtered. Let $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ be the projection. By Lemma 5.3.4.3, the Yoneda extension $(h_{\mathsf{C}^{\mathrm{op}}})_! F$ can be computed as the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{p^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{\mathrm{colim}} \mathsf{Grpd}_{\mathcal{B}},$$

and since both p^* and colim are U-continuous, we deduce that F is U-flat.

Conversely, suppose that F is U-flat. By Lemma 5.3.4.4, the commutative square

$$\begin{tikzcd} h_{\mathsf{C}_{/F}} \arrow[r, "\eta h_{\mathsf{C}_{/F}}"] \arrow[d, "\pi h_{\mathsf{C}_{/F}}"] & p^* p_! h_{\mathsf{C}_{/F}} \arrow[d, "p^* p_! \pi h_{\mathsf{C}_{/F}}"] \\ 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})} \pi_{\mathsf{C}_{/F}} \arrow[r, "\eta 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})} \pi_{\mathsf{C}_{/F}}"] \arrow[d, "\simeq"] & p^* p_! 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})} \pi_{\mathsf{C}_{/F}} \arrow[d, "\simeq"] \\ \mathrm{diag}_{\mathsf{C}_{/F}}(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})}) \arrow[r, "\mathrm{diag}_{\mathsf{C}_{/F}}(\eta 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})})"] & \mathrm{diag}_{\mathsf{C}_{/F}}(p^* F) \end{tikzcd}$$

is a pullback in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}))$. By Proposition 3.4.1.1 the composition of the two vertical maps on the left is a colimit cocone, hence so is the composition of the two vertical maps on the right, for $p^* p_!$ preserves all colimits.

Let $d\colon \mathsf{K} \to (\mathsf{C}_{/F})^{\mathrm{op}}$ be a diagram with $\mathsf{K} \in \mathsf{U}(1)$. By postcomposition with $\lim_{\mathsf{K}} d^* : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$, the above pullback square induces a cartesian square

$$\begin{tikzcd} \lim_{\mathsf{K}} d^* h_{\mathsf{C}_{/F}} \arrow[r] \arrow[d] & \lim_{\mathsf{K}} d^* p^* p_! h_{\mathsf{C}_{/F}} \arrow[d] \\ \mathrm{diag}_{\mathsf{C}_{/F}}(1_{\mathsf{Grpd}_{\mathcal{B}}}) \arrow[r] & \mathrm{diag}_{\mathsf{C}_{/F}}(\lim_{\mathsf{K}} d^* p^* F). \end{tikzcd}$$

We claim that the right vertical map in the this last diagram is still a colimit cocone. To see this, note that the equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{Grpd}_{\mathcal{B}}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathsf{Grpd}_{\mathcal{B}}))$$

carries the diagram $d^* p^* p_! h_{\mathsf{C}_{/F}}$ to the composition

$$\mathsf{K} \xrightarrow{d} (\mathsf{C}_{/F})^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} \mathsf{C}^{\mathrm{op}} \xrightarrow{h_{\mathsf{C}^{\mathrm{op}}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{p^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathsf{Grpd}_{\mathcal{B}}).$$

Now the functor $\lim_{\mathsf{K}}$ preserving the colimit of $d^* p^* p_! h_{\mathsf{C}_{/F}}$ is equivalent to the colimit functor $\mathrm{colim}_{\mathsf{C}_{/F}}$ preserving the limit of the diagram $p^* h_{\mathsf{C}^{\mathrm{op}}} p^{\mathrm{op}} d$ (cf. the argument in Remark 5.1.1.4). As the functor p^* commutes with all limits, this in turn follows once $\mathrm{colim}_{\mathsf{C}_{/F}} p^*$ preserves the limit of $h_{\mathsf{C}^{\mathrm{op}}} p^{\mathrm{op}} d$, which follows from the equivalence $\mathrm{colim}_{\mathsf{C}_{/F}} p^* \simeq (h_{\mathsf{C}^{\mathrm{op}}})_!(F)$ from Lemma 5.3.4.3 and the assumption that F is U-flat. As a consequence, we now deduce from Lemma 5.3.4.5 that the map

$$\underset{\mathsf{C}_{/F}}{\mathrm{colim}} \lim_{\mathsf{K}} d^* h_{\mathsf{C}_{/F}} \to 1_{\mathsf{Grpd}_{\mathcal{B}}}$$

must be an equivalence. By Proposition 5.1.2.5, this means that $(\mathsf{C}_{/F})^{\mathrm{gpd}}_{d^{\mathrm{op}}/} \simeq 1$. As d was chosen arbitrarily and as replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ allows us to derive the same conclusion for any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, (\mathsf{C}_{/F})^{\mathrm{op}})$ in context $A \in \mathcal{B}$, this shows that $\mathsf{C}_{/F}$ is weakly U-filtered and therefore U-filtered by soundness of U. Hence F is contained in $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. □

5.4. Presentable $\mathcal{B}$-categories

In this section we introduce and study *presentable* $\mathcal{B}$-categories. Classically, a (locally) presentable 1-category is one that is locally small and is generated by a small collection of κ-compact objects under small colimits [23]. In [49, § 5.5], Lurie generalised this concept to ∞-categories. In particular, his treatment contains a multitude of equivalent characterisations of presentability [49, Theorem 5.5.1.1]. One of the main goals of this section is to obtain a comparable result for $\mathcal{B}$-categories. As a starting point, we chose to *define* a presentable $\mathcal{B}$-category as a *Bousfield* localisation of a presheaf $\mathcal{B}$-category at a (small) subcategory. To make sense of this, we need to study the notion of *local objects* in a $\mathcal{B}$-category, which we do in Section 5.4.1. In Section 5.4.2, we formally define presentable $\mathcal{B}$-categories and prove our main result about various different characterisations of this condition (Theorem 5.4.2.5), building upon our work on accessible $\mathcal{B}$-categories. In Section 5.4.3, we discuss adjoint functor theorems for presentable $\mathcal{B}$-categories, and in Section 5.4.4 we construct large $\mathcal{B}$-categories of presentable $\mathcal{B}$-categories and show that these are complete and cocomplete. Finally, we discuss the notion of U-*sheaves* in Section 5.4.5: these are U-continuous functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$, where C is an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category and D is a large complete $\mathcal{B}$-category. We show that if D is presentable, such U-sheaves form a presentable $\mathcal{B}$-category as well, and that this provides yet another equivalent characterisation of the notion of presentability.

5.4.1. Local objects

Recall from Section 1.3.3 the definition of a localisation of a $\mathcal{B}$-category. If $j\colon \mathsf{S} \to \mathsf{D}$ is a functor of $\mathcal{B}$-categories, we obtain a localisation functor

$$L\colon \mathsf{D} \to \mathsf{S}^{-1}\mathsf{D} = \mathsf{D} \sqcup_{\mathsf{S}} \mathsf{S}^{\mathrm{gpd}}.$$

If E is an arbitrary $\mathcal{B}$-category, L satisfies the universal property that

$$L^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{D}, \mathsf{E}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

is fully faithful and identifies the domain with the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$ that is spanned by those functors $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{D}$ whose restriction along $\pi_A^*(j)$ factors through the inclusion $\pi_A^*\mathsf{D}^{\simeq} \hookrightarrow \pi_A^*\mathsf{D}$. We may now define:

Definition 5.4.1.1. If $\mathsf{S} \to \mathsf{D}$ is a functor, we define the associated $\mathcal{B}$-*category* $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *of* S-*local objects in* D as the full subcategory of D that is defined via the pullback

$$\begin{tikzcd} \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \arrow[r, hook] \arrow[d, hook, "i"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{D}) \arrow[d, hook, "L^*"] \\ \mathsf{D} \arrow[r, hook, "h"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$. We refer to an object $d\colon A \to \mathsf{D}$ as being S-*local* if it is contained in $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$.

Remark 5.4.1.2. If $A \in \mathcal{B}$ is an arbitrary object, we deduce from Proposition 1.2.5.4 and Remark 2.3.2.1 that there is a canonical equivalence

$$\pi_A^* \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \simeq \mathsf{Loc}_{\pi_A^*\mathsf{S}}(\pi_A^*\mathsf{D})$$

of full subcategories in $\pi_A^*\mathsf{D}$. In particular, this implies that an object $d\colon A \to \mathsf{D}$ is S-local if and only if its transpose $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{D}$ defines a $\pi_A^*\mathsf{S}$-local object.

Remark 5.4.1.3. Explicitly, an object $d\colon 1 \to \mathsf{D}$ is contained in $\mathsf{Loc}_{\mathsf{S}}$ precisely if the restriction of the presheaf $h(d)$ along j factors through the inclusion $\mathsf{Grpd}_{\mathcal{B}}^{\simeq} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$, which is the case if and only if for every map $s\colon e \to e'$ in S in context $A \in \mathcal{B}$ the morphism $j(s)^*\colon \operatorname{map}_{\mathsf{D}}(j(e'), \pi_A^* d) \to \operatorname{map}_{\mathsf{D}}(j(e), \pi_A^* d)$ is an equivalence of $\mathcal{B}_{/A}$-groupoids (cf. Proposition 1.3.1.4). By Remark 5.4.1.2, an analogous description holds for S-local objects in arbitrary context.

Remark 5.4.1.4. In the situation of Definition 5.4.1.1, we deduce from Proposition 1.3.3.15 that if $\mathsf{T} \hookrightarrow \mathsf{D}$ is the 1-*image* of the map $\mathsf{S} \to \mathsf{D}$ (in the sense of Definition 1.3.1.11), the canonical map $\mathsf{S}^{-1}\mathsf{D} \to \mathsf{T}^{-1}\mathsf{D}$ is an equivalence. Consequently, the induced map $\mathsf{Loc}_{\mathsf{T}}(\mathsf{D}) \to \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ must be an equivalence as well. Therefore, we may always assume that S is a *subcategory* of D.

Remark 5.4.1.5. Suppose that $(f_i\colon c_i \to d_i)_{i\in I}$ is a (small) family of maps in D, with $A_i \in \mathcal{B}$ being the context of f_i. By the discussion in Section 1.3.1, the subcategory $\mathsf{S} \hookrightarrow \mathsf{D}$ that is *generated* by this family is given by the 1-image of the induced map $\bigsqcup_i \Delta^1 \otimes A_i \to \mathsf{D}$. By combining Remark 5.4.1.3 and Remark 5.4.1.4, an object $d\colon 1 \to \mathsf{D}$ is S-local if and for each $i \in I$ the map

$$f_i^*\colon \operatorname{map}_{\mathsf{D}}(d_i, \pi_{A_i}^* d) \to \operatorname{map}_{\mathsf{D}}(c_i, \pi_{A_i}^* d)$$

is an equivalence in $\mathcal{B}_{/A_i}$.

The theory of local objects is intimately connected to the notion of *Bousfield localisations*, i.e. of reflective subcategories:

Proposition 5.4.1.6. *Let* D *be a* $\mathcal{B}$*-category and let* $L\colon \mathsf{D} \to \mathsf{C}$ *be a Bousfield localisation. Let* $\mathsf{S} = L^{-1}(\mathsf{C}^{\simeq}) \hookrightarrow \mathsf{D}$*. Then the inclusion* $\mathsf{C} \hookrightarrow \mathsf{D}$ *of* L *induces an equivalence* $\mathsf{C} \simeq \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *of full subcategories in* D*. Furthermore, if* D *is* U*-accessible and* L *is a* U*-accessible Bousfield localisation, there is a* small *subcategory* $\mathsf{T} \hookrightarrow \mathsf{S}$ *such that* $\mathsf{C} \simeq \mathsf{Loc}_{\mathsf{T}}(\mathsf{D})$.

Proof. We begin with the first statement. By Proposition 3.1.4.6, the functor $L\colon \mathsf{D} \to \mathsf{C}$ identifies C with the localisation $\mathsf{S}^{-1}\mathsf{D}$. In light of the very definition of $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$, the claim thus follows once we show that the commutative square

$$\begin{tikzcd} \mathsf{C} \arrow[r, hook, "h_{\mathsf{C}}"] \arrow[d, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, hook, "L^*"] \\ \mathsf{D} \arrow[r, hook, "h_{\mathsf{D}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

is a pullback. Using Remark 2.3.2.1, it will be enough to show that given any presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ for which $L^*(F)\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is representable by an object $d\colon 1 \to \mathsf{Grpd}_{\mathcal{B}}$, then F is representable as well. This immediately follows from the computation

$$F \simeq L_! L^* F \simeq L_! h_{\mathsf{D}}(d) \simeq h_{\mathsf{C}} L(d),$$

cf. Corollary 3.1.3.3. Now if D is U-accessible and L is a U-accessible Bousfield localisation, let us set $\mathsf{E} = \mathsf{D}^{\mathsf{U}\text{-cpt}}$ and $\mathsf{T} = i^{-1}(\mathsf{S}) \hookrightarrow \mathsf{E}$, where $i\colon \mathsf{E} \hookrightarrow \mathsf{D}$ is the inclusion. Since E is small by Proposition 5.3.3.1, so is T, and we obtain a full inclusion $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{Loc}_{\mathsf{T}}(\mathsf{D})$. We need to show that this is an equivalence. By Remark 5.4.1.2, it will be enough to show that every T-local object $d\colon 1 \to \mathsf{D}$ is already S-local. Let $\eta\colon \mathrm{id}_{\mathsf{D}} \to iL$ be the adjunction unit. We then obtain a map

$$\eta^*\colon \operatorname{map}_{\mathsf{D}}(iL(-), d) \to \operatorname{map}_{\mathsf{D}}(-, d),$$

and since d is T-local the restriction of η^* to E is an equivalence. But as both domain and codomain of this map are $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous when viewed as functors $\mathsf{D} \to \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}$, the fact that we have $\mathsf{D} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{E})$ immediately implies that η^* is already an equivalence, so that d is S-local. $\square$

In the situation of Proposition 5.4.1.6, the question naturally arises whether the converse is true: namely, whether the inclusion $i\colon \mathrm{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ always defines a Bousfield localisation (i.e. admits a left adjoint) for every $\mathcal{B}$-category D and every functor $\mathsf{S} \to \mathsf{D}$. In general, this is false, but there is a large class of $\mathcal{B}$-categories D and functors $\mathsf{S} \to \mathsf{D}$ for which this is nonetheless the case:

Proposition 5.4.1.7. *Let* D *be a* $\mathrm{Grpd}_{\mathcal{B}}$*-cocomplete large* $\mathcal{B}$*-category that takes values in the* ∞*-category* $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ *of presentable* ∞*-categories. Let furthermore* $i\colon \mathsf{S} \to \mathsf{D}$ *be a functor where* S *is small. Then the inclusion* $i\colon \mathrm{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ *admits a left adjoint and therefore exhibits* $\mathrm{Loc}_{\mathsf{S}}(\mathsf{D})$ *as a Bousfield localisation of* D*. Moreover, this Bousfield localisation is accessible.*

Proof. By Remark 1.3.3.16, we may assume without loss of generality that S is a subcategory of D, i.e. that i is a monomorphism. Let us first show that $i(A)\colon \mathrm{Loc}_{\mathsf{S}}(\mathsf{D})(A) \hookrightarrow \mathsf{D}(A)$ admits a left adjoint for every object $A \in \mathcal{B}$. Choose a small subcategory of generators $\mathcal{G} \hookrightarrow \mathcal{B}$. Then an object $d\colon A \to \mathsf{D}$ is contained in $\mathrm{Loc}_{\mathsf{S}}(\mathsf{D})(A)$ precisely if for every $g\colon G \to A$ with $G \in \mathcal{G}$ and every map $s\colon p \to q$ in $\mathsf{S}(G)$ the induced map

$$s^*\colon \mathrm{map}_{\mathsf{D}(G)}(q, g^*d) \to \mathrm{map}_{\mathsf{D}(G)}(p, g^*d)$$

is an equivalence in Ani (cf. Corollary 2.2.2.8). As g^* admits a left adjoint $g_!$, the object d is thus contained in $\mathrm{Loc}_{\mathsf{S}}(\mathsf{D})(A)$ if and only if d is local with respect to the set of maps

$$T_A = \bigcup_{G \to A} \{g_!(s) \mid s \in \mathsf{S}(G)^{\Delta^1}\}$$

in $\mathsf{D}(A)$. By construction, T_A is a small set, and since $\mathsf{D}(A)$ is by assumption a presentable ∞-category, we deduce from [49, Proposition 5.5.4.15] that $i(A)$ admits a left adjoint L_A and that $i(A)$ is accessible.

Next, we show that for every map $p\colon P \to A$ in $\mathcal{B}$ the map $L_B p^* \to p^* L_A$ is an equivalence. By Remark 3.1.2.10, we only need to show that $p^* L_G$ sends the adjunction unit of $L_A \dashv i(A)$ to an equivalence. Recall from [49, Section 5.5.4] that the set of maps in $\mathsf{D}(A)$ that is inverted by L_A coincides with the *strong saturation* of T_A, which is the smallest set of maps in $\mathsf{D}(A)$ containing T_A that is stable under pushouts, satisfies the two out of three property and is stable under small colimits in $\mathsf{D}(A)^{\Delta^1}$. Therefore the adjunction unit η is contained in the strong saturation of T_A, and since p^* commutes with colimits (being a morphism in $\mathrm{Pr}^{\mathrm{L}}_\infty$) we conclude that it will suffice to show that p^* sends maps in T_A to maps in the strong saturation of T_G. Let us therefore fix a map $g\colon G \to A$ with $G \in \mathcal{G}$ as well as a map $s \in \mathsf{S}(G)^{\Delta^1}$. Since D is $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete, we find $p^* g_!(s) \simeq h_! q^*(s)$, where h and q are defined via the pullback square

$$\begin{tikzcd} Q \arrow[r, "h"] \arrow[d, "q"] & P \arrow[d, "b"] \\ G \arrow[r, "g"] & A. \end{tikzcd}$$

Now $q^*(s)$ is a map in $\mathsf{S}(P)$ and therefore inverted by L_P, hence $h_! q^*(s)$ is inverted by L_B whenever $h_!$ sends maps in T_P to maps in the strong saturation of T_B, which is immediate by definition of T_P.

Finally, we may employ Proposition 3.1.2.9 to deduce that i admits a left adjoint L. Furthermore, as D is by assumption both $\mathrm{Grpd}_{\mathcal{B}}$- and LConst-cocomplete and therefore cocomplete (Corollary 3.5.4.2), and since every reflective subcategory of a cocomplete $\mathcal{B}$-category is cocomplete as well (Proposition 3.3.2.11), i being section-wise accessible already implies that i is accessible (see Corollary 5.3.2.5). □

Corollary 5.4.1.8. *Let* C *and* S *be (small)* $\mathcal{B}$*-categories and let* $j\colon \mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *be a functor. Then there is a sound doctrine* U *such that* $\mathrm{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ *is a* U*-accessible Bousfield localisation of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*. Conversely, any* U*-accessible Bousfield localisation of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *can be identified with* $\mathrm{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ *for some small* $\mathcal{B}$*-category* S *and some functor* $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*.*

Proof. By the straightening equivalence for right fibrations, for any $A \in \mathcal{B}$ there is a natural equivalence of ∞-categories $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A) \simeq \mathrm{RFib}(\mathsf{C} \times A)$, and since the right-hand side is a localisation of the presentable ∞-category $\mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}}$ at a small set of objects, we find that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is section-wise given by a presentable ∞-category. Moreover, if $s\colon B \to A$ is a map in $\mathcal{B}$, the functor $s^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(B)$ admits a right adjoint s_* by the theory of Kan extensions (Section 3.4.3) and therefore in particular commutes with small colimits. As $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is cocomplete, we can therefore apply Proposition 5.4.1.7 to deduce first claim. In light of Proposition 5.3.3.6, the second claim follows from Proposition 5.4.1.6. □

5.4.2. Presentability

In this section we define the concept of a presentable $\mathcal{B}$-category and discuss various characterisations of this notion.

Definition 5.4.2.1. A large $\mathcal{B}$-category is said to be *presentable* if there exist $\mathcal{B}$-categories C and S as well as a functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that D is equivalent to $\mathrm{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$.

Remark 5.4.2.2. In the situation of Definition 5.4.2.1, the fact that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is locally small implies that the 1-image S' of the functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (i.e. the subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is obtained by factoring the functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ into a strong epimorphism and a monomorphism) is small as well. In fact, by combining Proposition 1.3.3.9 and Proposition 2.3.1.3, it is clear that S' is locally small, hence Proposition 2.3.1.5 implies that S' is small whenever S'_0 is contained in $\mathcal{B}$, which follows in turn from the observation that S' is a subcategory of the essential image of $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, which is small by Lemma 2.3.1.6. As a consequence, Remark 5.4.1.4 shows that we may always assume that S is a subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Definition 5.4.2.3. We call a large $\mathcal{B}$-category C *section-wise accessible* if the associated sheaf takes values in the subcategory $\mathrm{Acc} \hookrightarrow \widehat{\mathrm{Cat}}_{\infty}$ of accessible ∞-categories. Analogously, we call C *section-wise presentable* if it factors through the inclusion $\mathrm{Pr}^{\mathrm{L}}_{\infty} \hookrightarrow \widehat{\mathrm{Cat}}_{\infty}$.

We now come to the main characterisation of presentable $\mathcal{B}$-categories. This will require the following lemma

Lemma 5.4.2.4. *Let D be a cocomplete $\mathcal{B}$-category and let κ be a $\mathcal{B}$-regular cardinal. Then an object $d\colon 1 \to \mathsf{D}$ is $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-compact if and only if for every $A \in \mathcal{B}^{\kappa\text{-cpt}}$ the object $\pi_A^* d \in \mathsf{D}(A)$ is κ-compact.*

Proof. By definition, d being $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-compact means that the functor

$$\mathrm{map}_{\mathsf{D}}(d, -)\colon \mathsf{D} \to \mathsf{Grpd}_{\mathcal{B}}$$

is $\mathsf{Filt}_{\mathrm{Cat}^{\kappa}_{\mathcal{B}}}$-cocontinuous. As this is a functor between cocomplete $\mathcal{B}$-categories, we deduce from Proposition 5.3.2.4 that this functor being $\mathsf{Filt}_{\mathrm{Cat}^{\kappa}_{\mathcal{B}}}$-cocontinuous precisely if it section-wise preserves κ-filtered colimits. Since Corollary 2.2.2.8 together with Remark 2.3.2.1 implies that we can recover $\mathrm{map}_{\mathsf{D}(A)}(\pi_A^*(d), -)$ as the composition

$$\mathsf{D}(A) \xrightarrow{\mathrm{map}_{\mathsf{D}}(d,-)(A)} \mathcal{B}_{/A} \xrightarrow{\Gamma_{\mathcal{B}_{/A}}} \mathrm{Ani},$$

we find that d being $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-compact implies that $\pi_A^*(d) \in \mathsf{D}(A)^{\kappa\text{-cpt}}$. Conversely, the fact that the inclusion $\mathcal{B} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathcal{B}^{\kappa\text{-cpt}})$ preserves κ-filtered colimits implies that the functor $\mathrm{map}_{\mathsf{D}}(d,-)(1)$ preserves κ-filtered colimits if and only if the functor $\mathrm{map}_{\mathcal{B}}(A, \mathrm{map}_{\mathsf{D}}(d,-)(1))$ does for every κ-compact A. In light of the commutative diagram

$$\begin{tikzcd} \mathsf{D}(1) \arrow[r, "\mathrm{map}_{\mathsf{D}}(d,-)(1)"] \arrow[d, "\pi_A^*"] & \mathcal{B} \arrow[d, "\pi_A^*"] \arrow[dr, "\mathrm{map}_{\mathcal{B}}(A,-)"] & \\ \mathsf{D}(A) \arrow[r, "\mathrm{map}_{\mathsf{D}}(d,-)(A)"] & \mathcal{B}_{/A} \arrow[r, "\Gamma_{\mathcal{B}_{/A}}"] & \mathrm{Ani}, \end{tikzcd}$$

we thus find that if $\pi_A^*(d) \in \mathsf{D}(A)^{\kappa\text{-cpt}}$ for every $A \in \mathcal{B}^{\kappa\text{-cpt}}$, then $\mathrm{map}_{\mathsf{D}}(d,-)(1)$ preserves κ-filtered colimits. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (for $A \in \mathcal{B}^{\kappa\text{-cpt}}$), the same argument moreover shows that $\mathrm{map}_{\mathsf{D}}(d,-)(A)$ preserves κ-filtered colimits. Now if $A \in \mathcal{B}$ is arbitrary, we may find a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ with each A_i κ-compact. Since the induced map $\mathcal{B}_{/A} \to \prod_i \mathcal{B}_{/A_i}$ is conservative and $\mathsf{D}(A) \to \prod_i \mathsf{D}(A_i)$ is cocontinuous, we conclude that $\mathrm{map}_{\mathsf{D}}(d,-)(A)$ preserves κ-filtered colimits even if A is not κ-compact. Together with Proposition 5.3.2.4, this yields the claim. ☐

Theorem 5.4.2.5. *For a large $\mathcal{B}$-category D, the following are equivalent:*

1. D *is presentable;*
2. D *arises as an accessible Bousfield localisation $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$ for some small $\mathcal{B}$-category C;*
3. D *is accessible and cocomplete;*
4. D *is cocomplete, and there is a $\mathcal{B}$-regular cardinal κ such that D is $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-accessible;*

5. D *is cocomplete and section-wise accessible;*

6. D *is* $\mathrm{Grpd}_{\mathcal{B}}$*-cocomplete and section-wise presentable.*

Proof. The fact that (1) and (2) are equivalent is an immediate consequence of Corollary 5.4.1.8. Now if we assume that (2) is satisfied, we may find a $\mathcal{B}$-regular cardinal κ such that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is $\mathsf{Filt}_{\mathrm{Cat}^{\kappa}_{\mathcal{B}}}$-cocontinuous (see Remark 5.3.2.2), which by Corollary 5.3.3.3 implies that D is $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-accessible. As any reflective subcategory of a cocomplete $\mathcal{B}$-category is cocomplete as well (Proposition 3.3.2.11), we conclude that (4) is satisfied. Trivially, (4) implies (3). Lastly, if $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ for some sound doctrine U and some $\mathcal{B}$-category C and if D is furthermore cocomplete, we deduce from Corollary 3.5.1.13 that the inclusion $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ admits a left adjoint, hence (3) implies (2).

To show that (2) implies (5), since Proposition 3.3.2.11 already shows that D is cocomplete, it remains to see that D is section-wise accessible. For every $A \in \mathcal{B}$, the ∞-category $\mathsf{D}(A)$ is a Bousfield localisation of the presentable ∞-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A) \simeq \mathrm{RFib}(\mathsf{C} \times A)$. Since Corollary 5.3.2.5 implies that this Bousfield localisation is *accessible*, one concludes that $\mathsf{D}(A)$ is an accessible ∞-category. Furthermore, since D is also complete, the functor $s^* \colon \mathsf{D}(A) \to \mathsf{D}(B)$ preserves colimits for any map $s \colon B \to A$ in $\mathcal{B}$, so it is in particular accessible. Thus D is section-wise accessible. The fact that (5) implies (6) is an immediate consequence of Corollary 3.5.4.2 and Proposition 3.3.2.7.

To complete the proof, we show that (6) implies (3). Since D is already assumed to be $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete and since the assumption that D takes values in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ in particular implies that D is LConst-cocomplete (see Proposition 3.3.2.7), we deduce from Corollary 3.5.4.2 that D is cocomplete. Therefore, we only need to show that D is U-accessible for some doctrine U. To that end, let us choose a small full subcategory $\mathcal{G} \hookrightarrow \mathcal{B}$ of generators (as in Remark 1.2.1.3). We can then find a regular cardinal κ such $\mathsf{D}(G)$ is κ-accessible for each $G \in \mathcal{G}$ and $s^* \colon \mathsf{D}(H) \to \mathsf{D}(G)$ preserves κ-compact objects for each $s \colon H \to G$ in $\mathcal{G}$. By enlarging κ if necessary, we can assume that κ is $\mathcal{B}$-regular (in the sense of Definition 5.2.2.3, cf. also Remark 5.2.2.5) and that every $G \in \mathcal{G}$ is κ-compact. Let $\mathsf{C} \hookrightarrow \mathsf{D}$ be the full subcategory spanned by the $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-compact objects in arbitrary context. By combining Remark 5.2.2.6 with Lemma 5.4.2.4, an object $d \colon G \to \mathsf{D}$ in context

$G \in \mathcal{G}$ is contained in C if and only if $d \in \mathsf{D}(G)^{\kappa\text{-cpt}}$. In particular, C is small. Since therefore every object in $\mathsf{D}(G)$ is obtained as a κ-filtered colimit of objects in $\mathsf{C}(G)$ and as the constant $\mathcal{B}_{/G}$-category associated with a κ-filtered ∞-category is $\mathrm{Cat}^{\kappa}_{\mathcal{B}_{/G}}$-filtered (see Lemma 5.2.2.19), we deduce from Proposition 5.3.3.2 that D is $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-accessible. □

We end this section by recording a few consequences of Theorem 5.4.2.5. We begin by noting that as Theorem 5.4.2.5 implies that every presentable $\mathcal{B}$-category is a reflective subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ for some $\mathcal{B}$-category C, we deduce from Proposition 3.3.2.11:

Corollary 5.4.2.6. *Every presentable $\mathcal{B}$-category is complete and cocomplete.* □

Corollary 5.4.2.7. *Let D be a presentable $\mathcal{B}$-category and let K be a $\mathcal{B}$-category. Then $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{D})$ is presentable.*

Proof. By Theorem 5.4.2.5, we may choose a $\mathcal{B}$-category C and a sound doctrine U such that D is a U-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. In light of Proposition 3.3.2.12, this implies that the large $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{D})$ is a U-accessible Bousfield localisation of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{K}^{\mathrm{op}} \times \mathsf{C})$, hence the result follows. □

Corollary 5.4.2.8. *Let D be a presentable $\mathcal{B}$-category and let $d\colon A \to \mathsf{D}$ be an arbitrary object. Then $\mathsf{D}_{/d}$ is a presentable $\mathcal{B}_{/A}$-category.*

Proof. We may assume that $A \simeq 1$ (cf. Remark 5.4.2.10 below). Using Proposition 3.3.2.13, one finds that $\mathsf{D}_{/d}$ is cocomplete. By Theorem 5.4.2.5, it therefore suffices to show that $\mathsf{D}_{/d}$ is accessible, which is a consequence of Proposition 5.3.1.6. □

Corollary 5.4.2.9. *Let D be a presentable $\mathcal{B}$-category and let $\mathsf{S} \to \mathsf{D}$ be a functor where S is small. Then there is a sound doctrine U such that $\mathrm{Loc}_{\mathsf{S}}(\mathsf{D})$ is a U-accessible Bousfield localisation of D. In particular, $\mathrm{Loc}_{\mathsf{S}}(\mathsf{D})$ is presentable.*

Proof. Since D is cocomplete by Corollary 5.4.2.6 and section-wise presentable by Theorem 5.4.2.5, the claim follows from Proposition 5.4.1.7. □

Remark 5.4.2.10. As yet another consequence of Theorem 5.4.2.5, the condition of a large $\mathcal{B}$-category to be presentable is a local condition: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then a $\mathcal{B}$-category D is presentable if and only if each $\pi_{A_i}^* \mathsf{D}$ is a presentable $\mathcal{B}_{/A_i}$-category. This follows from condition (3) in Theorem 5.4.2.5, together with cocompleteness being a local condition (cf. Remark 3.3.2.3) and Remark 5.3.3.5.

5.4.3. The adjoint functor theorem

Recall from Proposition 3.3.2.10 that any left adjoint functor between cocomplete large $\mathcal{B}$-categories is cocontinuous. Therefore, if D and E are cocomplete large $\mathcal{B}$-categories, there is a canonical inclusion

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{L}}(\mathsf{D}, \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D}, \mathsf{E}).$$

If D is presentable and E is locally small, then this inclusion is in fact an equivalence:

Proposition 5.4.3.1 (Adjoint functor theorem I)**.** *Let* D *and* E *be large* $\mathcal{B}$*-categories such that* D *is presentable and* E *is cocomplete and locally small. Then every cocontinuous functor* $f\colon \mathsf{D} \to \mathsf{E}$ *admits a right adjoint. In particular, there is an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{L}}(\mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D}, \mathsf{E})$$

of (large) $\mathcal{B}$*-categories.*

Proof. In light of Remark 5.4.2.10, it is clear that the second statement immediately follows from the first. Now choose $\mathcal{B}$-categories C and S as well as a functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that $\mathsf{D} \simeq \mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. If $f\colon \mathsf{D} \to \mathsf{E}$ is a cocontinuous functor, then $fL\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{E}$ is cocontinuous as well and therefore a left adjoint by Remark 3.5.1.4. To show that f admits a right adjoint, we therefore only need to verify that the right adjoint r of fL factors through D. Since $\mathsf{D} \simeq \mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ as full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ by Theorem 5.4.2.5, this is in turn equivalent to $h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})} r$ factoring through the functor

$$L^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})),$$

which is clear on account of r being right adjoint to fL. $\square$

Recall from Corollary 5.4.2.9 that if D is a presentable $\mathcal{B}$-category and $\mathsf{S} \to \mathsf{D}$ is a functor where S is small, the $\mathcal{B}$-category $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ is an accessible Bousfield localisation of D and therefore in particular presentable. We may now use Proposition 5.4.3.1 to derive a universal property of $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ among presentable $\mathcal{B}$-categories. To that end, recall from Section 1.3.3 that if E is another presentable $\mathcal{B}$-category, we denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$ that is spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$ for which the restriction of the associated functor $\pi_A^* \mathsf{D} \to \pi_A^* \mathsf{E}$ along $\pi_A^* \mathsf{S} \to \pi_A^* \mathsf{D}$ takes values in the subcategory $\pi_A^* \mathsf{E}^{\simeq}$. We will denote by $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$ its intersection with the full subcategory $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$. We now obtain:

Corollary 5.4.3.2. *Let* $\mathsf{S} \to \mathsf{D}$ *be a functor of* $\mathcal{B}$*-categories where* S *is small and* D *is presentable, and let* E *be another presentable* $\mathcal{B}$*-category. Then precomposition with the left adjoint* $L \colon \mathsf{D} \to \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}.$$

Proof. To begin with, note that as L is in particular a localisation functor (cf. Proposition 3.1.4.6), the universal property of localisations (Proposition 1.3.3.20) implies that

$$L^* \colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}) \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

is fully faithful. Therefore, it suffices to identify the essential image of L^* with $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$. Since the restriction of L along $\mathsf{S} \to \mathsf{D}$ takes values in $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})^{\simeq}$, it is clear that L^* takes values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$, so that it suffices to show that every object $A \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{S})_{\mathsf{S}}$ is contained in the essential image of L^*. By combining Remark 5.4.1.2, Remark 3.3.3.4 and Remark 1.3.3.18, it will be enough to verify that any cocontinuous functor $f \colon \mathsf{D} \to \mathsf{E}$ whose restriction along $\mathsf{S} \to \mathsf{D}$ takes values in $\mathsf{E}^{\simeq}$ factors through L. Note that the assumption on f precisely means that f factors through the localisation $l \colon \mathsf{D} \to \mathsf{S}^{-1}\mathsf{D}$, so that $f^* \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ factors through $l^* \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{D}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$. Since $f^* \simeq g_!$ where g is the right adjoint of f that is provided by Proposition 5.4.3.1, the very definition of $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ implies that g factors through the inclusion $i \colon \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ via a functor $g' \colon \mathsf{E} \to \mathsf{Loc}(\mathsf{D})$. Since the composite fi defines a left adjoint of g', the claim follows by passing to left adjoints. $\square$

There is also a dual version to Proposition 5.4.3.1 that classifies right adjoint functors between presentable $\mathcal{B}$-categories.

Proposition 5.4.3.3 (Adjoint functor theorem II)**.** *Let $f\colon \mathsf{D} \to \mathsf{E}$ be a functor between presentable $\mathcal{B}$-categories. Then the following are equivalent:*

1. *f admits a left adjoint;*
2. *f is continuous and accessible;*
3. *f is continuous and section-wise accessible.*

Proof. By Corollary 5.3.2.5, (2) and (3) are equivalent. Moreover, since Theorem 5.4.2.5 implies that f is section-wise given by a functor between presentable ∞-categories, the adjoint functor theorem for presentable ∞-categories [49, Corollary 5.5.2.9] shows that (1) implies (3). For the converse, note that the same result implies that $f(A)$ admits a left adjoint l_A for every $A \in \mathcal{B}$. By Proposition 3.1.2.9, it now suffices to see that the natural map $l_B s^* \to s^* l_A$ is an equivalence for every map $s\colon B \to A$ in $\mathcal{B}$. This is equivalent to seeing that the transpose map $f(A)s_* \to s_* f(B)$ that is given by passing to right adjoints is an equivalence. But this is just another way of saying that f is $\mathsf{Grpd}_{\mathcal{B}}$-continuous. □

5.4.4. The large $\mathcal{B}$-category of presentable $\mathcal{B}$-categories

Recall from Section 3.3.3 that we defined the (very large) $\mathcal{B}$-category $\mathsf{Cat}^{\mathrm{cc}}_{\widehat{\mathcal{B}}}$ of cocomplete large $\mathcal{B}$-categories as the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ which is determined by the subobject of $(\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ that is spanned by the cocontinuous functors between cocomplete large $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. By Remark 3.3.3.2 a functor of large $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Cat}^{\mathrm{cc}}_{\widehat{\mathcal{B}}}$ precisely if it is a cocontinuous functor between cocomplete large $\mathcal{B}$-categories. We may now define:

Definition 5.4.4.1. The large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories is defined as the full subcategory of $\mathsf{Cat}^{\mathrm{cc}}_{\widehat{\mathcal{B}}}$ that is spanned by the presentable $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. We denote by $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ the ∞-category of global sections of $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$.

Remark 5.4.4.2. As presentability is a local condition (Remark 5.4.2.10) and by Remark 3.3.3.2, a large $\mathcal{B}_{/A}$-category defines an object in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ if and only if it is presentable, and a functor between such large $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ if and only if it is cocontinuous. Consequently, the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ identifies $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ with the sheaf $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/-})$ on $\mathcal{B}$. In particular, one obtains a canonical equivalence $\pi_A^* \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$.

Remark 5.4.4.3. A priori, $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is a very large $\mathcal{B}$-category. However, note that the set of equivalence classes of presentable $\mathcal{B}$-categories is $\mathbf{V}$-small as it admits a surjection from the $\mathbf{V}$-small union

$$\bigsqcup_{\mathsf{C} \in \mathrm{Cat}(\mathcal{B})} \mathrm{Sub}_{\mathrm{small}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$$

where $\mathrm{Sub}_{\mathrm{small}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ denotes the $\mathbf{V}$-small poset of *small* subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ is furthermore locally $\mathbf{V}$-small, this shows that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is in fact only a large $\mathcal{B}$-category.

Recall from Section 4.5.2 that we denote by $\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}$ the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is determined by the subobject $L \hookrightarrow (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ of left adjoint functors. By Proposition 5.4.3.1, the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ factors through the inclusion $\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$. Suppose now that D and E are presentable $\mathcal{B}$-categories. By combining Corollary 3.2.6.5 with the fact that $L \hookrightarrow (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ is closed under equivalences and composition in the sense of Proposition 1.3.1.17 and by furthermore making use of Remark 3.3.3.4, we find that the inclusion

$$\mathrm{map}_{\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{D}, \mathsf{E}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}}(\mathsf{D}, \mathsf{E})$$

is obtained by applying the core $\mathcal{B}$-groupoid functor to the equivalence

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

from Proposition 5.4.3.1. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and using Remark 5.4.4.2, the same assertion holds for objects in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ in context $A \in \mathcal{B}$, so that we conclude:

Proposition 5.4.4.4. *The inclusion* $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}$ *is fully faithful.* □

Dually, let us denote by $\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}}$ the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is determined by the subobject $R \hookrightarrow (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ of right adjoint functors.

Definition 5.4.4.5. The $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories is defined as the full subcategory of $\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}}$ that is spanned by the presentable $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. We denote by $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ the underlying ∞-category of global sections.

Remark 5.4.4.6. As in Remark 5.4.4.2, a large $\mathcal{B}_{/A}$-category defines an object in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ if and only if it is presentable, and a functor between such large $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ if and only if it is a right adjoint. As a consequence, the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ corresponds to the sheaf $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B}_{/-})$ on $\mathcal{B}$ that is spanned by the presentable $\mathcal{B}_{/A}$-categories and right adjoint functors. In particular, one obtains a canonical equivalence $\pi_A^* \mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}} \simeq \mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$.

Proposition 5.4.4.7. *There is a canonical equivalence* $(\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}})^{\mathrm{op}} \simeq \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *that carries a right adjoint functor between presentable $\mathcal{B}$-categories to its left adjoint.*

Proof. By Proposition 4.5.2.1, there is such an equivalence $(\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}})^{\mathrm{op}} \simeq \mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}$, and since this functor necessarily acts as the identity on the underlying core $\mathcal{B}$-groupoids, it restricts to the desired equivalence by virtue of Proposition 5.4.4.4. □

Example 5.4.4.8. We are now in the position to provide a large class of examples of presentable $\mathcal{B}$-categories: recall from Construction 1.4.2.1 that there is a functor

$$- \otimes \mathrm{Grpd}_{\mathcal{B}} \colon \ \mathrm{Pr}^{\mathrm{R}}_{\infty} \to \mathrm{Cat}(\widehat{\mathcal{B}})$$

that sends a presentable ∞-category $\mathcal{E}$ to $\mathcal{E} \otimes \mathrm{Grpd}_{\mathcal{B}} = \mathcal{E} \otimes \mathcal{B}_{/-}$ (where $- \otimes -$ is Lurie's tensor product of presentable ∞-categories). By Example 3.5.4.6, the $\mathcal{B}$-category $\mathcal{E} \otimes \mathrm{Grpd}_{\mathcal{B}}$ is cocomplete, so that Theorem 5.4.2.5 implies that it is presentable as it takes values in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. Moreover, we deduce from Example 3.1.2.12 that whenever $g \colon \ \mathcal{E} \to \mathcal{E}'$ is a map in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$, the induced functor $g \otimes \mathrm{Grpd}_{\mathcal{B}}$ is a right adjoint. Consequently, we conclude that the functor $- \otimes \mathrm{Grpd}_{\mathcal{B}}$ takes values in $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$. In particular, by applying this observation to $\mathcal{E} = \mathrm{Cat}_{\infty}$, we find that $\mathsf{Cat}_{\mathcal{B}}$ is presentable.

Our next goal is to show that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is complete and cocomplete. For completeness, we first need a lemma:

Lemma 5.4.4.9. *The $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathsf{Grpd}_{\mathcal{B}}\text{-cc}}$ of $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete $\mathcal{B}$-categories is LConst-complete, and the inclusion*

$$\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathsf{Grpd}_{\mathcal{B}}\text{-cc}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$$

is LConst-continuous.

Proof. By Remark 3.3.3.2, it is enough to show that for any small ∞-category $\mathcal{K}$ and any functor $d\colon \mathcal{K} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathsf{Grpd}_{\mathcal{B}}\text{-cc}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$, the following two conditions are satisfied:

1. $\lim d$ is $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete;
2. for every $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete large $\mathcal{B}$-category C, a functor $f\colon \mathsf{C} \to \lim d$ is $\mathsf{Grpd}_{\mathcal{B}}$-cocontinuous precisely if the maps $\mathsf{C} \to \lim d \to d(k)$ are $\mathsf{Grpd}_{\mathcal{B}}$-cocontinuous for all $k \in \mathcal{K}$.

Recall from [50, Corollary 4.7.4.18] that the subcategory

$$\mathrm{Fun}^{\mathrm{LAdj}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \hookrightarrow \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$$

that is spanned by the right adjoint functors and the left adjointable squares (i.e. those commutative squares of ∞-categories whose associated mate transformation is an equivalence) admits small limits and that the inclusion preserves small limits. Let us fix a map $p\colon P \to A$ in $\mathcal{B}$. Now evaluation at p defines a functor $\mathrm{Cat}(\widehat{\mathcal{B}}) \to \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ that restricts to a map

$$\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathsf{Grpd}_{\mathcal{B}}\text{-cc}} \to \mathrm{Fun}^{\mathrm{LAdj}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty).$$

Since limits in $\mathrm{Cat}(\widehat{\mathcal{B}})$ are computed section-wise, this already shows that

$$p^*\colon \lim d(A) \to \lim d(P)$$

admits a left adjoint. Similarly, if $s\colon B \to A$ is a map in $\mathcal{B}$ and if $q\colon Q \to B$ denotes the pullback of p along s, evaluating large $\mathcal{B}$-categories at this pullback square yields a morphism $\Delta^1 \times \mathrm{Cat}(\widehat{\mathcal{B}})^{\mathsf{Grpd}_{\mathcal{B}}\text{-cc}} \to \mathrm{Fun}^{\mathrm{LAdj}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$. Consequently, applying $\lim d$ to the very same pullback square must yield a left-adjointable square of ∞-categories, which implies that condition (1) is satisfied. By the same argument, if C is $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete and if $f\colon \mathsf{C} \to \lim d$ is a functor, evaluating f at p yields a commutative square of ∞-categories that is left-adjointable if and only if the evaluation of the composition $\mathsf{C} \to \lim d \to d(k)$ at p is left-adjointable for all $k \in \mathcal{K}$. Hence (2) follows. □

Proposition 5.4.4.10. *The large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is complete, and the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is continuous.*

Proof. By the dual of Corollary 3.5.4.2, it suffices to show that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is both $\mathrm{Grpd}_{\mathcal{B}}$- and LConst-complete and that the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is both $\mathrm{Grpd}_{\mathcal{B}}$- and LConst-continuous. Using Remark 5.4.4.2, this follows once we show that whenever K is either given by the constant $\mathcal{B}$-category Λ^2_0 or by a $\mathcal{B}$-groupoid, the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ admits K-indexed limits and the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves K-indexed limits.

Let us first assume that $\mathsf{K} = \Lambda^2_0$, i.e. suppose that

$$\begin{tikzcd} \mathsf{Q} \arrow[r, "g"] \arrow[d, "q"] & \mathsf{P} \arrow[d, "p"] \\ \mathsf{D} \arrow[r, "f"] & \mathsf{C} \end{tikzcd}$$

is a pullback diagram in $\mathrm{Cat}(\widehat{\mathcal{B}})$ in which f and p are cocontinuous functors between presentable $\mathcal{B}$-categories. By Theorem 5.4.2.5, the cospan determined by f and p takes values in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. Therefore, [49, Proposition 5.5.3.13] implies that Q takes values in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ and that g and q are section-wise cocontinuous. Moreover, Lemma 5.4.4.9 shows that Q is $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete and that g and q are $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete. By again making use of Theorem 5.4.2.5, we thus conclude that the Q is presentable and that g and q are cocontinuous. Now if Z is another presentable $\mathcal{B}$-category, a similar argumentation shows that a functor $\mathsf{Z} \to \mathsf{Q}$ is cocontinuous if and only if its composition with both g and q are cocontinuous. In total, this shows that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ admits pullbacks and that the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves pullbacks.

Let us now assume $\mathsf{K} = \mathsf{G}$ for some $\mathcal{B}$-groupoid G. In order to show that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ has G-indexed limits and that the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves G-indexed limits, another application of Remark 5.4.4.2 allows us to reduce to showing that the adjunction

$$(\pi_G)_* \dashv \pi_G^*)) \colon \; \mathrm{Cat}(\widehat{\mathcal{B}}_{/G}) \rightleftarrows \mathrm{Cat}(\widehat{\mathcal{B}})$$

restricts to an adjunction between $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/G})$ and $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. Recall that on the level of $\widehat{\mathrm{Cat}}_\infty$-valued sheaves, the functor $(\pi_G)_*$ is given by precomposition with π_G^*. By combining the characterisation of presentable $\mathcal{B}$-categories as $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete $\mathrm{Pr}^{\mathrm{L}}_\infty$-valued sheaves (Theorem 5.4.2.5) with the explicit description of $\mathrm{Grpd}_{\mathcal{B}}$-cocompleteness from Proposition 3.3.2.5 and the section-wise characterisation of left adjoint functors (Proposition 3.1.2.9), it is therefore immediate that $(\pi_G)_*$ restricts to a functor $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/G}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. Moreover, the adjunction unit $\mathrm{id}_{\mathrm{Cat}(\widehat{\mathcal{B}})} \to (\pi_G)_*\pi_G^*$ is given by precomposition with the adjunction counit $(\pi_G)_!\pi_G^* \to \mathrm{id}_{\mathcal{B}}$, and the adjunction counit $\pi_G^*(\pi_G)_* \to \mathrm{id}_{\mathrm{Cat}(\widehat{\mathcal{B}}_{/A})}$ is given by precomposition with the adjunction unit $\mathrm{id}_{\mathcal{B}_{/A}} \to \pi_G^*(\pi_G)_!$. Thus, by the section-wise characterisation of left adjoint functors and the fact that presentable $\mathcal{B}$-categories are $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete, these two maps must also restrict in the desired way, hence the result follows. □

Proposition 5.4.4.11. *The large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ is complete, and the inclusion $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is continuous.*

Proof. As in the proof of Proposition 5.4.4.10, it suffices to show that for either $\mathsf{K} = \Lambda^2_0$ or $\mathsf{K} = \mathsf{G}$ for G a $\mathcal{B}$-groupoid, the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ admits K-indexed limits and the inclusion $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves K-indexed limits. The first case follows as in the proof of Proposition 5.4.4.10, by making use of the dual version of Lemma 5.4.4.9, [49, Theorem 5.5.3.18] and the fact that a continuous and section-wise accessible functor between presentable $\mathcal{B}$-categories admits a left adjoint (Proposition 5.4.3.3). The argument for the second case is carried out in a completely analogous way as the one in the proof of Proposition 5.4.4.10, the only difference being that one must use the $\mathrm{Grpd}_{\mathcal{B}}$-completeness of presentable $\mathcal{B}$-categories and not their $\mathrm{Grpd}_{\mathcal{B}}$-cocompleteness. □

Remark 5.4.4.12. As a consequence of Proposition 5.4.4.11, we can furthermore deduce that $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ is generated under pullbacks by presheaf $\mathcal{B}$-categories. In fact, if D is a presentable $\mathcal{B}$-category, we may find small $\mathcal{B}$-categories C and S and a functor $j \colon \mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ so that $\mathsf{D} \simeq \mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. By definition of the right-hand side, we therefore obtain a pullback square

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}^{\mathrm{gpd}}) \arrow[d, "\gamma^*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "j^* h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}) \end{tikzcd}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$, where $\gamma\colon \mathsf{S} \to \mathsf{S}^{\mathrm{gpd}}$ is the natural map. By Remark 3.5.1.4, the functor $j^* h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ is a right adjoint: its left adjoint is the left Kan extension $(h_{\mathsf{S}})_!(j)$ of j along the Yoneda embedding h_{S}. Since γ^* is a right adjoint as well, Proposition 5.4.4.11 implies that this diagram is a pullback square in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$.

Finally, by combining Proposition 5.4.4.10 and Proposition 5.4.4.11 with Proposition 5.4.4.7, we conclude:

Corollary 5.4.4.13. *Both* $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *and* $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ *are complete and cocomplete.* □

5.4.5. U-sheaves

The main goal in this section is to derived yet another characterisation of presentable $\mathcal{B}$-categories: that of $\mathcal{B}$-categories of U-*sheaves* on an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category. These are defined as follows:

Definition 5.4.5.1. Let U be an internal class and suppose that C is an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category. For any (not necessarily small) U-complete $\mathcal{B}$-category E, we denote by $\mathsf{Sh}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$ that is spanned by those presheaves $F\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$ (in arbitrary context $A \in \mathcal{B}$) that are $\pi_A^*\mathsf{U}$-continuous when viewed as functors $\pi_A^*\mathsf{C}^{\mathrm{op}} \to \pi_A^*\mathsf{E}$. We refer to such presheaves as U-*sheaves*. For the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$, we will simply call them *sheaves*, and we will write $\mathsf{Sh}_{\mathsf{E}}(\mathsf{C}) = \mathsf{Sh}^{\mathsf{Cat}_{\mathcal{B}}}_{\mathsf{E}}(\mathsf{C})$ for the associated $\mathcal{B}$-category

Remark 5.4.5.2. By Remark 3.3.3.4, if $A \in \mathcal{B}$ is an arbitrary object, we obtain a canonical equivalence $\pi_A^* \mathsf{Sh}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C}) \simeq \mathsf{Sh}^{\pi_A^*\mathsf{U}}_{\pi_A^*\mathsf{E}}(\pi_A^*\mathsf{C})$.

We first focus on $\mathsf{Grpd}_{\mathcal{B}}$-valued U-sheaves.

Proposition 5.4.5.3. *Let* D *be a presentable* $\mathcal{B}$*-category and let* $F\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ *be a presheaf on* D*. Then F is representable if and only if F is continuous. In particular, the Yoneda embedding induces an equivalence* $\mathsf{D} \simeq \mathsf{Sh}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{D})$.

Proof. By Remark 5.4.5.2, the first claim implies the second, and by Proposition 3.3.2.15, every representable functor is continuous, so that it suffices to prove that every continuous presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ is representable. Now F being continuous is equivalent to $F^{\mathrm{op}}\colon \mathsf{D} \to \mathrm{Grpd}_{\mathcal{B}}^{\mathrm{op}}$ being cocontinuous, which by Proposition 5.4.3.1 is in turn equivalent to it being a left adjoint. Hence F is continuous if and only if F is a right adjoint. Let $l\colon \mathrm{Grpd}_{\mathcal{B}} \to \mathsf{D}^{\mathrm{op}}$ be the left adjoint of F. Since the final $\mathcal{B}$-groupoid $1_{\mathrm{Grpd}_{\mathcal{B}}}\colon 1 \to \mathrm{Grpd}_{\mathcal{B}}$ corepresents the identity on $\mathrm{Grpd}_{\mathcal{B}}$, we find equivalences

$$F \simeq \mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(1_{\mathrm{Grpd}_{\mathcal{B}}}, F(-)) \simeq \mathrm{map}_{\mathsf{D}^{\mathrm{op}}}(l(1_{\mathrm{Grpd}_{\mathcal{B}}}), -) \simeq \mathrm{map}_{\mathsf{D}}(-, l(1_{\mathrm{Grpd}_{\mathcal{B}}})),$$

hence F is represented by $l(1_{\mathrm{Grpd}_{\mathcal{B}}})$. □

Next, we use Proposition 5.4.5.3 to deduce that whenever U is a *doctrine*, the $\mathcal{B}$-category of $\mathrm{Grpd}_{\mathcal{B}}$-valued U-sheaves on a small $\mathcal{B}$-category is presentable, and that it satisfies a universal property:

Proposition 5.4.5.4. *For any doctrine* U*, the large* $\mathcal{B}$*-category* $\mathrm{Sh}^{\mathsf{U}}_{\mathrm{Grpd}_{\mathcal{B}}}(\mathsf{C})$ *is presentable. Moreover, for any complete large* $\mathcal{B}$*-category* E*, restriction along the Yoneda embedding* h_{C} *induces an equivalence*

$$h_{\mathsf{C}}^*\colon\ \mathrm{Sh}_{\mathsf{E}}(\mathrm{Sh}^{\mathsf{U}}_{\mathrm{Grpd}_{\mathcal{B}}}(\mathsf{C})) \simeq \mathrm{Sh}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$$

of large $\mathcal{B}$*-categories.*

Proof. Fix a small full subcategory $\mathcal{G} \hookrightarrow \mathcal{B}$ of generators, and define the small set

$$R = \bigsqcup_{G\in\mathcal{G}} \{f\colon\ \operatorname{colim} h_{\mathsf{C}} d \to h_{\mathsf{C}} \operatorname{colim} d \mid d\colon \mathsf{K} \to \pi_G^*\mathsf{C},\ \mathsf{K}^{\mathrm{op}} \in \mathsf{U}(G)\}$$

(where each f is to be considered as a map in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context $G \in \mathcal{G}$). We let $\mathsf{S}_R \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the subcategory that is spanned by R. Note that since R is a small set, the subcategory S_R is small, so that $\mathsf{D} = \mathrm{Loc}_{\mathsf{S}_R}(\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}))$ is a presentable $\mathcal{B}$-category. Moreover, if E is an arbitrary complete large $\mathcal{B}$-category, the construction of S_R (together with the fact that the preservation of limits can be checked locally, see Remark 3.2.2.3) makes it evident that a cocontinuous functor $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{E}^{\mathrm{op}}$ carries the maps in S_R to equivalences precisely if its restriction to C is $\mathrm{op}(\mathsf{U})$-cocontinuous. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, the same assertion holds for any object $A \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}^{\mathrm{op}})$. As a consequence, the universal property of presheaf $\mathcal{B}$-categories implies that restriction along the Yoneda embedding h_{C} determines an equivalence of large $\mathcal{B}$-categories $h^*_{\mathsf{C}} \colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}^{\mathrm{op}})_{S_R} \simeq \underline{\mathsf{Fun}}^{\mathrm{op}(\mathsf{U})\text{-cc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}^{\mathrm{op}})$. Upon taking opposite $\mathcal{B}$-categories and using Corollary 5.4.3.2, one thus obtains an equivalence $(Lh_{\mathsf{C}})^* \colon \mathsf{Sh}_{\mathsf{E}}(\mathsf{D}) \simeq \mathsf{Sh}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$. By plugging in $\mathsf{E} = \mathsf{Grpd}_{\mathcal{B}}$ into this equivalence and using Proposition 5.4.5.3, one ends up with an equivalence $\mathsf{D} \simeq \mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})$ of full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, which completes the proof. ☐

Whenever U is a *sound* doctrine, we can identify the $\mathcal{B}$-category of U-sheaves on an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category C with the free $\mathsf{Filt}_{\mathsf{U}}$-cocompletion of C:

Proposition 5.4.5.5. *Let* U *be a sound internal class and let* C *be an* $\mathrm{op}(\mathsf{U})$-*cocomplete* $\mathcal{B}$-*category. Then there is an equivalence* $\mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *of full subcategories of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Proof. By Proposition 5.3.4.6 as well as Remark 5.4.5.2 and Remark 5.3.4.2, it suffices to show that a presheaf $F \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is U-flat if and only if F is U-continuous. As the inclusion $h_{\mathsf{C}^{\mathrm{op}}} \colon \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$ commutes with all limits that exist in C, the presheaf F being U-flat immediately implies that F is U-continuous. Conversely, suppose that F is U-continuous. By Proposition 5.3.4.6, it suffices to show that $\mathsf{C}_{/F}$ is weakly U-filtered. By applying Lemma 3.5.1.11 to the pullback square

$$\begin{tikzcd} \mathsf{C}_{/F} \arrow[r] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}}_{/1_{\mathsf{Grpd}_{\mathcal{B}}}} \arrow[d] \\ \mathsf{C} \arrow[r, "F^{\mathrm{op}}"] & \mathsf{Grpd}^{\mathrm{op}}_{\mathcal{B}}, \end{tikzcd}$$

(which satisfies the conditions of the lemma by Proposition 3.2.4.3), we conclude that $\mathsf{C}_{/F}$ is $\mathrm{op}(\mathsf{U})$-cocomplete, hence the claim follows from Example 5.1.2.4. ☐

Corollary 5.4.5.6. *Let* U *be a sound doctrine and let* C *be an* $\mathrm{op}(\mathsf{U})$-*cocomplete* $\mathcal{B}$-*category. Then* $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *is presentable. Moreover, for any cocomplete large* $\mathcal{B}$-*category* E, *restriction along the Yoneda embedding* h_{C} *induces an equivalence*

$$h^*_{\mathsf{C}} \colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{op}(\mathsf{U})\text{-cc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$$

of large $\mathcal{B}$-categories. □

Corollary 5.4.5.7. *Let* D *be a large* $\mathcal{B}$*-category. Then the following are equivalent:*

1. D *is presentable;*
2. *there is a sound doctrine* U *such that* D *is* U*-accessible and* $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ *is* $\mathrm{op}(\mathsf{U})$*-cocomplete;*
3. *there is a doctrine* U *and a small* $\mathrm{op}(\mathsf{U})$*-cocomplete* $\mathcal{B}$*-category* C *such that one has an equivalence* $\mathsf{D} \simeq \mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})$.

Proof. By combining Theorem 5.4.2.5 with Proposition 5.1.5.4, it is clear that (1) implies (2). If (2) is satisfied, Proposition 5.3.3.2 implies that $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small and that there is an equivalence $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{D}^{\mathsf{U}\text{-cpt}})$. In light of Proposition 5.4.5.5, this shows that (3) is satisfied. Finally, Proposition 5.4.5.4 shows that (3) implies (1). □

We complete this section by noting that as a consequence of the results that we have established so far, we may deduce that the $\mathcal{B}$-category of sheaves between presentable $\mathcal{B}$-categories is presentable as well:

Corollary 5.4.5.8. *For every two presentable* $\mathcal{B}$*-categories* D *and* E*, the* $\mathcal{B}$*-category* $\mathsf{Sh}_{\mathsf{E}}(\mathsf{D})$ *is presentable as well.*

Proof. By Corollary 5.4.5.7, there is a doctrine U and a small $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category C such that $\mathsf{D} \simeq \mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})$. Consequently, Proposition 5.4.5.4 gives rise to an equivalence $\mathsf{Sh}_{\mathsf{E}}(\mathsf{D}) \simeq \mathsf{Sh}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$. Therefore, it suffices to show that the right-hand side is presentable. Choose a small $\mathcal{B}$-category C' such that $\mathsf{E} \simeq \mathrm{Loc}_{\mathsf{S}'}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}'))$ for some $\mathsf{S}' \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')$ with S' small. We obtain a commutative square

$$\begin{tikzcd}
\mathsf{Sh}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C}) \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E}) \arrow[d, hook] \\
\mathsf{Sh}^{\mathsf{U}}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')}(\mathsf{C}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')).
\end{tikzcd}$$

We first claim that this square is a pullback. To see this, note that by Remark 5.4.5.2 and Remark 5.4.1.2, it will be enough to verify that a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{E}$ is U-continuous if $\mathsf{C}^{\mathrm{op}} \to \mathsf{E} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')$ is U-continuous. This is a straightforward consequence of the fact that fully faithful functors are conservative. To proceed, note that by Corollary 5.4.2.7, the vertical map on the right in the above diagram defines a map in $\mathrm{Pr}^{\mathrm{R}}_{\mathcal{B}}$. Using Proposition 5.4.4.11, the proof is thus complete once we verify that the lower horizontal map is a map in $\mathrm{Pr}^{\mathrm{R}}_{\mathcal{B}}$ as well. To see this, observe that by Lemma 5.5.1.3 below, we may identify this map with the inclusion

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}'))$$

that is induced by postcomposition with the inclusion $\mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As the latter is a map in $\mathrm{Pr}^{\mathrm{R}}_{\mathcal{B}}$ by Proposition 5.4.5.4, the claim thus follows by again appealing to Corollary 5.4.2.7. $\square$

5.5. The tensor product of presentable $\mathcal{B}$-categories

In [50], Lurie establishes a symmetric monoidal structure on the ∞-category $\mathrm{Cat}_{\infty}^{\mathcal{K}\text{-cc}}$ of $\mathcal{K}$-cocomplete ∞-categories with $\mathcal{K}$-cocontinuous functors, for any class $\mathcal{K}$ of ∞-categories. In particular, his construction gives rise to a symmetric monoidal structure on the ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ of presentable ∞-categories. In this section, our goal is to obtain a $\mathcal{B}$-categorical analogue of these results, i.e. to construct a symmetric monoidal structure on the ∞-category $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}$ of U-cocomplete $\mathcal{B}$-categories and U-cocontinuous functors, for any choice of internal class U, and in particular one for the ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ of presentable $\mathcal{B}$-categories. Our construction will be entirely analogous to the one in [50]: we will define the desired symmetric monoidal ∞-category $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ as the subcategory of the cartesian monoidal ∞-category $\mathrm{Cat}(\mathcal{B})^{\times} \to \mathrm{Fin}_*$ that is spanned by what we call U-*multilinear* functors. We define and study this concept in Section 5.5.1. In order to show that the map $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ that we end up with indeed defines a symmetric monoidal ∞-category, we will need a $\mathcal{B}$-categorical analogue of *cocompletions with relations*, which we discuss in Section 5.5.2. At last, we construct the desired symmetric monoidal structure on $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}$ in Section 5.5.3. In particular, our construction will yield a symmetric monoidal structure on the ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ of presentable $\mathcal{B}$-categories. In Section 5.5.4, we make use of this structure to identify $\mathcal{B}$-*modules* as a full subcategory of $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$.

Remark 5.5.0.1. The attentive reader may point out that a genuine $\mathcal{B}$-categorical version of Lurie's theory should entail developing a notion of *symmetric monoidal* $\mathcal{B}$-*categories* and to prove that the large $\mathcal{B}$-category $\mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ of U-cocomplete $\mathcal{B}$-categories admits a symmetric monoidal structure in this sense. This is certainly possible, see [56]. However, as developing this theory in its full generality would lead us too far away from the main theme of this thesis, we will content ourselves with a more slimmed-down version, which will be sufficient for our purposes.

5.5.1. Bilinear functors

Recall that a bilinear functor of cocomplete ∞-categories $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a functor that preserves small colimits separately in each variable. We will now introduce this notion in the internal setting. It will be useful to consider functors that only preserve certain (internal) classes of colimits in each variable, so that we arrive at the following general definition:

Definition 5.5.1.1. Let U and V be two internal classes of $\mathcal{B}$-categories. Suppose that C, D and E are $\mathcal{B}$-categories such that C is U-cocomplete, D is V-cocomplete and E is $\mathsf{U} \cup \mathsf{V}$-cocomplete. We will say that a functor $f\colon \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is (U, V)-*bilinear* if for any $A \in \mathcal{B}$ and any two objects $c\colon A \to \mathsf{C}$ and $d\colon A \to \mathsf{D}$ the functor

$$f(c,-)\colon \pi_A^*\mathsf{D} \xrightarrow{c\times \mathrm{id}} \pi_A^*\mathsf{C} \times \pi_A^*\mathsf{D} \xrightarrow{\pi_A^* f} \pi_A^*\mathsf{E}$$

is $\pi_A^*\mathsf{V}$-cocontinuous and the functor

$$f(-,d)\colon \pi_A^*\mathsf{C} \xrightarrow{\mathrm{id}\times d} \pi_A^*\mathsf{C} \times \pi_A^*\mathsf{D} \xrightarrow{\pi_A^* f} \pi_A^*\mathsf{E}$$

is $\pi_A^*\mathsf{U}$-cocontinuous. We write $\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ for the full subcategory spanned by the $(\pi_A^*\mathsf{U}, \pi_A^*\mathsf{V})$-bilinear functors for every $A \in \mathcal{B}$, and we write $\mathrm{Fun}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ for the underlying ∞-category of global sections. In the case where $\mathsf{U} = \mathsf{V} = \mathrm{Cat}_{\mathcal{B}}$ (and C,D and E are large), we will simply say that f is *bilinear* and write $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{bil}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ for the associated $\mathcal{B}$-category of bilinear functors (and likewise $\mathrm{Fun}_{\mathcal{B}}^{\mathrm{bil}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ for its underlying ∞-category of global sections).

Remark 5.5.1.2. In the situation of Definition 5.5.1.1, the fact that U- and V-cocontinuity are local conditions (Remark 3.3.2.3) implies that for any cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$, a functor f is (U, V)-bilinear if and only if for each i the functor $\pi_{A_i}^* f$ is $(\pi_{A_i}^* \mathsf{U}, \pi_{A_i}^* \mathsf{V})$-bilinear. In particular, an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ in context $A \in \mathcal{B}$ is contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})^{(\mathsf{U},\mathsf{V})}$ if and only if it defines a $(\pi_A^* \mathsf{U}, \pi_A^* \mathsf{V})$-bilinear functor, and there consequently is a canonical equivalence

$$\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{(\pi_A^* \mathsf{U}, \pi_A^* \mathsf{V})}(\pi_A^* \mathsf{C} \times \pi_A^* \mathsf{D}, \pi_A^* \mathsf{E})$$

of $\mathcal{B}_{/A}$-categories.

Lemma 5.5.1.3. *Let* U *and* V *be two internal classes and let* C, D *and* E *be* $\mathcal{B}$*-categories such that* C *is* U*-cocomplete,* D *is* V*-cocomplete and* E *is* $\mathsf{U} \cup \mathsf{V}$*-cocomplete. Then* $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})$ *is closed under* U*-colimits,* $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{E})$ *is closed under* V*-colimits, and there are natural equivalences*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \mathsf{E})).$$

Proof. By symmetry, it is enough to show that $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})$ is closed under U-colimits and to construct the first of the two equivalences. To begin with, we claim that a functor $f \colon \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is (U, V)-bilinear if and only if its transpose $f' \colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$ is U-cocontinuous and takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})$. To see this, note that for any $A \in \mathcal{B}$ and any object $c \colon A \to \mathsf{C}$, the functor $f'(c) \colon \pi_A^* \mathsf{D} \to \pi_A^* \mathsf{E}$ is by definition given by $f(c, -)$, which in turn implies that $f(c, -)$ is V-cocontinuous if and only if f' factors through the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{C}, \mathsf{E})$. Moreover, given any object $d \colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$, note that the functor $f(-, d)$ is given by the composite

$$\pi_A^* \mathsf{C} \xrightarrow{\pi_A^* f'} \pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{D}, \pi_A^* \mathsf{E}) \xrightarrow{d^*} \pi_A^* \mathsf{E}.$$

Therefore, Proposition 3.2.3.2 implies that f' is U-cocontinuous if and only if $f(-, d)$ is U-cocontinuous for all $d \colon A \to \mathsf{D}$ and all $A \in \mathcal{B}$. Hence the claim follows. In light of Remark 5.5.1.2 and Remark 3.3.3.4, this already implies we have a pullback square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})) \arrow[d, hook] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}).
\end{tikzcd}$$

To complete the proof, it is now enough to show that $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})$ is closed under U-colimits. In light of Remark 3.3.3.4, this follows once we show that for any $\mathsf{I} \in \mathsf{U}(1)$ we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})) \arrow[r] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E}) \arrow[d, hook] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})) \arrow[r, "\mathrm{colim}_{\mathsf{I}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}).
\end{tikzcd}$$

As $\mathrm{colim}_{\mathsf{I}}$ is cocontinuous, we get a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{E})) \arrow[r, hook] \arrow[d, "(\mathrm{colim}_{\mathsf{I}})_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{E})) \arrow[d, "(\mathrm{colim}_{\mathsf{I}})_*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}).
\end{tikzcd}$$

By what we have already shown above, we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \mathsf{E})) \arrow[r, "\simeq"] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{E})) \arrow[d, hook] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})) \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{E})).
\end{tikzcd}$$

Hence, upon combining the last two diagrams, we conclude that the colimit functor $\mathrm{colim}_{\mathsf{I}}$ restricts as desired. □

We now generalise the above situation to so-called *multilinear* functors. For the sake of simplicity, we will only do this in the case of one fixed internal class.

Definition 5.5.1.4. Let U be an internal class of $\mathcal{B}$-categories and suppose that $\mathsf{C}_1, \ldots, \mathsf{C}_n, \mathsf{E}$ are U-cocomplete $\mathcal{B}$-categories. A functor $f\colon \mathsf{C}_1 \times .... \times \mathsf{C}_n \to \mathsf{E}$ is said to be U-*multilinear* if for every $i = 1, \dots, n$ and all objects $c_j \colon A_j \to \mathsf{C}_j$ in context $A \in \mathcal{B}$ for $i \neq j$ the functor

$$\pi_A^* \mathsf{C}_i \xrightarrow{(c_1, \dots, \mathrm{id}, \dots, c_n)} \prod_{k=1}^{n} \pi_A^* \mathsf{C}_k \xrightarrow{f} \pi_A^* \mathsf{D}$$

is $\pi_A^* \mathsf{U}$-cocontinuous. We will write $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^{n} \mathsf{C}_k, \mathsf{E})$ for the full subcategory spanned by the $\pi_A^* \mathsf{U}$-multilinear functors for all $A \in \mathcal{B}$, and we furthermore denote the underlying ∞-category of global sections by $\mathrm{Fun}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^{n} \mathsf{C}_k, \mathsf{E})$.

Remark 5.5.1.5. By a similar argument as in Remark 5.5.1.2, the condition of a functor as in Definition 5.5.1.4 to be U-multilinear is local in $\mathcal{B}$, which implies that there is a canonical equivalence

$$\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^{n} \mathsf{C}_k, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^* \mathsf{U}\text{-mult}}(\prod_{k=1}^{n} \pi_A^* \mathsf{C}_k, \pi_A^* \mathsf{E})$$

for each $A \in \mathcal{B}$.

5.5.2. Cocompletion with relations

Let U be an internal class, and let C be a $\mathcal{B}$-category. In Section 3.5.1, we constructed the *free* U-*cocompletion* $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ of C, i.e. the universal U-cocomplete $\mathcal{B}$-category that is equipped with a functor $\mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. The goal of this section is to generalise this result by imposing that a chosen collection of cocones in C (that are indexed by objects of U) become colimit cocones in the free U-cocompletion. Our proof of this result is a straightforward adaptation of the discussion in [49, § 5.3.6].

Let us fix a small collection $R = (\bar{d}_i \colon \mathsf{K}_i^{\triangleright} \to \pi_{A_i}^* \mathsf{C})_{i \in I}$ of cocones with $A_i \in \mathcal{B}$ and $\mathsf{K}_i \in \mathsf{U}(A_i)$ for all $i \in I$. Let $\mathsf{S}_R \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the (non-full) subcategory that is spanned by the canonical maps $(f_i \colon \operatorname{colim} h_{\mathsf{C}} d_i \to h_{\mathsf{C}} \bar{d}_i(\infty))_{i \in I}$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (with each f_i being in context A_i for $i \in I$), where d_i denotes the restriction of $\bar{d}_i$ along the inclusion $\mathsf{K}_i \hookrightarrow \mathsf{K}_i^{\triangleright}$ and where $\infty \colon A_i \to \mathsf{K}_i^{\triangleright}$ denotes the cone point.

Let us set $\mathsf{D} = \mathrm{Loc}_{\mathsf{S}_R}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. By Corollary 5.4.1.8, the inclusion

$$i \colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$$

admits a left adjoint $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$. In particular, D is cocomplete (Proposition 3.3.2.11). We define the (large) $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{(\mathsf{U},R)}(\mathsf{C})$ as the smallest full subcategory of D that contains the essential image of $Lh_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{D}$ and that is closed under U-colimits in D, and we let $j_{\mathsf{C}}\colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$ be the map that is obtained by composing Lh_{C} with the inclusion.

Remark 5.5.2.1. Given any object $A \in \mathcal{B}$, we denote by $\pi_A^* R$ the set of cocones $(\pi_A^*(\bar{d}_i))_{i\in I}$. We then obtain an equivalence $\pi_A^* \mathsf{S}_R \simeq \mathsf{S}_{\pi_A^* R}$ of subcategories in $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$. Hence Remark 5.4.1.2 and the same argument as in the proof of Proposition 3.5.1.9 shows that one obtains a canonical equivalence of large $\mathcal{B}_{/A}$-categories

$$\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}^{(\pi_A^*\mathsf{U},\pi_A^* R)}(\pi_A^*\mathsf{C})$$

with respect to which $\pi_A^* j_{\mathsf{C}}$ corresponds to the map $j_{\pi_A^*\mathsf{C}}$.

For any U-cocomplete large $\mathcal{B}$-category E, we will denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})$ that arises as the pullback

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}) \arrow[d, "(h_{\mathsf{C}})_!"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E})_{\mathsf{S}_R} \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}).
\end{tikzcd}$$

We now obtain:

Proposition 5.5.2.2. *For every $i \in I$ the cocone $(j_{\mathsf{C}})_*(\bar{d}_i)$ is a colimit cocone in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$, and for every U-cocomplete large $\mathcal{B}$-category E, precomposition with j_{C} induces an equivalence*

$$j_{\mathsf{C}}^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}),\mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R.$$

Proof. Note that by construction of D, the map j_{C} carries each of the cocones $\bar{d}_i$ to a colimit cocone in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$, hence the first claim follows immediately. The proof of the second claim employs a similar strategy as in the proof of Theorem 3.5.1.12. First, if E is an arbitrary U-cocomplete $\mathcal{B}$-category, note that the Yoneda embedding induces a U-cocontinuous functor

$$\mathsf{E} \hookrightarrow \mathsf{E}' = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}}$$

into a cocomplete $\mathcal{B}$-category. By Corollary 5.4.3.2 and the universal property of presheaf $\mathcal{B}$-categories, we now obtain an equivalence

$$(Lh_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}')_R \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}')_{\mathsf{S}_R} \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}')$$

As the inclusion $\underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \mathsf{D}$ is by construction U-cocontinuous, we therefore obtain an induced inclusion

$$(j_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}')_R \hookrightarrow \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}').$$

Now if $f \colon \underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{E}'$ is a U-cocontinuous functor, precisely the same argument as the one employed in the proof of Theorem 3.5.1.12 shows that the adjunction counit $\epsilon \colon (j_{\mathsf{C}})_! j_{\mathsf{C}}^* f \to f$ is an equivalence and that f is therefore contained in the essential image of $(j_{\mathsf{C}})_!$. Together with Remark 5.5.2.1, this shows that $(j_{\mathsf{C}})_!$ is an equivalence. Finally, the same argumentation as in the proof of Theorem 3.5.1.12 also shows that this equivalence restricts to the desired equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})_R \simeq \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$. □

Remark 5.5.2.3. In the situation of Proposition 5.5.2.2, if U is assumed to be small (i.e. a *doctrine* in the terminology of Section 5.1.3) implies that $\underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C})$ is small as well. In fact, as D is locally small, the essential image of

$$Lh_{\mathsf{C}} \colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$$

is small (Lemma 2.3.1.6), hence we can make use of the same argument as in Remark 3.5.1.7 to deduce that $\underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C})$ must also be small.

We will now use Proposition 5.5.2.2 to construct the *universal* U-multilinear functor (in the sense of Definition 5.5.1.4). To that end, let us fix an internal class U and U-cocomplete $\mathcal{B}$-categories $\mathsf{C}^1, \dots, \mathsf{C}^n$.

Construction 5.5.2.4. Let $\mathcal{G} \subset \mathcal{B}$ be a small subcategory of generators (in the sense of Remark 1.2.1.3), and let us set

$$R_k = \bigsqcup_{G \in \mathcal{G}} \left\{ \bar{d} \colon \mathsf{K}^{\triangleright} \to \pi_G^* \mathsf{C}^k \mid \mathsf{K} \in \mathsf{U}(G),\ \bar{d} \text{ is a colimit cocone} \right\}$$

as well as

$$S_k = \bigsqcup_{G \in \mathcal{G}} \left\{ \bar{d} \colon \mathsf{K}^{\triangleright} \to \pi_G^* \underline{\mathsf{PSh}}^{(\mathsf{U},R_k)}_{\mathcal{B}}(\mathsf{C}^k) \mid \mathsf{K} \in \mathsf{U}(G),\ \bar{d} \text{ is a colimit cocone} \right\}.$$

Furthermore, let $\square_{k=1}^n R_k$ be the set of all diagrams of the form

$$(c_1, \dots, c_{l-1}, \mathrm{id}, c_{l+1}, \dots, c_n)\bar{d} \colon \mathsf{K}^{\triangleright} \to \pi_G^* \mathsf{C}^l \to \prod_{k=1}^n \pi_G^* \mathsf{C}^k$$

where $\bar{d}$ is an element of R_l and $c_k \colon G \to \mathsf{C}^k$ is an arbitrary object for each $k \neq l$. Let $\square_{k=1}^n S_k$ be defined analogously.

In the situation of Construction 5.5.2.4, observe that a functor $\prod_{k=1}^n \mathsf{C}_k \to \mathsf{E}$ (where E is an arbitrary U-cocomplete $\mathcal{B}$-category) is U-multilinear if and only if it is contained in the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}(\prod_{k=1}^n \mathsf{C}_i, \mathsf{E})_{\square_{k=1}^n R_k}$. Using Proposition 5.5.2.2, we thus conclude:

Proposition 5.5.2.5. *If $\mathsf{U} \hookrightarrow \mathsf{V}$ are internal classes and $\mathsf{C}_1, \dots, \mathsf{C}_n, \mathsf{E}$ are U-cocomplete $\mathcal{B}$-categories, the canonical functor*

$$j \colon \prod_{k=1}^n \mathsf{C}_i \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}, \square_{k=1}^n R_k)}(\prod_{k=1}^n \mathsf{C}_i)$$

induces an equivalence

$$j^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}, \square_{k=1}^n R_k)}(\prod_{i=1}^n \mathsf{C}_i), \mathsf{E}) \xrightarrow{\simeq} \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^n \mathsf{C}_k, \mathsf{E}).$$

of $\mathcal{B}$-categories. □

We conclude this section by noting that the construction of the universal U-multilinear map is *transitive*: suppose that $\mathsf{U} \hookrightarrow \mathsf{V}$ are internal classes, where U is a doctrine. Then, in the situation of Construction 5.5.2.4, the composition

$$\begin{aligned} \mathsf{C}^1 \times \dots \times \mathsf{C}^n &\to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U}, R_1)}(\mathsf{C}^1) \times \dots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U}, R_n)}(\mathsf{C}^n) \\ &\to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}, \square_{k=1}^n S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U}, R_1)}(\mathsf{C}^1) \times \dots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U}, R_n)}(\mathsf{C}^n)) \end{aligned}$$

carries each cocone in $\square_{k=1}^n R_k$ to a colimit cocone, hence Proposition 5.5.2.2 determines a functor

$$\phi \colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}, \square_{k=1}^n R_k)}(\mathsf{C}^1 \times \dots \times \mathsf{C}^n) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}, \square_{k=1}^n S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U}, R_1)}(\mathsf{C}^1) \times \dots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U}, R_n)}(\mathsf{C}^n)).$$

Proposition 5.5.2.6. *The map ϕ is an equivalence.*

Proof. Note that in light of Remark 5.5.2.3, the map ϕ is a well-defined morphism in the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}$ of V-cocomplete $\mathcal{B}$-categories and V-cocontinuous functors. By combining Yoneda's lemma with Remark 5.5.2.1 and Remark 3.3.3.4, the result thus follows once we verify that for every V-cocomplete $\mathcal{B}$-category E the restriction functor

$$\begin{array}{c}\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^{n}S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1)\times\cdots\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n)),\mathsf{E})\\ \big\downarrow{\scriptstyle\phi^*}\\ \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^{n}R_k)}(\mathsf{C}^1\times\cdots\times\mathsf{C}^n),\mathsf{E})\end{array}$$

is an equivalence. Using Proposition 5.5.2.2, this is in turn equivalent to the map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1)\times\cdots\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n),\mathsf{E})_{\square_{k=1}^{n}S_k}\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^1\times\cdots\times\mathsf{C}^n,\mathsf{E})_{\square_{k=1}^{n}R_k}$$

being an equivalence. We will use induction on n to show that this functor is an equivalence. If $n=1$, this is precisely the content of Proposition 5.5.2.2. For $n>1$, the construction of $\square_{k=1}^{n}R_k$ and $\square_{k=1}^{n}S_k$ together with Lemma 5.5.1.3 imply that the above map can be identified with the morphism

$$\begin{array}{c}\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1)\times\cdots\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_{n-1})}(\mathsf{C}^{n-1}),\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n),\mathsf{E}))_{\square_{k=1}^{n-1}S_k}\\ \big\downarrow\\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^1\times\cdots\times\mathsf{C}^{n-1},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^n,\mathsf{E})_{R_n})_{\square_{k=1}^{n-1}R_k}.\end{array}$$

As Proposition 5.5.2.2 implies that the map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^n,\mathsf{E})_{R_n}\to\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n),\mathsf{E})$$

is an equivalence, the claim thus follows by the induction hypothesis. □

5.5.3. The tensor product of U-cocomplete $\mathcal{B}$-categories

Throughout this section, let us fix a doctrine U. The goal of this section is to extend the results from [50, § 4.8.1] to the setting of $\mathcal{B}$-categories. Namely, we will construct a symmetric monoidal structure $\mathsf{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}$ on the ∞-category

$\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}$ of U-cocomplete $\mathcal{B}$-categories and U-cocontinuous functors. For this we will roughly follow the arguments in [50].

Let $p\colon \mathrm{Cat}(\mathcal{B})^{\times} \to \mathrm{Fin}_*$ be the cartesian monoidal structure on $\mathrm{Cat}(\mathcal{B})$. We define a subcategory $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}$ of $\mathrm{Cat}(\mathcal{B})^{\times}$ as follows: Let $f\colon x \to y$ be a morphism in $\mathrm{Cat}(\mathcal{B})^{\times}$, and assume that $p(f)$ is given by a map $\alpha\colon \langle n\rangle \to \langle m\rangle$ in the 1-category Fin_*. We now obtain equivalences $x \simeq (\mathsf{C}_1, \dots, \mathsf{C}_n)$ and $y \simeq (\mathsf{D}_1, \dots, \mathsf{D}_m)$ where the C_i and D_j are $\mathcal{B}$-categories, and the map f is determined by a collection of maps $f_j\colon \prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i \to \mathsf{D}_j$ for $j = 1, \dots, m$. We shall say that f is U-*multilinear* if the C_i and D_j are U-cocomplete and the functors f_j are U-multilinear. We let $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}$ be the subcategory of $\mathrm{Cat}(\mathcal{B})^{\times}$ that is spanned by the U-multilinear maps.

Lemma 5.5.3.1. *A map in* $\mathrm{Cat}(\mathcal{B})^{\times}$ *is contained in* $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}\otimes}$ *if and only if it is* U-*multilinear.*

Proof. We first show that U-multilinear maps are closed under composition. To that end, suppose that $f\colon x \to y$ and $f'\colon y \to z$ are U-multilinear maps, and consider the commutative diagram

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "h_y"'] & & y \arrow[rr, "f'"] \arrow[dr, "h_z"'] & & z \\
& x' \arrow[ur, "g_y"'] \arrow[dr, "h'_y"'] & & z' \arrow[ur, "g_z"'] & \\
& & y' \arrow[ur, "g'_y"'] & &
\end{tikzcd}$$

in which h_y, h'_y and h_z are cocartesian and the maps g_y, g'_y and g_z are sent to identity maps in Fin_*. Then $f'f$ being U-multilinear precisely means that $g_z g'_y$ is determined by a tuple of U-multilinear functors between U-cocomplete $\mathcal{B}$-categories. Unwinding the definitions, this follows immediately from the observation that U-multilinear functors compose in the expected way. Together with the fact that equivalences between U-cocomplete $\mathcal{B}$-categories are automatically U-cocontinuous, this already implies that the subspace of $\mathrm{Cat}(\mathcal{B})^{\times}_1$ that is spanned by the U-multilinear maps is closed under composition and equivalences in the sense of Proposition 1.3.1.17, hence the very same proposition (applied in the case $\mathcal{B} = \mathrm{Ani}$) proves the claim. $\square$

To proceed, let $\mathcal{M}_{\mathsf{U}}^{\otimes} \hookrightarrow \Delta^1 \times \mathrm{Cat}(\mathcal{B})^{\times}$ be the subcategory that is spanned by those maps $(\phi, f)\colon (\epsilon, \mathsf{C}) \to (\delta, \mathsf{D})$ in $\Delta^1 \times \mathrm{Cat}(\mathcal{B})^{\times}$ that satisfy:

1. if $\delta = 1$, then D is U-cocomplete, and
2. if furthermore $\epsilon = 1$ (so that necessarily $\phi = \mathrm{id}_1$), then f is U-multilinear.

Since clearly the maps that satisfy this condition are closed under equivalences and composition in the sense of Proposition 1.3.1.17, the very same proposition implies that a map in $\mathrm{Cat}(\mathcal{B})^{\times}$ is contained in $\mathcal{M}_{\mathsf{U}}^{\otimes}$ if and only if it satisfies the above condition.

By construction, the pullback of the composition

$$q\colon \mathcal{M}_{\mathsf{U}}^{\otimes} \hookrightarrow \Delta^1 \times \mathrm{Cat}(\mathcal{B})^{\times} \to \Delta^1 \times \mathrm{Fin}_*$$

along the inclusion $d^1\colon \mathrm{Fin}_* \hookrightarrow \Delta^1 \times \mathrm{Fin}_*$ recovers the cocartesian fibration $p\colon \mathrm{Cat}(\mathcal{B})^{\times} \to \mathrm{Fin}_*$, and the pullback of q along $d^0\colon \mathrm{Fin}_* \hookrightarrow \Delta^1 \times \mathrm{Fin}_*$ recovers the restriction of p to the subcategory $\mathrm{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$. We now obtain:

Proposition 5.5.3.2. *The composition $q\colon \mathcal{M}_{\mathsf{U}}^{\otimes} \hookrightarrow \Delta^1 \times \mathrm{Cat}(\mathcal{B})^{\times} \to \Delta^1 \times \mathrm{Fin}_*$ is a cocartesian fibration.*

Proof. Let us begin by fixing maps $\alpha\colon \langle n\rangle \to \langle m\rangle$ in the 1-category Fin_* and $\epsilon \leq \delta$ in the poset Δ^1, and let $x \in \mathcal{M}_{\mathsf{U}}^{\otimes}|_{(\epsilon,\langle n\rangle)}$ be an arbitrary object. Let us write $\mathsf{V}_0 = \varnothing$ and $\mathsf{V}_1 = \mathsf{U}$. By construction of $\mathcal{M}_{\mathsf{U}}^{\otimes}$, the object x corresponds to a tuple $(\epsilon, \mathsf{C}_1, \dots, \mathsf{C}_n)$ where $\mathsf{C}_1, \dots, \mathsf{C}_n$ are V_{ϵ}-cocomplete $\mathcal{B}$-categories. Let $f\colon x \to y$ be a cocartesian lift of α in $\mathrm{Cat}(\mathcal{B})^{\times}$. For each $j = 1, \dots, m$, we can make use of Construction 5.5.2.4 to define a map

$$g_j\colon \prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i \to \mathsf{D}_j = \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}_{\delta},\square_i R_i)}\Big(\prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i\Big),$$

and by setting $z = (\delta, \mathsf{D}_1, \dots, \mathsf{D}_m)$, precomposing the tuple $g = (\epsilon \leq \delta, g_1, \dots, g_m)$ with $(\mathrm{id}_{\epsilon}, f)$ defines a lift of $(\epsilon \leq \delta, \alpha)$ in $\mathcal{M}_{\mathsf{U}}^{\otimes}$. By Proposition 5.5.2.5, precomposition with each g_j induces an equivalence

$$g_j^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}_{\epsilon}\text{-mult}}\Big(\prod_i \mathsf{C}_i, \mathsf{E}\Big) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}_{\delta}\text{-cc}}(\mathsf{D}_j, \mathsf{E})$$

for every V_δ-cocomplete $\mathcal{B}$-category E. By construction of $\mathcal{M}_{\mathsf{U}}^{\otimes}$, Lemma 5.5.3.1 and Corollary 2.2.2.8, the underlying core $\mathcal{B}$-groupoids of both domain and codomain of g_j^* recover on global sections the mapping ∞-groupoids in the pullback of q along $\mathrm{id} \times \langle 1 \rangle \colon \Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$. Consequently, the functor $\mathcal{M}_{\mathsf{U}}^{\otimes} \times_{\Delta^1 \times \mathrm{Fin}_*} \Delta^1 \to \Delta^1$ that is obtained as the pullback of q along $(\epsilon \leq \delta, \alpha) \colon \Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$ must be a cocartesian fibration. In other words, q is a locally cocartesian fibration. Since Proposition 5.5.2.6 shows that the locally cocartesian maps are closed under composition, the result now follows. □

Corollary 5.5.3.3. *The functor* $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ *is a cocartesian fibration that gives rise to a symmetric monoidal structure on the* ∞*-category* $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}$.

Proof. Since the map $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ is a pullback of the functor q from Proposition 5.5.3.2, the same proposition immediately implies the first claim. Moreover, the straightforward observation that for every $n \geq 0$ the equivalence

$$\mathrm{Cat}(\mathcal{B})^{\times}_n \simeq \prod_{i=1}^{n} \mathrm{Cat}(\mathcal{B})^{\times}_1$$

restricts to an equivalence

$$\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}_n \simeq \prod_{i=1}^{n} \mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}_1$$

shows the second claim. □

Remark 5.5.3.4. By unstraightening the cocartesian fibration q from Proposition 5.5.3.2 we get a functor $\Delta^1 \to \mathrm{CMon}(\mathrm{Cat}(\mathcal{B}))$ and therefore a morphism of symmetric monoidal ∞-categories $L \colon \mathrm{Cat}(\mathcal{B})^{\times} \to \mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}$. Note that the pullback $\Delta^1 \times_{\Delta^1 \times \mathrm{Fin}_*} \mathcal{M}_{\mathsf{U}}^{\otimes} \to \Delta^1$ of q along $\mathrm{id} \times \langle 1 \rangle \colon \Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$ is also a *cartesian* fibration: in fact, by making use of Proposition 4.4.5.1 (in the case where $\mathcal{B} \simeq \mathrm{Ani}$), this follows from the straightforward observation that the adjunction $(d^1 \dashv s^0) \colon \Delta^1 \times \mathrm{Cat}(\mathcal{B}) \rightleftarrows \mathrm{Cat}(\mathcal{B})$ restricts to an adjunction $\Delta^1 \times_{\Delta^1 \times \mathrm{Fin}_*} \mathcal{M}_{\mathsf{U}}^{\otimes} \rightleftarrows \mathrm{Cat}(\mathcal{B})$. By Corollary 4.4.5.5, this means that L is the left adjoint of the inclusion $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ as provided by Corollary 3.5.1.14. In particular, we see that the $\mathcal{B}$-category underlying the tensor unit of $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}$ is equivalent to the free U-cocompletion of the point $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$.

Remark 5.5.3.5. By a similar argument as in Remark 5.5.3.4, the projection $\mathcal{M}_{\mathsf{U}}^{\otimes} \to \Delta^1$ is both cartesian and cocartesian. Therefore, one also obtains an adjunction

$$(L \dashv i)\colon\ \mathrm{Cat}(\mathcal{B})^{\times} \rightleftarrows \mathrm{Cat}(\mathcal{B})^{\mathsf{U},\otimes\text{-cc}}$$

in which i is simply the inclusion. Since the projection $\mathcal{M}_{\mathsf{U}}^{\otimes} \to \mathrm{Fin}_*$ carries every map in $\mathcal{M}_{\mathsf{U}}^{\otimes}$ that is cartesian over Δ^1 to an equivalence, we end up with a relative adjunction $\mathrm{Cat}(\mathcal{B})^{\times} \rightleftarrows \mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes}$ over Fin_*. As both maps are morphisms of ∞-operads, we thus obtain an induced adjunction

$$(L \dashv i)\colon\ \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}) \rightleftarrows \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})) \simeq \mathrm{Cat}(\mathcal{B})^{\otimes}$$

of ∞-categories. We refer to an object $\mathsf{C} \in \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B}))$ as *symmetric monoidal $\mathcal{B}$-category*. By unwinding the definitions, we see that such a symmetric monoidal $\mathcal{B}$-category C lies in $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}})$ if and only if C is U-cocomplete and the functor $-\otimes-\colon\ \mathsf{C}\times\mathsf{C} \to \mathsf{C}$ that is provided by the commutative algebra structure on C defines a U-bilinear map. In particular, it follows from Remark 5.5.3.4 that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ can be canonically equipped with the structure of a symmetric monoidal $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)^{\otimes}$ such that $-\otimes-\colon\ \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ is U-bilinear and that the canonical functor $1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)^{\otimes}$ induced by the adjunction unit is symmetric monoidal (i.e. arises from a map in $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B}))$). In particular, this means that we have a commutative diagram

$$\begin{tikzcd}
1\times 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \arrow[r, "-\otimes-"] & \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1).
\end{tikzcd}$$

By the universal property of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ and Lemma 5.5.1.3, there is a unique such functor $-\otimes-$, which must therefore coincide with the product functor $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$, see Proposition 3.5.3.8.

Example 5.5.3.6. Let $\mathcal{B} = \mathrm{PSh}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ and let $P \subseteq \mathcal{C}$ be a subcategory that is closed under pullbacks. Then P generates a local class W in $\mathrm{PSh}(\mathcal{C})$ and therefore a full subcategory $\mathrm{Grpd}_W \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$. Then Remark 3.3.2.9 implies together with Remark 5.5.3.5 that the ∞-category $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\mathrm{Grpd}_W\text{-cc}})$ is equivalent to the ∞-category of *pullback formalisms* in the sense of [21, §2.2].

By Remark 5.5.3.4 and Remark 5.5.3.5 the initial object of $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\mathsf{Grpd}_W\text{-cc}})$ is equivalent to the free Grpd_W-cocompletion of the point, equipped with the cartesian monoidal structure. Together with Example 3.5.3.6, this gives a new proof of [21, Theorem 3.25]. Furthermore, our proof yields a slightly more general result as it does not require that $\mathcal{C}$ is a 1-category.

We will now move one universe up and consider the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$ is the internal class of small $\mathcal{B}$-categories in $\mathsf{Cat}_{\widehat{\mathcal{B}}}$. By the above, we obtain a symmetric monoidal structure $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc},\otimes}$ on the very large $\mathcal{B}$-category $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}}$ of cocomplete large $\mathcal{B}$-categories and cocontinuous functors.

Proposition 5.5.3.7. *The tensor product* $-\otimes- \colon \mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}} \times \mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}} \to \mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}}$ *of cocomplete* $\mathcal{B}$*-categories restricts to a functor* $-\otimes- \colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \times \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. *Therefore,* $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *inherits the structure of a symmetric monoidal* ∞*-category.*

Proof. In light of the observation that the tensor unit in $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}}$ is given by the presentable $\mathcal{B}$-category $\mathsf{Grpd}_{\mathcal{B}}$, the second claim follows from [50, Proposition 2.2.1.1], so it suffices to show the first one. We need to show that if D and E are presentable then so is their tensor product $\mathsf{D} \otimes \mathsf{E}$. By Corollary 5.4.5.7, we may find a sound doctrine U and U-cocomplete (small) $\mathcal{B}$-categories C and C' such that $\mathsf{D} \simeq \mathsf{Sh}^{\mathrm{op}(\mathsf{U})}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})$ and $\mathsf{E} \simeq \mathsf{Sh}^{\mathrm{op}(\mathsf{U})}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C}')$. If X is an arbitrary cocomplete large $\mathcal{B}$-category, we compute

$$\begin{aligned}
\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D} \otimes \mathsf{E}, \mathsf{X}) &\simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{E}, \mathsf{X})) \\
&\simeq \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C}', \mathsf{X})) \\
&\simeq \underline{\mathsf{Fun}}^{\mathsf{U}\text{-mult}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{C}', \mathsf{X}) \\
&\simeq \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C} \otimes^{\mathsf{U}} \mathsf{C}', \mathsf{X}),
\end{aligned}$$

where the first and third equivalence are consequences of Lemma 5.5.1.3, the second equivalence follows from Corollary 5.4.5.6 and where $- \otimes^{\mathsf{U}} -$ denotes the tensor product in $\mathsf{Cat}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}$. Now U being a doctrine implies that the tensor product $\mathsf{C} \otimes^{\mathsf{U}} \mathsf{C}'$ is small (see Remark 5.5.2.3), hence another application of Corollary 5.4.5.6 gives rise to an equivalence $\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C} \otimes^{\mathsf{U}} \mathsf{C}', \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Sh}^{\mathrm{op}(\mathsf{U})}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C} \otimes^{\mathsf{U}} \mathsf{C}'), \mathsf{X})$. As the same corollary shows that $\mathsf{Sh}^{\mathrm{op}(\mathsf{U})}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C} \otimes^{\mathsf{U}} \mathsf{C}')$ is presentable and as all of the above equivalences are natural in X, the result follows. $\square$

Proposition 5.5.3.8. *Let* D *and* E *be presentable* $\mathcal{B}$*-categories. Then there is an equivalence of* $\mathcal{B}$*-categories*

$$\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{Sh}_{\mathsf{E}}(\mathsf{D}), \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{D} \times \mathsf{E}, \mathsf{X})$$

that is natural in $\mathsf{X} \in \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *and hence in particular an equivalence* $\mathsf{Sh}_{\mathsf{E}}(\mathsf{D}) \simeq \mathsf{D} \otimes \mathsf{E}$.

Proof. Let $\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(-,-) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ be the full subcategory spanned by the continuous functors in arbitrary context. We claim that we have a chain of equivalences

$$\begin{aligned}
\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{D} \times \mathsf{E}, \mathsf{X}) &\simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{E}, \mathsf{X}))\\
&\simeq \underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{E}, \mathsf{X})^{\mathrm{op}})^{\mathrm{op}}\\
&\simeq \underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E}))^{\mathrm{op}}\\
&\simeq \underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X}, \underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{E}))^{\mathrm{op}}\\
&\simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{E}), \mathsf{X})\\
&\simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{Sh}_{\mathsf{E}}(\mathsf{D}), \mathsf{X})
\end{aligned}$$

that are natural in E. The first equivalence follows from Lemma 5.5.1.3, the second and the last equivalences are obvious and the third and fifth equivalences follow from Proposition 5.4.3.3, so it remains to argue that the fourth equivalence holds.

We may choose a sound doctrine U such that $\mathsf{D} \simeq \mathsf{Sh}^{\mathsf{U}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{C})$ for some small U-cocomplete $\mathcal{B}$-category C (cf. Corollary 5.4.5.7). Using Corollary 5.4.5.6, we only need to see that the equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E}))$$

restricts to an equivalence

$$\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})) \simeq \underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X}, \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E}))$$

(where $\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(-,-)$ denotes the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ that is spanned by the $\pi_A^*\mathsf{U}$-continuous functors of $\mathcal{B}_{/A}$-categories, for all $A \in \mathcal{B}$). We already know from (the dual version of) Lemma 5.5.1.3 that we have an equivalence

$$\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})) \simeq \underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{X}, \underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})),$$

hence Proposition 5.4.3.3 together with Remark 3.3.3.4 and Remark 5.3.2.3 implies that the proof is finished once we verify that a functor $f\colon \mathsf{X} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$ is accessible if only if its transpose $f'\colon \mathsf{C}^{\mathrm{op}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})$ takes values in the full subcategory $\underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})$ spanned by the accessible functors. If f is accessible there is some sound doctrine U such that f is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous. But then it follows from Lemma 5.5.1.3 that f' takes values in the $\mathcal{B}$-category $\underline{\mathsf{Fun}}^{\mathsf{Filt}_{\mathsf{U}}\text{-cc}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})$, as desired. For the converse, suppose that f' takes values in $\underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})$. Let $z\colon \mathsf{C}_0 \to \mathsf{C}$ be the tautological object. Then $f'(z)\colon \pi^*_{\mathsf{C}_0}\mathsf{X} \to \pi^*_{\mathsf{C}_0}\mathsf{E}$ is $\pi^*_{\mathsf{C}_0}\mathsf{U}$-accessible for some sound doctrine U. Since *every* object in C is a pullback of z, this already shows that f' takes values in $\underline{\mathsf{Fun}}^{\mathsf{Filt}_{\mathsf{U}}\text{-cc}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})$, hence Lemma 5.5.1.3 shows that f is accessible. □

5.5.4. $\mathcal{B}$-modules as presentable$\mathcal{B}$-categories

By the discussion in the previous section, there is a symmetric monoidal functor

$$L\colon \mathrm{Cat}(\widehat{\mathcal{B}})^{\times} \to \mathrm{Cat}(\mathcal{B})^{\mathrm{cc},\otimes}$$

which is left adjoint of the inclusion $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc}} \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})$, so that the latter can be promoted to a lax symmetric monoidal functor $\mathrm{Cat}(\mathcal{B})^{\mathrm{cc},\otimes} \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})^{\times}$, see [50, Corollary 7.3.2.7]. In particular, we obtain a lax symmetric monoidal functor $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes} \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})^{\times}$. Moreover, as the global sections functor Γ preserves limits, it defines a symmetric monoidal functor $\mathrm{Cat}(\widehat{\mathcal{B}})^{\times} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}$. Since a multilinear functor in $\mathrm{Cat}(\mathcal{B})$ induces a multilinear functor on the underlying ∞-categories of global sections, it is evident that the induced map $\mathrm{Cat}(\mathcal{B})^{\mathrm{cc},\otimes} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}$ takes values in $\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{cc},\otimes} \hookrightarrow \widehat{\mathrm{Cat}}_{\infty}^{\times}$ and therefore defines a lax symmetric monoidal functor $\Gamma^{\mathrm{cc},\otimes}\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{cc},\otimes} \to \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{cc},\otimes}$. Upon restricting this functor to presentable $\mathcal{B}$-categories, we now end up with a lax symmetric monoidal functor $\Gamma^{\mathrm{cc},\otimes}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes} \to (\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\otimes}$ that in turn induces a map

$$\Gamma^{\mathrm{lin}}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \simeq \mathrm{Mod}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$$

(where $\mathcal{B}$ is regarded as the algebra in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ that is given by image of the trivial algebra $\mathrm{Grpd}_{\mathcal{B}}$ in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ along $\Gamma^{\mathrm{cc},\otimes}$, which is is precisely the *cartesian* monoidal structure on $\mathcal{B}$ as the product bifunctor $\mathrm{Grpd}_{\mathcal{B}} \times \mathrm{Grpd}_{\mathcal{B}} \to \mathrm{Grpd}_{\mathcal{B}}$ is bilinear, cf. Proposition 3.2.5.10).

The main goal of this section is to show that Γ^{lin} admits a fully faithful left adjoint that embeds $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ into $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ and to give an explicit description of this embedding. As a preliminary step, we need to show that the global sections functor $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_{\infty}$ admits a left adjoint. Recall from Example 5.4.4.8 that there is a functor $-\otimes \mathsf{Grpd}_{\mathcal{B}}\colon \mathrm{Pr}^{\mathrm{R}}_{\infty} \to \mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ that assigns to a presentable ∞-category $\mathcal{D}$ the presentable $\mathcal{B}$-category that is given by the sheaf $\mathcal{D}\otimes\mathcal{B}_{/-}$. Using Proposition 5.4.4.7, we may equivalently regard this map as a functor $\mathrm{Pr}^{\mathrm{L}}_{\infty} \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. We now obtain:

Proposition 5.5.4.1. *The functor* $-\otimes \mathsf{Grpd}_{\mathcal{B}}$ *is left adjoint to the global sections functor* $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_{\infty}$.

Proof. The composition $\Gamma\circ(-\otimes \mathsf{Grpd}_{\mathcal{B}})$ can be identified with the endofunctor $-\otimes\mathcal{B}\colon \mathrm{Pr}^{\mathrm{L}}_{\infty} \to \mathrm{Pr}^{\mathrm{L}}_{\infty}$, hence $\Gamma_*\colon \mathrm{Sh}_{\mathcal{B}}(-) \to \mathrm{id}_{\mathrm{Pr}^{\mathrm{R}}_{\infty}}$ defines a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Pr}^{\mathrm{L}}_{\infty}} \to -\otimes\mathcal{B}$ upon passing to opposite ∞-categories. We need to show that the composition

$$\mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})}(\mathcal{D}\otimes \mathsf{Grpd}_{\mathcal{B}}, \mathsf{E}) \to \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}_{\infty}}(\mathcal{D}\otimes\mathcal{B}, \Gamma\mathsf{E}) \xrightarrow{\eta^*_{\mathcal{D}}} \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}_{\infty}}(\mathcal{D}, \Gamma\mathsf{E}) \qquad (*)$$

is an equivalence. Choose a regular cardinal κ such that $\mathcal{D}\simeq \mathrm{Sh}^{\kappa}_{\mathrm{Ani}}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ that admits κ-small colimits. Using Proposition 5.4.5.4, we obtain an equivalence $\mathcal{D}\otimes \mathsf{Grpd}_{\mathcal{B}} \simeq \mathrm{Sh}^{\mathsf{LConst}_{\kappa}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathcal{C})$ with respect to which the map $\eta_{\mathcal{D}}$ corresponds to the left adjoint of $\Gamma_*\colon \mathrm{Sh}^{\kappa}_{\mathcal{B}}(\mathcal{C}) \to \mathrm{Sh}^{\kappa}_{\mathrm{Ani}}(\mathcal{C})$. Again using Proposition 5.4.5.4, we have equivalences

$$\begin{aligned}\mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})}(\mathcal{D}\otimes \mathsf{Grpd}_{\mathcal{B}}, \mathsf{E}) &\xrightarrow{(h^{\mathcal{B}}_{\mathcal{C}})^*} \mathrm{map}_{\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathsf{LConst}_{\kappa}\text{-cc}}}(\mathcal{C}, \mathsf{E})\\ &\simeq \mathrm{map}_{\widehat{\mathrm{Cat}}_{\infty}^{\kappa\text{-cc}}}(\mathcal{C}, \Gamma\mathsf{E})\\ &\xrightarrow{(h^{\mathrm{Ani}}_{\mathcal{C}})^*} \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}_{\infty}}(\mathcal{D}, \Gamma\mathsf{E})\end{aligned}$$

where $h^{\mathcal{B}}_{\mathcal{C}}$ is the Yoneda embedding in $\mathrm{Cat}(\widehat{\mathcal{B}})$ and $h^{\mathrm{Ani}}_{\mathcal{C}}$ is the Yoneda embedding in $\widehat{\mathrm{Cat}}_{\infty}$. On account of the commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "h^{\mathrm{Ani}}_{\mathcal{C}}"] \arrow[d] & \mathrm{Sh}^{\kappa}_{\mathrm{Ani}}(\mathcal{C}) \arrow[d, "\eta_{\mathcal{D}}"] \\ \Gamma\mathcal{C} \arrow[r, hook, "\Gamma(h^{\mathcal{B}}_{\mathcal{C}})"] & \mathrm{Sh}^{\kappa}_{\mathcal{B}}(\mathcal{C}) \end{tikzcd}$$

in which the vertical map on the left is the unit of $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ (see Lemma 4.4.4.6), the composition of the above chain of equivalences recovers the map in ($*$), hence the claim follows. □

Proposition 5.5.4.2. *The functor* $\Gamma^{\mathrm{lin}} \colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ *admits a fully faithful left adjoint.*

Proof. Note that since $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \simeq \mathrm{Pr}^{\mathrm{R}}(\mathcal{B})^{\mathrm{op}}$ it follows from Proposition 5.4.4.11 that the global sections functor $\Gamma \colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_{\infty}$ preserves colimits. So in light of Proposition 5.5.4.1 we may apply [50, Corollary 4.7.3.16] to the commutative triangle

$$\begin{tikzcd} \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \arrow[rr, "\Gamma^{\mathrm{lin}}"] \arrow[dr, "\Gamma"'] & & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \arrow[dl, "U"] \\ & \mathrm{Pr}^{\mathrm{L}}_{\infty} & \end{tikzcd}$$

(where U denotes the forgetful functor), which yields the claim. □

We will now give a more explicit description of the left adjoint from Proposition 5.5.4.2. To that end, observe that the functor

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{Grpd}_{\mathcal{B}}) \colon \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$$

takes values in $\mathrm{Pr}^{\mathrm{R}}_{\mathcal{B}}$ and therefore determines a limit-preserving map

$$\mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}(\mathcal{B}) \simeq (\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))^{\mathrm{op}}$$

which by postcomposition with Γ^{lin} results in a limit-preserving functor

$$\mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}}.$$

We now get a map

$$\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \xrightarrow{(-\otimes_{\mathcal{B}} \mathcal{B}_{/-})^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to (\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}_{\infty} \hookrightarrow \widehat{\mathrm{Cat}}_{\infty}$$

and hence by adjunction a functor $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to \mathrm{PSh}_{\widehat{\mathrm{Cat}}_{\infty}}(\mathcal{B})$.

Lemma 5.5.4.3. *The functor* $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to \mathrm{PSh}_{\widehat{\mathrm{Cat}}_{\infty}}(\mathcal{B})$ *factors through* $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ *and thus defines a functor*

$$- \otimes_{\mathcal{B}} \mathsf{Grpd}_{\mathcal{B}} \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}).$$

Proof. First we prove that the functor factors through $\mathrm{Cat}(\widehat{\mathcal{B}})$. This amounts to showing that the functor $\mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ is continuous for every $\mathcal{D} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$. As the functor $\mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)^{\mathrm{op}}$ preserves limits, this follows from the fact that $\mathcal{D} \otimes_{\mathcal{B}} -$, viewed as an endofunctor on $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)^{\mathrm{op}}$, preserve limits as well [50, Corollary 4.4.2.15]. Next, we show that the resulting $\mathcal{B}$-category $\mathcal{D} \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ is presentable. As it by construction takes values in $\mathrm{Pr}^{\mathrm{R}}_\infty$, Theorem 5.4.2.5 implies that it suffices to show that $\mathcal{D} \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ is $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete and that the transition functors are cocontinuous. Both statements follow from the observation that the functor $\mathcal{D} \otimes_{\mathcal{B}} - \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$ can be upgraded to an $(\infty, 2)$-functor (see [37, §4.4] for details) and that for any $s \colon B \to A$ in $\mathcal{B}$ the adjunction $s_! \dashv s^*$ is $\mathcal{B}$-linear, see [50, Corollary 7.3.2.7]. To finish the proof, it remains to see that for any map of $\mathcal{B}$-modules $\mathcal{D} \to \mathcal{E}$ the induced map $\mathcal{E} \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \to \mathcal{D} \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ admits a left adjoint. By construction, it has one section-wise, so it suffices to check that for any map $s \colon B \to A$ in $\mathcal{B}$ the induced lax square

$$\begin{tikzcd} \mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/B} \arrow[r] & \mathcal{E} \otimes_{\mathcal{B}} \mathcal{B}_{/B} \\ \mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/A} \arrow[r] \arrow[u] & \mathcal{E} \otimes_{\mathcal{B}} \mathcal{B}_{/A} \arrow[u] \end{tikzcd}$$

commutes. Using again $(\infty, 2)$-functoriality of the relative tensor product, this follows by essentially the same argument as in the proof of Lemma 3.2.2.12. $\square$

Remark 5.5.4.4. It also seems natural to consider the functor

$$\begin{aligned} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \times \mathcal{B}^{\mathrm{op}} &\xrightarrow{\mathrm{id} \times \mathcal{B}_{/-}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \times \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \\ &\xrightarrow{- \otimes_{\mathcal{B}} -} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \\ &\to \mathrm{Cat}_\infty \end{aligned}$$

which by transposition also gives rise to a functor $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. We expect that this functor takes values in $\mathrm{Cat}(\mathcal{B})$ and is equivalent to $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$. It is easy to see that for fixed $\mathcal{C} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$ the two resulting presheaves of categories on $\mathcal{B}$ have the same value on objects and morphisms. However, a proof that they agree as functors seems two require $(\infty, 2)$-categorical techniques that are not quite available yet.

Lemma 5.5.4.5. *The functor* $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *preserves colimits.*

Proof. As limits in $\mathrm{Cat}(\widehat{\mathcal{B}})$ are computed section-wise, it suffices to show that for every $A \in \mathcal{B}$ the functor

$$\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \xrightarrow{(-\otimes_{\mathcal{B}}\mathcal{B}_{/A})^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to (\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}_{\infty} \to \widehat{\mathrm{Cat}}_{\infty}$$

preserves limits, which is obvious. □

Proposition 5.5.4.6. *The functor* $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ *defines a left adjoint of* Γ^{lin}.

Proof. We show that $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ is equivalent to the left adjoint L of Γ^{lin} from Proposition 5.5.4.2. Let us denote by $- \otimes \mathcal{B} \colon \mathrm{Pr}^{\mathrm{L}}_{\infty} \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ the left adjoint to the forgetful functor. Then by the associativity of the relative tensor product ([50, Proposition 4.4.3.14] we have equivalences

$$(- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}) \circ (- \otimes \mathcal{B}) \simeq - \otimes \mathrm{Grpd}_{\mathcal{B}} \simeq L \circ (- \otimes \mathcal{B}) \tag{$*$}$$

of functors from $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ to $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. By [50, Remark 4.7.3.15] we may find a functor

$$F \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}_{\infty})$$

such that the composite

$$\begin{aligned}\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) &\xrightarrow{F} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}_{\infty})\\ &\xrightarrow{(-\otimes\mathcal{B})_*} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}))\\ &\xrightarrow{\mathrm{colim}_{\Delta^{\mathrm{op}}}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})\end{aligned}$$

is equivalent to the identity. From ($*$) and Lemma 5.5.4.5 it follows that the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}_{\infty}) \arrow[r, "(-\otimes\mathcal{B})_*"] \arrow[d, "(-\otimes\mathcal{B})_*"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})) \arrow[r, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] \arrow[d, "(-\otimes_{\mathcal{B}}\mathrm{Grpd}_{\mathcal{B}})_*"] & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \arrow[d, "-\otimes_{\mathcal{B}}\mathrm{Grpd}_{\mathcal{B}}"] \\
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})) \arrow[r, "L_*"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) \arrow[r, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] & \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})
\end{tikzcd}$$

commutes. Since L commutes with colimits as well, we get an equivalence $L \simeq (- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}})$, as desired. □

The functor $-\otimes_{\mathcal{B}}\mathrm{Grpd}_{\mathcal{B}}$ can be naturally extended to a strong monoidal functor. To see this, observe that since the global sections functor $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_{\infty}$ admits an extension to a lax monoidal functor $\Gamma^{\mathrm{cc},\otimes}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes} \to (\mathrm{Pr}^{\mathrm{L}})^{\otimes}$, the commutative diagram

$$\begin{tikzcd}
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \arrow[rr, "\Gamma^{\mathrm{lin}}"] \arrow[dr, "\Gamma"'] & & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \arrow[dl, "U"] \\
& \mathrm{Pr}^{\mathrm{L}}_{\infty} &
\end{tikzcd}$$

can be naturally extended to a diagram of lax monoidal functors. By passing to left adjoints, we thus obtain a commutative triangle

$$\begin{tikzcd}
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes} & & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\otimes} \arrow[ll, "-\otimes_{\mathcal{B}}\mathrm{Grpd}_{\mathcal{B}}"'] \\
& (\mathrm{Pr}^{\mathrm{L}})^{\otimes} \arrow[ul, "-\otimes\mathrm{Grpd}_{\mathcal{B}}"] \arrow[ur, "-\otimes\mathcal{B}"'] &
\end{tikzcd}$$

of *oplax* monoidal functors, see [34]. In order to show that the functor $-\otimes_{\mathcal{B}}\mathrm{Grpd}_{\mathcal{B}}$ is strong monoidal, it thus suffices to show that the natural map

$$(-\otimes_{\mathcal{B}} -)\otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \to (-\otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}})\otimes(-\otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}})$$

is an equivalence. As both sides of this map preserve colimits in both variables and since every $\mathcal{B}$-module can be written as a colimit of objects that are contained in the image of $-\otimes\mathcal{B}$, it suffices to show that the natural map

$$(-\otimes -)\otimes \mathrm{Grpd}_{\mathcal{B}} \to (-\otimes \mathrm{Grpd}_{\mathcal{B}})\otimes(-\otimes \mathrm{Grpd}_{\mathcal{B}})$$

is an equivalence, i.e. that $-\otimes\mathrm{Grpd}_{\mathcal{B}}$ is strong monoidal. Recall (e.g. from Remark 5.4.4.12) that every presentable ∞-category can be obtained as a pushout (in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$) of presheaf ∞-categories. The claim therefore follows from the observation that $-\otimes\mathrm{Grpd}_{\mathcal{B}}$ fits into a commutative square

$$\begin{tikzcd}
\mathrm{Cat}(\widehat{\mathcal{B}})^{\times} \arrow[d, "L"] & \widehat{\mathrm{Cat}}_{\infty}^{\times} \arrow[l, "\mathrm{const}_{\mathcal{B}}"'] \arrow[d, "L"] \\
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes} & (\mathrm{Pr}^{\mathrm{L}})^{\otimes} \arrow[l, "-\otimes\mathrm{Grpd}_{\mathcal{B}}"']
\end{tikzcd}$$

of oplax monoidal functors (which is again constructed from the associated commutative square of lax monoidal functors by passing to left adjoints) in which both vertical maps as well as $\mathrm{const}_{\mathcal{B}}$ are strong monoidal. We conclude:

Proposition 5.5.4.7. *The functor* $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \hookrightarrow \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *admits a natural enhancement to a strong monoidal functor.* □

The functor $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ being fully faithful raises the question what can be said about its essential image. First, we observe that there is an explicit criterion when a presentable $\mathcal{B}$-category arises from a $\mathcal{B}$-module:

Remark 5.5.4.8. Let C be a presentable $\mathcal{B}$-category. Then the unit of the adjunction $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \dashv \Gamma^{\mathrm{lin}}$ gives a canonical map $\Gamma^{\mathrm{lin}}(\mathsf{C}) \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}} \to \mathsf{C}$. For $A \in \mathcal{B}$ the induced map $\varepsilon(A) \colon \mathcal{B}_{/A} \otimes_{\mathcal{B}} \Gamma^{\mathrm{lin}}(\mathsf{C}) \to \mathsf{C}(A)$ is the map underlying the essentially unique map of $\mathcal{B}_{/A}$-modules that makes the diagram

$$\begin{tikzcd} \mathsf{C}(1) \otimes_{\mathcal{B}} \mathcal{B}_{/A} \arrow[r, "\varepsilon(A)"] & \mathsf{C}(A)) \\ \mathsf{C}(1) \arrow[u] \arrow[r, "\mathrm{id}"] & \mathsf{C}(1) \arrow[u, "\pi_A^*"'] \end{tikzcd}$$

commute. It follows that a presentable $\mathcal{B}$-category is in the essential image of $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$ if and only if $\varepsilon(A)$ is an equivalence for all $A \in \mathcal{B}$.

Using the criterion from Remark 5.5.4.8, we are now able to write down an example of a presentable $\mathcal{B}$-category that is *not* in the essential image of $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$. We learned about this example from David Gepner and Rune Haugseng.

Example 5.5.4.9. Let Fin be the category of finite sets and let $\mathcal{B} = \mathrm{PSh}(\mathrm{Fin})$. Let X be a set with more than one element that we consider as an object in $\mathcal{B}$ via the Yoneda embedding. Then $\underline{\mathsf{Fun}}_{\mathcal{B}}(X, \mathrm{Grpd}_{\mathcal{B}})$ is a presentable $\mathcal{B}$-category that is not in the essential image of $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$. In fact, by Remark 5.5.4.8 this would imply that the canonical map $\varepsilon(X)$ being an equivalence. In our specific situation $\varepsilon(X)$ is the canonical left adjoint functor

$$\mathrm{PSh}(\mathrm{Fin}_{/X}) \otimes_{\mathrm{PSh}(\mathrm{Fin})} \mathrm{PSh}(\mathrm{Fin}_{/X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X \times X}).$$

Explicitly this functor is constructed by applying $\mathrm{PSh}(-)$ to the augmented cosimplicial diagram

$$\mathrm{Fin}_{/X\times X} \to \mathrm{Fin}_{/X} \times \mathrm{Fin}_{/X} \rightrightarrows \mathrm{Fin}_{/X} \times \mathrm{Fin} \times \mathrm{Fin}_{/X} \cdots$$

and then taking the induced map

$$\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X\times X})$$

in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. Thus, upon passing to right adjoints, we conclude that if the $\mathcal{B}$-category $\underline{\mathrm{Fun}}_{\mathcal{B}}(X, \mathrm{Grpd}_{\mathcal{B}})$ is contained in the essential image of $- \otimes_{\mathcal{B}} \mathrm{Grpd}_{\mathcal{B}}$, the cosimplicial diagram

$$\mathrm{PSh}(\mathrm{Fin}_{/X\times X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin}_{/X}) \rightrightarrows \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin} \times \mathrm{Fin}_{/X}) \cdots$$

in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$ must be a limit diagram. We show that this cannot be true. Let us denote the map $\mathrm{Fin}_{/X\times X} \to \mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X}$ by f_n. It is given explicitly by the assignment

$$(A \to X \times X) \mapsto (A \to X \times X \xrightarrow{\mathrm{pr}_0} X, A, \dots, A, A \to X \times X \xrightarrow{\mathrm{pr}_1} X).$$

Now for any $n \geq 1$ the map $\mathrm{PSh}(\mathrm{Fin}_{/X\times X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X})$ is the functor of right Kan extension $(f_n^{\mathrm{op}})_*$ along f_n^{op}. Hence, if the above cosimplicial diagram is a limit cone, the counit of the adjunctions $(f_n^{\mathrm{op}})^* \dashv (f_n^{\mathrm{op}})_*$ yields an equivalence $\mathrm{colim}_{n\in\Delta^{op}}(f_n^{\mathrm{op}})^*(f_n^{\mathrm{op}})_* F \to F$ for any $F \in \mathrm{PSh}(\mathrm{Fin}_{/X\times X})$. For any object

$$a = (A \to X, B_1, \dots, B_n, C \to X) \in \mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X}$$

we can compute $(f_n^{\mathrm{op}})_* F(a)$ via the point-wise formula for right Kan extensions as a limit indexed by $(\mathrm{Fin}_{/X\times X})^{\mathrm{op}}_{a/}$. But $A \times B_1 \times \dots \times B_n \times X \to X \times X$ defines an initial object of this category, hence we find

$$F(A \to X \times X) \simeq \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} F(A \times A^n \times A \to X \times X).$$

In particular, this shows that the map $F(A\times A \to X\times X) \to F(A \to X\times X)$ induced by $A \to A \times A$ is a cover in Ani. By taking F to be the presheaf represented by the diagonal $X \to X \times X$, it in turn follows that the map

$$\mathrm{map}_{\mathrm{Fin}_{/X\times X}}(X \times X, X) \to \mathrm{map}_{\mathrm{Fin}_{/X\times X}}(X, X)$$

is surjective. In particular, there is a preimage of the identity $X \to X$. But since X has at least two elements there is no map α making the diagram

$$\begin{tikzcd} X \times X \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{id}"'] & & X \arrow[dl, "\Delta"] \\ & X \times X & \end{tikzcd}$$

commute, which yields the desired contradiction.

There is, however, a class of ∞-topoi $\mathcal{B}$ for which the functor $- \otimes_{\mathcal{B}}$ turns out to be essentially surjective: those that are generated by (-1)-truncated objects:

Proposition 5.5.4.10. *Assume that $\mathcal{B}$ is generated by (-1)-truncated objects under colimits. Then $- \otimes_{\mathcal{B}} \mathsf{Grpd}_{\mathcal{B}}$ is an equivalence.*

Proof. By Proposition 5.5.4.2 and Proposition 5.5.4.6 it remains to show essential surjectivity. Since $- \otimes_{\mathcal{B}} \mathsf{Grpd}_{\mathcal{B}}$ preserves colimits and every presentable $\mathcal{B}$-category is a pushout of presheaf $\mathcal{B}$-categories (see Remark 5.4.4.12) it suffices to see that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is in the essential image for any small $\mathcal{B}$-category C. Furthermore, we can write C as a colimit of $\mathcal{B}$-categories of the form $\Delta^n \otimes U$, where $U \in \mathcal{B}$ is (-1)-truncated. Since the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ that is determined by the universal property of presheaf $\mathcal{B}$-categories is a (partial) left adjoint (see Corollary 3.5.1.14) and therefore preserves colimits, it suffices to see that the $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n \otimes U)$ is in the essential image. Since $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)$ is also symmetric monoidal by Remark 5.5.3.4, we have a canonical equivalence

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n \otimes U) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n) \otimes \underline{\mathsf{PSh}}_{\mathcal{B}}(U).$$

Furthermore, we may compute

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n) \simeq \mathrm{PSh}(\Delta^n) \otimes \mathsf{Grpd}_{\mathcal{B}} \simeq (\mathrm{PSh}(\Delta^n) \otimes \mathsf{Grpd}_{\mathcal{B}}) \otimes_{\mathcal{B}} \mathsf{Grpd}_{\mathcal{B}},$$

and since $- \otimes_{\mathcal{B}} \mathsf{Grpd}_{\mathcal{B}}$ is symmetric monoidal by Proposition 5.5.4.7, it thus suffices to see that $\underline{\mathsf{PSh}}_{\mathcal{B}}(U)$ is in the essential image. By Remark 5.5.4.8, it follows that we need to check that for any $A \in \mathcal{B}$ the canonical map

$$\mathcal{B}_{/A} \otimes_{\mathcal{B}} \underline{\mathsf{PSh}}_{\mathcal{B}}(U)(1) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(U)(A)$$

of $\mathcal{B}_{/A}$-modules is an equivalence. Since $\mathcal{B}$ is generated under colimits by (-1)-truncated objects, we may assume that $A = V$ is also (-1)-truncated. Thus, we have to show that the canonical map

$$\mathcal{B}_{/V} \otimes_{\mathcal{B}} \mathcal{B}_{/U} \to \mathcal{B}_{/U\times V}$$

is an equivalence. For this, note that because U is (-1)-truncated, we have a canonical commutative square

$$\begin{tikzcd} \Delta^1 \arrow[r, "U\to 1"] \arrow[d, "\mathrm{id}_1"'] & \mathcal{B} \arrow[d, "-\times U"] \\ \mathcal{B} \arrow[r, "-\times U"'] & \mathcal{B}_{/U} \end{tikzcd}$$

By adjunction and the universal property of presheaf ∞-categories, this induces a commutative square

$$\begin{tikzcd} \mathrm{PSh}(\Delta^1) \otimes \mathcal{B} \arrow[r, "(U\to 1)\otimes\mathcal{B}"] \arrow[d, "(\mathrm{id}_1)\otimes\mathcal{B}"'] & \mathcal{B} \arrow[d, "-\times U"] \\ \mathcal{B} \arrow[r, "-\times U"'] & \mathcal{B}_{/U} \end{tikzcd}$$

in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$. We claim that this square is a pushout. For this it suffices to see that the underlying square in $\mathrm{Pr}^{\mathrm{L}}_\infty$ is a pushout, i.e. it is a pullback after passing to right adjoints. The right adjoint of $\mathrm{id}_1 \otimes \mathcal{B}$ is simply the diagonal map $\mathcal{B} \to \mathcal{B}^{\Delta^1}$, and the right adjoint of $(U \to 1) \otimes \mathcal{B}$ sends an object $A \in \mathcal{B}$ to the arrow

$$A \to \underline{\mathrm{Hom}}_{\mathcal{B}}(U, A).$$

Thus, we may identify the pullback, with the full subcategory of $\mathcal{B}$ spanned by those objects for which the canonical map $A \to \underline{\mathrm{Hom}}_{\mathcal{B}}(U, A)$ is an equivalence. But because U is (-1)-truncated, this subcategory is canonically equivalent to $\mathcal{B}_{/U}$, so that the above square is indeed a pushout. Repeating the same argument with $\mathcal{B}_{/V}$ in place of $\mathcal{B}$ and $U \times V$ in place of U, we get a similar pushout in $\mathrm{Mod}_{\mathcal{B}_{/U}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$ with $\mathcal{B}_{/U\times V}$ in the lower right corner. But applying $-\otimes_{\mathcal{B}} \mathcal{B}_{/V}$ to the above square, we also get a pushout

$$\begin{tikzcd}
\mathrm{PSh}(\Delta^1) \otimes \mathcal{B}_{/V} \arrow[r, "(U\times V\to V)\otimes\mathcal{B}_{/V}"] \arrow[d, "(\mathrm{id}_1)\otimes\mathcal{B}_{/V}"'] & \mathcal{B}_{/V} \arrow[d] \\
\mathcal{B}_{/V} \arrow[r] & \mathcal{B}_{/U} \otimes_{\mathcal{B}} \mathcal{B}_{/V}
\end{tikzcd}$$

and thus an equivalence of $\mathcal{B}_{/V}$-modules $\mathcal{B}_{/U\times V} \simeq \mathcal{B}_{/U} \otimes_{\mathcal{B}} \mathcal{B}_{/V}$. Furthermore this equivalence is by construction compatible with the canonical map from $\mathcal{B}$. Thus it is indeed the map of Remark 5.5.4.8, and the claim follows. $\square$

6. $\mathcal{B}$-topoi

In this chapter, we develop the theory of *$\mathcal{B}$-topoi*, i.e. of the analogue of ∞-topoi themselves in the $\mathcal{B}$-categorical world. There are several equivalent ways to approach the subject of ∞-topoi: via the Giraud axioms, as left exact and accessible localisations of presheaf ∞-categories, or via the notion of *descent*. We firmly believe that it is the latter concept that is the distinguishing element in the theory of higher topoi. Consequently, our definition of a $\mathcal{B}$-topos will be that of a presentable $\mathcal{B}$-category that satisfies a suitable $\mathcal{B}$-categorical analogue of the descent property.

We will begin this chapter by setting up the theory of descent for $\mathcal{B}$-categories in Section 6.1. In Section 6.2, we then proceed by developing the main concepts of $\mathcal{B}$-topos theory. Using the results from Section 6.1, we characterize $\mathcal{B}$-topoi in terms of an internal version of the Giraud axioms as well as via explicit sheaf-theoretic criteria, see Theorem 6.2.1.5. We furthermore prove that any $\mathcal{B}$-topos can be presented by a left exact and accessible Bousfield localisation of a presheaf $\mathcal{B}$-category (Theorem 6.2.3.1).

We then establish one of the main results of this thesis: relating the theory of $\mathcal{B}$-topoi with that of *relative ∞-topoi over* $\mathcal{B}$, by which we simply mean geometric morphisms of ∞-topoi with fixed codomain $\mathcal{B}$. In 1-topos theory, these two notions are well-known to be entirely equivalent to one another [60]. In Theorem 6.2.5.1, we prove the same result in the ∞-categorical context.

The correspondence between $\mathcal{B}$-topoi on the one hand and geometric morphisms into $\mathcal{B}$ on the other allows us to seamlessly translate between properties of $\mathcal{B}$-topoi and properties of geometric morphisms of ∞-topoi. For example, it allows us to derive a formula for the pullback of ∞-topoi by means of the tensor product of presentable $\mathcal{B}$-categories, see Proposition 6.2.7.1. It will also imply that the localisation functor of every subtopos of $\mathcal{B}$ admits a quite explicit description in terms of an internal colimits, akin to Lurie's sheafification formula for sheaf ∞-topoi (Proposition 6.2.10.14).

In the final section of this chapter (Section 6.3), we study *localic* $\mathcal{B}$-topoi, which are the analogue of localic ∞-topoi in internal higher category theory. This will require setting up the basic theory of internal locales, which we call *$\mathcal{B}$-locales*, within our framework. We associate to every such $\mathcal{B}$-locale its *localic $\mathcal{B}$-topos of sheaves*, and we show that every localic $\mathcal{B}$-topos is of this form. If the base ∞-topos is itself localic, we then show that the theory of localic $\mathcal{B}$-topoi is equivalent to that of localic ∞-topoi with a structure map into $\mathcal{B}$ (Proposition 6.3.6.1). In light of this correspondence, the $\mathcal{B}$-topos of sheaves on a $\mathcal{B}$-locale corresponds to the ∞-topos of sheaves on the underlying locale of global sections.

6.1. Descent

Recall that if $\mathcal{C}$ is an ∞-category with pullbacks, then the codomain fibration $d_0\colon \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a cartesian fibration and therefore classified by a functor $\mathcal{C}_{/-}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. If $\mathcal{C}$ furthermore has all colimits, one says that $\mathcal{C}$ satisfies *descent* if $\mathcal{C}_{/-}$ preserves limits [49, § 6.1.3]. The goal of this section is to discuss an analogous concept for $\mathcal{B}$-categories. We begin in Section 6.1.1 with some preliminaries on pullbacks in $\mathcal{B}$-categories, before we define the descent property in Section 6.1.2. In Section 6.1.3, we bring this condition into a more explicit form using the notion of cartesian transformations. As we later want to compare descent with a $\mathcal{B}$-categorical version of the Giraud axioms, we use the remainder of this section to relate the descent property with the notions of universality (Section 6.1.4) and disjointness (Section 6.1.5) of colimits as well as effectivity of groupoid objects (Section 6.1.6).

6.1.1. Pullbacks in $\mathcal{B}$-categories

The goal of this section is to investigate the relation between the existence of pullbacks in a $\mathcal{B}$-category and adjunctions between its slices. Everything discussed in this section is well-known for ∞-categories, and our proofs are straightforward adaptations of their ∞-categorical counterparts.

Lemma 6.1.1.1. *Let* C *be a* $\mathcal{B}$*-category and let* $c\colon 1 \to \mathsf{C}$ *be an arbitrary object. For any map* $f\colon d \to c$*, there is a pullback square*

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}_{/c}}(-, f) \arrow[r, "(\pi_c)_!"] \arrow[d] & \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), d) \arrow[d, "f_*"] \\
\mathrm{diag}(1_{\mathsf{Grpd}_{\mathcal{B}}}) \arrow[r] & \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), c)
\end{tikzcd}$$

in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/c})$*.*

Proof. Since $(\pi_c)_!$ is a right fibration and therefore a cartesian fibration in which every map is a cartesian morphism (see the discussion in Section 4.1.2), the claim follows from the very definition of cartesian morphisms and the fact that id_c is final in $\mathsf{C}_{/c}$. □

Proposition 6.1.1.2. *Let* C *be a* $\mathcal{B}$*-category with a final object* $1_{\mathsf{C}}\colon 1 \to \mathsf{C}$*. Then the following are equivalent:*

1. C *admits finite products;*
2. *for every* $A \in \mathcal{B}$ *and every object* $c\colon 1 \to \pi_A^*\mathsf{C}$*, the forgetful functor* $(\pi_c)_!\colon (\pi_A^*\mathsf{C})_{/c} \to \pi_A^*\mathsf{C}$ *admits a right adjoint* π_c^**.*

Moreover, if either of these conditions is satisfied, the composition $(\pi_c)_!\pi_c^*$ *is equivalent to the endofunctor* $- \times c$ *on* $\pi_A^*\mathsf{C}$*.*

Proof. Let us first assume that C admits finite products. As this implies that π_A^* admits finite products as well, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and C with $\pi_A^*\mathsf{C}$, so that we can reduce to $A \simeq 1$. Suppose that $d\colon 1 \to \mathsf{C}$ is an arbitrary object. On account of the equivalence $1_{\mathsf{C}} \times c \simeq c$, we have a commutative square

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}"] \arrow[d, "d_0"] & 1 \arrow[r, "\mathrm{id}_c"] \arrow[d, "1_{\mathsf{C}}"'] & \mathsf{C}_{/c} \arrow[d, "(\pi_c)_!"] \\
\Delta^1 \arrow[r, "\pi_d"] & \mathsf{C} \arrow[r, "-\times c"] \arrow[ur, dashed, "\pi_c^*"] & \mathsf{C}
\end{tikzcd}$$

(in which $\pi_d\colon d \to 1_{\mathsf{C}}$ denotes the unique map), and since 1_{C} is final, the lift π_c^* exists. Note that the projection $\mathrm{pr}_0\colon -\times c \to \mathrm{id}_{\mathsf{C}}$ defines a map $\epsilon\colon (\pi_c)_!\pi_c^* \to \mathrm{id}_{\mathsf{C}}$.

Now the fact that π_c^* by construction preserves final objects implies that this functor carries the unique map $\pi_d \colon d \to 1_{\mathsf{C}}$ to the unique map $\pi_{\pi_c^*(d)} \colon \pi_c^*(d) \to \mathrm{id}_c$. As this implies that the image of $\pi_{\pi_c^*(d)}$ along $(\pi_c)_!$ recovers the projection $\mathrm{pr}_1 \colon d \times \pi_A^*(c) \to c$, the commutative square

$$\begin{tikzcd} \mathrm{map}_{\mathsf{C}}(-, (\pi_c)_! \pi_c^*(d)) \arrow[r, "\epsilon_*"] \arrow[d, "(\pi_c)_!(\pi_{\pi_c^*(d)})_*"] & \mathrm{map}_{\mathsf{C}}(-, d) \arrow[d] \\ \mathrm{map}_{\mathsf{C}}(-, c) \arrow[r] & \mathrm{diag}(1_{\mathsf{Grpd}_{\mathcal{B}}}) \end{tikzcd}$$

is a pullback in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Together with Lemma 6.1.1.1, this shows that the composition

$$\mathrm{map}_{\mathsf{C}_{/c}}(-, \pi_c^*(d)) \xrightarrow{(\pi_c)_!} \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), (\pi_c)_! \pi_c^*(d)) \xrightarrow{\epsilon_*} \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), d)$$

is an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and C with $\pi_A^* \mathsf{C}$, the same assertion is true for any object $d \colon A \to \mathsf{C}$. Hence π_c^* is right adjoint to $(\pi_c)_!$.

Conversely, suppose that for every $A \in \mathcal{B}$ and every $c \colon 1 \to \pi_A^* \mathsf{C}$ the map $(\pi_c)_!$ admits a right adjoint π_c^*. Our goal is to show that C admits finite products. By induction, it suffices to consider binary products. Given any pair of objects $(c, d) \colon A \to \mathsf{C} \times \mathsf{C}$, we need to show that the presheaf $\mathrm{map}_{\pi_A^* \mathsf{C}}(\mathrm{diag}(-), (c, d))$ is representable. By replacing C with $\pi_A^* \mathsf{C}$ and c, d with their transpose, we may again assume that $A \simeq 1$. Let us show that the object $(\pi_c)_! \pi_c^*(d)$ represents this presheaf. Note that there is a pullback square

$$\begin{tikzcd} \mathrm{map}_{\mathsf{C} \times \mathsf{C}}(\mathrm{diag}(-), (c, d)) \arrow[r] \arrow[d] & \mathrm{map}_{\mathsf{C}}(-, d) \arrow[d] \\ \mathrm{map}_{\mathsf{C}}(-, c) \arrow[r] & \mathrm{diag}(1_{\mathsf{Grpd}_{\mathcal{B}}}). \end{tikzcd}$$

To complete the proof, it therefore suffices to show that the maps

$$\mathrm{map}_{\mathsf{C}}(-, c) \xleftarrow{((\pi_c)_! \pi_{\pi_c^*(d)})_*} \mathrm{map}_{\mathsf{C}}(-, (\pi_c)_! \pi_c^*(d)) \xrightarrow{\epsilon_*} \mathrm{map}_{\mathsf{C}}(-, d)$$

exhibit $\mathrm{map}_{\mathsf{C}}(-, (\pi_c)_! \pi_c^*(d))$ as a product of $\mathrm{map}_{\mathsf{C}}(-, c)$ and $\mathrm{map}_{\mathsf{C}}(-, d)$. By the object-wise criterion for equivalences and Corollary 3.1.1.9, this follows once we show that for every $z\colon 1 \to \mathsf{C}$ the commutative square

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}}(z, (\pi_c)_! \pi_c^*(d)) \arrow[r, "\epsilon_*"] \arrow[d, "((\pi_c)_! \pi_{\pi_c^*(d)})_*"] & \mathrm{map}_{\mathsf{C}}(z, d) \arrow[d] \\
\mathrm{map}_{\mathsf{C}}(z, c) \arrow[r] & 1
\end{tikzcd}$$

is a pullback square in $\mathcal{B}$. By descent in $\mathcal{B}$ and Lemma 6.1.1.1, this is the case as soon as we verify that for any map $f\colon \pi_A^*(z) \to \pi_A^*(c)$ in context $A \in \mathcal{B}$ the composition

$$\mathrm{map}_{\mathsf{C}_{/c}}(f, \pi_A^* \pi_c^*(d)) \xrightarrow{(\pi_c)_!} \mathrm{map}_{\mathsf{C}}(\pi_A^*(z), \pi_A^*(\pi_c)_! \pi_c^*(d)) \xrightarrow{\pi_A^*(\epsilon)_*} \mathrm{map}_{\mathsf{C}}(\pi_A^*(z), \pi_A^*(d))$$

is an equivalence. Since this is just the adjunction property of $(\pi_c)_! \dashv \pi_c^*$, the claim follows. $\square$

Remark 6.1.1.3. In the situation of Proposition 6.1.1.2, the proof shows that in light of the equivalence $(\pi_c)_! \pi_c^* \simeq - \times c$, the counit of the adjunction $(\pi_c)_! \dashv \pi_c^*$ can be identified with $\mathrm{pr}_0 \colon - \times c \to \mathrm{id}_{\pi_A^* \mathsf{C}}$. Similarly, if $d \to c$ is an arbitrary map in context $A \in \mathcal{B}$, the unit $d \to \pi_c^*(\pi_c)_!$ is characterised by the condition that the composition

$$(\pi_c)_! d \to (\pi_c)_! \pi_c^*(\pi_c)_! d \simeq ((\pi_c)_! d) \times c \to (\pi_c)_! d$$

is equivalent to the identity. It is thus determined by the map $(\pi_c)_! d \to ((\pi_c)_! d) \times c$ that is given by the identity on the first factor and the structure map $d \to c$ on the second factor.

Corollary 6.1.1.4. *For any $\mathcal{B}$-category C, the following are equivalent:*

1. *C admits pullbacks;*
2. *for every map $f\colon c \to d$ in C in context $A \in \mathcal{B}$, the projection $f_! \colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ admits a right adjoint f^*.*

Moreover, if either of these conditions are satisfied, then the composition $f_! f^*$ can be identified with the pullback functor $- \times_d c$.

Proof. In light of Proposition 6.1.1.2, it will be enough to show that C admits pullbacks if and only if for every object $c: A \to \mathsf{C}$ the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/c}$ admits binary products. Using Example 3.2.1.13, this is easily reduced to the corresponding statement for ∞-categories, which appears as [18, Theorem 6.6.9]. $\square$

Corollary 6.1.1.5. *Let $(l \dashv r): \mathsf{C} \to \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories, and suppose that C admits pullbacks. Then for any object $d: A \to \mathsf{D}$ in D in context $A \in \mathcal{B}$, the induced functor $l_{/d}: \mathsf{D}_{/d} \to \mathsf{C}_{/l(c)}$ admits a right adjoint r_d that is explicitly given by the composition*

$$r_d: \mathsf{C}_{/l(d)} \xrightarrow{r_{/l(d)}} \mathsf{D}_{/rl(d)} \xrightarrow{(\eta d)^*} \mathsf{D}_{/d}$$

in which $(\eta d)^*$ is the pullback functor along the adjunction unit $\eta d: d \to rl(d)$.

Proof. Since C has pullbacks, Corollary 6.1.1.4 shows that the functor $(\eta d)^*$ indeed exists and is right adjoint to the projection $(\eta d)_!$. Now by Proposition 3.1.1.15, the functor $r_{/l(d)}: \mathsf{C}_{/l(d)} \to \mathsf{D}_{/rl(d)}$ admits a left adjoint $l_{r(d)}$ that is given by the composition $(\epsilon l(d))_! l_{/rl(d)}$. Therefore, the functor r_d is right adjoint to the composition

$$D_{/d} \xrightarrow{(\eta d)_!} \mathsf{D}_{/rl(d)} \xrightarrow{l_{/rl(d)}} C_{/lrl(d)} \xrightarrow{(\epsilon l(d))_!} \mathsf{C}_{/l(d)}.$$

It now suffices to notice that on account of the triangle identities, this functor is equivalent to $l_{/d}$. $\square$

Next, our goal is to describe the property that a $\mathcal{B}$-category C admits pullbacks in terms of the codomain fibration $d_0: \mathsf{C}^{\Delta^1} \to \mathsf{C}$. We begin with establishing that this map is always a cocartesian fibration:

Lemma 6.1.1.6. *For any $\mathcal{B}$-category C, the codomain fibration $d_0: \mathsf{C}^{\Delta^1} \to \mathsf{C}$ is a cocartesian fibration.*

Proof. In light of the identification $\Delta^1 \times \Delta^1 \simeq \Delta^2 \sqcup_{\Delta^1} \Delta^2$, the restriction functor

$$\mathrm{res}_{d_0}: \mathsf{C}^{\Delta^1 \times \Delta^1} \to \mathsf{C}^{\Delta^1} \downarrow_{\mathsf{C}} \mathsf{C}$$

can be identified with the map

$$\mathsf{C}^{\Delta^2 \sqcup_{\Delta^1} \Delta^2} \to \mathsf{C}^{\Delta^2}$$

that is given by precomposition with the inclusion $\Delta^2 \hookrightarrow \Delta^2 \sqcup_{\Delta^1} \Delta^2$ of the first summand. The latter admits a retraction

$$(\mathrm{id}, s_2) \colon \Delta^2 \sqcup_{\Delta^1} \Delta^2 \to \Delta^2$$

which is right adjoint to the inclusion. Hence precomposition with this retraction defines the desired fully faithful left adjoint lift_{d_1} of res_{d_1}. □

By the $\mathcal{B}$-categorical straightening equivalence (Theorem 4.4.3.1), the cocartesian fibration $d_0 \colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ gives rise to a functor $\mathsf{C}_{/-} \colon \mathsf{C} \to \mathsf{Cat}_{\mathcal{B}}$. Note that the map $(d_1, d_0) \colon \mathsf{C}^{\Delta^1} \to \mathsf{C} \times \mathsf{C}$ can be regarded as a morphism of cocartesian fibrations over C, where we regard the codomain as a cocartesian fibration over C by virtue of the projection onto the second factor. Therefore, one obtains an induced map $\mathsf{C}_{/-} \to \mathrm{diag}(\mathsf{C})$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$, where $\mathrm{diag}(\mathsf{C})$ is the constant functor with value C. Equivalently, we may regard $\mathsf{C}_{/-}$ as a functor $\mathsf{C} \to (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$. By construction, if $c \colon A \to \mathsf{C}$ is an arbitrary object in context $A \in \mathcal{B}$, the induced map $\mathsf{C}_{/c} \to \pi_A^* \mathsf{C}$ is precisely given by the projection $(\pi_c)_!$ and therefore in particular a right fibration. Thus, the functor $\mathsf{C}_{/-}$ takes values in $\mathsf{RFib}_{\mathsf{C}}$. In particular, this implies that for any map $f \colon c \to d$ in C (in arbitrary context), the induced functor $\mathsf{C}_{/c} \to \mathsf{C}_{/d}$ is a right fibration. On account of the orthogonality between right fibrations and final functors, this map is uniquely determined by the image of the final object id_c. As it is moreover evident from the construction of $\mathsf{C}_{/-}$ that the image of id_c is given by f, we thus conclude that $\mathsf{C}_{/-}$ acts on maps by carrying f to the functor $f_! \colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ that is obtained as the image of f under the Yoneda embedding $\mathsf{C} \hookrightarrow \mathsf{RFib}_{\mathsf{C}}$.

Proposition 6.1.1.7. *Let* C *be a* $\mathcal{B}$*-category. Then the following are equivalent:*

1. *The codomain fibration* $d_0 \colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ *is a cartesian fibration;*
2. *for every map* $f \colon c \to d$ *in* C*, the functor* $f_! \colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ *admits a right adjoint* f^**;*
3. C *admits pullbacks.*

Proof. By combining Lemma 6.1.1.6 and Lemma 4.5.2.2 with Corollary 4.4.5.5, the functor d_0 is a cartesian fibration if and only if for every map $f \colon c \to d$ in C in context $A \in \mathcal{B}$ the functor $f_! \colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ admits a right adjoint f^*. Hence (1) and (2) are equivalent. The fact that (2) and (3) are equivalent has already been established in Corollary 6.1.1.4. □

Remark 6.1.1.8. In the situation of Proposition 6.1.1.7, note that if $d_0 \colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ is a cartesian fibration, then a morphism in $\sigma \colon \Delta^1 \otimes A \to \mathsf{C}^{\Delta^1}$ in context $A \in \mathcal{B}$ is cartesian if and only if it transposes to a pullback square in C. In fact, by Remark 4.1.2.8 the map σ is cartesian precisely if it is in the essential image of $\mathrm{lift}_{d_0} \colon \mathsf{C} \downarrow_{\mathsf{C}} \mathsf{C}^{\Delta^1} \hookrightarrow \mathsf{C}^{\Delta^1 \times \Delta^1}$. Unwinding the definitions, we can identify this map with the inclusion

$$\iota_* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Lambda^2_2, \mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}((\Lambda^2_2)^{\triangleleft}, \mathsf{C}),$$

so that the claim follows from Proposition 3.4.4.1.

6.1.2. The definition of descent

In order to define the descent property of a $\mathcal{B}$-category C, we first need to construct the functor $\mathsf{C}_{/-} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$. As we have the straightening equivalence for cartesian fibrations at our disposal (Theorem 4.4.3.1), we may proceed in the same fashion as in [49].

If C is a $\mathcal{B}$-category with pullbacks, Proposition 6.1.1.7 implies that the functor $d_0 \colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ is a cartesian fibration. By applying the straightening functor to this map, we therefore obtain a functor $\mathsf{C}_{/-} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$. By the discussion in Section 4.5.2, this functor is equivalently obtained by observing that the straightening of the *cocartesian* fibration $\mathsf{C}^{\Delta^1} \to \mathsf{C}$ takes values in $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}}$ and by applying the equivalence $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}} \simeq (\mathsf{Cat}^{\mathrm{R}}_{\mathcal{B}})^{\mathrm{op}}$ from Proposition 4.5.2.1. We may now define:

Definition 6.1.2.1. Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with finite limits. We say that C *satisfies* U*-descent* if the functor $\mathsf{C}_{/-} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is $\mathrm{op}(\mathsf{U})$-continuous. If X is a cocomplete large $\mathcal{B}$-category, we simply say that C satisfies descent if C satisfies $\mathsf{Cat}_{\mathcal{B}}$-descent.

Remark 6.1.2.2. The property of a U-cocomplete $\mathcal{B}$-category C with pullbacks to satisfy U-descent is local in $\mathcal{B}$: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then C satisfies U-descent if and only if $\pi_{A_i}^* \mathsf{C}$ satisfies $\pi_{A_i}^* \mathsf{U}$-descent for all i. This follows immediately from the locality of U-continuity (Remark 3.3.2.3) and from Remark 1.4.2.4.

Example 6.1.2.3. Let $\mathcal{K}$ be a class of ∞-categories and let C be an $\mathsf{LConst}_{\mathcal{K}}$-cocomplete $\mathcal{B}$-category with pullbacks (where $\mathsf{LConst}_{\mathcal{K}}$ is the $\mathcal{B}$-category of locally $\mathcal{K}$-constant $\mathcal{B}$-categories, see Example 3.3.1.4). Then C satisfies $\mathsf{LConst}_{\mathcal{K}}$-descent if and only if for all $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ satisfies $\mathcal{K}$-descent. In fact, by Corollary 4.4.4.11 the composition $\Gamma_{\mathcal{B}_{/A}} \circ \mathsf{C}_{/-}(A)$ recovers the functor

$$\mathsf{C}(A)_{/-} \colon \mathsf{C}(A)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}.$$

Consequently, if $s \colon B \to A$ is an arbitrary map in $\mathcal{B}$, postcomposing $\mathsf{C}_{/-}(A)$ with the evaluation functor $\mathrm{ev}_B \colon \mathrm{Cat}(\mathcal{B}_{/A}) \to \widehat{\mathrm{Cat}}_{\infty}$ recovers the composition $\mathsf{C}_{/-}(B) \circ s^*$. Using that C is $\mathsf{LConst}_{\mathcal{K}}$-cocomplete, we find that the restriction functor $s^* \colon \mathsf{C}(A)^{\mathrm{op}} \to \mathsf{C}(B)^{\mathrm{op}}$ is $\mathrm{op}(\mathcal{K})$-continuous, and since limits in $\mathrm{Cat}(\mathcal{B}_{/A})$ are detected section-wise, the claim follows.

6.1.3. Cartesian transformations

The main goal of this section is to obtain a more explicit description of the descent property which will rely on the notion of *cartesian* morphisms of functors:

Definition 6.1.3.1. Let I and C be $\mathcal{B}$-categories such that C admits pullbacks. We say that a map $\phi \colon d \to d'$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in context $1 \in \mathcal{B}$ is *cartesian* if for every map $i \to i'$ in I in context $A \in \mathcal{B}$ the induced commutative square

$$\begin{tikzcd} d(i) \arrow[r] \arrow[d] & d'(i) \arrow[d] \\ d(i') \arrow[r] & d'(i') \end{tikzcd}$$

is a pullback in $\mathsf{C}(A)$. A map $d \to d'$ in context $A \in \mathcal{B}$ is called cartesian if it is cartesian when viewed as a map in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{I}, \pi_A^* \mathsf{C})$ in context $1 \in \mathcal{B}_{/A}$. We denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ that is spanned by the cartesian maps in arbitrary context $A \in \mathcal{B}$.

Remark 6.1.3.2. In the situation of Definition 6.1.3.1, the property of a map $\phi d \to d'$ in context $A \in \mathcal{B}$ being cartesian is local in $\mathcal{B}$: if $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ is a cover in $\mathcal{B}$, then ϕ is cartesian if and only if each $s_i^*(\phi)$ is. In fact, by unwinding the definition, this follows from the fact that the property of a commutative square being a pullback is local in that sense. As a consequence, every object of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})^{\mathrm{cart}}_{/d}$ in context A encodes a cartesian map $d \to d'$, and there is a canonical equivalence $\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})^{\mathrm{cart}}_{/d} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{I},\pi_A^*\mathsf{C})^{\mathrm{cart}}_{/\pi_A^* d}$ of $\mathcal{B}_{/A}$-categories.

Lemma 6.1.3.3. *Let* I *and* C *be* $\mathcal{B}$*-categories, and suppose that* C *admits pullbacks. Then a map* $d \to d'$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ *(in arbitrary context) is cartesian if and only if the associated object in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}^{\Delta^1})$ *is contained in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},(\mathsf{C}^{\Delta^1})_\#)$*, where* $(\mathsf{C}^{\Delta^1})_\# \hookrightarrow \mathsf{C}^{\Delta^1}$ *is the subcategory that is spanned by the cartesian morphisms over* $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$.

Proof. This follows immediately from the description of the cartesian morphisms in C^{Δ^1} as pullback squares in C, see Remark 6.1.1.8. □

The main goal of this section is to prove the following description of the descent property:

Proposition 6.1.3.4. *Let* C *be a cocomplete* $\mathcal{B}$*-category with pullbacks and let* $d\colon \mathsf{I} \to \mathsf{C}$ *be a diagram that admits a colimit in* C*. Let* $\overline{d}\colon \mathsf{I}^{\triangleright} \to \mathsf{C}$ *be the corresponding colimit cocone. Then the functor* $\mathsf{C}_{/-}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$ *carries* $\overline{d}$ *to a limit cone in* $\mathsf{Cat}_{\mathcal{B}}$ *if and only if the restriction map* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright},\mathsf{C})_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})_{/d}$ *restricts to an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright},\mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})^{\mathrm{cart}}_{/d}$$

of $\mathcal{B}$*-categories.*

The main idea for the proof of Proposition 6.1.3.4 is to identify the left-hand side of the equivalence with $\mathsf{C}_{/\operatorname{colim} d}$ and the right-hand side with $\lim \mathsf{C}_{/d(-)}$. In order to do so, we will need the formula for limits in $\mathsf{Cat}_{\mathcal{B}}$ that we derived in Proposition 4.5.1.2. For the convenience of the reader, we will briefly recall the main setup from Chapter 4. The ∞-topos of *marked simplicial objects* in $\mathcal{B}$ is defined as $\mathcal{B}^+_\Delta = \mathrm{Fun}(\Delta_+^{\mathrm{op}},\mathcal{B})$, where Δ_+ denotes the marked simplex 1-category. Precomposition with the inclusion $\Delta \hookrightarrow \Delta_+$ induces a forgetful functor $(-)|_\Delta\colon \mathcal{B}^+_\Delta \to \mathcal{B}_\Delta$ which admits a left adjoint $(-)^\flat$ and a right adjoint $(-)^\sharp$. Every cartesian fibration $p\colon \mathsf{P} \to \mathsf{C}$ can be equivalently encoded by a *marked cartesian fibration* $p^\natural\colon \mathsf{P}^\natural \to \mathsf{C}^\sharp$, where $\mathsf{P}^\natural$ is the marked simplicial object that is obtained from P by marking the cartesian arrows and where a marked cartesian fibration is by definition a map that is internally right orthogonal to the collection of *marked right anodyne maps* (see Definition 4.2.2.1). Now if $d\colon \mathsf{I}^{\mathrm{op}} \to \mathrm{Cat}_{\mathcal{B}}$ is a functor and if $p\colon \mathsf{P} \to \mathsf{I}$ is the associated cartesian fibration, one obtains a canonical equivalence

$$\lim d \simeq (\underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(\mathsf{I}^\sharp, \mathsf{P}^\natural)_{/\mathsf{I}^\sharp})|_\Delta,$$

where the right-hand side is defined via the pullback diagram

$$\begin{tikzcd} (\underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(\mathsf{I}^\sharp, \mathsf{P}^\natural)_{/\mathsf{I}^\sharp})|_\Delta \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(\mathsf{I}^\sharp, \mathsf{P}^\natural)|_\Delta \arrow[d] \\ 1 \arrow[r, "\mathrm{id}_{\mathsf{I}^\sharp}"] & \underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(\mathsf{I}^\sharp, \mathsf{I}^\sharp)|_\Delta \end{tikzcd}$$

(in which $\underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(-,-)$ denotes the internal hom in $\mathcal{B}^+_\Delta$).

Lemma 6.1.3.5. *Let K be a simplicial object in $\mathcal{B}$ and let $p\colon \mathsf{P} \to \mathsf{C}$ be a cartesian fibration. Then the canonical map $K^\flat \to K^\sharp$ of marked simplicial objects in $\mathcal{B}$ induces a fully faithful functor $\underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(K^\sharp, \mathsf{P}^\natural)|_\Delta \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(K^\flat, \mathsf{P}^\natural)|_\Delta$ of $\mathcal{B}$-categories.*

Proof. Let M be the marked simplicial object in $\mathcal{B}$ that fits into the pushout square

$$\begin{tikzcd} (\Delta^0 \sqcup \Delta^0)^\flat \otimes K^\flat \arrow[r] \arrow[d] & (\Delta^0 \sqcup \Delta^0)^\flat \otimes K^\sharp \arrow[d] \\ (\Delta^1)^\flat \otimes K^\flat \arrow[r] & M. \end{tikzcd}$$

Unwinding the definitions, we need to show that $\mathsf{P}^\natural$ is internally local with respect to the induced map $\phi\colon M \to (\Delta^1)^\flat \otimes K^\sharp$. Since $\mathsf{C}^\sharp$ is easily seen to be internally local with respect to ϕ, this follows once we show that $p^\natural$ is internally right orthogonal to this map. We therefore need to verify that ϕ is marked right anodyne. Writing K as a colimit of objects of the form $\Delta^n \otimes A$, we may assume that $K = \Delta^n \otimes A$. Moreover, since marked right anodyne morphisms are closed under products, we can assume that $A \simeq 1$. Using that the two maps $(I^n)^\flat \hookrightarrow (\Delta^n)^\flat$ and $(I^n)^\sharp \hookrightarrow (\Delta^n)^\sharp$ that are induced by the spine inclusions are marked right anodyne, we may further reduce this to $K = \Delta^1$. In this case, one can apply Lemma 4.2.2.3 to deduce that ϕ is an equivalence. Hence the claim follows. □

Lemma 6.1.3.6. *Let $s\colon B \to A$ be a map in $\mathcal{B}$, and let $P \to A$ be an arbitrary map. Let $\eta_s\colon \operatorname{id}_{\mathcal{B}_{/A}} \to s_* s^*$ be the adjunction unit. Then the value of the natural transformation $(\pi_A)_* \xrightarrow{(\pi_A)_*\eta_s} (\pi_A)_* s_* s^* \simeq (\pi_B)_* s^*$ at an object $p\colon P \to A$ in $\mathcal{B}_{/A}$ can be identified with the map*

$$\underline{\operatorname{Hom}}_{\mathcal{B}}(A, P)_{/A} \to \underline{\operatorname{Hom}}_{\mathcal{B}}(B, s^* P)_{/B}$$

that is induced by precomposition with s. Here $\underline{\operatorname{Hom}}_{\mathcal{B}}(A, P)_{/A}$ is the fibre of the morphism $\underline{\operatorname{Hom}}_{\mathcal{B}}(A, P) \to \underline{\operatorname{Hom}}_{\mathcal{B}}(A, A)$ over id_A, and $\underline{\operatorname{Hom}}_{\mathcal{B}}(B, s^* P)_{/B}$ is the fibre of the morphism $\underline{\operatorname{Hom}}_{\mathcal{B}}(B, s^* P) \to \underline{\operatorname{Hom}}_{\mathcal{B}}(B, B)$ over id_B.

Proof. Since the morphism $\operatorname{id} \times s\colon - \times B \to - \times A$ can be identified with the composition

$$(\pi_B)_! \pi_B^* \xrightarrow{\simeq} (\pi_A)_! s_! s^* \pi_A^* \xrightarrow{(\pi_A)_! \epsilon_s \pi_A^*} (\pi_A)_! \pi_A^*$$

(in which ϵ_s is the counit of the adjunction $s_! \dashv s^*$), it follows by adjunction that the map $(\pi_A)_* \eta_s \pi_A^*$ can be identified with $s^*\colon \underline{\operatorname{Hom}}_{\mathcal{B}}(A, -) \to \underline{\operatorname{Hom}}_{\mathcal{B}}(B, -)$. Now if $p\colon P \to A$ is any map, the unique morphism $p \to \operatorname{id}_{\mathcal{B}_{/A}}$ in $\mathcal{B}_{/A}$ is the pullback of $\pi_A^*(\pi_A)_! p \to \pi_A^*(\pi_A)_! 1_{\mathcal{B}_{/A}}$. Together with naturality of η_s, this implies that the map $(\pi_A)_* \eta_s(p)$ fits into a commutative diagram

$$\begin{tikzcd}
 & s_* s^* (\pi_A)_*(p) \arrow[rr] \arrow[dd] & & \underline{\operatorname{Hom}}_{\mathcal{B}}(B, P) \arrow[dd, "p_*"] \\
(\pi_A)_*(p) \arrow[ur, "(\pi_A)_*\eta_s(p)"] \arrow[rr] \arrow[dd] & & \underline{\operatorname{Hom}}_{\mathcal{B}}(A, P) \arrow[ur, "s^*"] \arrow[dd, "p_*"] & \\
 & 1 \arrow[rr, "s"] & & \underline{\operatorname{Hom}}_{\mathcal{B}}(B, A) \\
1 \arrow[ur] \arrow[rr, "\operatorname{id}_A"] & & \underline{\operatorname{Hom}}_{\mathcal{B}}(A, A) \arrow[ur, "s^*"] &
\end{tikzcd}$$

in which the front and the back square are pullbacks. As the fibre of

$$p_*\colon \underline{\operatorname{Hom}}_{\mathcal{B}}(B, P) \to \underline{\operatorname{Hom}}_{\mathcal{B}}(B, A)$$

over s can be identified with $\underline{\operatorname{Hom}}_{\mathcal{B}}(B, s^* P)_{/B}$, the claim follows. $\square$

Proof of Proposition 6.1.3.4. Let $\iota\colon \mathsf{I} \hookrightarrow \mathsf{I}^{\triangleleft}$ be the inclusion. Since $\mathsf{Cat}_{\mathcal{B}}$ is complete, the theory of Kan extensions gives rise to an adjunction

$$(\iota^* \dashv \iota_*)\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathsf{Cat}_{\mathcal{B}}) \rightleftarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Cat}_{\mathcal{B}}),$$

see Section 3.4.3. Given any diagram $\overline{h}\colon \mathsf{I}^{\triangleleft} \to \mathsf{Cat}_{\mathcal{B}}$, we will let $h = \iota^*\overline{h}$, and we denote by $\eta_{\overline{h}}\colon \overline{h} \to \iota_* h$ the adjunction unit. Now let us set $\overline{h} = \mathsf{C}_{/\overline{d}(-)}$, so that we get $h = \mathsf{C}_{/d(-)}$. Furthermore, let $p\colon \mathsf{P} \to \mathsf{I}^{\triangleright}$ be the pullback of $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ along $\overline{d}$, and let $q\colon \mathsf{Q} \to \mathsf{I}$ be the pullback of d_0 along d. According to Proposition 4.5.1.2 and Lemma 6.1.3.6 (applied to the ∞-topos $\mathcal{B}^+_{\Delta}$ and the map $\iota^{\#}$), the canonical map $\lim \eta_{\overline{h}}\colon \lim \overline{h} \to \lim \iota_* h$ can be identified with the functor

$$\underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}((\mathsf{I}^{\triangleright})^{\#}, \mathsf{P}^{\natural})_{/(\mathsf{I}^{\triangleright})^{\#}}|_{\Delta} \to \underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}(\mathsf{I}^{\#}, \mathsf{Q}^{\natural})_{/\mathsf{I}^{\#}}|_{\Delta} \qquad (*)$$

that is induced by precomposition with the inclusion $\iota\colon \mathsf{I} \hookrightarrow \mathsf{I}^{\triangleright}$. As $\mathsf{C}_{/-}$ preserving the limit of d is therefore equivalent to $(*)$ being an equivalence, we only need to identify this map with the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$.

To see this, first note that there is an equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}((-)^{\flat}, -)|_{\Delta} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(-, (-)|_{\Delta}).$$

Therefore, we obtain a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}((\mathsf{I}^{\triangleright})^{\#}, \mathsf{P}^{\natural})_{/(\mathsf{I}^{\triangleright})^{\#}}|_{\Delta} \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}((\mathsf{I}^{\triangleright})^{\#}, \mathsf{P}^{\natural})|_{\Delta} \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}((\mathsf{I}^{\triangleright})^{\#}, (\mathsf{C}^{\Delta^1})^{\natural})|_{\Delta} \arrow[d, hook] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{P}) \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C}^{\Delta^1}) \arrow[d] \\
1 \arrow[r, "\mathrm{id}_{\mathsf{I}^{\triangleright}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{I}^{\triangleright}) \arrow[r, "\overline{d}_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})
\end{tikzcd}$$

in which the upper three vertical maps are induced by precomposition with the canonical map $(\mathsf{I}^{\triangleright})^{\flat} \to (\mathsf{I}^{\triangleright})^{\#}$. By Lemma 6.1.3.5, they are fully faithful. Furthermore, all but the upper right square are pullbacks. But since the map $(I^{\triangleright})^{\flat} \to (I^{\triangleright})^{\#}$ is internally right orthogonal to every map that is contained in the image of $(-)^{\#}\colon \mathcal{B}_{\Delta} \hookrightarrow \mathcal{B}^+_{\Delta}$, it must also be internally right orthogonal to $\mathsf{P}^{\natural} \to (\mathsf{C}^{\Delta^1})^{\natural}$ as the latter is the pullback of $\overline{d}^{\#}$. Therefore, the upper right square must also be a pullback. Note, furthermore, that since the map $(-)^{\flat} \to (-)^{\#}$ is an equivalence when restricted along the inclusion $\mathcal{B} \hookrightarrow \mathcal{B}_{\Delta}$, the upper right inclusion in the above diagram identifies the domain with the essential image of the map $\underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}(\mathsf{I}^{\triangleright}, (\mathsf{C}^{\Delta^1})_{\#}) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{B}^+_{\Delta}}(\mathsf{I}^{\triangleright}, \mathsf{C}^{\Delta^1})$. Therefore, Lemma 6.1.3.3 implies that there is an equivalence $\underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}((\mathsf{I}^{\triangleright})^\sharp, \mathsf{P}^\natural)_{/(\mathsf{I}^{\triangleright})^\sharp}|_\Delta \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\bar{d}}$. By an analogous argument, one obtains an equivalence $\underline{\mathrm{Hom}}_{\mathcal{B}^+_\Delta}(\mathsf{I}^\sharp, \mathsf{Q}^\natural)_{/\mathsf{I}^\sharp}|_\Delta \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$, hence the claim follows. □

Remark 6.1.3.7. In the situation of Proposition 6.1.3.4, let $\infty \colon 1 \to \mathsf{I}^{\triangleleft}$ be the cone point, and let

$$(\infty^* \dashv \infty_*) \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathsf{Cat}_{\mathcal{B}}) \rightleftarrows \mathsf{Cat}$$

be the induced adjunction. Let $\eta \colon \mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathsf{Cat}_{\mathcal{B}})} \to \infty_*\infty^*$ be the adjunction unit. By the same argument as in the proof of Proposition 6.1.3.4, evaluating $\lim_{\mathsf{I}^{\triangleleft}} \eta$ at the cone $\mathsf{C}_{/\bar{d}}$ recovers the restriction map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\bar{d}} \to \mathsf{C}_{/c}.$$

Since $\lim_{\mathsf{I}^{\triangleleft}}$ can be identified with ∞^* (owing to $\infty \colon 1 \to \mathsf{I}^{\triangleleft}$ being initial), the triangle identities imply that this map must be an equivalence. Furthermore, note that by Corollary 6.1.1.5 the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\bar{d}} \to \mathsf{C}_{/c}$ admits a right adjoint that is is given by the composition

$$\mathsf{C}_{/c} \xrightarrow{\mathrm{diag}_{/c}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\,\mathrm{diag}(c)} \xrightarrow{\eta^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\bar{d}}$$

(where $\eta \colon \bar{d} \to \mathrm{diag}(c)$ denotes the adjunction unit). Now if $c' \to c$ is a map in C, Corollary 6.1.1.4 implies that the counit $\eta_!\eta^* \,\mathrm{diag}(c') \to \mathrm{diag}(c')$ of the adjunction $\eta_! \dashv \eta^*$ is given by the pullback of η along $\mathrm{diag}(c') \to \mathrm{diag}(c)$. Since evaluation at ∞ preserves pullbacks and since $\mathrm{diag}_{/c}$ is fully faithful, we conclude that evaluating the counit of the adjunction $\mathsf{C}_{/c} \rightleftarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\bar{d}}$ at $c' \to c$ must result in an equivalence. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the same argument, we conclude that the entire counit must be an equivalence, so that the functor $\mathsf{C}_{/c} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\bar{d}}$ is fully faithful. Now combining the evident observation that this map takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\bar{d}}$ with the fact that the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\bar{d}} \to \mathsf{C}_{/c}$ is an equivalence, one concludes that the inclusion $\mathsf{C}_{/c} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\bar{d}}$ identifies $\mathsf{C}_{/c}$ with $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\bar{d}}$. In particular, the inclusion $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\bar{d}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\bar{d}}$ admits a left adjoint.

Remark 6.1.3.8. In the situation of Remark 6.1.3.7, let $\overline{d}' \to \overline{d}$ be a map in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$, and let us set $c' = \infty^*(\overline{d}')$. Then the unit of the adjunction

$$\mathsf{C}_{/c} \rightleftarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}}$$

evaluates at $\overline{d}' \to \overline{d}$ to the natural map $\overline{d}' \to \eta^* \operatorname{diag}(c')$. Therefore, the map $\overline{d}' \to \overline{d}$ is cartesian precisely if the square

$$\begin{tikzcd}
\overline{d}' \arrow[r] \arrow[d] & \operatorname{diag}(c') \arrow[d] \\
\overline{d} \arrow[r] & \operatorname{diag}(c)
\end{tikzcd}$$

is a pullback. As the functor $(\iota^*, \infty^*) \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \times \mathsf{C}$ is conservative (on account of the map $(\iota, \infty) \colon \mathsf{I} \sqcup 1 \twoheadrightarrow \mathsf{I}^{\triangleright}$ being essentially surjective) and as the image of the above square along ∞^* is always a pullback, the map $\overline{d}' \to \overline{d}$ is cartesian precisely if the square

$$\begin{tikzcd}
d' \arrow[r] \arrow[d] & \operatorname{diag}(c') \arrow[d] \\
d \arrow[r] & \operatorname{diag}(c)
\end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ is a pullback.

Suppose that U is an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. Given any $\mathsf{I} \in \mathsf{U}(1)$ and any diagram $d \colon \mathsf{I} \to \mathsf{C}$ with colimit cocone $\overline{d} \colon \mathsf{I}^{\triangleright} \to \mathsf{C}$, Proposition 3.1.1.15 implies that the functor

$$\iota^*_{/\overline{d}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/d}$$

has a left adjoint that is given by $(\iota_!)_{/d}$. Combining this observation with Remark 6.1.3.7, we thus end up with a left adjoint $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$ to the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ that we will refer to as the *glueing functor*. In light of Proposition 6.1.3.4, the functor $\mathsf{C}_{/-}$ preserves the limit of d precisely if both unit and counit of this adjunction are equivalences, i.e. if both the restriction functor and the glueing functor are fully faithful. We may therefore split up the notion of U-descent into two separate conditions:

Definition 6.1.3.9. Let U be an internal class and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. We say that C has *faithful* U*-descent* if for every $A \in \mathcal{B}$, every $\mathsf{I} \in \mathsf{U}(A)$ and every diagram $d \colon \mathsf{I} \to \pi_A^* \mathsf{C}$ with colimit cocone $\overline{d} \colon \mathsf{I}^{\triangleright} \to \pi_A^* \mathsf{C}$, the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\triangleright}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/d}$ is fully faithful. We say that C has *effective* U*-descent* if for every $A \in \mathcal{B}$, every $\mathsf{I} \in \mathsf{U}(A)$ and every diagram $d \colon \mathsf{I} \to \pi_A^* \mathsf{C}$ with colimit cocone $\overline{d} \colon \mathsf{I}^{\triangleright} \to \pi_A^* \mathsf{C}$, the glueing functor $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/d} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\triangleright}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$ is fully faithful. If C is a cocomplete large $\mathcal{B}$-category, we simply say that C has faithful/effective descent if it has faithful/effective $\mathrm{Cat}_{\mathcal{B}}$-descent.

Remark 6.1.3.10. As a consequence of Remark 6.1.3.2, the property of C having faithful/effective U-descent is local in $\mathcal{B}$, in the sense that whenever $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, the $\mathcal{B}$-category C satisfies faithful/effective U-descent precisely if for each i the $\mathcal{B}_{/A_i}$-category $\pi_{A_i}^* \mathsf{C}$ has faithful/effective $\pi_{A_i}^* \mathsf{U}$-descent.

By unwinding how the unit and counit of the adjunction

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \rightleftarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$$

are computed (cf. Remark 6.1.1.3) and by using Remark 6.1.3.8, we may characterise the notion of faithful and effective U-descent as follows:

Proposition 6.1.3.11. *Let* U *be an internal class and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category with pullbacks. Then the following are equivalent:*

1. C *has faithful* U*-descent;*

2. *for every* $A \in \mathcal{B}$*, every* $\mathsf{I} \in \mathsf{U}(A)$ *and every cartesian map* $\overline{d}' \to \overline{d}$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\triangleright}, \pi_A^* \mathsf{C})$ *in which* $\overline{d}$ *is a colimit cocone,* $\overline{d}'$ *is a colimit cocone as well;*

3. *for every* $A \in \mathcal{B}$*, every* $\mathsf{I} \in \mathsf{U}(A)$ *and every pullback diagram*

$$\begin{tikzcd} d' \arrow[r] \arrow[d] & \operatorname{diag}(c') \arrow[d, "\operatorname{diag}(g)"] \\ d \arrow[r, "\eta"] & \operatorname{diag}\operatorname{colim} d \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})$ *in which* η *is the unit of the adjunction* $\operatorname{colim} \dashv \operatorname{diag}$*, the transpose map* $\operatorname{colim} d' \to c'$ *is an equivalence.* $\square$

Proposition 6.1.3.12. *Let* U *be an internal class and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category with pullbacks. Then the following are equivalent:*

1. C *has effective* U*-descent;*

2. *for every* $A \in \mathcal{B}$*, every* $\mathsf{I} \in \mathsf{U}(A)$ *and every cartesian map* $d' \to d$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})$*, the induced map between colimit cocones* $\overline{d'} \to \overline{d}$ *is cartesian as well;*

3. *for every* $A \in \mathcal{B}$*, every* $\mathsf{I} \in \mathsf{U}(A)$ *and every cartesian map* $d' \to d$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})$*, the naturality square*

$$\begin{tikzcd} d' \arrow[r, "\eta"] \arrow[d] & \operatorname{diag}(\operatorname{colim} d') \arrow[d] \\ d \arrow[r, "\eta"] & \operatorname{diag} \operatorname{colim} d \end{tikzcd}$$

is a pullback. □

Corollary 6.1.3.13. *Let* S *be a local class of maps in* $\mathcal{B}$ *and let* C *be an* Grpd_S*-cocomplete* $\mathcal{B}$*-category with pullbacks. Then the following are equivalent:*

1. C *has* Grpd_S*-descent;*

2. *for every map* $p \colon P \to A$ *in* S *the functor* $p_! : \mathsf{C}(P) \to \mathsf{C}(A)$ *is a right fibration;*

3. *for every map* $p \colon P \to A$ *in* S *the functor* $(p_!)_{/1_{\mathsf{C}(P)}} : \mathsf{C}(P) \to \mathsf{C}(A)_{/p_!(1_{\mathsf{C}(P)})}$ *is an equivalence.*

Proof. Since $(p_!)_{/1_{\mathsf{C}(P)}}$ is always final, this functor is an equivalence if and only if it is a right fibration, which is in turn equivalent to $p_!$ being a right fibration. Hence (2) and (3) are equivalent conditions. Now in light of the adjunction $p_! \dashv p^*$, a map $f \colon c' \to c$ in $\mathsf{C}(P)$ is cartesian with respect to $p_!$ precisely if the naturality square

$$\begin{tikzcd} c' \arrow[r] \arrow[d, "f"] & p^* p_! c' \arrow[d, "p^* p_! f"] \\ c \arrow[r] & p^* p_! c \end{tikzcd}$$

is a pullback. Therefore, Proposition 6.1.3.12 and the fact that *every* map of diagrams indexed by a $\mathcal{B}$-groupoid is cartesian imply that C has effective Grpd_S-descent if and only if for every map $p\colon P \to A$ in S, every morphism in $\mathsf{C}(P)$ is cartesian with respect to $p_!$. By the same observation, Proposition 6.1.3.11 shows that C has faithful Grpd_S-descent if and only if for every map $p\colon P \to A$ in S, every object $c \in \mathsf{C}(P)$ and every morphism $g\colon c'' \to p_!(c)$ in $\mathsf{C}(A)$, the pullback of $p^*(g)$ along the adjunction unit $c \to p^*p_!c$ defines a cartesian lift of g. In other words, C has faithful Grpd_S-descent if and only if $p_!$ is a cartesian fibration. Hence (1) and (2) are equivalent. □

For the next corollary, recall from Section 5.2.1 that if $\mathcal{K}$ is a class of ∞-categories and S is a local class of morphisms in $\mathcal{B}$, we denote by $\mathsf{Cat}_{\mathcal{B}}^{\langle\mathcal{K},S\rangle}$ the *left regularisation* of the internal class $\mathsf{LConst}_{\mathcal{K}} \cup \mathsf{Grpd}_S$, i.e. the smallest internal class that contains $\Delta \cup \mathsf{LConst}_{\mathcal{K}} \cup \mathsf{Grpd}_S$ and that is closed under $\mathsf{LConst}_{\mathrm{op}(\mathcal{K})} \cup \mathsf{Grpd}_S$-colimits in $\mathsf{Cat}_{\mathcal{B}}$ (where $\mathrm{op}(\mathcal{K})$ is the image of $\mathcal{K}$ under $(-)^{\mathrm{op}}\colon \mathsf{Cat}_\infty \simeq \mathsf{Cat}_\infty$). We now obtain:

Corollary 6.1.3.14. *Let $\mathcal{K}$ be a class of ∞-categories and let S be a local class of maps in $\mathcal{B}$. Let C be a $\mathsf{Cat}_{\mathcal{B}}^{\langle\mathcal{K},S\rangle}$-cocomplete $\mathcal{B}$-category with pullbacks. Then $\mathsf{Cat}_{\mathcal{B}}$ satisfies $\mathsf{Cat}_{\mathcal{B}}^{\langle\mathcal{K},S\rangle}$-descent if and only if*

1. *for all $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ satisfies $\mathcal{K}$-descent, and*
2. *for every map $p\colon P \to A$ in S the functor $p_!$ is a right fibration.*

Proof. By Proposition 5.1.3.4, the $\mathcal{B}$-category C has $\mathsf{Cat}_{\mathcal{B}}^{\langle\mathcal{K},S\rangle}$-descent if and only if it satisfies both $\mathsf{LConst}_{\mathcal{K}}$- and Grpd_S-descent. By Example 6.1.2.3 the first condition is equivalent to (1), and by Corollary 6.1.3.13 the second one is equivalent to (2). □

Example 6.1.3.15. Let S be a local class of morphisms in $\mathcal{B}$ that is closed under pullbacks in $\mathrm{Fun}(\Delta^1, \mathcal{B})$ and that is *left cancellable*, i.e. satisfies the condition that whenever there is a composable pair of morphisms f and g in $\mathcal{B}$ for which g is contained in S, then gf is contained in S if and only if f is. Then the associated subuniverse $\mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ is closed under pullbacks and under Grpd_S-colimits, and for every map $s\colon B \to A$ in S the functor $s_!\colon \mathsf{Grpd}_S(B) \to \mathsf{Grpd}_S(A)$ is a right fibration. Hence Grpd_S has Grpd_S-descent. These conditions are for example satisfied if S is the right complement of a factorisation system.

6.1.4. Universality of colimits

The goal of this section is to establish that the notion of faithful U-descent is equivalent to *universality* of U-colimits:

Definition 6.1.4.1. Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. We say that U-colimits are *universal* in C if for every map $f\colon c \to d$ in C in context $A \in \mathcal{B}$ the functor $f^* : \mathsf{C}_{/d} \to \mathsf{C}_{/c}$ is $\pi_A^*\mathsf{U}$-cocontinuous. If C is a cocomplete large $\mathcal{B}$-category, we simply say that colimits are universal in C if $\mathsf{Cat}_{\mathcal{B}}$-colimits are universal in C.

Remark 6.1.4.2. In the situation of Definition 6.1.4.1, note that by Proposition 3.3.2.13 the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/c} \simeq (\pi_A^*\mathsf{C})_{/\bar{c}}$ (where $\bar{c} : 1 \to \pi_A^*\mathsf{C}$ is the transpose of c) is $\pi_A^*\mathsf{U}$-cocomplete for every $c\colon A \to \mathsf{C}$. Therefore, asking for f^* to be $\pi_A^*\mathsf{U}$-cocontinuous makes sense.

Remark 6.1.4.3. The condition that U-colimits are universal in C is local in $\mathcal{B}$: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then U-colimits are universal in C if and only if $\pi_{A_i}^*\mathsf{U}$-colimits are universal in $\pi_{A_i}^*\mathsf{C}$ for each i. This is easily seen using the fact that U-cocontinuity is a local condition (Remark 3.3.2.3).

Example 6.1.4.4. Let $\mathcal{K}$ be a class of ∞-categories and let C be an $\mathsf{LConst}_{\mathcal{K}}$-cocomplete $\mathcal{B}$-category with pullbacks. Then $\mathsf{LConst}_{\mathcal{K}}$-colimits are universal in C if and only if $\mathcal{K}$-colimits are universal in $\mathsf{C}(A)$ for all $A \in \mathcal{B}$. In fact, this follows immediately from the observation that for every map $f\colon c \to d$ in C in context $A \in \mathcal{B}$ and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $f^*(B)$ can be identified with $(s^*f)^* : \mathsf{C}(B)_{/d} \to \mathsf{C}(B)_{/c}$.

Proposition 6.1.4.5. *Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. Then U-colimits are universal in C if and only if C has faithful U-descent.*

Proof. Let I be an object in $\mathsf{U}(1)$, and let $f\colon c' \to c$ be an arbitrary map in C in context $1 \in \mathcal{B}$. Suppose that $d\colon \mathsf{I} \to \mathsf{C}_{/c}$ is a diagram with colimit cocone $\bar{d}\colon d \to \operatorname{diag}\operatorname{colim} d$, which we may equivalently regard as a diagram

$\bar{d}\colon \mathsf{I} \to \mathsf{C}_{/\operatorname{colim} d}$. Let $g\colon \operatorname{colim} d \to c$ be the induced map. On account of the fact that the (vertical) mate of the commutative square

$$\begin{tikzcd} \mathsf{C}_{/f^*(\operatorname{colim} d)} \arrow[r, "(f^*(g))_!"] \arrow[d, "(g^*f)_!"] & \mathsf{C}_{/c'} \arrow[d, "f_!"] \\ \mathsf{C}_{/\operatorname{colim} d} \arrow[r, "g_!"] & \mathsf{C}_{/c} \end{tikzcd}$$

commutes and since the horizontal maps in this diagram are conservative and U-cocontinuous, the functor f^* preserves the colimit of d if and only if the functor $(g^*f)^*$ preserves the colimit of $\bar{d}\colon \mathsf{I} \to \mathsf{C}_{/\operatorname{colim} d}$. By (the proof of) Proposition 3.2.4.3, the colimit of the latter is the final object in $\mathsf{C}_{/\operatorname{colim} d}$. Therefore, in order to show that C has U-universal colimits, it suffices to consider those diagrams in $\mathsf{C}_{/c}$ whose colimit is the final object.

Now on account of the commutative square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/\operatorname{diag}(c')} \arrow[r, "\simeq"] \arrow[d, "\operatorname{diag}(f)_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c'}) \arrow[d, "(f_!)_*"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/\operatorname{diag}(c)} \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c}) \end{tikzcd}$$

we may identify $\operatorname{diag}(f)^*$ with $(f^*)_*$. Therefore, if $d \to \operatorname{diag}(c)$ is a colimit cocone, the upper horizontal equivalence in the above diagram identifies its pullback $d' \to \operatorname{diag}(c')$ along $\operatorname{diag}(f)$ with the composition $\mathsf{I} \xrightarrow{\bar{d}} \mathsf{C}_{/c} \xrightarrow{f^*} \mathsf{C}_{/c'}$. Thus, by again using Proposition 3.2.4.3, the map $d' \to \operatorname{diag}(c')$ is a colimit cocone if and only if the colimit of $f^*\bar{d}$ is the final object in $\mathsf{C}_{/c'}$, which is in turn equivalent to f^* preserving the colimit of $\bar{d}$. As replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ allows us to arrive at the same conclusion for any $\mathsf{I} \in \mathsf{U}(A)$, Proposition 6.1.3.11 yields the claim. □

Example 6.1.4.6. Let S be a local class of morphisms in $\mathcal{B}$ and let C be a Grpd_S-cocomplete $\mathcal{B}$-category with pullbacks. Then Grpd_S-colimits are universal in C if and only if for every map $p\colon P \to A$ in S the functor $p_!\colon \mathsf{C}(P) \to \mathsf{C}(A)$ is a cartesian fibration. In fact, in light of Proposition 6.1.4.5 this follows from the argument in the proof of Corollary 6.1.3.13.

We end this section by relating universality of colimits with the property of being *locally cartesian closed*:

Proposition 6.1.4.7. *Let* X *be a presentable* $\mathcal{B}$*-category. Then* X *has* $\mathrm{Grpd}_{\mathcal{B}}$*-universal colimits if and only if for every object* $x\colon A \to \mathsf{X}$*, the* $\mathcal{B}_{/A}$*-category* $(\pi_A^*\mathsf{X})_{/x}$ *is cartesian closed, which is to say that there exists a bifunctor*

$$\underline{\mathrm{Hom}}_{\pi_A^*\mathsf{X}}(-,-)\colon (\pi_A^*\mathsf{X})_{/x}^{\mathrm{op}} \times (\pi_A^*\mathsf{X})_{/x} \to (\pi_A^*\mathsf{X})_{/x}$$

that fits into an equivalence

$$\mathrm{map}_{(\pi_A^*\mathsf{X})_{/x}}(- \times -, -) \simeq \mathrm{map}_{(\pi_A^*\mathsf{X})_{/x}}(-, \underline{\mathrm{Hom}}_{\pi_A^*\mathsf{X}}(-,-)).$$

Proof. It will be enough to show that X is cartesian closed if and only if for every $y\colon A \to \mathsf{X}$ the functor $(\pi_y)^*\colon \pi_A^*\mathsf{X} \to (\pi_A^*\mathsf{X})_{/y}$ is $\mathrm{Grpd}_{\mathcal{B}_{/A}}$-cocontinuous. Using Remark 6.1.4.3, we can assume that $A \simeq 1$. Recall that the forgetful functor $(\pi_y)_!\colon \mathsf{X}_{/y} \to \mathsf{Y}$ is $\mathrm{Grpd}_{\mathcal{B}}$-cocontinuous (Proposition 3.3.2.13). As this functor is moreover a right fibration and therefore in particular conservative, we find that π_y^* is $\mathrm{Grpd}_{\mathcal{B}}$-cocontinuous if and only if the composition $(\pi_y)_!\pi_y^*$ is. Together with Proposition 6.1.1.2, this shows that π_y^* being $\mathrm{Grpd}_{\mathcal{B}}$-cocontinuous is equivalent to $y \times -$ being $\mathrm{Grpd}_{\mathcal{B}}$-cocontinuous. As X is presentable, this is in turn equivalent to $y \times -$ having a right adjoint $\underline{\mathrm{Hom}}_{\mathsf{X}}(y,-)$ (see Proposition 5.4.3.1). Clearly, this holds if X is cartesian closed. Conversely, if $y \times -$ admits a right adjoint for all $y\colon A \to \mathsf{X}$, then $\mathrm{map}_{\mathsf{X}}(- \times -, -)$, viewed as a functor $\mathsf{X}^{\mathrm{op}} \times \mathsf{X} \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{X})$, takes values in $\mathsf{X} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{X})$ and therefore gives rise to the desired internal hom. $\square$

6.1.5. Disjoint colimits

If $\mathcal{C}$ is an ∞-category with pullbacks and finite coproducts, one says that a coproduct $c_0 \sqcup c_1$ in $\mathcal{C}$ is *disjoint* if the fibre product $c_i \times_{c_0 \sqcup c_1} c_j$ is equivalent to c_i if $i = j$ and the initial object otherwise. In this section, our goal is to study an internal analogue of this concept. In fact, we will define what it means for arbitrary $\mathcal{B}$-*groupoidal* colimits to be disjoint. To that end, recall that if S is an arbitrary local class of morphisms in $\mathcal{B}$, the associated subuniverse Grpd_S is contained in the free Grpd_S-cocompletion $\underline{\mathrm{PSh}}_{\mathcal{B}}^{\mathrm{Grpd}_S}(1)$, cf. Example 3.5.3.5. Therefore, if C is an arbitrary Grpd_S-cocomplete $\mathcal{B}$-category, the tensoring bifunctor

$$- \otimes -\colon \mathrm{Grpd}_S \times \mathsf{C} \to \mathsf{C}$$

(see Proposition 3.5.3.8) is well-defined. Furthermore, note that S is closed under diagonals if and only if for every $\mathsf{G} \in \mathrm{Grpd}_S(A)$ and every pair of objects $g, g' : A \rightrightarrows \mathsf{G}$ the mapping $\mathcal{B}_{/A}$-groupoid $\mathrm{map}_{\mathsf{G}}(g, g')$ is contained in $\mathrm{Grpd}_S(A)$ as well. Therefore, we may define:

Definition 6.1.5.1. Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Grpd_S-cocomplete $\mathcal{B}$-category with pullbacks. If $\mathsf{G} \in \mathrm{Grpd}_S(1)$ is an arbitrary object, we say that G-*indexed colimits are disjoint in* C if for all diagrams $d\colon \mathsf{G} \to \mathsf{C}$ and for every pair of objects g, g' in G in context $1 \in \mathcal{B}$ the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{G}}(g, g') \otimes d(g) \arrow[r] \arrow[d] & d(g') \arrow[d] \\
d(g) \arrow[r] & \operatorname{colim} d
\end{tikzcd}$$

is a pullback. We say that Grpd_S-*colimits are disjoint in* C if for all $A \in \mathcal{B}$ and all $\mathsf{G} \in \mathsf{U}(A)$ all G-indexed colimits are disjoint in $\pi_A^* \mathsf{C}$.

Remark 6.1.5.2. In the situation of Definition 6.1.5.1, let $\bar{d}\colon \mathsf{G}^{\triangleright} \to \mathsf{C}$ be the colimit cocone associated with d. Then the commutative square in the definition is obtained by transposing the commutative diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{G}^{\triangleright}}(g, g') \arrow[r, "\bar{d}"] \arrow[d] & \mathrm{map}_{\mathsf{C}}(d(g), d(g')) \arrow[d] \\
\mathrm{map}_{\mathsf{G}^{\triangleright}}(g, \infty) \arrow[r, "\bar{d}"] & \mathrm{map}_{\mathsf{C}}(d(g), \operatorname{colim} d)
\end{tikzcd}$$

across the equivalence $\mathrm{map}_{\mathsf{C}}(- \otimes -, -) \simeq \mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(-, \mathrm{map}_{\mathsf{C}}(-, -))$, noting that since $\iota\colon \mathsf{G} \hookrightarrow \mathsf{G}^{\triangleright}$ is fully faithful the upper left corner can be identified with $\mathrm{map}_{\mathsf{G}}(g, g')$ and since $\infty : 1 \to \mathsf{G}^{\triangleright}$ is final the lower left corner is equivalent to the final object.

Example 6.1.5.3. Let us unwind Definition 6.1.5.1 in the case where $\mathcal{B} = \mathrm{Ani}$ and where $\mathsf{G} = \{0, 1\}$. Then a diagram $d\colon \{0, 1\} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is simply given by a pair (c_0, c_1) of objects in $\mathcal{C}$, and its colimit is the coproduct $c_0 \sqcup c_1$.

Furthermore, the square in Definition 6.1.5.1 is explicitly given by

$$\begin{tikzcd} \mathrm{map}_{\{0,1\}}(i,j) \times c_i \arrow[r] \arrow[d] & c_j \arrow[d] \\ c_i \arrow[r] & c_0 \sqcup c_1 \end{tikzcd}$$

(for $i, j \in \{0, 1\}$) and is therefore a pullback for all pairs (i, j) precisely if coproducts are disjoint in $\mathcal{C}$ in the usual sense.

Remark 6.1.5.4. The property of Grpd_S-colimits to be disjoint in C is a local condition. More precisely, if $\mathsf{G} \in \mathsf{U}(1)$ is an arbitrary object and if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then G-indexed colimits are disjoint in C if and only if $\pi_{A_i}^* \mathsf{G}$-indexed colimits are disjoint in $\pi_{A_i}^* \mathsf{C}$ for all i. This follows immediately from the fact that both limits and colimits are determined locally, cf. Remark 3.2.1.7. As a consequence, if Grpd_S is generated by a family of objects $(\mathsf{G}_i : A_i \to \mathrm{Grpd}_S)_i$, then Grpd_S-colimits are disjoint in C precisely if G_i-indexed colimits are disjoint in $\pi_{A_i}^* \mathsf{C}$ for all i.

Example 6.1.5.5. Let S be the local class of morphisms in $\mathcal{B}$ that is generated by $\varnothing$, 1 and $2 = 1 \sqcup 1$. Then S is closed under diagonals. By using Remark 6.1.5.4 and Example 6.1.5.3, one finds that Grpd_S-colimits are disjoint in C if and only if coproducts are disjoint in $\mathsf{C}(A)$ for all $A \in \mathcal{B}$.

The main goal of this section is to show:

Proposition 6.1.5.6. *Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Grpd_S-cocomplete $\mathcal{B}$-category with pullbacks in which Grpd_S-colimits are universal. Then C has effective Grpd_S-descent if and only if Grpd_S-colimits are disjoint.*

In order to prove Proposition 6.1.5.6, we will need a more explicit description of the notion of disjoint Grpd_S-colimits. The key input is the following construction:

Construction 6.1.5.7. Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Grpd_S-cocomplete $\mathcal{B}$-category with pullbacks. Suppose that $p : P \to A$ is a map in S, and let $c : P \to \mathsf{C}$ be an arbitrary object.

Let $\eta\colon c \to p^* p_! c$ be the adjunction unit, and consider the pullback square

$$\begin{tikzcd} z \arrow[r] \arrow[d] & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\eta)"] \\ \mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^*(\eta)"] & \mathrm{pr}_0^* p^* p_!(c) \end{tikzcd}$$

in $\mathsf{C}(P \times_A P)$ (where we implicitly identify $\mathrm{pr}_0^* p^* \simeq \mathrm{pr}_1^* p^*$). Note that if $\Delta_p\colon P \to P \times_A P$ is the diagonal map, the pullback of the above square along Δ_p yields the pullback square

$$\begin{tikzcd} c \times_{p^* p_!(c)} c \arrow[r] \arrow[d] & c \arrow[d] \\ c \arrow[r] & p^* p_!(c) \end{tikzcd}$$

in $\mathsf{C}(P)$. Therefore, the diagonal map $c \to c \times_{p^* p_!(c)} c$ transposes to a map $\delta_p(c)\colon (\Delta_p)_!(c) \to z$.

Proposition 6.1.5.8. *Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Grpd_S-cocomplete $\mathcal{B}$-category with pullbacks. Then Grpd_S-colimits are disjoint in C if and only if for all maps $p\colon P \to A$ and all objects $c\colon P \to \mathsf{C}$ the map $\delta_p(c)$ from Construction 6.1.5.7 is an equivalence.*

Proof. By identifying $p\colon P \to A$ with a $\mathcal{B}_{/A}$-groupoid G, the object $c\colon P \to \mathsf{C}$ corresponds to a diagram $d\colon \mathsf{G} \to \pi_A^* \mathsf{C}$. Also, the two maps $\mathrm{pr}_0, \mathrm{pr}_1\colon P \times_A P \rightrightarrows P$ correspond to objects g and g' in G in context $P \times_A P$. In light of these identifications, the two cospans

$$\begin{tikzcd} & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\eta_P)"] \\ \mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^*(\eta_P)"] & \mathrm{pr}_0^* p^* p_!(c) \end{tikzcd} \qquad \begin{tikzcd} & d(g') \arrow[d] \\ d(g) \arrow[r] & \mathrm{pr}_0^* p^*(\operatorname{colim} d) \end{tikzcd}$$

(in context $P \times_A P$) are translated into each other. Next, we note that pulling back g and g' along the diagonal $\Delta\colon P \to P \times_A P$ recovers the tautological object τ of G (i.e. the one corresponding to id_P). As there is a section $\mathrm{id}_\tau\colon P \to \mathrm{map}_{\mathsf{G}}(\tau, \tau)$, we thus obtain a commutative diagram

$$\begin{tikzcd}
d(\tau) \arrow[dr] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \mathrm{map}_{\mathsf{G}}(\tau,\tau) \otimes d(\tau) \arrow[r] \arrow[d] & d(\tau) \arrow[d] \\
& d(\tau) \arrow[r] & p^*(\operatorname{colim} d)
\end{tikzcd}$$

in context P. Observe that value of the unit of the adjunction

$$(\Delta_! \dashv \Delta^*) \colon \mathcal{B}_{/P} \rightleftarrows \mathcal{B}_{/P \times_A P}$$

at the final object precisely recovers the map $\mathrm{id}_\tau \colon P \to \mathrm{map}_{\mathsf{G}}(\tau, \tau)$. Hence, as the functor

$$- \otimes d(\tau) \colon \pi_P^* \mathrm{Grpd}_{\mathcal{B}} \to \pi_P^* \mathsf{C}$$

is by construction $\pi_P^* \mathsf{U}$-cocontinuous, one finds that $d(\tau) \to \mathrm{map}_{\mathsf{G}}(\tau, \tau) \otimes d(\tau)$ can be identified with the unit of the adjunction

$$(\Delta_! \dashv \Delta^*) \colon \mathsf{C}(P) \rightleftarrows \mathsf{C}(P \times_A P).$$

But this precisely means that the transpose map $\Delta_! d(\tau) \to \mathrm{map}_{\mathsf{G}}(g, g') \otimes d(g)$ must be an equivalence. As $d(\tau)$ is simply c, we therefore find that the two diagrams

$$\begin{tikzcd}
\Delta_!(c) \arrow[r] \arrow[d] & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\eta)"] \\
\mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^*(\eta)"] & \mathrm{pr}_0^* \, p^* p_!(c)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{map}_{\mathsf{G}}(g, g') \otimes d(g) \arrow[r] \arrow[d] & d(g') \arrow[d] \\
d(g) \arrow[r] & \mathrm{pr}_0^* \, p^*(\operatorname{colim} d)
\end{tikzcd}$$

are equivalent. To complete the proof, we still need to show that the right square being cartesian is equivalent to Grpd_S-colimits being disjoint in C. Certainly, this is a necessary condition since this square is precisely of the form as in Definition 6.1.5.1 (after identifying $\mathrm{pr}_0^* \, p^*(\operatorname{colim} d)$ with $\operatorname{colim} \mathrm{pr}_0^* \, p^* d$ and regarding g and g' as objects of $\mathrm{pr}_0^* \, p^* \mathsf{G}$ in context $1 \in \mathcal{B}_{/P \times_A P}$). The converse follows from the observation that *every* pair of objects $h, h' \colon A \rightrightarrows \mathsf{G}$ must be a pullback of g and g', i.e. that the above diagram is the *universal* one. $\square$

Proof of Proposition 6.1.5.6. We will freely make use of the setup from Proposition 6.1.5.8. Therefore, let us fix a map $p\colon P \to A$ in S, and let us denote the unit and counit of the associated adjunction $p_! \dashv p^*$ by η_p and ϵ_p, respectively. We first assume that C has effective Grpd_S-descent. Choose an arbitrary object $c\colon P \to \mathrm{C}$ and consider the pullback

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d] & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\eta_p)"] \\ \mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^*(\eta_p)"] & \mathrm{pr}_0^* p^* p_!(c) \end{tikzcd}$$

in $\mathrm{C}(P \times_A P)$. By making use of the commutative diagram

$$\begin{tikzcd} \mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^* \eta_p c"] \arrow[dr, "\eta_{\mathrm{pr}_1} \mathrm{pr}_0^*(c)"'] & \mathrm{pr}_1^* p^* p_!(c) \arrow[d, "\mathrm{pr}_1^*(\alpha)"] \\ & \mathrm{pr}_1^*(\mathrm{pr}_1)_! \mathrm{pr}_0^*(c), \end{tikzcd}$$

(where α is an equivalence owing to C having Grpd_S-colimits), we may identify the above square with the pullback square

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d] & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\alpha\eta_p)"] \\ \mathrm{pr}_0^*(c) \arrow[r, "\eta_{\mathrm{pr}_1} \mathrm{pr}_0^*(c)"] & \mathrm{pr}_1^*(\mathrm{pr}_1)_! \mathrm{pr}_0^*(c). \end{tikzcd}$$

Since by assumption Grpd_S-colimits are universal in C, Proposition 6.1.4.5 implies that C has faithful Grpd_S-descent. Hence Proposition 6.1.3.11 implies that the transpose $(\mathrm{pr}_1)_!(z) \to c$ of g must be an equivalence. Together with the commutative diagram

$$\begin{tikzcd} (\mathrm{pr}_1)_!\Delta_!(c) \arrow[r] \arrow[d, "\simeq"] \arrow[rr, bend left, "(\mathrm{pr}_1)_!(\delta_p(c))"] & (\mathrm{pr}_1)_!\Delta_!\Delta^*(z) \arrow[r, "(\mathrm{pr}_1)_!\epsilon_\Delta z"] \arrow[d, "\simeq"] & (\mathrm{pr}_1)_!(z) \arrow[d] \\ c \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \Delta^*(z) \arrow[r, "\Delta^*(g)"] & c, \end{tikzcd}$$

this observation implies that $(\mathrm{pr}_1)_!(\delta_p(c))$ is an equivalence. But since C has effective and faithful Grpd_S-descent, Corollary 6.1.3.13 implies that $(\mathrm{pr}_1)_!$ is a right fibration and therefore in particular conservative. Hence $\delta_p(c)$ is already an equivalence.

Conversely, suppose that Grpd_S-colimits in C are disjoint and let $f\colon c \to d$ be an arbitrary map in C in context P. Consider the diagram

$$\begin{tikzcd}
c \arrow[dr, "\phi"] \arrow[drr, bend left, "\eta_P c"] \arrow[ddr, bend right, "f"'] & & \\
& e \arrow[r] \arrow[d] & p^* p_!(c) \arrow[d, "p^* p_!(f)"] \\
& d \arrow[r, "\eta d"] & p^* p_!(d)
\end{tikzcd}$$

in $\mathsf{C}(P)$ in which the square is a pullback. By Proposition 6.1.3.12, the result follows once we show that ϕ is an equivalence. We now obtain a pullback diagram

$$\begin{tikzcd}
& x \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{pr}_1^*(c) \arrow[dl] \arrow[dd] \\
\mathrm{pr}_0^*(e) \arrow[rr] \arrow[dd] & & \mathrm{pr}_0^* p^* p_!(c) \arrow[dd] & \\
& y \arrow[rr] \arrow[dl] & & \mathrm{pr}_1^*(d) \arrow[dl] \\
\mathrm{pr}_0^*(d) \arrow[rr] & & \mathrm{pr}_0^* p^* p_!(d) &
\end{tikzcd} \tag{$*$}$$

in $\mathrm{Fun}(\Delta^1, \mathsf{C}(P \times_A P))$ in which the front square is obtained by applying pr_0^* to the pullback square in the previous diagram and the right square is given by applying pr_1^* to the outer square in the previous diagram. Note that by applying the functor Δ^* to this cube, we obtain a commutative diagram

$$\begin{tikzcd}
& c \arrow[dr] \arrow[drrr, bend left, "\mathrm{id}"] \arrow[ddl, bend right, "\phi"'] \arrow[ddd] & & & \\
& & \Delta^*(x) \arrow[rr] \arrow[dll] \arrow[ddd] & & c \arrow[dl] \arrow[ddd] \\
e \arrow[rrr] \arrow[ddd] & & & p^* p_!(c) \arrow[ddd] & \\
& d \arrow[dr] \arrow[drrr, bend left, "\mathrm{id}"] \arrow[ddl, bend right, "\mathrm{id}"'] & & & \\
& & \Delta^*(y) \arrow[rr] \arrow[dll] & & d \arrow[dl] \\
d \arrow[rrr] & & & p^* p_!(d) &
\end{tikzcd}$$

in which the cube defines a pullback in $\mathrm{Fun}(\Delta^1, \mathsf{C}(P))$. Now by disjointness of Grpd_S-colimits, the map $\Delta_!(d) \to y$ must be an equivalence. Since this map fits into a commutative diagram

$$\begin{tikzcd} d \arrow[r, "\eta' d"] \arrow[dr] & \Delta^*\Delta_!(c) \arrow[d, "\simeq"] \\ & \Delta^*(y) \end{tikzcd}$$

(in which η' denotes the unit of the adjunction $\Delta_! \dashv \Delta^*$) and since we have a pullback square

$$\begin{tikzcd} c \arrow[r] \arrow[d] & \Delta^*(x) \arrow[d] \\ d \arrow[r] & \Delta^*(y), \end{tikzcd}$$

the assumption that Grpd_S-colimits are universal in C and Proposition 6.1.3.11 imply that the transpose map $\Delta_!(c) \to x$ is an equivalence as well. By the argument in the beginning of the proof, applied to the top square in $(*)$, the commutative diagram

$$\begin{tikzcd} \mathrm{pr}_1^*(c) \arrow[r, "\mathrm{pr}_1^*\eta_p c"] \arrow[dr, "\eta_{\mathrm{pr}_0}\mathrm{pr}_1^*(c)"'] & \mathrm{pr}_0^* p^* p_!(c) \arrow[d, "\simeq"] \\ & \mathrm{pr}_0^*(\mathrm{pr}_0)_! \mathrm{pr}_1^*(c) \end{tikzcd}$$

implies that the map $(\mathrm{pr}_0)_!(x) \to e$ is an equivalence too. Taken together, we thus conclude that the composition

$$c \xrightarrow{\simeq} (\mathrm{pr}_0)_!\Delta_!(c) \to (\mathrm{pr}_0)_!(x) \to e$$

is an equivalence. By its very construction, this map can be identified with ϕ, hence the claim follows. $\square$

6.1.6. Effective groupoid objects

In this section we briefly review the notion of *groupoid objects* and their relation to descent (as discussed in [49, § 6.1]) in the context of $\mathcal{B}$-category theory.

Definition 6.1.6.1. Let C be a $\mathcal{B}$-category with pullbacks. A *groupoid object* in C is a functor $G_\bullet \colon \Delta^{\mathrm{op}} \to \mathsf{C}$ such that for all $n \geq 0$ and every decomposition $\langle n \rangle \simeq \langle k \rangle \sqcup_{\langle 0 \rangle} \langle l \rangle$ the map $G_n \to G_k \times_{G_0} G_l$ is an equivalence. We denote by $\mathrm{Seg}^{\simeq}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C})$ spanned by the groupoid objects in $\pi_A^* \mathsf{C}$ for every $A \in \mathcal{B}$.

Definition 6.1.6.2. Let C be a $\mathcal{B}$-category that admits Δ^{op}-indexed colimits and pullbacks. We say that a groupoid object $G_\bullet$ in C is *effective* if the map $G_1 \to G_0 \times_{\operatorname{colim} G_\bullet} G_0$ is an equivalence in C. We denote by $\mathrm{Seg}^{\simeq}_{\mathrm{eff}}(\mathsf{C})$ the full subcategory of $\mathrm{Seg}^{\simeq}(\mathsf{C})$ that is spanned by the effective groupoid objects in $\pi_A^* \mathsf{C}$ for every $A \in \mathcal{B}$. We say that *groupoid objects are effective* in C if the inclusion $\mathrm{Seg}^{\simeq}_{\mathrm{eff}}(\mathsf{C}) \hookrightarrow \mathrm{Seg}^{\simeq}(\mathsf{C})$ is an equivalence.

Remark 6.1.6.3. Since the property of a map being an equivalence is local in $\mathcal{B}$, it follows immediately from the definition that an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C})$ is contained in $\mathrm{Seg}^{\simeq}(\mathsf{C})$ if and only if it encodes a groupoid object in $\pi_A^* \mathsf{C}$, which is in turn equivalent to its transpose $\Delta^{\mathrm{op}} \to \mathsf{C}(A)$ being a groupoid object in the conventional sense. An analogous remark can be made for effective groupoid objects. In particular, groupoid objects are effective in C if and only if they are effective in $\mathsf{C}(A)$ for each $A \in \mathcal{B}$.

Let Pos be the 1-category of posets, which we always identify with 0-categories. Observe that the functor $(-)^{\triangleright} \colon \mathrm{Pos} \to \mathrm{Pos}$ that freely adjoins a final object to a partially ordered set restricts to a functor $(-)^{\triangleright} \colon \Delta^{\triangleleft} \to \Delta$, and the map $\mathrm{id}_{\mathrm{Pos}} \hookrightarrow (-)^{\triangleright}$ restricts to a map $\mathrm{id}_{\Delta} \to (-)^{\triangleright}\iota$, where $\iota \colon \Delta \hookrightarrow \Delta^{\triangleleft}$ is the inclusion. By precomposition, we thus obtain a functor

$$(-)_{+1} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}((\Delta^{\mathrm{op}})^{\triangleright}, \mathsf{C})$$

together with a morphism $\iota^*(-)_{+1} \to \mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C})}$. Now using Remark 6.1.6.3, one finds:

Proposition 6.1.6.4 ([49, Lemma 6.1.3.7 and Remark 6.1.3.18]). *Let C be a $\mathcal{B}$-category that admits Δ^{op}-indexed colimits and pullbacks, and let $G_\bullet \colon \Delta^{\mathrm{op}} \to \mathsf{C}$ be a simplicial object. Then $G_{\bullet+1}$ is a colimit cocone, and $G_\bullet$ is a groupoid object if and only if the morphism of functors $\iota^* G_{\bullet+1} \to G_\bullet$ is cartesian.* □

By combining Proposition 6.1.6.4 with Proposition 6.1.3.12, we conclude:

Corollary 6.1.6.5. *Let* U *be the internal class that is spanned by* $\Delta^{\mathrm{op}}\colon 1 \to \mathrm{Cat}_{\mathcal{B}}$ *and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category with pullbacks that has effective* U*-descent. Then groupoid objects are effective in* C*.* □

6.2. Foundations of $\mathcal{B}$-topos theory

In this section we develop the basic theory of $\mathcal{B}$-topoi. We begin in Section 6.2.1 by giving an axiomatic definition of this concept using the notion of descent that has been established in the previous section. By unwinding the descent condition, we furthermore establish an explicit characterisation of $\mathcal{B}$-topoi in terms of the underlying $\widehat{\mathrm{Cat}}_\infty$-valued sheaves on $\mathcal{B}$. In Section 6.2.2, we construct the *free* $\mathcal{B}$-topos on an arbitrary $\mathcal{B}$-category, which we use in Section 6.2.3 to establish a characterisation of $\mathcal{B}$-topoi as left exact and accessible Bousfield localisations of presheaf $\mathcal{B}$-categories. In Section 6.2.4, we make use of this characterisation to show that the $\mathcal{B}$-category of $\mathcal{B}$-topoi is tensored and powered over $\mathrm{Cat}_{\mathcal{B}}$. In Section 6.2.5, we prove that $\mathcal{B}$-topoi are entirely determined by their global sections, in the sense that the ∞-category of $\mathcal{B}$-topoi is equivalent to that of geometric morphisms of ∞-topoi with codomain $\mathcal{B}$. Having this simple description of $\mathcal{B}$-topoi at our disposal, it is straightforward to construct limits and colimits of $\mathcal{B}$-topoi, which is the topic of Section 6.2.6. Also, we provide an explicit formula for the coproduct of $\mathcal{B}$-topoi in Section 6.2.7, which in particular yields a formula for the pushout in $\mathrm{Top}^{\mathrm{L}}_\infty$. In Section 6.2.8, we discuss a $\mathcal{B}$-categorical version of Diaconescu's theorem for $\mathcal{B}$-topoi, from which we deduce a universal property of étale $\mathcal{B}$-topoi in Section 6.2.9. Lastly, we discuss *subterminal* $\mathcal{B}$-topoi in Section 6.2.10, where we derive a general formula for left exact localisations in terms of internal colimits.

6.2.1. Definition and characterisation of $\mathcal{B}$-topoi

In this section we introduce the notion of a $\mathcal{B}$*-topos* and prove several equivalent characterisations of this concept.

Recall from Proposition 5.2.3.5 that a $\mathcal{B}$-category C admits finite limits if and only if for all $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits finite limits and for each map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves finite limits. Similarly, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between such $\mathcal{B}$-categories preserves finite limits precisely if it does so section-wise. We may now define:

Definition 6.2.1.1. A large $\mathcal{B}$-category X is a $\mathcal{B}$-*topos* if it is presentable and satisfies descent. A functor $f^*\colon \mathsf{X} \to \mathsf{Y}$ between $\mathcal{B}$-topoi is called an *algebraic morphism* if f is cocontinuous and preserves finite limits. A functor $f_*\colon \mathsf{Y} \to \mathsf{X}$ between $\mathcal{B}$-topoi is called a *geometric morphism* if f_* admits a left adjoint f^* that defines an algebraic morphism.

The large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ of $\mathcal{B}$-topoi is defined as the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is spanned by the algebraic morphisms between $\mathcal{B}_{/A}$-topoi, for all $A \in \mathcal{B}$. Dually, the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ of $\mathcal{B}$-topoi is defined as the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is spanned by the geometric morphisms between $\mathcal{B}_{/A}$-topoi, for all $A \in \mathcal{B}$. We denote by $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ and $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$, respectively, the underlying ∞-categories of global sections.

If X and Y are $\mathcal{B}$-topoi, we will denote by $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y})$ that is spanned by the algebraic morphisms $\pi_A^*\mathsf{X} \to \pi_A^*\mathsf{Y}$ for each $A \in \mathcal{B}$. We define the $\mathcal{B}$-category $\underline{\mathsf{Fun}}^{\mathrm{geom}}_{\mathcal{B}}(\mathsf{Y}, \mathsf{X})$ of geometric morphisms in the evident dual way.

Remark 6.2.1.2. The fact that both $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ and $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ are large and not very large follows from Remark 5.4.4.3.

Remark 6.2.1.3. The subobject of $(\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ that is spanned by the algebraic morphisms between $\mathcal{B}_{/A}$-topoi (for each $A \in \mathcal{B}$) is stable under composition and equivalences in the sense of Proposition 1.3.1.17. Since moreover cocontinuity and the property that a functor preserves finite limits are local conditions (Remark 3.3.2.3) and on account of Remark 6.2.1.8 below, we conclude that a map $A \to (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ is contained in $(\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})_1$ if and only if it defines an algebraic morphism between $\mathcal{B}_{/A}$-topoi. In particular, if X and Y are $\mathcal{B}_{/A}$-topoi, the image of the monomorphism

$$\mathrm{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{X}, \mathsf{Y}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{X}, \mathsf{Y}) \tag{$*$}$$

is spanned by the algebraic morphisms. Moreover, the sheaf associated to $\mathrm{Top}^{\mathrm{L}}_{\mathcal{B}}$ is given by sending $A \in \mathcal{B}$ to the subcategory $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}}_{/A})$, and there is consequently a canonical equivalence $\pi_A^* \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}_{/A}}$. Analogous observations can be made for the $\mathcal{B}$-category $\mathrm{Top}^{\mathrm{R}}_{\mathcal{B}}$.

By the same argument, we have a canonical equivalence

$$\pi_A^* \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y}) \simeq \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{X}, \pi_A^* \mathsf{Y})$$

for all $\mathcal{B}$-topoi X and Y and all $A \in \mathcal{B}$. Furthermore, by using Corollary 3.2.6.5, we deduce that the inclusion in $(*)$ is obtained by applying the core $\mathcal{B}$-groupoid functor to the inclusion of $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y})$ into $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y})$. Again, analogous observations can be made for geometric morphisms.

By restricting the equivalence $\mathrm{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}} \simeq (\mathrm{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}})^{\mathrm{op}}$ from Proposition 4.5.2.1, one finds:

Proposition 6.2.1.4. *There is an equivalence* $(\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \simeq \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ *that acts as the identity on objects and that carries an algebraic morphism to its right adjoint.* □

Let us denote by $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty}$ the subcategory of $\mathrm{Top}^{\mathrm{L}}_{\infty}$ that is spanned by the étale algebraic morphisms (i.e. those that are of the form $\pi_U^* \colon \mathcal{X} \to \mathcal{X}_{/U}$ for some ∞-topos $\mathcal{X}$ and some $U \in \mathcal{X}$). By [49, Theorem 6.3.5.13], this ∞-category admits small limits, and the inclusion $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty} \hookrightarrow \mathrm{Top}^{\mathrm{L}}_{\infty}$ preserves small limits. The main goal of this section is to prove the following characterisation of $\mathcal{B}$-topoi:

Theorem 6.2.1.5. *For a large $\mathcal{B}$-category* X*, the following are equivalent:*

1. X *is a $\mathcal{B}$-topos;*
2. X *satisfies the* internal Giraud axioms*:*
   a) X *is presentable;*
   b) X *has universal colimits;*
   c) *groupoid objects in* X *are effective;*
   d) $\mathrm{Grpd}_{\mathcal{B}}$*-colimits in* X *are disjoint.*
3. X *is* $\mathrm{Grpd}_{\mathcal{B}}$*-cocomplete and takes values in* $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty}$*;*

4. X *is a* $\mathrm{Top}_\infty^{\mathrm{L,ét}}$*-valued sheaf that preserves pushouts.*

Remark 6.2.1.6. It is crucial to include the condition that all $\mathrm{Grpd}_{\mathcal{B}}$-groupoidal colimits are disjoint into the internal Giraud axioms, instead of just all coproducts. As a concrete example, let κ be an uncountable regular cardinal and let $\mathcal{C} \hookrightarrow \mathrm{Cat}_\infty$ be the subcategory spanned by the κ-small ∞-categories and *cocartesian* fibrations between them. Let us set $\mathcal{B} = \mathrm{PSh}(\mathcal{C})$ and let $\mathsf{X} \in \mathrm{Cat}(\widehat{\mathcal{B}})$ be the large $\mathcal{B}$-category that is determined by the presheaf $\mathrm{PSh}(-)\colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. Since X takes values in $\mathrm{Top}_\infty^{\mathrm{L}}$ and since cocartesian fibrations are *smooth* [49, Proposition 4.1.2.15], we deduce from Theorem 5.4.2.5, Remark 6.1.6.3 and Example 6.1.5.5 that X is presentable, has effective groupoid objects and that coproducts in X are disjoint. Moreover, again by using that cocartesian fibrations are smooth, one easily finds that X has universal colimits. Yet, the $\mathcal{B}$-category X cannot be a $\mathcal{B}$-topos since the transition functors are in general not étale.

Before we prove Theorem 6.2.1.5, let us us first record the following immediate consequence:

Corollary 6.2.1.7. *The universe* $\mathrm{Grpd}_{\mathcal{B}}$ *is a* $\mathcal{B}$*-topos.* □

Remark 6.2.1.8. As another consequence of Theorem 6.2.1.5, a large $\mathcal{B}$-category X is a $\mathcal{B}$-topos if and only if there is a cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ such that for all i the large $\mathcal{B}_{/i}$-category $\pi_{A_i}^* \mathsf{X}$ is a $\mathcal{B}_{/A_i}$-topos. In fact, this most easily follows from part (3) of the theorem, together with the fact that $\mathrm{Grpd}_{\mathcal{B}}$-cocompleteness can be checked locally (Remark 3.3.2.3).

The proof of Theorem 6.2.1.5 requires the following lemma:

Lemma 6.2.1.9. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "h^*"] \arrow[d, "f^*"] & \mathcal{Z} \arrow[d, "g^*"] \\ \mathcal{Y} \arrow[r, "k^*"] & \mathcal{W} \end{tikzcd}$$

be a commutative square in $\mathrm{Top}_\infty^{\mathrm{L}}$*, and suppose that* h^* *and* k^* *are étale. Then the square is a pushout in* $\mathrm{Top}_\infty^{\mathrm{L}}$ *if and only if the mate transformation* $k_! g^* \to f^* h_!$ *is an equivalence.*

Proof. Since h^* is étale, we may replace $\mathcal{Z}$ with $\mathcal{X}_{/U}$ and h^* with π_U^*, where we set $U = h_!(1_{\mathcal{Z}})$. By using [49, Remark 6.3.5.8], the pushout of π_U^* along f^* is given by the commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\pi_U^*"] \arrow[d, "f^*"] & \mathcal{X}_{/U} \arrow[d, "f^*_{/U}"] \\
\mathcal{Y} \arrow[r, "\pi^*_{f^*(U)}"] & \mathcal{Y}_{/f^*(U)}.
\end{tikzcd}$$

It is immediate that the mate of this square is an equivalence, so it suffices to prove the converse. Since k^* is étale, we may replace k^* with $\pi_V^* \colon \mathcal{Y} \to \mathcal{Y}_{/V}$, where $V = k_!(1_{\mathcal{W}})$. By [49, Remark 6.3.5.7], the induced map $\mathcal{Y}_{/f^*(U)} \to \mathcal{Y}_{/V}$ is uniquely determined by a morphism $V \to f^*(U)$ in $\mathcal{Y}$. Unwinding the definitions, this map is precisely the value of the mate transformation $(\pi_V)_! g^* \to f^*(\pi_U)_!$ at $1_{\mathcal{X}_{/U}}$ and therefore an equivalence. Hence the functor $\mathcal{Y}_{/f^*(U)} \to \mathcal{Y}_{/V}$ must be an equivalence as well, which finishes the proof. □

Proof of Theorem 6.2.1.5. Let X be a $\mathcal{B}$-topos. By combining Proposition 6.1.4.5 and Proposition 6.1.5.6 with Corollary 6.1.6.5, we find that X satisfies the internal Giraud axioms, so that (1) implies (2). If X satisfies the internal Giraud axioms, then X being presentable implies that it is $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete. Moreover, Example 6.1.4.4 and Example 6.1.5.5 together with Remark 6.1.6.3 imply that $\mathsf{X}(A)$ satisfies the ∞-categorical Giraud axioms in the sense of [49] for all $A \in \mathcal{B}$, so that each $\mathsf{X}(A)$ is an ∞-topos. Now by Proposition 6.1.5.6 and Proposition 6.1.4.5, the $\mathcal{B}$-category X has $\mathsf{Grpd}_{\mathcal{B}}$-descent, hence Corollary 6.1.3.13 implies that for every map $s \colon B \to A$ in $\mathcal{B}$ the functor $s_!$ is a right fibration. This implies that s^* is an étale geometric morphism, hence (3) holds. The equivalence between (3) and (4), on the other hand, is an immediate consequence of Lemma 6.2.1.9. Finally, if X satisfies condition (3), then X is presentable (see Theorem 5.4.2.5), hence the claim follows from Corollary 6.1.3.14. □

6.2.2. Free $\mathcal{B}$-topoi

The goal of this section is to construct a partial left adjoint to the inclusion $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is defined on the full subcategory $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ and that carries a $\mathcal{B}$-category C to the associated *free* $\mathcal{B}$-topos $\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$. To that end, first note that if $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ denotes the subcategory spanned by the left exact (i.e. $\mathsf{Fin}_{\mathcal{B}_{/A}}$-continuous) functors between $\mathcal{B}_{/A}$-categories with finite limits for all $A \in \mathcal{B}$, then the dual of Corollary 3.5.1.14 implies that the inclusion admits a left adjoint $(-)^{\mathrm{lex}} \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}}$ that carries a $\mathcal{B}$-category C to its free $\mathsf{Fin}_{\mathcal{B}}$-completion $\mathsf{C}^{\mathrm{lex}}$. Moreover, the same result implies that we have a functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(-) \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ that is obtained by restricting the free cocompletion functor $\mathsf{Cat}_{\widehat{\mathcal{B}}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc}}$ in the appropriate way. By combining these two constructions, we thus end up with a well-defined functor $\mathsf{Grpd}_{\mathcal{B}}[-] = \underline{\mathsf{PSh}}_{\mathcal{B}}((-)^{\mathrm{lex}}) \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$. Our goal is to show:

Proposition 6.2.2.1. *For any $\mathcal{B}$-category C, the large $\mathcal{B}$-category $\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$ is a $\mathcal{B}$-topos. Moreover, if X is another $\mathcal{B}$-topos, precomposition with the canonical map $\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$ induces an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}], \mathsf{X}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$$

of $\mathcal{B}$-categories.

The proof of Proposition 6.2.2.1 requires a few preparations and will be given at the end of this section. For now, let us record a few consequences of this result.

Corollary 6.2.2.2. *The functor $\mathsf{Grpd}_{\mathcal{B}}[-]$ takes values in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$ and fits into an equivalence*

$$\mathrm{map}_{\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}}(\mathsf{Grpd}_{\mathcal{B}}[-], -) \simeq \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-, -)$$

of bifunctors $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Top}_{\mathcal{B}}^{\mathrm{L}} \to \mathsf{Grpd}_{\widehat{\mathcal{B}}}$.

Proof. Note that if $A \in \mathcal{B}$ is an arbitrary object, we deduce from Proposition 3.5.1.9 that the base change of the canonical map $\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$ along π_A^* can be identified with the canonical map $\pi_A^*\mathsf{C} \to \mathsf{Grpd}_{\mathcal{B}}[[]\mathcal{B}_{/A}]\pi_A^*\mathsf{C}$. Thus, in light of Remark 6.2.1.3, the result is an immediate consequence of Proposition 6.2.2.1. □

Corollary 6.2.2.2 already implies the existence of certain colimits in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$:

Corollary 6.2.2.3. *For any diagram $d \colon \mathsf{I} \to \mathsf{Cat}_{\mathcal{B}}$, the induced cocone*

$$\mathsf{Grpd}_{\mathcal{B}}[d(-)] \to \mathsf{Grpd}_{\mathcal{B}}[\operatorname{colim} d]$$

is a colimit cocone in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$.

Proof. Combine Proposition 6.2.2.1 with Proposition 3.2.5.8. $\square$

By combining Corollary 6.2.2.3 with the evident equivalence $1 \simeq \varnothing^{\mathrm{lex}}$, we in particular obtain:

Corollary 6.2.2.4. *The universe* $\mathsf{Grpd}_{\mathcal{B}}$ *defines an initial object in* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. $\square$

In light of Corollary 6.2.2.4, we may now define:

Definition 6.2.2.5. Let X be a $\mathcal{B}$-topos. Then the unique algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{X}$ is referred to as the *constant sheaf functor*, and its right adjoint $\Gamma_{\mathsf{X}} \colon \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}$ is called the *global sections functor*.

Remark 6.2.2.6. If X is a $\mathcal{B}$-topos, then the global sections functor Γ_{X} is equivalent to $\mathrm{map}_{\mathsf{X}}(1_{\mathsf{X}}, -)$, where 1_{X} denotes the final object in X. In fact, since the unique algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{X}$ is left exact and since $\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}, -) \simeq \mathrm{id}_{\mathsf{Grpd}_{\mathcal{B}}}$ by Proposition 2.2.2.4, this follows immediately from the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}$.

We now turn to the proof of Proposition 6.2.2.1. As a first step, we need to establish that presheaf $\mathcal{B}$-categories are $\mathcal{B}$-topoi:

Proposition 6.2.2.7. *For every $\mathcal{B}$-category C, the large $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is a $\mathcal{B}$-topos.*

Proof. Since $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is presentable, we only need to show that it satisfies descent. Let us first show that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ has universal colimits. Let therefore $f \colon F \to G$ be an arbitrary map of presheaves on C in context $A \in \mathcal{B}$. By Remark 2.3.2.1, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$, so that we can assume that $A \simeq 1$. By Lemma 3.4.1.4 there are equivalences $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})$ and $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/G} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/G})$ with respect to which the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/G})$ that corresponds to $f_!$ carries the final presheaf on $\mathsf{C}_{/F}$ to the presheaf that classifies the right fibration $f_! \colon \mathsf{C}_{/F} \to \mathsf{C}_{/G}$. As the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/G})$ is a morphism of right fibrations over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, this map is uniquely specified by the image of the final object. We thus conclude that this functor must be equivalent to the functor of left Kan extension along $f_! \colon \mathsf{C}_{/F} \to \mathsf{C}_{/G}$. Its right adjoint is simply given by precomposition with $f_!$, which defines a cocontinuous functor. Hence

$f^* : \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})_{/G} \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ must be cocontinuous as well, and we conclude that $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ has universal colimits.

To conclude the proof, we need to show that $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ has effective descent. By Proposition 6.1.3.12, this is equivalent to the condition that for every $A \in \mathcal{B}$, every small $\mathcal{B}_{/A}$-category I and every cartesian map $d' \to d$ in $\underline{\mathrm{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}))$, the naturality square

$$\begin{tikzcd} d' \arrow[r, "\eta"] \arrow[d] & \mathrm{diag}(\operatorname{colim} d') \arrow[d] \\ d \arrow[r, "\eta"] & \mathrm{diag}(\operatorname{colim} d) \end{tikzcd}$$

is a pullback. Upon replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$, we may assume without loss of generality $A \simeq 1$. Moreover, since limits and colimits in functor $\mathcal{B}$-categories are detected object-wise (Proposition 3.2.3.2), we can reduce to $\mathsf{C} \simeq 1$. In this case, the result follows from Corollary 6.2.1.7. ☐

Next, we need to establish an internal analogue of the well-known statement that left exact functors with values in an ∞-topos are equivalently *flat* functors [49, Proposition 6.1.5.2]. The key ingredient to this result is the following lemma:

Lemma 6.2.2.8. *Let* C *be a* $\mathcal{B}$*-category, let* X *be a* $\mathcal{B}$*-topos and let* $f\colon \mathsf{C} \to \mathsf{X}$ *be a functor. Suppose that the Yoneda extension* $h_!(f) : \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ *of this functor preserves the limit of every cospan in* $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ *(in arbitrary context* $A \in \mathcal{B}$*) that is contained in the essential image of the Yoneda embedding* $h\colon \mathsf{C} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$*. Then* $h_!(f)$ *preserves pullbacks.*

Proof. Suppose that

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ G \arrow[r] & F \end{tikzcd}$$

is a pullback square in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$. We need to show that the image of this square along $(h_{\mathsf{C}})_!(f)$ is a pullback in X. By combining Remark 2.3.2.1 and Remark 3.4.3.2, we may assume without loss of generality that the above square is in context $1 \in \mathcal{B}$.

Let us first show that the claim is true whenever F is representable by an object $c\colon 1 \to \mathsf{C}$. In this case, Lemma 3.4.1.4 implies that there is an equivalence

$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/c})$ with respect to which the composition $(h_! f)(\pi_{h(c)})_!$ can be identified with the left Kan extension of $f(\pi_c)_!$ along the Yoneda embedding $\mathsf{C}_{/c} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/c})$. Therefore, by replacing C with $\mathsf{C}_{/c}$, one can assume that $F \simeq 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$. Now the product functor $G \times - \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ being cocontinuous (by Proposition 6.2.2.7) implies that the canonical map

$$h_! f(G \times -) \to h_! f(G) \times h_! f(-)$$

is a morphism between cocontinuous functors. On account of the universal property of presheaf $\mathcal{B}$-categories, this means that we may further reduce to the case where H is representable. By the same argument, the presheaf G can also be assumed to be representable. In this case, the claim follows from the assumption on $h_!(f)$.

We now turn to the general case. By Proposition 3.4.1.1, there is a diagram $d \colon \mathsf{I} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that $F \simeq \operatorname{colim} d$ and such that d takes values in $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Let us write $\overline{d}$ for the associated colimit cocone. In light of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}}$ from Remark 6.1.3.7 and by identifying the above pullback square with a diagram in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$, we obtain a pullback diagram

$$\begin{tikzcd} \overline{q} \arrow[r] \arrow[d] & \overline{p} \arrow[d] \\ \overline{g} \arrow[r] & \overline{d} \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}}$. By the above and the fact that limits in functor $\mathcal{B}$-categories can be computed object-wise (Proposition 3.2.3.2), the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/d} \xrightarrow{(h_! f)_*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{X})_{/(h_! f)_* d}$$

carries the above pullback diagram of cocones to a pullback and therefore in particular to a diagram in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{X})^{\mathrm{cart}}_{/(h_! f)_* d}$. By using descent in X and in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (cf. Proposition 6.2.2.7) together with the fact that $h_!(f)$ is cocontinuous, this implies that the functor $(h_! f)_* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{X})_{/\overline{d}}$ preserves the above pullback. Upon evaluating the latter at the cone point $\infty \colon 1 \to \mathsf{I}^{\triangleright}$, we recover the image of the original pullback square along $h_!(f)$, hence the claim follows. $\square$

Proposition 6.2.2.9. *Let* C *be a* $\mathcal{B}$*-category with finite limits, and let* X *be a* $\mathcal{B}$*-topos. Then a functor* $f\colon \mathsf{C} \to \mathsf{X}$ *preserves finite limits if and only if its left Kan extension* $h_!(f)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ *preserves finite limits.*

Proof. Since the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ preserves finite limits, it is clear that the condition is sufficient. Conversely, suppose that f preserves finite limits. Since the final object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is contained in C, it is clear that $h_!(f)$ preserves final objects. We therefore only need to show that this functor also preserves pullbacks, which is an immediate consequence of Lemma 6.2.2.8. □

By combining Proposition 6.2.2.9 with the universal property of presheaf $\mathcal{B}$-categories and Remark 6.2.1.3 and Remark 3.3.3.4, we now conclude:

Corollary 6.2.2.10. *For any* $\mathcal{B}$*-category* C *with finite limits and any* $\mathcal{B}$*-topos* X*, the functor of left Kan extension along the Yoneda embedding* $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *gives rise to an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{X}),$$

where $\underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ *is the full subcategory of* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ *that is spanned by the left exact functors in arbitrary context.* □

Proof of Proposition 6.2.2.1. Combine Proposition 6.2.2.7 with Corollary 6.2.2.10 and the universal property of free $\mathsf{Fin}_{\mathcal{B}}$-completion, cf. Theorem 3.5.1.12. □

6.2.3. Presentations of $\mathcal{B}$-topoi

Recall from Definition 5.3.3.4 that a Bousfield localisation $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$ is said to be *accessible* if the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous for some choice of *sound doctrine* U (see Definition 5.1.3.8 and Definition 5.1.2.7). We will say that the localisation is *left exact* if L preserves finite limits. The main goal of this section is to prove the following characterisation of $\mathcal{B}$-topoi:

Theorem 6.2.3.1. *A large* $\mathcal{B}$*-category* X *is a* $\mathcal{B}$*-topos if and only if there is a* $\mathcal{B}$*-category* C *such that* X *arises as a left exact and accessible localisation of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*.*

The proof of Theorem 6.2.3.1 relies on the following two lemmas:

Lemma 6.2.3.2. *Suppose that*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\pi_U^*"] \arrow[d, "L"] & \mathcal{X}_{/U} \arrow[d, "L'"] \\
\mathcal{Y} \arrow[r, "h^*"] & \mathcal{Z}
\end{tikzcd}$$

is a commutative square in $\mathrm{Top}_\infty^{\mathrm{L}}$ *in which L and L' are Bousfield localisations. Suppose furthermore that h^* admits a left adjoint $h_!$ and that the mate transformation $\phi\colon h_!L' \to L(\pi_U)_!$ is an equivalence. Then h^* is étale.*

Proof. We would like to apply [49, Proposition 6.3.5.11], which says that the functor h^* is étale precisely if $h_!$ is conservative and if for every map $f\colon W \to V$ in $\mathcal{Y}$ and every object $P \in \mathcal{Z}_{/h^*(V)}$, the canonical map

$$\alpha\colon h_!(h^*(W) \times_{h^*(V)} P) \to W \times_V h_!(P)$$

is an equivalence.

Let us begin by showing that $h_!$ is conservative. To that end, note that if $f\colon V \to W$ is a map in $\mathcal{X}_{/U}$ such that $L(\pi_U)_!(f)$ is an equivalence, then $L'(f)$ is an equivalence. In fact, since the adjunction unit of $(\pi_U)_! \dashv \pi_U^*$ exhibits f as a pullback of $\pi_U^*(\pi_U)_!(f)$, the localisation functor L' being left exact implies that $L'(f)$ is a pullback of $L'\pi_U^*(\pi_U)_!(f) \simeq h^*L(\pi_U)_!(f)$. Since the latter is an equivalence, the claim follows. Applying this observation to a map f that is contained in $\mathcal{Z}$ and using the assumption that the mate transformation $\phi\colon h_!L' \to L(\pi_U)_!$ is an equivalence, we deduce that $h_!$ is indeed conservative.

To conclude the proof, we show that the map α is an equivalence. From the map $f\colon W \to V$ in $\mathcal{Y}$ we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathcal{X}_{/V} \arrow[rr, "(\pi_U^*)_{/V}"] \arrow[dl, "L_{/V}"'] \arrow[dd, "f^*"] & & \mathcal{X}_{/U\times V} \arrow[dl, "L'_{/\pi_U^*(V)}"'] \arrow[dd, "(\pi_U^* f)^*"] \\
\mathcal{Y}_{/V} \arrow[rr, "h^*_{/V}"] \arrow[dd, "f^*"] & & \mathcal{Z}_{/h^*(V)} \arrow[dd, "(h^* f)^*"] & \\
 & \mathcal{X}_{/W} \arrow[rr, "(\pi_U^*)_{/W}"] \arrow[dl, "L_{/W}h^*_{/W}"'] & & \mathcal{X}_{/U\times W} \arrow[dl, "L'_{/\pi_U^*(W)}"] \\
\mathcal{Y}_{/W} \arrow[rr] & & \mathcal{Z}_{/h^*(W)} &
\end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{L}}$ in which all of the four maps pointing to the right admit a left adjoint. Note that α being an equivalence for all $P \in \mathcal{Z}_{/h^*(V)}$ precisely means that the front square is left adjointable (i.e. has an invertible mate transformation). Now since the mate $\phi\colon h_! i' \to i(\pi_U)_!$ is by assumption an equivalence, it follows that both the top and the bottom square in the above diagram are left adjointable. Since π_U^* is an étale algebraic morphism, the back square is left adjointable as well. Therefore, by combining the functoriality of the mate construction with the fact that the four maps in the above diagram pointing to the front are localisation functors and thus in particular essentially surjective, we conclude that the front square must be left adjointable as well, as desired. □

Lemma 6.2.3.3. *Let* D *be a presentable* $\mathcal{B}$*-category. Then there exists a sound doctrine* U *such that* D *is* U*-accessible and* $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ *is closed under finite limits in* D.

Proof. Since D is presentable, there exists a $\mathcal{B}$-category C and a sound doctrine U such that D arises as a U-accessible Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$ (cf. Theorem 5.4.2.5). In particular, for every sound doctrine V that contains U, the $\mathcal{B}$-category D is V-accessible (Corollary 5.3.3.3). Therefore, for any cardinal κ we can always find a $\mathcal{B}$-regular cardinal $\tau \geq \kappa$ such that D is $\mathsf{Cat}_{\mathcal{B}}^{\tau}$-accessible. By Remark 5.2.2.5, we can always choose τ such that $\tau \gg \kappa$. Now D being presentable implies that D is section-wise accessible (Theorem 5.4.2.5). Therefore, if $\mathcal{G} \hookrightarrow \mathcal{B}$ is a small generating subcategory, we may find a regular cardinal κ such that $\mathsf{D}(G)$ is κ-accessible for all $G \in \mathcal{G}$. Let us choose a $\mathcal{B}$-regular cardinal $\tau \gg \kappa$ such that

1. $\mathcal{G}$ is contained in $\mathcal{B}^{\tau\text{-cpt}}$;
2. D is $\mathsf{Cat}_{\mathcal{B}}^{\tau}$-accessible;
3. $\mathsf{D}(G)^{\kappa\text{-cpt}}$ is τ-small for all $G \in \mathcal{G}$.

Then [49, Proposition 5.4.7.4] implies that the inclusion $\mathsf{D}(G)^{\tau\text{-cpt}} \hookrightarrow \mathsf{D}(G)$ is closed under finite limits for all $G \in \mathcal{G}$. Recall from Corollary 2.2.2.8 that for every object $d\colon A \to \mathsf{D}$ the mapping functor $\mathrm{map}_{\mathsf{D}(A)}(d, -)$ can be identified with the composition

$$\mathsf{D}(A) \xrightarrow{\mathrm{map}_{\mathsf{D}}(d,-)(A)} \mathcal{B}_{/A} \xrightarrow{\Gamma_{\mathcal{B}_{/A}}} \mathrm{Ani}\,.$$

By combining this observation with Proposition 5.3.2.4 and the fact that $\mathcal{B}$ is generated by $\mathcal{G}$, we find that for any $G \in \mathcal{G}$ an object $d\colon G \to \mathsf{D}$ is contained in $\mathsf{D}^{\mathrm{Cat}^{\tau}_{\mathcal{B}}\text{-cpt}}$ if and only if for every $H \in \mathcal{G}$ and every map $s\colon H \to G$ the object $s^*(d) \in \mathsf{D}(H)$ is contained in $\mathsf{D}(H)^{\tau\text{-cpt}}$. Since s^* commutes with limits, this implies that the inclusion $\mathsf{D}^{\mathrm{Cat}^{\tau}_{\mathcal{B}}\text{-cpt}} \hookrightarrow \mathsf{D}$ is closed under finite limits. □

Proof of Theorem 6.2.3.1. Suppose first that X is a left exact and U-accessible localisation of $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$, and let us show that X is a $\mathcal{B}$-topos. We would like to apply Theorem 6.2.1.5. First, note that by choosing a $\mathcal{B}$-regular cardinal κ such that $\mathsf{U} \hookrightarrow \mathrm{Cat}^{\kappa}_{\mathcal{B}}$, we may assume that X is a $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-accessible Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})$. Therefore, for every $A \in \mathcal{B}$ the ∞-category $\mathsf{X}(A)$ is a κ-accessible and left exact Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C})(A)$, and since the latter is an ∞-topos by Proposition 6.2.2.7, it follows that $\mathsf{X}(A)$ is an ∞-topos as well. Moreover, if $s\colon B \to A$ is a map in $\mathcal{B}$, the fact that X is a presentable $\mathcal{B}$-category (see Theorem 5.4.2.5) implies that $s^*\colon \mathsf{X}(A) \to \mathsf{X}(B)$ is continuous and cocontinuous and therefore in particular an algebraic morphism that admits a left adjoint $s_!\colon \mathsf{X}(B) \to \mathsf{X}(A)$. We are therefore in the situation of Lemma 6.2.3.2 and may thus conclude that s^* is an étale algebraic morphism. Theorem 6.2.1.5 thus implies that X is a $\mathcal{B}$-topos.

Conversely, suppose that X is a $\mathcal{B}$-topos. Then X is presentable, consequently Lemma 6.2.3.3 implies that there exists a sound doctrine U such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. Then we may identify $\mathsf{X} \simeq \underline{\mathrm{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{X}^{\mathsf{U}\text{-cpt}})$, and since X is cocomplete the inclusion $\mathsf{X} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{X}^{\mathsf{U}\text{-cpt}})$ admits a left adjoint $L\colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{X}^{\mathsf{U}\text{-cpt}}) \to \mathsf{X}$ which is obtained as the left Kan extension of the inclusion $\mathsf{X}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{X}$ (see Corollary 3.5.1.13). By Proposition 6.2.2.9, the functor L is left exact, hence the claim follows. □

Corollary 6.2.3.4. *For any $\mathcal{B}$-topos X and any $\mathcal{B}$-category D, the functor $\mathcal{B}$-category $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{X})$ is again a $\mathcal{B}$-topos.*

Proof. Choose a left exact and accessible Bousfield localisation $L\colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$. Then the postcomposition functor $L_*\colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}^{\mathrm{op}}) \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{X})$ is again an accessible and left exact Bousfield localisation, hence the claim follows. □

Corollary 6.2.3.5. *A large $\mathcal{B}$-category* X *is a $\mathcal{B}$-topos if and only if there is a $\mathcal{B}$-category* C *and a left exact and accessible Bousfield localisation* $L\colon \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \to \mathsf{X}$.

Proof. By Theorem 6.2.3.1, it suffices to show that every presheaf $\mathcal{B}$-topos arises as a left exact and accessible Bousfield localisation of a free $\mathcal{B}$-topos. But if C is a $\mathcal{B}$-category, the fact that $i\colon \mathsf{C} \hookrightarrow \mathsf{C}^{\mathrm{lex}}$ is fully faithful implies that $i_*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$ is fully faithful too (by the dual of Theorem 3.4.3.5), hence i^* is a left exact and accessible Bousfield localisation. □

Corollary 6.2.3.6. *Any $\mathcal{B}$-topos* X *is a pushout of free $\mathcal{B}$-topoi.*

Proof. By Corollary 6.2.3.5, we may choose a small $\mathcal{B}$-category C and a left exact and accessible Bousfield localisation $L\colon \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \to \mathsf{X}$. We can therefore find a small subcategory $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$ such that L induces an equivalence $\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]) \simeq \mathsf{X}$ (see Theorem 5.4.2.5). Note that a functor $\mathsf{X} \to \mathsf{Y}$ between $\mathcal{B}$-topoi is an algebraic morphism if and only if its precomposition with L is one (this is easily deduced from Remark 6.2.1.3 and the explicit computation of colimits in a Bousfield localisation, cf. Proposition 3.2.2.14. Therefore, by combining Corollary 5.4.3.2 with Proposition 6.2.2.1, we deduce that the induced square

$$\begin{tikzcd} \mathsf{Grpd}_{\mathcal{B}}[\mathsf{W}] \arrow[r] \arrow[d] & \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \arrow[d] \\ \mathsf{Grpd}_{\mathcal{B}}\left[\mathsf{W}^{\mathrm{gpd}}\right] \arrow[r] & \mathsf{X} \end{tikzcd}$$

is a pushout in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. □

6.2.4. The $\mathsf{Cat}_{\widehat{\mathcal{B}}}$-enrichment of $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$

Recall from Proposition 3.2.6.3 that $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ is *cartesian closed*, i.e. that forming functor $\mathcal{B}$-categories defines a bifunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{op}} \times \mathsf{Cat}_{\widehat{\mathcal{B}}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ and therefore in particular a bifunctor $(\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$. Let us denote by $p\colon \mathsf{P} \to (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ the unstraightening of the latter (in the sense of Theorem 4.4.3.1). Explicitly, an object $A \to \mathsf{P}$ is given by a functor $\pi_A^*\mathsf{X} \to \pi_A^*\mathsf{Y}$ between $\mathcal{B}_{/A}$-topoi. Let $\mathsf{Q} \hookrightarrow \mathsf{P}$ be the full subcategory that is spanned by those objects that correspond to algebraic morphisms. By Lemma 6.2.4.1 below, the induced functor $q\colon \mathsf{Q} \to (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is a cocartesian fibration as well and therefore classified by a bifunctor $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(-,-)\colon (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$.

Lemma 6.2.4.1. *Let $p\colon \mathsf{P} \to \mathsf{C}$ be a cocartesian fibration of $\mathcal{B}$-categories. Let $\mathsf{Q} \hookrightarrow \mathsf{P}$ be a full subcategory such that for each map $f\colon c \to d$ in C in context $A \in \mathcal{B}$ the induced functor $f_!\colon \mathsf{P}|_c \to \mathsf{P}|_d$ restricts to a functor $\mathsf{Q}|_c \to \mathsf{Q}|_d$. Then the induced functor $q\colon \mathsf{Q} \to \mathsf{C}$ is a cocartesian fibration as well, and the inclusion $\mathsf{Q} \hookrightarrow \mathsf{P}$ is a cocartesian functor.*

Proof. Using Proposition 4.1.2.7, it will be enough to show that for any cocartesian lift $\phi\colon x \to y$ of f in P in which x is contained in $\mathsf{Q}|_c$, the object y is contained in $\mathsf{Q}|_d$. But this immediately follows from the assumptions, using Remark 4.4.5.4. □

Definition 6.2.4.2. We define the *functor of $\mathcal{B}$-points* as the functor

$$\underline{\mathsf{Pt}}_{\mathcal{B}} = \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(-, \mathsf{Grpd}_{\mathcal{B}})\colon \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}.$$

Recall that if C is a $\mathcal{B}$-category and X is a $\mathcal{B}$-topos, then $\mathsf{X}^{\mathsf{C}} = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ is a $\mathcal{B}$-topos as well (Corollary 6.2.3.4). Moreover, as precomposition and postcomposition preserves all limits and colimits, the bifunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ restricts to a bifunctor $(-)^{(-)}\colon \mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ which we refer to as the *powering* of $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ over $\mathsf{Cat}_{\mathcal{B}}$. This terminology is justified by the following proposition:

Proposition 6.2.4.3. *The powering bifunctor $(-)^{(-)}$ fits into an equivalence*

$$\mathrm{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(-, (-)^{(-)}) \simeq \mathrm{map}^{\mathrm{alg}}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)).$$

Proof. If C is a $\mathcal{B}$-category and X and Y are $\mathcal{B}$-topoi, then Lemma 5.5.1.3 and its dual imply that a functor $\mathsf{X} \to \mathsf{Y}^{\mathsf{C}}$ defines an algebraic morphism if and only if the transpose functor $\mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y})$ takes values in $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X}, \mathsf{Y})$. By replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$ (which is made possible by Remark 6.2.1.3 and Proposition 1.2.5.4) , one obtains that the same is true for any object $A \to \mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. Hence, the equivalence

$$\mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(i(-), \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)) \simeq \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-, \underline{\mathsf{Fun}}_{\mathcal{B}}(i(-),-))$$

of functors $\mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times \mathsf{Cat}^{\mathrm{op}}_{\widehat{\mathcal{B}}} \times \mathsf{Cat}_{\widehat{\mathcal{B}}} \to \mathsf{Grpd}_{\widehat{\mathcal{B}}}$ (where $i\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is the inclusion) restricts in the desired way. □

Corollary 6.2.4.4. *The functor of $\mathcal{B}$-points $\underline{\mathrm{Pt}}_{\mathcal{B}}$ is a partial right adjoint of the functor*

$$\mathrm{Grpd}_{\mathcal{B}}^{(-)} \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Top}_{\mathcal{B}}^{\mathrm{R}},$$

in the sense that there is an equivalence

$$\mathrm{map}_{\mathrm{Top}_{\mathcal{B}}^{\mathrm{R}}}(\mathrm{Grpd}_{\mathcal{B}}^{(-)}, -) \simeq \mathrm{map}_{\mathrm{Cat}_{\widehat{\mathcal{B}}}}(-, \underline{\mathrm{Pt}}_{\mathcal{B}}(-))$$

of functors $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathrm{Top}_{\mathcal{B}}^{\mathrm{R}} \to \mathrm{Grpd}_{\widehat{\mathcal{B}}}$. □

In light of Corollary 6.2.4.4, it is reasonable to define:

Definition 6.2.4.5. If C is a $\mathcal{B}$-category, we refer to the $\mathcal{B}$-topos $\mathsf{C}^{\mathrm{disc}} = \mathrm{Grpd}_{\mathcal{B}}^{\mathsf{C}}$ as the *discrete* $\mathcal{B}$-topos associated with C.

Lastly, we note that the large $\mathcal{B}$-category $\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}$ is also *tensored* over $\mathrm{Cat}_{\mathcal{B}}$:

Proposition 6.2.4.6. *There is a bifunctor* $- \otimes - \colon \mathrm{Cat}_{\mathcal{B}} \times \mathrm{Top}_{\mathcal{B}}^{\mathrm{L}} \to \mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}$ *that fits into an equivalence*

$$\mathrm{map}_{\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}}(- \otimes -, -) \simeq \mathrm{map}_{\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}}(-, (-)^{(-)})$$

of functors $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{op}} \times (\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}})^{\mathrm{op}} \times \mathrm{Top}_{\mathcal{B}}^{\mathrm{L}} \to \mathrm{Grpd}_{\widehat{\mathcal{B}}}$.

Proof. As an immediate consequence of the constructions, if C and D are $\mathcal{B}$-categories, we obtain a chain of equivalences

$$\begin{aligned} \mathrm{map}_{\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}}(\mathrm{Grpd}_{\mathcal{B}}[\mathsf{C}], (-)^{\mathsf{D}}) &\simeq \mathrm{map}_{\mathrm{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{C}, \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{D}, -)) \\ &\simeq \mathrm{map}_{\mathrm{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{C} \times \mathsf{D}, -) \\ &\simeq \mathrm{map}_{\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}}(\mathrm{Grpd}_{\mathcal{B}}[\mathsf{C} \times \mathsf{D}], -), \end{aligned}$$

which implies that the functor $\mathrm{map}_{\mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}}(\mathsf{X}, (-)^{\mathsf{D}})$ is representable whenever X is in the image of $\mathrm{Grpd}_{\mathcal{B}}[-]$. But since every $\mathcal{B}$-topos is a pushout of such $\mathcal{B}$-topoi (see Corollary 6.2.3.6), this functor must be representable for *any* $\mathcal{B}$-topos X. As by Remark 6.2.1.3 the same argument shows that this is the case for every object $\mathsf{X} \colon A \to \mathrm{Top}_{\mathcal{B}}^{\mathrm{L}}$ and every $\mathcal{B}_{/A}$-category D, the result follows. □

6.2.5. Relative ∞-topoi as $\mathcal{B}$-topoi

By Theorem 6.2.1.5 and the evident fact that an algebraic morphism between $\mathcal{B}$-topoi induces an algebraic morphism of ∞-topoi upon taking global sections, we obtain a functor $\Gamma\colon\ \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Top}^{\mathrm{L}}_{\infty}$. By making use of the fact that the universe $\mathsf{Grpd}_{\mathcal{B}}$ is an initial object in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ (Corollary 6.2.2.4), this functor factors through the projection $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/} \to \mathrm{Top}^{\mathrm{L}}$, so that we end up with a functor

$$\Gamma\colon\ \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}.$$

The main goal in this section is to prove:

Theorem 6.2.5.1. *The global sections functor* $\Gamma\colon\ \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}$ *is an equivalence of* ∞*-categories.*

Remark 6.2.5.2. Theorem 6.2.5.1 implies that the datum of a $\mathcal{B}$-topos X is equivalent to that of a geometric morphism $f_*\colon\ \mathcal{X} \to \mathcal{B}$. We will refer to the latter as the geometric morphism that is *associated* with X.

Remark 6.2.5.3. The inverse to the equivalence $\Gamma\colon\ \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \simeq (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}$ from Theorem 6.2.5.1 can be described explicitly as follows: Given an algebraic morphism $f^*\colon\ \mathcal{B} \to \mathcal{X}$, recall that we get an induced functor $f_*\colon\ \mathrm{Cat}(\widehat{\mathcal{X}}) \to \mathrm{Cat}(\widehat{\mathcal{B}})$. Then Theorem 6.2.1.5 easily implies that the large $\mathcal{B}$-category $\mathsf{X} = f_*(\mathsf{Grpd}_{\mathcal{X}})$ is a $\mathcal{B}$-topos (since the associated sheaf on $\mathcal{B}$ is simply given by $\mathcal{X}_{/f^*(-)}$). Moreover, the functor f^* induces a map $\mathcal{B}_{/-} \to \mathcal{X}_{/f^*(-)}$ of sheaves on $\mathcal{B}$ that recovers the unique algebraic morphism $\mathrm{const}_{\mathsf{X}}\colon\ \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{X}$. This implies that X is the image of $f^*\colon\ \mathcal{B} \to \mathcal{X}$ under the equivalence from Theorem 6.2.5.1.

The proof of Theorem 6.2.5.1 requires a few preparations. We begin with the following lemma:

Lemma 6.2.5.4. *Let* $\mathcal{C}$ *be an* ∞*-category with an initial object* $\varnothing_{\mathcal{C}}$*, and let* $\mathcal{D}$ *be an* ∞*-category that admits pushouts. Then the evaluation functor* $\mathrm{ev}_{\varnothing_{\mathcal{C}}}\colon\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ *is a cocartesian fibration. Moreover, a morphism* $\phi\colon F \to G$ *in* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is cocartesian if and only if for every map* $f\colon c \to c'$ *in* $\mathcal{C}$ *the induced commutative* *square*

$$\begin{tikzcd} F(c) \arrow[r, "\alpha(c)"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha(c')"] & G(c') \end{tikzcd}$$

is a pushout in $\mathcal{D}$.

Proof. Note that the diagonal functor $\mathrm{diag}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ defines a left adjoint to $\mathrm{ev}_{\varnothing_{\mathcal{C}}}$. Therefore, we deduce from [32, Proposition 4.51] that a map $\alpha\colon F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is cocartesian if and only if for every $c \in \mathcal{C}$ the square

$$\begin{tikzcd} F(\varnothing_{\mathcal{C}}) \arrow[r, "\alpha(\varnothing_{\mathcal{C}})"] \arrow[d] & G(\varnothing_{\mathcal{C}}) \arrow[d] \\ F(c) \arrow[r, "\alpha(c)"] & G(c) \end{tikzcd}$$

is a pushout in $\mathcal{D}$. The assumption that $\mathcal{D}$ admits pushouts guarantees that there are enough such cocartesian maps, see [32, Corollary 4.52]. □

By Lemma 6.2.5.4, the global sections functor $\Gamma\colon \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B}) \to \mathrm{Top}^{\mathrm{L}}_{\infty}$ is a cocartesian fibration and therefore determines a left fibration

$$\Gamma\colon \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \to \mathrm{Top}^{\mathrm{L}}_{\infty},$$

where $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ is the subcategory that is spanned by the cocartesian morphisms. Moreover, observe that by Theorem 6.2.1.5 we may regard the ∞-category $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ as a (non-full) subcategory of $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$. Now the key step towards the proof of Theorem 6.2.5.1 consists of the following proposition:

Proposition 6.2.5.5. *The (non-full) inclusion* $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ *fits into a commutative diagram*

$$\begin{tikzcd} \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \arrow[r, hook] \arrow[dr, hook] & \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \arrow[d, hook] \\ & \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B}) \end{tikzcd}$$

in which the horizontal map is fully faithful. Moreover, if X *is a* $\mathcal{B}$*-topos and if* $\mathsf{X} \to F$ *is a map in* $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}}$*, then* F *is contained in* $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$*.*

Proof. If $f^* \colon \mathsf{X} \to \mathsf{Y}$ is an algebraic morphism between $\mathcal{B}$-topoi, Lemma 6.2.1.9 and the fact that f^* is cocontinuous imply that for every map $s \colon B \to A$ in $\mathcal{B}$ the induced commutative square

$$\begin{tikzcd} \mathsf{X}(A) \arrow[r, "f^*(A)"] \arrow[d, "s^*"] & \mathsf{Y}(A) \arrow[d, "s^*"] \\ \mathsf{X}(B) \arrow[r, "f^*(B)"] & \mathsf{Y}(B) \end{tikzcd}$$

is a pushout in $\mathrm{Top}^{\mathrm{L}}_{\infty}$. By Lemma 6.2.5.4, this means that the underlying map of $\mathrm{Top}^{\mathrm{L}}_{\infty}$-valued presheaves on $\mathcal{B}$ defines a cocartesian morphism over $\mathrm{Top}^{\mathrm{L}}_{\infty}$. Hence the inclusion $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ factors through the inclusion $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$. To finish the proof, it now suffices to show that for any cocartesian morphism $f \colon \mathsf{X} \to F$ of $\mathrm{Top}^{\mathrm{L}}_{\infty}$-valued presheaves on $\mathcal{B}$, the presheaf F is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ and the map f defines an algebraic morphism of $\mathcal{B}$-topoi. Since f is a cocartesian morphism and since étale algebraic morphisms are closed under pushouts in $\mathrm{Top}^{\mathrm{L}}_{\infty}$, we find that for every $s \colon B \to A$ in $\mathcal{B}$ the induced functor $s^* \colon F(A) \to F(B)$ is an étale algebraic morphism of ∞-topoi. Moreover, the pasting lemma for pushouts (and the fact that X is a $\mathcal{B}$-topos) imply that $F \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty}$ preserves pushouts. Hence Theorem 6.2.1.5 implies that F must be contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ whenever F is a sheaf. But if $d \colon I \to \mathcal{B}$ is an arbitrary diagram, then we deduce from [50, Corollary 4.7.4.18] that the commutative square

$$\begin{tikzcd} \mathsf{X}(\operatorname{colim} d) \arrow[r] \arrow[d] & F(\operatorname{colim} d) \arrow[d] \\ \lim \mathsf{X} \circ d \arrow[r] & \lim F \circ d \end{tikzcd}$$

is left adjointable and therefore a pushout in $\mathrm{Top}^{\mathrm{L}}_{\infty}$, using Lemma 6.2.1.9. Hence, since the left vertical map is an equivalence, so is the right one, which means that F is a sheaf. Finally, since f is already section-wise given by an algebraic morphism of ∞-topoi, the map defines an algebraic morphism of $\mathcal{B}$-topoi precisely if it is $\mathrm{Grpd}_{\mathcal{B}}$-cocontinuous, which again follows from Lemma 6.2.1.9. $\square$

Corollary 6.2.5.6. *The global sections functor* $\Gamma\colon \mathsf{Top}^{\mathrm{L}}(\mathcal{B}) \to \mathsf{Top}^{\mathrm{L}}_{\infty}$ *is a left fibration.*

Proof. By the first part of Proposition 6.2.5.5, every map in $\mathsf{Top}^{\mathrm{L}}(\mathcal{B})$ is cocartesian. By its second part, if X is a $\mathcal{B}$-topos and $f^*\colon \Gamma(\mathsf{X}) \to \mathcal{Z}$ is an arbitrary algebraic morphism, the codomain of the cocartesian lift $\mathsf{X} \to F$ of f^* in $\mathrm{PSh}_{\mathsf{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ is again a $\mathcal{B}$-topos. Hence the claim follows. □

Proof of Theorem 6.2.5.1. By Corollary 6.2.5.6, the functor $\Gamma\colon \mathsf{Top}^{\mathrm{L}}(\mathcal{B}) \to \mathsf{Top}^{\mathrm{L}}_{\infty}$ is a left fibration, hence so is the functor $\Gamma\colon \mathsf{Top}^{\mathrm{L}}(\mathcal{B}) \to (\mathsf{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}$. Since this functor carries the initial object $\mathsf{Grpd}_{\mathcal{B}}$ to the initial object $\mathrm{id}_{\mathcal{B}}$, it must be an initial functor as well. Hence Γ is an equivalence. □

6.2.6. Limits and colimits of $\mathcal{B}$-topoi

In this section, we discuss how one can construct limits and colimits in the $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ of $\mathcal{B}$-topoi. The construction of *limits* in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is rather easy: they are simply computed in $\mathsf{Cat}_{\widehat{\mathcal{B}}}$. This is analogous to how limits are computed in the $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories, cf. Proposition 5.4.4.10. The proof of this statement follows along similar lines as well.

Proposition 6.2.6.1. *The large $\mathcal{B}$-category* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ *is complete, and the inclusion* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ *is continuous.*

Proof. As in the proof of Proposition 5.4.4.10, it will be enough to show that whenever K is either given by the constant $\mathcal{B}$-category Λ^2_0 or by a $\mathcal{B}$-groupoid, the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ admits K-indexed limits and the inclusion $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves K-indexed limits.

We begin with the case were K is a $\mathcal{B}$-groupoid. Let us set $A = \mathsf{K}_0$. Since the functor $(\pi_A)_*\colon \mathsf{Cat}(\widehat{\mathcal{B}}_{/A}) \to \mathsf{Cat}(\widehat{\mathcal{B}})$ is given by precomposition with π_A^*, Theorem 6.2.1.5 implies that $(\pi_A)_*$ takes objects in $\mathsf{Top}^{\mathrm{L}}(\mathcal{B}_{/A})$ to objects in $\mathsf{Top}^{\mathrm{L}}(\mathcal{B})$. Furthermore it easily follows from Corollary 3.5.4.4 that $(\pi_A)_*$ therefore defines a functor $\mathsf{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \to \mathsf{Top}^{\mathrm{L}}(\mathcal{B})$. Moreover, since the adjunction unit $\mathrm{id}_{\mathsf{Cat}(\widehat{\mathcal{B}})} \to (\pi_A)_*\pi_A^*$ is given by precomposition with the adjunction counit $(\pi_A)_!\pi_A^* \to \mathrm{id}_{\mathcal{B}}$ and vice versa for the adjunction counit, the same argument shows together with the fact that $\mathcal{B}$-topoi are $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete that these two maps must also restrict in the desired way. Hence $(\pi_A)_* \colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ defines a right adjoint of π_A^*.

Now let us assume that $\mathsf{K} = \Lambda_0^2$, i.e. let

$$\begin{tikzcd} \mathsf{X} \times_{\mathsf{Z}} \mathsf{Y} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & \mathsf{Y} \arrow[d, "g"] \\ \mathsf{X} \arrow[r, "f"] & \mathsf{Z} \end{tikzcd}$$

be a pullback square in $\mathrm{Cat}(\widehat{\mathcal{B}})$ in which the cospan in the lower right corner is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. By Proposition 5.4.4.10 this square defines a pullback in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$, and [49, Proposition 6.3.2.3] implies that both pr_0 and pr_1 preserve finite limits. Hence the above pullback square is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ whenever $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ satisfies descent. But the codomain fibration $(\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y})^{\Delta^1} \to \mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ can be identified with the pullback of the cospan

$$\mathrm{pr}_0^*(\mathsf{X}^{\Delta^1}) \to \mathrm{pr}_0^* f^*(\mathsf{Z}^{\Delta^1}) \leftarrow \mathrm{pr}_1^*(\mathsf{Y}^{\Delta^1})$$

of cartesian fibrations over $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$, which implies that we may identify $(\mathsf{X} \times_{\mathsf{Z}} \mathsf{Z})_{/-}$ with the pullback $\mathsf{X}_{/\mathrm{pr}_0^*(-)} \times_{\mathsf{Z}_{/\mathrm{pr}_0^* f^*(-)}} \mathsf{Y}_{/\mathrm{pr}_1^*(-)}$ in $\underline{\mathrm{Fun}}_{\mathcal{B}}((\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y})^{\mathrm{op}}, \mathrm{Cat}_{\widehat{\mathcal{B}}})$. Since all four functors in the initial pullback square are continuous, we conclude that $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ satisfies descent provided that continuous functors are closed under pullbacks in $\underline{\mathrm{Fun}}_{\mathcal{B}}((\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y})^{\mathrm{op}}, \mathrm{Cat}_{\widehat{\mathcal{B}}})$, which follows immediately from the fact that limit functors are themselves continuous (see the proof of Lemma 5.1.5.3 for more details). We complete the proof by showing that if we are given another $\mathcal{B}$-topos E and algebraic morphisms $h \colon \mathsf{E} \to \mathsf{X}$ and $k \colon \mathsf{E} \to \mathsf{Z}$ together with an equivalence $f \circ h \simeq g \circ k$, the induced map $\mathsf{E} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{Z}$ is an algebraic morphism as well. That this map is cocontinuous follows from Proposition 5.4.4.10, and that it preserves finite limits is a consequence of the fact that this property can be checked section-wise. □

As a consequence of Proposition 6.2.6.1, we can now upgrade the equivalence from Theorem 6.2.5.1 to a *functorial* one:

Corollary 6.2.6.2. *Let* $(\mathrm{Top}_\infty^{\mathrm{L}})_{(\mathcal{B}_{/-})/}$ *be the* $\widehat{\mathrm{Cat}}_\infty$*-valued presheaf on* $\mathcal{B}$ *whose associated cocartesian fibration on* $\mathcal{B}^{\mathrm{op}}$ *is given by the pullback of the domain* *fibration* $d_1\colon \operatorname{Fun}(\Delta^1, \mathrm{Top}^{\mathrm{L}}_\infty) \to \mathrm{Top}^{\mathrm{L}}_\infty$ *along* $\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}_\infty$. *Then this presheaf is a sheaf whose associated large* $\mathcal{B}$*-category is equivalent to* $\mathrm{Top}^{\mathrm{L}}_{\mathcal{B}}$.

Proof. To begin with, note that the functor $(\mathrm{Grpd}_{\mathcal{B}})_{/-}\colon \mathrm{Grpd}_{\mathcal{B}}^{\mathrm{op}} \to \mathrm{Cat}_{\widehat{\mathcal{B}}}$ takes values in $\mathrm{Top}^{\mathrm{L}}_{\mathcal{B}}$ (see the discussion before Definition 6.2.9.1 below). Thus, by combining descent with Proposition 6.2.6.1, we obtain an $\mathrm{Grpd}_{\mathcal{B}}$-continuous functor $\mathrm{Grpd}_{\mathcal{B}}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}_{\mathcal{B}}$. Hence, the underlying map of $\widehat{\mathrm{Cat}}_\infty$-valued presheaves on $\mathcal{B}$ can be regarded as a morphism in $\operatorname{Fun}^{\mathrm{LAdj}}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ (in the sense of [50, § 4.7.4]). On account of the equivalence

$$\operatorname{Fun}^{\mathrm{LAdj}}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty) \simeq \operatorname{Fun}^{\mathrm{RAdj}}(\mathcal{B}, \widehat{\mathrm{Cat}}_\infty)$$

from [50, Corollary 4.7.4.18] that is furnished by passing to right adjoints, we thus obtain a morphism of functors $\Phi\colon (\mathcal{B}_{/-})^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/-})$ in which the functoriality on both sides is given by the right adjoints of the transition functors. Let $\eta\colon \phi \to \operatorname{diag}_{\mathcal{B}}(\phi(1))$ be the commutative square in $\operatorname{Fun}(\mathcal{B}, \widehat{\mathrm{Cat}}_\infty)$ that is obtained from the unit of the adjunction

$$\mathrm{ev}_1 \dashv \operatorname{diag}_{\mathcal{B}}\colon \operatorname{Fun}(\mathcal{B}, \widehat{\mathrm{Cat}}_\infty) \rightleftarrows \widehat{\mathrm{Cat}}_\infty.$$

We may regard η as a morphism in $\operatorname{Fun}(\mathcal{B}, \widehat{\mathrm{Cat}}_\infty^{\Delta^1})$. Note that for every map $s\colon B \to A$ in $\mathcal{B}$ one has a commutative triangle

$$\begin{tikzcd}
\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/B}) \arrow[rr, "s_*"] \arrow[dr, "\Gamma_{\mathcal{B}_{/B}}"'] & & \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \arrow[dl, "\Gamma_{\mathcal{B}_{/A}}"] \\
& \mathrm{Top}^{\mathrm{L}}_\infty, &
\end{tikzcd}$$

hence Corollary 6.2.5.6 implies that s_* is a left fibration. As the functor

$$s_!^{\mathrm{op}}\colon \mathcal{B}_{/B}^{\mathrm{op}} \to \mathcal{B}_{/A}^{\mathrm{op}}$$

is a left fibration too, the map η thus defines a morphism in $\operatorname{Fun}(\mathcal{B}, \mathrm{LFib})$ (where LFib is the full subcategory of $\operatorname{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ that is spanned by the left fibrations). Explicitly, this morphism carries $A \in \mathcal{B}$ to the commutative square $\eta(A)\colon \phi(A) \to \phi(1)$. Now observe that the domain of η is contained in the fibre

$$\mathrm{LFib}(\mathcal{B}^{\mathrm{op}}) \hookrightarrow \mathrm{LFib}$$

and the codomain is contained in the fibre

$$\mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B})) \hookrightarrow \mathrm{LFib}\,.$$

Moreover, for each $A \in \mathcal{B}$ the functor $\Phi(A)\colon (\mathcal{B}_{/A})^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A})$ carries the final object in $\mathcal{B}_{/A}$ to the initial object $\mathcal{B}_{/A} \in \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A})$ (see Corollary 6.2.2.4), hence Φ is section-wise initial. Altogether, these observations imply that

$$\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/-})\colon \mathcal{B} \to \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B}))$$

is equivalent to the composition of $(\mathcal{B}_{/-})^{\mathrm{op}}\colon \mathcal{B} \to \mathrm{LFib}(\mathcal{B}^{\mathrm{op}})$ (which is just the Yoneda embedding) with the functor of left Kan extension

$$\Phi(1)_!\colon \mathrm{LFib}(\mathcal{B}^{\mathrm{op}}) \to \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B}))$$

along $\Phi(1)\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. But the latter composition is equivalent to the composition

$$\mathcal{B} \xrightarrow{\Phi(1)^{\mathrm{op}}} \mathrm{Top}^{\mathrm{L}}(\mathcal{B})^{\mathrm{op}} \xhookrightarrow{h_{\mathrm{Top}^{\mathrm{L}}(\mathcal{B})^{\mathrm{op}}}} \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B})).$$

By making use of the commutative diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Phi(1)"] \arrow[rr, bend left, "\mathcal{B}_{/-}"] & \mathrm{Top}^{\mathrm{L}}(\mathcal{B})^{\mathrm{op}} \arrow[r, "\Gamma"] \arrow[d, hook] & (\mathrm{Top}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \arrow[d, hook] \\
& \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B})) \arrow[r, "\Gamma_!"] & \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}_{\infty}),
\end{tikzcd}$$

the claim now follows. □

Remark 6.2.6.3. Corollary 6.2.6.2 implies that for any map $s\colon B \to A$ in $\mathcal{B}$ the transition functor $s^*\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/B})$ can be identified with the pushout functor

$$- \sqcup_{\mathcal{B}_{/A}} \mathcal{B}_{/B}\colon (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/A}/} \to (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/B}/}.$$

As opposed to limits in $\mathrm{Top}^{\mathrm{L}}_{\mathcal{B}}$, general colimits of $\mathcal{B}$-topoi can *not* be computed on the underlying $\mathcal{B}$-categories, not even after passing to the opposite $\mathcal{B}$-category $\mathrm{Top}^{\mathrm{R}}_{\mathcal{B}}$. The existence of constant colimits follows easily from Theorem 6.2.5.1:

Lemma 6.2.6.4. *The large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is LConst-cocomplete.*

Proof. In light of Remark 6.2.6.3, this follows from the fact that for any map $s\colon B \to A$ in $\mathcal{B}$ the ∞-categories $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/A}/}$ and $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/B}/}$ have colimits by [49, Proposition 6.3.4.6] and $- \sqcup_{\mathcal{B}_{/A}} \mathcal{B}_{/B}$ preserves all colimits. □

Lemma 6.2.6.5. *The $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete.*

Proof. By Remark 6.2.1.3, it suffices to show that whenever $d\colon \mathsf{G} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is a diagram indexed by a $\mathcal{B}$-groupoid G, the functor $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G},\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})}(d, \mathrm{diag}(-))$ is corepresentable. Note that we have an equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, -) \simeq (\pi_{\mathsf{G}})_* \pi_{\mathsf{G}}^*$. Therefore, Corollary 6.2.3.6 implies that we can assume that d is in the image of $\mathsf{Grpd}_{\mathcal{B}}[-]_* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Cat}_{\widehat{\mathcal{B}}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})$. In this case, the claim follows from Corollary 6.2.2.3. □

Proposition 6.2.6.6. *The $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is cocomplete.*

Proof. By Corollary 3.5.4.2, this follows from Lemma 6.2.6.4 and Lemma 6.2.6.5. □

6.2.7. A formula for the coproduct of $\mathcal{B}$-topoi

The goal of this section is to give an explicit description of the coproduct in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. To that end, recall that by the discussion in Section 5.5.3 the ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ of presentable $\mathcal{B}$-categories is symmetric monoidal. Explicitly, if D and E are presentable $\mathcal{B}$-categories, their tensor product $\mathsf{D} \otimes \mathsf{E}$ is equivalent to the $\mathcal{B}$-category $\mathsf{Sh}_{\mathsf{E}}(\mathsf{D})$ of E-valued sheaves on D (i.e. the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{E})$ spanned by the continuous functors $\pi_A^* \mathsf{D}^{\mathrm{op}} \to \pi_A^* \mathsf{E}$ for each $A \in \mathcal{B}$). In light of this identification, the proof of Proposition 5.5.3.8 shows that if $f^* \colon \mathsf{D} \to \mathsf{D}'$ and $g^* \colon \mathsf{E} \to \mathsf{E}'$ are maps in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{B}}$ with right adjoints f_* and g_*, then the functor $\mathrm{id} \otimes f^* \colon \mathsf{D} \otimes \mathsf{E} \to \mathsf{D} \otimes \mathsf{E}'$ can be identified with the left adjoint of the postcomposition functor $(f_*)_* \colon \mathsf{Sh}_{\mathsf{E}'}(\mathsf{D}) \to \mathsf{Sh}_{\mathsf{E}}(\mathsf{D})$, and the functor $g^* \otimes \mathrm{id} \colon \mathsf{D} \otimes \mathsf{E} \to \mathsf{D}' \otimes \mathsf{E}$ can be identified with the left adjoint of the precomposition functor $(g^*)^* \colon \mathsf{Sh}_{\mathsf{E}}(\mathsf{D}') \to \mathsf{Sh}_{\mathsf{E}}(\mathsf{D})$.

Proposition 6.2.7.1. *If* X *and* Y *are* $\mathcal{B}$*-topoi, then their tensor product* $\mathsf{X} \otimes \mathsf{Y}$ *is a* $\mathcal{B}$*-topos as well. Moreover, the cospan*

$$\mathsf{X} \simeq \mathsf{X} \otimes \mathrm{Grpd}_{\mathcal{B}} \xrightarrow{\mathrm{id} \otimes \mathrm{const}_{\mathsf{Y}}} \mathsf{X} \otimes \mathsf{Y} \xleftarrow{\mathrm{const}_{\mathsf{X}} \otimes \mathrm{id}} \mathrm{Grpd}_{\mathcal{B}} \otimes \mathsf{Y} \simeq \mathsf{Y}$$

exhibits $\mathsf{X} \otimes \mathsf{Y}$ *as the coproduct of* X *and* Y *in* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$.

Combining the above result with Proposition 5.5.4.10, we obtain the following generalisation of [6, Corollary 1.10]:

Corollary 6.2.7.2. *Assume that* $\mathcal{B}$ *is generated under colimits by* (-1)*-truncated objects. Then for* $\mathcal{X}, \mathcal{Y} \in \mathrm{Top}^{\mathrm{L}}_{\mathcal{B}/}$ *the canonical map*

$$\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y} \to \mathcal{X} \sqcup_{\mathcal{B}} \mathcal{Y}$$

is an equivalence.

The proof of Proposition 6.2.7.1 requires a few preparations and will be given at the end of this section. First, let us observe that this result provides an explicit formula for the pushout of ∞-topoi:

Corollary 6.2.7.3. *Given a cospan* $\mathcal{X} \xleftarrow{f^*} \mathcal{Z} \xrightarrow{g^*} \mathcal{Y}$ *in* $\mathrm{Top}^{\mathrm{L}}_{\infty}$*, there is a canonical equivalence*

$$\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y} \simeq \mathrm{Fun}^{\mathrm{cont}}_{\mathcal{Z}}(f_*(\mathrm{Grpd}_{\mathcal{X}})^{\mathrm{op}}, g_* \mathrm{Grpd}_{\mathcal{Y}})$$

where the right-hand side is the full subcategory of $\mathrm{Fun}_{\mathcal{Z}}(f_*(\mathrm{Grpd}_{\mathcal{X}})^{\mathrm{op}}, g_* \mathrm{Grpd}_{\mathcal{Y}})$ *that is spanned by the continuous functors.* □

Remark 6.2.7.4. In light of Corollary 6.2.7.3, the ∞-topos $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ admits the following explicit description: It is the full subcategory of the ∞-category of natural transformations between the two $\widehat{\mathrm{Cat}}_{\infty}$-valued sheaves $\mathcal{X}_{/f^*(-)}$ and $\mathcal{Y}_{/g^*(-)}$ on $\mathcal{Z}$ that is spanned spanned by those maps $\phi \colon (\mathcal{X}_{/f^*(-)})^{\mathrm{op}} \to \mathcal{Y}_{/g^*(-)}$ which satisfy that

1. the functor $\phi(A)$ preserves limits for all $A \in \mathcal{Z}$, and

2. for any map $s \colon B \to A$ in $\mathcal{Z}$ the canonical lax square

$$\begin{tikzcd}
(\mathcal{X}_{/f^*(B)})^{\mathrm{op}} \arrow[r, "\phi(B)"] \arrow[d, "f^*(s)_!"'] \arrow[dr, phantom, "\nearrow"] & \mathcal{Y}_{/g^*(B)} \arrow[d, "g^*(s)_*"] \\
(\mathcal{X}_{/f^*(A)})^{\mathrm{op}} \arrow[r, "\phi(A)"'] & \mathcal{Y}_{/g^*(A)}
\end{tikzcd}$$

commutes. □

Admittedly, the description of the pushout of ∞-topoi in Remark 6.2.7.4 is rather unwieldy in general. However, we can paint a more concrete picture in the following case:

Example 6.2.7.5. Let X be a $\mathcal{B}$-topos and let C be an arbitrary $\mathcal{B}$-category. Then Proposition 6.2.7.1 implies that the commutative square

$$\begin{tikzcd} \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{const}_{\mathsf{X}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "(\mathrm{const}_{\mathsf{X}})_*"] \\ \mathsf{X} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}) \end{tikzcd}$$

is a pushout in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. Furthermore, if $f\colon \mathcal{X} \to \mathcal{B}$ is the geometric morphism associated to X, then the lower horizontal map can be identified with the image of $\mathrm{diag}\colon \mathsf{Grpd}_{\mathcal{X}} \to \underline{\mathsf{PSh}}_{\mathcal{X}}(f^*\mathsf{C})$ along f_*.

We now turn to the proof of Proposition 6.2.7.1. It is a straightforward adaption of the proof presented in [4, §2.3] to the setting of $\mathcal{B}$-categories. We begin with the following lemma:

Lemma 6.2.7.6. *Let C and D be $\mathcal{B}$-categories with finite limits. Then precomposition with the canonical maps $\mathrm{id}_{\mathsf{C}} \times 1_{\mathsf{D}}\colon \mathsf{C} \to \mathsf{C} \times \mathsf{D}$ and $1_{\mathsf{C}} \times \mathrm{id}_{\mathsf{D}}\colon \mathsf{D} \to \mathsf{C} \times \mathsf{D}$ induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \times \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

for any $\mathcal{B}$-category E with finite limits. In other words, these two maps exhibit $\mathsf{C} \times \mathsf{D}$ as the coproduct of C and D in $\mathsf{Cat}^{\mathrm{lex}}_{\mathcal{B}}$.

Proof. The composition

$$\begin{aligned} \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \times \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}) &\xrightarrow{\mathrm{pr}_0^* \times \mathrm{pr}_1^*} \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \times \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \\ &\xrightarrow{\simeq} \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E} \times \mathsf{E}) \\ &\xrightarrow{(-\times-)_*} \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \end{aligned}$$

defines an inverse. □

The rough strategy of the proof of Proposition 6.2.7.1 is to first prove the claim for free $\mathcal{B}$-topoi, which will follow from Lemma 6.2.7.6. In order to reduce the general case to this setting we need to understand the compatibility of tensor products with localisations:

Lemma 6.2.7.7. *Suppose that* C *and* D *are presentable* $\mathcal{B}$*-categories and that* $\mathsf{W} \hookrightarrow \mathsf{C}$ *and* $\mathsf{S} \hookrightarrow \mathsf{D}$ *are small subcategories. Let* $\mathsf{C}' \hookrightarrow \mathsf{C}$ *be a small full subcategory that exhibits* C *as the free* $\mathsf{Filt}_{\mathsf{U}}$*-cocompletion of* C' *for some sound doctrine* U*. Let* $\mathsf{D}' \hookrightarrow \mathsf{D}$ *be chosen similarly. We write* $\tau\colon \mathsf{C} \times \mathsf{D} \to \mathsf{C} \otimes \mathsf{D}$ *for the universal bilinear functor. Let us set* $\mathsf{W} \boxtimes \mathsf{S} = (\mathsf{W} \times (\mathsf{D}')^{\simeq}) \sqcup ((\mathsf{C}')^{\simeq} \times \mathsf{S})$*. Then the canonical map* $\mathsf{C} \otimes \mathsf{D} \to \mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \otimes \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *induces an equivalence*

$$\mathsf{Loc}_{\mathsf{W}\boxtimes\mathsf{S}}(\mathsf{C} \otimes \mathsf{D}) \xrightarrow{\simeq} \mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \otimes \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),$$

where the left-hand side is the $\mathcal{B}$*-category of local objects with respect to the map* $(\tau,\tau)\colon \mathsf{W} \boxtimes \mathsf{S} \to \mathsf{C} \otimes \mathsf{D}$*.*

Proof. Let E be any other presentable $\mathcal{B}$-category and let $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})_{\mathsf{W}\boxtimes\mathsf{S}}$ be the full subcategory of $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ that is spanned by those bilinear functors $\pi_A^*\mathsf{C} \times \pi_A^*\mathsf{D} \to \pi_A^*\mathsf{E}$ (in arbitrary context $A \in \mathcal{B}$) whose precomposition with $\pi_A^*(\mathsf{W} \boxtimes \mathsf{S}) \to \pi_A^*\mathsf{C} \times \pi_A^*\mathsf{D}$ factors through $\pi_A^*\mathsf{E}^{\simeq}$. By combining the universal property of the tensor product with Corollary 5.4.3.2, we now obtain a chain of equivalences

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}\boxtimes\mathsf{S}}(\mathsf{C} \otimes \mathsf{D}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E})_{\mathsf{W}\boxtimes\mathsf{S}} \simeq \underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})_{\mathsf{W}\boxtimes\mathsf{S}}.$$

Note that a bilinear functor $f\colon \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is contained in $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})_{\mathsf{W}\boxtimes\mathsf{D}}$ if and only if

1. for any $c\colon A \to \mathsf{C}'$ in context $A \in \mathcal{B}$ the functor $\pi_A^*\mathsf{S} \hookrightarrow \pi_A^*\mathsf{D} \xrightarrow{f(c,-)} \pi_A^*\mathsf{E}$ factors through $\pi_A^*\mathsf{E}^{\simeq}$, and
2. for any $d\colon A \to \mathsf{D}'$ in context $A \in \mathcal{B}$ the functor $\pi_A^*\mathsf{W} \hookrightarrow \pi_A^*\mathsf{C} \xrightarrow{f(-,d)} \pi_A^*\mathsf{E}$ factors through $\pi_A^*\mathsf{E}^{\simeq}$.

Let $f'\colon \mathsf{C} \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$ be the transpose of f. By Lemma 5.5.1.3, f' is cocontinuous. Now the first condition is equivalent to the composition

$$\mathsf{C}' \hookrightarrow \mathsf{C} \xrightarrow{f'} \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

taking values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E})$. Note that the inclusion

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

is given by precomposition with $\mathsf{D} \to \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ and is therefore cocontinuous. Since C' generates C under $\mathsf{Filt}_{\mathsf{U}}$-colimits, it follows that (1) is equivalent to f' being contained in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}))$. Similarly, if $f'' \colon \mathsf{D} \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ is the other transpose of f, condition (2) is equivalent to f'' taking values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}}(\mathsf{C}), \mathsf{E})$. Therefore, the naturality of the equivalence in Lemma 5.5.1.3 implies that f satisfies (1) and (2) if and only if f is contained in the full subcategory $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \times \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E})$. As the same argument can be carried out for bilinear functors in arbitrary context, this shows that the equivalence $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ restricts to an equivalence

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W} \boxtimes \mathsf{S}}(\mathsf{C} \otimes \mathsf{D}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \times \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}),$$

which proves the claim. □

A similar argument as above shows the following:

Lemma 6.2.7.8. *Let* C *and* D *be presentable* $\mathcal{B}$*-categories and let* $\mathsf{W} \hookrightarrow \mathsf{C}$ *and* $\mathsf{S} \hookrightarrow \mathsf{D}$ *be small subcategories. Then the commutative square*

$$\begin{tikzcd}
\mathsf{C} \otimes \mathsf{D} \arrow[r] \arrow[d] & \mathsf{C} \otimes \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \arrow[d] \\
\mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \otimes \mathsf{D} \arrow[r] & \mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \otimes \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})
\end{tikzcd}$$

is a pushout in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$. □

Proof of Proposition 6.2.7.1. To simplify notation, we shall write $i_0 = \mathrm{id} \otimes \mathrm{const}_{\mathsf{Y}}$ as well as $i_1 = \mathrm{const}_{\mathsf{X}} \otimes \mathrm{id}$. First, let us show the claim in the special case where $\mathsf{X} = \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]$ and $\mathsf{Y} = \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}]$. In this situation, we have an equivalence $\mathsf{X} \otimes \mathsf{Y} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{lex}} \times \mathsf{D}^{\mathrm{lex}})$ with respect to which the functors i_0 and i_1 are given by left Kan extension along the two maps

$$\mathrm{id} \times 1_{\mathsf{D}^{\mathrm{lex}}} \colon \mathsf{C}^{\mathrm{lex}} \to \mathsf{C}^{\mathrm{lex}} \times \mathsf{D}^{\mathrm{lex}} \qquad \text{and} \qquad 1_{\mathsf{C}^{\mathrm{lex}}} \times \mathrm{id} \colon \mathsf{D}^{\mathrm{lex}} \to \mathsf{C}^{\mathrm{lex}} \times \mathsf{D}^{\mathrm{lex}}.$$

By Lemma 6.2.7.6, the latter two functors exhibit $\mathsf{C}^{\mathrm{lex}} \times \mathsf{D}^{\mathrm{lex}}$ as the coproduct $\mathsf{C}^{\mathrm{lex}} \sqcup \mathsf{D}^{\mathrm{lex}}$ in $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{lex}}$. As the functor $(-)^{\mathrm{lex}}$ is a left adjoint and thus preserves coproducts, we end up with an equivalence $\mathsf{X} \otimes \mathsf{Y} \simeq \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C} \sqcup \mathsf{D}]$ with respect to which i_0 and i_1 correspond to the image of the inclusions $\mathsf{C} \hookrightarrow \mathsf{C} \sqcup \mathsf{D}$ and $\mathsf{D} \hookrightarrow \mathsf{C} \sqcup \mathsf{D}$ along the functor $\mathsf{Grpd}_{\mathcal{B}}[-]$. The claim thus follows from Corollary 6.2.2.3.

In the general case, Corollary 6.2.3.5 implies that we may choose left exact and accessible Bousfield localisations $L\colon \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \to \mathsf{X}$ and $L'\colon \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}] \to \mathsf{Y}$. By Lemma 6.2.7.8 we have a pushout square

$$\begin{tikzcd} \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}] \arrow[r] \arrow[d] & \mathsf{X} \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}] \arrow[d] \\ \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathsf{Y} \arrow[r] & \mathsf{X} \otimes \mathsf{Y} \end{tikzcd}$$

in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. The upper horizontal functor is equivalent to the functor

$$L_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{D}^{\mathrm{lex}})^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}]) \to \underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{D}^{\mathrm{lex}})^{\mathrm{op}}, \mathsf{X})$$

and therefore a left exact and accessible Bousfield localisation. By symmetry, the same holds for the left vertical functor. Thus $\mathsf{X} \otimes \mathsf{Y}$ is equivalent to the intersection of two accessible and left exact Bousfield localisations of $\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}]$ and therefore by [49, Lemma 6.3.3.4] in particular a $\mathcal{B}$-topos. Since the square

$$\begin{tikzcd} \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \arrow[r] \arrow[d, "i_0"'] & \mathsf{X} \arrow[d, "i_0"'] \\ \mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}] \arrow[r] & \mathsf{X} \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}] \end{tikzcd}$$

commutes, it follows that $i_0\colon \mathsf{X} \to \mathsf{X} \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}]$ is left exact. Since $i_0\colon \mathsf{X} \to \mathsf{X} \otimes \mathsf{Y}$ factors as the composite $\mathsf{X} \xrightarrow{i_0} \mathsf{X} \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}] \to \mathsf{X} \otimes \mathsf{Y}$ it is therefore also left exact. The same argument shows that $i_1\colon \mathsf{Y} \to \mathsf{X} \otimes \mathsf{Y}$ is left exact. Finally, note that L and L' induce a commutative square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{X} \otimes \mathsf{Y}, \mathsf{Z}) \arrow[r, hook] \arrow[d, "(i_0^*, i_1^*)"] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}], \mathsf{Z}) \arrow[d, "\simeq"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{X}, \mathsf{Z}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{Y}, \mathsf{Z}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{Grpd}_{\mathcal{B}}[\mathsf{C}], \mathsf{Z}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\mathsf{Grpd}_{\mathcal{B}}[\mathsf{D}], \mathsf{Z}) \end{tikzcd}$$

for any $\mathcal{B}$-topos Z. As the right vertical map being an equivalence implies that (i_0^*, i_1^*) is fully faithful, it thus suffices to see that this functor is also essentially surjective. Using Remark 6.2.1.3, it will be enough to show that for any two algebraic morphisms $f\colon \mathsf{X} \to \mathsf{Z}$ and $g\colon \mathsf{Y} \to \mathsf{Z}$ the induced map $\mathrm{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathrm{Grpd}_{\mathcal{B}}[\mathsf{D}] \to \mathsf{Z}$ factors through $\mathrm{Grpd}_{\mathcal{B}}[\mathsf{C}] \otimes \mathrm{Grpd}_{\mathcal{B}}[\mathsf{D}] \to \mathsf{X} \otimes \mathsf{Y}$. This is a direct consequence of Lemma 6.2.7.7. □

6.2.8. Diaconescu's theorem

In classical category theory, Diaconescu's theorem states that for any 1-category $\mathcal{C}$ and any 1-topos $\mathcal{X}$, a functor $f\colon \mathcal{C} \to \mathcal{X}$ is *internally flat* if and only if its left Kan extension $h_! f\colon \mathrm{PSh}_{\mathrm{Set}}(\mathcal{C}) \to \mathcal{X}$ preserves finite limits, see for example [39, Theorem B.3.2.7]. Here f being internally flat precisely means that its internal unstraightening results in a *filtered* internal category in $\mathcal{X}$. For ∞-categories, a comparable result has been proved by Lurie [49, Proposition 6.1.5.2] in the special case where the ∞-category $\mathcal{C}$ already admits finite limits. In the general case, Raptis and Schäppi proved Diaconescu's theorem under the assumption that the codomain $\mathcal{X}$ is a hypercomplete ∞-topos [67].

The main goal of this section is to establish a general version of Diaconescu's theorem for $\mathcal{B}$-topoi and therefore also a general version of Diaconescu's theorem for ∞-topoi, without any hypercompleteness assumptions. To that end, let us say that a presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ on an arbitrary $\mathcal{B}$-category C is *flat* if it is $\mathsf{Fin}_{\mathcal{B}}$-flat in the sense of Definition 5.3.4.1. We will denote by $\underline{\mathsf{Flat}}_{\mathcal{B}}(\mathsf{C})$ the associated $\mathcal{B}$-category of flat functors. Recall from Proposition 5.2.3.7 that the doctrine $\mathsf{Fin}_{\mathcal{B}}$ is sound. Therefore, Proposition 5.3.4.6 implies:

Proposition 6.2.8.1. *For any $\mathcal{B}$-category C, a functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ is flat if and only if the $\mathcal{B}$-category $\mathsf{C}_{/F}$ is filtered.* □

Diaconescu's theorem for $\mathcal{B}$-topoi can now be stated as follows:

Theorem 6.2.8.2. *Let X be a $\mathcal{B}$-topos and let $f_*\colon \mathcal{X} \to \mathcal{B}$ be the associated geometric morphism. Let C be an arbitrary $\mathcal{B}$-category. Then precomposition with the Yoneda embedding induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{X}) \simeq f_* \underline{\mathsf{Flat}}_{\mathcal{X}}(f^* \mathsf{C}^{\mathrm{op}}).$$

Specialising to the case where $\mathcal{B} \simeq \mathrm{Ani}$, Theorem 6.2.8.2 implies:

Corollary 6.2.8.3. *For any small ∞-category $\mathcal{C}$, a functor $f\colon \mathcal{C} \to \mathcal{B}$ is flat if and only if its Yoneda extension $h_! f\colon \mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}) \to \mathcal{B}$ preserves finite limits. In particular, the functor of left Kan extension along $h_{\mathcal{C}}$ induces an equivalence*

$$h_! \colon \mathrm{Flat}_{\mathcal{B}}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Fun}^{\mathrm{alg}}(\mathrm{PSh}_{\mathrm{Ani}}(\mathcal{C}), \mathcal{B})$$

of ∞-categories. □

Remark 6.2.8.4. Corollary 6.2.8.3 can be used to define morphisms of general ∞-sites: if $\mathcal{C}$ and $\mathcal{D}$ are ∞-sites, a functor $f\colon \mathcal{C} \to \mathcal{D}$ is a morphism of ∞-sites if the associated functor $f' \colon \mathcal{C} \to \mathrm{Sh}(\mathcal{D})$ (which is obtained by composing f with the sheafified Yoneda embedding $Lh \colon \mathcal{D} \to \mathrm{Sh}(\mathcal{D})$) is flat and if for every covering $(c_i \to c)_{i\in I}$ in $\mathcal{C}$ the induced functor $\bigsqcup_i f'(c_i) \to f'(c)$ is a cover in $\mathrm{Sh}(\mathcal{D})$. Using this definition, Corollary 6.2.8.3 and [49, Lemma 6.2.3.19] imply that every morphism of ∞-sites $f\colon \mathcal{C} \to \mathcal{D}$ induces an algebraic morphism $F\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$.

The proof of Theorem 6.2.8.2 relies on the following two elementary lemmas:

Lemma 6.2.8.5. *Let $f\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism. Suppose that $p \colon \mathsf{P} \to \mathsf{C}$ is a left fibration of $\mathcal{X}$-categories that is classified by a functor $g \colon \mathsf{C} \to \mathsf{Grpd}_{\mathcal{X}}$. Then the left fibration $f_*(p)$ of $\mathcal{B}$-categories is classified by the composition*

$$f_*\mathsf{C} \xrightarrow{f_*(g)} \mathsf{X} \xrightarrow{\Gamma_{\mathsf{X}}} \mathsf{Grpd}_{\mathcal{B}}\,.$$

Proof. Since f_* commutes with pullbacks and with powering by ∞-categories (cf. Remark 1.2.5.6), the image of the universal left fibration $(\mathsf{Grpd}_{\mathcal{X}})_{1/} \to \mathsf{Grpd}_{\mathcal{X}}$ along f_* can be identified with $(\pi_{1_{\mathsf{X}}})_! \colon \mathsf{X}_{1_{\mathsf{X}}/} \to \mathsf{X}$ and is therefore classified by $\mathrm{map}_{\mathsf{X}}(1_{\mathsf{X}}, -) \simeq \Gamma_{\mathsf{X}}$. Hence the claim follows. □

Lemma 6.2.8.6. *Let* X *be a $\mathcal{B}$-topos and let $f\colon \mathcal{X} \to \mathcal{B}$ be the corresponding geometric morphism. Then for any $\mathcal{B}$-category* C*, there is a commutative square*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "h_{\mathsf{C}}"] \arrow[d, "\eta"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "(\mathrm{const}_{\mathsf{X}})_*"] \\ f_* f^* \mathsf{C} \arrow[r, "f_*(h_{f^*\mathsf{C}})"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}). \end{tikzcd}$$

Proof. Transposing the Yoneda embedding $h_{f^*\mathsf{C}} \colon f^*\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{X}}(f^*\mathsf{C})$ across the adjunction $f^* \dashv f_*$ yields the composition

$$\mathsf{C} \xrightarrow{\eta} f^* f_* \mathsf{C} \xhookrightarrow{f_*(h_{f^*\mathsf{C}})} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X})$$

in which η is the adjunction unit. By transposing the above map across the adjunction $\mathsf{C}^{\mathrm{op}} \times - \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, -)$, one ends up with the functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \xrightarrow{\eta} f_* f^*(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}) \xrightarrow{f_*(\mathrm{map}_{f^*\mathsf{C}})} \mathsf{X}.$$

On the other hand, the transpose of the composition

$$\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X})$$

yields

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \xrightarrow{\mathrm{map}_{\mathsf{C}}} \mathsf{Grpd}_{\mathcal{B}} \xrightarrow{\mathrm{const}_{\mathsf{X}}} \mathsf{X},$$

so it suffices to show that these two functors are equivalent. By Lemma 6.2.8.5 the functor $\mathrm{map}_{f_* f^*\mathsf{C}}$ is equivalent to the composition

$$\Gamma_{\mathsf{X}} \circ f_*(\mathrm{map}_{f^*\mathsf{C}}) \colon f_*\mathsf{C}^{\mathrm{op}} \times f_*\mathsf{C} \to \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}.$$

As a consequence, the morphism of functors $\mathrm{map}_{\mathsf{C}} \to \mathrm{map}_{f_* f^*\mathsf{C}} \circ \eta$ that is induced by the action of η on mapping $\mathcal{B}$-groupoids determines a morphism $\mathrm{map}_{\mathsf{C}} \to \Gamma_{\mathsf{X}} \circ f_*(\mathrm{map}_{f^*\mathsf{C}}) \circ \eta$ which in turn transposes to a map

$$\mathrm{const}_{\mathsf{X}} \circ \mathrm{map}_{\mathsf{C}} \to f_*(\mathrm{map}_{f^*\mathsf{C}}) \circ \eta.$$

To show that this is an equivalence, it will be enough to show that it induces an equivalence when evaluated at (τ, τ), where τ is the tautological object in C, i.e. the one given by the identity of C_0. But by construction, the resulting map is simply the transpose of $\eta \colon \mathsf{C}_1 \to f_* f^* \mathsf{C}_1$ across the adjunction $f^* \dashv f_*$ and therefore an equivalence, as desired. $\square$

Proof of Theorem 6.2.8.2. To begin with, we note that the universal property of presheaf $\mathcal{B}$-categories together with Remark 6.2.1.3 and Remark 5.3.4.2 implies that it suffices to show that a functor $g \colon \mathsf{C} \to \mathsf{X}$ transposes to a flat functor

$g' \colon f^*\mathsf{C} \to \mathrm{Grpd}_{\mathcal{X}}$ if and only if its Yoneda extension $(h_{\mathsf{C}})_!(g) \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ preserves finite limits. Note that by Lemma 6.2.8.6 (and the fact that base change along f_* preserves cocontinuity), we have a commutative diagram

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "(\mathrm{const}_{\mathsf{X}})_*"] \arrow[dr, bend left, "(h_{\mathsf{C}})_!(g)"] & \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}) \arrow[r, "f_*(h_{f^*\mathsf{C}})_!(g')"] & \mathsf{X}. \end{tikzcd}$$

Therefore, g' being flat immediately implies that $(h_{\mathsf{C}})_!(g)$ is an algebraic morphism, so it suffices to consider the converse implication. Suppose therefore that the left Kan extension $(h_{\mathsf{C}})_!(g)$ preserves finite limits. We wish to show that the functor $(h_{f^*\mathsf{C}})_!(g')$ preserves finite limits as well. In light of the previous commutative diagram and the fact that $(\mathrm{const}_{\mathsf{X}})_*$ preserves finite limits, it is clear that it preserves the final object, so we only need to consider the case of pullbacks. By Lemma 6.2.2.8, we may reduce to pullbacks of cospans in $\underline{\mathsf{PSh}}_{\mathcal{X}}(f^*\mathsf{C})$ (in arbitrary context $U \in \mathcal{X}$) which are contained in the essential image of the Yoneda embedding $h_{f^*\mathsf{C}}$. Since any such cospan is determined by a map $U \to (f^*\mathsf{C})^{\Lambda^2_0}$, it factors through the core inclusion $\tau_{f^*\mathsf{C}} \colon f^*(\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1) \to (f^*\mathsf{C})^{\Lambda^2_0}$, which we may regard as the *tautological* cospan. Therefore, it is enough to show that the pullback of $\tau_{f^*\mathsf{C}}$ is preserved by $(h_{f^*\mathsf{C}})_!(g')$. As this diagram is in context $f^*(\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1)$, we may make use of the adjunction $f^* \dashv f_*$ to regard $\tau_{f^*\mathsf{C}}$ as a cospan in $f_*f^*\mathsf{C}$ in context $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$. As such, it is precisely the cospan that arises as the image of the tautological cospan τ_{C} in C (i.e. the one given by the core inclusion $\tau_{\mathsf{C}} \colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \hookrightarrow \mathsf{C}^{\Lambda^2_0}$) along $\eta \colon \mathsf{C} \to f_*f^*\mathsf{C}$. By again making use of Lemma 6.2.8.6, we thus conclude that the image of $\tau_{f^*\mathsf{C}}$ along $f_*(h_{f^*\mathsf{C}}) \colon f_*f^*\mathsf{C} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X})$ can be identified with the image of τ_{C} along the composition $(\mathrm{const}_{\mathsf{X}})_* \circ h_{\mathsf{C}}$. In particular, the cospan $f_*(h_{f^*\mathsf{C}})(\tau_{f^*\mathsf{C}})$ is contained in the image of $(\mathrm{const}_{\mathsf{X}})_*$, hence the above commutative diagram yields the claim. □

In the remainder of this section we will explain how our version of Diaconescu's theorem for ∞-topoi (Corollary 6.2.8.3) relates to that of Raptis and Schäppi [67] when $\mathcal{B}$ is *hypercomplete*. More precisely, in [67, Theorem 1.1 (3)] Raptis and Schäppi give an explicit characterisation of flat functors $\mathcal{C} \to \mathcal{X}$ valued in a hypercomplete ∞-topos $\mathcal{X}$, and a priori it is not clear how to relate this description to our substantially less explicit characterisation of flat functors in terms of internal filteredness (Proposition 6.2.8.1). Therefore, our goal is to recover the description in [67, Theorem 1.1 (3)] from Proposition 6.2.8.1. To that end, suppose that there is a left exact and accessible localisation $L\colon \mathrm{PSh}(\mathcal{D}) \to \mathcal{B}$ for some small ∞-category $\mathcal{D}$, and let $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{D})$ be its right adjoint. We denote by $\mathcal{C}_{f/} \to \mathcal{C}$ the left fibration (in $\mathrm{Cat}(\mathcal{B})$) that is classified by $f\colon \mathcal{C} \to \mathrm{Grpd}_{\mathcal{B}}$. By definition, it sits inside a pullback square

$$\begin{tikzcd}
\mathcal{C}_{f/} \arrow[r] \arrow[d] & (\mathrm{Grpd}_{\mathcal{B}})_{1/} \arrow[d] \\
\mathcal{C} \arrow[r, "f"] & \mathrm{Grpd}_{\mathcal{B}}
\end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$. If $\mathcal{B}$ is hypercomplete, we deduce from Proposition 6.2.8.1 and Proposition 5.2.3.14 that f being flat is equivalent to $(\mathcal{C}_{f/})^{\mathrm{op}}$ being quasi-filtered. In order to obtain a more explicit understanding of the latter condition, let us first consider the constant presheaf $\underline{\mathcal{C}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ with value $\mathcal{C}$ and compute the pullback

$$\begin{tikzcd}
\underline{\mathcal{C}}_{f'/} \arrow[r] \arrow[d] & i((\mathrm{Grpd}_{\mathcal{B}})_{1/}) \arrow[d] \\
\underline{\mathcal{C}} \arrow[r, "f'"] & i(\mathrm{Grpd}_{\mathcal{B}})
\end{tikzcd}$$

in $\mathrm{Cat}(\mathrm{PSh}(\mathcal{D})) \simeq \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. Here f' is the transpose of $f\colon \mathcal{C} \to \mathrm{Grpd}_{\mathcal{B}}$ across the adjunction $L \dashv i$. Note that $L(\underline{\mathcal{C}}_{f'/}) \simeq \mathcal{C}_{f/}$ since L is left exact. Upon evaluating the previous pullback square at any $d \in \mathcal{D}$, we obtain a commutative rectangle

$$\begin{tikzcd}
\underline{\mathcal{C}}_{f'/}(d) \arrow[r] \arrow[d] & \mathcal{B}_{Ld/} \arrow[r] \arrow[d] & \mathcal{B}_{L(d)L(d)} \arrow[d] \\
\mathcal{C} \arrow[r, "f"] & \mathcal{B} \arrow[r, "\pi_{L(d)}^*"] & \mathcal{B}_{/Ld}
\end{tikzcd}$$

where the lower composite is is equivalent to $f'(d)$ and all squares are pullback squares. Here $\mathcal{B}_{L(d)L(d)}$ denotes the ∞-category of pointed objects in $\mathcal{B}_{/L(d)}$. It follows that we can explicitly describe $\underline{\mathcal{C}}_{f'/}(d)$ as the pullback in the left square such that for any map $s\colon d \to e$ in $\mathcal{D}$ the functor $s^*\colon \underline{\mathcal{C}}_{f'/}(e) \to \underline{\mathcal{C}}_{f'/}(d)$ is induced by pulling back the canonical functor $s^* : \mathcal{B}_{Le/} \to \mathcal{B}_{Ld/}$ along f. To proceed, we now need the following lemma that characterises those Cat_∞-valued presheaves on $\mathcal{D}$ which yield quasi-filtered $\mathcal{B}$-categories upon sheafification:

Lemma 6.2.8.7. *Let* $\mathsf{C} \in \mathrm{Cat}(\mathrm{PSh}(\mathcal{D}))$. *Then* $L\mathsf{C}$ *is a quasi-filtered* $\mathcal{B}$*-category if and only if for any finite* ∞*-category* $\mathcal{K}$*, any* $d \in \mathcal{D}$ *and any map* $\beta \colon \mathcal{K} \to \mathsf{C}(d)$ *there exist morphisms* $(s_i : d_i \to d)_i$ *such that* $(Ls_i) : \bigsqcup_i L(d_i) \twoheadrightarrow L(d)$ *is a cover in* $\mathcal{B}$*, and there are maps* $\alpha_i : \mathcal{K}^{\triangleright} \to \mathsf{C}(d_i)$ *for every* i *that fit into commutative diagrams*

$$\begin{tikzcd} \mathcal{K}^{\triangleright} \arrow[r, "\alpha_i"] & \mathsf{C}(d_i) \\ \mathcal{K} \arrow[u] \arrow[r, "\beta"] & \mathsf{C}(d) \arrow[u, "s_i^*"'] \end{tikzcd}$$

of ∞*-categories.*

Proof. The if part of the statement is a direct consequence of Proposition A.2.0.2. For the converse we note that for any finite ∞-category $\mathcal{K}$ the canonical map

$$L\underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}, \mathsf{C})^{\simeq} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, L\mathsf{C})^{\simeq}$$

is an equivalence. Now for some $\beta \colon \mathcal{K} \to \mathsf{C}(d)$ corresponding via Yoneda's lemma to a morphism $d \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{C})^{\simeq}$ this shows that the projection map

$$\mathrm{pr}_1 : d \times_{\underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}, \mathsf{C})^{\simeq}} \underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}^{\triangleright}, \mathsf{C})^{\simeq} \to d$$

becomes a cover after applying L. We now pick a cover

$$(t_i) : \bigsqcup_i d_i \to d \times_{\underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}, \mathsf{C})^{\simeq}} \underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}^{\triangleright}, \mathsf{C})^{\simeq}$$

in $\mathrm{PSh}(\mathcal{D})$ by representables. Then the $s_i = \mathrm{pr}_1 \circ t_i$ yield a cover after applying L, and by Yoneda's lemma every s_i gives a commutative square as in the claim. □

By combining Lemma 6.2.8.7 with the discussion preceding it, we recover the following characterisation of flat functors in the hypercomplete case:

Proposition 6.2.8.8 ([67, Definition 3.1 and Theorem 3.5])**.** *Suppose that* $\mathcal{B}$ *is hypercomplete, and let* $f \colon \mathcal{C} \to \mathcal{B}$ *be a functor. Then* f *is flat if and only if for every*

$d \in \mathcal{D}$, every functor $\alpha\colon \mathcal{K} \to \mathcal{C}$ (where $\mathcal{K}$ is a finite ∞-category) and every map $\overline{\beta}\colon \mathcal{K}^{\triangleleft} \to \mathcal{B}$ with cone point $L(d)$ such that $f\alpha \simeq \overline{\beta}|_{\mathcal{K}}$, there are maps $(s_i : d_i \to d)_i$ in $\mathcal{D}$ such that

1. *$L(s_i) : \bigsqcup_i L(d_i) \twoheadrightarrow d$ is a cover in $\mathcal{B}$;*
2. *for each i there is a cocone $\overline{\alpha}_i : \mathcal{K}^{\triangleleft} \to \mathcal{C}$ extending α, together with a morphism of cones $h : \Delta^1 \diamond \mathcal{K} \to \mathcal{B}$ from the cocone $\overline{\beta} \circ s_i$ (which is given by composing the cone point of β with s_i) to $f \circ \overline{\alpha}_i$.* □

6.2.9. Étale $\mathcal{B}$-topoi

By Theorem 6.2.5.1, geometric morphisms $f_* : \mathcal{X} \to \mathcal{B}$ are in correspondence with $\mathcal{B}$-topoi $f_*(\mathsf{Grpd}_{\mathcal{X}})$. In this section, we study those $\mathcal{B}$-topoi that correspond to *étale* geometric morphisms. To prepare our discussion, note that Corollary 6.2.4.4 implies that the functor $(-)^{\mathrm{disc}} = \mathsf{Grpd}_{\mathcal{B}}^{(-)} : \mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ from Definition 6.2.4.5 is cocontinuous. Moreover, as this functor carries the final object $1_{\mathsf{Grpd}_{\mathcal{B}}}$ to $\mathsf{Grpd}_{\mathcal{B}}$ itself, the universal property of $\mathsf{Grpd}_{\mathcal{B}}$ implies that we have a functorial equivalence $(-)^{\mathrm{disc}} \simeq (\mathsf{Grpd}_{\mathcal{B}})_{/-}$. In particular, the functor $(\mathsf{Grpd}_{\mathcal{B}})_{/-}$ takes values in $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ too. We may therefore define:

Definition 6.2.9.1. A $\mathcal{B}$-topos X is *étale* if there is an equivalence $\mathsf{X} \simeq (\mathsf{Grpd}_{\mathcal{B}})_{/\mathsf{G}}$ for some $\mathcal{B}$-groupoid G.

In [49, Proposition 6.3.5.5], Lurie proved a universal property for étale geometric morphisms of ∞-topoi. In light of Theorem 6.2.5.1, such étale geometric morphisms precisely correspond to étale $\mathcal{B}$-topoi. The main goal of this section is to discuss how Lurie's result can also be deduced from Diaconescu's theorem. To that end, note that if X is a $\mathcal{B}$-topos with associated geometric morphism $f_* : \mathcal{X} \to \mathcal{B}$ and if G is a $\mathcal{B}$-groupoid, the fact that we may identify $\mathsf{X} \simeq f_*(\mathsf{Grpd}_{\mathcal{X}})$ implies that precomposition with the Yoneda embedding $h_{\mathsf{G}} : \mathsf{G} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Grpd}_{\mathcal{B}}) \simeq (\mathsf{Grpd}_{\mathcal{B}})_{/\mathsf{G}}$ induces a map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{Grpd}_{\mathcal{B}})_{/\mathsf{G}}, \mathsf{X}) \to f_*\underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{G}, \mathsf{Grpd}_{\mathcal{X}}) \simeq \mathsf{X}_{/\,\mathrm{const}_{\mathsf{X}}\mathsf{G}}.$$

The universal property of étale $\mathcal{B}$-topoi can now be formulated as follows:

Proposition 6.2.9.2. *Let* G *be a* $\mathcal{B}$*-groupoid and let* X *be a* $\mathcal{B}$*-topos. Precomposition with the Yoneda embedding* h_{G} *induces a fully faithful functor*

$$h_{\mathrm{G}}^* \colon \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}((\mathsf{Grpd}_{\mathcal{B}})_{/\mathrm{G}}, \mathsf{X}) \hookrightarrow \mathsf{X}_{/\operatorname{const}_{\mathsf{X}} \mathrm{G}}$$

that fits into a pullback square

$$\begin{tikzcd} \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}((\mathsf{Grpd}_{\mathcal{B}})_{/\mathrm{G}}, \mathsf{X}) \arrow[r, hook, "h_{\mathrm{G}}^*"] \arrow[d] & \mathsf{X}_{/\operatorname{const}_{\mathsf{X}} \mathrm{G}} \arrow[d, "(\pi_{\operatorname{const}_{\mathsf{X}} \mathrm{G}})_!"] \\ 1 \arrow[r, hook, "1_{\mathsf{X}}"] & \mathsf{X}. \end{tikzcd}$$

In particular, there is a canonical equivalence

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}((\mathsf{Grpd}_{\mathcal{B}})_{/\mathrm{G}}, \mathsf{X}) \simeq \operatorname{map}_{\mathsf{X}}(1_{\mathsf{X}}, \operatorname{const}_{\mathsf{X}} \mathrm{G}).$$

The proof of Proposition 6.2.9.2 requires the following lemma:

Lemma 6.2.9.3. *For any* $\mathcal{B}$*-groupoid* G*, the full embedding* $\mathrm{G} \hookrightarrow (\mathsf{Grpd}_{\mathcal{B}})_{/\mathrm{G}}$ *that is obtained by combining the Yoneda embedding* h_{G} *with the equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{G}, \mathsf{Grpd}_{\mathcal{B}}) \simeq (\mathsf{Grpd}_{\mathcal{B}})_{/G}$$

fits into a pullback square

$$\begin{tikzcd} \mathrm{G} \arrow[r, hook] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})_{/\mathrm{G}} \arrow[d, "(\pi_{\mathrm{G}})_!"] \\ 1 \arrow[r, hook, "1_{\mathsf{Grpd}_{\mathcal{B}}}"] & \mathsf{Grpd}_{\mathcal{B}}. \end{tikzcd}$$

Proof. Since we have a commutative diagram

$$\begin{tikzcd} (\mathsf{Grpd}_{\mathcal{B}})_{/\mathrm{G}} \arrow[r, "\simeq"] \arrow[dr, "(\pi_{\mathrm{G}})_!"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{G}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "\operatorname{colim}_{\mathrm{G}}"] \\ & \mathsf{Grpd}_{\mathcal{B}}, \end{tikzcd}$$

the claim follows once we show that a copresheaf $F \colon \mathrm{G} \to \mathsf{Grpd}_{\mathcal{B}}$ is representable if and only if $\operatorname{colim}_{\mathrm{G}} F \simeq 1_{\mathsf{Grpd}_{\mathcal{B}}}$. But F is representable if and only if $\mathrm{G}_{/F}$ admits an initial object, and since the latter is a $\mathcal{B}$-groupoid, this is in turn equivalent to $\mathrm{G}_{/F} \simeq 1$. Since by Proposition 3.2.5.1 we have $\mathrm{G}_{/F} \simeq \operatorname{colim}_{\mathrm{G}} F$, the claim follows. $\square$

Proof of Proposition 6.2.9.2. Let $f_* : \mathcal{X} \to \mathcal{B}$ be the geometric morphism that corresponds to the $\mathcal{B}$-topos $\mathcal{X}$. Since for every $U \in \mathcal{X}$ an $\mathcal{X}_{/U}$-groupoid is filtered if and only if it is final (see Remark 5.2.3.3), the Yoneda embedding $h_{f^*\mathsf{G}} \colon f^*\mathsf{G} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{G}, \mathsf{Grpd}_{\mathcal{X}})$ induces an equivalence $f^*\mathsf{G} \simeq \underline{\mathsf{Flat}}_{\mathcal{X}}(f^*\mathsf{G})$. By combining this observation with Theorem 6.2.8.2, we thus find that precomposition with the Yoneda embedding $\mathsf{G} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Grpd}_{\mathcal{B}})$ yields an equivalence

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}((\mathsf{Grpd}_{\mathcal{B}})_{/\mathsf{G}}, \mathsf{X}) \simeq f_*(f^*\mathsf{G}).$$

Hence the claim follows from Lemma 6.2.9.3. □

Corollary 6.2.9.4. *The functor* $(\mathsf{Grpd}_{\mathcal{B}})_{/-} \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *factors through a cocontinuous and fully faithful embedding* $(\mathsf{Grpd}_{\mathcal{B}})_{/-} \colon \mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ *whose essential image is spanned by the étale $\mathcal{B}$-topoi.*

Proof. It is clear that this functor takes values in $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$, and by combining the descent property of $\mathsf{Grpd}_{\mathcal{B}}$ with Proposition 6.2.6.1, this functor must be cocontinuous. It therefore suffices to show that it is fully faithful. As we have seen above, we may identify $(\mathsf{Grpd}_{\mathcal{B}})_{/-}$ with the restriction of the functor $(-)^{\mathrm{disc}} \colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ from Section 6.2.4 along the inclusion $\mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$. Using Corollary 6.2.4.4, the claim thus follows once we show that for every $\mathcal{B}$-groupoid G the (partial) adjunction unit $\mathsf{G} \to \underline{\mathsf{Pt}}_{\mathcal{B}}(\mathsf{G}^{\mathrm{disc}})$ is an equivalence. By construction, this map is obtained by the transpose of the evaluation map $\mathrm{ev} \colon \mathsf{G} \times \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$, which by Yoneda's lemma is precisely the inverse of the equivalence from Proposition 6.2.9.2. This finishes the proof. □

Remark 6.2.9.5. The functor $(\mathsf{Grpd}_{\mathcal{B}})_{/-} \colon \mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ also preserves finite limits. In fact, this is clear for the final object, and the case of binary products is an immediate consequence of the formula from Example 6.2.7.5 (together with the fact that the étale base change of this functor along π_A^* recovers the functor $(\mathsf{Grpd}_{\mathcal{B}_{/A}})_{/-}$). This is already enough to deduce that $(\mathsf{Grpd}_{\mathcal{B}})_{/-}$ preserves pullbacks: in fact, since Corollary 6.2.6.2 and Corollary 6.1.3.13 imply that $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ has $\mathsf{Grpd}_{\mathcal{B}}$-descent, this follows from the argument in the second part of the proof of Lemma 6.2.2.8.

6.2.10. Subterminal $\mathcal{B}$-topoi

The goal of this section is to study *subterminal* $\mathcal{B}$-topoi. To begin with, observe that if $f_* \colon \mathcal{X} \to \mathcal{B}$ and $g_* \colon \mathcal{Y} \to \mathcal{B}$ are geometric morphism where f_* is fully faithful, then the formula that we derived in Section 6.2.7 immediately implies that the geometric morphism $\mathcal{X} \times_{\mathcal{B}} \mathcal{Y} \to \mathcal{Y}$ (whose domain is the pullback in $\mathrm{Top}^{\mathrm{R}}_{\infty}$) is fully faithful as well. Thus, we may define:

Definition 6.2.10.1. A $\mathcal{B}$-topos X is said to be *subterminal* if the global sections functor Γ_{X} is fully faithful, or equivalently if the associated geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$ is fully faithful.

By Theorem 6.2.5.1, any subterminal $\mathcal{B}$-topos X determines and is determined by a left exact and accessible Bousfield localisation of $\mathcal{B}$ and therefore in particular by a class of maps S in $\mathcal{B}$ for which $\Gamma(\mathsf{X}) \simeq \mathrm{Loc}_S(\mathcal{B})$. The main goal of this section is to characterise those collections of maps S that arise from and give rise to a subterminal $\mathcal{B}$-topos X in this way, and to describe the associated endofunctor

$$\Gamma_{\mathsf{X}} \operatorname{const}_{\mathsf{X}} \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$$

by an explicit colimit formula in terms of S, akin to Lurie's sheafification formula from [49, § 6.2.2].

We begin with the following definition:

Definition 6.2.10.2. Let $d \colon \mathsf{I} \to \mathsf{Grpd}_{\mathcal{B}}$ be a functor of $\mathcal{B}$-categories, where I is small. We define the *+-construction* $(-)^+_d \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ relative to d as the composition

$$\mathsf{Grpd}_{\mathcal{B}} \xhookrightarrow{h_{\mathsf{Grpd}_{\mathcal{B}}}} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{d^*} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \xrightarrow{\operatorname{colim}_{\mathsf{C}^{\mathrm{op}}}} \mathsf{Grpd}_{\mathcal{B}},$$

i.e. by the formula $(-)^+_d = \operatorname{colim}_{\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{\mathsf{Grpd}_{\mathcal{B}}}(d(-), -)$.

Remark 6.2.10.3. If I is *cofiltered*, i.e. if I^{op} is filtered, then the +-construction $(-)^+_d$ is left exact.

Remark 6.2.10.4. If I is cofiltered, then the diagonal functor

$$\operatorname{diag}_{\mathsf{I}^{\mathrm{op}}} \colon \mathsf{Grpd}_{\mathcal{B}} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I})$$

is fully faithful (which follows from I being weakly contractible, see Remark 5.2.3.3, and from the explicit formula of the colimit in $\mathrm{Grpd}_{\mathcal{B}}$ from Proposition 3.2.5.1). Therefore, by applying the limit functor $\lim_{\mathsf{I}^{\mathrm{op}}}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I}) \to \mathrm{Grpd}_{\mathcal{B}}$ to the adjunction unit $\mathrm{id} \to \mathrm{diag}_{\mathsf{I}^{\mathrm{op}}} \operatorname{colim}_{\mathsf{I}^{\mathrm{op}}}$, we end up with a natural map $\lim_{\mathsf{I}^{\mathrm{op}}} \to \operatorname{colim}_{\mathsf{I}^{\mathrm{op}}}$. Now suppose furthermore that the colimit of $d\colon \mathsf{I} \to \mathrm{Grpd}_{\mathcal{B}}$ is the final object $1_{\mathrm{Grpd}_{\mathcal{B}}}\colon 1_{\mathcal{B}} \to \mathrm{Grpd}_{\mathcal{B}}$. Then the composition

$$\mathrm{Grpd}_{\mathcal{B}} \xhookrightarrow{h_{\mathrm{Grpd}_{\mathcal{B}}}} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{B}}) \xrightarrow{d^*} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I}) \xrightarrow{\lim_{\mathsf{I}^{\mathrm{op}}}} \mathrm{Grpd}_{\mathcal{B}}$$

is equivalent to the identity: in fact, this follows from the observation that its left adjoint is given by the composition of $\mathrm{diag}_{\mathsf{I}^{\mathrm{op}}}$ with the Yoneda extension of d (see Remark 3.5.1.4) and therefore preserves final objects. Thus, we obtain a natural map $\phi\colon \mathrm{id}_{\mathrm{Grpd}_{\mathcal{B}}} \to (-)^+_d$.

To proceed, fix a (small) cofiltered $\mathcal{B}$-category I and a functor $d\colon \mathsf{I} \to \mathrm{Grpd}_{\mathcal{B}}$ whose colimit is the final object. Since I is small, there is a $\mathcal{B}$-regular cardinal κ such that the essential image of d is contained in the full subcategory $\mathrm{Grpd}^{\kappa}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ determined by the local class of relatively κ-compact objects in $\mathcal{B}$ (cf. Proposition 5.2.2.11). We will call such a $\mathcal{B}$-regular cardinal κ *adapted to d*. We will identify κ with the linearly ordered set of ordinals $< \kappa$. Using transfinite induction, we may now construct a diagram $T^d_\bullet\colon \kappa \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})$ by setting $T^d_0 = \mathrm{id}$, by defining the map $T_\tau \to T_{\tau+1}$ to be the morphism $\phi\colon T^d_\tau \to (T^d_\tau)^+_d$ from Remark 6.2.10.4 and finally by setting $T^d_\tau = \operatorname{colim}_{\tau'<\tau} T^d_{\tau'}$ whenever τ is a limit ordinal.

Definition 6.2.10.5. Let $d\colon \mathsf{I} \to \mathrm{Grpd}_{\mathcal{B}}$ be a functor whose colimit is the final object and whose domain is a cofiltered small $\mathcal{B}$-category. We define the *sheafification functor* $(-)^{\mathrm{sh}}_d$ relative to the functor $d\colon \mathsf{I} \to \mathrm{Grpd}_{\mathcal{B}}$ as the colimit $(-)^{\mathrm{sh}}_d = \operatorname{colim}_{\tau<\kappa} T^d_\tau$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})$, where κ is an arbitrary $\mathcal{B}$-regular cardinal that is adapted to d.

Remark 6.2.10.6. A priori, the sheafification functor $(-)^{\mathrm{sh}}_d$ depends on the choice of $\mathcal{B}$-regular cardinal κ. However, since d takes values in $\mathrm{Grpd}^{\kappa}_{\mathcal{B}}$ and therefore in κ-compact objects in $\mathrm{Grpd}_{\mathcal{B}}$ (see Corollary 5.2.2.23), and since κ (when viewed as a linearly ordered set) is κ-filtered, one can show that whenever $\tau \geq \kappa$ is another

$\mathcal{B}$-regular cardinal, the sheafification functor that is constructed with respect to τ is equivalent to the one constructed with respect to κ.

Remark 6.2.10.7. In the situation of Definition 6.2.10.5, the sheafification functor $(-)^{\mathrm{sh}}_d$ is left exact since by Remark 6.2.10.4 it is a filtered colimit of left exact functors (see the argument in the proof of Lemma 5.1.5.3).

Example 6.2.10.8. Let S be a bounded local class of morphisms in $\mathcal{B}$ which is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$, and let $\iota\colon \mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ be the associated inclusion. Then Grpd_S is small and closed under finite limits in $\mathsf{Grpd}_{\mathcal{B}}$. In particular, Grpd_S is cofiltered by Proposition 5.2.3.7 and contains the final object of $\mathsf{Grpd}_{\mathcal{B}}$, so that the sheafification functor $(-)^{\mathrm{sh}}_{\iota}$ is well-defined.

Example 6.2.10.9. Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism, and let S and ι be as in Example 6.2.10.8. Then the functor $\mathrm{const}_{f_*(\mathsf{Grpd}_{\mathcal{X}})}\,\iota\colon \mathsf{Grpd}_S \to f_*(\mathsf{Grpd}_{\mathcal{X}})$ transposes to a map $\iota' : f^*(\mathsf{Grpd}_S) \to \mathsf{Grpd}_{\mathcal{X}}$ of $\mathcal{X}$-categories. As $\mathrm{const}_{f_*(\mathsf{Grpd}_{\mathcal{X}})}\,\iota$ preserves the final object, its colimit is $1_{f_*(\mathsf{Grpd}_{\mathcal{X}})}$, hence the colimit of ι' is the final object as well. Moreover, the fact that Grpd_S is a cofiltered $\mathcal{B}$-category implies that $f^*\,\mathsf{Grpd}_{\mathcal{X}}$ is a cofiltered $\mathcal{X}$-category: in fact, the colimit functor $\mathrm{colim}_{f^*(\mathsf{Grpd}_S)^{\mathrm{op}}} : \underline{\mathsf{PSh}}_{\mathcal{X}}(f^*(\mathsf{Grpd}_S)) \to \mathsf{Grpd}_{\mathcal{X}}$ preserves finite limits if and only if the underlying functor of ∞-categories $\mathrm{PSh}_{\mathcal{X}}(f^*(\mathsf{Grpd}_S)) \to \mathcal{X}$ preserves finite limits, and as the latter can be identified with the global sections of

$$\mathrm{colim}_{\mathsf{Grpd}_S^{\mathrm{op}}} : \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_S^{\mathrm{op}}, f_*(\mathsf{Grpd}_{\mathcal{X}})) \to f_*(\mathsf{Grpd}_{\mathcal{X}}),$$

the claim follows from the fact that filtered colimits commute with finite limits in every $\mathcal{B}$-topos (which one can see by reducing to the case of a presheaf $\mathcal{B}$-topos where it readily follows from the definitions). Thus, we are in the situation of Definition 6.2.10.5, so that $(-)^{\mathrm{sh}}_{\iota'}$ is well-defined.

Construction 6.2.10.10. Suppose that S is a bounded local class of maps in $\mathcal{B}$. Since S is bounded, there is a $\mathcal{B}$-regular cardinal κ adapted to $\iota\colon \mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$. Let $S^{\kappa} \subset S$ be the class of maps in S between κ-compact objects. Let $E \hookrightarrow (\mathsf{Grpd}_{\mathcal{B}})_1$ be the subobject that is spanned by the maps $f\colon P \to Q$ in $\mathcal{B}_{/A}$ (for arbitrary $A \in \mathcal{B}$) for which $(\pi_A)_!(f)$ is contained in S and the two maps $P \to A$ and $Q \to A$ are relatively κ-compact. Since both S and the class of relatively κ-compact maps are local, a map f in $\mathsf{Grpd}_{\mathcal{B}}$ in context A is contained in E if and only if $(\pi_A)_!(f) \in S$ and both $P \to A$ and $Q \to A$ are relatively κ-compact. In particular, E is small. We define $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ as the subcategory that is generated by E in the sense of Definition 1.3.1.14. Note that as E is small, the subcategory W is small as well.

We can now state the first main result of this section:

Proposition 6.2.10.11. *Let S be a bounded local class of morphisms in $\mathcal{B}$ such that S is closed under finite limits in* $\mathrm{Fun}(\Delta^1, \mathcal{B})$. *Let* $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ *be as in Construction 6.2.10.10. Then* $\mathsf{X} = \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$ *is a subterminal $\mathcal{B}$-topos with the property that* $\Gamma(\mathsf{X}) \simeq \mathrm{Loc}_S(\mathcal{B})$. *Moreover, the adjunction unit $\eta \colon \mathrm{id} \to \Gamma_{\mathsf{X}} \mathrm{const}_{\mathsf{X}}$ can be identified with the map* $\mathrm{id} \to (-)^{\mathrm{sh}}_{\iota}$, *where $\iota \colon \mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ is the inclusion.*

Lemma 6.2.10.12. *Let S be a bounded local class of morphisms in $\mathcal{B}$ and let* $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ *be as in Construction 6.2.10.10. Then there is an equivalence*

$$\Gamma(\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})) \simeq \mathrm{Loc}_S(\mathcal{B})$$

of full subcategories in $\mathcal{B}$.

Proof. By Corollary 5.4.1.8, the inclusion $i \colon \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}}) \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ admits a left adjoint L that exhibits $\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$ as an accessible Bousfield localisation of $\mathsf{Grpd}_{\mathcal{B}}$. Note that every map in S can be written as a colimit of maps in S^{κ}, which implies that every map in S is inverted by L. Consequently, we have an inclusion $\Gamma(\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})) \hookrightarrow \mathrm{Loc}_S(\mathcal{B})$, so that the claim follows once we verify that every S-local object $G \in \mathcal{B}$ is W-local. This amounts to showing that for every $A \in \mathcal{B}$ and and every map $s \colon P \to Q$ in $\mathcal{B}_{/A}$ for which $(\pi_A)_!(f) \in S$ and both $P \to A$ and $Q \to A$ are relatively κ-compact, the map

$$s^* \colon \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(Q, \pi_A^* G) \to \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(P, \pi_A^* G)$$

is an equivalence (cf. Remark 5.4.1.5). By Proposition 3.2.5.11, we may identify this morphism with the map

$$\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(Q, \pi_A^* G) \to \underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, \pi_A^* G).$$

By evaluating the latter at any object $B \to A$ in $\mathcal{B}_{/A}$, we recover the morphism

$$\mathrm{map}_{\mathcal{B}}(Q \times_A B, G) \to \mathrm{map}_{\mathcal{B}}(P \times_A B, G),$$

which is indeed an equivalence as the maps in S are closed under base change. Hence G is W-local, as claimed. □

Before we can prove Proposition 6.2.10.11, we first need to make a few remarks on the *internal hom* of a $\mathcal{B}$-topos X. Recall from Proposition 6.1.4.7 that colimits being universal in X precisely means that X is *cartesian closed.* We denote by

$$\underline{\mathrm{Hom}}_{\mathsf{X}}(-,-)\colon \mathsf{X}^{\mathrm{op}} \times \mathsf{X} \to \mathsf{X}$$

the internal hom of X that results from this observation. Note that if $f_*\colon \mathcal{X} \to \mathcal{B}$ is the geometric morphism associated with X, we deduce from combining Proposition 3.2.5.10 with Corollary 3.1.1.9 and Remark 2.1.2.5 that $\underline{\mathrm{Hom}}_{\mathsf{X}}(-,-)$ is explicitly given by the image of the bifunctor of $\mathcal{X}$-categories

$$\mathrm{map}_{\mathsf{Grpd}_{\mathcal{X}}}(-,-)\colon \ \mathsf{Grpd}_{\mathcal{X}}^{\mathrm{op}} \times \mathsf{Grpd}_{\mathcal{X}} \to \mathsf{Grpd}_{\mathcal{X}}$$

along f_*.

Remark 6.2.10.13. If X is a $\mathcal{B}$-topos, then the composition $\Gamma_{\mathsf{X}} \circ \underline{\mathrm{Hom}}_{\mathsf{X}}(-,-)$ recovers the mapping bifunctor $\mathrm{map}_{\mathsf{X}}(-,-)$. In fact, as Remark 6.2.2.6 implies that Γ_{X} is corepresented by 1_{X}, we deduce that there is a pullback square

$$\begin{tikzcd}
\mathsf{X}_{1_{\mathsf{X}}/} \arrow[r] \arrow[d] & (\mathsf{Grpd}_{\mathcal{B}})_{1_{\mathsf{Grpd}_{\mathcal{B}}}/} \arrow[d] \\
\mathsf{X} \arrow[r, "\Gamma_{\mathsf{X}}"] & \mathsf{Grpd}_{\mathcal{B}} .
\end{tikzcd}$$

On the other hand, Remark 2.1.2.5 implies that there also is a pullback square

$$\begin{tikzcd}
\mathrm{Tw}(\mathsf{X}) \arrow[r] \arrow[d, "p_{\mathsf{X}}"] & \mathsf{X}_{1_{\mathsf{X}}/} \arrow[d] \\
\mathsf{X}^{\mathrm{op}} \times \mathsf{X} \arrow[r, "\underline{\mathrm{Hom}}_{\mathsf{X}}^{\mathcal{B}}(-,-)"] & \mathsf{X}.
\end{tikzcd}$$

By pasting these two pullback squares together, the claim follows.

Proof of Proposition 6.2.10.11. Let κ be the cardinal and $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ be the subcategory from Construction 6.2.10.10, and let us denote by

$$(L \dashv i) \colon \mathsf{Grpd}_{\mathcal{B}} \rightleftarrows \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$$

the associated Bousfield localisation provided by Corollary 5.4.1.8. In light of Lemma 6.2.10.12, we only need to show that $\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$ is a subterminal $\mathcal{B}$-topos and to identify the adjunction unit $\mathrm{id} \to iL$ with the sheafification map $\mathrm{id} \to (-)^{\mathrm{sh}}_{\iota}$. In light of Theorem 6.2.3.1 and Example 6.2.10.8, the former claim is implied by the latter one, so that the proof is finished once we identify $\mathrm{id} \to iL$ with $\mathrm{id} \to (-)^{\mathrm{sh}}_{\iota}$.

We only need to show that for every object $G \colon A \to \mathsf{Grpd}_{\mathcal{B}}$ in arbitrary context $A \in \mathcal{B}$, the object G^{sh}_{ι} is W-local and the map $G \to G^{\mathrm{sh}}_{\iota}$ is inverted by the localisation functor L. Note that as $\mathcal{B}$ is generated by its κ-compact objects (as κ is assumed to be $\mathcal{B}$-regular), we may assume that A is κ-compact. In this case, note that κ is also $\mathcal{B}_{/A}$-regular and adapted to $\pi_A^*(\iota)$ (by Remark 5.2.2.6) and that we may identify the base change of $(-)^{+}_{\iota}$ along π_A^* with $(-)^{+}_{\pi_A^*(\iota)} \colon \mathsf{Grpd}_{\mathcal{B}_{/A}} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$. Therefore, we may also identify the base change of $(-)^{\mathrm{sh}}_{\iota}$ along π_A^* with $(-)^{\mathrm{sh}}_{\pi_A^*(\iota)}$. Together with Remark 5.4.1.2, this implies that we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and G with its transpose $\hat{G} \colon 1_{\mathcal{B}_{/A}} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$, so that we may assume without loss of generality that $A \simeq 1_{\mathcal{B}}$.

We first show that the map $G \to G^{\mathrm{sh}}_{\iota}$ is inverted by L, for which it will be enough to show that the map $\phi \colon G \to G^{+}_{\iota}$ is inverted by L, or equivalently that the map

$$\phi^* \colon \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(G^{+}_{\iota}, i(-)) \to \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(G, i(-))$$

is an equivalence. Note that by the triangle identities, the map

$$\lim_{\mathsf{Grpd}_S^{\mathrm{op}}} \to \operatorname*{colim}_{(\mathsf{Grpd}_S)^{\mathrm{op}}}$$

can be identified with the composition

$$\lim_{\mathsf{Grpd}_S^{\mathrm{op}}} \simeq \operatorname*{colim}_{\mathsf{Grpd}_S^{\mathrm{op}}} \mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} \lim_{\mathsf{Grpd}_S^{\mathrm{op}}} \to \operatorname*{colim}_{\mathsf{Grpd}_S^{\mathrm{op}}}$$

where the first map is induced by the counit of $\mathrm{colim}_{(\mathsf{Grpd}_S)^{\mathrm{op}}} \dashv \mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}}$ and the second map is induced by the counit of $\mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} \dashv \lim_{\mathsf{Grpd}_S^{\mathrm{op}}}$. Moreover, observe that as $\lim_{\mathsf{Grpd}_S^{\mathrm{op}}}$ can be identified with evaluation at the final object $1_{\mathsf{Grpd}_{\mathcal{B}}} \colon 1 \to \mathsf{Grpd}_S$, this functor is cocontinuous and therefore given by the left Kan extension of its restriction along the Yoneda embedding

$$h_{\mathsf{Grpd}_S} \colon \mathsf{Grpd}_S \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Grpd}_S).$$

As the restriction of $\lim_{\mathsf{Grpd}_S^{\mathrm{op}}}$ along h_{Grpd_S} yields $\mathrm{map}_{\mathsf{Grpd}_S}(1_{\mathsf{Grpd}_{\mathcal{B}}}, -)$ and the latter is equivalent to the inclusion ι, it follows that the left adjoint of $\lim_{\mathsf{Grpd}_S^{\mathrm{op}}}$ is given by $\iota^* h_{\mathsf{Grpd}_{\mathcal{B}}}$ (see Remark 3.5.1.4). Altogether, these observations imply that we may decompose ϕ^* into the chain of morphisms

$$\begin{aligned} \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(G_\iota^+, i(-)) &\simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\operatorname*{colim}_{\mathsf{Grpd}_S^{\mathrm{op}}} \iota^* h_{\mathsf{Grpd}_{\mathcal{B}}}(G), i(-)) \\ &\simeq \mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Grpd}_S)}(\iota^* h_{\mathsf{Grpd}_{\mathcal{B}}}(G), \operatorname*{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} i(-)) \\ &\to \mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Grpd}_S)}(\iota^* h_{\mathsf{Grpd}_{\mathcal{B}}}(G), \iota^* h_{\mathsf{Grpd}_{\mathcal{B}}} \lim_{\mathsf{Grpd}_S^{\mathrm{op}}} \operatorname*{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} i(-)) \\ &\simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(G, i(-)) \end{aligned}$$

in which the penultimate map is induced by composition with the adjunction unit $\mathrm{id} \to \iota^* h_{\mathsf{Grpd}_{\mathcal{B}}} \lim_{\mathsf{Grpd}_S^{\mathrm{op}}}$ and where the last equivalence follows from both $\mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}}$ and $\iota^* h_{\mathsf{Grpd}_{\mathcal{B}}}$ being fully faithful functors. Hence, it suffices to show that the map

$$\mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} i \to \iota^* h_{\mathsf{Grpd}_{\mathcal{B}}} \lim_{\mathsf{Grpd}_S^{\mathrm{op}}} \mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} i$$

is an equivalence, i.e. that $\mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} i$ takes value in the essential image of $\iota^* h_{\mathsf{Grpd}_{\mathcal{B}}}$. To see this, note that as the restriction of $L \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$ to Grpd_S factors through $1_{\mathsf{Grpd}_{\mathcal{B}}} \colon 1 \hookrightarrow \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$, it follows that we may identify $\iota^* h_{\mathsf{Grpd}_{\mathcal{B}}} i \simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\iota(-), i(-))$ with the transpose of the composition

$$\mathsf{Grpd}_S^{\mathrm{op}} \times \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{\mathrm{pr}_1} \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{\mathrm{map}_{\mathsf{Loc}_{\mathsf{W}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}, -)} \mathsf{Grpd}_{\mathcal{B}},$$

which is precisely $\mathrm{diag}_{\mathsf{Grpd}_S^{\mathrm{op}}} i$, as desired.

We finish the proof by showing that G_ι^{sh} is W-local. By the same reduction steps as above, it is enough to show that for every map $s \colon P \to Q$ in S between

κ-compact objects, the map $s^*\colon \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(Q, G_\iota^{\mathrm{sh}}) \to \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(P, G_\iota^{\mathrm{sh}})$ is an equivalence. Note that in light of the adjunction

$$(\pi_Q)_! \dashv \pi_Q^*\colon (\mathrm{Grpd}_{\mathcal{B}})_{/Q} \rightleftarrows \mathrm{Grpd}_{\mathcal{B}},$$

the map s^* can be interpreted as the morphism

$$s^*\colon \operatorname{map}_{(\mathrm{Grpd}_{\mathcal{B}})_{/Q}}(Q, \pi_Q^* G_\iota^{\mathrm{sh}}) \to \operatorname{map}_{(\mathrm{Grpd}_{\mathcal{B}})_{/Q}}(P, \pi_Q^* G_\iota^{\mathrm{sh}}).$$

By Remark 6.2.10.13, we can identify $\operatorname{map}_{(\mathrm{Grpd}_{\mathcal{B}})_{/Q}}(-,-)$ with the global sections of the internal hom $\underline{\mathrm{Hom}}_{(\mathrm{Grpd}_{\mathcal{B}})_{/Q}}(-,-)$. Hence we may as well show that the map

$$s^*\colon \underline{\mathrm{Hom}}_{(\mathrm{Grpd}_{\mathcal{B}})_{/Q}}(Q, \pi_Q^* G_\iota^{\mathrm{sh}}) \to \underline{\mathrm{Hom}}_{(\mathrm{Grpd}_{\mathcal{B}})_{/Q}}(P, \pi_Q^* G_\iota^{\mathrm{sh}})$$

is an equivalence. In other words, by replacing $\mathcal{B}$ with $\mathcal{B}_{/Q}$, we can reduce to the case where $Q \simeq 1$. Thus, what is left to show is that $t^*\colon G_\iota^{\mathrm{sh}} \to \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(P, G_\iota^{\mathrm{sh}})$ is an equivalence for every κ-compact $P\colon 1 \to \mathrm{Grpd}_{\mathcal{S}}$, where $t\colon G \to 1_{\mathrm{Grpd}_{\mathcal{B}}}$ is the terminal map. Note that by Corollary 5.2.2.23, the fact that P is κ-compact even implies that P is $\mathrm{Cat}_{\mathcal{B}}^{\kappa}$-compact when viewed as an object of $\mathrm{Grpd}_{\mathcal{B}}$. Hence, the map t^* can be identified with the colimit

$$\operatorname*{colim}_{\tau<\kappa} t_\tau^*\colon \operatorname*{colim}_{\tau<\kappa} T_\tau^\iota G \to \operatorname*{colim}_{\tau<\kappa} \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(P, T_\tau^\iota G).$$

To show that this map is an equivalence, it will be sufficient to prove that for every ordinal $\tau < \kappa$ the map $\operatorname{colim}_{n\in\mathbb{N}} t_{\tau+n}^*$ is one. To see the latter claim, observe that for every $H \in \mathcal{B}$ we have a commutative diagram

$$\begin{tikzcd}
H \arrow[rr, "t^*"] \arrow[d] & & \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(P, H) \arrow[d, "\alpha"] \\
\operatorname*{colim}_{(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}})_{P/}} \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(\iota(\pi_P)_!(-), H) \arrow[rr, "\simeq"] \arrow[d, "\simeq"] & & \operatorname*{colim}_{(\mathrm{Grpd}_{\mathcal{S}})^{\mathrm{op}}} \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(\iota(-), H) \arrow[d, "\beta"] \\
\operatorname*{colim}_{(\mathrm{Grpd}_{\mathcal{S}})^{\mathrm{op}}} \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(\iota(-), H) \arrow[rr, "t^*"] & & \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(P, \operatorname*{colim}_{(\mathrm{Grpd}_{\mathcal{S}})^{\mathrm{op}}} \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(\iota(-), H)).
\end{tikzcd}$$

Here the composition of the two vertical maps on the left and right can be identified by ϕ and ϕ_*, respectively. Moreover, the equivalences in this diagram are induced by the initial map $(\pi_P)_! \colon (\mathsf{Grpd}_S)_{/P} \to \mathsf{Grpd}_S$, the map α is induced by $P \colon 1 \to \mathsf{Grpd}_S$ and β is given by the composition

$$\begin{aligned}\operatorname*{colim}_{(\mathsf{Grpd}_S)^{\mathrm{op}}} \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\iota(-), H) &\to \operatorname*{colim}_{(\mathsf{Grpd}_S)^{\mathrm{op}}} \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(P, \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\iota(-), H)) \\ &\to \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(P, \operatorname*{colim}_{(\mathsf{Grpd}_S)^{\mathrm{op}}} \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\iota(-), H)).\end{aligned}$$

By substituting $H = T^{\iota}_{\tau+n} G$ for any $n \in \mathbb{N}$, we deduce that the map $t^*_{\tau+n} \to t^*_{\tau+n+1}$ is a retract of an equivalence and must therefore be an equivalence itself. $\square$

We finish this section with the following converse of Proposition 6.2.10.11:

Proposition 6.2.10.14. *Let* X *be a subterminal* $\mathcal{B}$*-topos. Then there is a bounded local class S that is closed under finite limits in* $\mathsf{Fun}(\Delta^1, \mathcal{B})$ *such that* $\Gamma(\mathsf{X}) \simeq \mathsf{Loc}_S(\mathcal{B})$. *Moreover, for* any *such local class S, we obtain an equivalence* $\mathsf{X} \simeq \mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$, *where* W *is as in Construction 6.2.10.10, and the adjunction unit* $\eta \colon \mathrm{id} \to \Gamma_{\mathsf{X}} \mathrm{const}_{\mathsf{X}}$ *can be identified with the map* $\mathrm{id} \to (-)^{\mathrm{sh}}_{\iota}$*, where* $\iota \colon \mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ *is the inclusion.*

Proof. Let us denote by $j_* \colon \mathcal{X} \hookrightarrow \mathcal{B}$ the geometric morphism associated with X. We begin by proving the second statement, i.e. suppose that S is a bounded local class closed under finite limits in $\mathsf{Fun}(\Delta^1, \mathcal{B})$ such that $\mathcal{X} \simeq \mathsf{Loc}_S(\mathcal{B})$. Let $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ be as in Construction 6.2.10.10. By Lemma 6.2.10.12, we may identify $\Gamma(\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}}))$ with $\mathcal{X}$. As Proposition 6.2.10.11 moreover implies that $\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}})$ is a subterminal $\mathcal{B}$-topos, Theorem 6.2.5.1 implies that we must necessarily have $\mathsf{Loc}_{\mathsf{W}}(\mathsf{Grpd}_{\mathcal{B}}) \simeq \mathsf{X}$. Hence the same proposition gives rise to the desired identification of the adjunction unit $\eta \colon \mathrm{id} \to \Gamma_{\mathsf{X}} \mathrm{const}_{\mathsf{X}}$.

To complete the proof, it is therefore enough to show that there always exists a bounded local class S that is closed under finite limits in $\mathsf{Fun}(\Delta^1, \mathcal{B})$ such that $\mathcal{X} \simeq \mathsf{Loc}_S(\mathcal{B})$. To that end, choose a $\mathcal{B}$-regular cardinal κ for which Γ_{X} is $\mathsf{Filt}_{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}$-cocontinuous. We let S be the class of relatively κ-compact maps in $\mathcal{B}$ that are inverted by j^*. Since by Proposition 5.2.2.11 the class of relatively κ-compact maps in $\mathcal{B}$ is local, we find that S is local as well. Moreover, S is closed under finite limits in $\mathsf{Fun}(\Delta^1, \mathcal{B})$ as j^* is left exact and as κ-compact objects in $\mathcal{B}$ are closed under finite limits (by choice of κ). Since S is inverted by j, we already have an inclusion $\mathcal{X} \hookrightarrow \mathsf{Loc}_S(\mathcal{B})$, so that it suffices to prove that every S-local object in $\mathcal{B}$ is contained in $\mathcal{B}$. Since $\mathcal{X}$ is a κ-accessible localisation of $\mathcal{B}$ (using Proposition 5.3.2.4) and $\mathcal{B}$ itself is κ-accessible, we deduce from the proof of [49, Proposition 5.5.4.2] (or alternatively the proof of Proposition 5.4.1.6 applied in the case $\mathcal{B} = \mathrm{Ani}$) that $\mathcal{X}$ is the Bousfield localisation at the class S' of those maps in $\mathcal{B}$ between κ-compact objects which are inverted by j. Since every such map must be relatively κ-compact (using that κ-compact objects are closed under finite limits in $\mathcal{B}$), every such map is contained in S. Hence the claim follows. □

Remark 6.2.10.15. We can use our understanding of subterminal $\mathcal{B}$-topoi to obtain a quite explicit understanding of pushouts in $\mathrm{Top}^{\mathrm{L}}_{\infty}$ in which one of the two maps is a Bousfield localisation. In fact, suppose that $f_* : \mathcal{X} \to \mathcal{B}$ and $i_* : \mathcal{Z} \hookrightarrow \mathcal{B}$ be geometric morphisms, where i_* is fully faithful. Then $i_*(\mathsf{Grpd}_{\mathcal{Z}})$ is a subterminal $\mathcal{B}$-topos, so that Proposition 6.2.10.14 implies that we can find a bounded local class S that is closed under compositions and finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$ such that $\mathcal{Z} = \mathrm{Loc}_S(\mathcal{B})$. Let $\overline{f^*S}$ denote the smallest local class of maps in $\mathcal{X}$ that contains $f^*(S)$. Then we claim that the functor $j_* : \mathcal{Z} \times_{\mathcal{B}} \mathcal{X} \hookrightarrow \mathcal{X}$ exhibits $\mathcal{Z} \times_{\mathcal{B}} \mathcal{X}$ as the Bousfield localisation of $\mathcal{X}$ at $\overline{f^*S}$. To see this, note that Proposition 6.2.7.1 and Lemma 6.2.7.7 imply that the morphism in $\mathrm{Top}^{\mathrm{R}}_{\mathcal{B}}$ corresponding to j_* is given by $\mathsf{Loc}_{\mathsf{W} \boxtimes f_*(\mathsf{Grpd}_{\mathcal{X}})^{\simeq}}(\mathsf{Grpd}_{\mathcal{B}} \otimes f_*(\mathsf{Grpd}_{\mathcal{X}})) \hookrightarrow f_*(\mathsf{Grpd}_{\mathcal{X}})$, where $\mathsf{W} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ is the subcategory from Construction 6.2.10.10. Since $\mathsf{Grpd}_{\mathcal{B}} \otimes f_*(\mathsf{Grpd}_{\mathcal{X}}) \simeq f_*(\mathsf{Grpd}_{\mathcal{X}})$, the left-hand side can be identified with the full subcategory of local objects with respect to

$$\mathsf{W} \times f_*(\mathsf{Grpd}_{\mathcal{X}})^{\simeq} \xrightarrow{\mathrm{const}_{i_* \mathsf{Grpd}_{\mathcal{X}}}(-) \times -} f_*(\mathsf{Grpd}_{\mathcal{X}}).$$

By the same argument as in the proof of Lemma 6.2.10.12, this means that an object $U \in \mathcal{X}$ is contained in $\mathcal{Z} \times_{\mathcal{B}} \mathcal{X}$ if and only if it is local with respect to every map in $\mathcal{X}$ of the form

$$f^*(s) \times_{f^*A} X \colon\ f^*(P) \times_{f^*A} X \to f^*(Q) \times_{f^*A} X$$

where $s \colon P \to Q$ is a map in $\mathsf{W}(A)$ and X is an arbitrary object in $\mathcal{X}_{/f^*A}$. By construction of W, the map $(\pi_A)_!(f)$ is contained in S, which in turn implies that $f^*(s) \times_{f^*A} X$ is in $\overline{f^*S}$. Hence the claim follows.

6.3. Localic $\mathcal{B}$-topoi

In higher topos theory, the 1-category of *locales* (with left exact left adjoints as maps) arises as a coreflective subcategory of the ∞-category $\mathrm{Top}_\infty^{\mathrm{L}}$ of ∞-topoi. The inclusion is given by sending a locale L to the ∞-topos $\mathrm{Sh}(L)$ of sheaves on L, and the coreflection carries an ∞-topos $\mathcal{X}$ to the locale $\mathrm{Sub}(\mathcal{X}) = \mathrm{Sub}_{\mathcal{X}}(1_{\mathcal{X}})$ of subterminal (i.e. (-1)-truncated) objects in $\mathcal{X}$. An ∞-topos $\mathcal{X}$ is said to be localic if it is equivalent to $\mathrm{Sh}(\mathrm{Sub}(\mathcal{X}))$.

In this section, we give an exposition of the analogous story in the world of $\mathcal{B}$-topoi. We do not aim to provide a comprehensive study of localic $\mathcal{B}$-topoi, but rather restrict our attention to those aspects of the theory that allow us to *define* the notion of a localic $\mathcal{B}$-topos and to provide an external characterisation of this concept in the case where $\mathcal{B}$ is itself localic.

We begin in Section 6.3.1 and Section 6.3.2 by providing the necessary background material on $\mathcal{B}$-posets. In Section 6.3.3 we define and characterise $\mathcal{B}$-locales, and in Section 6.3.4 we construct the $\mathcal{B}$-topos of sheaves on a $\mathcal{B}$-locale, which we use in Section 6.3.5 to show that $\mathcal{B}$-locales are a coreflective localisation of $\mathcal{B}$-topoi. In Section 6.3.6 we discuss how localic $\mathcal{B}$-topoi correspond to localic ∞-topoi over $\mathcal{B}$ in the case where $\mathcal{B}$ is itself localic.

6.3.1. $\mathcal{B}$-posets

Recall that an ∞-category $\mathcal{C}$ is (equivalent to) a poset precisely if if for all objects c and d in $\mathcal{C}$ the mapping ∞-groupoid $\mathrm{map}_{\mathcal{C}}(c, d)$ is (-1)-truncated. In this section we discuss a generalisation of this concept to $\mathcal{B}$-categories.

Recall that the class of (-1)-truncated maps in $\mathcal{B}$ is precisely the collection of morphisms that are internally right orthogonal to the map $1 \sqcup 1 \to 1$ (see Example 1.1.5.10).Since the internal saturation of the latter map is given by the covers in $\mathcal{B}$ and therefore closed under pullbacks in $\mathcal{B}$, the resulting factorisation system is a *modality*, so that the class of (-1)-truncated maps is local (see Example 3.1.4.5) We may therefore define:

Definition 6.3.1.1. The subuniverse $\mathrm{Sub}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ of *subterminal* $\mathcal{B}$*-groupoids* is the full subcategory of $\mathrm{Grpd}_{\mathcal{B}}$ that is determined by the local class of (-1)-

truncated morphisms in $\mathcal{B}$. A $\mathcal{B}_{/A}$-groupoid G is said to be a subterminal $\mathcal{B}_{/A}$-groupoid if it defines an object of $\mathrm{Sub}_{\mathcal{B}}$ (in context A).

Remark 6.3.1.2. As the functor $(\pi_A)_! \colon \mathcal{B}_{/A} \to \mathcal{B}$ creates pullbacks, a map $P \to B$ in $\mathcal{B}_{/A}$ defines an object $B \to \mathrm{Sub}_{\mathcal{B}_{/A}}$ if and only if the underlying map in $\mathcal{B}$ defines an object $(\pi_A)_!(B) \to \mathrm{Sub}_{\mathcal{B}}$. As a consequence, the equivalence $\pi_A^* \mathrm{Grpd}_{\mathcal{B}} \simeq \mathrm{Grpd}_{\mathcal{B}_{/A}}$ restricts to an equivalence $\pi_A^* \mathrm{Sub}_{\mathcal{B}} \simeq \mathrm{Sub}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$.

Proposition 6.3.1.3. *The $\mathcal{B}$-category* $\mathrm{Sub}_{\mathcal{B}}$ *is an accessible Bousfield localisation of* $\mathrm{Grpd}_{\mathcal{B}}$ *(in the sense of Definition 5.3.3.4) and therefore in particular presentable.*

Proof. By Example 3.1.4.5 the subcategory $\mathrm{Sub}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ is reflective. Since the inclusion $\mathrm{Sub}_{\mathcal{B}} \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ is section-wise accessible, we deduce from Corollary 5.3.2.5 implies that the localisation must be accessible. □

Definition 6.3.1.4. A $\mathcal{B}$-category C is said to be a $\mathcal{B}$-*poset* if the mapping bifunctor $\mathrm{map}_{\mathsf{C}}$ takes values in $\mathrm{Sub}_{\mathcal{B}}$. The full subcategory of $\mathrm{Cat}_{\mathcal{B}}$ that is spanned by the $\mathcal{B}_{/A}$-posets for each $A \in \mathcal{B}$ is denoted by $\mathrm{Pos}_{\mathcal{B}}$ and its underlying ∞-category of global sections by $\mathrm{Pos}(\mathcal{B})$.

Remark 6.3.1.5. A $\mathcal{B}$-category C is a $\mathcal{B}$-poset precisely if the map $\mathsf{C}_1 \to \mathsf{C}_0 \times \mathsf{C}_0$ is (-1)-truncated. In fact, since this map exhibits C_1 as the mapping $\mathcal{B}_{/\mathsf{C}_0\times\mathsf{C}_0}$-groupoid between the two objects $\mathrm{pr}_0, \mathrm{pr}_1 \colon \mathsf{C}_0 \times \mathsf{C}_0 \rightrightarrows \mathsf{C}$, this is clearly a necessary condition. The fact that it is also sufficient follows from the observation that every mapping $\mathcal{B}_{/A}$-groupoid of C is a pullback of this map.

Remark 6.3.1.6. Since the class of (-1)-truncated maps in $\mathcal{B}$ is local, we deduce from Remark 6.3.1.5 that if $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ is a cover in $\mathcal{B}$, a $\mathcal{B}_{/A}$-category $\mathsf{C} \colon A \to \mathrm{Cat}_{\mathcal{B}}$ defines a $\mathcal{B}_{/A}$-poset if and only if the $\mathcal{B}_{/A_i}$-category $s_i^*\mathsf{C}$ defines a $\mathcal{B}_{/A_i}$-poset for every i. In particular, *every* object of $\mathrm{Pos}_{\mathcal{B}}$ in context $A \in \mathcal{B}$ encodes a $\mathcal{B}_{/A}$-poset, and one has a canonical equivalence $\pi_A^* \mathrm{Pos}_{\mathcal{B}} \simeq \mathrm{Pos}_{\mathcal{B}_{/A}}$.

Remark 6.3.1.7. Remark 6.3.1.5 and the fact that a map is (-1)-truncated precisely if it is so section-wise imply that a $\mathcal{B}$-category C is a $\mathcal{B}$-poset if and only if $\mathsf{C}(A)$ is an poset for every $A \in \mathcal{B}$. Together with Remark 6.3.1.6, this implies that we obtain an equivalence $\mathrm{Pos}_{\mathcal{B}} \simeq \mathrm{Pos} \otimes \mathrm{Grpd}_{\mathcal{B}}$ (where Pos is the 1-category of posets).

Remark 6.3.1.8. Recall that if C is a $\mathcal{B}$-category, then the map $s_0 \colon \mathsf{C}_0 \to \mathsf{C}_1$ is a monomorphism in $\mathcal{B}$ (see Example 1.3.1.3). Together with Remark 6.3.1.5, this implies that if P is a $\mathcal{B}$-poset, then each P_n is contained in the underlying 1-topos $\mathrm{Disc}(\mathcal{B}) \hookrightarrow \mathcal{B}$ of 0-truncated objects. Consequently, we may identify $\mathrm{Pos}(\mathcal{B})$ with the full subcategory of $\mathrm{Disc}(\mathcal{B})_\Delta$ that is spanned by the complete Segal objects P for which the map $\mathsf{P}_1 \to \mathsf{P}_0 \times \mathsf{P}_0$ is a monomorphism. Hence $\mathrm{Pos}(\mathcal{B})$ is equivalent to the 1-category of internal posets in the 1-topos $\mathrm{Disc}(\mathcal{B})$ in the sense of [39, § B2.3].

6.3.2. Presentable $\mathcal{B}$-posets

In this section we study *presentable* $\mathcal{B}$-posets. We begin with the following definition:

Definition 6.3.2.1. If C is a $\mathcal{B}$-category, the full subcategory $\mathsf{Sub}(\mathsf{C}) \hookrightarrow \mathsf{C}$ of *subterminal objects* is defined as the pullback $\mathsf{C} \times_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$.

Remark 6.3.2.2. If C is a $\mathcal{B}$-category and $A \in \mathcal{B}$ is an arbitrary object, then Remark 6.3.1.2 and Remark 5.4.5.2 imply that there is a canonical equivalence $\pi_A^* \mathsf{Sub}(\mathsf{C}) \simeq \mathsf{Sub}(\pi_A^* \mathsf{C})$ of full subcategories in $\pi_A^* \mathsf{C}$.

Proposition 6.3.2.3. *For every presentable $\mathcal{B}$-category D there is a canonical equivalence $\mathsf{Sub}(\mathsf{D}) \simeq \mathsf{D} \otimes \mathsf{Sub}_{\mathcal{B}}$ of full subcategories in D. In particular, $\mathsf{Sub}(\mathsf{D})$ is an accessible Bousfield localisation of D and therefore presentable as well.*

Proof. Recall from Proposition 5.4.5.3 that the Yoneda embedding $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ identifies D with the full subcategory $\mathsf{Sh}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{D}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ that is spanned by the continuous functors (in arbitrary context). Therefore, it will be enough to show that the square

$$\begin{tikzcd}
\mathsf{D} \otimes \mathsf{Sub}_{\mathcal{B}} \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \arrow[d, hook] \\
\mathsf{Sh}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{D}) \arrow[r, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})
\end{tikzcd}$$

is a pullback. Together with Remark 5.4.5.2 and Remark 6.3.1.2, this means that we only need to check that a functor $F \colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Sub}_{\mathcal{B}}$ is continuous if and only if its postcomposition with the inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ is. This follows immediately from the fact that the inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ has a left adjoint and is therefore continuous and conservative. □

For every presentable $\mathcal{B}$-category D, we will denote the left adjoint of the inclusion $\mathsf{Sub}(\mathsf{D}) \hookrightarrow \mathsf{D}$ by $(-)^{\mathrm{Sub}}$ and refer to it as the *subterminal truncation functor*.

Example 6.3.2.4. If C is an arbitrary $\mathcal{B}$-category, then Proposition 6.3.2.3 provides us with an equivalence of $\mathcal{B}$-categories

$$\mathsf{Sub}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}).$$

In light of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \mathsf{RFib}_{\mathsf{C}}$, we may identify $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ with the full subcategory of $\mathsf{RFib}_{\mathsf{C}}$ spanned by the right fibrations $p\colon \mathsf{P} \to \pi_A^*\mathsf{C}$ (in arbitrary context $A \in \mathcal{B}$) which are fully faithful, i.e. which are *sieves* in the $\mathcal{B}_{/A}$-category $\pi_A^*\mathsf{C}$. To see the latter claim, first note that by Remark 6.3.2.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ so that we can assume that $A \simeq 1$. In this case, since $p_0\colon \mathsf{P}_0 \to \mathsf{C}_0$ can be identified with the image of the tautological object $\mathsf{C}_0 \to \mathsf{C}$ along the functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ that classifies p and since every object in C (in arbitrary context) arises as a pullback of the tautological object, F takes values in $\mathsf{Sub}_{\mathcal{B}}$ if and only if p_0 is a monomorphism. Therefore it suffices to show that p_0 is monic if and only if p is fully faithful. This follows from considering the commutative diagram

$$\begin{tikzcd}
\mathsf{P}_1 \arrow[r, "\mathrm{id}"] \arrow[d, "(d_1,d_0)"] & \mathsf{P}_1 \arrow[r] \arrow[d] & \mathsf{C}_1 \arrow[d, "(d_1,d_0)"] \\
\mathsf{P}_0 \times \mathsf{P}_0 \arrow[r, "p_0\times\mathrm{id}"] & \mathsf{C}_0 \times \mathsf{P}_0 \arrow[r, "\mathrm{id}\times p_0"] & \mathsf{C}_0 \times \mathsf{C}_0
\end{tikzcd}$$

in which the right square is a pullback as p is a right fibration and where the left square is a pullback since p_0 is a monomorphism.

Furthermore, the above observation implies that we may compute the subterminal truncation functor $(-)^{\mathrm{Sub}}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{Sub}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ on the level of right fibrations by taking essential images: If $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is a presheaf, then F^{Sub} classifies the essential image of the right fibration $\mathsf{C}_{/F} \to \mathsf{C}$. In fact, this is a consequence of the straightforward observation that the essential image is still a right fibration.

By definition, if C is a $\mathcal{B}$-category, then $\mathrm{Sub}(\mathsf{C})$ is a $\mathcal{B}$-poset. Our next goal is to show that if C is presentable, then $\mathrm{Sub}(\mathsf{C})$ can be characterised as the *largest* accessible Bousfield localisation of C with that property.

Lemma 6.3.2.5. *Let* $(l \dashv r)\colon \mathsf{D} \rightleftarrows \mathsf{C}$ *be an adjunction of* $\mathcal{B}$*-categories. Then there is a commutative square*

$$\begin{tikzcd} \mathsf{Sub}(\mathsf{D}) \arrow[r, "r"] \arrow[d, hook] & \mathsf{Sub}(\mathsf{C}) \arrow[d, hook] \\ \mathsf{D} \arrow[r, "r"] & \mathsf{C} \end{tikzcd}$$

which is a pullback when r *is fully faithful.*

Proof. Unwinding the definitions, it will be enough to show that we have a commutative square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \arrow[r, "r_!"] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \arrow[d, hook] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r, "r_!"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

that is a pullback when r is fully faithful. The existence of this square follows from observing that $r_!$ can be identified with l^*. If r is moreover fully faithful, then l is a localisation and therefore in particular essentially surjective. As essentially surjective functors are internally left orthogonal to fully faithful functors and as the inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ is fully faithful, the second claim follows. □

Proposition 6.3.2.6. *Let* D *be a presentable* $\mathcal{B}$*-category and* P *be a presentable* $\mathcal{B}$*-poset. Then composition with the inclusion* $\mathsf{Sub}_{\mathcal{B}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ *induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{P}, \mathsf{Sub}(\mathsf{D})) \hookrightarrow \underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{P}, \mathsf{D}).$$

Proof. By Remark 6.3.2.2 and Remark 3.1.3.6, it will be enough to show that every right adjoint functor $\mathsf{P} \to \mathsf{D}$ factors through $\mathsf{Sub}(\mathsf{D})$. This follows immediately from Lemma 6.3.2.5. □

In light of Remark 6.3.2.2, Remark 3.1.3.6 and Remark 5.4.4.6, Proposition 6.3.2.6 implies:

Corollary 6.3.2.7. *The full subcategory of* $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *that is spanned by the presentable* $\mathcal{B}_{/A}$*-posets for all* $A \in \mathcal{B}$ *is reflective, with the left adjoint given by sending a presentable* $\mathcal{B}$*-category* D *to* $\mathsf{Sub}(\mathsf{D})$. □

Remark 6.3.2.8. Lemma 6.3.2.5 furthermore implies that if D is a presentable $\mathcal{B}$-category and $d\colon 1 \to \mathsf{D}$ is an arbitrary object, then d is subterminal if and only if the diagonal $d \to d \times d$ is an equivalence. In fact, by choosing a presentation of D as an accessible Bousfield localisation of a presheaf $\mathcal{B}$-category and making use of Lemma 6.3.2.5, we may assume that $\mathsf{D} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ and hence that d can be identified with a right fibration over C. Then the claim follows immediately from Example 6.3.2.4. In particular, this observation implies that $\mathsf{Sub}(\mathsf{D})$ can be identified with the sheaf $\mathsf{Sub}(\mathsf{D}(-))$ on $\mathcal{B}$.

Finally, we arrive at the following characterisation of presentable $\mathcal{B}$-posets:

Proposition 6.3.2.9. *For an (a priori large)* $\mathcal{B}$*-category* D*, the following are equivalent:*

1. D *is a presentable* $\mathcal{B}$*-poset;*
2. $\mathsf{D} \simeq \mathsf{Sub}(\mathsf{E})$ *for some presentable* $\mathcal{B}$*-category* E*;*
3. D *is small and cocomplete;*
4. D *is small, and the Yoneda embedding* $h\colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ *admits a left adjoint;*
5. D *is a small* $\mathcal{B}$*-poset, and the Yoneda embedding* $h\colon \mathsf{D} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ *admits a left adjoint.*

Proof. (1) and (2) are equivalent by Corollary 6.3.2.7. If D is a presentable $\mathcal{B}$-poset, then Lemma 6.3.2.5 combined with Example 6.3.2.4 shows that there is a small $\mathcal{B}$-category C such that D arises as a Bousfield localisation of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ for some small $\mathcal{B}$-category C. Hence, as the latter is small, so is D. Therefore (1) implies (3). Conversely, every small and cocomplete $\mathcal{B}$-category is presentable (this follows from our characterisation of presentable $\mathcal{B}$-categories as the accessible and cocomplete ones, see Theorem 5.4.2.5), and since every small and cocomplete ∞-category is a poset, we conclude by employing Remark 6.3.1.7 that (3) implies (1). Furthermore, (3) and (4) are equivalent by the universal property of presheaf $\mathcal{B}$-categories (see Corollary 3.5.1.13). Finally, (4) implies (5) by Lemma 6.3.2.5 (and since we already know that (3) forces D to be a $\mathcal{B}$-poset), and (5) implies (3) since $\mathrm{Sub}_{\mathcal{B}}$ is cocomplete. □

We end this section with the observation that all colimits in a presentable $\mathcal{B}$-poset are $\mathcal{B}$-groupoidal and can be computed by an explicit formula:

Proposition 6.3.2.10. *Let* D *be a presentable* $\mathcal{B}$*-poset and let* $d\colon \mathsf{I} \to \mathsf{D}$ *be a diagram. Then the inclusion* $\mathsf{I}^{\simeq} \to \mathsf{I}$ *induces an equivalence* $\operatorname{colim} d|_{\mathsf{I}^{\simeq}} \simeq \operatorname{colim} d$*. Moreover, this colimit can be explicitly computed as*

$$\operatorname{colim} d \simeq \bigvee_{\substack{i\colon G \to \mathsf{I} \\ G \in \mathcal{G}}} (\pi_G)_!(d(i)),$$

where $\mathcal{G} \hookrightarrow \mathcal{B}$ *is a small dense full subcategory and where the right-hand side denotes the join in the poset* $\Gamma_{\mathcal{B}}(\mathsf{D})$*.*

Proof. Consider the full subcategory $\mathcal{E} \subset \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ that is spanned by those diagrams $d\colon \mathsf{I} \to \mathsf{D}$ for which the inclusion $\mathsf{I}^{\simeq} \to \mathsf{I}$ induces an equivalence $\operatorname{colim} d|_{\mathsf{I}^{\simeq}} \simeq \operatorname{colim} d$. To prove the first statement, we need to show that we have $\mathcal{E} = \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$.

To that end, first observe that if $h_{\mathsf{D}}\colon \mathsf{D} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{D})$ denotes the Yoneda embedding, then $\operatorname{colim} h_{\mathsf{D}} d$ classifies the right fibration $p\colon \mathsf{P} \to \mathsf{D}$ that arises from factoring d into a final functor and a right fibration. In other words, p is the image of d under the localisation functor $L\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}} \to \mathrm{RFib}(\mathsf{D})$ from Lemma 3.5.4.9, and we may compute the colimit of d by applying the left adjoint $l\colon \mathrm{RFib}(\mathsf{D}) \to \mathsf{D}$ (which exists by Proposition 6.3.2.9) to p. Since both l and L are cocontinuous, it follows that for every ∞-category $\mathcal{K}$ and every diagram $\phi\colon \mathcal{K} \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ with colimit $d\colon \mathsf{I} \to \mathsf{D}$, we have a canonical equivalence $\operatorname{colim} lL\phi \simeq \operatorname{colim} d$.

Now let $\phi^{\simeq}\colon \mathcal{K} \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ be the composition of ϕ with the core $\mathcal{B}$-groupoid functor $(-)^{\simeq}$. We then obtain a natural comparison map $\operatorname{colim} \phi^{\simeq} \to \mathsf{I}^{\simeq}$ which has the property that the composition of this map with the inclusion $\mathsf{I}^{\simeq} \to \mathsf{I}$ can be identified with the colimit of the canonical morphism $\phi^{\simeq} \to \phi$. As a consequence, we obtain maps

$$\operatorname{colim} lL\phi^{\simeq} \to \operatorname{colim} d|_{\mathsf{I}^{\simeq}} \to \operatorname{colim} d$$

in which the composition can be identified with $\operatorname{colim} lL\phi^{\simeq} \to \operatorname{colim} lL\phi$. Hence, if ϕ takes values in $\mathcal{E}$, the latter map is an equivalence, which implies that the map $\operatorname{colim} d|_{\mathsf{I}^{\simeq}} \to \operatorname{colim} d$ is one as well since D is a poset. Thus, we conclude that $\mathcal{E}$ is closed under colimits in $\mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$.

Consequently, as every $\mathcal{B}$-category can be written as a colimit of $\mathcal{B}$-categories of the form $G \otimes \Delta^n$ for $G \in \mathcal{B}$ and $n \in \mathbb{N}$ (cf. Remark 1.2.1.3), it suffices to see that every diagram of the form $d\colon G \otimes \Delta^n \to \mathsf{D}$ is in $\mathcal{E}$. To that end, note that the colimit of a diagram $d\colon G \otimes \Delta^n \to \mathsf{D}$ is given by applying $(\pi_G)_! \colon \mathsf{D}(G) \to \mathsf{D}(1)$ to the colimit of the transposed map $d' \colon \Delta^n \to \mathsf{D}(G)$, which is simply $d'(n)$. Likewise, the colimit of the induced diagram $\bigsqcup_n G = (G \otimes \Delta^n)^{\simeq} \to \mathsf{D}$ is given by applying $(\pi_G)_!$ to the supremum of the objects $d'(i)$ for $i \in \Delta^n$. Since $d'(i) \leq d'(n)$ for all $i \in \Delta^n$, we deduce that $d \in \mathcal{E}$, as desired.

As for the second statement of the proposition, note that we have an equivalence

$$\mathsf{I}^{\simeq} \simeq \operatorname*{colim}_{\substack{i\colon G \to \mathsf{I} \\ G \in \mathcal{G}}} G$$

since $\mathcal{G}$ is dense in $\mathcal{B}$. Thus the description of $\operatorname{colim} d|_{\mathsf{I}^{\simeq}}$ follows from the observation at the beginning of the proof. $\square$

6.3.3. $\mathcal{B}$-locales

In this section we define what it means for a $\mathcal{B}$-poset to be a *$\mathcal{B}$-locale* and provide a first characterisation of this notion.

Definition 6.3.3.1. A $\mathcal{B}$-category L is said to be a *$\mathcal{B}$-locale* if

1. L is a $\mathcal{B}$-poset,
2. L is presentable, and
3. colimits are universal in L (in the sense of Definition 6.1.4.1).

A functor $f\colon K \to L$ between $\mathcal{B}$-locales is called an algebraic morphism of $\mathcal{B}$-locales if it is cocontinuous and preserves finite limits. We let $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Pos}_{\mathcal{B}}$ be the subcategory that is spanned by the algebraic morphisms of $\mathcal{B}_{/A}$-locales for all $A \in \mathcal{B}$.

Remark 6.3.3.2. In the situation of Definition 6.3.3.1, note that colimits are universal in L if and only if for every $A \in \mathcal{B}$ and every $U\colon A \to \mathsf{L}$ the functor $U \times -\colon \pi_A^* \mathsf{L} \to \pi_A^* \mathsf{L}$ is cocontinuous. In fact, note that for any map $U \to V$ in L in context A, the fact that L is a $\mathcal{B}$-poset implies that the diagram

$$\begin{tikzcd}
(\pi_A^* \mathsf{L})_{/V} \arrow[r, "U \times_V -"] \arrow[d, "(\pi_V)_!"] & (\pi_A^* \mathsf{L})_{/V} \arrow[d, "(\pi_V)_!"] \\
\pi_A^* \mathsf{L} \arrow[r, "U \times -"] & \pi_A^* \mathsf{L}
\end{tikzcd}$$

commutes. Therefore, the composition $(\pi_V)_!(U \times_V -)$ is cocontinuous. As $(\pi_V)_!$ is conservative, this implies that $U \times_V -$ is already cocontinuous.

Remark 6.3.3.3. By Remark 6.3.1.6, Remark 5.4.2.10 and Remark 6.1.4.3, the condition of a $\mathcal{B}$-category to be a $\mathcal{B}$-locale is local in $\mathcal{B}$: for every cover $\bigsqcup A_i \twoheadrightarrow 1$ in $\mathcal{B}$, a $\mathcal{B}$-category L is a $\mathcal{B}$-locale if and only if the $\mathcal{B}_{/A_i}$-category $\pi_{A_i}^* \mathsf{L}$ is a $\mathcal{B}_{/A_i}$-locale.

Remark 6.3.3.4. The subobject of $(\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ that is spanned by the algebraic morphisms between $\mathcal{B}_{/A}$-locales (for each $A \in \mathcal{B}$) is stable under composition and equivalences in the sense of Proposition 1.3.1.17. Since moreover cocontinuity and the property that a functor preserves finite limits are local conditions and on account of Remark 6.3.3.3, we conclude that a map $A \to (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ is contained in $(\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}})_1$ if and only if it defines an algebraic morphism between $\mathcal{B}_{/A}$-locales. In particular, if L and M are $\mathcal{B}_{/A}$-locales, the image of the monomorphism

$$\mathrm{map}_{\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{L}, \mathsf{M}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{L}, \mathsf{M})$$

is spanned by the algebraic morphisms, and there is furthermore a canonical equivalence $\pi_A^* \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}_{/A}}$.

Remark 6.3.3.5. By Remark 6.3.1.8, it is easy to see that $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{Pos}(\mathcal{B})$ can be identified with the category of internal locales in $\mathrm{Disc}(\mathcal{B})$ in the sense of [39, § C1.6]. In other words, our notion of an internal locale coincides with the classical one.

Lemma 6.3.3.6. *Let D be a presentable $\mathcal{B}$-category with universal colimits, and let $l\colon \mathsf{D} \to \mathsf{L}$ be a Bousfield localisation that preserves binary products. Suppose furthermore that L is a $\mathcal{B}$-poset. Then L is a $\mathcal{B}$-locale.*

Proof. We need to show that colimits are universal in L, i.e. that for every $A \in \mathcal{B}$ and every object $U\colon A \to \mathsf{L}$ the functor $U \times - \colon \pi_A^* \mathsf{L} \to \pi_A^* \mathsf{L}$ is cocontinuous, or equivalently has a right adjoint. Now l preserving binary products implies that $U \times -$ carries every map in D (in arbitrary context) that is inverted by l to one that is inverted by l as well. Hence the functor $\underline{\mathrm{Hom}}_{\mathsf{D}}(i(U), i(-))$ (where $\underline{\mathrm{Hom}}_{\mathsf{D}}(-,-)$ is the internal hom in D) takes values in L, which yields the claim. $\square$

Proposition 6.3.3.7. *For a $\mathcal{B}$-category L, the following are equivalent:*

1. *L is a $\mathcal{B}$-locale.*
2. a) *L takes values in the 1-category $\mathrm{Loc}^{\mathrm{L}}$ of locales;*
   b) *L is $\mathsf{Grpd}_{\mathcal{B}}$-cocomplete;*
   c) *for every map $s\colon B \to A$ in $\mathcal{B}$, the functor $s_! \colon \mathsf{L}(B) \to \mathsf{L}(A)$ is a cartesian fibration.*
3. *L is small, and the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ admits a left adjoint which preserves finite products;*
4. *L is a small $\mathcal{B}$-poset, and the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ admits a left exact left adjoint.*

Proof. First, we show that (1) and (2) are equivalent. To that end, if L is a $\mathcal{B}$-locale, then for each $A \in \mathcal{B}$ the ∞-category $\mathsf{L}(A)$ is a presentable poset in which colimits are universal. Therefore, $\mathsf{L}(A)$ is a locale. Moreover, L being cocomplete implies that for every map $s\colon B \to A$ in $\mathcal{B}$ the transition functor $s^* \colon \mathsf{L}(A) \to \mathsf{L}(B)$ is cocontinuous. Likewise, L having finite limits implies that s^* preserves finite limits.

Therefore (2a) follows. Moreover, condition (2b) is part of the definition of a $\mathcal{B}$-locale, and condition (2c) is a reformulation of the condition that $\mathrm{Grpd}_{\mathcal{B}}$-colimits are universal in L (see Example 6.1.4.6). Conversely, if the three conditions in (2) are satisfied, then L is $\mathrm{Grpd}_{\mathcal{B}}$-cocomplete and section-wise presentable, so that Theorem 5.4.2.5 implies that L is presentable. By Remark 6.3.1.7, the assumption that L is section-wise given by a poset implies that L is a $\mathcal{B}$-poset. Finally, the fact that L takes values in Loc implies that LConst-colimits are universal in L (see Example 6.1.4.4), so that it suffices to verify that $\mathrm{Grpd}_{\mathcal{B}}$-colimits are universal in L as well. Again, this is a consequence of Example 6.1.4.6.

Next, if L is a $\mathcal{B}$-locale, then Proposition 6.3.2.9 implies that the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ has a left adjoint l. Moreover, as colimits are universal in L and as $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ is generated by L under colimits, the comparison map $l(- \times -) \to l(-) \times l(-)$ is an equivalence already when its restriction to L is one, which is trivially true. Hence (1) implies (3). If we assume (3), then Proposition 6.3.2.9 implies that L is a small $\mathcal{B}$-poset and that the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ has a left adjoint. Explicitly, this left adjoint arises as the restriction of the left adjoint $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \mathsf{L}$ and therefore preserves finite products. But since pullbacks in $\mathcal{B}$-posets coincide with binary products, this is already enough to conclude that this functor is left exact. Hence (4) follows. Finally, if (4) holds, then L is presentable by Proposition 6.3.2.9. Moreover, using Lemma 6.3.3.6 it will be enough to show that the subterminal truncation functor $(-)^{\mathrm{Sub}} \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ preserves binary products, which is an immediate consequence of Example 6.3.2.4. □

Using Proposition 6.3.3.7, it is now easy to show that the $\mathcal{B}$-poset of subterminal objects in a $\mathcal{B}$-topos is a $\mathcal{B}$-locale. More precisely, one has:

Proposition 6.3.3.8. *The functor* $\mathrm{Sub} \colon \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *from Corollary 6.3.2.7 restricts to a functor* $\mathrm{Sub} \colon \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$.

Proof. By combining Remark 6.3.2.2 and Remark 6.3.3.4, it is enough to show that for every algebraic morphism $f^* \colon \mathsf{X} \to \mathsf{Y}$ of $\mathcal{B}$-topoi the induced map $\mathrm{Sub}(f^*) \colon \mathrm{Sub}(\mathsf{X}) \to \mathrm{Sub}(\mathsf{Y})$ is an algebraic morphism of $\mathcal{B}$-locales. First, let us show that $\mathrm{Sub}(\mathsf{X})$ (and therefore by symmetry also $\mathrm{Sub}(\mathsf{Y})$) is a $\mathcal{B}$-locale. To that end, choose a presentation of X as a left exact and accessible Bousfield localisation

$L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$. Then Remark 6.3.2.8 implies that L restricts to a left exact and accessible Bousfield localisation $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \to \mathsf{Sub}(\mathsf{X})$, hence the claim follows from Proposition 6.3.3.7. Second, since we already know that $\mathsf{Sub}(f^*)$ is cocontinuous, it is enough to show that it is left exact as well. But on account of Remark 6.3.2.8, this functor arises as the restriction of f^* to subterminal objects, which immediately yields the claim. □

6.3.4. Sheaves on a $\mathcal{B}$-locale

In Proposition 6.3.3.8, we saw that the functor $\mathsf{Sub}(-)$ lets us pass from $\mathcal{B}$-topoi from $\mathcal{B}$-locales. The goal of this section is to discuss how one can conversely associate to every $\mathcal{B}$-locale a $\mathcal{B}$-topos: that of *sheaves* on the $\mathcal{B}$-locale.

Definition 6.3.4.1. Let L be a $\mathcal{B}$-locale and let $U\colon A \to \mathsf{L}$ be an object. A *covering* of U is a diagram $d\colon \mathsf{G} \to \pi_A^*\mathsf{L}$ with colimit U, where G is a $\mathcal{B}_{/A}$-groupoid.

Remark 6.3.4.2. Explicitly, a covering of U is given by a map $s\colon B \to A$ in $\mathcal{B}$ together with an object $V\colon B \to \mathsf{L}$ such that $s_!(V) \simeq U$.

Example 6.3.4.3. Let L be a $\mathcal{B}$-locale and $U\colon A \to \mathsf{L}$ be an object. Then every covering $(j_i\colon U_i \to U)_{i\in I}$ in $\mathsf{L}(A)$ (in the conventional sense) can be regarded as a covering in the sense of Definition 6.3.4.1 by setting $\mathsf{G} = I$ and $d = (j_i)_{i\in I}$.

Recall from Proposition 6.3.3.7 that if L is a $\mathcal{B}$-locale, then the associated Yoneda embedding $h_{\mathsf{L}}\colon \mathsf{L} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ admits a (left exact) left adjoint l. We shall denote by $\eta\colon \mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})} \to h_{\mathsf{L}} l$ the adjunction unit.

Definition 6.3.4.4. Let L be a $\mathcal{B}$-locale and let $d\colon \mathsf{G} \to \pi_A^*\mathsf{L}$ be a covering of an object $U\colon A \to \mathsf{L}$. Then the induced map $\eta \operatorname{colim} h_{\mathsf{L}} d\colon S_d = \operatorname{colim} h_{\mathsf{L}} d \hookrightarrow h_{\mathsf{L}}(U)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ is referred to as the *covering sieve* associated with d.

Remark 6.3.4.5. Let L be a $\mathcal{B}$-locale and $d\colon \mathsf{G} \to \pi_A^*\mathsf{L}$ be a covering of an object $U\colon A \to \mathsf{L}$. Then, for every map $s\colon B \to A$ in $\mathcal{B}$, we obtain an equivalence $s^*S_d \simeq S_{s^*d}$ that commutes with the canonical equivalence $s^*h_{\mathsf{L}}(U) \simeq h_{\mathsf{L}}(s^*U)$.

Definition 6.3.4.6. If L is a $\mathcal{B}$-locale, we denote by

$$\mathsf{Cov} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$$

the subcategory that is spanned by the covering sieves in arbitrary context.

Remark 6.3.4.7. For every $A \in \mathcal{B}$, one may identify

$$\pi_A^* \operatorname{Cov} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\pi_A^* \mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}_{/A}})$$

with the subcategory of covering sieves in $\pi_A^* \mathsf{L}$.

Remark 6.3.4.8. The $\mathcal{B}$-category Cov is small. In fact, first note that by Remark 6.3.4.5, the subcategory $\operatorname{Cov} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ is already spanned by all covering sieves of objects in context $G \in \mathcal{G}$, where $\mathcal{G}$ is a small full subcategory of $\mathcal{B}$ that generates $\mathcal{B}$ under colimits. Furthermore, since L is small, the collection of all coverings of objects in fixed context G is parametrised by a small set. Hence, the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(L)^{\Delta^1}$ that is spanned by the covering sieves must be small. In light of the very construction of a subcategory from a collection of morphisms (see Definition 1.3.1.14), the claim thus follows from the fact that the 1-image of a small $\mathcal{B}$-category in a locally small $\mathcal{B}$-category must also be small (being a subcategory of the essential image, which is small by Lemma 2.3.1.6).

Definition 6.3.4.9. Let L be a $\mathcal{B}$-locale. We define the $\mathcal{B}$-category $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ of *sheaves* on L to be the Bousfield localisation $\mathsf{Loc}_{\mathsf{Cov}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}))$ of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$. We will furthermore denote the underlying ∞-category of global sections of $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ by $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$.

Remark 6.3.4.10. By Remark 6.3.4.7 and Remark 5.4.1.2, for every $A \in \mathcal{B}$ there is a canonical equivalence $\pi_A^* \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{L})$ of full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{L})$.

Remark 6.3.4.11. If L is a $\mathcal{B}$-locale, then Proposition 6.3.3.7 implies that the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ admits a left adjoint $l \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \mathsf{L}$. By construction, this functor carries Cov into $\mathsf{L}^{\simeq}$. By Corollary 5.4.3.2, this implies that l factors through the sheafification functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$. By passing to right adjoints, this implies that the Yoneda embedding factors through the inclusion $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$, which means that every representable presheaf on L is already a sheaf.

The main goal of this section is to prove that $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is a $\mathcal{B}$-topos. More precisely, we will show:

Proposition 6.3.4.12. *For any $\mathcal{B}$-locale L, the localisation functor*

$$\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathrm{Sh}}_{\mathcal{B}}(\mathsf{L})$$

preserves finite limits. In particular, $\underline{\mathrm{Sh}}_{\mathcal{B}}(\mathsf{L})$ is a $\mathcal{B}$-topos.

The proof of Proposition 6.3.4.12 is based on the following three lemmas:

Lemma 6.3.4.13. *For every $\mathcal{B}$-locale L, the ∞-category $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ is the Bousfield localisation of $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$ at the set*

$$W = \{(\pi_A)_!(S) \hookrightarrow (\pi_A)_! h(U) \mid A \in \mathcal{B},\ U\colon A \to \mathsf{L},\ S \hookrightarrow h(U) \textit{ covering sieve}\}$$

of morphisms in $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$.

Proof. A presheaf $F\colon \mathsf{L}^{\mathrm{op}} \to \mathrm{Grpd}_{\mathcal{B}}$ is a sheaf if and only if for every $A \in \mathcal{B}$ and every covering sieve $S \hookrightarrow h(U)$ in context A the morphism

$$\phi\colon \ \mathrm{map}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})}(h(U), \pi_A^* F) \to \mathrm{map}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})}(S, \pi_A^* F)$$

is an equivalence in $\mathcal{B}_{/A}$. Recall from Corollary 2.2.2.8 that if $s\colon B \to A$ is a map in $\mathcal{B}$, then on local sections over A the map ϕ recovers the morphism

$$\mathrm{map}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(B)}(s^* h(U), \pi_B^* F) \to \mathrm{map}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(B)}(s^* S, \pi_B^* F)$$

of mapping ∞-groupoids, which by adjunction can in turn be identified with the map

$$\mathrm{map}_{\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})}((\pi_B)_! s^* h(U), F) \to \mathrm{map}_{\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})}(\pi_B)_! s^* S, F).$$

Hence F is a sheaf precisely if the latter map is an equivalence for every covering sieve $S \hookrightarrow h(U)$ in context A and every map $s\colon B \to A$ in $\mathcal{B}$. Together with Remark 6.3.4.5, this yields the claim. $\square$

Lemma 6.3.4.14. *Let L be a locale and let $S \hookrightarrow h(U)$ be a covering sieve on an object $U\colon A \to \mathsf{L}$. Then for every map $V \to U$ in $\mathsf{L}(A)$ the map $h(V) \times_{h(U)} S \hookrightarrow h(V)$ is a covering sieve.*

Proof. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume without loss of generality that $A \simeq 1$. Now if $d\colon \mathsf{G} \to \mathsf{L}$ is a covering of U giving rise to the covering sieve S, then universality of colimits in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$ and the fact that h preserves limits implies that $S \times_{h(U)} h(V)$ is the colimit of the diagram

$$\mathsf{G} \xrightarrow{d} \mathsf{L} \xrightarrow{-\times V} \mathsf{L} \xhookrightarrow{h} \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}}).$$

Since universality of colimits in L implies that the diagram $d(-) \times V \colon \mathsf{G} \to \mathsf{L}$ is a covering of V, the claim follows. □

Lemma 6.3.4.15. *Let L be a $\mathcal{B}$-locale and let $S_0 \hookrightarrow h(U)$ and $S_1 \hookrightarrow h(U)$ be covering sieves on an object $U \colon A \to \mathsf{L}$. Then $S_0 \times_{h(U)} S_1 \hookrightarrow h(U)$ is a covering sieve as well.*

Proof. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume without loss of generality that $A \simeq 1$. Let $d_0 : \mathsf{G}_0 \to \mathsf{L}$ be a covering giving rise to the covering sieve S_0, and let $d_1 : \mathsf{G}_1 \to \mathsf{L}$ be a covering giving rise to S_1. Define $\mathsf{G} = \mathsf{G}_0 \times \mathsf{G}_1$ and let $d \colon \mathsf{G} \to \mathsf{L}$ be the diagram given by the composition

$$\mathsf{G}_0 \times \mathsf{G}_1 \xrightarrow{d_0 \times d_1} \mathsf{L} \times \mathsf{L} \xrightarrow{-\times-} \mathsf{L}.$$

Then we have $\operatorname{colim} d \simeq U$ since colimits are universal in L and since $U \times U \simeq U$ in L. Therefore, it is enough to show that the induced map $\operatorname{colim} h_{\mathsf{L}} d \hookrightarrow h(U)$ in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$ can be identified with $S_0 \times_{h(U)} S_1 \hookrightarrow h(U)$. This follows from the fact that $h_{\mathsf{L}} d$ is given by the composition

$$\mathsf{G}_0 \times \mathsf{G}_1 \xrightarrow{h_{\mathsf{L}} d_0 \times h_{\mathsf{L}} d_1} \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}}) \times \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}}) \xrightarrow{-\times-} \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$$

and the very same argument as above, using that colimits are universal in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$ as well. □

Proof of Proposition 6.3.4.12. Since the localisation is already accessible by Proposition 5.4.1.7, the second claim follows from the first by Theorem 6.2.3.1. To prove the first, given $A \in \mathcal{B}$, let $T'(A)$ be the class of monomorphisms $f \colon G \hookrightarrow H$ in the ∞-topos $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(A)$ satisfying the condition that for every map $s \colon B \to A$ in $\mathcal{B}$, every $U \colon B \to \mathsf{L}$ and every map $h(U) \to s^* H$ the pullback $s^* G \times_{s^* H} h(U) \hookrightarrow h(U)$ is a covering sieve in context B. Then $T'(A)$ has the following properties:

1. the maps in $T'(A)$ are closed under pullbacks in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(A)$;

2. the maps in $T'(A)$ are closed under finite limits in $\mathrm{Fun}(\Delta^1, \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(A))$;

3. every map in $T'(A)$ is inverted by $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(A) \to \underline{\mathrm{Sh}}_{\mathcal{B}}(\mathsf{L})(A)$;

4. every covering sieve in context $A \in \mathcal{B}$ is contained in $T'(A)$.

In fact, the first property is evident, and the second property follows from combining the first one with Lemma 6.3.4.15. Property (3) follows from the observation that by descent in the $\mathcal{B}_{/A}$-topos $\underline{\mathrm{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{L})$, every map in $T'(A)$ is a ($\mathcal{B}_{/A}$-internal) colimit of covering sieves, which implies (using Remark 6.3.4.10) that it is inverted by the localisation functor $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(A) \to \underline{\mathrm{Sh}}_{\mathcal{B}}(\mathsf{L})(A)$. The last property is an immediate consequence of Lemma 6.3.4.14.

Let us set $T' = \bigcup_{A \in \mathcal{B}} (\pi_A)_! T'(A)$ and let T be the smallest local class of morphisms in $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$ that contains T'. Explicitly, a map $f\colon F \to G$ is contained in T precisely if there is a cover $\bigsqcup_i G_i \twoheadrightarrow G$ in $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$ such that for each i the pullback $G_i \times_G F \to G_i$ is in T'. Then T is bounded since it only contains monomorphisms. Moreover, T is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathrm{PSh}_{\mathcal{B}}(\mathsf{L}))$. To see this, the fact that every map in T is locally (in the ∞-topos $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$) contained in T' implies that it suffices to show that for every cospan $f_0 \to f \leftarrow f_1$ with f_0 and f_1 in T', their pullback is in T' as well. Note that if $s\colon B \to A$ is a map in $\mathcal{B}$ and if g is a map in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L})(B)$ such that $s_!(g) \in T'(A)$, we have $g \in T'(B)$. Therefore, we may assume that both f_0 and f_1 are in $T'(A)$ for some $A \in \mathcal{B}$. In this case, the claim immediately follows from properties (1) and (2) of $T'(A)$. The same argument moreover shows that every map in T is inverted by the localisation functor $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L}) \to \mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ as it can be written as a Δ^{op}-indexed colimit of coproducts of maps in T'.

By employing Lemma 6.3.4.13 and property (4) above, we now conclude that there is an equivalence $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L}) \simeq \mathrm{Loc}_T(\mathrm{PSh}_{\mathcal{B}}(\mathsf{L}))$ of Bousfield localisations of $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$. Using Proposition 6.2.10.11, we thus obtain that $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L}) \to \mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ is left exact. In light of Remark 6.3.4.10, this is already sufficient to conclude that the entire functor of $\mathcal{B}$-categories $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathrm{Sh}}_{\mathcal{B}}(\mathsf{L})$ is left exact. $\square$

6.3.5. The localic reflection of $\mathcal{B}$-topoi

In the previous section, we introduced the $\mathcal{B}$-topos of sheaves on a $\mathcal{B}$-locale. In this section, we show that this construction is the universal way to attach a

$\mathcal{B}$-topos to a $\mathcal{B}$-locale. More precisely, we show:

Proposition 6.3.5.1. *Suppose that* L *is a* $\mathcal{B}$*-locale. Then the Yoneda embedding* $h\colon \mathsf{L} \hookrightarrow \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *induces an equivalence* $\mathsf{L} \simeq \mathsf{Sub}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}))$*, and for every* $\mathcal{B}$*-topos* X *precomposition with* h *induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}), \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{L}, \mathsf{Sub}(\mathsf{X})),$$

where the left-hand side denotes the $\mathcal{B}$*-category of algebraic morphisms between the* $\mathcal{B}$*-topoi* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *and* X *and the right-hand side denotes the* $\mathcal{B}$*-category of algebraic morphisms between the* $\mathcal{B}$*-locales* L *and* $\mathsf{Sub}(\mathsf{X})$*.*

Proof. We begin by showing the first claim. To that end, Lemma 6.3.2.5 implies that a sheaf $F\colon \mathsf{L}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is subterminal if and only if it takes values in $\mathsf{Sub}_{\mathcal{B}}$. Together with Remark 2.3.2.1, Remark 6.3.2.2 and Remark 6.3.4.10, this implies that the first claim follows once we verify that every such sheaf $F\colon \mathsf{L}^{\mathrm{op}} \to \mathsf{Sub}_{\mathcal{B}}$ is representable. Note that by Example 6.3.2.4, the associated right fibration $p\colon \mathsf{L}_{/F} \to \mathsf{L}$ is fully faithful. Let $U\colon 1 \to \mathsf{L}$ be the colimit of p. We then obtain a canonical map $F \to h(U)$ in $\mathsf{Sub}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}))$. To show the claim, it is therefore enough to produce a map in the opposite direction, which by Yoneda's lemma is equivalent to show that $F(U) \simeq 1_{\mathsf{Grpd}_{\mathcal{B}}}$. To see this, note that by Proposition 6.3.2.10 the object U is the colimit of the restriction of p to $\mathsf{G} = (\mathsf{L}_{/F})^{\simeq}$. In other words, we have a covering of U given by $p|_{\mathsf{G}}$. Let $S \hookrightarrow h(U)$ be the associated covering sieve. Then, since F is a sheaf, we obtain an equivalence $F(U) \simeq \mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(S, F)$. To complete the proof of the first claim, we therefore need to show that the right-hand side can be identified with $1_{\mathsf{Grpd}_{\mathcal{B}}}$. But as F is subterminal, we may in turn identify $\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(S, F)$ with $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})}(S, F) \simeq \lim F p|_{\mathsf{G}}$. Thus, the claim follows once we show that $Fp|_{\mathsf{G}}\colon \mathsf{G} \to \mathsf{Sub}_{\mathcal{B}}$ is final in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Sub}_{\mathcal{B}})$. Note that the associated object $P \hookrightarrow \mathsf{G}$ in $\mathsf{Sub}(\mathcal{B}_{/\mathsf{G}})$ is explicitly obtained as the fibre of $p\colon \mathsf{L}_{/F} \to \mathsf{Grpd}_{\mathcal{B}}$ over $p|_{\mathsf{G}}$. Thus, the inclusion $\mathsf{G} \hookrightarrow \mathsf{L}_{/F}$ induces a section $\mathsf{G} \to P$, which immediately yields the claim.

We now show the second claim. Let $l\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ be the localisation functor. We now have maps

$$\begin{tikzcd}
\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}),\mathsf{X}) \arrow[dr, hook, "l^*"] & & \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{L},\mathsf{Sub}(\mathsf{X})) \arrow[dl, hook'] \\
& \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}),\mathsf{X}) \xrightarrow{\simeq} \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{L},\mathsf{Sub}(\mathsf{X})) &
\end{tikzcd}$$

in which the fact that l^* is fully faithful follows from l being a localisation functor, and where the equivalence in the middle follows from Corollary 6.2.2.10 and the straightforward observation that by Remark 6.3.2.8 every left exact functor $\pi_A^*\mathsf{L} \to \pi_A^*\mathsf{X}$ necessarily factors through $\pi_A^*\,\mathsf{Sub}(\mathsf{X})$. Thus, by using Remark 6.3.4.10, Remark 6.3.2.2 and Remark 3.3.3.4 together with Corollary 5.4.3.2, the claim follows once we show that a left exact functor $f\colon \mathsf{L} \to \mathsf{Sub}(\mathsf{X})$ is cocontinuous if and only if the left Kan extension $h_!(if)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \mathsf{X}$ (where $i\colon \mathsf{Sub}(\mathsf{X}) \hookrightarrow \mathsf{X}$ is the inclusion) carries Cov into $\mathsf{X}^{\simeq}$. To see this, note that as $h_!(if)$ is an algebraic morphism, it restricts to a map $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \to \mathsf{Sub}(\mathsf{X})$ which is cocontinuous as well. Consequently, for every object $U\colon A \to \mathsf{L}$ and every covering $d\colon \mathsf{G} \to \pi_A^*\mathsf{L}$ of U, the image of the associated covering sieve $S_d \hookrightarrow h(U)$ along $h_!(if)$ can be identified with the canonical morphism $\operatorname{colim} fd \to f(U)$. In other words, $h_!(if)$ carries Cov into $\mathsf{X}^{\simeq}$ precisely if f is $\mathsf{Grpd}_{\mathcal{B}}$-cocontinuous. But by using Proposition 6.3.2.10, this already implies that f is cocontinuous. Hence the claim follows. □

Corollary 6.3.5.2. *The $\mathcal{B}$-category $\mathsf{Loc}^{\mathsf{L}}_{\mathcal{B}}$ is a coreflective subcategory of $\mathsf{Top}^{\mathsf{L}}_{\mathcal{B}}$, where the inclusion is given by carrying a $\mathcal{B}$-locale to its associated sheaf $\mathcal{B}$-topos and the coreflection sends a $\mathcal{B}$-topos to its underlying $\mathcal{B}$-locale of subterminal objects.*

Proof. In light of Remark 2.3.2.1 and Remark 6.3.4.10, this follows immediately from combining Proposition 6.3.3.8 with Proposition 6.3.5.1. □

Definition 6.3.5.3. A $\mathcal{B}$-topos X is *localic* if it is contained in the essential image of the fully faithful functor $\underline{\mathsf{Sh}}_{\mathcal{B}}(-)\colon \mathsf{Loc}^{\mathsf{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Top}^{\mathsf{L}}_{\mathcal{B}}$.

6.3.6. Localic $\mathcal{B}$-topoi as relative locales

Observe that as a consequence of Corollary 6.3.5.2 and Corollary 6.2.2.4, the $\mathcal{B}$-locale $\mathsf{Sub}_{\mathcal{B}}$ is the *initial* $\mathcal{B}$-locale. Since the global sections functor $\Gamma_{\mathcal{B}}$ restricts to a functor $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Loc}^{\mathrm{L}}$ (by Proposition 6.3.3.7), we thus obtain an induced functor $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Loc}^{\mathrm{L}}_{\mathrm{Sub}(\mathcal{B})/}$.

Proposition 6.3.6.1. *If $\mathcal{B}$ is a localic ∞-topos, then the functor*

$$\Gamma\colon\ \mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Loc}^{\mathrm{L}}_{\mathrm{Sub}(\mathcal{B})/}$$

is an equivalence of ∞-categories.

Proof. Since by Remark 6.3.3.5 the ∞-category $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B})$ can be identified with the 1-category of internal locales in $\mathrm{Disc}(\mathcal{B})$, the statement reduces to the analogous result in 1-topos theory, see [39, Theorem C1.6.3]. □

Corollary 6.3.6.2. *If $\mathcal{B}$ is a localic ∞-topos, then for every $\mathcal{B}$-locale L, the ∞-topos $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ can be canonically identified with $\mathrm{Sh}(\Gamma\mathsf{L})$.*

Proof. As a result of Remark 6.3.2.8, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \arrow[r, "\Gamma"] \arrow[d, "\mathrm{Sub}"] & (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/} \arrow[d, "\mathrm{Sub}"] \\
\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \arrow[r, "\Gamma"] & \mathrm{Loc}^{\mathrm{L}}_{\mathrm{Sub}(\mathcal{B})/}
\end{tikzcd}$$

(where we note that as $\mathrm{Loc}^{\mathrm{L}}$ is a 1-category coherence issues do not arise). Therefore, the claim follows from Proposition 6.3.6.1 and the fact that by Theorem 6.2.5.1 the upper horizontal map is an equivalence as well. □

Remark 6.3.6.3. The inverse to the equivalence from Proposition 6.3.6.1 can be described explicitly: given an algebraic morphism of locales $f^*\colon\ \mathrm{Sub}(\mathcal{B}) \to L$, let $f^*\colon \mathcal{B} \to \mathrm{Sh}(L)$ be the associated algebraic morphism of ∞-topoi. Then $f_* \mathrm{Sub}_{\mathrm{Sh}(L)}$ is a $\mathcal{B}$-locale (as can be easily verified using Proposition 6.3.3.7), and the canonical algebraic morphism $\mathrm{Sub}_{\mathcal{B}} \to f_* \mathrm{Sub}_{\mathrm{Sh}(L)}$ recovers the map $f^*\colon\ \mathrm{Sub}(\mathcal{B}) \to L$ upon passing to global sections. Therefore, we can identify $f_* \mathrm{Sub}_{\mathrm{Sh}(L)}$ as the image of $f^*\colon\ \mathrm{Sub}(\mathcal{B}) \to L$ under the equivalence from Proposition 6.3.6.1. Explicitly, this $\mathcal{B}$-locale can be described as the sheaf $L_{/f^*(-)}$ on $\mathrm{Sub}(\mathcal{B})$, i.e. the $\widehat{\mathrm{Cat}}_{\infty}$-valued functor that is classified by the cartesian fibration $\mathrm{Sub}(\mathcal{B}) \times_L \mathrm{Fun}(\Delta^1, \mathsf{L}) \to \mathrm{Sub}(\mathcal{B})$.

7. Smooth and proper geometric morphisms

Suppose that

$$\begin{tikzcd} Q \arrow[r, "k"] \arrow[d, "q"] & P \arrow[d, "p"] \\ Y \arrow[r, "h"] & X \end{tikzcd}$$

is a pullback square of topological spaces. The *proper base change theorem* asserts that, provided all of the spaces in the above square are locally compact Hausdorff and the map p is *proper*, the induced square

$$\begin{tikzcd} \mathrm{Sh}(Q) \arrow[r, "k_*"] \arrow[d, "q_*"] & \mathrm{Sh}(P) \arrow[d, "p_*"] \\ \mathrm{Sh}(Y) \arrow[r, "h_*"] & \mathrm{Sh}(X) \end{tikzcd}$$

of ∞-topoi is horizontally left adjointable, in the sense that the natural map $h^* p_* \to q_* k^*$ is an equivalence. Dually, the *smooth base change* theorem asserts that provided p is a *topological submersion*, the induced square of ∞-topoi is vertically left adjointable.

In [49, § 7.3], Lurie defined the notion of a *proper morphism* of ∞-topoi by turning the proper base change theorem into a definition:

Definition 7.0.0.1 ([49, Definition 7.3.1.4])**.** Let $p_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. We say that f_* is *proper* if for every commutative diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "g'_*"] \arrow[d, "q'_*"'] & \mathcal{Y} \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{A}' \arrow[r, "f'_*"'] & \mathcal{A} \arrow[r, "f_*"'] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ in which both squares are cartesian, the left square is left adjointable, in the sense that the mate transformation $(f')^* q_* \to q'_*(g')^*$ is an equivalence.

Example 7.0.0.2. It follows from [49, Proposition 7.3.12 and Corollary 7.3.2.13] that any *closed immersion* of ∞-topoi, in the sense of [49, Definition 7.3.2.6], is proper.

Example 7.0.0.3. Let $p\colon Y \to X$ be a proper and separated morphism of topological spaces. In Section 7.3.2 we will prove that then the geometric morphism $p_*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is proper, generalising a result of Lurie [49, Theorem 7.3.1.16].

Since properness in topology ought to capture a relative notion of *compactness*, the natural question arises in what sense ∞-toposic properness can be thought of in the same way. There is a well-established notion of *compactness* in higher topos theory: an ∞-topos $\mathcal{X}$ is said to be *compact* if the global sections functor $\Gamma_{\mathcal{X}}$ commutes with filtered colimits. Since the ∞-topos Ani of ∞-groupoid is the final object in $\mathrm{Top}_\infty^{\mathrm{R}}$, one would expect that this is the case precisely of $\Gamma_{\mathcal{X}}$ is proper. However, it is not clear at all how this would follow from the definitions. One of the main goals in this chapter is to establish this result. More generally, in light of the correspondence between geometric morphisms $f_*\colon \mathcal{X} \to \mathcal{B}$ and $\mathcal{B}$-topoi that we established in the previous chapter, one should expect that f_* is proper precisely if the associated $\mathcal{B}$-topos is compact in the evident $\mathcal{B}$-categorical sense: that is, if the *internal* global sections functor $\Gamma_{f_*(\mathsf{Grpd}_{\mathcal{X}})}\colon f_*(\mathsf{Grpd}_{\mathcal{X}}) \to \mathsf{Grpd}_{\mathcal{B}}$ preserves filtered colimits, i.e. is Filt-cocontinuous. The 1-toposic analogue of this statement has been shown to be true by Moerdijk and Vermeulen [61]. In Theorem 7.2.5.1, we will establish its ∞-categorical version.

One can define the notion of a *smooth* geometric morphism in the evident dual way:

Definition 7.0.0.4. Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. We say that f_* is *smooth* if for every commutative diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "k'_*"] \arrow[d, "g'_*"] & \mathcal{Y} \arrow[r, "k_*"] \arrow[d, "g_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{A}' \arrow[r, "h'_*"] & \mathcal{A} \arrow[r, "h_*"] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ in which both squares are pullbacks, the mate $g^* h'_* \to k'_*(g')^*$ is an equivalence.

Example 7.0.0.5. Every étale geometric morphism is smooth. In fact, this follows immediately from the explicit description of pullbacks of such geometric morphisms in $\mathrm{Top}_\infty^{\mathrm{R}}$, see [49, Remark 6.3.5.8].

Example 7.0.0.6. Every *shape submersion* $f\colon Y \to X$ of topological spaces induces a smooth geometric morphism $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. We prove this claim in Section 7.3.1 below.

Again, in light of the correspondence between geometric morphisms into $\mathcal{B}$ and $\mathcal{B}$-topoi, the question arises which $\mathcal{B}$-toposic property the smoothness condition corresponds to. In 1-topos theory, it is a classical result (see [39, Corollary C.3.3.16]) that smooth geometric morphisms are precisely the *locally connected* maps, i.e. those geometric morphisms $f_* : \mathcal{X} \to \mathcal{B}$ for which the left adjoint f^* has a further left adjoint $f_!$ that satisfies a projection formula. In other words, a geometric morphism of 1-topoi $f_* : \mathcal{X} \to \mathcal{B}$ is smooth precisely if it exhibits $\mathcal{X}$ as a *locally connected* $\mathcal{B}$*-topos*. In Theorem 7.1.3.1, we establish the ∞-toposic analogue of this statement: we show that a geometric morphism $f_* : \mathcal{X} \to \mathcal{B}$ is smooth precisely if the associated $\mathcal{B}$-topos is *locally contractible*.

We begin this chapter with the discussion of locally contractible $\mathcal{B}$-topoi and their relation to smooth geometric morphisms in Section 7.1. In Section 7.2.1, we study compact $\mathcal{B}$-topoi and their relation to proper geometric morphisms. Lastly, in Section 7.3 we discuss how our characterisations of smooth and proper maps of ∞-topoi can be used to detect when a continuous map of topological spaces induces a smooth or proper map between the associated sheaf ∞-topoi.

7.1. Locally contractible $\mathcal{B}$-topoi

An ∞-topos $\mathcal{X}$ is said to be *locally contractible* if the constant sheaf functor $\mathrm{const}_{\mathcal{X}} : \mathrm{Ani} \to \mathcal{X}$ admits a left adjoint $\pi_{\mathcal{X}} : \mathcal{X} \to \mathrm{Ani}$ which is to be thought of as the functor that carries an object $U \in \mathcal{X}$ to its *homotopy type* (or *shape*) $\pi_{\mathcal{X}}(U)$. In 1-topos theory, the corresponding notion is that of a *locally connected*

1-topos $\mathcal{E}$, in which the additional left adjoint carries an object $U \in \mathcal{E}$ to its set of connected components $\pi_0(E)$.

In this section, we study the analogous concept for $\mathcal{B}$-topoi. We begin in Section 7.1.1 by defining the notion of a locally contractible $\mathcal{B}$-topos and providing a few characterisations of this concept. In Section 7.1.2, we show that every locally contractible $\mathcal{B}$-topos is generated by its *contractible objects* in a quite strong sense. Finally, in Section 7.1.3 we provide a characterisation of locally contractible $\mathcal{B}$-topoi in terms of *smoothness* of the associated geometric morphisms.

7.1.1. Local contractibility in $\mathcal{B}$-topos theory

The goal of this section is to define the condition of a $\mathcal{B}$-topos to be locally contractible and to derive a few explicit characterisations of this concept. We begin with the following definition, which is a straightforward generalisation of the notion of a locally contractible ∞-topos to the world of $\mathcal{B}$-topoi:

Definition 7.1.1.1. A $\mathcal{B}$-topos X is *locally contractible* if the unique algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{X}$ admits a left adjoint $\pi_{\mathsf{X}} \colon \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}$. We call a geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$ *locally contractible* if $f_*(\mathsf{Grpd}_{\mathcal{X}})$ is a locally contractible $\mathcal{B}$-topos, in which case we denote by $f_!$ the additional left adjoint of f^* (i.e. the functor $\Gamma(\pi_{f_*(\mathsf{Grpd}_{\mathcal{X}})})$).

Remark 7.1.1.2. As the property of a functor being a right adjoint is local in $\mathcal{B}$ (see Remark 3.1.3.6) and by making use of Remark 6.2.1.3, we find that for any cover $\bigsqcup_i A_i \to 1$ in $\mathcal{B}$, the $\mathcal{B}$-topos X is locally contractible if and only if for every i the $\mathcal{B}_{/A_i}$-topos $\pi_{A_i}^* \mathsf{X}$ is locally contractible.

Remark 7.1.1.3. Explicitly, a geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$ is locally contractible precisely if $f^* \colon \mathcal{B} \to \mathcal{X}$ admits a left adjoint $f_! \colon \mathcal{X} \to \mathcal{B}$ such that for every map $s \colon B \to A$ in $\mathcal{B}$ the induced map

$$f_!(f^*B \times_{f^*A} -) \to B \times_A f_!(-)$$

is an equivalence. In fact, this follows from the section-wise characterisation of internal adjunctions (Proposition 3.1.2.9) together with the fact that if f^* admits a left adjoint $f_!$, one obtains an induced left adjoint $(f_!)_A$ of $f^*_{/A} \colon \mathcal{B}_{/A} \to \mathcal{X}_{/f^*A}$ for every $A \in \mathcal{B}$ which is simply given by the composition

$$\mathcal{X}_{/f^*A} \xrightarrow{(f_!)_{/A}} \mathcal{B}_{/f_!f^*A} \xrightarrow{\epsilon_!} \mathcal{B}_{/A}$$

(in which $\epsilon \colon f_!f^* \to \mathrm{id}_{\mathcal{B}}$ is the adjunction counit).

Example 7.1.1.4. Every étale $\mathcal{B}$-topos is locally contractible. More precisely, one can characterise the class of étale $\mathcal{B}$-topoi as those locally contractible $\mathcal{B}$-topoi X for which the additional left adjoint π_{X} is a conservative functor. This is an immediate consequence of [49, Proposition 6.3.5.11].

Recall from Theorem 6.2.5.1 and Remark 6.2.5.3 that every $\mathcal{B}$-topos X corresponds uniquely to a geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$ such that X can be recovered by $f_*(\mathrm{Grpd}_{\mathcal{X}})$. The goal of this section is to characterise the property that X is locally contractible in terms of the geometric morphism f_*. To that end, recall that a product-preserving functor $g \colon \mathcal{C} \to \mathcal{D}$ between cartesian closed ∞-categories is said to be cartesian closed if the natural map $g(\underline{\mathrm{Hom}}(-,-)) \to \underline{\mathrm{Hom}}(g(-), g(-))$ (in which $\underline{\mathrm{Hom}}(-,-)$ denotes the internal hom in $\mathcal{C}$ and $\mathcal{D}$, respectively) is an equivalence. If $\mathcal{C}$ and $\mathcal{D}$ are even *locally* cartesian closed and g preserves finite limits, one says that g is locally cartesian closed if the induced functor $g_{/c} \colon \mathcal{C}_{/c} \to \mathcal{D}_{/g(d)}$ is cartesian closed for every $c \in \mathcal{C}$. We now obtain:

Proposition 7.1.1.5. *Let X be a $\mathcal{B}$-topos and let $f_* \colon \mathcal{X} \to \mathcal{B}$ be the associated ∞-topos over $\mathcal{B}$. Then the following are equivalent:*

1. *X is locally contractible;*
2. *the unique algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \mathrm{Grpd}_{\mathcal{B}} \to \mathsf{X}$ is $\mathrm{Grpd}_{\mathcal{B}}$-continuous;*
3. *the functor $\mathrm{const}_{\mathsf{X}} \colon f^*(\mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Grpd}_{\mathcal{X}}$ (which is obtained by transposing the algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \mathrm{Grpd}_{\mathcal{B}} \to \mathsf{X}$ across the adjunction $f^* \dashv f_*$) is fully faithful.*
4. *the functor $f^* \colon \mathcal{B} \to \mathcal{X}$ is locally cartesian closed.*

Before we can prove Proposition 7.1.1.5, we need the following lemma:

Lemma 7.1.1.6. *Let* X *be a* $\mathcal{B}$*-topos and let* $f_* \colon \mathcal{X} \to \mathcal{B}$ *be the associated* ∞*-topos over* $\mathcal{B}$*. Let* $A \in \mathcal{B}$ *be an arbitrary object and let* $P \to A$ *and* $Q \to A$ *be two* $\mathcal{B}_{/A}$*-groupoids. Then the morphism of* $\mathcal{X}_{/f^*A}$*-groupoids*

$$\mathrm{map}_{f^*(\mathsf{Grpd}_{\mathcal{B}})}(P, Q) \to \mathrm{map}_{\mathsf{Grpd}_{\mathcal{X}}}(\mathrm{const}_{\mathsf{X}}(P), \mathrm{const}_{\mathsf{X}}(Q))$$

that is induced by the action of $\mathrm{const}_{\mathsf{X}}$ *recovers the map*

$$f^*(\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, Q)) \to \underline{\mathrm{Hom}}_{\mathcal{X}_{/f^*A}}(f^*P, f^*Q).$$

Proof. Using Remark 6.2.1.3, we may assume without loss of generality that $A \simeq 1$. Furthermore, by transposing across the adjunction $f^* \dashv f_*$, it suffices to show that the map

$$\underline{\mathrm{Hom}}_{\mathcal{B}}(P, Q) \to f_*\underline{\mathrm{Hom}}_{\mathcal{X}}(f^*P, f^*Q)$$

can be identified with the morphism of $\mathcal{B}$-groupoids

$$\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{G}, \mathsf{H}) \to \mathrm{map}_{\mathsf{X}}(\mathrm{const}_{\mathsf{X}}(\mathsf{G}), \mathrm{const}_{\mathsf{X}}(\mathsf{H})).$$

Now the latter can be identified with

$$\eta_* \colon \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{G}, \mathsf{H}) \to \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{G}, \Gamma_{\mathsf{X}}\, \mathrm{const}_{\mathsf{X}}\, \mathsf{H})$$

(where $\Gamma_{\mathsf{X}} \colon \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}$ denotes the unique geometric morphism and where η is the adjunction unit). We can also identify the former map with

$$\eta_* \colon \underline{\mathrm{Hom}}_{\mathcal{B}}(P, Q) \to \underline{\mathrm{Hom}}_{\mathcal{X}}(P, f_*f^*Q).$$

by using the equivalence $f_*\underline{\mathrm{Hom}}(f^*P, f^*Q) \simeq \underline{\mathrm{Hom}}(P, f_*f^*P)$. Therefore, the claim follows from Proposition 3.2.5.11. $\square$

Proof of Proposition 7.1.1.5. Since $\mathrm{const}_{\mathsf{X}}$ is cocontinuous and preserves finite limits, one deduces from Proposition 3.5.4.1 and the adjoint functor theorem (Proposition 5.4.3.1) that (1) and (2) are equivalent. In light of Lemma 7.1.1.6, it is moreover clear that (3) and (4) are equivalent. To complete the proof, we will show that (2) and (4) are equivalent. To that end, given any map $p \colon P \to A$ in $\mathcal{B}$, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{/f^*A} \arrow[r, "p^*"] & \mathcal{X}_{/f^*P} \arrow[r, "p_!"] & \mathcal{X}_{/f^*A} \\
\mathcal{B}_{/A} \arrow[u, "f^*_{/A}"] \arrow[r, "p^*"] & \mathcal{B}_{/P} \arrow[u, "f^*_{/P}"] \arrow[r, "p_!"] & \mathcal{B}_{/A}. \arrow[u, "f^*_{/A}"]
\end{tikzcd}$$

Given $q\colon Q \to A$, the natural map $f^*\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, Q) \to \underline{\mathrm{Hom}}_{\mathcal{X}_{/f^*A}}(f^*P, f^*Q)$ is precisely obtained by evaluating the (horizontal) mate of the composite square at q. Since the horizontal mate of the left square being an equivalence (for every such map p) precisely means that $\mathrm{const}_{\mathsf{X}}$ is $\mathrm{Grpd}_{\mathcal{B}}$-continuous, we only need to show that the mate of the left square is an equivalence if and only if the mate of the composite square is one. One direction is trivial. As for the other direction, if we know that the map $\phi\colon f^*_{/A}p_* \to p_*f^*_{/P}$ is an equivalence for every object in the image of $p^*\colon \mathcal{B}_{/A} \to \mathcal{B}_{/P}$, then the entire map has to be an equivalence since every object in $\mathcal{B}_{/P}$ can be written as a pullback of such objects and since both domain and codomain of ϕ preserves finite limits. □

7.1.2. Contractible objects

A topological space X is by definition locally contractible if it admits a basis of contractible open subsets. A priori, the definition of a locally contractible $\mathcal{B}$-topos does not appear to be related to this condition at all. In this section, we reconcile the two notions by showing that local contractibility of a $\mathcal{B}$-topos can be characterised by the property of it being generated under colimits by its *contractible objects*. We begin with the following definition:

Definition 7.1.2.1. Let X be a $\mathcal{B}$-topos. An object $U\colon A \to \mathsf{X}$ is said to be *contractible* if the functor $\mathrm{map}_{\mathsf{X}}(U, \mathrm{const}_{\mathsf{X}}(-))\colon \mathrm{Grpd}_{\mathcal{B}_{/A}} \to \mathrm{Grpd}_{\mathcal{B}_{/A}}$ is an equivalence. We define the full subcategory $\mathrm{Contr}(\mathsf{X}) \hookrightarrow \mathsf{X}$ as the fibre of the functor

$$\mathrm{const}_{\mathsf{X}}^* h_{\mathsf{X}^{\mathrm{op}}}^{\mathrm{op}}\colon \mathsf{X} \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathrm{Grpd}_{\mathcal{B}})^{\mathrm{op}} \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})^{\mathrm{op}}$$

over the identity $\mathrm{id}\colon \mathrm{Grpd}_{\mathcal{B}} \to \mathrm{Grpd}_{\mathcal{B}}$.

Remark 7.1.2.2. Note that as $\mathrm{Grpd}_{\mathcal{B}}$ is the initial $\mathcal{B}$-topos, we find that the inclusion of the identity $\mathrm{id}_{\mathrm{Grpd}_{\mathcal{B}}}\colon 1 \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})$ determines a fully faithful functor that identifies the domain with $\underline{\mathrm{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{B}}, \mathrm{Grpd}_{\mathcal{B}})$. Therefore, the functor $\mathrm{Contr}(\mathsf{X}) \hookrightarrow \mathsf{X}$ is indeed fully faithful. Moreover, as the universal property of $\mathrm{Grpd}_{\mathcal{B}_{/A}}$ implies that a functor $\mathrm{Grpd}_{\mathcal{B}_{/A}} \to \mathrm{Grpd}_{\mathcal{B}_{/A}}$ is an equivalence if and only if it is equivalent to the identity, we find that an object $U\colon A \to \mathsf{X}$ is contained in $\mathrm{Contr}(\mathsf{X})$ if and only if it is contractible.

Remark 7.1.2.3. If $A \in \mathcal{B}$ is an arbitrary object, we may combine Remark 6.2.1.3 with Remark 2.3.2.1 and Proposition 1.2.5.4 to deduce that there is a canonical equivalence

$$\pi_A^* \operatorname{Contr}(\mathsf{X}) \simeq \operatorname{Contr}(\pi_A^* \mathsf{X})$$

of full subcategories in $\pi_A^* \mathsf{X}$.

Remark 7.1.2.4. In the situation of Definition 7.1.2.1, suppose that X is locally contractible. Then we obtain an equivalence $\operatorname{const}_{\mathsf{X}}^* h_{\mathsf{X}^{\mathrm{op}}}^{\mathrm{op}} \simeq h_{\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}}^{\mathrm{op}} \pi_{\mathsf{X}}$. Since $h_{\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}}^{\mathrm{op}}$ is fully faithful and since the identity on $\mathsf{Grpd}_{\mathcal{B}}$ is corepresented by $1_{\mathsf{Grpd}_{\mathcal{B}}}$ (see Proposition 2.2.2.4), we thus find that $\operatorname{Contr}(\mathsf{X})$ arises as the fibre of $\pi_{\mathsf{X}} \colon \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}$ over $1_{\mathsf{Grpd}_{\mathcal{B}}} \colon 1 \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$. In particular, this means that an object $U \colon A \to \mathsf{X}$ is contractible precisely if $\pi_{\mathsf{X}}(U) \colon A \to \mathsf{Grpd}_{\mathcal{B}}$ transposes to the final object in $\mathsf{Grpd}_{\mathcal{B}_{/A}}$.

For the remainder of this section, fix a $\mathcal{B}$-topos X. Recall from Lemma 6.2.3.3 that we may always find a sound doctrine U such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. We will denote by $\operatorname{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ the intersection of $\operatorname{Contr}(\mathsf{X})$ with $\mathsf{X}^{\mathsf{U}\text{-cpt}}$. The main goal of this section is to prove the following proposition:

Proposition 7.1.2.5. *Let X be a $\mathcal{B}$-topos and let U be a sound doctrine such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. Then the following are equivalent:*

1. *X is locally contractible;*
2. *the left Kan extension*

   $$h_!(j) \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\operatorname{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})) \to \mathsf{X}$$

   of $j \colon \operatorname{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ along the Yoneda embedding defines a left exact and accessible Bousfield localisation;
3. *X is generated by $\operatorname{Contr}(\mathsf{X})$ under colimits.*

The proof of Proposition 7.1.2.5 is based on the following two lemmas:

Lemma 7.1.2.6. *Let $j\colon \mathsf{C} \hookrightarrow \mathsf{X}$ be a (small) full subcategory such that the identity on X is the left Kan extension of j along itself. Then j is flat.*

Proof. We need to show that $h_!(j)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ preserves finite limits. By virtue of Proposition 3.4.1.1, the final object $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}\colon 1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is given by the colimit of the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, hence $h_!(j)(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})})$ is the colimit of j. But as the left Kan extension of j along itself is by assumption the identity on X, the formula from Remark 3.4.3.6 implies that every object $U\colon 1 \to \mathsf{X}$ is the colimit of the composition $\mathsf{C}_{/U} \hookrightarrow \mathsf{X}_{/U} \to \mathsf{X}$. In particular, the final object in X must be the colimit of j itself. Hence $h_!(j)$ preserves the final object. To complete the proof, it therefore suffices to show that $h_!(j)$ also preserves pullbacks. By Lemma 6.2.2.8 and in light of Remark 3.4.3.2 and Remark 2.3.2.1, it will be enough to show that if σ is an arbitrary cospan in C in context $1 \in \mathcal{B}$, the functor $h_!(j)$ preserves the pullback P of $h(\sigma)$. In other words, we need to prove that the induced functor $h_!(j)_*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(\sigma)} \to \mathsf{X}_{/j(\sigma)}$ preserves the final object. Let $Q\colon 1 \to \mathsf{X}$ be the pullback of $j(\sigma)$. We then have a commutative diagram

$$\begin{tikzcd}
\mathsf{C}_{/P} \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/P} \arrow[r] \arrow[d, "\simeq"] & \mathsf{X}_{/Q} \arrow[d, "\simeq"] \\
\mathsf{C}_{/\sigma} \arrow[r, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(\sigma)} \arrow[r, "h_!(j)_*"] & \mathsf{X}_{/j(\sigma)}
\end{tikzcd}$$

in which the upper right horizontal functor can be identified with the composition of $h_!(j)_{/P}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/P} \to \mathsf{X}_{/h_!(j)(P)}$ with the forgetful functor $\mathsf{X}_{/h_!(j)(P)} \to \mathsf{X}_{/Q}$ along the natural map $h_!(j)(P) \to Q$. As both of these functors are cocontinuous (see Corollary 6.1.1.5 and Proposition 6.1.1.2), the upper right horizontal functor must be cocontinuous as well. By combining this observation with the identification $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/P} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/P})$ from Lemma 3.4.1.4 and the universal property of presheaf $\mathcal{B}$-categories, we conclude that this functor is equivalent to the Yoneda extension of the inclusion $\mathsf{C}_{/P} \hookrightarrow \mathsf{X}_{/Q}$. Therefore, by using the first part of the proof, it will be enough to show that the identity on $\mathsf{X}_{/j(\sigma)}$ is the left Kan extension of the inclusion $\mathsf{C}_{/\sigma} \hookrightarrow \mathsf{X}_{/j(\sigma)}$ along itself. By using the criterion from Remark 3.4.3.6 together with the fact that any slice $\mathcal{B}$-category over $\mathsf{X}_{/j(\sigma)}$ can be identified with a slice $\mathcal{B}$-category over X, this in turn follows from the assumption that the identity on X is the left Kan extension of j along itself. $\square$

Lemma 7.1.2.7. *If* X *is locally contractible, the identity on* X *is the left Kan extension of the inclusion* $\mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ *along itself.*

Proof. Note that we have inclusions $\mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{X}$ in which the left Kan extension of the second inclusion along itself is the identity on X. Consequently, it will be enough to show that the left Kan extension of the inclusion $\mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ along the inclusion $\mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}^{\mathsf{U}\text{-cpt}}$ recovers the inclusion $\mathsf{X}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{X}$. By using Remark 3.4.3.6 and Remark 7.1.2.3, this follows once we verify that for any U-compact object $U\colon 1 \to \mathsf{X}$, the colimit of the induced inclusion $\mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U} \hookrightarrow \mathsf{X}_{/U}$ is the final object. Now observe that the functor $(\pi_{\mathsf{X}})_{/U}\colon \mathsf{X}_{/U} \to (\mathrm{Grpd}_{\mathcal{B}})_{/\pi_{\mathsf{X}}(U)}$ restricts to a functor $(\pi_{\mathsf{X}})_{/U}\colon \mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U} \to \pi_{\mathsf{X}}(U)$. We claim that the right adjoint $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)} : (\mathrm{Grpd}_{\mathcal{B}})_{/\pi_{\mathsf{X}}(U)} \to \mathsf{X}_{/U}$ (which is constructed by composing the functor $(\mathrm{const}_{\mathsf{X}})_{/\pi_{\mathsf{X}}(U)} : (\mathrm{Grpd}_{\mathcal{B}})_{/\pi_{\mathsf{X}}(U)} \to \mathsf{X}_{/\,\mathrm{const}_{\mathsf{X}}\,\pi_{\mathsf{X}}(U)}$ with the pullback morphism $\eta^* : \mathsf{X}_{/\,\mathrm{const}_{\mathsf{X}}\,\pi_{\mathsf{X}}(U)} \to \mathsf{X}_{/U}$ along the adjunction unit $\eta : U \to \mathrm{const}_{\mathsf{X}}\,\pi_{\mathsf{X}}(U)$, see Corollary 6.1.1.5) restricts to a map $\pi_{\mathsf{X}}(U) \to \mathrm{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U}$. In fact, by making use of Remark 7.1.2.3, it will be enough to verify that if $x\colon 1 \to \pi_{\mathsf{X}}(U)$ is an arbitrary object in context $1 \in \mathcal{B}$, its image along $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}$ is U-compact and contractible. By construction, this object fits into a pullback square

$$\begin{tikzcd}
(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x) \arrow[r] \arrow[d] & 1_{\mathsf{X}} \arrow[d, "\mathrm{const}_{\mathsf{X}}(x)"] \\
U \arrow[r, "\eta"] & \mathrm{const}_{\mathsf{X}}\,\pi_{\mathsf{X}}(U).
\end{tikzcd}$$

Note that both π_{X} and $\mathrm{const}_{\mathsf{X}}$ are left adjoint to $\mathrm{Filt}_{\mathsf{U}}$-cocontinuous functors and therefore preserve U-compact objects. In combination with our assumption that the full subcategory of U-compact objects in X is closed under finite limits, we thus find that $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x)$ is U-compact too. Furthermore, note that we may regard η as an object in $\mathcal{X}_{/f^*(\pi_{\mathsf{X}}(U))} = \mathsf{X}(\pi_{\mathsf{X}}(U))$, i.e. as an *object* in X in context $\pi_{\mathsf{X}}(U)$. As such, η is contractible: in fact, by Remark 7.1.1.3 the object $\pi_{\mathsf{X}}(\eta) \in \mathrm{Grpd}_{\mathcal{B}}(\pi_{\mathsf{X}}(U))$ is explicitly computed as the composition

$$\pi_{\mathsf{X}}(U) \xrightarrow{\pi_{\mathsf{X}}(\eta)} \pi_{\mathsf{X}}\,\mathrm{const}_{\mathsf{X}}\,\pi_{\mathsf{X}}(U) \xrightarrow{\epsilon} \pi_{\mathsf{X}}(U)$$

(where in the first map η is regarded as a *morphism* in X in context $1 \in \mathcal{B}$ and where ϵ is the counit of the adjunction $\pi_{\mathsf{X}} \dashv \mathrm{const}_{\mathsf{X}}$), hence the claim follows from the triangle identities. Now viewing η as an object in X in context $\pi_{\mathsf{X}}(U)$, the above pullback square exhibits the global object $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x) \in \mathsf{X}(1) = \mathcal{X}$ as the image of $\eta \in \mathsf{X}(\pi_{\mathsf{X}}(U))$ along the transition map $x^* : \mathsf{X}(\pi_{\mathsf{X}}(U)) \to \mathsf{X}(1)$. Therefore, η being a contractible object implies that $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x)$ must be contractible as well. Thus we conclude that $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x)$ is contained in $\mathsf{Contr}^{\text{U-cpt}}(\mathsf{X})_{/U}$, as claimed.

So far, our arguments have shown that we have a commutative square

$$\begin{tikzcd} \mathsf{Contr}^{\text{U-cpt}}(\mathsf{X})_{/U} \arrow[r, hook] & \mathsf{X}_{/U} \\ \pi_{\mathsf{X}}(U) \arrow[u] \arrow[r, hook] & (\mathsf{Grpd}_{\mathcal{B}})_{/\pi_{\mathsf{X}}(U)}. \arrow[u, "(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}"'] \end{tikzcd}$$

Since the vertical maps in this diagram are right adjoints, they are in particular final. Since furthermore $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}$ is cocontinuous, the colimit of the upper horizontal map is the image of the colimit of the lower horizontal map along $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}$. To complete the proof, it is therefore enough to prove that the colimit of the lower horizontal map is the final object in $(\mathsf{Grpd}_{\mathcal{B}})_{/\pi_{\mathsf{X}}(U)}$. But this is simply the statement that $\pi_{\mathsf{X}}(U)$ is the colimit of the constant diagram $\pi_{\mathsf{X}}(U) \to 1 \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ with value $1_{\mathsf{Grpd}_{\mathcal{B}}}$, which is clear. □

Proof of Proposition 7.1.2.5. Let us first assume that X is locally contractible. By combining Lemma 7.1.2.7 and Lemma 7.1.2.6, the map

$$h_!(j) : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\text{U-cpt}}(\mathsf{X})) \to \mathsf{X}$$

is left exact, so it suffices that this functor is a Bousfield localisation. Since it is cocontinuous, it has a right adjoint r (which is automatically accessible). The counit of this adjunction carries an object $U : A \to \mathsf{X}$ to the canonical map from the colimit of $(\pi_A^* \mathsf{C})_{/U} \to \pi_A^* \mathsf{X}$ to U. By again using Lemma 7.1.2.7, this map is an equivalence, hence (2) follows. Trivially, (2) implies (3). Finally, suppose that (3) holds, i.e. X is the smallest full subcategory of itself that contains $\mathsf{Contr}(\mathsf{X})$ and that is closed under $\mathsf{Cat}_{\mathcal{B}}$-colimits. Now consider the commutative diagram

$$\begin{tikzcd} \mathsf{Contr}(\mathsf{X}) \arrow[r, hook] \arrow[d] & \mathsf{P} \arrow[r, hook] \arrow[d] & \mathsf{X} \arrow[d, "\mathrm{const}_{\mathsf{X}}^* h_{\mathsf{X}^{\mathrm{op}}}^{\mathrm{op}}"] \\ 1 \arrow[r, hook, "1_{\mathsf{Grpd}_{\mathcal{B}}}"] & \mathsf{Grpd}_{\mathcal{B}} \arrow[r, hook, "h^{\mathrm{op}}_{\mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}} \end{tikzcd}$$

in which both squares are pullbacks. Since $h^{\mathrm{op}}_{\mathsf{Grpd}^{\mathrm{op}}_{\mathcal{B}}}$ and $h^{\mathrm{op}}_{\mathsf{X}^{\mathrm{op}}}$ are cocontinuous functors (Proposition 3.3.2.15) the inclusion $\mathsf{P} \hookrightarrow \mathsf{X}$ is closed under $\mathsf{Grpd}_{\mathcal{B}}$-colimits (see Lemma 3.5.1.11) and must therefore be an equivalence. Hence $\mathrm{const}^*_{\mathsf{X}}\, h^{\mathrm{op}}_{\mathsf{X}^{\mathrm{op}}}$ factors through the Yoneda embedding $\mathsf{Grpd}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}})^{\mathrm{op}}$, which precisely means that $\mathrm{const}_{\mathsf{X}}$ has a left adjoint π_{X}. Hence X is locally contractible. □

7.1.3. Classification of smooth geometric morphisms

In this section, our goal is to prove that smooth geometric morphisms precisely correspond to the locally contractible ones:

Theorem 7.1.3.1. *Let* X *be a* $\mathcal{B}$*-topos and let* $f_* \colon \mathcal{X} \to \mathcal{B}$ *be the associated geometric morphism. Then* X *is locally contractible if and only if* f_* *is smooth.*

The proof of Theorem 7.1.3.1 relies on a few reduction steps. Our first goal is to establish that the property of a geometric morphism to be locally contractible is stable under taking *powers* by $\mathcal{B}$-categories: Recall from Proposition 6.2.4.3 that the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ admits a powering bifunctor

$$(-)^{(-)} \colon \; \mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}\,.$$

We now find:

Lemma 7.1.3.2. *Let* C *be a* $\mathcal{B}$*-category and let* X *be a locally contractible* $\mathcal{B}$*-topos. Then the geometric morphism* $(\Gamma_{\mathsf{X}})_* \colon \mathsf{X}^{\mathsf{C}} \to \mathsf{Grpd}^{\mathsf{C}}_{\mathcal{B}}$ *exhibits* $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ *as a locally contractible* $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$*-topos.*

Proof. Since the algebraic morphism associated with $(\Gamma_{\mathsf{X}})_*$ is given by $(\mathrm{const}_{\mathsf{X}})_*$, the functor $(\pi_{\mathsf{X}})_*$ defines a further left adjoint of $(\mathrm{const}_{\mathsf{X}})_*$. Therefore, Remark 7.1.1.3 implies that we only need to show that for every map $F \to G$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$ and every map $H \to (\mathrm{const}_{\mathsf{X}})_*(G)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$, the canonical morphism

$$(\pi_{\mathsf{X}})_*((\mathrm{const}_{\mathsf{X}})_* F \times_{(\mathrm{const}_{\mathsf{X}})_* G} H) \to F \times_H (\pi_{\mathsf{X}})_* H$$

is an equivalence. It will be enough to show that this map becomes an equivalence after being evaluated at an arbitrary object $c \colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$. In light of Remark 7.1.1.2 and Proposition 1.2.5.4, we can replace $\mathcal{B}$ with $\mathcal{B}_{/A}$, so that we can reduce to the case $A \simeq 1$. But as pullbacks in functor $\mathcal{B}$-categories are computed object-wise (Proposition 3.2.3.2) and as evaluating the unit and counit of the adjunction $(\pi_{\mathsf{X}})_* \dashv (\mathrm{const}_{\mathsf{X}})_*$ at c recovers the unit and counit of the adjunction $\pi_{\mathsf{X}} \dashv \mathrm{const}_{\mathsf{X}}$, the claim follows from the assumption that X is locally contractible and Remark 7.1.1.3. $\square$

Before we can prove Theorem 7.1.3.1, we also need the following result:

Lemma 7.1.3.3. *Let* X *be a locally contractible* $\mathcal{B}$*-topos and let* U *be a sound doctrine such that* X *is* U*-accessible and* $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ *is closed under finite limits in* X*. Then the diagonal map* $\mathrm{diag}\colon \mathsf{Grpd}_{\mathcal{B}} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}))$ *takes values in* $\mathsf{X} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}))$.

Proof. Let $L \dashv i\colon \mathsf{X} \rightleftarrows \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}))$ be the adjunction induced by the Yoneda extension of the inclusion $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$, and let η be its unit. We need to show that the induced morphism $\eta\,\mathrm{diag}\colon\ \mathrm{diag} \to iL\,\mathrm{diag}$ is an equivalence. As we may check this object-wise and by using Remark 7.1.1.2, Remark 5.1.5.2, Remark 5.3.1.2 and Remark 7.1.2.3 together with Proposition 1.2.5.4, we only need to show that for any object $U\colon 1 \to \mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})$ the map $U^*\eta\,\mathrm{diag}\colon U^*\,\mathrm{diag} \to U^*iL\,\mathrm{diag}$ is an equivalence in $\mathsf{Grpd}_{\mathcal{B}}$. Note that we have a chain of equivalences

$$U^*iL\,\mathrm{diag} \simeq \mathrm{map}_{\mathsf{X}}(U, \mathrm{const}_{\mathsf{X}}) \simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(\pi_{\mathsf{X}}(U), -).$$

As $\pi_{\mathsf{X}}(U) \simeq 1_{\mathsf{Grpd}_{\mathcal{B}}}$, we thus find that $U^*iL\,\mathrm{diag} \simeq \mathrm{id}$. Since also $U^*\,\mathrm{diag}$ is equivalent to the identity and since the universal property of $\mathsf{Grpd}_{\mathcal{B}}$ implies that

$$\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{Grpd}_{\mathcal{B}}, \mathsf{Grpd}_{\mathcal{B}})}(\mathrm{id}, \mathrm{id}) \simeq 1_{\mathsf{Grpd}_{\mathcal{B}}},$$

the claim follows. $\square$

Proof of Theorem 7.1.3.1. Suppose first that f_* is smooth. Then f_* in particular satisfies condition (2) of Proposition 7.1.1.5 and is therefore locally contractible. To prove the converse direction, suppose that we have two pullback squares

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "k'_*"] \arrow[d, "g'_*"] & \mathcal{Y} \arrow[r, "k_*"] \arrow[d, "g_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{A}' \arrow[r, "h'_*"] & \mathcal{A} \arrow[r, "h_*"] & \mathcal{B} \end{tikzcd}$$

of ∞-topoi in which f_* is locally contractible. By viewing $\mathcal{A}$ as a $\mathcal{B}$-topos and using Theorem 6.2.3.1, we may factor h_* into a composition

$$\mathcal{A} \hookrightarrow \mathsf{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathcal{B}.$$

Since the commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}) \arrow[r, "\lim"] \arrow[d, "(\Gamma_{\mathsf{X}})_*"] & \mathsf{X} \arrow[d, "\Gamma_{\mathsf{X}}"] \\
\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "\lim"] & \mathsf{Grpd}_{\mathcal{B}}
\end{tikzcd}$$

is a pullback in $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ (see Example 6.2.7.5) and on account of Lemma 7.1.3.2, this allows us to reduce to the case where h_* is already an embedding. But then k_* must be an embedding as well, so that the mate of the left square is an equivalence if and only if the mates of the right one and the composite one are equivalences. Hence, to complete the proof, it will be enough to show that if we are given any pullback square

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "k_*"] \arrow[d, "g_*"] & \mathcal{X} \arrow[d, "f_*"] \\
\mathcal{A} \arrow[r, "h_*"] & \mathcal{B}
\end{tikzcd}$$

in which f_* is locally contractible, the mate transformation $f^*h_* \to g_*k^*$ is an equivalence. By the same argument as above (and the fact that $\mathrm{const}_{\mathsf{X}}$ is continuous), we can moreover still assume that h_* and k_* are embeddings. To proceed, we make use of Proposition 7.1.2.5 to obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, hook, "k_*"] \arrow[d, hook] & \mathcal{X} \arrow[d, hook] \\
\mathsf{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\text{U-cpt}}(\mathsf{X})^{\mathrm{op}}, h_*(\mathsf{Grpd}_{\mathcal{A}})) \arrow[r, hook] \arrow[d, "\lim"] & \mathsf{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\text{U-cpt}}(\mathsf{X})^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[d, "\lim"] \\
\mathcal{A} \arrow[r, hook, "h_*"] & \mathcal{B}
\end{tikzcd}$$

in which both squares are pullbacks. Since the mate of the lower square is evidently an equivalence and since Lemma 7.1.3.3 implies that the diagonal map

$$\mathrm{diag}\colon \mathcal{B} \to \mathsf{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\text{U-cpt}}(\mathsf{X})^{\mathrm{op}}, \mathsf{Grpd}_{\mathcal{B}})$$

takes values in $\mathcal{X}$, it will be enough to show that the diagonal map

$$\mathrm{diag}\colon \mathcal{A} \to \mathrm{Fun}_{\mathcal{B}}(\mathrm{Contr}^{\text{U-cpt}}(\mathsf{X})^{\mathrm{op}}, h_*(\mathrm{Grpd}_{\mathcal{A}}))$$

likewise takes values in $\mathcal{Y}$. Let us therefore pick an arbitrary object $A \in \mathcal{A}$. By [49, Lemmas 6.3.3.4], the upper square in the above diagram is a pullback square of ∞-categories, hence it suffices to show that the image of $\mathrm{diag}(A)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathrm{Contr}^{\text{U-cpt}}(\mathsf{X})^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}})$ is contained in $\mathcal{X}$. But as the mate of the lower square is an equivalence, this latter object is equivalent to $\mathrm{diag}\, h_*(A)$, hence another application of Lemma 7.1.3.3 yields the claim. □

7.2. Compact $\mathcal{B}$-topoi

The goal of this section is to study the concept of a *compact* $\mathcal{B}$-topos and to show that this notion is equivalent to the condition that the associated geometric morphism is proper. In Section 7.2.1, we give the definition of a compact $\mathcal{B}$-topos and discuss a few examples. In Section 7.2.2, we study how certain compactness conditions on $\mathcal{B}$-*locales* lead to their associated localic $\mathcal{B}$-topoi being compact. We will spend Section 7.2.3 to Section 7.2.5 to prove the aforementioned result that compact $\mathcal{B}$-topoi precisely correspond to proper geometric morphisms. More precisely, in Section 7.2.3 and Section 7.2.4, we discuss two auxiliary steps that are required for the proof: the ∞-toposic cone construction and the fact that compact geometric morphisms commute with left exact localisations of ∞-topoi. Lastly, we put everything together in Section 7.2.5 to finish the proof. Finally, in Section 7.2.6 we discuss a variant of this result in which we allow coefficients in an arbitrary compactly generated ∞-category.

7.2.1. Compactness in $\mathcal{B}$-topos theory

Recall that an ∞-topos $\mathcal{X}$ is said to be *compact* if the global sections functor $\Gamma_{\mathcal{X}}\colon \mathcal{X} \to \mathrm{Ani}$ preserves filtered colimits. In this section, we study the $\mathcal{B}$-toposic analogue of this notion.

Definition 7.2.1.1. A $\mathcal{B}$-topos X is said to be *compact* if the global sections functor $\Gamma_{\mathsf{X}}\colon \mathsf{X} \to \mathrm{Grpd}_{\mathcal{B}}$ preserves filtered colimits, i.e. is Filt-cocontinuous. We say that a geometric morphism $p_* : \mathcal{X} \to \mathcal{B}$ is compact if the associated $\mathcal{B}$-topos $p_*(\mathsf{Grpd}_{\mathcal{X}})$ is compact.

Example 7.2.1.2. If $A \in \mathcal{B}$ is an arbitrary object, then the associated étale geometric morphism $(\pi_A)_* : \mathcal{B}_{/A} \to \mathcal{B}$ is compact if and only if A is *internally compact* in $\mathcal{B}$, i.e. if the functor $\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(A, -) \colon \mathsf{Grpd}_{\mathcal{B}} \to \mathsf{Grpd}_{\mathcal{B}}$ preserves filtered colimits. To see this, first note that $(\pi_A)_*$ corresponds to the étale $\mathcal{B}$-topos $\underline{\mathsf{Fun}}_{\mathcal{B}}(A, \mathsf{Grpd}_{\mathcal{B}})$ (see Section 6.2.9) and the unique geometric morphism into $\mathsf{Grpd}_{\mathcal{B}}$ is given by the limit functor $\lim_A \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(A, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ (as its left adjoint diag_A is again a right adjoint and therefore preserves all limits. Moreover, since $A \simeq \mathrm{colim}_A \mathrm{diag}_A 1_{\mathsf{Grpd}_{\mathcal{B}}}$, the adjunctions $\mathrm{colim}_A \dashv \mathrm{diag}_A \dashv \lim_A$ imply that we obtain an identification

$$\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(A, -) \simeq \mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}, \lim_A \mathrm{diag}_A(-)) \simeq \lim_A \mathrm{diag}_A(-)$$

(since $\mathrm{map}_{\mathsf{Grpd}_{\mathcal{B}}}(1_{\mathsf{Grpd}_{\mathcal{B}}}, -)$ is equivalent to the identity, see Proposition 2.2.2.4). Hence $\lim_A$ preserving filtered colimits implies that A is internally compact. To see the converse, note that by Corollary 5.2.3.8, the object A being internally compact is equivalent to A being locally constant with compact values. If this is the case, then the fact that Filt-cocontinuity can be checked locally in $\mathcal{B}$ (see Remark 3.3.2.3) allows us to reduce to the case where A is constant with compact value. In other words, A is a retract of a finite $\mathcal{B}$-groupoid, so that we may further reduce to the case where A is already finite. In this case, $\lim_A$ preserves filtered colimits by the very definition of filteredness.

Example 7.2.1.3. Let X be a subterminal $\mathcal{B}$ topos such that the associated geometric morphism $j_* : \mathcal{X} \hookrightarrow \mathcal{B}$ is a *closed immersion* in the sense of [49, Definition 7.3.2.6]. Then X is compact. In fact, since by [49, Corollary 7.3.2.13] the geometric morphism $\mathsf{X}(A) \hookrightarrow \mathcal{B}_{/A}$ is a closed immersion for every $A \in \mathcal{B}$, this follows once we verify that for every filtered $\mathcal{B}$-category I, the composition

$$\mathsf{Fun}_{\mathcal{B}}(\mathsf{I}, \mathsf{X}) \xhookrightarrow{(\Gamma_{\mathsf{X}})_*} \mathsf{Fun}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{\mathrm{colim}} \mathcal{B}$$

takes values in $\mathcal{X}$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathrm{diag}"] \arrow[d, hook, "j_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \mathsf{X}) \arrow[r, "\lim"] \arrow[d, hook, "(\Gamma_{\mathsf{X}})_*"] & \mathcal{X} \arrow[d, hook, "j_*"] \\ \mathcal{B} \arrow[r, "\mathrm{diag}"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "\lim"] & \mathcal{B} \end{tikzcd}$$

in which both squares are pullbacks in $\mathrm{Top}^{\mathrm{R}}_{\infty}$ by Example 6.2.7.5 and the fact that diag is fully faithful as I is filtered and therefore in particular weakly contractible (see Remark 5.2.3.3). Now as j_* is a closed immersion, there is an object $U \in \mathrm{Sub}_{\mathcal{B}}(1)$ such that $\mathcal{X} \simeq \mathcal{B}_{\setminus U}$, where the right-hand side denotes the full subcategory of $\mathcal{B}$ spanned by the objects $A \in \mathcal{B}$ for which there is a map $U \to A$ for which $(\mathrm{id}, g)\colon U \to U \times A$ is an equivalence. Then we deduce from [49, Proposition 7.3.2.12] that $(\Gamma_{\mathsf{X}})_*$ identifies $\mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \mathsf{X})$ with $\mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}})_{\setminus \mathrm{diag}(U)}$. Since $\mathrm{colim}\colon \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathcal{B}$ preserves finite limits, we now deduce that whenever $d\colon \mathsf{I} \to \mathsf{Grpd}_{\mathcal{B}}$ takes values in X, the $\mathrm{colim}\, d$ must be contained in $\mathcal{X}$. Hence X is compact.

Warning 7.2.1.4. In the context of Definition 7.2.1.1, it is essential that we require for the entire global sections functor $\Gamma_{\mathsf{X}}\colon \mathsf{X} \to \mathsf{Grpd}_{\mathcal{B}}$ to be Filt-cocontinuous instead of just asking for its underlying geometric morphism p_* to preserve filtered colimits. In fact, if $A \in \mathcal{B}$ is an arbitrary object, we saw in Example 7.2.1.2 that $(\pi_A)_*\colon \mathcal{B}_{/A} \to \mathcal{B}$ is compact if and only if A is internally compact. On the other hand, $(\pi_A)_*$ preserves filtered colimits if and only if the functor $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, -)\colon \mathcal{B} \to \mathcal{B}$ preserves filtered colimits. By Proposition 3.2.5.11, A being internally compact implies that $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, -)$ preserves filtered colimits, but the converse is not true in general. For example, if X is a coherent space, then any quasi-compact open $U \subset X$ defines an object in the ∞-topos $\mathrm{Sh}(X)$ satisfying the latter condition (since quasi-compact opens in X define compact objects in $\mathrm{Sh}(X)$ and generate this ∞-topos under colimits). On the other hand, U is in general quite far from being locally constant and can therefore not be internally compact.

Remark 7.2.1.5. Let $\mathcal{X}$ be a 1-localic ∞-topos. If $\mathcal{X}$ is compact, the associated 1-topos $\mathrm{Disc}(\mathcal{X})$ of 0-truncated objects in $\mathcal{X}$ is *tidy* in the sense of [61]. However, the converse it not true in general. For example, any coherent 1-topos is tidy, but 1-localic coherent ∞-topoi are not compact in general. An explicit counterexample is $\mathrm{Spec}(\mathbb{R})_{\acute{e}t} \simeq \mathrm{Fun}(B(\mathbb{Z}/2\mathbb{Z}), \mathrm{Ani})$, which cannot be tidy since $B(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{RP}^{\infty}$ is not compact in Ani.

Example 7.2.1.6. Any ∞-topos of the form $\mathrm{Sh}^{\tau}(\mathcal{C})$ where $\mathcal{C}$ is an ∞-category with an initial and a terminal object and τ a topology generated by a cd-structure is compact. This follows since under these assumption $\mathrm{Sh}^{\tau}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ commutes with filtered colimits and $\mathrm{PSh}(\mathcal{C})$ is always compact if $\mathcal{C}$ has a terminal object. Examples of such topologies from algebraic geometry include the Zariski-, Nisnevich- and cdh-topology.

7.2.2. Compactness of localic $\mathcal{B}$-topoi

In this section, we study how certain compactness properties of $\mathcal{B}$-locales are inherited by their associated localic $\mathcal{B}$-topoi, which will lead us to an important class of compact $\mathcal{B}$-topoi. To that end, if D is a presentable $\mathcal{B}$-category, we shall say that D is *compactly generated* if the inclusion $\mathsf{D}^{\mathrm{cpt}} \hookrightarrow \mathsf{D}$ of the full subcategory of compact objects induces via left Kan extension an equivalence $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{cpt}}) \simeq \mathsf{D}$. We will furthermore say that D is *compactly assembled* if D is a retract (in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$) of a compactly generated $\mathcal{B}$-category. We may now define:

Definition 7.2.2.1. A $\mathcal{B}$-locale L is said to be *locally coherent* if it is compactly generated and if $\mathsf{L}^{\mathrm{cpt}}$ is closed under binary products in L. We say that L is *coherent* if it is locally coherent and 1_{L} is compact.

Furthermore, L is said to be *(locally) stably compact* if it is a retract in $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$ of a (locally) coherent $\mathcal{B}$-locale.

Remark 7.2.2.2. Since the existence and preservation of limits is local in $\mathcal{B}$ (Remark 3.3.2.3) and one has $\pi_A^*\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}}) \simeq \underline{\mathsf{Ind}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{L}^{\mathrm{cpt}})$ by Remark 5.3.1.2 and Remark 5.1.5.2 for every $A \in \mathcal{B}$, we deduce that for any cover $(\pi_{A_i})\colon \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$, a $\mathcal{B}$-locale L is (locally) coherent if and only if $\pi_{A_i}^*\mathsf{L}$ is a (locally) coherent $\mathcal{B}_{/A_i}$-locale for every i.

The main goal of this section is to show:

Proposition 7.2.2.3. *If* L *is a locally coherent* $\mathcal{B}$*-locale, then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *is compactly generated. If* L *is moreover coherent, then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *is a compact* $\mathcal{B}$*-topos.*

Before we prove Proposition 7.2.2.3, let us record the following important consequence of this result:

Corollary 7.2.2.4. *If* L *is locally stably compact, then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *is compactly assembled. If* L *is even stably compact, then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *is a compact* $\mathcal{B}$*-topos.*

Proof. Choose a locally coherent $\mathcal{B}$-locale L' such that L is a retract of L' in $\mathsf{Loc}^{\mathsf{L}}_{\mathcal{B}}$. Applying the functor $\underline{\mathsf{Sh}}_{\mathcal{B}}(-)$, we thus obtain that $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is a retract of $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}')$ in $\mathsf{Top}^{\mathsf{L}}_{\mathcal{B}}$. By Proposition 7.2.2.3, the latter is compactly generated, hence $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is compactly assembled. If L is even stably compact, we may choose L' to be coherent. Then Proposition 7.2.2.3 implies that $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}')$ is compact. Now as $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is a retract of $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}')$ by algebraic morphisms, we deduce that $\Gamma_{\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})}$ is a retract in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}), \mathsf{Grpd}_{\mathcal{B}})$ of a Filt-cocontinuous functor, which by Lemma 5.1.5.7 implies that $\Gamma_{\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})}$ is Filt-cocontinuous as well. Hence $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is compact. □

We now turn to the proof of Proposition 7.2.2.3. We will need the notion of a *finitary sheaf*:

Definition 7.2.2.5. Let P be a $\mathcal{B}$-poset with finite colimits and binary products. A presheaf $F\colon \mathsf{P}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is said to be a *finitary sheaf* if

1. $F(\varnothing_{\mathsf{P}}) \simeq 1_{\mathsf{Grpd}_{\mathcal{B}}}$;
2. for every two objects $U, V\colon A \rightrightarrows \mathsf{P}$ in arbitrary context $A \in \mathcal{B}$, the commutative square
$$\begin{array}{ccc} F(U \vee V) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \wedge V) \end{array}$$
is a pullback.

We let $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ be the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ that is spanned by those presheaves $\pi_A^*\mathsf{P}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ (in arbitrary context $A \in \mathcal{B}$) which are finitary sheaves on $\pi_A^*\mathsf{P}$.

Remark 7.2.2.6. As preservation of (co)limits is a local property (Remark 3.3.2.3), we deduce that for every cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ a presheaf $F\colon \mathsf{P}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ is a finitary sheaf if and only if the presheaf $\pi_{A_i}^*(F)$ is a finitary sheaf on $\pi_A^*\mathsf{P}$. In particular, an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ is contained in $\underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{P})$ if and only if it transposes to a finitary sheaf on $\pi_A^*\mathsf{P}$, and we obtain a canonical equivalence of $\mathcal{B}$-categories $\pi_A^*\underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{P}) \simeq \underline{\mathsf{Sh}}_{\mathcal{B}_{/A}}^{\mathrm{fin}}(\pi_A^*\mathsf{P})$ for every $A \in \mathcal{B}$.

Recall from Section 5.4.5 that if C is a Filt-cocomplete $\mathcal{B}$-category, we denote by $\mathsf{Sh}_{\mathsf{Grpd}_{\mathcal{B}}}^{\mathsf{Filt}}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by the Filt-*sheaves*, i.e. by those functors $\pi_A^*\mathsf{C}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ (in arbitrary context $A \in \mathcal{B}$) whose opposite is Filt-cocontinuous. We now obtain the following characterisation of sheaves on a $\mathcal{B}$-locale:

Proposition 7.2.2.7. *Let* L *be a* $\mathcal{B}$*-locale. Then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{L}) \cap \mathsf{Sh}_{\mathsf{Grpd}_{\mathcal{B}}}^{\mathsf{Filt}}(\mathsf{L})$ *as full subcategories in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$.

Before proving Proposition 7.2.2.7, we need the following small lemma:

Lemma 7.2.2.8. *The inclusion* $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ *preserves filtered colimits*

Proof. Using Remark 6.3.2.2 and Example 6.3.2.4, it suffices to show that for every filtered $\mathcal{B}$-category I, the functor $\mathrm{colim}_{\mathsf{I}} : \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathsf{Grpd}_{\mathcal{B}}$ restricts to subterminal objects. By Remark 6.3.2.8, this is an immediate consequence of $\mathrm{colim}_{\mathsf{I}}$ being left exact. □

Proof. By combining Remark 7.2.2.6, Remark 6.3.4.10 and Remark 5.4.5.2, we need to show that for every $A \in \mathcal{B}$, a presheaf $F\colon \pi_A^*\mathsf{L}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}$ is a sheaf if and only if

1. $F^{\mathrm{op}} : \pi_A^*\mathsf{L} \to \mathsf{Grpd}_{\mathcal{B}_{/A}}^{\mathrm{op}}$ is $\pi_A^*\,\mathsf{Filt}$-cocontinuous;
2. $F(\varnothing_{\pi_A^*\mathsf{L}}) \simeq 1_{\mathsf{Grpd}_{\mathcal{B}_{/A}}}$;
3. for every two objects $U, V\colon B \rightrightarrows \mathsf{L}$ in arbitrary context $B \in \mathcal{B}_{/A}$, the commutative square

$$\begin{tikzcd} F(U \vee V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \wedge V) \end{tikzcd}$$

is a pullback.

By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume that $A \simeq 1$. Now suppose first that F is a sheaf. To show that (1) is satisfied, we need to verify that for every diagram $d\colon \mathsf{I} \to \pi_A^* \mathsf{L}$ where I is a filtered $\mathcal{B}_{/A}$-category and $A \in \mathcal{B}$ is arbitrarily chosen, the natural map

$$(\pi_A^* F)(\operatorname{colim} d) \to \lim(\pi_A^* F)d$$

is an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may again assume without loss of generality that $A \simeq 1$. As I is filtered, we deduce from Lemma 7.2.2.8 that $\operatorname{colim} h_{\mathsf{L}} d$ is subterminal. Now by Proposition 6.3.2.10, we may replace I by $\mathsf{I}^{\simeq}$ and can thus assume that I is a $\mathcal{B}$-groupoid. Therefore, the sheaf condition implies that we obtain an equivalence

$$F(\operatorname{colim} d) \simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(\operatorname{colim} h_{\mathsf{L}} d, F) \simeq \lim Fd.$$

This shows that F is a Filt-sheaf. Condition (2) follows from the observation that as $\varnothing_{\mathsf{L}}$ is the colimit of the unique diagram $\varnothing \to \mathsf{L}$, we obtain an equivalence

$$F(\varnothing_{\mathsf{L}}) \simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(\varnothing_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}, F) \simeq 1_{\mathsf{Grpd}_{\mathcal{B}}}$$

Lastly, to show that condition (3) is met, we may again replace $\mathcal{B}$ with $\mathcal{B}_{/B}$ and can therefore assume that $B \simeq 1$. Now as $U \vee V$ is the coproduct of U and V in L, the claim follows from the fact that the pushout $h_{\mathsf{L}}(U) \sqcup_{h_{\mathsf{L}}(U \wedge V)} h_{\mathsf{L}}(V)$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ computes the coproduct of $h_{\mathsf{L}}(U)$ and $h_{\mathsf{L}}(V)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$.

Conversely, suppose that F satisfies the three conditions. To show that F is a sheaf, we need to verify that for every covering $d\colon \mathsf{G} \to \pi_A^* \mathsf{L}$ of an object $U\colon A \to \mathsf{L}$, the functor $\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(-, F)$ carries the induced covering sieve $S_d \hookrightarrow h(U)$ to an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may again assume that $A \simeq 1$. First, let us show the claim in the case where G is finite, i.e. a locally constant sheaf of finite ∞-groupoids (see Proposition 5.2.3.4). Upon passing to a suitable cover, we can assume that G is (the constant $\mathcal{B}$-category associated with) a finite ∞-groupoid. Since L is a $\mathcal{B}$-poset, we can even assume that G is a finite set. By induction, it suffices to cover the cases $\mathsf{G} = \varnothing$ and $\mathsf{G} = 1 \sqcup 1$. By the above argumentation, these two cases follow immediately from conditions (2) and (3).

For the general case, let $\mathsf{Fin}_{\mathcal{B}}$ be the internal class of finite $\mathcal{B}$-categories. Since $\mathsf{Fin}_{\mathcal{B}}$ being a sound and regular doctrine (Proposition 5.2.3.7) implies that it has the decomposition property (see Section 5.1.4), we may find a filtered $\mathcal{B}$-category I and a diagram $k\colon \mathsf{I} \to \mathsf{Fin}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ such that $\mathsf{G} \simeq \operatorname{colim} k$. Note that since G is a $\mathcal{B}$-groupoid and since the groupoidification of a finite $\mathcal{B}$-category is a finite $\mathcal{B}$-groupoid, postcomposing k with the groupoidification functor yields a diagram $k' : \mathsf{I} \to \mathsf{Grpd}_{\mathcal{B}} \cap \mathsf{Fin}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ that also has colimit G. Therefore, we deduce from Proposition 3.5.4.10 and by making use of the subterminal truncation functor $(-)^{\mathrm{Sub}} : \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ that there is a diagram $d' : \mathsf{I} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ such that (a) we have $\operatorname{colim} d' \simeq \operatorname{colim} h_{\mathsf{L}} d$ and such that (b) for every object $i\colon A \to \mathsf{I}$ in arbitrary context $A \in \mathcal{B}$ there is a finite $\mathcal{B}_{/A}$-groupoid H_i together with a diagram $d_i : \mathsf{H}_i \to \pi_A^* \mathsf{L}$ such that $d'(i) \simeq \operatorname{colim} h_{\mathsf{L}} d_i$. From (a) we deduce that if $l\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \to \mathsf{L}$ is the left adjoint of the Yoneda embedding, the unit of the adjunction $l \dashv h_{\mathsf{L}}$ determines morphisms

$$\operatorname{colim} h_{\mathsf{L}} d \simeq \operatorname{colim} d' \xrightarrow{\alpha} \operatorname{colim} h_{\mathsf{L}} l d' \xrightarrow{\beta} h_{\mathsf{L}}(\operatorname{colim} l d') \simeq h_{\mathsf{L}}(\operatorname{colim} d)$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$. As I is filtered, Lemma 7.2.2.8 implies that the colimit in the middle is already the colimit in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$. Thus condition (1) implies that $\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(-, F)$ carries β to an equivalence. To finish the proof, it is therefore enough to show that this functor also sends α to an equivalence. For this, we only need to show that for every object $i\colon A \to \mathsf{I}$ in arbitrary context $A \in \mathcal{B}$ the map $d'(i) \to h_{\mathsf{L}} l d'(i)$ is sent to an equivalence. By (b), we find that $d'(i)$ is of the form $\operatorname{colim} h_{\mathsf{L}} d_i$ for some diagram $d_i : \mathsf{H}_i \to \pi_A^* \mathsf{L}$ where H_i is a finite $\mathcal{B}_{/A}$-groupoid. Since this case has already been shown above, the result follows. □

Lemma 7.2.2.9. *Let L be a locally coherent $\mathcal{B}$-locale and let $F\colon \mathsf{L}^{\mathrm{op}} \to \mathsf{Grpd}_{\mathcal{B}}$ be a Filt-sheaf on L. Then F is a sheaf on L if and only if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ is a finitary sheaf on $\mathsf{L}^{\mathrm{cpt}}$.*

Proof. By Proposition 7.2.2.7, we need to show that F is a finitary sheaf on L if and only if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ is a finitary sheaf on $\mathsf{L}^{\mathrm{cpt}}$. As L is locally coherent and therefore $\mathsf{L}^{\mathrm{cpt}}$ is closed under binary products in L, it is clear that the condition is necessary. Moreover, as $\mathsf{L}^{\mathrm{cpt}}$ contains the initial object, it is clear that F satisfies condition (1) of the definition of a finitary sheaf if and only if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ does. Therefore, we only need to show that if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ is a finitary sheaf, then for every pair of objects $U, V\colon A \rightrightarrows \mathsf{L}$, the map $F(U \vee V) \to F(U) \times_{F(U \wedge V)} F(V)$ is an equivalence. Using Remark 7.2.2.6 and Remark 5.1.5.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Note that it follows from the bifunctoriality of $- \wedge -$ that for a fixed U, both the map $U \wedge V \to V$ and the map $U \wedge V \to U$ are natural in V, i.e. define morphisms in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$. Therefore, we obtain a cospan $\mathrm{diag}(U) \leftarrow U \wedge - \to \mathrm{id}_{\mathsf{L}}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$ (where $\mathrm{diag}\colon \mathsf{L} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$ is the diagonal map). By taking the colimit of this diagram, we end up with a commutative square

$$\begin{tikzcd} U \wedge - \arrow[r] \arrow[d] & \mathrm{id}_{\mathsf{L}} \arrow[d] \\ \mathrm{diag}(U) \arrow[r] & U \vee - \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$. Since colimits are universal in L, the functor $U \wedge -$ is cocontinuous. Furthermore, the functor $\mathrm{diag}(U)$ is Filt-cocontinuous: in fact, as it can be identified with $U \wedge \mathrm{diag}(1_{\mathsf{L}})(-)$, it suffices to see that $\mathrm{diag}(1_{\mathsf{L}}) \simeq 1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L},\mathsf{L})}$ is Filt-cocontinuous. As in the proof of Lemma 5.1.5.3, this is a consequence of the fact that filtered colimits in L are left exact, which is easily shown using Lemma 7.2.2.8 and the fact that L is a left exact localisation of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$, see Proposition 6.3.3.7. Thus, as Filt-cocontinuous functors are clearly closed under pushouts in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$, the above commutative diagram is a square of Filt-cocontinuous functors. By again using that filtered colimits in L are left exact, this observation now implies that by postcomposition with (the opposite of) F, we end up with a morphism $F(U \vee -) \to F(U) \times_{F(U \wedge -)} F(-)$ of Filt-cocontinuous functors $\mathsf{L} \to \mathsf{Grpd}_{\mathcal{B}}^{\mathrm{op}}$. Since $\mathsf{L} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$, the universal property of $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ thus implies that this morphism is an equivalence already when its restriction to $\mathsf{L}^{\mathrm{cpt}}$ is one. Together with our assumption on F, it follows that if U is compact, then the map $F(U \vee V) \to F(U) \times_{F(U \wedge V)} F(V)$ is an equivalence for *all* $V\colon 1 \to \mathsf{L}$. By symmetry and the fact that the context of U and V has been arbitrarily chosen, this now implies that the morphism $F(- \vee V) \to F(-) \times_{F(- \wedge V)} F(V)$ is an equivalence when restricted to $\mathsf{L}^{\mathrm{cpt}}$ and must therefore be an equivalence on all of L. Hence the claim follows. □

Proposition 7.2.2.10. *For any locally coherent $\mathcal{B}$-locale L, restriction along the* *inclusion* $\mathsf{L}^{\mathrm{cpt}} \hookrightarrow \mathsf{L}$ *induces an equivalence*

$$\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}}).$$

Proof. By Proposition 7.2.2.7, we have an identification

$$\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}) \cap \mathsf{Sh}^{\mathsf{Filt}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{L})$$

of full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$. In particular, we obtain an inclusion

$$\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \hookrightarrow \mathsf{Sh}^{\mathsf{Filt}}_{\mathsf{Grpd}_{\mathcal{B}}}(\mathsf{L}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$$

(where we use that we have $\mathsf{L} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ and the universal property of $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$). Using Remark 6.3.4.10 and Lemma 7.2.2.9, we now find that an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ is contained in $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ if and only if its transpose restricts to a finitary sheaf on $\pi_A^* \mathsf{L}^{\mathrm{cpt}}$. In other words, we obtain the desired equivalence $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$. □

Lemma 7.2.2.11. *Let* P *be a poset with finite colimits and binary products. Then* $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ *is closed under* Filt*-colimits in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$*.*

Proof. We need to show that for every diagram $d \colon \mathsf{I} \to \pi_A^* \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ in context $A \in \mathcal{B}$, where I is a filtered $\mathcal{B}_{/A}$-category, the colimit of d is contained in $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$. Using Remark 7.2.2.6, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. We may compute the colimit of d as the composition

$$\mathsf{P}^{\mathrm{op}} \xrightarrow{d'} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \xrightarrow{\mathrm{colim}_{\mathsf{I}}} \mathsf{Grpd}_{\mathcal{B}}$$

where d' is the transpose of $d \colon \mathsf{I} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$. As I is filtered, the functor on the right preserves finite limits. Moreover, the assumption that d takes values in $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ together with Lemma 5.5.1.3 implies that d' is a $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}})$-valued finitary sheaf on P. Hence the claim follows. □

Proof of Proposition 7.2.2.3. Suppose first that L is locally coherent. Then Proposition 7.2.2.10 implies that restriction along $\mathsf{L}^{\mathrm{cpt}} \hookrightarrow \mathsf{L}$ gives rise to an equivalence $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$. Together with Lemma 7.2.2.11, this implies that the inclusion $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ is Filt-cocontinuous. Hence, its left adjoint exhibits

$\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ as a $\mathsf{Fin}_{\mathcal{B}}$-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$, so that Proposition 5.3.3.6 and Corollary 5.3.3.3 together imply that $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is compactly generated. Now if L is even coherent, the $\mathcal{B}$-category $\mathsf{L}^{\mathrm{cpt}}$ has a final object. Therefore, $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ is a compact $\mathcal{B}$-topos. As the inclusion $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ is Filt-cocontinuous, this immediately implies that $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is compact as well. $\square$

7.2.3. The toposic cone

Every topological space X admits a closed immersion into a contractible and locally contractible space Y, for example by setting $Y = C(X)$, where $C(X)$ is the *cone* of X. In this section, we discuss a $\mathcal{B}$-toposic analogue of this observation, which will give rise to a factorisation of every geometric morphism into a compact and a locally contractible one. To that end, if X is a $\mathcal{B}$-topos, recall that the *comma $\mathcal{B}$-category* $\mathsf{X} \downarrow_{\mathsf{X}} \mathsf{Grpd}_{\mathcal{B}}$ is defined via the pullback square

$$\begin{tikzcd} \mathsf{X} \downarrow_{\mathsf{X}} \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "e^*"] \arrow[d, "j^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X}) \arrow[d, "d_0"] \\ \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\mathrm{const}_{\mathsf{X}}"'] & \mathsf{X} \end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$, where $\mathrm{const}_{\mathsf{X}}$ denotes the unique algebraic morphism $\mathsf{Grpd}_{\mathcal{B}} \to \mathsf{X}$, i.e. the left adjoint of Γ_{X}. By Proposition 6.2.6.1, this is a pullback diagram in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$, so that $\mathsf{X} \downarrow_{\mathsf{X}} \mathsf{Grpd}_{\mathcal{B}}$ is a $\mathcal{B}$-topos and j^* and e^* are algebraic morphisms.

Definition 7.2.3.1. For any $\mathcal{B}$-topos X, we refer to the $\mathcal{B}$-topos $\mathsf{X} \downarrow_{\mathsf{X}} \mathsf{Grpd}_{\mathcal{B}}$ as its *$\mathcal{B}$-toposic right cone* and denote it by $\mathsf{X}^{\triangleright}$.

If X is a $\mathcal{B}$-topos, let $i^* : \mathsf{X}^{\triangleright} \to \mathsf{X}$ be the algebraic morphism that is obtained by composing the functor $d_1 : \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X}) \to \mathsf{X}$ with the upper horizontal map in the defining pullback square of $\mathsf{X}^{\triangleright}$.

Remark 7.2.3.2. Suppose that X is a $\mathcal{B}$-topos and let $f_* : \mathcal{X} \to \mathcal{B}$ be the associated geometric morphism of ∞-topoi. Then the ∞-topos $\mathrm{Cone}(f) = \Gamma_{\mathcal{B}}(\mathsf{X}^{\triangleright})$ recovers the comma ∞-category $\mathcal{X} \downarrow_{\mathcal{X}} \mathcal{B}$ and is therefore the *recollement* of $\mathcal{B}$ and $\mathcal{X}$ along f^* in the sense of [50, § A.8]. In particular, $j_* : \mathcal{B} \to \mathrm{Cone}(f)$ is an open and $i_* : \mathcal{X} \to \mathrm{Cone}(f)$ a closed immersion of ∞-topoi. In particular, the latter is a compact geometric morphism (by Example 7.2.1.3).

Remark 7.2.3.3. In the situation of Remark 7.2.3.2, the ∞-topos $\mathrm{Cone}(f)$ sits inside a pushout square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f_*"] \arrow[d, "\mathrm{id} \otimes s^0"'] & \mathcal{B} \arrow[d] \\ \mathcal{X} \otimes \Delta^1 \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$, where $\mathcal{X} \otimes \Delta^1$ denotes the tensoring in $\mathrm{Top}_\infty^{\mathrm{R}}$ over Cat_∞. Therefore $\mathrm{Cone}(f)$ is to be thought of as the mapping cone of f_*.

Recall from Section 7.1 that a $\mathcal{B}$-topos X is said to be *locally contractible* if the algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \mathrm{Grpd}_{\mathcal{B}} \to \mathsf{X}$ has a left adjoint π_{X}. The following proposition expresses the fact that the $\mathcal{B}$-toposic right cone $\mathsf{X}^{\triangleright}$ is contractible and locally contractible (in the $\mathcal{B}$-toposic sense):

Proposition 7.2.3.4. *For every $\mathcal{B}$-topos X, the $\mathcal{B}$-topos $\mathsf{X}^{\triangleright}$ is locally contractible, and the additional left adjoint π_{X} of $\mathrm{const}_{\mathsf{X}^{\triangleright}}$ is equivalent to j^*. In particular, π_{X} preserves finite limits.*

Proof. Since $s_0 \colon \mathsf{X} \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X})$ is right adjoint to d_0 (for example by using Proposition 3.1.1.14) and by the dual of Lemma 3.4.4.2, the functor j_* is the pullback of s_0 along ϵ^*. Since s_0 is cocontinuous and as $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ preserves limits, this implies that j_* must be cocontinuous as well and therefore equivalent to $\mathrm{const}_{\mathsf{X}^{\triangleright}}$ (by the universal property of $\mathrm{Grpd}_{\mathcal{B}}$). As this shows that j^* is left adjoint to $\mathrm{const}_{\mathsf{X}^{\triangleright}}$, the claim follows. $\square$

Corollary 7.2.3.5. *Every geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$ can be factored into a closed immersion followed by a contractible and locally contractible geometric morphism.*

Proof. By Remark 7.2.3.2, the geometric morphism $i_* \colon \mathcal{X} \to \mathrm{Cone}(f)$ is a closed immersion, and it follows from Proposition 7.2.3.4 that the geometric morphism $h_* \colon \mathrm{Cone}(f) \to \mathcal{B}$ is contractible and locally contractible. $\square$

Remark 7.2.3.6. For 1-topoi, the factorisation in Corollary 7.2.3.5 appears in the proof of [39, Theorem C.3.3.14].

7.2.4. Compact geometric morphisms and localisations

The goal of this section is to establish that compact geometric morphisms commute with localisations of subtopoi:

Proposition 7.2.4.1. *Consider a pullback square in* $\mathrm{Top}_\infty^{\mathrm{R}}$

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "p'_*"] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{Z} \arrow[r, "j_*"'] & \mathcal{B} \end{tikzcd}$$

where j_* is fully faithful and p_* is compact. Then the mate natural transformation $p'_*(j')^* \to j^* p_*$ is an equivalence.

Intuitively, Proposition 7.2.4.1 should hold because the localisation functor $(j')^* \colon \mathcal{X} \to \mathcal{Z}'$ is given by an (internally) filtered colimit. Indeed, we may pick a bounded local class S' of morphisms in $\mathcal{X}$ that is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{X})$ such that $(j')^*$ exhibits $\mathcal{Z}'$ as the Bousfield localisation of $\mathcal{X}$ at S' (see Proposition 6.2.10.14). We denote by $\iota' \colon \mathrm{Grpd}_{S'} \hookrightarrow \mathrm{Grpd}_{\mathcal{X}}$ the associated full subcategory. Then S' being bounded implies that $\mathrm{Grpd}_{S'}$ is a small $\mathcal{X}$-category, and S' being closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{X})$ implies that $\mathrm{Grpd}_{S'}$ has finite limits and is therefore *cofiltered* by Proposition 5.2.3.7 (i.e. its opposite $\mathrm{Grpd}_{S'}^{\mathrm{op}}$ is filtered). Now by the formula in Proposition 6.2.10.14, we find that for every $X \in \mathcal{X}$ we obtain a canonical equivalence

$$j'_*(j')^*(X) \simeq X_{\iota'}^{\mathrm{sh}} = \operatorname*{colim}_{\tau<\kappa} T_\tau^{\iota'} X$$

where κ is a suitably large regular cardinal and $T_\tau^{\iota'} X$ is defined recursively by the condition that we have $T_{\tau+1}^{\iota'} X = \mathrm{colim}_{\mathrm{Grpd}_{S'}^{\mathrm{op}}} \mathrm{map}_{\mathrm{Grpd}_{\mathcal{X}}}(\iota'(-), T_\tau^{\iota'} X)$ and that $T_\tau^{\iota'} X = \mathrm{colim}_{\tau'<\tau} T_{\tau'}^{\iota'} X$ when τ is a limit ordinal. In particular, the endofunctor $j'_*(j')^*$ is given by an (iterated) *filtered* colimit. So intuitively, p_* being compact should imply that this functor carries $j'_*(j')^*(X)$ to $j_* j^* p_*(X)$, which precisely means that the mate transformation $p'_*(j')^*(X) \to j^* p_*(X)$ is an equivalence. However, we have to be a bit careful at this point: the above formula for $j'_*(j')^*$ exhibits $j'_*(j')^*(X)$ as a filtered colimit *internal to* $\mathcal{X}$, whereas p_* being compact only implies that this functor commutes with filtered colimits *internal to* $\mathcal{B}$. Hence, the main challenge is to rewrite the above formula in terms of a filtered colimit internal to $\mathcal{B}$.

Observation 7.2.4.2. Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism, I a $\mathcal{B}$-category and C an $\mathcal{X}$-category. On account of the commutative diagram

$$\begin{tikzcd} f_*\mathsf{C} \arrow[r, "\mathrm{diag}_{\mathsf{I}}"] \arrow[dr, "f_*(\mathrm{diag}_{f^*\mathsf{I}})"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\mathsf{C}) \arrow[d, "\simeq"] \\ & f_*\underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{I}, \mathsf{C}), \end{tikzcd}$$

the $\mathcal{B}$-category $f_*\mathsf{C}$ admits I-indexed colimits if and only if the $\mathcal{X}$-category C admits $f^*\mathsf{I}$-indexed colimits, and we may identify $\mathrm{colim}_{\mathsf{I}} : \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\mathsf{C}) \to f_*\mathsf{C}$ with the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\mathsf{C}) \simeq f_*\underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{I}, \mathsf{C}) \xrightarrow{f_*(\mathrm{colim}_{f^*\mathsf{I}})} f_*\mathsf{C}.$$

By passing to global sections, this implies that for every diagram $d \colon \mathsf{I} \to f_*\mathsf{C}$ with transpose $\bar{d} : f^*\mathsf{I} \to \mathsf{C}$, we have a canonical equivalence $\mathrm{colim}^{\mathcal{B}}_{\mathsf{I}}\, d \xrightarrow{\simeq} \mathrm{colim}^{\mathcal{X}}_{f^*\mathsf{I}}\, \bar{d}$ in the ∞-category $\Gamma_{\mathcal{X}}(\mathsf{C}) = \Gamma_{\mathcal{B}}(f_*\mathsf{C})$ (where the superscripts emphasise internal to which ∞-topos the colimits are taken). We will repeatedly use this observation throughout this section.

Suppose now that S is a bounded local class of morphisms in $\mathcal{B}$ that is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$, and let $\iota \colon \mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ be the associated (cofiltered) full subcategory. If $f_* : \mathcal{X} \to \mathcal{B}$ is a geometric morphism, we let $\iota' : f^*(\mathsf{Grpd}_S) \to \mathsf{Grpd}_{\mathcal{X}}$ be the functor of $\mathcal{X}$-categories that arises from transposing $\mathrm{const}_{f_*(\mathsf{Grpd}_{\mathcal{X}})}\, \iota \colon \mathsf{Grpd}_S \to f_*(\mathsf{Grpd}_{\mathcal{X}})$ across the adjunction $f^* \dashv f_*$. By Example 6.2.10.9, this is a cofiltered $\mathcal{X}$-category, and the colimit of ι' is the final object in $\mathsf{Grpd}_{\mathcal{X}}$. Therefore, we are in the situation of Definition 6.2.10.5 and thus obtain an endofunctor $(-)^{\mathrm{sh}}_{\iota'} : \mathsf{Grpd}_{\mathcal{X}} \to \mathsf{Grpd}_{\mathcal{X}}$ via $(-)^{\mathrm{sh}}_{\iota'} = \mathrm{colim}_{\tau<\kappa} T^{\iota'}_{\tau}$, where κ is a suitable $\mathcal{X}$-regular cardinal and where $T^{\iota'}_{\bullet} : \kappa \to \mathrm{Fun}_{\mathcal{X}}(\mathsf{Grpd}_{\mathcal{X}}, \mathsf{Grpd}_{\mathcal{X}})$ is defined via transfinite induction by setting $T^{\iota'}_0 = \mathrm{id}$, by defining the map $T^{\iota'}_{\tau} \to T^{\iota'}_{\tau+1}$ to be the morphism $\phi \colon T^{\iota'}_{\tau} \to (T_{\tau})^{+}_{\iota'} = \mathrm{colim}_{f^*(\mathsf{Grpd}_S)^{\mathrm{op}}}\, \mathrm{map}_{\mathsf{Grpd}_{\mathcal{X}}}(\iota'(-), -)$ from Remark 6.2.10.4 and finally by setting $T^{\iota'}_{\tau} = \mathrm{colim}_{\tau'<\tau} T^{\iota'}_{\tau'}$ whenever τ is a limit ordinal. We will slightly abuse notation and also denote by $(-)^{\mathrm{sh}}_{\iota'}$ the underlying endofunctor on $\mathcal{X}$ that is obtained by passing to global sections. It will always be clear from the context which variant we refer to.

Proposition 7.2.4.3. *Consider a pullback square* $\mathcal{Q}$ *in* $\mathrm{Top}^{\mathrm{R}}_{\infty}$

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "g_*"'] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z} \arrow[r, "j_*"'] & \mathcal{B} \end{tikzcd}$$

where j_* *(and therefore also* j'_**) is fully faithful. Let* S *be a bounded local class of morphisms, closed under finite limits in* $\mathrm{Fun}(\Delta^1, \mathcal{B})$*, such that* j^* *is the Bousfield localisation at* S *(such a local class always exists by Proposition 6.2.10.14), and let* $\iota\colon \mathsf{Grpd}_S \hookrightarrow \mathsf{Grpd}_{\mathcal{B}}$ *be the associated full subcategory. Then we obtain an equivalence* $j'_*(j')^* \simeq (-)^{\mathrm{sh}}_{\iota'}$*, where* $\iota' : f^*(\mathsf{Grpd}_S) \to \mathsf{Grpd}_{\mathcal{X}}$ *is the transpose of* $\mathrm{const}_{f_*(\mathsf{Grpd}_{\mathcal{X}})}\, \iota$*.*

Remark 7.2.4.4. The above proposition can be thought of as an ∞-toposic version of [39, Theorem C.3.3.14].

We first prove this proposition in a special case:

Lemma 7.2.4.5. *Consider a pullback square* $\mathcal{Q}$ *in* $\mathrm{Top}^{\mathrm{R}}_{\infty}$

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "h'_*"'] & \mathcal{X} \arrow[d, "h_*"] \\ \mathcal{Z} \arrow[r, "j_*"'] & \mathcal{B} \end{tikzcd}$$

where j_* *is fully faithful and* h_* *is locally contractible such that the additional left adjoint* $h_!$ *of* h^* *preserves finite limits. Let* S *and* ι *be as in Proposition 7.2.4.3. Then there is an equivalence* $j'_*(j')^* \simeq (-)^{\mathrm{sh}}_{\iota'}$*, where* $\iota' : h^*(\mathsf{Grpd}_S) \to \mathsf{Grpd}_{\mathcal{X}}$ *is the transpose of* $\mathrm{const}_{h_*(\mathsf{Grpd}_{\mathcal{X}})}\, \iota$*.*

Proof. By Proposition 7.1.1.5, the functor ι' is fully faithful, and since h_* is locally contractible the $\mathcal{X}$-category $h^*(\mathsf{Grpd}_S)$ is given by the sheaf $\mathsf{Grpd}_S(h_!(-))$. It follows that a map $s\colon X \to Y$ in $\mathcal{X}$ defines an object of $h^*(\mathrm{Grpd}_S)(Y)$ if and only if $h_!(s) \in S$ and the square

$$\begin{array}{ccc} X & \longrightarrow & h^*h_!X \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & h^*h_!Y \end{array}$$

is a pullback. Let W be the class of maps in $\mathcal{X}$ that satisfies these two conditions. Then, since $h_!$ is cocontinuous and preserves finite limits, it easily follows that W is local. Hence we find $h^*(\mathrm{Grpd}_S) = \mathrm{Grpd}_W$ as full subcategories of $\mathrm{Grpd}_{\mathcal{X}}$. Moreover, by the explicit description of W, it is clear that W is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{X})$. Thus, by appealing to Proposition 6.2.10.14, we only need to verify that $\mathcal{Z}'$ is the Bousfield localisation of $\mathcal{X}$ at W. We know from Remark 6.2.10.15 that $\mathcal{Z}' \hookrightarrow \mathcal{X}$ is obtained as the Bousfield localisation of $\mathcal{X}$ at the smallest local class $\overline{h^*S}$ that contains the image h^*S of S along h^*. Since we clearly have $h^*S \subset W$, this immediately implies $W = \overline{f^*S}$, hence the claim follows. □

Lemma 7.2.4.6. *Let $p_* : \mathcal{X} \to \mathcal{B}$ be a compact geometric morphism and let $\iota\colon \mathsf{I} \to \mathrm{Grpd}_{\mathcal{B}}$ be a functor where I is cofiltered and where* $\operatorname{colim} \iota$ *is the final object. Let $\iota' : p^*\mathsf{I} \to \mathrm{Grpd}_{\mathcal{X}}$ be the transpose of* $\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\, \iota\colon \mathsf{I} \to p_*(\mathrm{Grpd}_{\mathcal{X}})$. *Then there is an equivalence $p_*(-)^{\mathrm{sh}}_{\iota'} \simeq (-)^{\mathrm{sh}}_{\iota} p_*$.*

Proof. Since $(-)^{\mathrm{sh}}_{\iota'}$ and $(-)^{\mathrm{sh}}_{\iota}$ are obtained as filtered colimits of iterations of $(-)^{+}_{\iota'}$ and $(-)^{+}_{\iota}$, respectively, and as p_* commutes with filtered colimits, it suffices to produce an equivalence $p_*(-)^{+}_{\iota'} \simeq (-)^{+}_{\iota} p_*$. Now for every $X \in \mathcal{X}$, we have a natural chain of equivalences

$$\begin{aligned}
(p_*X)^{+}_{\iota} &= \operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{\mathrm{Grpd}_{\mathcal{B}}}(\iota(-), \Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})}X) \\
&\simeq \operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\, \iota(-), X) \\
&\simeq \operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\underline{\mathrm{Hom}}^{\mathcal{B}}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\, \iota(-), X)) \\
&\simeq \Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \underline{\mathrm{Hom}}^{\mathcal{B}}_{p_*(\mathrm{Grpd}_{\mathcal{B}})}(\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\, \iota(-), X)) \\
&\simeq \Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\operatorname{colim}^{\mathcal{X}}_{p^*\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\iota'(-), X)) \\
&\simeq p_* X^{+}_{\iota'}
\end{aligned}$$

where the third step follows from Remark 6.2.10.13, the fourth step is a consequence of the fact that $\Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})}$ preserves filtered colimits and the fifth step follows from Observation 7.2.4.2. Hence the result follows. ☐

Proof of Proposition 7.2.4.3. Using Proposition 7.2.3.4, we may factor the pullback square $\mathcal{Q}$ into two squares

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{X} \arrow[d, "i_*"] \\ \mathcal{Z}'' \arrow[r, "j''_*"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{Y} \arrow[d, "h_*"] \\ \mathcal{Z} \arrow[r, "j_*"] & \mathcal{B} \end{tikzcd}$$

where h_* is as in Lemma 7.2.4.5 and i_* is a closed immersion (and therefore compact, see Example 7.2.1.3). By Lemma 7.2.4.5, we have an equivalence $j''_*(j'')^* \simeq (-)^{\mathrm{sh}}_{\iota''}$, where $\iota'' : h^*(\mathrm{Grpd}_S) \to \mathrm{Grpd}_{\mathcal{Y}}$ is obtained as the transpose of $\mathrm{const}_{h_*(\mathrm{Grpd}_{\mathcal{Y}})}\,\iota$. Furthermore, since i_* is a closed immersion and therefore proper (by Example 7.0.0.2), the upper square is horizontally left adjointable. Therefore, we have an equivalence $j'_*(j')^* \simeq i^* j''_*(j'')^* i_*$ and therefore $j'_*(j')^* \simeq i^*(-)^{\mathrm{sh}}_{\iota''} i_*$. Now as i_* is compact, we may apply Lemma 7.2.4.6 to deduce $(-)^{\mathrm{sh}}_{\iota''} i_* \simeq i_*(-)^{\mathrm{sh}}_{\iota'}$, which yields the claim. ☐

We are finally ready to prove Proposition 7.2.4.1:

Proof of Proposition 7.2.4.1. It suffices to construct a natural equivalence

$$p_* j'_*(j')^* \simeq j_* j^* p_*.$$

Pick a local class S in $\mathcal{B}$, as in Proposition 7.2.4.3, and let $\iota \colon \mathrm{Grpd}_S \hookrightarrow \mathrm{Grpd}_{\mathcal{B}}$ be the associated full subcategory. Furthermore, we let $\iota' : p^*(\mathrm{Grpd}_S) \to \mathrm{Grpd}_{\mathcal{X}}$ be the transpose of $\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\,\iota \colon \mathrm{Grpd}_S \to p_*(\mathrm{Grpd}_{\mathcal{X}})$. We then have equivalences $j_* j^* \simeq (-)^{\mathrm{sh}}_{\iota}$ (by Proposition 6.2.10.14) and $j'_*(j')^* \simeq (-)^{\mathrm{sh}}_{\iota'}$ (by Proposition 7.2.4.3). Hence the claim follows from Lemma 7.2.4.6. ☐

7.2.5. Classification of proper geometric morphism

The goal of this section is to put together the preparations made in Section 7.2.3 and Section 7.2.4 in order to show:

Theorem 7.2.5.1. *A geometric morphism $p_* : \mathcal{X} \to \mathcal{B}$ is proper if and only if it is compact.*

One direction of Theorem 7.2.5.1 is (almost) trivial:

Lemma 7.2.5.2. *Let $p_* : \mathcal{X} \to \mathcal{B}$ be a proper geometric morphism. Then p_* is compact.*

Proof. Let us denote by $\mathsf{X} = p_*(\mathrm{Grpd}_{\mathcal{X}})$ the $\mathcal{B}$-topos that corresponds to p_*. Note that if $A \in \mathcal{B}$ is an arbitrary object, the induced morphism $\mathcal{X}_{/p^*A} \to \mathcal{B}_{/A}$ is proper as well. As this is the geometric morphism which corresponds to the $\mathcal{B}_{/A}$-topos $\pi_A^* \mathsf{X}$, we may (after replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$) reduce to the case where we have to show that if I is a filtered $\mathcal{B}$-category, then Γ_{X} preserves I-filtered colimits. We now obtain a commutative diagram

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathrm{diag}"] \arrow[d, "\Gamma_{\mathsf{X}}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{X}) \arrow[r, "\lim"] \arrow[d, "(\Gamma_{\mathsf{X}})_*"'] & \mathsf{X} \arrow[d, "\Gamma_{\mathsf{X}}"] \\
\mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \arrow[r, "\lim"] & \mathsf{Grpd}_{\mathcal{B}}
\end{tikzcd}$$

in $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$ in which both squares are pullbacks (see Example 6.2.7.5). As for every $A \in \mathcal{B}$ the geometric morphism $\Gamma_{\mathsf{X}}(A) : \mathcal{X}_{/p^*A} \to \mathcal{B}_{/A}$ is proper, it follows that the mate of the left square is an equivalence. This precisely means that Γ_{X} commutes with I-indexed colimits, as desired. □

The converse direction of Theorem 7.2.5.1 takes far more effort. We begin with the following small but useful observation:

Lemma 7.2.5.3. *Let*

$$\begin{tikzcd}
\mathsf{Q} \arrow[r, "g_*"] \arrow[d, "q_*"] & \mathsf{P} \arrow[d, "p_*"] \\
\mathsf{Y} \arrow[r, "f_*"] & \mathsf{X}
\end{tikzcd}$$

be a commutative square in $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$. Then the mate $\phi \colon f^* p_* \to g_* q^*$ is an equivalence if and only if it induces an equivalence on global sections.

Proof. Since the condition is clearly necessary, it suffices to show that it is sufficient too. To that end, we need to show that for any object $A \in \mathcal{B}$, the horizontal mate $\phi(A)$ of the back square in the commutative diagram

$$\begin{tikzcd}
 & \mathsf{Q}(A) \arrow[rr, "g_*(A)"] \arrow[dl, "(\pi_A)_*"'] \arrow[dd, "q_*(A)" near end] & & \mathsf{P}(A) \arrow[dl, "(\pi_A)_*"] \arrow[dd, "p_*(A)"] \\
\mathsf{Q}(1) \arrow[rr, "g_*(1)" near end, crossing over] \arrow[dd, "q_*(1)"'] & & \mathsf{P}(1) & \\
 & \mathsf{Y}(A) \arrow[rr, "f_*(A)"' near start] \arrow[dl, "(\pi_A)_*"'] & & \mathsf{X}(A) \arrow[dl, "(\pi_A)_*"] \\
\mathsf{Y}(1) \arrow[rr, "f_*(1)"'] & & \mathsf{X}(1) \arrow[from=uu, "p_*(1)" near start, crossing over] &
\end{tikzcd}$$

is an equivalence, given that the mate $\phi(1)$ of the front square is one. But since the horizontal mate of both the left and the right square is an equivalence, it follows that $\phi(A)$ is an equivalence when evaluated at any object in the image of π_A^*. Since $\mathsf{X}(A)$ is étale over $\mathsf{X}(1)$, every object in $\mathsf{X}(A)$ is a pullback of objects that are contained in the image of π_A^*. Therefore, the claim follows from the fact that $\phi(A)$ is a morphism of left exact functors. □

In order to prove Theorem 7.2.5.1, we in particular need to show that compact morphisms are stable under pullback. In fact it will suffice to prove this in a special case (see Corollary 7.2.5.6), which we will turn to now.

Lemma 7.2.5.4. *Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. Suppose we are given a commutative square*

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "q_*"] \arrow[d, "g_*"'] & \mathcal{X} \arrow[d, "f_*"] \\
\mathcal{Z} \arrow[r, "p_*"'] & \mathcal{B}
\end{tikzcd}$$

whose horizontal mate is an equivalence and such that g_* is compact. Then, for every filtered $\mathcal{B}$-category I, the functor $p^* : \mathcal{B} \to \mathcal{Z}$ carries the horizontal mate of *the commutative square*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\mathrm{diag}"] \arrow[d, "f_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, f_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[d, "(\Gamma_{f_*(\mathrm{Grpd}_{\mathcal{X}})})_*"] \\
\mathcal{B} \arrow[r, "\mathrm{diag}"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}})
\end{tikzcd}$$

to an equivalence.

Proof. Note that we have a commutative diagram of ∞-topoi

$$\begin{tikzcd}
& \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, f_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[dd, "(f_*)_*"] & & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, p_* g_*(\mathrm{Grpd}_{\mathcal{W}})) \arrow[ll, "(q_*)_*"'] \arrow[dd, "(g_*)_*"] \\
\mathcal{X} \arrow[ur, "\mathrm{diag}"] \arrow[dd, "f_*"'] & & \mathcal{W} \arrow[ll, "q_*"'] \arrow[ur, "\mathrm{diag}"] \arrow[dd, "g_*"] & \\
& \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \mathrm{Grpd}_{\mathcal{B}}) & & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, p_*(\mathrm{Grpd}_{\mathcal{Z}})) \arrow[ll, "(p_*)_*"] \\
\mathcal{B} \arrow[ur, "\mathrm{diag}"] & & \mathcal{Z} \arrow[ll, "p_*"] \arrow[ur, "\mathrm{diag}"] &
\end{tikzcd}$$

where the horizontal mates of the front and the back square are invertible (the latter using Lemma 7.2.5.3 and the 2-functoriality of $\mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, -)$). Furthermore, the adjunction $p^* \dashv p_*$ allows us to identify the right square with

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\mathrm{diag}"] \arrow[d, "g_*"] & \mathrm{Fun}_{\mathcal{Z}}(p^*\mathsf{I}, g_*(\mathrm{Grpd}_{\mathcal{W}})) \arrow[d, "(g_*)_*"] \\
\mathcal{Z} \arrow[r, "\mathrm{diag}"] & \mathrm{Fun}_{\mathcal{Z}}(p^*\mathsf{I}, \mathrm{Grpd}_{\mathcal{Z}})
\end{tikzcd}$$

whose horizontal mate is invertible since g_* was assumed to be compact and $p^*\mathsf{I}$ is filtered. Therefore, the functoriality of mates implies that the functor $p^* : \mathcal{B} \to \mathcal{Z}$ carries the horizontal mate of the left square to the mate of the right square, which is an equivalence. $\square$

As a consequence of Lemma 7.2.5.4, we obtain that compactness can be checked locally on the base in the following strong sense:

Proposition 7.2.5.5. *Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. Assume that there is a family of commutative squares*

$$\begin{tikzcd} \mathcal{W}_i \arrow[r] \arrow[d, "g^i_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z}_i \arrow[r, "p^i_*"] & \mathcal{B} \end{tikzcd}$$

whose mates are equivalences and in which each $(p^i)^*$ is conservative and each g^i_* is compact. Then f_* is compact.

Proof. First, let us verify that for any filtered $\mathcal{B}$-category I the mate of the commutative square

$$\begin{tikzcd} f_*(\mathsf{Grpd}_{\mathcal{X}}) \arrow[r, "\mathrm{diag}"] \arrow[d, "f_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*(\mathsf{Grpd}_{\mathcal{X}})) \arrow[d, "(\Gamma_{f_*(\mathsf{Grpd}_{\mathcal{X}})})_*"] \\ \mathsf{Grpd}_{\mathcal{B}} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}}) \end{tikzcd}$$

is an equivalence. By Lemma 7.2.5.3 it suffices to see this on global sections. Our assumptions guarantee that we can check that the mate is an equivalence after applying each $(p^i)^* : \mathcal{B} \to \mathcal{Z}?i$. But then the claim follows from Lemma 7.2.5.4. Now if $A \in \mathcal{B}$ and I is a filtered $\mathcal{B}_{/A}$-category, we observe that the square obtained by pulling back along $(\pi_A)_* : \mathcal{B}_{/A} \to \mathcal{B}$ again satisfies the assumptions of the proposition. Thus we can replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ in the first part of the proof, and the result follows. $\square$

Corollary 7.2.5.6. *Let $p_* : \mathcal{X} \to \mathcal{B}$ be a compact geometric morphism and let* C *be a $\mathcal{B}$-category. Then the geometric morphism*

$$(\Gamma_{p_*(\mathsf{Grpd}_{\mathcal{X}})})_* : \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, p_*(\mathsf{Grpd}_{\mathcal{X}})) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$$

is again compact.

Proof. The core inclusion $\iota : \mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$ gives rise to a geometric morphism

$$\iota_* : \mathcal{B}_{/\mathsf{C}^{\simeq}} \simeq \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\simeq}, \mathsf{Grpd}_{\mathcal{B}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Grpd}_{\mathcal{B}})$$

whose left adjoint is given by restriction along ι and is therefore conservative (which is easily seen using the straightening equivalence for left fibrations and Proposition 2.1.1.12). Since in the commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{/p^*(\mathsf{C}^\simeq)} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, p_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[r, "\lim"] \arrow[d] & \mathcal{X} \arrow[d, "p_*"] \\
\mathcal{B}_{/\mathsf{C}^\simeq} \arrow[r] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}}) \arrow[r, "\lim"] & \mathcal{B}
\end{tikzcd}$$

both squares are pullbacks (the one on the right by Example 6.2.7.5), it follows that the left vertical morphism is compact as an étale base change of a compact morphism. As a consequence, the left square satisfies the assumptions of Proposition 7.2.5.5, which immediately yields the claim. □

Proof of Theorem 7.2.5.1. Suppose that $p_* : \mathcal{X} \to \mathcal{B}$ is a compact geometric morphism. First, we show that for any pullback square

$$\begin{tikzcd}
\mathcal{Z}' \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\
\mathcal{Z} \arrow[r, "f_*"] & \mathcal{B}
\end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ the mate natural transformation $q_* g^* \to f^* p_*$ is invertible. To see this, we factor the above square as

$$\begin{tikzcd}
\mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "q_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[r, "\lim_{\mathsf{C}^{\mathrm{op}}}"] \arrow[d, "(p_*)_*"] & \mathcal{X} \arrow[d, "p_*"'] \\
\mathcal{Z} \arrow[r, "j_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}}) \arrow[r, "\lim_{\mathsf{C}^{\mathrm{op}}}"] & \mathcal{B}.
\end{tikzcd}$$

(using again Example 6.2.7.5). It is clear that the mate of the right square is an equivalence, hence it suffices to show the claim for the left square. In other words, by Corollary 7.2.5.6 we may reduce to the case where f_* is already fully faithful, which follows from Proposition 7.2.4.1.

To complete the proof, we now have to show that given a second pullback

$$\begin{tikzcd}
\mathcal{W}' \arrow[r, "s_*"] \arrow[d, "\bar{q}_*"] & \mathcal{Z}' \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\
\mathcal{W} \arrow[r, "r_*"] & \mathcal{Z} \arrow[r, "f_*"] & \mathcal{B}
\end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ the mate of the left square is an equivalence. For this we again use the factorisation from above and consider the diagram

$$\begin{tikzcd} \mathcal{W}' \arrow[r, "s_*"] \arrow[d, "q'_*"] & \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "q_*"'] & \mathrm{Fun}_{\mathcal{B}}(\mathrm{C}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[d, "(p_*)_*"] \\ \mathcal{W} \arrow[r, "r_*"] & \mathcal{Z} \arrow[r, "j_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathrm{C}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}}) \end{tikzcd}$$

By Corollary 7.2.5.6 the geometric morphism $(p_*)_*$ is compact. Together with what we have already shown so far, this implies that both the outer square and the right square is left adjointable. As j'_* is fully faithful it now immediately follows that the left square is also left adjointable, as desired. $\square$

7.2.6. $\mathcal{E}$-compact $\mathcal{B}$-topoi

The goal of this section is to discuss a generalisation of Theorem 7.2.5.1 where we allow coefficients in an arbitrary compactly generated ∞-category $\mathcal{E}$. The proof is essentially the same as the one of Theorem 7.2.5.1, however this level of generality allows us to apply the result to a wider range of examples.

Definition 7.2.6.1. Let $\mathcal{E}$ be a presentable ∞-category. Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. We say that f_* is *$\mathcal{E}$-proper* if for every diagram

$$\begin{tikzcd} \mathcal{W}' \arrow[r, "t'_*"] \arrow[d, "h_*"'] & \mathcal{W} \arrow[r, "t_*"] \arrow[d, "g_*"'] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z}' \arrow[r, "s'_*"'] & \mathcal{Z} \arrow[r, "s_*"'] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ in which both squares are pullbacks, the square

$$\begin{tikzcd} \mathcal{W}' \otimes \mathcal{E} \arrow[r, "g_* \otimes \mathcal{E}"] \arrow[d, "p'_* \otimes \mathcal{E}"] & \mathcal{W} \otimes \mathcal{E} \arrow[d, "f_* \otimes \mathcal{E}"] \\ \mathcal{Z}' \otimes \mathcal{E} \arrow[r, "q_* \otimes \mathcal{E}"] & \mathcal{Z} \otimes \mathcal{E} \end{tikzcd}$$

is horizontally left adjointable. Here $- \otimes - : \mathrm{Pr}_\infty^{\mathrm{R}} \times \mathrm{Pr}_\infty^{\mathrm{R}} \to \mathrm{Pr}_\infty^{\mathrm{R}}$ denotes Lurie's tensor product of presentable ∞-categories.

There is a natural way to enhance Lurie's tensor products to presentable $\mathcal{B}$-categories, which we will need to formulate our version of compactness with coefficients:

Construction 7.2.6.2. Recall from Construction 1.4.2.1 and Example 5.4.4.8 that there is a functor $-\otimes \mathsf{Grpd}_{\mathcal{B}} \colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ that sends a presentable ∞-category $\mathcal{E}$ to the $\mathcal{B}$-category

$$\mathcal{E} \otimes \mathsf{Grpd}_{\mathcal{B}} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}; \quad A \mapsto \mathcal{E} \otimes \mathcal{B}_{/A}.$$

We can therefore define a presentable $\mathcal{B}$-category $\mathsf{C} \otimes \mathcal{E} \coloneqq \mathsf{C} \otimes^{\mathcal{B}} (\mathcal{E} \otimes \mathsf{Grpd}_{\mathcal{B}})$ for every presentable $\mathcal{B}$-category C. Here $- \otimes^{\mathcal{B}} -$ denotes the tensor product of presentable $\mathcal{B}$-categories introduced in Section 5.5.3. In particular, $- \otimes \mathcal{E}$ defines a functor $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$.

Remark 7.2.6.3. If I is a $\mathcal{B}$-category, C a presentable $\mathcal{B}$-category and $\mathcal{E}$ is a presentable ∞-category, it follows from the explicit description of the tensor product of presentable $\mathcal{B}$-categories from Proposition 5.5.3.8 that we have a canonical equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \otimes \mathcal{E} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C} \otimes \mathcal{E})$$

and therefore in particular an equivalence $\mathsf{C}(A) \otimes \mathcal{E} \simeq (\mathsf{C} \otimes \mathcal{E})(A)$ for every $A \in \mathcal{B}$.

Definition 7.2.6.4. Let $p_* \colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism and $\mathcal{E}$ a presentable ∞-category. Then p_* is called $\mathcal{E}$-compact if

$$\Gamma_{p_*(\mathsf{Grpd}_{\mathcal{X}})} \otimes \mathcal{E} \colon p_*(\mathsf{Grpd}_{\mathcal{X}}) \otimes \mathcal{E} \to \mathsf{Grpd}_{\mathcal{B}} \otimes \mathcal{E}$$

commutes with filtered colimits.

We now come to the main result of this section, the $\mathcal{E}$-linear version of Theorem 7.2.5.1:

Theorem 7.2.6.5. *Let $p_* \colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism and $\mathcal{E}$ a compactly generated ∞-category. Then p_* is $\mathcal{E}$-proper if and only if it is $\mathcal{E}$-compact.*

Remark 7.2.6.6. More generally one could define $p_* : \mathcal{X} \to \mathcal{B}$ to be E-compact for a presentable $\mathcal{B}$-category E whenever $p_*(\mathsf{Grpd}_{\mathcal{X}}) \otimes^{\mathcal{B}} \mathsf{E} \to \mathsf{E}$ commutes with filtered colimits. Similarly, one can also define a notion of E-properness. Then the analogue of Theorem 7.2.6.5 still holds whenever E is compactly generated (in the $\mathcal{B}$-categorical sense). We decided to only prove the result in the case where $\mathsf{E} = \mathsf{Grpd}_{\mathcal{B}} \otimes \mathcal{E}$, since the proof is slightly less technical and since this case already contains most examples of interest.

Remark 7.2.6.7. Let $\mathcal{E}$ be a compactly generated ∞-category and X a $\mathcal{B}$-topos. Then for any $A \in \mathcal{B}$ we may identify the tensor product $\mathsf{X}(A) \otimes \mathcal{E}$ with the ∞-category $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathsf{X}(A))$ (where $\mathcal{E}^{\mathrm{cpt}} \hookrightarrow \mathcal{E}$ is the full subcategory of compact objects). Furthermore, since for any map $s : B \to A$ the transition functor $s^* : \mathsf{X}(A) \to \mathsf{X}(B)$ is a left exact left adjoint, it follows that we may identify the transition map $s^* \otimes \mathcal{E} : (\mathsf{X} \otimes \mathcal{E})(A) \to (\mathsf{X} \otimes \mathcal{E})(B)$ with the functor

$$\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathsf{X}(A)) \to \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathsf{X}(B))$$

given by postcomposition with s^*. Now let $f_* : \mathsf{X} \to \mathsf{Y}$ be a geometric morphism of $\mathcal{B}$-topoi. Since f_* and f^* are both left exact if follows as in [30, Observation 2.9] that the induced morphism $f_* \otimes \mathcal{E}$ is given by pointwise postcomposition with f_* and its left adjoint is given by postcomposition with f^*.

We begin by establishing the $\mathcal{E}$-linear analogue of Corollary 7.2.5.6. This requires a few preparations:

Proposition 7.2.6.8. *Let $f_* : \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi and let $\mathcal{E}$ be a compactly generated ∞-category. Assume that there exists a family of commutative squares*

$$\begin{tikzcd} \mathcal{W}_i \arrow[r] \arrow[d, "g^i_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z}_i \arrow[r, "p^i_*"] & \mathcal{B} \end{tikzcd}$$

such that for every $A \in \mathcal{B}$ the functor $(- \times_{\mathcal{B}} \mathcal{B}_{/A}) \otimes \mathcal{E}$ carries these squares to a left adjointable square, each $(p^i)^* \otimes \mathcal{E}$ is conservative, and each g^i_* is $\mathcal{E}$-compact. Then f_* is $\mathcal{E}$-compact.

Proof. The proof is essentially the same as the one of Proposition 7.2.5.5. We first check that for every filtered $\mathcal{B}$-category I the mate of the commutative square

$$\begin{tikzcd}
f_*(\mathsf{Grpd}_{\mathcal{X}}) \otimes \mathcal{E} \arrow[r, "\mathrm{diag}"] \arrow[d, "f_* \otimes \mathcal{E}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*(\mathsf{Grpd}_{\mathcal{X}}) \otimes \mathcal{E}) \arrow[d, "(\Gamma_{f_*(\mathsf{Grpd}_{\mathcal{X}})} \otimes \mathcal{E})_*"] \\
\mathsf{Grpd}_{\mathcal{B}} \otimes \mathcal{E} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Grpd}_{\mathcal{B}} \otimes \mathcal{E})
\end{tikzcd}$$

is an equivalence. Let us first show that the mate

$$\operatorname*{colim}_{\mathsf{I}}(\Gamma_{f_*(\mathsf{Grpd}_{\mathcal{X}})} \otimes \mathcal{E})_* \to (f_* \otimes \mathcal{E}) \operatorname*{colim}_{\mathsf{I}}$$

is an equivalence after passing to global sections. For this, it suffices to see that the mate is an equivalence after composing with each $(p^i)^* \otimes \mathcal{E}$, which follows from an $\mathcal{E}$-linear version of Lemma 7.2.5.4 that can be shown in exactly the same way. To see that the mate is an equivalence after evaluating at $A \in \mathcal{B}$, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and the above square by its base change along π_A^* to reduce to the case treated above. Finally, we have to see that for every $A \in \mathcal{B}$ the functor of $\mathcal{B}_{/A}$-categories

$$\pi_A^*(f_*(\mathsf{Grpd}_{\mathcal{X}}) \otimes \mathcal{E}) \colon \pi_A^*(f_*(\mathsf{Grpd}_{\mathcal{X}}) \otimes \mathcal{E}) \to \pi_A^*(\mathsf{Grpd}_{\mathcal{B}} \otimes \mathcal{E})$$

commutes with colimits indexed by filtered $\mathcal{B}_{/A}$-categories. But this follows again from the above after replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$. □

Remark 7.2.6.9. Note that in Proposition 7.2.6.8, we require that the assumptions also hold locally on $\mathcal{B}$, while for the version without coefficients (Proposition 7.2.5.5) this was automatic by Lemma 7.2.5.3. To illustrate why Lemma 7.2.5.3 may fail when using coefficients, consider the example where $\mathcal{E} = \mathrm{Sub}(\mathrm{Ani}) \simeq \Delta^1$ is the ∞-category of (-1)-truncated spaces. Then, a square

$$\begin{tikzcd}
\mathcal{W}_i \arrow[r] \arrow[d, "(g^i)_*"] & \mathcal{Y} \arrow[d, "f_*"] \\
\mathcal{Z}_i \arrow[r, "p^i_*"] & \mathcal{B}
\end{tikzcd}$$

being horizontally left adjointable after tensoring with Sub(Ani) simply means that the mate transformation is an equivalence on (-1)-truncated objects in $\mathcal{Y}$.

However, after passing to a slice $\mathcal{X}_{/X}$, the mate transformations now involves (-1)-truncated objects in $\mathcal{X}_{/X}$, i.e. subobjects of X. These need not be (-1)-truncated in general, therefore there is no reason for the mate transformation to be an equivalence.

Remark 7.2.6.10. The proof of Proposition 7.2.6.8 shows that more generally we do not need the existence of such squares for every $A \in \mathcal{B}$, but it suffices to find these for a set of objects $A_i \in \mathcal{B}$ that generates $\mathcal{B}$ under colimits.

Lemma 7.2.6.11. *For any $\mathcal{B}$-category C and any geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$, there is a commutative square*

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{const}_{f_*(\mathrm{Grpd}_{\mathcal{X}})}"] & \mathrm{LFib}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "f^*"] \\ \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[r, "\simeq"] & \mathrm{LFib}_{\mathcal{X}}(f^*\mathsf{C}) \end{tikzcd}$$

where f^* is the restriction of $f^* \colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}} \to \mathrm{Cat}(\mathcal{X})_{/f^*\mathsf{C}}$ to left fibrations and the horizontal equivalences are induced by the straightening equivalences for left fibrations (internal to both $\mathcal{B}$ and $\mathcal{X}$). Moreover, this commutative square is natural in C.

Proof. This is shown in exactly the same fashion as Lemma 6.2.8.5. □

Corollary 7.2.6.12. *Let $f_* \colon \mathcal{X} \to \mathcal{B}$ be an $\mathcal{E}$-compact geometric morphism and* C *a $\mathcal{B}$-category. Then the geometric morphism*

$$(\Gamma_{f_*(\mathrm{Grpd}_{\mathcal{X}})})_* \colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*(\mathrm{Grpd}_{\mathcal{X}})) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})$$

is $\mathcal{E}$-compact.

Proof. Pick any $F \in \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})$ and let $G = \mathrm{const}_{f_*(\mathrm{Grpd}_{\mathcal{X}})} F$. Furthermore, let $\mathsf{C}_{F/} \to \mathsf{C}$ be the left fibration associated to F via the straightening equivalence.

We then deduce from Lemma 7.2.6.11 that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})_{/F} \arrow[r, "(\mathrm{const}_{f_*(\mathrm{Grpd}_{\mathcal{X}})})_*"] \arrow[d, "\simeq"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})_{/G} \arrow[d, "\simeq"] \\
\mathrm{LFib}_{\mathcal{B}}(\mathsf{C})_{/(\mathsf{C}_{F/})} \arrow[r, "f^*"] \arrow[d, "\simeq"] & \mathrm{LFib}_{\mathcal{X}}(f^*\mathsf{C})_{/f^*(\mathsf{C}_{F/})} \arrow[d, "\simeq"] \\
\mathrm{LFib}_{\mathcal{B}}(\mathsf{C}_{F/}) \arrow[r, "f^*"] \arrow[d, "\simeq"] & \mathrm{LFib}_{\mathcal{X}}(f^*(\mathsf{C}_{F/})) \arrow[d, "\simeq"] \\
\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, \mathrm{Grpd}_{\mathcal{B}}) \arrow[r, "(\mathrm{const}_{f_*(\mathrm{Grpd}_{\mathcal{X}})})_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, f_*(\mathrm{Grpd}_{\mathcal{X}}))
\end{tikzcd}$$

which is natural in C. Thus, by passing to right adjoints, the base change of $(\Gamma_{f_*(\mathrm{Grpd}_{\mathcal{X}})})_*$ along the geometric morphism

$$(\pi_F)_* \colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})_{/F} \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})$$

can be identified with the geometric morphism

$$(\Gamma_{f_*(\mathrm{Grpd}_{\mathcal{X}})})_* \colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, f_*(\mathrm{Grpd}_{\mathcal{X}})) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, \mathrm{Grpd}_{\mathcal{B}}).$$

Also, the base change of the right adjoint of the restriction functor

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\simeq}, \mathrm{Grpd}_{\mathcal{B}}) \simeq \mathcal{B}_{/\mathsf{C}^{\simeq}}$$

along $(\pi_A)_*$ can be identified with the right adjoint of the restriction functor

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, \mathrm{Grpd}_{\mathcal{B}}) \to \mathcal{B}_{/(\mathsf{C}_{F/})^{\simeq}}$$

(using that the pullback of $\mathsf{C}_{F/} \to \mathsf{C}$ along $\mathsf{C}^{\simeq} \to \mathsf{C}$ is $(\mathsf{C}_{F/})^{\simeq}$, see Corollary 1.3.3.5). Consequently, we conclude that the pullback square

$$\begin{tikzcd}
\mathcal{X}_{/f^*\mathsf{C}^{\simeq}} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[d] \\
\mathcal{B}_{/\mathsf{C}^{\simeq}} \arrow[r] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathrm{Grpd}_{\mathcal{B}})
\end{tikzcd}$$

satisfies the assumptions of Proposition 7.2.6.8. Thus the claim follows. □

Proof of Theorem 7.2.6.5: By Remark 7.2.6.3, the same proof as in Lemma 7.2.5.2 shows that an $\mathcal{E}$-proper morphism is $\mathcal{E}$-compact. Hence it remains to prove the converse. By Corollary 7.2.6.12, the same reduction steps as in the proof of Theorem 7.2.5.1 imply that it suffices to see that for every pullback square of $\mathcal{B}$-topoi

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{Z} \arrow[r, "j_*"] & \mathcal{B} \end{tikzcd}$$

in which j_* is fully faithful, the square is left adjointable after tensoring with $\mathcal{E}$. By Remark 7.2.6.7, it suffices to see that the square

$$\begin{tikzcd} \operatorname{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \arrow[r, "(j'_*(j')^*)_*"] \arrow[d, "(p_*)_*"'] & \operatorname{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \arrow[d, "(p_*)_*"] \\ \operatorname{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \arrow[r, "(j_*j^*)_*"] & \operatorname{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \end{tikzcd}$$

commutes. We pick a local class S of maps in $\mathcal{B}$ as in Proposition 7.2.4.3, so that we obtain equivalences $j'_*(j')^* \simeq (-)^{\mathrm{sh}}_{\iota'}$ and $j_*j^* \simeq (-)^{\mathrm{sh}}_{\iota}$. Now since the two inclusions

$$\operatorname{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \hookrightarrow \operatorname{Fun}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X})$$

and

$$\operatorname{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \hookrightarrow \operatorname{Fun}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})$$

both preserve filtered colimits and since colimits in functor ∞-categories are computed object-wise, it follows that $(j'_*(j')^*)_*$ and $(j_*j^*)_*$ are given by the κ-fold iteration of postcomposition with the functors $(-)^+_{\iota'}$ and $(-)^+_{\iota}$, respectively. Therefore, it suffices to provide an equivalence $(p_*(-)^+_{\iota'})_* \simeq ((-)^+_{\iota} p_*)_*$.

To obtain such an equivalence, note that Remark 7.2.6.3 implies that we may identify the map $\operatorname{colim}_{\mathrm{Grpd}_S^{\mathrm{op}}} \otimes \mathcal{E}$ with

$$\operatorname*{colim}_{\mathrm{Grpd}_S^{\mathrm{op}}} \colon \underline{\operatorname{Fun}}_{\mathcal{B}}(\mathrm{Grpd}_S^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}} \otimes \mathcal{E}) \to \mathrm{Grpd}_{\mathcal{B}} \otimes \mathcal{E}.$$

Therefore, we deduce that postcomposition with $(-)^+_{\iota}$ is equivalently given by the composition

$$\begin{aligned}\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \to \mathrm{Fun}^{\mathrm{lex}}&((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{PSh}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}))\\ &\simeq \mathrm{Fun}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}, \mathrm{Grpd}_{\mathcal{B}} \otimes \mathcal{E})\\ &\xrightarrow{\mathrm{colim}} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})\end{aligned}$$

in which the first functor is given by postcomposition with (the global sections of)

$$\mathrm{map}_{\mathrm{Grpd}_{\mathcal{B}}}(\iota(-), -)\colon\ \mathrm{Grpd}_{\mathcal{B}} \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}).$$

Similarly, postcomposition with $(-)^{+}_{\iota'}$ can be identified with the composition

$$\begin{aligned}\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}&((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{Fun}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}})))\\ &\simeq \mathrm{Fun}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}}) \otimes \mathcal{E})\\ &\xrightarrow{\mathrm{colim}} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X})\end{aligned}$$

where the first functor is given by postcomposition with (the global sections of)

$$\underline{\mathrm{Hom}}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\,\iota(-), -)\colon\ p_*(\mathrm{Grpd}_{\mathcal{X}}) \to \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}}))$$

since this is precisely the map we obtain when composing the global sections of

$$\mathrm{map}_{\mathrm{Grpd}_{\mathcal{X}}}(\iota'(-), -)\colon\ \mathrm{Grpd}_{\mathcal{X}} \to \underline{\mathrm{PSh}}_{\mathcal{X}}(p^*(\mathrm{Grpd}_{\mathcal{S}}))$$

with the equivalence

$$\mathrm{PSh}_{\mathcal{X}}(p^*(\mathrm{Grpd}_{\mathcal{S}})) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}})).$$

Thus, since $\Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})} \otimes \mathcal{E}$ commutes with $\mathrm{colim}_{\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}}$, it is enough to provide a commutative diagram

$$\begin{tikzcd}\mathcal{X} \arrow[r, "\underline{\mathrm{Hom}}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}(\mathrm{const}_{p_*(\mathrm{Grpd}_{\mathcal{X}})}\,\iota(-),-)"] \arrow[d, "p_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}^{\mathrm{op}}, p_*(\mathrm{Grpd}_{\mathcal{X}})) \arrow[d, "(\Gamma_{p_*(\mathrm{Grpd}_{\mathcal{X}})})_*"] \\ \mathcal{B} \arrow[r, "\mathrm{map}_{\mathrm{Grpd}_{\mathcal{X}}}(\iota(-),-)"] & \mathrm{PSh}_{\mathcal{B}}(\mathrm{Grpd}_{\mathcal{S}}),\end{tikzcd}$$

which is evident from Remark 6.2.10.13. □

Example 7.2.6.13. For a scheme X, let us denote by $X_{\text{ét}}^{\text{hyp}}$ the ∞-topos of étale hypersheaves of spaces on X. If $f\colon X \to S$ is a proper morphism of schemes, then the geometric morphism $f_* : X_{\text{ét}}^{\text{hyp}} \to S_{\text{ét}}^{\text{hyp}}$ is $\mathrm{D}(R)$-proper for any torsion ring R. In fact, since $X_{\text{ét}}^{\text{hyp}}$ has enough points by [51, Theorem A.4.0.5], the family of all points $\bar{s}_* : \mathcal{S} \to X_{\text{ét}}^{\text{hyp}}$ yields a family of jointly conservative functors $\bar{s}^* \otimes \mathrm{D}(R)$. Furthermore, proper base change for unbounded derived categories of étale sheaves (see [19, Theorem 1.2.1]) implies that the squares

$$\begin{tikzcd}
X_{\bar{s},\text{ét}}^{\text{hyp}} \arrow[r] \arrow[d] & X_{\text{ét}}^{\text{hyp}} \arrow[d] \\
\mathcal{S} \arrow[r] & S_{\text{ét}}^{\text{hyp}}
\end{tikzcd}$$

are left adjointable after applying $- \otimes \mathrm{D}(R)$. Finally, [19, Corollary 1.1.15] implies that $X_{\bar{s},\text{ét}}^{\text{hyp}}$ is $\mathrm{D}(R)$-compact, so that we may apply Proposition 7.2.6.8 and Theorem 7.2.6.5 to conclude that f_* is $\mathrm{D}(R)$-proper.

Definition 7.2.6.14. We call a geometric morphism $f_* : \mathcal{Y} \to \mathcal{X}$ *n-proper* if it is $\mathcal{S}_{\leq n}$-proper, where $\mathrm{Ani}_{\leq n}$ denotes the ∞-category of n-truncated spaces. We call f_* *almost proper* if it is n-proper for all n.

Example 7.2.6.15. Recall that by [50, Example 4.8.1.22] one has $\mathcal{X} \otimes \mathcal{S}_{\leq n} \simeq \mathcal{X}_{\leq n}$. Thus it follows from [51, Proposition A.2.3.1] and Theorem 7.2.6.5 that for an n-coherent ∞-topos $\mathcal{X}$ the geometric morphism $\Gamma_* : \mathcal{X} \to \mathcal{S}$ is n-proper. In particular it is almost proper if $\mathcal{X}$ is coherent. However it is not proper in general (see Remark 7.2.1.5).

Example 7.2.6.16. A geometric morphism $f_* : \mathcal{X} \to \mathcal{B}$ is Set-proper if and only if the underlying morphism of 1-topoi is tidy in the sense of [61, § 3].

7.3. Smooth and proper maps in topology

In this section, we will apply our classification of smooth and proper geometric morphisms to obtain examples of such maps coming from topology. In Section 7.3.1, we show that *shape submersions* are a class of continuous maps for which the associated geometric morphisms of sheaf ∞-topoi is smooth. In Section 7.3.2, we show that *proper and separated* continuous maps induce proper geometric morphisms between their associated sheaf ∞-topoi.

7.3.1. Shape submersions

In this section, we show that every *shape submersion* of topological spaces induces a smooth geometric morphism between their associated sheaf ∞-topoi. Barring this result, we make no claim to originality in this section. All ideas are adopted from [82].

Definition 7.3.1.1. A topological space X is called *essential* if $\mathrm{Sh}(X)$ is locally contractible.

Example 7.3.1.2 ([82, Corollary 3.19])**.** Any topological space X that is hypercomplete and locally contractible (meaning it admits a basis of contractible open subsets) is essential. If the basis of contractible open subsets can be chosen to be closed under finite intersections (i.e. if X admits a *good cover*), one can moreover omit the hypercompleteness assumption. In fact, by [7, Corollaries A.7 and A.8] these assumptions guarantee that the inclusion of the poset of contractible open subsets $\mathrm{Open}^{\mathrm{contr}}(X) \subset \mathrm{Open}(X)$ into the poset of all open subsets of X gives rise to an equivalence

$$\mathrm{Sh}(\mathrm{Open}^{\mathrm{contr}}(X)) \simeq \mathrm{Sh}(X),$$

so that in particular every sheaf on X can be written as a colimit of contractible open subsets of X. Consequently, Proposition 7.1.2.5 implies the claim once we verify that every contractible open subset of X defines a contractible object in $\mathrm{Sh}(X)$. Unwinding the definitions, this precisely means that whenever U is a contractible topological space, the functor $\mathrm{const}_{\mathrm{Sh}(U)} \colon \mathrm{Ani} \to \mathrm{Sh}(U)$ is fully faithful, which follows from [50, Remark A.4.7].

Definition 7.3.1.3. A continuous map $f \colon Y \to X$ of topological spaces is a *shape submersion* if there is an open cover $Y = \bigcup_i V_i$ and for each i an essential space Y_i

such that $f(V_i)$ is open and we have commutative diagram

$$\begin{tikzcd} f(V_i)\times Y_i \arrow[r, "\simeq"] \arrow[dr, "\mathrm{pr}_0"'] & V_i \arrow[r, hook] \arrow[d, "f|_{V_i}"] & Y \arrow[d, "f"] \\ & f(V_i) \arrow[r, hook] & X. \end{tikzcd}$$

Example 7.3.1.4 ([82, Example 3.22])**.** Every *topological submersion* is a shape submersion.

Theorem 7.3.1.5. *If $f\colon Y \to X$ is a shape submersion, then $f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is smooth.*

Proof. In light of Remark 7.1.1.3, we deduce from [82, Corollary 3.26] that f_* is locally contractible, hence the result follows from Theorem 7.1.3.1. □

7.3.2. Proper and separated maps

Recall that a map $p: Y \to X$ of topological spaces is called *proper* if it is universally closed (i.e. if every pullback of p is a closed map). In [49, Theorem 7.3.1.6], Lurie shows that every such proper map in which Y is moreover assumed to be *completely regular* (i.e. a subspace of a compact Hausdorff space) induces a proper morphism between the associated sheaf ∞-topoi. Since (toposic) properness is an entirely relative notion, it is somewhat surprising that there are constraints on the space Y and not just on the map p. Instead, by relativising the fact that the ∞-topos of sheaves on a compact Hausdorff space is *compact* [49, Corollary 7.3.4.12], one would expect that every proper and *separated* map $p: Y \to X$ of topological spaces gives rise to a proper morphism of ∞-topoi (where p being separated means that the diagonal $Y \to Y \times_X Y$ is closed). In this section, our goal is to show that this is indeed the case:

Theorem 7.3.2.1. *Let $p: Y \to X$ be a proper and separated map of topological spaces. Then the induced geometric morphism $p_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is proper.*

Remark 7.3.2.2. A continuous map $p: Y \to X$ is separated as soon as Y is Hausdorff. Since any completely regular topological space is Hausdorff, it follows that Theorem 7.3.2.1 subsumes [49, Theorem 7.3.1.6].

Example 7.3.2.3. It follows from [39, Example C.3.4.1] that the separatedness assumption in Theorem 7.3.2.1 cannot be dropped. We briefly recall the example for the convenience of the reader. Consider the topological space Y that is given by taking two copies of the interval $[0, 1]$ and identifying both copies of x for $0 < x < 1$. Then Y is compact, but $\mathrm{Sh}(Y)$ is not. Indeed, consider the sequence that takes $n \in \mathbb{N}$ to the sheaf represented by the map $Y_n \to Y$ in which Y_n is given by two copies of $[0, 1]$ where we identify both copies of x for $2^{-n} < x < 1 - 2^{-n}$. We note that all the maps $Y_n \to Y_{n+1}$ and $Y_n \to Y$ are local homeomorphisms, which implies that the colimit of the sheaves represented by $(Y_n \to Y)_{n\in\mathbb{N}}$ is the sheaf represented by $\operatorname{colim}_n Y_n = Y$. In particular, we have $\Gamma_Y(\operatorname{colim}_n Y_n) = 1$, but since $\operatorname{colim}_n \Gamma_Y(Y_n) = \varnothing$, the global sections functor Γ_Y does not commute with filtered colimits.

Before we prove Theorem 7.3.2.1, let us record that it implies the proper base change theorem in topology, at least for sober spaces:

Corollary 7.3.2.4. *For every pullback square*

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "q"] & P \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

of sober topological spaces in which p is proper and separated, the induced commutative square

$$\begin{tikzcd} \mathrm{Sh}(Q) \arrow[r, "g_*"] \arrow[d, "q_*"] & \mathrm{Sh}(P) \arrow[d, "p_*"] \\ \mathrm{Sh}(Y) \arrow[r, "f_*"] & \mathrm{Sh}(X) \end{tikzcd}$$

is horizontally left adjointable.

Proof. Using Theorem 7.3.2.1, it suffices to show that the second square is a pullback in $\mathrm{Top}^{\mathrm{R}}_{\infty}$, or equivalently that the underlying square of locales is a pullback. The latter fact follows from combining [40, Corollary 3.6] with [40, Lemmas 2.1]. □

We now move on to the proof of Theorem 7.3.2.1. First, let us give an informal outline of our strategy. We begin with the special case where X is the point, so that Y is a compact Hausdorff space. Let $\mathrm{Open}(Y)$ be the associated locale of open subsets in Y. It is then a classical fact that Y is a retract (in the category of locales) of a *coherent locale* (see for example [39, § C4.1]). In fact, recall that an *ideal* in $\mathrm{Open}(Y)$ is a full subposet which is downward closed (i.e. a sieve) and that is closed under finite unions. Then the poset of ideals $\mathrm{Id}(\mathrm{Open}(Y))$ is a coherent locale: its quasi-compact objects are precisely the principal ideals generated by the opens in Y (i.e. those of the form $\{U \in \mathrm{Open}(Y) \mid U \subset U_0\}$ for some $U_0 \subset Y$), which are clearly closed under finite meets and generate $\mathrm{Id}(\mathrm{Open}(Y))$ under arbitrary joins. Using that Y is compact Hausdorff and therefore in particular locally compact, the canonical map $\mathrm{Id}(\mathrm{Open}(Y)) \to \mathrm{Open}(Y)$ (which takes an ideal to its union) admits a section sending $U \subset Y$ to the ideal $\{V \in \mathrm{Open}(Y) \mid \overline{V} \subset U\}$. Moreover, both the map $\mathrm{Id}(\mathrm{Open}(Y)) \to \mathrm{Open}(Y)$ and its section define morphisms in the category of locales. Hence, one obtains that $\mathrm{Open}(Y)$ is a retract of a coherent locale. Consequently, the ∞-topos $\mathrm{Sh}(Y)$ is a retract of the ∞-topos of sheaves on a coherent locale, which by [49, Corollary 7.3.5.4] implies that $\mathrm{Sh}(Y)$ is a retract of a compact ∞-topos and therefore compact as well. Finally, by applying Theorem 7.2.5.1, the result follows (see also Remark 7.3.2.5.)

We will prove the general case in exactly the same way. The only difference is that all steps now have to be carried out internally in $\mathrm{Sh}(X)$. More specifically, if now $p\colon Y \to X$ is a proper and separated map of topological spaces, we obtain a $\mathrm{Sh}(X)$-locale $\mathrm{Open}_X(Y)$ by means of the sheaf $U \mapsto \mathrm{Open}(p^*(U))$ on X. Recall from Remark 6.3.3.5 that such a $\mathrm{Sh}(\mathcal{X})$-locale is equivalently an internal locale in the 1-topos $\mathrm{Sh}_{\mathrm{Set}}(X)$. Now by a result of Johnstone [40], the above proof that the locale of opens on a compact Hausdorff space is a retract of a coherent locale can be interpreted internally in any 1-topos. Consequently, one obtains that $\mathrm{Open}_X(Y)$ is a retract (in the category of internal locales in $\mathrm{Sh}_{\mathrm{Set}}(X)$) of a *coherent* internal locale. Thus, by using (1) that we can functorially assign to each internal locale L in $\mathrm{Sh}_{\mathrm{Set}}(X)$ a $\mathrm{Sh}(X)$-topos $\underline{\mathrm{Sh}}_{\mathrm{Sh}(X)}(\mathsf{L})$, and that (2) this assignment carries coherent internal locales to compact $\mathrm{Sh}(X)$-topoi, we can derive the desired result from Theorem 7.2.5.1. As we have already developed all the necessary ingredients in previous sections, the proof is now remarkably short:

Proof of Theorem 7.3.2.1. Let $p^*\colon \mathrm{Open}(X) \to \mathrm{Open}(Y)$ be the algebraic morphism of locales that carries $U \subset X$ to $p^{-1}(U) \subset Y$. By Proposition 6.3.6.1, this map corresponds to a $\mathrm{Sh}(X)$-locale $\mathsf{Open}_X(Y)$ that is explicitly given by the sheaf $U \mapsto \mathrm{Open}(p^{-1}(U))$ (see Remark 6.3.6.3). In light of Remark 6.3.3.5, we can equivalently regard $\mathrm{Sh}(X)$-locales as internal locales in the 1-topos $\mathrm{Sh}_{\mathrm{Set}}(X)$. Therefore, we deduce from [40, Proposition 1.2 and Lemma 2.1] that $\mathsf{Open}_X(Y)$ is a *stably compact* $\mathrm{Sh}(X)$-locale (in the sense of Definition 7.2.2.1), i.e. a retract of a coherent $\mathrm{Sh}(X)$-locale in $\mathrm{Loc}^{\mathrm{L}}(\mathrm{Sh}(X))$. By Corollary 7.2.2.4, this implies that the $\mathrm{Sh}(X)$-topos $\underline{\mathrm{Sh}}_{\mathrm{Sh}(X)}(\mathsf{Open}_X(Y))$ is compact. As this $\mathrm{Sh}(X)$-topos recovers the geometric morphism $p_* \colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ when passing to global sections (Corollary 6.3.6.2), the claim now follows from Theorem 7.2.5.1. □

Remark 7.3.2.5. If one assumes that the topological space Y is *completely regular* (see [49, Definition 7.3.1.12]), one can alternatively apply a number of geometric reduction steps, as in the proof of [49, Theorem 7.3.16], to reduce to the case where $X = *$ and then use that any compact Hausdorff space is a retract of a coherent topological space, as outlined above. The author learned about this proof strategy for Theorem 7.3.2.1 from Ko Aoki. In comparison, Lurie shows that $\mathrm{Sh}(Y)$ is compact in [49, Corollary 7.3.4.12] by using the theory of $\mathcal{K}$-sheaves.

Remark 7.3.2.6. If $p \colon Y \to X$ is only assumed to be *locally* proper (see Definition 7.3.2.7 below for a precise definition), the same argumentation as in the proof of Theorem 7.3.2.1 shows that the $\mathrm{Sh}(X)$-topos $\underline{\mathrm{Sh}}_{\mathrm{Sh}(X)}(\mathsf{Open}_X(Y))$ is *compactly assembled*. Therefore, by suitably internalising the arguments in [51, § 21.1.6] (or alternatively those in [4]), one can deduce that p_* is *exponentiable* (i.e. that $- \times_{\mathrm{Sh}(X)} \mathrm{Sh}(Y) \colon \mathrm{Top}^{\mathrm{R}}_{\infty} \to \mathrm{Top}^{\mathrm{R}}_{\infty}$ has a right adjoint). Moreover, this result implies that the stable ∞-category $\mathrm{Sh}_{\mathrm{Sp}}(Y)$ of spectral sheaves on Y is a dualisable $\mathrm{Sh}_{\mathrm{Sp}}(X)$-module.

For the convenience of the reader, we will use the remainder of this section to provide a self-contained proof of Johnstone's result that every proper and separated map $Y \to X$ of topological spaces gives rise to a stably compact $\mathrm{Sh}(X)$-locale $\mathsf{Open}_X(Y)$. With future applications in mind, we will prove a slightly more general statement about *locally* proper and separated maps of topological spaces (which Johnstone also mentions in [39, § C4.1] but never explicitly spells out).

We begin by recalling the definition of a locally proper map from [74]:

Definition 7.3.2.7. A continuous map $f\colon Y \to X$ of topological spaces is said to be *locally proper* if for every $y \in Y$ and every open neighbourhood V of y there is a neighbourhood $K \subset V$ of y and an open neighbourhood U of $f(y)$ such that $f(K) \subset U$ and such that the induced map $K \to U$ is proper (i.e. universally closed).

Remark 7.3.2.8. The property of a map $f\colon Y \to X$ being locally proper and separated is local in the target: if $X = \bigcup_i U_i$ is an open covering, then f is locally proper and separated if and only if each of the restrictions $f^{-1}(U_i) \to U_i$ has that property [74, Lemma 2.7].

Remark 7.3.2.9. Every proper and separated morphism is also locally proper [74, Proposition 2.12]. This is the relative version of the fact that compact Hausdorff spaces are locally compact as well.

Remark 7.3.2.10. In the situation of Definition 7.3.2.7, if f is separated and locally proper, then for every $y \in Y$ and every open neighbourhood V of y there is an open neighbourhood $V' \subset V$ and an open neighbourhood U of $f(y)$ such that $f(V') \subset U$ and such that the closure of V' in $f^{-1}(U)$ is proper over U. In fact, f being separated implies that its restriction $f^{-1}(U) \to U$ is separated as well. Therefore, [74, Lemma 9.12] implies that if $K \subset V$ is as in Definition 7.3.2.7, then K is closed in $f^{-1}(U)$. Hence the closure of the interior of K (again in $f^{-1}(U)$) is a closed subset of K and therefore also proper over U.

To proceed, recall that if $f\colon Y \to X$ is a map of topological spaces, we obtain an algebraic morphism of locales $f^*\colon \mathrm{Open}(X) \to \mathrm{Open}(Y)$, where $\mathrm{Open}(X)$ and $\mathrm{Open}(Y)$ denote the locales of open subsets of X and Y, respectively. By Proposition 6.3.6.1, f^* gives rise to a $\mathrm{Sh}(X)$-locale $\mathsf{Open}_X(Y)$ that is explicitly given by the sheaf on X that carries an open $U \in \mathrm{Open}(X)$ to the locale $\mathrm{Open}(f^{-1}(U))$ (see Remark 6.3.6.3). Recall, furthermore, that we refer to a $\mathcal{B}$-locale L as (locally) stably compact if it arises as a retract in $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$ of a (locally) coherent $\mathcal{B}$-locale (see Definition 7.2.2.1). Our goal is to show:

Proposition 7.3.2.11. *If $f\colon Y \to X$ is a locally proper and separated morphism of topological spaces, then $\mathsf{Open}_X(Y)$ is a locally stably compact $\mathrm{Sh}(X)$-locale. If f is even proper, then $\mathsf{Open}_X(Y)$ is stably compact.*

The proof of Proposition 7.3.2.11 requires a few preparations first. To begin with, we need to construct a candidate for a (locally) coherent $\mathrm{Sh}(X)$-locale of which $\mathrm{Open}_X(Y)$ is a retract. We will use the following general observation:

Proposition 7.3.2.12. *Let* L *be a* $\mathcal{B}$*-locale and let* $j\colon \mathsf{P} \hookrightarrow \mathsf{L}$ *be a full subposet that is closed under binary products and finite colimits. Then*

1. *the left Kan extension* $h_!(j)\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \to \mathsf{L}$ *is cocontinuous;*
2. $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ *is a locally coherent* $\mathcal{B}$*-locale which is coherent if* P *contains the final object of* L*;*
3. *a right fibration over* P *(in arbitrary context* $A \in \mathcal{B}$*) is contained in the essential image of the inclusion* $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \hookrightarrow \mathsf{RFib}_{\mathsf{P}}$ *if and only if it is the inclusion of a sieve in* $\pi_A^*\mathsf{P}$ *(i.e. a fully faithful right fibration) that is closed under finite colimits.*

The proof of Proposition 7.3.2.12 requires the following lemma:

Lemma 7.3.2.13. *Let* C *be a* $\mathcal{B}$*-poset with finite colimits and let* $p\colon \mathsf{P} \hookrightarrow \mathsf{C}$ *be a sieve (i.e. a fully faithful right fibration). Then* P *is filtered if and only if it is closed under finite colimits in* C*.*

Proof. It will be sufficient to show that whenever $d\colon \mathsf{K} \to \mathsf{P}$ is a finite diagram, then $\mathsf{P}_{d/}$ admits an initial object which is carried to the initial object in $\mathsf{C}_{pd/}$ along the induced functor $p_*\colon \mathsf{P}_{d/} \to \mathsf{C}_{pd/}$. Note that $p\colon \mathsf{P} \to \mathsf{C}$ being a sieve implies that p_* is one as well. Now since C has finite colimits, $\mathsf{C}_{pd/}$ admits an initial object $\mathrm{colim}(pd)$. Since p_* is a right fibration, the inclusion $\mathsf{P}_{d/}|_{\mathrm{colim}(pd)} \hookrightarrow \mathsf{P}_{d/}$ of p_* over $\mathrm{colim}(pd)$ is initial (cf. Proposition 2.1.4.9). Since P is assumed to be filtered, we furthermore have $(\mathsf{P}_{d/})^{\mathrm{gpd}} \simeq 1$. Therefore, we must have $\mathsf{P}_{d/}|_{\mathrm{colim}(pd)} \simeq 1$ as this is already a subterminal $\mathcal{B}$-groupoid (since p is fully faithful, see Example 6.3.2.4). Hence $\mathsf{P}_{d/}$ admits an initial object which is preserved by p_*. □

Proof of Proposition 7.3.2.12. The fact that P has finite colimits implies that the $\mathcal{B}$-category $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ is presentable and that $h_!(j)\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \to \mathsf{L}$ is cocontinuous (Corollary 5.4.5.6), which shows (1).

To show (2), since $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ is by definition compactly generated and since we may always identify $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})^{\mathrm{cpt}} \simeq \mathsf{P}$ (as P is a $\mathcal{B}$-poset), we only need to verify that $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ is indeed a $\mathcal{B}$-locale. To that end, note that $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ being presentable implies that the inclusion $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P}) \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})$ admits a left adjoint $l\colon \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P}) \to \underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ (see Corollary 3.5.1.13). Moreover, Lemma 7.2.2.8 implies that the inclusion $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{P}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}}) \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})$ preserves filtered colimits. Therefore, $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ must be contained in $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathsf{P}^{\mathrm{op}}, \mathrm{Sub}_{\mathcal{B}})$ and is therefore in particular a $\mathcal{B}$-poset. Hence, we only need to check that l preserves binary products (see Lemma 6.3.3.6). This is equivalent to $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ being an exponential ideal in $\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})$, i.e. that for every object $F\colon A \to \underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ and every object $G\colon A \to \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})$ (in arbitrary context $A \in \mathcal{B}$), the internal hom $\underline{\mathrm{Hom}}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})}(G, F)$ is contained in $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$. By using Proposition 3.5.1.9, we can assume that $A \simeq 1$. Upon writing G as a colimit of representables and using that the inclusion $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P}) \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})$ is continuous, we may assume without loss of generality that G is itself representable by an object $U\colon 1 \to \mathsf{P}$. Thus, Yoneda's lemma and the fact that the Yoneda embedding is continuous imply that $\underline{\mathrm{Hom}}_{\underline{\mathrm{PSh}}_{\mathcal{B}}(\mathsf{P})}(G, F)$ can be identified with the presheaf $F(U \times -)$. Note that by Proposition 5.4.5.5 a presheaf is contained in $\underline{\mathrm{Ind}}_{\mathcal{B}}(\mathsf{P})$ if and only if it carries finite colimits in P to limits. Thus, as F by assumption has this property and since colimits are universal in L, the claim follows.

Lastly, in light of Example 6.3.2.4 and Remark 5.3.3.5, statement (3) is an immediate consequence of Lemma 7.3.2.13. □

In light of Proposition 7.3.2.12, our task is now to find a full subposet of $\mathrm{Open}_X(Y)$ that is closed under binary products and finite colimits. To that end, note that the datum of an object in $\mathrm{Open}_X(Y)$ in context $U \subset X$ is precisely given by an open subset $V \subset f^{-1}(U)$. With that in mind, we may now define:

Definition 7.3.2.14. Let $f\colon Y \to X$ be a locally proper and separated map of topological spaces. We say that an object $V \subset f^{-1}(U)$ has *proper closure* if its closure $\overline{V}$ in $f^{-1}(U)$ is proper over U. We define the subposet $\mathrm{Open}_X^{\mathrm{pc}}(Y) \hookrightarrow \mathrm{Open}_X(Y)$ as the full subposet of $\mathrm{Open}_X(Y)$ that is spanned by these objects.

Remark 7.3.2.15. In the situation of Definition 7.3.2.14, note that f being separated implies that if $\overline{V}$ is proper over U, then $\overline{V}$ is also closed in Y (see [74, Lemma 9.12]). Therefore, $\overline{V}$ is also the closure of V in Y in this case.

A priori, the subposet $\mathrm{Open}_X^{\mathrm{pc}}(Y)$ is only *spanned* by the objects with proper closure, so there could potentially be more objects. Our next result shows that this cannot happen:

Lemma 7.3.2.16. *An object $V \subset f^{-1}(U)$ in $\mathrm{Open}_X(Y)$ is contained in $\mathrm{Open}_X^{\mathrm{pc}}(Y)$ if and only if it has proper closure.*

Proof. By definition, the condition is sufficient, so it suffices to prove that it is also necessary. This amounts to showing that the property of having proper closure is *local* on the target: if $U = \bigcup_i U_i$ is a covering and if $\overline{V \cap f^{-1}(U_i)} \subset f^{-1}(U_i)$ is proper over U_i, then $\overline{V} \subset f^{-1}(U)$ is proper over U. Since properness is local on the target [74, § 9.5], this follows from the identity $\overline{V \cap f^{-1}(U_i)} = \overline{V} \cap f^{-1}(U_i)$. □

Remark 7.3.2.17. Note that if $U \subset X$ is an arbitrary open subset, we may identify $\mathrm{Sh}(X)_{/U}$ with $\mathrm{Sh}(U)$. In light of this identification, the $\mathrm{Sh}(U)$-locale $\pi_U^* \mathrm{Open}_X(Y)$ can be identified with $\mathrm{Open}_U(f^{-1}(U))$. Moreover, Lemma 7.3.2.16 implies that we obtain a canonical equivalence $\pi_U^* \mathrm{Open}_X^{\mathrm{pc}}(Y) \simeq \mathrm{Open}_U^{\mathrm{pc}}(f^{-1}(U))$ of full subposets in $\mathrm{Open}_U(f^{-1}(U))$ (see also Remark 7.3.2.8).

Having an explicit description of the full subposet $\mathrm{Open}_X^{\mathrm{pc}}(Y) \hookrightarrow \mathrm{Open}_X(Y)$, we now proceed by showing that it satisfies the conditions of Proposition 7.3.2.12:

Lemma 7.3.2.18. $\mathrm{Open}_X^{\mathrm{pc}}(Y)$ *is closed under binary products and finite colimits in* $\mathrm{Open}_X(Y)$.

Proof. Since the map $\emptyset \to X$ is always proper, Lemma 7.3.2.16 implies that it is enough to show that for every two objects $V \subset f^{-1}(U)$ and $V' \subset f^{-1}(U)$ whose closure (in $f^{-1}(U)$) is proper over U, both $\overline{V \cup V'} \to U$ and $\overline{V \cap V'} \to U$ are proper. The first map is proper by [74, § 9.7] and the fact that union and closure commute. The second map is proper as it can be decomposed into the composition $\overline{V \cap V'} \hookrightarrow \overline{V} \to U$ where the first map is a closed embedding (hence proper) and the second map is proper by assumption. □

Proposition 7.3.2.19. *The* $\mathrm{Sh}(X)$*-category* $\underline{\mathrm{Ind}}_{\mathrm{Sh}(X)}(\mathrm{Open}_X^{\mathrm{pc}}(Y))$ *is a locally coherent* $\mathrm{Sh}(X)$*-locale, and the left Kan extension* $\underline{\mathrm{Ind}}_{\mathrm{Sh}(X)}(\mathrm{Open}_X^{\mathrm{pc}}(Y)) \to \mathrm{Open}_X(Y)$ *of the inclusion is a Bousfield localisation.*

Proof. In light of Lemma 7.3.2.18, the first claim follows from Proposition 7.3.2.12, so that it suffices to show the second one. We need to prove that the counit of the adjunction

$$\mathsf{Open}_X(Y) \rightleftarrows \underline{\mathsf{Ind}}_{\mathrm{Sh}(X)}(\mathsf{Open}_X^{\mathrm{pc}}(Y))$$

is an equivalence. By making use of Remark 7.3.2.17 and Remark 5.3.1.2, it will be enough to check this on a global object $V \subset Y$. By Remark 3.4.3.6, this amounts to showing that V is the colimit of the diagram $\mathsf{Open}_X^{\mathrm{pc}}(Y)_{/V} \to \mathsf{Open}_X(Y)$. Using Proposition 6.3.2.10, we only need to verify that $V \simeq \bigcup_{V' \subset f^{-1}(U) \cap V} V'$, where U runs though all open subsets of X and V' runs through all objects in $\mathsf{Open}_X^{\mathrm{pc}}(Y)(U)$ which are contained in V. This is an immediate consequence of the fact that Y is locally proper and separated over X (see Remark 7.3.2.10). □

The following Lemma is a suitable relative analogue of the fact that in a locally compact Hausdorff space, every open covering of a compact subset has a finite refinement:

Lemma 7.3.2.20. *Let $V \subset f^{-1}(U)$ be an object in $\mathsf{Open}_X^{\mathrm{pc}}(Y)(U)$, choose an arbitrary family $(V_j' \subset f^{-1}(U_j))_{j \in J}$ of objects in $\mathsf{Open}_X(Y)$ and suppose that $\overline{V} \subset \bigcup_{j \in J} V_j'$. Then there is a covering $U = \bigcup_i U_i$ in X such that for each i there is a finite subset $J_i \subset J$ such that $U_i \subset U_j$ for all $j \in J_i$ and such that $\overline{V} \cap f^{-1}(U_i) \subset \bigcup_{j \in J_i} V_j'$.*

Proof. In light of Remark 7.3.2.17, we may replace Y/X by $f^{-1}(U)/U$ and each $V_j' \subset f^{-1}(U_j)$ by its intersection $V_j' \cap f^{-1}(U) \subset f^{-1}(U_j \cap U)$ and can thus assume without loss of generality that $U = X$. Now since $\overline{V}$ is proper over X, its fibre $\overline{V}|_x$ over every $x \in X$ is compact (as being proper is stable under base change). Therefore, for each $x \in X$ we have a finite subset $J_x \subset J$ such that $\overline{V}|_x \subset \bigcup_{j \in J_x} V_j'$. We can assume that $x \in U_j$ for all $j \in J_x$, since otherwise $V_j'|_x$ would be empty. Now let Z be the complement of $\bigcup_{j \in J_x} V_j'$ in Y. Then $\overline{V} \cap Z$ is closed in $\overline{Y}$, hence $f(\overline{V} \cap Z)$ is closed in X (as proper maps are always closed). By construction, a point $x' \in X$ is contained in $f(\overline{V} \cap Z)$ precisely if $\overline{V}|_{x'}$ is not contained in $\bigcup_{j \in J_x} V_j'$. Therefore, if U is the complement of $f(\overline{V} \cap Z)$ in X, then U contains precisely those points $x' \in X$ for which $\overline{V}|_{x'} \subset \bigcup_{j \in J_x} V_j'$. In other words, we have $\overline{V} \cap f^{-1}(U) \subset \bigcup_{j \in J_x} V_j'$. Since $x \in U$, we may shrink U if necessary so that it is contained in $\bigcap_{j \in J_x} U_x$. Now the claim follows. □

Proof of Proposition 7.3.2.11. By Proposition 7.3.2.19, the left Kan extension

$$l\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{Open}_X^{\mathrm{pc}}(Y)) \to \mathsf{Open}_X(Y)$$

is a Bousfield localisation. Therefore, we only need to show that l admits a left adjoint λ which preserves finite limits.

We begin by showing that $l(X)$ has a left adjoint λ_X. On account of Proposition 7.3.2.12, this amounts to showing that for every $V \subset Y$, there is sieve $\lambda_X(V)\colon \mathsf{P} \hookrightarrow \mathsf{Open}_X^{\mathrm{pc}}(Y)$ which is closed under finite colimits such that for every other sieve $q\colon \mathsf{Q} \hookrightarrow \mathsf{Open}_X^{\mathrm{pc}}(Y)$ with the same property and for which $V \subset \operatorname{colim} q$, we have $\mathsf{P} \hookrightarrow \mathsf{Q}$. We define P to be the full subposet of $\mathsf{Open}_X^{\mathrm{pc}}(Y)$ which is spanned by those $V' \subset f^{-1}(U)$ whose closure is contained in V. This property is clearly local in X, so that every object of P in context $U \subset X$ will be of this form. Moreover, if $V'' \subset V'$ and V' is in $\mathsf{P}(U)$, so is V''. Therefore, $\mathsf{P} \hookrightarrow \mathsf{Open}_X^{\mathrm{pc}}(Y)$ is a sieve. Furthermore, P is closed under finite colimits. Now let $V' \subset f^{-1}(U)$ be an arbitrary object in P in context $U \subset X$ and let $q\colon \mathsf{Q} \hookrightarrow \mathsf{Open}_X^{\mathrm{pc}}(Y)$ be a sieve which is closed under finite colimits such that $V \subset \operatorname{colim} q$. We need to show that V' is contained in Q. By assumption, the closure $\overline{V'}$ is contained in $\operatorname{colim} q$. Using Proposition 6.3.2.10, we may identify

$$\operatorname{colim} q \simeq \bigcup_{\substack{V'' \in \mathsf{Q}(U) \\ U \subset X}} V''.$$

Therefore, Lemma 7.3.2.20 implies that there is a covering $U = \bigcup_{i \in I} U_i$ such that for each i there are finitely many $V''_{i_1}, \dots, V''_{i_n} \in \mathsf{Q}(U_i)$ with the property that $V' \cap f^{-1}(U_i) \subset \bigcup_{j=1}^n V''_{i_j}$. As Q is closed under finite colimits, the right-hand side is contained in $\mathsf{Q}(U_i)$. Consequently, V' is locally contained in Q and must therefore also be globally contained in Q.

Now by carrying out the above argument with $f|_{f^{-1}(U)}$ in place of f, Remark 7.3.2.17 implies that $l(U)$ admits a left adjoint λ_U for every $U \subset X$. Furthermore, for every pair of opens $U \subset U' \subset X$ and every $V' \subset f^{-1}(U')$, it follows readily from the constructions that the restriction of $\lambda_{U'}(V')$ to U can be identified with $\lambda_U(V' \cap f^{-1}(U))$. Therefore, we deduce from Corollary 3.1.2.11 that l admits a left adjoint, as desired. It is then clear from its explicit construction that this left adjoint preserves finite limits.

Lastly, if f is proper, then $\mathsf{Open}_X^{\mathrm{pc}}(Y)$ contains the final object of $\mathsf{Open}_X(Y)$, which immediately implies that $\mathsf{Open}_X(Y)$ is stably compact. $\square$

Appendix

A.1. The marked simplex ∞-category

Recall that we denote by $\sigma_0 : \langle 1 \rangle \to \langle 0 \rangle$ the unique map in Δ. Let us also denote by $\sigma_0 : \Delta^1 \to \Delta$ the associated functor that picks out this map.

Definition A.1.0.1. The *marked simplex ∞-category* Δ_+ is defined as the pushout

$$\begin{tikzcd} \Delta^1 \arrow[r, hook, "d_{\{0,2\}}"] \arrow[d, "\sigma_0"] & \Delta^2 \arrow[d, "\nu"] \\ \Delta \arrow[r, hook, "\iota"] & \Delta_+ \end{tikzcd}$$

in Cat_∞.

Note that the functor $\iota\colon \Delta \hookrightarrow \Delta_+$ is fully faithful by Lemma 3.4.4.3. Note, furthermore, that an object in Δ_+ is either of the form $\iota\langle n \rangle$ for some $\langle n \rangle \in \Delta$ or the image of $\{1\} \in \Delta^2$ along the map $\nu\colon \Delta^2 \to \Delta_+$, which we will denote by $+$.

Observe that the functor $d_{\{0,2\}}$ admits both a left adjoint $s_{\{1,2\}}$ and a right adjoint $s_{\{0,1\}}$. The pushout of these adjoints along $\Delta^1 \hookrightarrow \Delta$ then define a left adjoint $\flat : \Delta_+ \to \Delta$ and a right adjoint $\sharp : \Delta_+ \to \Delta$ to the inclusion ι. This follows from the following lemma and its dual version, applied to the case $\mathcal{B} = \mathrm{Ani}$:

Lemma A.1.0.2. *If*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "h"] \arrow[d, hook, "f"] & \mathsf{E} \arrow[d, hook, "g"] \\ \mathsf{D} \arrow[r, "k"] & \mathsf{F} \end{tikzcd}$$

is a pushout square in $\mathrm{Cat}(\mathcal{B})$ *such that f is fully faithful and admits a left adjoint l, then the pushout of l along k defines a left adjoint of g.*

Proof. Let $l\colon \mathsf{D} \to \mathsf{C}$ be the left adjoint of f and let $\eta : \Delta^1 \otimes \mathsf{D} \to \mathsf{D}$ be the adjunction unit. We define $l' : \mathsf{F} \to \mathsf{E}$ to be the pushout of l along k. Consider the commutative diagram

$$\begin{tikzcd}
 & \Delta^1 \otimes \mathsf{E} \arrow[rr, dashed, "\mathrm{id}\otimes g"] \arrow[rrrr, bend left, "s^0\otimes g"] \arrow[dd, "s^0"] & & \Delta^1 \otimes \mathsf{F} \arrow[rr, dashed, "s^0\otimes\mathrm{id}"] \arrow[dd, dashed, "\eta'"] & & \mathsf{F} \arrow[dd, "l'"] \\
\Delta^1 \otimes \mathsf{C} \arrow[ur, "\mathrm{id}\otimes h"] \arrow[rr, "\mathrm{id}\otimes f"] \arrow[dd, "s^0"] & & \Delta^1 \otimes \mathsf{D} \arrow[ur, dashed, "\mathrm{id}\otimes k"] \arrow[rr, "s^0\otimes\mathrm{id}"] \arrow[dd, "\eta"] & & \mathsf{D} \arrow[ur, "k"] \arrow[dd, "l"] & \\
 & \mathsf{E} \arrow[rr, "g"] & & \mathsf{F} \arrow[rr, "l'"] & & \mathsf{E} \\
\mathsf{C} \arrow[ur, "h"] \arrow[rr, "f"] & & \mathsf{D} \arrow[ur, "k"] \arrow[rr, "l"] & & \mathsf{C} \arrow[ur, "h"] &
\end{tikzcd}$$

which is constructed as the left Kan extension of its solid part. Since $\eta \circ (d^1 \otimes \mathrm{id})$ is equivalent to the identity and $\eta \circ (d^0 \otimes \mathrm{id})$ is equivalent to fl, the same must be true when replacing η by η' and fl by gl'. In other words, η' encodes a map $\mathrm{id} \to gl'$. The two squares in the back of the above diagram now precisely express the two conditions that $\eta' k$ and $l'\eta'$ are equivalences. Using Corollary 3.1.4.3, this shows that l' is left adjoint to g. $\square$

A priori, the ∞-category Δ_+ need not be a 1-category, so it is not clear that this definition recovers the usual marked simplex 1-category (which can be defined as the homotopy 1-category of Δ_+, i.e. as the pushout $\Delta \sqcup_{\Delta^1} \Delta^2$ in Cat_1). The main goal of this appendix is to show that this is nonetheless the case.

Proposition A.1.0.3. *The ∞-category Δ_+ is a 1-category.*

To show Proposition A.1.0.3, note that since $+ \in \Delta_+$ is the only object that is not contained in the essential image of the inclusion $\iota\colon \Delta \hookrightarrow \Delta_+$, it suffices to show that the two functors $\mathrm{map}_{\Delta_+}(+,-)$ and $\mathrm{map}_{\Delta_+}(-,+)$ take values in sets. Furthermore, by making use of the adjunctions $\flat \dashv \iota$ and $\iota \dashv \sharp$, there are equivalences

$$\mathrm{map}_{\Delta_+}(\iota\langle n\rangle, +) \simeq \mathrm{map}_{\Delta}(\langle n\rangle, \langle 1\rangle)$$
$$\mathrm{map}_{\Delta_+}(+, \iota\langle n\rangle) \simeq \mathrm{map}_{\Delta}(\langle 0\rangle, \langle n\rangle)$$

for all $n \geq 0$. Consequently, we only need to show that $\mathrm{map}_{\Delta_+}(+,+)$ is a set. We will do so by explicitly constructing a simplicial model of this ∞-groupoid using the approach via necklaces due to Dugger and Spivak [22]. We will review the Dugger-Spivak model in Appendix A.1.1, and we will use it to compute the mapping ∞-groupoid $\mathrm{map}_{\Delta_+}(+,+)$ in Appendix A.1.2

A.1.1. The Dugger-Spivak model for mapping ∞-groupoids

A *necklace* is defined to be a simplicial set T of the form

$$T = \Delta^{n_0} \vee \cdots \vee \Delta^{n_k}$$

with $n_i \geq 0$ and where in each wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$. Note that in the case $n_i = 1$ for all $i = 0, \dots, k$, the above necklace is precisely the k-spine I^k. Every necklace is naturally bi-pointed by its initial and final vertex and will therefore be regarded as an object in the 1-category $(\mathrm{Set}_\Delta)_{\partial\Delta^1/}$. We let Nec be the full subcategory of $(\mathrm{Set}_\Delta)_{\partial\Delta^1/}$ that is spanned by the necklaces. Now if S is a simplicial set and if $s, t \in S$ are vertices, we denote by $S_{(s,t)}$ the associated bi-pointed simplicial set. The main input to our proof of Proposition A.1.0.3 is the following theorem:

Theorem A.1.1.1 ([22, Theorem 1.2]). *Let S be a simplicial set and let $S \to \mathcal{C}$ be a fibrant replacement in the Joyal model structure on* Set_Δ*. Given two vertices $s, t \in S$, the mapping ∞-groupoid* $\mathrm{map}_{\mathcal{C}}(s,t)$ *is equivalent to* $(\mathrm{Nec}_{/S_{(s,t)}})^{\mathrm{gpd}}$.

Now let K be the simplicial set that is defined by the pushout

$$\begin{tikzcd} \Delta^1 \arrow[r, hook, "d_{\{0,2\}}"] \arrow[d, hook, "\sigma_0"] & \Delta^2 \arrow[d, hook] \\ \Delta \arrow[r, hook] & K \end{tikzcd}$$

in Set_Δ, where we implicitly identify the three 1-categories with their associated nerves. We will again denote by $+$ the image of $\{1\} \in \Delta^2$ in K, and we will implicitly identify Δ with its image in K. Any fibrant replacement of K in the Joyal model structure on Set_Δ will be a model for Δ_+, so that we may regard K as an explicit simplicial model of the marked simplex ∞-category. As a consequence, by making use of Theorem A.1.1.1, the ∞-groupoid $\mathrm{map}_{\Delta_+}(+,+)$ is presented by the nerve of the 1-category $\mathrm{Nec}_{/K_{(++)}}$. Proposition A.1.0.3 will thus be an immediate consequence of the following proposition:

Proposition A.1.1.2. *The ∞-groupoid $(\mathrm{Nec}_{/K_{(++)}})^{\mathrm{gpd}}$ is a set.*

In order to prove Proposition A.1.1.2, we need to understand the 1-category $\mathrm{Nec}_{/K_{(++)}}$ in more detail. This is the content of the next section.

A.1.2. Necklaces in the simplicial model of Δ_+

By construction, there is a unique non-degenerate edge $\alpha\colon \langle 1\rangle \to +$ in K with codomain $+$. Similarly, there is a unique non-degenerate edge $\beta\colon + \to \langle 0\rangle$ in K with domain $+$. Therefore, an arbitrary object $f\colon \Delta^{n_0} \vee \cdots \vee \Delta^{n_k} \to K$ in $\mathrm{Nec}_{/K_{(+,+)}}$ satisfies exactly one of the two disjoint conditions:

1. for all $0 \leq i \leq k$, the n_i-simplex $\sigma_k : \Delta^{n_k} \to K$ factors through $+ : \Delta^0 \to K$;
2. there are indices $0 \leq l < r \leq k$ such that
   a) for all $i < l$ and all $i > r$ the simplex $\sigma_i : \Delta^{n_i} \to K$ factors through $+ : \Delta^0 \to K$,
   b) the simplex $\sigma_l : \Delta^{n_l} \to K$ factors uniquely into a surjection $\Delta^{n_l} \twoheadrightarrow \Delta^1$ followed by $\beta\colon \Delta^1 \to K$,
   c) the simplex $\sigma_r : \Delta^{n_r} \to K$ factors uniquely into a surjection $\Delta^{n_r} \twoheadrightarrow \Delta^1$ followed by $\alpha\colon \Delta^1 \to K$.

We say that an object in $\mathrm{Nec}_{/K_{(++)}}$ is *degenerate* if it satisfies condition (1), and *non-degenerate* otherwise.

Lemma A.1.2.1. *There are no maps between a degenerate and a non-degenerate object in* $\mathrm{Nec}_{/K_{(++)}}$.

Proof. Let us fix a degenerate object $f\colon \Delta^{n_0} \vee \cdots \vee \Delta^{n_k} \to K$ and a non-degenerate object $g : \Delta^{n_0} \vee \cdots \vee \Delta^{n_l} \to K$. Note that there is always a unique map from f to the degenerate object $+ : \Delta^0 \to K$. Therefore, if there were a map $g \to f$ in $\mathrm{Nec}_{/K_{(+,+)}}$, we would in particular obtain a map $g \to +$, which would however imply that

g is degenerate. Conversely, note that if we set $n = \sum_{i=0}^{k} n_i$, the inclusion of the spine $I^{n_i} \hookrightarrow \Delta^{n_i}$ for all $i = 0, \dots k$ induces an inclusion $I^n \hookrightarrow \Delta^{n_0} \vee \dots \vee \Delta^{n_k}$ of bi-pointed simplicial sets that in turn gives rise to a degenerate object $f' : I^n \to K$ in $\mathrm{Nec}_{/K_{(+,+)}}$. Therefore, any map $f \to g$ in $\mathrm{Nec}_{/K_{(+,+)}}$ restricts to a map $f' \to g$, which is clearly not possible as this would imply that the image of f' in K contains objects that are different from $+$. $\square$

As a consequence of Lemma A.1.2.1, there is a decomposition

$$\mathrm{Nec}_{/K_{(+,+)}} \simeq \mathrm{Nec}^{\mathrm{deg}}_{/K_{(+,+)}} \sqcup \mathrm{Nec}^{\mathrm{nondeg}}_{/K_{(+,+)}}$$

of $\mathrm{Nec}_{/K_{(+,+)}}$ into its degenerate and non-degenerate part. Together with the fact that the groupoidification functor $(-)^{\mathrm{gpd}} : \mathrm{Cat}_\infty \to \mathrm{Ani}$ commutes with colimits, this implies that we may treat the degenerate and the non-degenerate part of $\mathrm{Nec}_{/K_{(+,+)}}$ separately.

The computation of the groupoidification of $\mathrm{Nec}^{\mathrm{deg}}_{/K_{(+,+)}}$ is easy: as observed in the proof of Lemma A.1.2.1, this category has a final object $+ : \Delta^0 \to K$, which immediately implies:

Lemma A.1.2.2. *There is an equivalence* $(\mathrm{Nec}^{\mathrm{deg}}_{/K_{(+,+)}})^{\mathrm{gpd}} \simeq 1$. $\square$

In order to compute the groupoidification of $\mathrm{Nec}^{\mathrm{nondeg}}_{/K_{(+,+)}}$, on the other hand, we need one additional step. Recall that we denote by $\beta : \Delta^1 \to K$ the map that picks out the unique 1-simplex $+ \to \langle 0 \rangle$. We obtain an evident functor

$$\beta \vee - : \mathrm{Nec}_{/K_{(0,+)}} \to \mathrm{Nec}^{\mathrm{nondeg}}_{/K_{(+,+)}}, \quad (f : T \to K) \mapsto (\beta \vee f : \Delta^1 \vee T \to K).$$

Lemma A.1.2.3. *The functor $\beta \vee -$ is homotopy final.*

Proof. Let us fix an arbitrary object $f : \Delta^{n_0} \vee \dots \vee \Delta^{n_k} \to K$ in $\mathrm{Nec}^{\mathrm{nondeg}}_{/K_{(+,+)}}$. By Quillen's theorem A, it suffices to show that the category $(\mathrm{Nec}_{/K_{(0,+)}})_{f/}$ admits an initial object. Recall that we denote by $0 \leq l$ the largest index such that for all $i < l$ the simplex $\sigma_i : \Delta^{n_i} \to K$ factors through $+ : \Delta^0 \to K$ and such that σ_l factors into a surjection $\tau : \Delta^{n_l} \twoheadrightarrow \Delta^1$ followed by $\beta : \Delta^1 \to K$. We may therefore construct a map

$$\Delta^{n_0} \vee \dots \vee \Delta^{n_k} \to \Delta^1 \vee \Delta^{n_{l+1}} \vee \dots \vee \Delta^{n_k}$$

over K that sends Δ^{n_i} to the initial object for all $i < l$, that sends Δ^{n_l} to Δ^1 via the degeneracy map τ and that acts as the identity on the remaining summands. This map defines the desired initial object in $(\mathrm{Nec}_{/K_{(0,+)}})_{f/}$. $\square$

Proof of Proposition A.1.1.2. By Lemma A.1.2.2, the ∞-groupoid $(\mathrm{Nec}^{\mathrm{deg}}_{/K_{(+,+)}})^{\mathrm{gpd}}$ is a set. By Lemma A.1.2.3, the functor $\beta \vee -$ induces an equivalence

$$(\mathrm{Nec}^{\mathrm{nondeg}}_{/K_{(+,+)}})^{\mathrm{gpd}} \simeq (\mathrm{Nec}_{/K_{(\langle 0\rangle,+)}})^{\mathrm{gpd}}.$$

Since the right-hand side is equivalent to $\mathrm{map}_{\Delta_+}(\langle 0\rangle, +)$ by Theorem A.1.1.1, this is a set as well. $\square$

A.2. Locally constant sheaves

This section is devoted to the study of locally constant sheaves in ∞-topoi. For the entire section, let us fix a compactly generated ∞-category $\mathcal{E}$. Recall that we write $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ for the tensor product $\mathcal{B} \otimes \mathcal{E}$ of presentable ∞-categories. By applying $- \otimes \mathcal{E}$ to the constant sheaf functor $\mathrm{const}_{\mathcal{B}} \colon \mathcal{S} \to \mathcal{B}$ we obtain an adjunction

$$(\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}) \colon \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}) \rightleftarrows \mathcal{E}.$$

Similarly, by applying $- \otimes \mathcal{E}$ to the adjunction $(\pi_A^* \dashv (\pi_A)_*) \colon \mathcal{B}_{/A} \rightleftarrows \mathcal{B}$ for some $A \in \mathcal{B}$, we obtain an induced adjunction

$$(\pi_A^* \dashv (\pi_A)_*) \colon \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}_{/A}) \rightleftarrows \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}).$$

Furthermore, if there is an accessible left exact localisation $L \dashv i \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ we get an induced localisation $L_{\mathcal{E}} \dashv i_{\mathcal{E}} \colon \mathrm{PSh}_{\mathcal{E}}(\mathcal{C}) \to \mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$.

Definition A.2.0.1. Let us fix the following terminology:

1. We call $\mathrm{const}_{\mathcal{B}}(K)$ the *constant sheaf associated to* $K \in \mathcal{E}$. The objects in the essential image of $\mathrm{const}_{\mathcal{B}}$ are called *constant* $\mathcal{E}$*-valued sheaves.*

2. We call an $\mathcal{E}$-valued sheaf F *constant with compact values if* it is of the form $\mathrm{const}_{\mathcal{B}}(K)$ for some compact object $K \in \mathcal{E}$.

3. An $\mathcal{E}$-valued sheaf F is *locally constant* if there is a cover $(\pi_{A_i}) : \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ such that for every i the $\mathcal{E}$-valued sheaf $\pi_{A_i}^* F \in \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}_{/A_i})$ is constant.

4. We call an $\mathcal{E}$-valued sheaf F *locally constant with compact values* if we can find a cover $(s_i)_i : \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ such that $s_i^* F$ is constant with compact values.

5. We denote by $\mathrm{LConst}^{\mathcal{E}}(\mathcal{B})$ the full subcategory of $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ spanned by the locally constant sheaves, and by $\mathrm{LConst}^{\mathcal{E}}_{\mathrm{cpt}}(\mathcal{B})$ the full subcategory spanned by the locally constant sheaves with compact values.

The key result that we will show in this section is the following:

Proposition A.2.0.2. *Suppose that $L : \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ is a left exact and accessible localisation, and let F be an $\mathcal{E}$-valued presheaf on $\mathcal{C}$. Then for any $c \in \mathcal{C}$ and any map $K \to L_{\mathcal{E}} F(c)$ there is a collection of morphisms $(s_i : c_i \to c)_{i \in I}$ in $\mathcal{C}$ such that $(Ls_i) : \bigsqcup_i L(c_i) \twoheadrightarrow L(c)$ is a cover in $\mathcal{B}$ and for any $i \in I$ the composite $K \to L_{\mathcal{E}} F(c) \xrightarrow{s_i^*} L_{\mathcal{E}} F(c_i)$ factors as a composite $K \xrightarrow{m_i} F(c_i) \xrightarrow{\varepsilon_F(c_i)} L_{\mathcal{E}} F(c_i)$ for some $m_i : K \to F(c_i)$.*

As an immediate consequence we obtain the following:

Corollary A.2.0.3. *Let $f : \mathrm{const}_{\mathcal{B}}(K) \to \mathrm{const}_{\mathcal{B}}(M)$ be a morphism in $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ where K is compact. Then there is a cover $(\pi_{A_i}) : \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ and maps $f_i : K \to M$ in $\mathcal{E}$ for each i such that $\pi_{A_i}^* f$ is equivalent to $\mathrm{const}_{\mathcal{B}_{/A_i}}(f_i)$.*

Proof. We may pick a left exact accessible localisation $L : \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ where $\mathcal{C}$ has a final object 1. The morphism f corresponds to a map

$$\tilde{f} : K \to \Gamma\, \mathrm{const}_{\mathcal{B}}(M) = \mathrm{const}_{\mathcal{B}}(M)(1).$$

By Proposition A.2.0.2 we may now find a covering $(\pi_{Lc_i}) : \bigsqcup_i L(c_i) \twoheadrightarrow 1$ and commutative squares

$$\begin{tikzcd} M = \underline{M}(c_i) \arrow[r] & \mathrm{const}_{\mathcal{B}}(M)(c_i) \\ K \arrow[u, "m_i"] \arrow[r, "\tilde{f}"] & \mathrm{const}_{\mathcal{B}}(M)(1) \arrow[u, "\pi_{Lc_i}^*"'] \end{tikzcd}$$

where $\underline{M}$ denotes the constant M-valued presheaf. Let $f_i = \mathrm{const}_{\mathcal{B}}(m_i)$. Then the above square translates to the statement that $\pi_{Lc_i}^*(f_i)$ is equivalent to $\mathrm{const}_{\mathcal{B}_{/Lc_i}}(f_i)$, and the claim follows. $\square$

Corollary A.2.0.4. *The full subcategory* $\mathrm{LConst}_{\mathrm{cpt}}^{\mathcal{E}}(\mathcal{B}) \hookrightarrow \mathcal{E} \otimes \mathcal{B}$ *is closed under finite colimits and retracts.*

Proof. We start by showing the claim about finite colimits. Since $\mathrm{LConst}_{\mathrm{cpt}}^{\mathcal{E}}(\mathcal{B})$ contains the final object it suffices to see that it is closed under pushouts. So let us consider a span $F \leftarrow G \rightarrow H$ in $\mathrm{LConst}_{\mathrm{cpt}}^{\mathcal{E}}(\mathcal{B})$. We may pass to a cover in $\mathcal{B}$ to assume that F, G and H are constant. Thus by Corollary A.2.0.3 we may further pass to a cover so that we can assume that the span above is given by applying $\mathrm{const}_{\mathcal{B}}$ to a span in $\mathcal{E}^{\mathrm{cpt}}$. So the claim follows since $\mathcal{E}^{\mathrm{cpt}}$ is closed under finite colimits and $\mathrm{const}_{\mathcal{B}}$ preserves finite colimits. The proof that $\mathrm{LConst}_{\mathrm{cpt}}^{\mathcal{E}}(\mathcal{B})$ is closed under retracts proceeds in the same way. $\square$

In order to prove Proposition A.2.0.2, we first need to treat the special case where $\mathcal{E} = \mathrm{Ani}$ and $K = 1$:

Lemma A.2.0.5. *Let* $F \in \mathrm{PSh}(\mathcal{C})$ *and let* $f\colon 1 \to LF(c)$ *be a map for some* $c \in \mathcal{C}$. *Then there is a collection of morphisms* $(s_i : c_i \to c)_{i \in I}$ *in* $\mathcal{C}$ *such that* $(Ls_i) : \bigsqcup_i L(c_i) \twoheadrightarrow L(c)$ *is a cover in* $\mathcal{B}$ *and maps* $m_i : 1 \to F(c_i)$ *for each* i *such that* $s_i^* f$ *is equivalent to the composite*

$$1 \xrightarrow{m_i} F(c_i) \xrightarrow{\varepsilon_F(c_i)} LF(c_i).$$

Proof. We pick a cover $(t_j) : \bigsqcup_j d_j \twoheadrightarrow F$ in $\mathrm{PSh}(\mathcal{C})$. Consider the pullback square

$$\begin{tikzcd} \bigsqcup_j A_j \arrow[r] \arrow[d] & \bigsqcup_i d_j \arrow[d] \\ c \arrow[r, "f"'] & LF \end{tikzcd}$$

in $\mathrm{PSh}(\mathcal{C})$. Covering each $A_j \in \mathrm{PSh}(\mathcal{C})$ with representables c_k^j then yields the desired collection of maps $(s_k^j) : \bigsqcup_{j,k} c_k^j \twoheadrightarrow c$. $\square$

To reduce the general case to the above lemma we use the ideas of [30, § 2]. Indeed, the fact that $\mathcal{E}$ is by assumption compactly generated means that we may identify $\mathcal{E}$ with $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{Ani})$. Consequently, we obtain an equivalence

$$\mathrm{Sh}_{\mathcal{B}}(\mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}).$$

In light of these identifications, the adjunction $L_{\mathcal{E}} \dashv i_{\mathcal{E}} \colon \mathrm{PSh}_{\mathcal{E}}(\mathcal{C}) \to \mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ translates into the adjunction $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{PSh}(\mathcal{C})) \rightleftarrows \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})$ that is obtained by postcomposition with $L \dashv i$. An analogous observation shows that for $c \in \mathcal{C}$ the evaluation functor $\mathrm{ev}_c^{\mathcal{E}} \colon \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}) \to \mathcal{E}$ is equivalent to the functor

$$\mathrm{ev}_{c,*} \colon \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \to \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{Ani})$$

given by composing with $\mathrm{ev}_c \colon \mathcal{B} \to \mathrm{Ani}$.

Proof of Proposition A.2.0.2. Since K is compact, the preceding discussion together with Yoneda's lemma allows us to identify $K \to L_{\mathcal{E}} F(c)$ with a map

$$f \colon 1 \to L_{\mathcal{E}} F(c)(K) \simeq L(F(K))(c).$$

Therefore we are in the situation of Lemma A.2.0.5 and get a collection of morphisms $(s_i \colon c_i \to c)_{i \in I}$ in $\mathcal{C}$ such that $(Ls_i) \colon \bigsqcup_i L(c_i) \twoheadrightarrow L(c)$ is a cover in $\mathcal{B}$ and maps $n_i \colon 1 \to F(K)(c)$ such that for each i we have a commutative square

$$\begin{tikzcd}
F(K)(c_i) \arrow[r] & L(F(K))(c_i) \\
1 \arrow[u, "n_i"] \arrow[r, "f"] & L(F(K))(c). \arrow[u, "s_i^*"']
\end{tikzcd}$$

Via Yoneda's lemma the maps n_i now yield the desired maps $m_i \colon K \to F(c_i)$. $\square$

Bibliography

[1] J. Adámek, F. Borceux, S. Lack, and J. Rosický. A classification of accessible categories. *Journal of Pure and Applied Algebra*, 175(1-3):7–30, 2002.

[2] M. Anel, G. Biedermann, E. Finster, and A. Joyal. A generalized Blakers–Massey theorem. *Journal of Topology*, 13(4):1521–1553, 2020.

[3] M. Anel and A. Joyal. Topo-logie. In *New spaces in mathematics—formal and conceptual reflections*, pages 155–257. Cambridge Univ. Press, Cambridge, 2021.

[4] M. Anel and D. Lejay. Exponentiable higher toposes. *arXiv preprint*, Feb. 2018, arXiv:1802.10425.

[5] M. Anel and C. L. Subramaniam. Small object arguments, plus-construction, and left-exact localizations. *arXiv preprint*, Apr. 2020, arXiv:2004.00731.

[6] K. Aoki. The sheaves-spectrum adjunction. *arXiv preprint*, 2023, arXiv:2302.04069.

[7] K. Aoki. Tensor triangular geometry of filtered objects and sheaves. *Mathematische Zeitschrift*, 303(3):27, 2023. Id/No 62.

[8] M. Artin, A. Grothendieck, J. L. Verdier, N. Bourbaki, P. Deligne, and B. Saint-Donat, editors. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, volume 269 of *Lect. Notes Math.* Springer, Cham, 1972.

[9] M. Artin, A. Grothendieck, J. L. Verdier, P. Deligne, and B. Saint-Donat, editors. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie*

des topos et cohomologie étale des schémas (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki , P. Deligne, B. Saint-Donat. Tome 2. Exposes V à VIII, volume 270 of *Lect. Notes Math.* Springer, Cham, 1972.

[10] D. Ayala and J. Francis. Fibrations of ∞-categories. *Higher Structures*, 4(1):168–265, 2020.

[11] J. C. Baez and J. Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. *Advances in Mathematics*, 135(2):145–206, 1998.

[12] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. Parametrized higher category theory and higher algebra: A general introduction. *arXiv preprint*, Aug. 2016, arXiv:1608.03654.

[13] C. Barwick, S. Glasman, and P. Haine. Exodromy. *arXiv preprint*, Aug. 2020, arXiv:1807.03281.

[14] C. Barwick and P. Haine. Pyknotic objects, i. basic notions. *arXiv preprint*, Apr. 2019, arXiv:1904.09966.

[15] U. Buchholtz and J. Weinberger. Synthetic fibered $(\infty, 1)$-category theory. *Higher Structures*, 7(1):74–165, 2023.

[16] O. Caramello and R. Zanfa. Relative topos theory via stacks. *arXiv preprint*, July 2021, arXiv:2107.04417.

[17] H. Chu and R. Haugseng. Enriched homotopy-coherent structures. *arXiv preprint*, Aug. 2023, arXiv:2308.11502.

[18] D.-C. Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019.

[19] D.-C. Cisinski and F. Déglise. Étale motives. *Compositio Mathematica*, 152(3):556–666, 2016.

[20] R. Diaconescu. Change of base for toposes with generators. *Journal of Pure and Applied Algebra*, 6(3):191–218, 1975.

[21] B. Drew and M. Gallauer. The universal six-functor formalism. *Annals of K-Theory*, 7(4):599–649, 2022.

[22] D. Dugger and D. I. Spivak. Rigidification of quasi-categories. *Algebraic & Geometric Topology*, 11(1):225–261, 2011.

[23] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[24] D. Gepner and R. Haugseng. Enriched ∞-categories via non-symmetric ∞-operads. *Advances in Mathematics*, 279:575 – 716, 2015.

[25] D. Gepner, R. Haugseng, and T. Nikolaus. Lax colimits and free fibrations in ∞-categories. *Documenta Mathematica*, 22:1225–1266, 2017.

[26] D. Gepner and J. Kock. Univalence in locally cartesian closed ∞-categories. *Forum Mathematicum*, 29(3):617–652, 2017.

[27] J. Giraud. Classifying topos. In *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N.S., 1971)*, Lecture Notes in Math., Vol. 274, pages 43–56. Springer, Berlin, 1972.

[28] S. Glasman. A spectrum-level Hodge filtration on topological Hochschild homology. *Selecta Mathematica*, 22(3):1583–1612, 2016.

[29] J. W. Gray. *Formal category theory: adjointness for 2-categories*, volume 391 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 1974.

[30] P. Haine. From nonabelian basechange to basechange with coefficients. *arXiv preprint*, Sept. 2022, arXiv:2108.03545.

[31] R. Haugseng. Rectification of enriched ∞-categories. *Algebraic & Geometric Topology*, 15(4):1931–1982, 2015.

[32] R. Haugseng. The higher Morita category of $\mathbb{E}_n$-algebras. *Geometry & Topology*, 21(3):1631–1730, 2017.

[33] R. Haugseng. On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories. *Higher Structures*, 5(1):244–281, 2021.

[34] R. Haugseng, F. Hebestreit, S. Linskens, and J. Nuiten. Lax monoidal adjunctions, two-variable fibrations and the calculus of mates. *Proceedings of the London Mathematical Society. Third Series*, 127(4):889–957, 2023.

[35] K. Hilman. Parametrised presentability over orbital categories. *arXiv preprint*, Feb. 2022, arXiv:2202.02594.

[36] V. Hinich. Lectures on infinity categories. *arXiv preprint*, Sept. 2017, arXiv:1709.06271.

[37] M. Hoyois, S. Scherotzke, and N. Sibilla. Higher traces, noncommutative motives, and the categorified chern character. *Advances in Mathematics*, 309:97–154, 2017.

[38] P. Johnstone. The associated sheaf functor in an elementary topos. *Journal of Pure and Applied Algebra*, 4:231–242, 1974.

[39] P. Johnstone. *Sketches of an Elephant - A Topos theory compendium.* Oxford Logic Guides. The Clarendon Press Oxford University Press, 2002.

[40] P. T. Johnstone. The Gleason cover of a topos, II. *Journal of Pure and Applied Algebra*, 22(3):229–247, 1981.

[41] A. Joyal. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra*, 175(1-3):207–222, 2002.

[42] A. Joyal. Notes on quasi-categories. preprint, 2008. https://www.math.uchicago.edu/~may/IMA/Joyal.pdf.

[43] A. Joyal. The theory of quasi-categories and its applications. In *Advanced Course on Simplicial Methods in Higher Categories*, volume 2. Centre de Recerca Matemàtica, 2008.

[44] A. Joyal and M. Tierney. Quasi-categories vs Segal spaces. *Contemporary Mathematics*, 431(277-326):10, 2007.

[45] G. M. Kelly and R. Street. Review of the elements of 2-categories. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 75–103. Springer, 2006.

[46] C. Lair. Catégories modelables et catégories esquissables. *Diagrammes*, 6:L1–L20, 1981.

[47] F. W. Lawvere. *Functorial semantics of algebraic theories*. PhD thesis, Columbia University New York, 1963.

[48] F. W. Lawvere. Quantifiers and sheaves. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 329–334. Gauthier-Villars, Paris, 1971.

[49] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[50] J. Lurie. *Higher algebra*. preprint, 2017. https://www.math.ias.edu/~lurie/papers/HA.pdf.

[51] J. Lurie. *Spectral algebraic geometry*. preprint, 2018. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[52] J. Lurie. Kerodon, 2024. https://kerodon.net.

[53] M. Makkai and R. Paré. *Accessible categories: the foundations of categorical model theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[54] L. Martini. Yoneda's lemma for internal higher categories. *arXiv preprint*, Mar. 2021, arXiv:2103.17141.

[55] L. Martini. Cocartesian fibrations and straightening internal to an ∞-topos. *arXiv preprint*, Apr. 2022, arXiv:2204.00295.

[56] L. Martini and S. Wolf. Presentable categories internal to an ∞-topos. *arXiv preprint*, Sept. 2022, arXiv:2209.05103.

[57] L. Martini and S. Wolf. Internal higher topos theory. *arXiv preprint*, Mar. 2023, arXiv:2303.06437.

[58] L. Martini and S. Wolf. Proper morphisms of ∞-topoi. *arXiv preprint*, Nov. 2023, arXiv:2311.08051.

[59] L. Martini and S. Wolf. Colimits and cocompletions in internal higher category theory. *Higher Structures*, 8(1):97–192, 2024.

[60] J. L. Moens. *Caractérisation des topos de faisceaux sur un site interne à un topos.* PhD thesis, Univ. catholique de Louvain, 1982.

[61] I. Moerdijk and J. J. C. Vermeulen. Proper maps of toposes. *Memoirs of the American Mathematical Society*, 148(705):x+108, 2000.

[62] D. Nardin. A universally non hypercomplete ∞-topos via Goodwillie calculus? Answer to a mathoverflow question. https://mathoverflow.net/q/300682.

[63] D. Nardin. Parametrized higher category theory and higher algebra: Exposé iv – stability with respect to an orbital ∞-category. *arXiv preprint*, Aug. 2016, arXiv:1608.07704.

[64] D. Nardin and J. Shah. Parametrized and equivariant higher algebra. *arXiv preprint*, Feb. 2022, arXiv:2203.00072.

[65] A. Pitts. On product and change of base for toposes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 26(1):43–61, 1985.

[66] M. Ramzi. An elementary proof of the naturality of the Yoneda embedding. *Proceedings of the American Mathematical Society*, 151(10):4163–4171, 2023.

[67] G. Raptis and D. Schäppi. Flat functors in higher topos theory. *arXiv preprint*, Aug. 2022, arXiv:2208.13897.

[68] N. Rasekh. Univalence in higher category theory. *arXiv preprint*, Mar. 2021, arXiv:2103.12762.

[69] N. Rasekh. Cartesian fibrations and representability. *Homology, Homotopy and Applications*, 24(2):135–161, 2022.

[70] C. Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.

[71] C. Rezk. Generalizing accessible ∞-categories. Unpublished note, 2021. https://faculty.math.illinois.edu/~rezk/accessible-cat-thoughts.pdf.

[72] E. Riehl and M. Shulman. A type theory for synthetic ∞-categories. *Higher Structures*, 1(1):147–224, 2017.

[73] E. Riehl and D. Verity. *Elements of ∞-category theory*, volume 194 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2022.

[74] O. M. Schnürer and W. Soergel. Proper base change for separated locally proper maps. *Rendiconti del Seminario Matematico della Università di Padova*, 135:223–250, 2016.

[75] P. Scholze. Condensed mathematics. Lecture notes based on joint work with D. Clausen, 2019. https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[76] J. Shah. Parametrized higher category theory ii: Universal constructions. *arXiv preprint*, Sept. 2021, arXiv:2109.11954.

[77] J. Shah. Parametrized higher category theory. *Algebraic & Geometric Topology*, 23(2):509–644, 2023.

[78] M. Shulman. All $(\infty, 1)$-toposes have strict univalent universes. *arXiv preprint*, Apr. 2019, arXiv:1904.07004.

[79] T. Streicher. Fibered categories a la Jean Bénabou. *arXiv preprint*, Jan. 2018, arXiv:1801.02927.

[80] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[81] M. Vergura. Localization theory in an ∞-topos. *arXiv preprint*, July 2019, arXiv:1907.03836.

[82] M. Volpe. The six operations in topology. *arXiv preprint*, Oct. 2021, arXiv:2110.10212.

[83] M. Volpe. Verdier duality on conically smooth stratified spaces. *arXiv preprint*, June 2022, arXiv:2206.02728.

[84] J. Weinberger. Internal sums for synthetic fibered $(\infty, 1)$-categories. *arXiv preprint*, May 2022, arXiv:2205.00386.

[85] J. Weinberger. Strict stability of extension types. *arXiv preprint*, Mar. 2022, arXiv:2203.07194.

[86] J. Weinberger. A synthetic perspective on $(\infty, 1)$-category theory: Fibrational and semantic aspects. *arXiv preprint*, Feb. 2022, arXiv:2202.13132.

[87] J. Weinberger. Two-sided cartesian fibrations of synthetic $(\infty, 1)$-categories. *arXiv preprint*, Apr. 2022, arXiv:2204.00938.

[88] J. Weinberger. Generalized Chevalley criteria in simplicial homotopy type theory. *arXiv preprint*, Mar. 2024, arXiv:2403.08190.

[89] S. Wolf. The pro-étale topos as a category of pyknotic presheaves. *Documenta Mathematica*, 27:2067–2106, 2022.