CURVATURE OF THE BASE MANIFOLD OF A MONGE-AMPÈRE FIBRATION AND ITS EXISTENCE

XUEYUAN WAN AND XU WANG

Abstract. In this paper, we consider a special relative Kähler fibration that satisfies a homogenous Monge-Ampère equation, which is called a Monge-Ampère fibration. There exist two canonical types of generalized Weil-Petersson metrics on the base complex manifold of the fibration. For the second generalized Weil-Petersson metric, we obtain an explicit curvature formula and prove that the holomorphic bisectional curvature is non-positive, the holomorphic sectional curvature, the Ricci curvature, and the scalar curvature are all bounded from above by a negative constant. For a holomorphic vector bundle over a compact Kähler manifold, we prove that it admits a projectively flat Hermitian structure if and only if the associated projective bundle fibration is a Monge-Ampère fibration. In general, we can prove that a relative Kähler fibration is Monge-Ampère if and only if an associated infinite rank Higgs bundle is Higgs-flat. We also discuss some typical examples of Monge-Ampère fibrations.

Contents

2020 Mathematics Subject Classification. 32Q05, 32G05, 53C55.

Key words and phrases. Monge-Ampère fibrations, Generalized Weil-Petersson metrics, Negative curvature, Projectively flat, Higgs-flat.

Xueyuan Wan is partially supported by the National Natural Science Foundation of China (grant no. 12101093) and Scientific Research Foundation of the Chongqing University of Technology.

Introduction

The curvature property of the moduli space of a holomorphic family of compact complex manifolds is an important research topic in complex geometry. For the moduli space of curves, there exists a classical Weil-Petersson metric, which is Kähler [1, Theorem 4], and the Ricci curvature, the holomorphic sectional curvature and the scalar curvature are negative [2, §10, Theorem], the holomorphic bisectional curvature is also negative [20, Theorem 1.3]. There are also other curvature properties for the Weil-Petersson metric, such as negative sectional curvature [34, Theorem 5] [43, Theorem 4.5], strongly-negative curvature in the sense of Siu [26, Theorem 1], dual Nakano negative [19, Theorem 4.1], non-positive Riemannian sectional curvature operator [44, Theorem 1.1], etc. One can refer to [20] for the relations among these curvature properties of the Weil-Petersson metric. Moreover, by deriving an explicit formula for the curvature of the Weil-Petersson metric, S. Wolpert proved that the holomorphic sectional curvature, the Ricci curvature, and the scalar curvature are all bounded above by a negative constant [43, Lemma 4.6].

For the moduli space of compact Kähler-Einstein manifolds, there is a canonical metric, i.e., the generalized Weil-Petersson metric, which can be proved to be Kähler [17, Theorem 12.3]. For the case of negative first Chern class, Y.-T. Siu [30] computed the curvature of the generalized Weil-Petersson metric and obtained a criterion on the negativity of the holomorphic bisectional curvature of the metric [30, Theorem 5.5]. In [27], G. Schumacher considered the case of Kähler-Einstein manifolds with nonzero Ricci curvature k and also gave an explicit formula [27, Theorem 1]. As an application, for $k > 0$, he proved that the holomorphic sectional curvature and Ricci curvature of the generalized Weil-Petersson metric are bounded from below by a negative constant [27, Corollary 1]. For the moduli space of Calabi-Yau manifolds, G. Schumacher [25] and G. Tian [32] showed that the generalized Weil-Petersson metric is Kähler. A. Nannicini [24, proof of Theorem 1] and A. N. Todorov [33] computed the curvature tensor of the generalized Weil-Petersson metric (two simple proofs of the curvature formula were given by C.-L. Wang [37, Theorem 2.1] who also showed that both the holomorphic bisectional curvature and the Ricci curvature are bounded from below by a negative constant). In [22], Z. Lu and X. Sun obtained an explicit formula for the curvature of partial Hodge metric [22, Theorem 1.1]. In the case of the moduli space of Calabi-Yau fourfolds, they proved that the holomorphic bisectional curvature of the partial metric with a special factor (which is precisely the Hodge metric (up to a constant)) is non-positive, the Ricci curvature and the holomorphic sectional curvature are all bounded above by a negative constant [22, Theorem 1.2]. For the general case, Z. Lu constructed a Hodge metric and proved that its holomorphic bisectional curvature is non-positive, the Ricci curvature and holomorphic sectional curvature are negative away from zero by a constant number [21, Theorem 5.1]. For other related results, one can refer to [8, 23, 28], etc.

In this paper, we will study the curvature properties of the base complex manifold of a Monge-Ampère fibration[1], see Definition 1.8. In [10], D. Burns considered the curvature of a Monge-Ampère foliation with only one-dimensional leaves (a local version of a Monge-Ampère fibration) and obtained that the curvature is bounded from above by a negative constant [10,

[1]The name of the Monge-Ampère fibration was firstly given by Professor Bo Berndtsson.

Theorem 3.1]. A related negative curvature property for the space of all compatible almost complex structures was proven by Smolentsev in [31]. Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. It is called a Monge-Ampère fibration if $\omega^{n+1} = 0$, where n denotes the dimension of each fiber. If the Kodaira-Spencer map is injective, then one can define two kinds of generalized Weil-Petersson metrics on the base complex manifold $\mathcal{B}$, i.e., ω_{WP} and $\omega_{\mathcal{WP}}$, see Section 1.3 for their definitions. The generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$ is defined by the ω-Kodaira-Spencer tensor κ_j without taking harmonic projection, so we always have $\omega_{\mathcal{WP}} \geq \omega_{\mathrm{WP}}$. Our main result is the following curvature formula for the generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$.

Theorem 0.1. *Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a Monge-Ampère fibration with injective Kodaira-Spencer map. Then the metric $\omega_{\mathcal{WP}}$ is Kähler and its curvature is given by*

$$R_{j\bar{k}l\bar{m}} = -\langle \overline{\kappa_m}\kappa_j, \overline{\kappa_l}\kappa_k \rangle - \langle \kappa_j \overline{\kappa_m}, \kappa_k \overline{\kappa_l} \rangle - \langle \mathrm{H}^{\perp}(L_{V_l}\kappa_j), \mathrm{H}^{\perp}(L_{V_m}\kappa_k) \rangle,$$

where κ_j is the ω-Kodaira-Spencer tensor, see Definition 1.10; V_l denotes the horizontal lift of $\frac{\partial}{\partial t^l}$, see (1.2); the operator L denotes the Lie derivative, $\mathrm{H}^{\perp}$ denotes the orthogonal projection from $A^{0,1}(X_t, T_{X_t})$ to $\mathrm{Span}\{\kappa_i\}^{\perp}$.

By using the above curvature formula, one can obtain some immediate consequences on various negativity results of different types of curvature.

Corollary 0.2. *Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a Monge-Ampère fibration with injective Kodaira-Spencer map. The holomorphic bisectional curvature of the generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$ satisfies*

$$R(\xi, \overline{\xi}, \eta, \overline{\eta}) \leq -\frac{2}{n}|X_t|^{-1}|\langle \eta, \xi \rangle_{\mathcal{WP}}|^2$$

for any two vectors η, ξ in $T_t\mathcal{B}$, where $|X_t| := \int_{X_t} \frac{\omega_t^n}{n!}$ denotes the volume of each fiber. In particular, we have the following negativity results of curvature[2]*:*

(i) *The holomorphic bisectional curvature is non-positive, and is negative if $\langle \eta, \xi \rangle_{\mathcal{WP}} \neq 0$;*
(ii) *The holomorphic sectional curvature and the Ricci curvature are both bounded from above by $-\frac{2}{n}|X_t|^{-1}$, the scalar curvature is bounded from above by $-\frac{2}{n}|X_t|^{-1}\dim \mathcal{B}$.*

Naturally, one may wonder what kind of relative Kähler fibration becomes a Monge-Ampère fibration. In particular, for a holomorphic vector bundle over a compact complex manifold $\mathcal{B}$, there is a canonical relative Kähler fibration $p : P(E) \to \mathcal{B}$ with each fiber a projective space, where $P(E) := (E - \{0\})/\mathbb{C}^*$ denotes the projectivization of E. A natural question is for which holomorphic vector bundles E, the associated projective bundle fibration $p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration. For this question, we have:

Theorem 0.3. *Let E be a holomorphic vector bundle over a compact Kähler manifold $\mathcal{B}$. Then the following statements are equivalent:*

1) *E admits a projectively flat Hermitian structure;*
2) *$p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration.*

[2]The negativity results of curvature are also obtained by Professor Bo Berndtsson independently using a different method based on the holomorphic motion structure of the fibration (see [7]).

For the case of $\dim \mathcal{B} = 1$*, both are equivalent to the polystability of* E*.*

In Section 4.2, we shall introduce a finite rank Higgs bundle structure associated with a (non-proper) Monge-Ampère fibration over the space $\mathcal{J}(V, \omega)$ of all ω-compatible complex structures on a symplectic vector space (V, ω). This construction also suggests to introduce a certain infinite rank Higgs bundle for a general relative Kähler fibration. Let $\mathcal{A} := \{\mathcal{A}_t\}_{t\in\mathcal{B}}$ be the space of smooth differential forms on X_t. Denote by Γ the space of all smooth sections of $\mathcal{A}$, see (3.18). With respect to the relative Kähler form ω, there exists a Lie derivative connection ∇ on $(\mathcal{A}, \Gamma)$, see (3.19), which induces a Chern connection D on $(\mathcal{A}, \Gamma)$, see (3.20), such that $\nabla - D = \theta + \bar{\theta}$ for a Higgs field θ, where $\theta := \sum dt^j \otimes \kappa_j$. Denoting by $(\mathcal{A}, \Gamma, D, \theta)$ the associated Higgs bundle, we have:

Theorem 0.4. *A relative Kähler fibration* $p : (\mathcal{X}, \omega) \to \mathcal{B}$ *is a Monge-Ampère fibration if and only if the following associated infinite rank Higgs bundle*

$$(\mathcal{A}, \Gamma, D, \theta)$$

is Higgs-flat (cf. Proposition 3.15), where each fiber $\mathcal{A}_t$ *denotes the space of smooth differential forms on* X_t*.*

We also discuss some typical examples of Monge-Ampère fibrations, which are also the motivations for studying such kind of relative Kähler fibration. For example, the family of elliptic curves, finite rank Higgs bundle version of a (non-proper) Monge-Ampère fibration, and various kinds of geodesics.

This article is organized as follows. In Section 1, we review some basic definitions and facts on the relative Kähler fibrations, Monge-Ampère fibrations, and two types of generalized Weil-Petersson metrics. In Section 2, we will compute the curvature of the generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$, and we will prove Theorem 0.1 and Corollary 0.2. In Section 3, we will consider the existence of Monge-Ampère fibrations. In Section 3.1, we will show that a holomorphic vector bundle admits a projectively flat Hermitian structure if and only if the associated projective bundle fibration is a Monge-Ampère fibration, and prove Theorem 0.3. In Section 3.2, we will prove that a relative Kähler fibration is Monge-Ampère if and only if an associated infinite rank Higgs bundle is Higgs-flat, and prove Theorem 0.4. The last section will give some typical examples of Monge-Ampère fibrations.

Acknowledgements. We would like to thank Bo Berndtsson and Ya Deng for several useful discussions about the topics of this paper. We also would like to thank the anonymous reviewers for their comments that helped improve the paper.

1. Preliminaries

In this section, we will review some basic definitions and facts on the relative Kähler fibrations, Monge-Ampère fibrations, and two types of generalized Weil-Petersson metrics.

1.1. **Relative Kähler fibrations.** Let $\mathcal{X}$ and $\mathcal{B}$ be two complex manifolds.

Definition 1.1. We call a proper holomorphic submersion $p : (\mathcal{X}, \omega) \to \mathcal{B}$ between two complex manifolds *a relative Kähler fibration* if ω is a real, smooth, d-closed $(1,1)$-form on $\mathcal{X}$ and ω is positive on each fiber $X_t := p^{-1}(t)$ of p.

Definition 1.2. Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. By vertical vector fields, we mean vector fields on $\mathcal{X}$ that are tangent to the fibers, a vector field V on $\mathcal{X}$ is said to be *horizontal* with respect to ω if

$$\omega(V, W) = 0$$

for every vertical W.

The relative Kähler form ω defines a natural inner product (not semi-positive in general) such that

$$\langle V, W\rangle_\omega = \omega(V, J\overline{W}), \tag{1.1}$$

where J denotes the complex structure on $\mathcal{X}$. We say that V is *orthogonal* to W with respect to ω if $\langle V, W\rangle_\omega = 0$. Thus a vector field is horizontal if and only if it is orthogonal to all vertical vector fields.

Definition 1.3. Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration, and let v be a vector field on $\mathcal{B}$. A vector field V on $\mathcal{X}$ is said to be *a horizontal lift* of v with respect to ω if V is horizontal and $p_*(V) = v$.

For the horizontal lift of a vector field, we have the following proposition (see e.g. [8, Section 4.1]).

Proposition 1.4. *Every vector field on $\mathcal{B}$ has a unique horizontal lift. Horizontal lift of a $(1,0)$-vector field (resp. $(0,1)$-vector field) is still a $(1,0)$-vector field (resp. $(0,1)$-vector field).*

Let $\{t^j\}$ be a holomorphic local coordinate system on $\mathcal{B}$. Since p is a holomorphic fibration, we can find ζ^α such that $\{t^j, \zeta^\alpha\}$ is a holomorphic local coordinate system on $\mathcal{X}$. Since ω is a closed $(1,1)$ form, we write it locally as $\omega = i\partial\overline{\partial}\phi$ for some local real function ϕ. Then we know that each

$$V_j := \frac{\partial}{\partial t^j} - \sum_{\beta=1}^{n} \phi_{j\bar\beta}\phi^{\bar\beta\alpha}\frac{\partial}{\partial \zeta^\alpha}, \quad \phi_{j\bar\beta} := \frac{\partial^2\phi}{\partial t^j \partial\bar\zeta^\beta}, \tag{1.2}$$

is a horizontal lift of $\frac{\partial}{\partial t^j}$, where $(\phi^{\bar\beta\alpha})$ denotes the inverse matrix of $(\phi_{\alpha\bar\beta})$ and $\phi_{\alpha\bar\beta} := \frac{\partial^2\phi}{\partial\zeta^\alpha\partial\bar\zeta^\beta}$, n denotes the complex dimension of each fiber. Denote

$$c_{j\bar k} := \langle V_j, V_k\rangle_\omega = \phi_{j\bar k} - \sum_{\alpha,\beta=1}^{n} \phi_{j\bar\beta}\phi^{\bar\beta\alpha}\phi_{\alpha\bar k}, \ c(\omega) := i\sum_{j,k=1}^{\dim\mathcal{B}} c_{j\bar k}\, dt^j \wedge d\bar t^k. \tag{1.3}$$

We call $c_{j\bar k}$ the *geodesic curvatures* and $c(\omega)$ the *geodesic curvature form.* A direct calculation shows that

$$\omega = i\partial\bar\partial\phi = c(\omega) + \omega_{\mathcal{X}/\mathcal{B}}, \quad \omega_{\mathcal{X}/\mathcal{B}} := i\sum_{\alpha,\beta=1}^{n} \phi_{\alpha\bar\beta}\delta\zeta^\alpha \wedge \delta\bar\zeta^\beta, \tag{1.4}$$

where $\delta\zeta^\alpha = d\zeta^\alpha + \sum_{\beta,j}\phi_{j\bar\beta}\phi^{\bar\beta\alpha}dt^j$. The following proposition is a generalization of [39, Lemma 6.1].

Proposition 1.5. *Let $\{V_j\}$ be the vector fields defined in (1.2), $\dim X_t = n$. Then*

(1) $[V_j, V_k] = 0$;
(2) $(\omega - c(\omega))^{n+1} = 0$;
(3) $[V_j, \bar V_k] \rfloor (\omega|_{X_t}) = i(dc_{j\bar k})|_{X_t}$;
(4) $[V_j, \bar V_k] \equiv 0$ *for all* j, k *if and only if* $d(c(\omega)) = 0$.

Proof. (1) By a direct computation, we know that $[V_j, V_k]$ are vertical. Since ω is non-degenerate on fibers, it is enough to prove that $[V_j, V_k]\rfloor\omega = 0$ on fibers. Notice that

$$[V_j, V_k]\rfloor\omega = (L_{V_j}V_k)\rfloor\omega = L_{V_j}(V_k\rfloor\omega) - V_k\rfloor L_{V_j}\omega,$$

and by (1.2) we have

$$V_j\rfloor\omega = i\sum_{l=1}^{\dim\mathcal{B}} c_{j\bar l}\, d\bar t^l. \tag{1.5}$$

By using the Cartan formula, we get

$$[V_j, V_k]\rfloor\omega = i\sum_{l=1}^{\dim\mathcal{B}}(V_j\rfloor dc_{k\bar l})\, d\bar t^l - i\sum_{l=1}^{\dim\mathcal{B}}(V_k\rfloor dc_{j\bar l})\, d\bar t^l. \tag{1.6}$$

Thus $[V_j, V_k]\rfloor\omega = 0$ on fibers, and so $[V_j, V_k] = 0$.

(2) From (1.4), one has

$$(\omega - c(\omega))^{n+1} = \omega^{n+1}_{\mathcal{X}/\mathcal{B}} = \left(i\sum_{\alpha,\beta=1}^{n}\phi_{\alpha\bar\beta}\delta\zeta^\alpha\wedge\delta\bar\zeta^\beta\right)^{n+1} = 0.$$

(3) Notice that

$$[V_j, \bar V_k]\rfloor\omega = (L_{V_j}\bar V_k)\rfloor\omega = L_{V_j}(\bar V_k\rfloor\omega) - \bar V_k\rfloor L_{V_j}\omega$$

and combining with (1.5) we have

$$[V_j, \bar V_k]\rfloor\omega = i\, dc_{j\bar k} - i\sum_{l=1}^{\dim\mathcal{B}}(V_j\rfloor dc_{l\bar k})\, dt^l - i\sum_{l=1}^{\dim\mathcal{B}}(\bar V_k\rfloor dc_{j\bar l})\, d\bar t^l.$$

Since $[V_j, \bar V_k]$ is vertical, so

$$[V_j, \bar V_k]\rfloor(\omega|_{X_t}) = \left([V_j, \bar V_k]\rfloor\omega\right)|_{X_t} = i(dc_{j\bar k})|_{X_t},$$

which proves (3).

(4) By (3), we know that $dc(\omega) = 0$ gives $[V_j, \bar V_k] \equiv 0$. For the opposite direction, assume that $[V_j, \bar V_k] \equiv 0$ all for j, k, then by (3), we know that $c_{j\bar k}$ depends only on $t \in \mathcal{B}$, thus by (1) and (1.6), we have

$$0 = [V_j, V_k]\rfloor\omega = i\sum_{l=1}^{\dim\mathcal{B}}\frac{\partial c_{k\bar l}}{\partial t^j}\, d\bar t^l - i\sum_{l=1}^{\dim\mathcal{B}}\frac{\partial c_{j\bar l}}{\partial t^k}\, d\bar t^l,$$

which implies that $c(\omega)$ is d-closed. Thus $dc(\omega) = 0$.

□

Remark 1.6. From (1) and (4) in Proposition 1.5, the horizontal distribution of a relative Kähler fibration is integrable if and only if each geodesic curvature $c_{j\bar{k}}$ is constant on fibers, which determines a differentiable trivialization of the fibration.

Remark 1.7. If the geodesic curvature form $c(\omega)$ depends only on the base $\mathcal{B}$, then

$$\int_{\mathcal{X}/\mathcal{B}} \frac{\omega^{n+1}}{(n+1)!} = \int_{\mathcal{X}/\mathcal{B}} c(\omega) \wedge \frac{\omega^n_{\mathcal{X}/\mathcal{B}}}{n!} = c(\omega) \int_{X_t} \frac{(\omega|_{X_t})^n}{n!} = c(\omega)|X_t|,$$

where $|X_t| := \int_{X_t} \frac{(\omega|_{X_t})^n}{n!}$ denotes the volume of each fiber. Hence

$$c(\omega) = \frac{1}{|X_t|} \int_{\mathcal{X}/\mathcal{B}} \frac{\omega^{n+1}}{(n+1)!},$$

which is d-closed.

1.2. **Monge-Ampère fibrations.** In this subsection, we will give the definition of a Monge-Ampère fibration.

Definition 1.8. A relative Kähler fibration $p : (\mathcal{X}, \omega) \to \mathcal{B}$ is said to be *Monge-Ampère* (we say that ω is *a Monge-Ampère form*) if ω solves the homogeneous complex Monge–Ampère equation, i.e.

$$\omega^{n+1} \equiv 0,$$

where n denotes the dimension of the fibers. In general, a proper holomorphic submersion $p : (\mathcal{X}, \omega_{\mathcal{X}}) \to (\mathcal{B}, \omega_{\mathcal{B}})$ between two Kähler manifolds is said to be *Monge-Ampère* if

$$(\omega_{\mathcal{X}} - p^*\omega_{\mathcal{B}})^{n+1} \equiv 0,$$

(in which case we know $\omega_{\mathcal{X}} - p^*\omega_{\mathcal{B}}$ is *a Monge-Ampère form*). A proper holomorphic submersion $p : \mathcal{X} \to \mathcal{B}$ is called a Monge-Ampère fibration if there exists a Monge-Ampère form ω on $\mathcal{X}$.

Remark 1.9.
(1) By Proposition 1.5, for a relative Kähler fibration $p : (\mathcal{X}, \omega) \to \mathcal{B}$, $d\omega' = 0$ if and only if $[V_j, \bar{V}_k] \equiv 0$ all for j, k, where $\omega' = \omega - c(\omega)$. Thus ω' is a Monge-Ampère form if and only if the horizontal distribution associated with ω is integrable.
(2) A relative Kähler fibration $p : (\mathcal{X}, \omega) \to \mathcal{B}$ is a Monge-Ampère fibration if and only if

$$\begin{aligned} 0 = \omega^{n+1} &= (c(\omega) + \omega_{\mathcal{X}/\mathcal{B}})^{n+1} \\ &= (n+1)\omega^n_{\mathcal{X}/\mathcal{B}} \wedge c(\omega) + \sum_{i=2}^{n+1} C^i_{n+1} \omega^{n+1-i}_{\mathcal{X}/\mathcal{B}} \wedge c(\omega)^i, \end{aligned}$$

which is equivalent to $c(\omega) \equiv 0$.
(3) If $p : (\mathcal{X}, \omega) \to \mathcal{B}$ is a Monge-Ampère form, then the d-closed $(1,1)$-form $\int_{\mathcal{X}/\mathcal{B}} \omega^{n+1}$ vanishes.

1.3. **Generalized Weil-Petersson metrics.** In this subsection, by using the relative Kähler form ω, we shall define two types of generalized Weil-Petersson metrics on the base manifold of a Monge-Ampère fibration.

Definition 1.10. Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. Let V_j (defined in (1.2)) be the horizontal lift of $\frac{\partial}{\partial t^j}$ with respect to ω. We call

$$\kappa_j := (\overline{\partial} V_j)|_{X_t}$$

the *ω-Kodaira–Spencer tensor* on X_t.

From the above definition, one sees that each ω-Kodaira–Spencer tensor κ_j is a $\overline{\partial}$-closed T_{X_t}-valued $(0,1)$-form on X_t. By using the ω-Kodaira–Spencer tensor, the generalized Weil-Petersson metric can be given as follows, see [14, Definition 7.1].

Definition 1.11. Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. We call the following metric on $\mathcal{B}$ defined by

$$\left\langle \frac{\partial}{\partial t^j}, \frac{\partial}{\partial t^k} \right\rangle_{\mathrm{WP}} (t) := \int_{X_t} \langle \kappa_j^h, \kappa_k^h \rangle_{\omega_t} \frac{\omega_t^n}{n!}, \quad \omega_t := \omega|_{X_t},$$

the *generalized Weil-Petersson metric* on $\mathcal{B}$, where κ_j^h denotes the ω_t harmonic representative of the Kodaira–Spencer class $[\kappa_j]$.

On the other hand, one can take the L^2-inner product of the *ω-Kodaira–Spencer tensors* κ_j directly (without taking the harmonic projection), which gives the following definition of generalized Weil-Petersson metrics, see [14, Section 8].

Definition 1.12. Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. We can define another kind of generalized Weil-Petersson metric on $\mathcal{B}$ by

$$\left\langle \frac{\partial}{\partial t^j}, \frac{\partial}{\partial t^k} \right\rangle_{\mathcal{WP}} (t) := \int_{X_t} \langle \kappa_j, \kappa_k \rangle_{\omega_t} \frac{\omega_t^n}{n!}, \quad \omega_t := \omega|_{X_t},$$

where κ_j are ω-Kodaira–Spencer tensors.

One may note that the generalized Weil-Petersson metric $\langle \cdot, \cdot \rangle_{\mathcal{WP}}$ is *bigger* than $\langle \cdot, \cdot \rangle_{\mathrm{WP}}$. In particular, if the Kodaira-Spencer map is injective, then both kinds of generalized Weil-Petersson metrics must be non-degenerated.

Remark 1.13. It is proved in [40] that if the relative cotangent bundle is $(n-1)$-semi-positive, then the bisectional curvature of the *generalized Weil-Petersson metric* is semi-negative. But in general, it is not easy to find such fibrations with $(n-1)$-semi-positive relative cotangent bundle. The main theme of this paper is to use the *generalized Weil-Petersson metric* $\langle \cdot, \cdot \rangle_{\mathcal{WP}}$ to study the curvature properties of the base manifold of a Monge-Ampère fibration.

2. Curvature of the generalized Weil-Petersson metric

Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration, i.e., ω is a real and smooth d-closed $(1,1)$-form on $\mathcal{X}$, and is positive on each fiber $X_t := p^{-1}(t)$. By $\bar{\partial}$-Poincaré Lemma, there exists a local weight, say ϕ, such that

$$\omega = i\partial\bar{\partial}\phi.$$

Let $\{t^j, \zeta^\alpha\}$ denote a holomorphic local coordinate system on $\mathcal{X}$ such that $p(t,\zeta) = t$. Then

$$\omega = i\left(\phi_{\alpha\bar\beta} d\zeta^\alpha \wedge d\bar\zeta^\beta + \phi_{i\bar\beta} dt^j \wedge d\bar\zeta^\beta + \phi_{\alpha\bar k} d\zeta^\alpha \wedge d\bar t^k + \phi_{j\bar k} dt^j \wedge d\bar t^k\right),$$

where $\phi_{A\bar B} := \partial_A \partial_{\bar B}\phi$. In this section, we will use the summation convention of Einstein. Recall the canonical horizontal lift of $\frac{\partial}{\partial t^j}$ is given by

$$V_j := \frac{\partial}{\partial t^j} - \phi_{j\bar\beta}\phi^{\bar\beta\alpha}\frac{\partial}{\partial\zeta^\alpha},$$

and recall the ω-Kodaira-Spencer tensor on X_t is given by

$$\kappa_j := (\bar\partial V_j)|_{X_t}.$$

The generalized Weil-Petersson metric $\langle\cdot,\cdot\rangle_{\mathcal{WP}}$ is then defined by

$$\left\langle \frac{\partial}{\partial t^j}, \frac{\partial}{\partial t^k}\right\rangle_{\mathcal{WP}}(t) := \int_{X_t} \langle \kappa_j, \kappa_k\rangle_{\omega_t} \frac{\omega_t^n}{n!}, \quad \omega_t = \omega|_{X_t}.$$

Denote

$$\omega_{\mathcal{WP}} = i G_{j\bar k} dt^j \wedge d\bar t^k, \quad G_{j\bar k} := \left\langle \frac{\partial}{\partial t^j}, \frac{\partial}{\partial t^k}\right\rangle_{\mathcal{WP}}.$$

With respect to ω, recall that the geodesic curvature form is given by

$$c(\omega) = i c_{j\bar k} dt^j \wedge d\bar t^k, \quad c_{j\bar k} := \langle V_j, V_k\rangle_\omega = \phi_{j\bar k} - \phi_{j\bar\beta}\phi^{\alpha\bar\beta}\phi_{\alpha\bar k}.$$

If each fiber X_t is compact, Fujiki and Schumacher [14] obtained the following expression on the generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$, see also [36, Lemma 3.8 (3.43)] for its proof.

Theorem 2.1 ([14, Theorem 8.1])**.** *The following identity holds*

$$\omega_{\mathcal{WP}} = i\int_{\mathcal{X}/\mathcal{B}} R^{K_{\mathcal{X}/\mathcal{B}}} \wedge \frac{\omega^n}{n!} + \int_{\mathcal{X}/\mathcal{B}} \rho c(\omega) \wedge \frac{\omega^n}{n!}, \tag{2.1}$$

where $R^{K_{\mathcal{X}/\mathcal{B}}} = \partial\bar\partial \log\det\phi$, $\rho = -\phi^{\alpha\bar\beta}\partial_\alpha\partial_{\bar\beta}\log\det\phi$ *is the saclar curvature,* $\det\phi := \det(\phi_{\alpha\bar\beta})$, $\int_{\mathcal{X}/\mathcal{B}}$ *denotes fiber integration (see e.g.* [28, Section 2.1] *for fiber integration).*

As a corollary, one has

Corollary 2.2. *If* $p : (\mathcal{X}, \omega) \to \mathcal{B}$ *is a Monge-Ampère fibration, then*

$$\omega_{\mathcal{WP}} = i\int_{\mathcal{X}/\mathcal{B}} R^{K_{\mathcal{X}/\mathcal{B}}} \wedge \frac{\omega^n}{n!}. \tag{2.2}$$

In particular, $\omega_{\mathcal{WP}}$ *is d-closed.*

Now we will follow Schumacher's method [27] to calculate the curvature of generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$. Let T_{X_t} denote the holomorphic tangent bundle of X_t, and denote by $T^{\mathbb{C}}_{X_t} = T_{X_t} \oplus \overline{T_{X_t}}$ the complexified tangent bundle. For any two tensors

$$\Phi = \Phi^A_B dx^B \otimes \frac{\partial}{\partial x^A}, \quad \Psi = \Psi^A_B dx^B \otimes \frac{\partial}{\partial x^A} \in A^1(X_t, T^{\mathbb{C}}_{X_t}) \simeq A^0(X_t, \mathrm{End}(T^{\mathbb{C}}_{X_t})),$$

where x^A, x^B are taken $\{\zeta^\alpha, \bar\zeta^\beta\}$. We define

$$\Phi \cdot \Psi := \mathrm{Tr}(\Phi\Psi) = \Phi^A_B \Psi^B_A.$$

For any vector field V, we denote by L_V the Lie derivative along V. For the tensor $\Phi = \Phi^A_B dx^B \otimes \frac{\partial}{\partial x^A} \in A^1(X_t, T^{\mathbb{C}}_{X_t})$, one has

$$L_V\Phi = \left(L_V\Phi^A_B\right)\frac{\partial}{\partial x^A}\otimes dx^B, \tag{2.3}$$

where

$$\begin{aligned} L_V\Phi^A_B &= V(\Phi^A_B) - \Phi^C_B\frac{\partial V^A}{\partial x^C} + \Phi^A_C\frac{\partial V^C}{\partial x^B} \\ &= \nabla_V(\Phi^A_B) - \Phi^C_B\nabla_C V^A + \Phi^A_C\nabla_B V^C. \end{aligned} \tag{2.4}$$

Here ∇_C denotes the covariant derivative along $\partial/\partial x^C$ with respect to some Hermitian metric. Since Lie derivative commutes with contraction and satisfies Leibniz's rule for tensors, so

$$L_V(\Phi\cdot\Psi) = (L_V\Phi)\cdot\Psi + \Phi\cdot(L_V\Psi).$$

Denote

$$\kappa_j = A^\alpha_{j\bar\beta}d\bar\zeta^\beta\otimes\frac{\partial}{\partial\zeta^\alpha}, \quad A^\alpha_{j\bar\beta} = -\partial_{\bar\beta}(\phi_{j\bar\gamma}\phi^{\bar\gamma\alpha}).$$

By a direct calculation, one has

$$A^\alpha_{j\bar\beta} = A^\sigma_{j\bar\gamma}\phi^{\bar\gamma\alpha}\phi_{\sigma\bar\beta}, \tag{2.5}$$

(see e.g. [36, (3.12)]). Then

$$\langle\kappa_j,\kappa_k\rangle_{\omega_t} = A^\alpha_{j\bar\beta}\overline{A^\sigma_{k\bar\gamma}}\phi^{\gamma\bar\beta}\phi_{\alpha\bar\sigma} = A^\alpha_{j\bar\beta}\overline{A^\beta_{k\bar\alpha}} = \kappa_j\cdot\overline{\kappa_k}.$$

The first variation of the generalized Weil-Petersson metric is

$$\begin{aligned} \frac{\partial G_{j\bar k}}{\partial t^l} &= \frac{\partial}{\partial t^l}\int_{X_t}\kappa_j\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!} \\ &= \int_{X_t}(L_{V_l}\kappa_j)\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!} + \int_{X_t}\kappa_j\cdot L_{V_l}\overline{\kappa_k}\frac{\omega_t^n}{n!} + \int_{X_t}\kappa_j\cdot\overline{\kappa_k}L_{V_j}\frac{\omega_t^n}{n!} \\ &= \int_{X_t}(L_{V_l}\kappa_j)\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!} + \int_{X_t}\kappa_j\cdot L_{V_l}\overline{\kappa_k}\frac{\omega_t^n}{n!}, \end{aligned} \tag{2.6}$$

where the second equality follows from [28, Lemma 1], the last equality holds by [27, Lemma 2.2 (2)]. From [27, Lemma 2.3] or (2.3), (2.4), one has

$$\begin{aligned} L_{V_l}\overline{\kappa_k} &= L_{V_l}(\overline{A^\beta_{k\bar\alpha}}d\zeta^\alpha\otimes\frac{\partial}{\partial\bar\zeta^\beta}) \\ &= -(c_{l\bar k})^{;\bar\beta}{}_\alpha d\zeta^\alpha\otimes\frac{\partial}{\partial\bar\zeta^\beta} - A^\gamma_{l\bar\beta}\overline{A^\beta_{k\bar\alpha}}d\zeta^\alpha\otimes\frac{\partial}{\partial\zeta^\gamma} + \overline{A^\beta_{k\bar\alpha}}A^\alpha_{l\bar\delta}d\bar\zeta^\delta\otimes\frac{\partial}{\partial\bar\zeta^\beta} \\ &= -(c_{l\bar k})^{;\bar\beta}{}_\alpha d\zeta^\alpha\otimes\frac{\partial}{\partial\bar\zeta^\beta} - \kappa_l\overline{\kappa_k} + \overline{\kappa_k}\kappa_l. \end{aligned} \tag{2.7}$$

Thus

$$\begin{aligned} \int_{X_t}\kappa_j\cdot L_{V_l}\overline{\kappa_k}\frac{\omega_t^n}{n!} &= -\int_{X_t}A^\alpha_{j\bar\beta}(c_{l\bar k})^{;\bar\beta}{}_\alpha\frac{\omega_t^n}{n!} \\ &= -\int_{X_t}(A^\alpha_{j\bar\beta})_{;\alpha}^{;\bar\beta}c_{l\bar k}\frac{\omega_t^n}{n!} = \int_{X_t}(V_j\rho)c_{l\bar k}\frac{\omega_t^n}{n!}, \end{aligned} \tag{2.8}$$

where the last equality follows from

$$\begin{aligned}(A^{\alpha}_{j\bar{\beta}})_{;\alpha}^{\ ;\bar{\beta}} &= (A^{\alpha}_{j\bar{\beta}})_{;\alpha\gamma}\phi^{\gamma\bar{\beta}} = -\phi_{j\bar{\sigma};\bar{\beta}\alpha\gamma}\phi^{\bar{\sigma}\alpha}\phi^{\gamma\bar{\beta}} \\ &= -(\phi_{j\bar{\sigma};\alpha\bar{\beta}} + R_{\alpha\bar{\sigma}\tau\bar{\beta}}\phi^{\tau\bar{\delta}}\phi_{j\bar{\delta}})_{;\gamma}\phi^{\bar{\sigma}\alpha}\phi^{\gamma\bar{\beta}} \\ &= -(\partial_j\phi_{\alpha\bar{\sigma}})_{;\bar{\beta}\gamma}\phi^{\bar{\sigma}\alpha}\phi^{\gamma\bar{\beta}} - (R_{\alpha\bar{\sigma}\tau\bar{\beta}}\phi^{\tau\bar{\delta}}\phi_{j\bar{\delta}})_{;\gamma}\phi^{\bar{\sigma}\alpha}\phi^{\gamma\bar{\beta}} \\ &= -\partial_j\partial_\gamma\partial_{\bar{\beta}}\log\det\phi\phi^{\gamma\bar{\beta}} + (\partial_\tau\partial_{\bar{\beta}}\log\det\phi\phi^{\tau\bar{\delta}}\phi_{j\bar{\delta}})_{;\gamma}\phi^{\gamma\bar{\beta}} \\ &= -\partial_j\rho + \partial_\gamma\partial_{\bar{\beta}}\log\det\phi\partial_j\phi^{\gamma\bar{\beta}} + \partial_\tau\partial_{\bar{\beta}}\log\det\phi\phi^{\tau\bar{\delta}}\phi_{j\gamma\bar{\delta}}\phi^{\gamma\bar{\beta}} \\ &\quad + (\partial_\tau\partial_{\bar{\beta}}\log\det\phi)_{;\gamma}\phi^{\tau\bar{\delta}}\phi_{j\bar{\delta}}\phi^{\gamma\bar{\beta}} \\ &= -\partial_j\rho + (\partial_\gamma\partial_{\bar{\beta}}\log\det\phi)_{;\tau}\phi^{\tau\bar{\delta}}\phi_{j\bar{\delta}}\phi^{\gamma\bar{\beta}} = -V_j\rho.\end{aligned}$$

On the other hand, by (2.3) and (2.4), one has

$$\begin{aligned}L_{V_l}\kappa_j &= (L_{V_l}\kappa_j)^{\alpha}_{\bar{\beta}}d\bar{\zeta}^{\beta}\otimes\frac{\partial}{\partial\zeta^{\alpha}} \\ &= \left(\partial_l(A^{\alpha}_{j\bar{\beta}}) - \phi_{l\bar{\gamma}}\phi^{\bar{\gamma}\sigma}A^{\alpha}_{j\bar{\beta};\sigma} + A^{\sigma}_{j\bar{\beta}}\phi_{l\sigma\bar{\gamma}}\phi^{\bar{\gamma}\alpha}\right)d\bar{\zeta}^{\beta}\otimes\frac{\partial}{\partial\zeta^{\alpha}}.\end{aligned} \tag{2.9}$$

By a direct calculation, one has

$$(L_{V_l}\kappa_j)^{\alpha}_{\bar{\beta}} = (L_{V_l}\kappa_j)^{\tau}_{\bar{\delta}}\phi^{\bar{\delta}\alpha}\phi_{\tau\bar{\beta}}. \tag{2.10}$$

In fact, by (2.5), one has

$$\begin{aligned}(L_{V_l}\kappa_j)^{\alpha}_{\bar{\beta}} &= \partial_l(A^{\alpha}_{j\bar{\beta}}) - \phi_{l\bar{\gamma}}\phi^{\bar{\gamma}\sigma}A^{\alpha}_{j\bar{\beta};\sigma} + A^{\sigma}_{j\bar{\beta}}\phi_{l\sigma\bar{\gamma}}\phi^{\bar{\gamma}\alpha} \\ &= \partial_l(A^{\alpha}_{j\bar{\beta}}) - A^{\sigma}_{j\bar{\gamma}}\phi_{\sigma\bar{\beta}}\partial_l\phi^{\bar{\gamma}\alpha} - (\phi_{l\bar{\gamma}}\phi^{\bar{\gamma}\sigma}A^{\tau}_{j\bar{\delta};\sigma})\phi^{\bar{\delta}\alpha}\phi_{\tau\bar{\beta}} \\ &= \partial_l A^{\sigma}_{j\bar{\gamma}}\phi_{\sigma\bar{\beta}}\phi^{\bar{\gamma}\alpha} + A^{\sigma}_{j\bar{\gamma}}\partial_l\phi_{\sigma\bar{\beta}}\phi^{\bar{\gamma}\alpha} - (\phi_{l\bar{\gamma}}\phi^{\bar{\gamma}\sigma}A^{\tau}_{j\bar{\delta};\sigma})\phi^{\bar{\delta}\alpha}\phi_{\tau\bar{\beta}} \\ &= (\partial_l(A^{\tau}_{j\bar{\delta}}) - \phi_{l\bar{\gamma}}\phi^{\bar{\gamma}\sigma}A^{\tau}_{j\bar{\delta};\sigma} + A^{\sigma}_{j\bar{\delta}}\phi_{l\sigma\bar{\gamma}}\phi^{\bar{\gamma}\tau})\phi^{\bar{\delta}\alpha}\phi_{\tau\bar{\beta}} \\ &= (L_{V_l}\kappa_j)^{\tau}_{\bar{\delta}}\phi^{\bar{\delta}\alpha}\phi_{\tau\bar{\beta}},\end{aligned}$$

which completes the proof of (2.10). Combining with (2.5), we have

$$\int_{X_t}(L_{V_l}\kappa_j)\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!} = \langle L_{V_l}\kappa_j, \kappa_k\rangle. \tag{2.11}$$

Here

$$\langle\cdot,\cdot\rangle := \int_{X_t}\langle\cdot,\cdot\rangle_{\omega_t}\frac{\omega_t^n}{n!}$$

denotes the global L^2-inner product. Substituting (2.8) and (2.11) into (2.6), we obtain

Proposition 2.3. *Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration with compact fibers. The first variation of the generalized Weil-Petersson metric is*

$$\frac{\partial G_{j\bar{k}}}{\partial t^l} = \langle L_{V_l}\kappa_j, \kappa_k\rangle + \int_{X_t}(V_j\rho)c_{l\bar{k}}\frac{\omega_t^n}{n!}.$$

In particular, if ρ is a constant or ω is a Monge-Ampère form (i.e. $c_{l\bar{k}} = 0$), then

$$\frac{\partial G_{j\bar{k}}}{\partial t^l} = \langle L_{V_l}\kappa_j, \kappa_k\rangle = \int_{X_t}(L_{V_l}\kappa_j)\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!}. \tag{2.12}$$

Now we compute the second variation of the generalized Weil-Petersson metric for a Monge-Ampère fibration. Since $[L_{\bar{V}_m}, L_{V^l}] = L_{[\bar{V}_m, V_l]}$ and by (2.12), so

$$\begin{aligned}\frac{\partial^2 G_{j\bar{k}}}{\partial t^l \partial\bar{t}^m} &= \frac{\partial}{\partial\bar{t}^m}\int_{X_t}(L_{V_l}\kappa_j)\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!}\\ &= \int_{X_t}(L_{\bar{V}_m}L_{V_l}\kappa_j)\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!} + \int_{X_t}L_{V_l}\kappa_j\cdot L_{\bar{V}_m}\overline{\kappa_k}\frac{\omega_t^n}{n!}\\ &= \int_{X_t}L_{[\bar{V}_m,V_l]}\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!} + \frac{\partial}{\partial t^l}\int_{X_t}L_{\bar{V}_m}\kappa_j\cdot\overline{\kappa_k}\frac{\omega_t^n}{n!}\\ &\quad -\int_{X_t}L_{\bar{V}_m}\kappa_j\cdot L_{V_l}\overline{\kappa_k}\frac{\omega_t^n}{n!} + \int_{X_t}L_{V_l}\kappa_j\cdot L_{\bar{V}_m}\overline{\kappa_k}\frac{\omega_t^n}{n!}\\ &= -\int_{X_t}L_{\bar{V}_m}\kappa_j\cdot L_{V_l}\overline{\kappa_k}\frac{\omega_t^n}{n!} + \int_{X_t}L_{V_l}\kappa_j\cdot L_{\bar{V}_m}\overline{\kappa_k}\frac{\omega_t^n}{n!},\end{aligned} \tag{2.13}$$

where the last equality holds by (2.8) and using [27, Lemma 2.6],

$$[\bar{V}_m, V_l] = -(c_{l\bar{m}})^{;\alpha}\frac{\partial}{\partial\zeta^\alpha} + (c_{l\bar{m}})^{;\bar{\beta}}\frac{\partial}{\partial\bar{\zeta}^\beta},$$

which vanishes in the case of Monge-Ampère fibration.

From (2.7), one has

$$\begin{aligned}\int_{X_t}L_{\bar{V}_m}\kappa_j\cdot L_{V_l}\overline{\kappa_k}\frac{\omega_t^n}{n!} &= \int_{X_t}(-\overline{\kappa_m}\kappa_j + \kappa_j\overline{\kappa_m})\cdot(-\kappa_l\overline{\kappa_k} + \overline{\kappa_k}\kappa_l)\frac{\omega_t^n}{n!}\\ &= -\int_M(\mathrm{Tr}(\overline{\kappa_m}\kappa_j\overline{\kappa_k}\kappa_l) + \mathrm{Tr}(\kappa_j\overline{\kappa_m}\kappa_l\overline{\kappa_k}))\frac{\omega_t^n}{n!}\\ &= -\langle\overline{\kappa_m}\kappa_j, \overline{\kappa_l}\kappa_k\rangle - \langle\kappa_j\overline{\kappa_m}, \kappa_k\overline{\kappa_l}\rangle.\end{aligned} \tag{2.14}$$

By (2.9) and (2.10), one has

$$\int_{X_t}L_{V_l}\kappa_j\cdot L_{\bar{V}_m}\overline{\kappa_k}\frac{\omega_t^n}{n!} = \langle L_{V_l}\kappa_j, L_{V_m}\kappa_k\rangle. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13), we have

$$\frac{\partial^2 G_{j\bar{k}}}{\partial t^l\partial\bar{t}^m} = \langle\overline{\kappa_m}\kappa_j, \overline{\kappa_l}\kappa_k\rangle + \langle\kappa_j\overline{\kappa_m}, \kappa_k\overline{\kappa_l}\rangle + \langle L_{V_l}\kappa_j, L_{V_m}\kappa_k\rangle. \tag{2.16}$$

Denote by $\mathrm{H} : A^{0,1}(X_t, T_{X_t}) \to \mathrm{Span}\{\kappa_i\}$ the orthogonal projection. By Proposition 2.3, one has

$$G^{p\bar{q}}\frac{\partial G_{j\bar{q}}}{\partial t^l}\frac{\partial G_{p\bar{k}}}{\partial\bar{t}^m} = G^{p\bar{q}}\langle L_{V_l}\kappa_j, \kappa_q\rangle\langle\kappa_p, L_{V_m}\kappa_k\rangle = \langle\mathrm{H}(L_{V_l}\kappa_j), \mathrm{H}(L_{V_m}\kappa_k)\rangle. \tag{2.17}$$

From (2.16) and (2.17), we obtain

Theorem 2.4. *The curvature of generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$ for a Monge-Ampère fibration is*

$$\begin{aligned}R_{j\bar{k}l\bar{m}} &= -\frac{\partial^2 G_{j\bar{k}}}{\partial t^l \partial \bar{t}^m} + G^{p\bar{q}} \frac{\partial G_{j\bar{q}}}{\partial t^l} \frac{\partial G_{p\bar{k}}}{\partial \bar{t}^m} \\ &= -\langle \overline{\kappa_m}\kappa_j, \overline{\kappa_l}\kappa_k\rangle - \langle \kappa_j\overline{\kappa_m}, \kappa_k\overline{\kappa_l}\rangle - \langle \mathrm{H}^{\perp}(L_{V_l}\kappa_j), \mathrm{H}^{\perp}(L_{V_m}\kappa_k)\rangle.\end{aligned}$$

Here $\mathrm{H}^{\perp}$ denotes the orthogonal projection from $A^{0,1}(X_t, T_{X_t})$ to $\mathrm{Span}\{\kappa_i\}^{\perp}$.

Remark 2.5. For a general relative Kähler fibration, we can also obtain the curvature of generalized Weil-Petersson metric $\langle\cdot,\cdot\rangle_{\mathcal{WP}}$. For more details, one can refer to [35, Section 4].

For any two vectors $\xi = \xi^j \frac{\partial}{\partial t^j}, \eta = \eta^j \frac{\partial}{\partial t^j}$ in $T_t\mathcal{B}$, we denote

$$\kappa_\xi = \kappa_j \xi^j, \quad \kappa_\eta = \kappa_j \eta^j.$$

From Theorem 2.4, the holomorphic bisectional curvature satisfies

$$\begin{aligned}R(\xi,\bar{\xi},\eta,\bar{\eta}) &:= R_{j\bar{k}l\bar{m}}\xi^j\bar{\xi}^k\eta^l\bar{\eta}^m \\ &\leq -\langle \overline{\kappa_\eta}\kappa_\xi, \overline{\kappa_\eta}\kappa_\xi\rangle - \langle \kappa_\xi\overline{\kappa_\eta}, \kappa_\xi\overline{\kappa_\eta}\rangle \\ &= -2\langle \overline{\kappa_\eta}\kappa_\xi, \overline{\kappa_\eta}\kappa_\xi\rangle.\end{aligned} \tag{2.18}$$

Note that

$$\langle \overline{\kappa_\eta}\kappa_\xi, \overline{\kappa_\eta}\kappa_\xi\rangle \geq \frac{1}{n}\left|\sum_{\beta=1}^n (\kappa_\eta\overline{\kappa_\xi})_\beta^\beta\right|^2 = \frac{1}{n}\left|\mathrm{Tr}(\kappa_\eta\overline{\kappa_\xi})\right|^2. \tag{2.19}$$

In fact, by taking a normal coordinate system around a fixed point, one can assume that $\phi_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$ at this point. Hence

$$\begin{aligned}\langle \overline{\kappa_\eta}\kappa_\xi, \overline{\kappa_\eta}\kappa_\xi\rangle &= (\overline{\kappa_\eta}\kappa_\xi)_{\bar{\beta}}^{\bar{\gamma}}(\kappa_\eta\overline{\kappa_\xi})_\alpha^\tau\phi^{\alpha\bar{\beta}}\phi_{\tau\bar{\gamma}} \\ &= \sum_{\beta,\gamma=1}^n (\overline{\kappa_\eta}\kappa_\xi)_{\bar{\beta}}^{\bar{\gamma}}(\kappa_\eta\overline{\kappa_\xi})_\beta^\gamma \geq \sum_{\beta=1}^n |(\kappa_\eta\overline{\kappa_\xi})_\beta^\beta|^2 \\ &\geq \frac{1}{n}\left(\sum_{\beta=1}^n |(\kappa_\eta\overline{\kappa_\xi})_\beta^\beta|\right)^2 \geq \frac{1}{n}\left|\sum_{\beta=1}^n (\kappa_\eta\overline{\kappa_\xi})_\beta^\beta\right|^2 = \frac{1}{n}\left|\mathrm{Tr}(\kappa_\eta\overline{\kappa_\xi})\right|^2.\end{aligned}$$

By (2.19), we have

$$\begin{aligned}\langle \overline{\kappa_\eta}\kappa_\xi, \overline{\kappa_\eta}\kappa_\xi\rangle &= \int_{X_t} \langle \overline{\kappa_\eta}\kappa_\xi, \overline{\kappa_\eta}\kappa_\xi\rangle \frac{\omega_t^n}{n!} \\ &\geq \int_{X_t} \frac{1}{n}\left|\mathrm{Tr}(\kappa_\eta\overline{\kappa_\xi})\right|^2 \frac{\omega_t^n}{n!} \\ &\geq \frac{1}{n}\left(\int_{X_t} \left|\mathrm{Tr}(\kappa_\eta\overline{\kappa_\xi})\right| \frac{\omega_t^n}{n!}\right)^2 \left(\int_{X_t} \frac{\omega_t^n}{n!}\right)^{-1} \\ &\geq \frac{1}{n}|\langle\eta,\xi\rangle_{\mathcal{WP}}|^2|X_t|^{-1},\end{aligned} \tag{2.20}$$

where $|X_t| := \int_{X_t} \frac{\omega_t^n}{n!}$ denotes the volume of each fiber. From (2.18) and (2.20), we obtain

$$R(\xi, \overline{\xi}, \eta, \overline{\eta}) \leq -\frac{2}{n}|X_t|^{-1}|\langle \eta, \xi\rangle_{\mathcal{WP}}|^2. \tag{2.21}$$

From (2.21), we obtain the holomorphic bisectional curvature of the generalized Weil-Petersson metric is non-positive, and is negative if ξ and η are not orthogonal to each other. The holomorphic sectional curvature satisfies

$$\frac{R(\xi, \overline{\xi}, \xi, \overline{\xi})}{\|\xi\|^4} \leq -\frac{2}{n}|X_t|^{-1}.$$

The Ricci curvature satisfies

$$\begin{aligned}\frac{\mathrm{Ric}(\xi, \overline{\xi})}{\|\xi\|^2} &= \frac{\sum_{j=1}^{\dim \mathcal{B}} R(\xi, \overline{\xi}, e_j, \overline{e_j})}{\|\xi\|^2} \\ &\leq -\frac{2}{n}|X_t|^{-1}\frac{\sum_{j=1}^{\dim \mathcal{B}}|\langle e_j, \xi\rangle_{\mathcal{WP}}|^2}{\|\xi\|^2} = -\frac{2}{n}|X_t|^{-1},\end{aligned}$$

where $\{e_j\}$ is an orthonormal basis with respect to the generalized Weil-Petersson metric. The scalar curvature satisfies

$$\sum_{j=1}^{\dim \mathcal{B}} \mathrm{Ric}(e_j, \overline{e_j}) \leq -\frac{2}{n}|X_t|^{-1}\dim \mathcal{B}.$$

In a word, we obtain

Corollary 2.6. *For a Monge-Ampère fibration $p : (\mathcal{X}, \omega) \to \mathcal{B}$, the holomorphic bisectional curvature of generalized Weil-Petersson metric $\omega_{\mathcal{WP}}$ satisfies*

$$R(\xi, \overline{\xi}, \eta, \overline{\eta}) \leq -\frac{2}{n}|X_t|^{-1}|\langle \eta, \xi\rangle_{\mathcal{WP}}|^2.$$

for any two vectors η, ξ in $T_t\mathcal{B}$, where $|X_t| := \int_{X_t} \frac{\omega_t^n}{n!}$ denotes the volume of each fiber. In particular,

(i) *Holomorphic bisectional curvature is non-positive, and is negative if $\langle \eta, \xi\rangle_{\mathcal{WP}} \neq 0$;*
(ii) *Holomorphic sectional curvature and Ricci curvature are both bounded from above by $-\frac{2}{n}|X_t|^{-1}$, the scalar curvature is bounded from above by $-\frac{2}{n}|X_t|^{-1}\dim \mathcal{B}$.*

3. Existence of Monge-Ampère fibrations

In this section, we will discuss some existence results on the Monge-Ampère fibrations.

3.1. **Projectively flat vector bundles.** From [16, Corollary 1.2.7, Proposition 1.2.8], a complex vector bundle E is projectively flat if it admits a projectively flat connection, i.e. the curvature satisfies

$$R = \alpha \mathrm{Id}_E \tag{3.1}$$

for some 2-form α. For a holomorphic Hermitian vector bundle (E, h), it is called projectively flat if the Chern curvature of h satisfies (3.1) for some $(1,1)$-form α (see e.g. the proof of [16, Proposition 4.1.11]).

Definition 3.1. Let $\pi : E \to \mathcal{B}$ be a holomorphic vector bundle of rank r over a complex manifold $\mathcal{B}$, we say that the holomorphic vector bundle E admits a *projectively flat Hermitian structure* if there exists a Hermitian metric h such that (E,h) is projectively flat.

Let $\{s_\alpha\}_{\alpha=1}^r$ denote a local holomorphic frame of E, $r = \operatorname{rank} E$, and $\{s^\alpha\}_{\alpha=1}^r$ denote the dual frame of $\{s_\alpha\}$, $h_{\alpha\bar\beta} := h(s_\alpha, s_\beta)$ and $(h^{\bar\beta\alpha})$ be the inverse matrix of $(h^{\bar\beta\alpha})$. Then the Chern curvature is given by

$$\begin{aligned}
R &= R^\alpha_{\beta j\bar k} s_\alpha \otimes s^\beta \otimes dt^j \wedge d\bar t^k \\
&= h^{\bar\gamma\alpha} R_{\beta\bar\gamma j\bar k} s_\alpha \otimes s^\beta \otimes dt^j \wedge d\bar t^k \\
&= h^{\bar\gamma\alpha}(-\partial_j\partial_{\bar k} h_{\beta\bar\gamma} + \partial_j h_{\beta\bar\sigma}\partial_{\bar k} h_{\tau\bar\gamma} h^{\bar\sigma\tau}) s_\alpha \otimes s^\beta \otimes dt^j \wedge d\bar t^k \in A^{1,1}(\mathcal{B}, \operatorname{End} E).
\end{aligned}$$

The Ricci curvature is given by

$$\operatorname{Ric} := \operatorname{Tr} R = \bar\partial\partial \log\det h,$$

which is a d-closed $(1,1)$-form on $\mathcal{B}$. If (E,h) is projectively flat, i.e. it satisfies (3.1), by taking trace to both sides of (3.1), then $\alpha = \frac{1}{r}\operatorname{Ric}$. Thus, (E,h) is projectively flat if and only if

$$R = \frac{1}{r}\operatorname{Ric} \cdot \operatorname{Id}_E. \tag{3.2}$$

Let $P(E) := (E - \{0\})/\mathbb{C}^*$ be the projectivization of the vector bundle E, and consider the projective bundle fibration $p : P(E) \to \mathcal{B}$.

Proposition 3.2. *If $\pi : E \to \mathcal{B}$ admits a projectively flat Hermitian structure, then $p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration.*

Proof. With respect to the local frame $\{s_\alpha\}_{\alpha=1}^r$ of E, we denote by

$$(t; v) = (t^1, \cdots, t^{\dim \mathcal{B}}; v^1, \cdots, v^r)$$

the local holomorphic coordinates of the complex manifold E, which represents the point $v^\alpha s_\alpha \in E$. Then one can define a norm on E by

$$H(v) := h(v^\alpha s_\alpha, v^\beta s_\beta) = h_{\alpha\bar\beta} v^\alpha \bar v^\beta.$$

From [11, Lemma 1.3], one has

$$\partial\bar\partial \log H = -R_{\alpha\bar\beta j\bar k}\frac{v^\alpha \bar v^\beta}{H} dz^j \wedge d\bar z^k + \frac{\partial^2 \log H}{\partial v^\alpha \partial \bar v^\beta}\delta v^\alpha \wedge \delta \bar v^\beta, \tag{3.3}$$

where $\delta v^\alpha := dv^\alpha + v^\beta h^{\bar\gamma\alpha}\partial_j h_{\beta\bar\gamma} dt^j$. By condition, (E,h) is projectively flat, i.e. it satisfies (3.2), so

$$R_{\alpha\bar\beta j\bar k} dz^j \wedge d\bar t^k = \frac{1}{r}\operatorname{Ric} \cdot h_{\alpha\bar\beta}. \tag{3.4}$$

Substituting (3.4) into (3.3), one has

$$\partial\bar\partial \log H = -\frac{1}{r}\operatorname{Ric} + \frac{\partial^2 \log H}{\partial v^\alpha \partial \bar v^\beta}\delta v^\alpha \wedge \delta \bar v^\beta. \tag{3.5}$$

Now we define the following d-closed real $(1,1)$-form on $P(E)$ by

$$\omega := i(\partial\bar{\partial}\log H + \frac{1}{r}\mathrm{Ric}).$$

Then ω is a relative Kähler form. Indeed, for any $t \in \mathcal{B}$, by taking a normal coordinates system around t, $h_{\alpha\bar{\beta}}(t) = \delta_{\alpha\bar{\beta}}$, then

$$\omega|_{P(E_t)} = i(\partial\bar{\partial}\log H + \frac{1}{r}\mathrm{Ric})|_{P(E_t)} = i\partial\bar{\partial}\log\sum_{\alpha=1}^{r}|v^\alpha|^2 > 0,$$

which is exactly the Fubini-Study metric on $P(E_t) = \mathbb{P}^{r-1}$, so we conclude that ω is relative Kähler. From (3.5), one has

$$\omega = i\frac{\partial^2 \log H}{\partial v^\alpha \partial \bar{v}^\beta}\delta v^\alpha \wedge \delta\bar{v}^\beta,$$

which vanishes along the tautological direction, i.e. $\frac{\partial^2 \log H}{\partial v^\alpha \partial \bar{v}^\beta}v^\alpha \bar{v}^\beta = 0$. It follows that $\omega^r = 0$. Thus ω is a Monge-Ampère form, and $p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration. □

Let $p : (P(E), \omega) \to \mathcal{B}$ be a Monge-Ampère fibration over a compact Kähler manifold $\mathcal{B}$. Denote by $\omega_{\mathcal{B}}$ a Kähler metric on $\mathcal{B}$, by taking a large $C > 0$, one concludes that $\omega + Cp^*\omega_{\mathcal{B}}$ is a Kähler metric on $P(E)$, so $P(E)$ is a compact Kähler manifold. Let $\mathcal{O}_{P(E)}(1)$ denote the hyperplane line bundle over $P(E)$. Then

Proposition 3.3. *There exist a constant $k \in \mathbb{R}$ and a d-closed real $(1,1)$-form α on $\mathcal{B}$ such that*

$$[\omega] = kc_1(\mathcal{O}_{P(E)}(1)) + [p^*\alpha]. \tag{3.6}$$

Here $[\bullet]$ denotes the de Rham cohomology class.

Proof. Note that the de Rham cohomology class of $P(E)$ satisfies

$$H^*_{dR}(P(E), \mathbb{R}) = H^*_{dR}(\mathcal{B}, \mathbb{R})[x]/(x^r + c_1(E)x^{r-1} + \cdots + c_r(E)),$$

where $x = c_1(\mathcal{O}_{P(E)}(1))$ (see e.g. [9, (20.7)]), so

$$H^2_{dR}(P(E), \mathbb{R}) = p^*H^2_{dR}(\mathcal{B}, \mathbb{R}) \oplus \mathbb{R}x.$$

Let $H^*_{dR}(P(E), \mathbb{C})$ denote the de Rham cohomology with complex coefficients. By Hodge decomposition theorem (see e.g. [42, Theorem 5.1]), one has

$$H^2_{dR}(P(E), \mathbb{C}) = H^{2,0}_{\bar{\partial}}(P(E)) \oplus H^{1,1}_{\bar{\partial}}(P(E)) \oplus H^{0,2}_{\bar{\partial}}(P(E))$$

where $H^{*,*}_{\bar{\partial}}(P(E))$ denotes the Dolbeault cohomology. Since $x \in H^2_{dR}(P(E), \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(P(E))$, so

$$\begin{aligned} H^2_{dR}(P(E), \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(P(E)) &= p^*H^2_{dR}(\mathcal{B}, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(P(E)) \oplus \mathbb{R}x \\ &= p^*(H^2_{dR}(\mathcal{B}, \mathbb{R}) \cap H^{1,1}_{\bar{\partial}}(\mathcal{B})) \oplus \mathbb{R}x, \end{aligned}$$

where the last equality follows from the Hodge decomposition theorem for the compact Kähler manifold $\mathcal{B}$. Since $[\omega] \in H^2_{dR}(P(E),\mathbb{R}) \cap H^{1,1}_{\bar\partial}(P(E))$, and note that any element in $H^2_{dR}(\mathcal{B},\mathbb{R}) \cap H^{1,1}_{\bar\partial}(\mathcal{B})$ is represented by a d-closed real $(1,1)$-form on $\mathcal{B}$, so

$$[\omega] = kx + [p^*\alpha] = kc_1(\mathcal{O}_{P(E)}(1)) + [p^*\alpha]$$

for some $k \in \mathbb{R}$ and some d-closed real $(1,1)$-form α on $\mathcal{B}$. □

Since ω is a relative Kähler form, so $k > 0$. By the $\partial\bar\partial$-lemma for compact Kähler manifolds (see e.g. [16, Proposition 1.7.24]), there exists a metric $e^{-\psi}$ on $\mathcal{O}_{P(E)}(1)$ such that its curvature satisfies

$$i\partial\bar\partial\psi = \frac{1}{k}(\omega - p^*\alpha).$$

By the condition $\omega^r = 0$, the geodesic curvature form $c(\psi)$ satisfies

$$c(\psi) := c(i\partial\bar\partial\psi) = -\frac{1}{k}p^*\alpha. \tag{3.7}$$

Now we denote

$$L := \mathcal{O}_{P(E)}(1) \otimes K^{-1}_{P(E)/\mathcal{B}} = \mathcal{O}_{P(E)}(r+1) \otimes p^* \det E, \tag{3.8}$$

where the second equality follows from [18, Proposition 2.2]. Since

$$c_1(\det E) = -p_*\left(c_1(\mathcal{O}_{P(E)}(1))^r\right)$$

(see e.g. [13, Section 3.2]), so there exists a metric h_1 on $\det E$ such that

$$c_1(\det E, h_1) = -\int_{P(E)/\mathcal{B}} \left(\frac{i}{2\pi}\partial\bar\partial\psi\right)^r = -\frac{r}{(2\pi)^r}\int_{X_t} c(\psi)(i\partial\bar\partial\psi)^{r-1}_{|X_t} = \frac{r\alpha}{2\pi k}, \tag{3.9}$$

where the last equality follows from (3.7) and noting $\int_{X_t}(\frac{i}{2\pi}\partial\bar\partial\psi)^{r-1}_{|X_t} = 1$. From (3.8), the induced metric on L is

$$e^{-\phi} = e^{-(r+1)\psi} \cdot p^*h_1.$$

The curvature of $e^{-\phi}$ is

$$\partial\bar\partial\phi = (r+1)\partial\bar\partial\psi + p^*\bar\partial\partial \log h_1. \tag{3.10}$$

By (3.7), (3.9) and (3.10), one has

$$\begin{aligned} c(\phi) &= (r+1)c(\psi) + ip^*\bar\partial\partial\log h_1 \\ &= (r+1)(-\frac{1}{k}p^*\alpha) + 2\pi p^*c_1(\det E, h_1) \\ &= -\frac{1}{k}p^*\alpha. \end{aligned} \tag{3.11}$$

By [29, Lemma 5.37], one knows that

$$E^* = p_*(\mathcal{O}_{P(E)}(1)) = p_*(L \otimes K_{P(E)/\mathcal{B}}).$$

Following Berndtsson (cf. [4, 6]), one can define the following L^2-metric on the direct image bundle E^*: for any $u \in E_t^* \equiv H^0(X_t, (L \otimes K_{P(E)/\mathcal{B}})|_{X_t})$, $t \in \mathcal{B}$, then

$$\|u\|^2 = \int_{X_t} |u|^2 e^{-\phi}. \tag{3.12}$$

Note that u can be written locally as $u = f dv \wedge e = f dv^1 \wedge \cdots \wedge dv^n \otimes e$, where e is a local holomorphic frame for $L|_{X_t}$, and so locally

$$|u|^2 e^{-\phi} := i^{n^2} |f|^2 |e|^2 dv \wedge d\bar{v} = i^{n^2} |f|^2 e^{-\phi} dv \wedge d\bar{v}.$$

Theorem 3.4 ([6, Theorem 1.2])**.** *For any $t \in \mathcal{B}$ and let $u \in E_t^*$, one has*

$$\langle iR^{E^*} u, u \rangle = \int_{X_t} c(\phi)|u|^2 e^{-\phi} + \langle (1 + \Box')^{-1} \kappa_j \cdot u, \kappa_k \cdot u \rangle i dt^j \wedge d\bar{t}^k, \tag{3.13}$$

where R^{E^*} denotes the curvature of the Chern connection on E^* with respect to the L^2 metric defined above, here $\Box' = \nabla' \nabla'^* + \nabla'^* \nabla$ is the Laplacian on $L|_{X_t}$-valued forms on X_t defined by the $(1,0)$-part of the Chern connection on $L|_{X_t}$.

Let $\{u_\alpha\}, 1 \leq \alpha \leq r$, be a local holomorphic frame of E^*, and set

$$G_{\alpha\bar{\beta}} = \langle u_\alpha, u_\beta \rangle = \int_{X_t} u_\alpha \overline{u_\beta} e^{-\phi}.$$

By taking trace to both sides of (3.13) and using (3.11), we have

$$i\mathrm{Ric}^{E^*} = -\frac{r}{k}\alpha + \langle (1 + \Box')^{-1} \kappa_j \cdot u_\alpha, \kappa_k \cdot u_\beta \rangle G^{\alpha\bar{\beta}} i dt^j \wedge d\bar{t}^k \geq -\frac{r}{k}\alpha, \tag{3.14}$$

where the above equality holds if and only if $\kappa_j = 0$ for all $1 \leq j \leq \dim \mathcal{B}$. From (3.9), one has

$$[i\mathrm{Ric}^{E^*}] = 2\pi c_1(E^*) = \left[-\frac{r}{k}\alpha\right]. \tag{3.15}$$

Combining (3.14) with (3.15) shows that $i\mathrm{Ric}^{E^*} = -\frac{r}{k}\alpha$ and thus

$$\kappa_j \equiv 0 \tag{3.16}$$

on $P(E)$. Since the generalized Weil-Petersson metrics with respect to ω and $i\partial\bar{\partial}\phi$ are the same, so $\omega_{\mathcal{WP}} \equiv 0$ on $\mathcal{B}$. Substituting (3.16) into (3.13), we get

$$\langle iR^{E^*} u, u \rangle = \int_{X_t} c(\phi)|u|^2 e^{-\phi} = -\frac{\alpha}{k}\|u\|^2,$$

which is equivalent to $R^{E^*} = i\frac{\alpha}{k}\mathrm{Id}_{E^*}$. Thus, with respect to the dual metric of the L^2-metric (3.12), the Chern curvature R^E is given by

$$R^E = -i\frac{\alpha}{k}\mathrm{Id}_E,$$

which implies that E is projectively flat.

Theorem 3.5. *If $p : (P(E), \omega) \to \mathcal{B}$ is a Monge-Ampère fibration over a compact Kähler manifold $\mathcal{B}$, then E admits a projectively flat Hermitian structure, and $\omega_{\mathcal{WP}} \equiv 0$ on $\mathcal{B}$.*

From [16, (2.3.4), (2.3.5) and Proposition 2.3.1 (b)], we obtain

Corollary 3.6. *If $p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration over a compact Kähler manifold $\mathcal{B}$, then*

(i) $c(E) = \left(1 + \frac{c_1(E)}{r}\right)^r$;
(ii) $\mathrm{ch}(\mathrm{End}(E)) = r^2$.

For the case of $\mathcal{B}$ is a compact Riemann surface, $\dim \mathcal{B} = 1$. Put

$$\mu(E) = \frac{\int_{\mathcal{B}} c_1(E)}{\mathrm{rank}(E)}.$$

Recall that E is said to be stable (resp. semi-stable) in the sense of Mumford if for every proper subbundle E' of E, $0 < \mathrm{rank}(E') < \mathrm{rank}(E)$, we have

$$\mu(E') < \mu(E), \quad (resp.\ \ \mu(E') \le \mu(E)). \tag{3.17}$$

E is called polystable if $E = \oplus E_i$ with E_i stable vector bundles all of the same slope $\mu(E) = \mu(E_i)$, see e.g. [15, Section 4.B]. Thus

Theorem 3.7. *Let E be a holomorphic vector bundle over a compact Kähler manifold $\mathcal{B}$. Let $P(E) := (E - \{0\})/\mathbb{C}^*$ be the projectivization of E. Then the following are equivalent:*

1) *E admits a projectively flat Hermitian structure;*
2) *$p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration.*

For the case of $\dim \mathcal{B} = 1$, both are equivalent to the polystability of E.

Proof. Now it suffices to prove the last part. Assume that $\dim \mathcal{B} = 1$, i.e. $\mathcal{B}$ is a compact Riemann surface. By [16, Proposition 5.2.3], (E, h) is projectively flat if and only if (E, h) is weak Hermitian-Einstein , i.e. $\Lambda_{\omega_{\mathcal{B}}} R^E = \varphi \mathrm{Id}_E$ for some function φ. By a conformal change (see e.g. [16, Proposition 4.2.4]), E admits a weak Hermitian-Einstein metric if and only if E admits a Hermitian-Einstein metric. Thus, E admits a Hermitian-Einstein metric if and only if E admits a projectively flat Hermitian metric, which is equivalent to that $p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration. All are equivalent to the polystability of E (see e.g. [15, Theorem 4.B.9]). The proof is complete. □

Remark 3.8. In [3], T. Aikou considered the projectively flat holomorphic vector bundle from the view of complex Finsler geometry, and proved that E admits a projectively flat Hermitian metric if and only if the projective bundle $p : P(E) \to \mathcal{B}$ is a flat Kähler fibration (see [3, Theorem 3.2]), where a Kähler fibration $p : \mathcal{X} \to \mathcal{B}$ with a smooth family of Kähler metrics $\{\Pi_z\}_{z \in \mathcal{B}}$ is said to be flat if, at each point $z \in \mathcal{B}$, there exists an open neighborhood U of z so that we can choose Kähler potentials for Π_z which is independent of $z \in U$, see [3, Definition 1.2]. Combining with Proposition 2.3 and Theorem 2.4, in the case that $\mathcal{B}$ is a compact Kähler manifold, the projective bundle $p : P(E) \to \mathcal{B}$ is a Monge-Ampère fibration if and only if it is a flat Kähler fibration.

Remark 3.9. After our paper [35] was submitted to arXiv, by using the negativity of direct image bundles [5, Section 3], S. Finski [12, Theorem 5.1] obtained another kind of description of the projectively flat holomorphic vector bundles, i.e., E admits a projectively flat Hermitian structure if and only if the class

$$\Lambda_E := c_1\left(\mathcal{O}_{P(E^*)}(1)\right) - \frac{1}{r}p^*c_1(E)$$

is semi-positive. In fact, if E admits a projectively flat Hermitian structure, so is E^*. By [35, Proposition 6.2, (6.6)], one knows that Λ_E is semi-positive. Conversely, if Λ_E is semi-positive, let α be a semi-positive form in the class Λ_E, then

$$\int_{P(E^*)} \alpha^r \wedge p^*\omega_0^{m-1} = \int_{P(E^*)} \Lambda_E^r \wedge p^*\omega_0^{m-1} \geq 0,$$

where ω_0 is a Kähler form on $\mathcal{B}$, $\dim \mathcal{B} = m$. On the other hand,

$$\begin{aligned}\int_{P(E^*)} \Lambda_E^r \wedge p^*\omega_0^{m-1} &= \int_{P(E^*)} c_1\left(\mathcal{O}_{P(E^*)}(1)\right)^r \wedge p^*\omega_0^{m-1} \\ &\quad - \int_{P(E^*)} c_1\left(\mathcal{O}_{P(E^*)}(1)\right)^{r-1} \wedge p^*c_1(E) \wedge p^*\omega_0^{m-1} \\ &= \int_{\mathcal{B}} c_1(E)\wedge\omega_0^{m-1} - \int_{\mathcal{B}} c_1(E)\wedge\omega_0^{m-1} = 0,\end{aligned}$$

which follows that $\alpha^r \wedge p^*\omega_0^{m-1} = 0$ since α is semi-positive, which is equivalent to $\alpha^r = 0$, i.e. α is a Monge-Ampère form. By [35, Theorem B] or Theorem 3.7, E admits a projectively flat Hermitian structure.

3.2. **Infinite rank flat Higgs bundles.** Firstly, we will recall the notion of quasi-vector bundles, and one can refer to an early version of [8].

Definition 3.10 (Quasi-vector bundle)**.** Let $A := \{A_t\}_{t\in\mathcal{B}}$ be a family of $\mathbb{C}$-vector spaces over a smooth manifold $\mathcal{B}$. Let Γ be a $C^\infty(\mathcal{B})$-submodule of the space of all sections of A. We call Γ a smooth quasi-vector bundle structure on V if each vector of the fiber A_t extends to a section in Γ locally near t.

Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. Let E be a holomorphic vector bundle over $\mathcal{X}$ with smooth Hermitian metric h_E. We write

$$X_t := p^{-1}(t), \quad E_t := E_{X_t}, \quad h_{E_t} := h_E|_{E_t}.$$

For each $t \in \mathcal{B}$, denote by $\mathcal{A}^{p,q}(E_t)$ the space of all smooth E_t-valued (p,q)-forms on X_t. Put

$$\mathcal{A}^{p,q} := \{\mathcal{A}^{p,q}(E_t)\}_{t\in\mathcal{B}}.$$

Denote by $\mathcal{A}^{p,q}(E)$ the space of smooth E-valued (p,q)-forms on $\mathcal{X}$. Let us define

$$\Gamma^{p,q} := \{u : t \mapsto u^t \in \mathcal{A}^{p,q}(E_t) : \exists\ \mathbf{u} \in \mathcal{A}^{p,q}(E),\ \mathbf{u}|_{X_t} = u^t,\ \forall\ t \in \mathcal{B}\}. \tag{3.18}$$

We call $\mathbf{u}$ a *smooth representative* of $u \in \Gamma^{p,q}$. Since p is a proper smooth submersion, we know that each $\Gamma^{p,q}$ defines a quasi-vector bundle structure on $\mathcal{A}^{p,q}$. Consider

$$(\mathcal{A}^k, \Gamma^k) := \oplus_{p+q=k}(\mathcal{A}^{p,q}, \Gamma^{p,q}).$$

We know that the fiber of $\mathcal{A}^k$ can be written as

$$\mathcal{A}^k(E_t) = \oplus_{p+q=k}\mathcal{A}^{p,q}(E_t),$$

which is the space of all E-valued smooth k-forms on X_t. For every $u \in \Gamma^k$, let us define

$$\nabla u := \sum dt^j \otimes [d^E, \delta_{V_j}]\mathbf{u} + \sum d\bar{t}^j \otimes [d^E, \delta_{\bar{V}_j}]\mathbf{u}, \tag{3.19}$$

where each V_j denotes the horizontal lift of $\partial/\partial t^j$ with respect to ω and

$$d^E := \overline{\partial} + \partial^E,$$

denotes the Chern connection on (E, h_E).

Definition 3.11. In this paper we shall identify u with its smooth representative $\mathbf{u}$. We call ∇ the Lie derivative connection on $(\mathcal{A}^k, \Gamma^k)$ with respect to ω.

For each p, q with $p + q = k$, ∇ induces a connection, say D, on $(\mathcal{A}^{p,q}, \Gamma^{p,q})$. For bidegree reason, we have

$$Du := \sum dt^j \otimes [\partial^E, \delta_{V_j}]\mathbf{u} + \sum d\bar{t}^j \otimes [\overline{\partial}, \delta_{\bar{V}_j}]\mathbf{u}, \qquad \forall\ u \in \Gamma^{p,q}. \tag{3.20}$$

The associated second fundamental form can be written as

$$(\nabla - D)u = \sum dt^j \otimes \kappa_j \cdot \mathbf{u} + \sum d\bar{t}^j \otimes \overline{\kappa_j} \cdot \mathbf{u},$$

where each

$$\kappa_j : \mathbf{u} \mapsto \kappa_j \cdot \mathbf{u},$$

denotes the action of the Kodaira–Spencer tensor κ_j on u.

Definition 3.12. We call

$$\theta := \sum dt^j \otimes \kappa_j,$$

the Higgs field associated to $(\mathcal{A}^k, \Gamma^k, \omega)$.

By Theorem 5.6 in [41] (or an early version of [8]), we know that

Proposition 3.13. *D defines a Chern connection on each $(\mathcal{A}^{p,q}, \Gamma^{p,q})$ and each $\overline{\kappa_j} = \kappa_j^*$.*

The curvature of the Lie derivative connection is

$$\nabla^2 u = \sum (dt^j \wedge d\bar{t}^k) \otimes [[d^E, \delta_{V_j}], [d^E, \delta_{\bar{V}_k}]]\mathbf{u}. \tag{3.21}$$

For bidegree reason, it gives the following curvature formula for the induced Chern connection

$$D^2 u = \nabla^2 u - \sum (dt^j \wedge d\bar{t}^k) \otimes [\kappa_j, \overline{\kappa_k}] \cdot \mathbf{u}. \tag{3.22}$$

Together with the following Lie derivative identity (see Proposition 4.2 in [39])

$$[[d^E, \delta_{V_j}], [d^E, \delta_{\bar{V}_k}]]\mathbf{u} = [d^E, \delta_{[V_j, \bar{V}_k]}]\mathbf{u} + \Theta^E(V_j, \bar{V}_k)\mathbf{u}, \tag{3.23}$$

where $\Theta^E := (d^E)^2$ denotes the Chern curvature of (E, h_E), (3.22) and (3.23) imply

Theorem 3.14. *For every $u \in \Gamma^{p,q}$, write*

$$D^2 u = \sum (dt^j \wedge d\bar{t}^k) \otimes \Theta_{j\bar{k}} u,$$

then the Chern curvature operators $\Theta_{j\bar{k}}$ satisfy

$$(\Theta_{j\bar{k}} u, u) = ([d^E, \delta_{[V_j, \bar{V}_k]}]\mathbf{u}, u) + (\Theta^E(V_j, \bar{V}_k)\mathbf{u}, u) + (\kappa_j u, \kappa_k u) - (\overline{\kappa_k} u, \overline{\kappa_j} u).$$

Proposition 3.15. *Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a Monge-Ampère fibration. If $\Theta^E \equiv 0$ then*

i) $\nabla^2 = 0$;
ii) $\theta^2 = 0$;
iii) $D\theta + \theta D = 0$.

In particular, each $(\mathcal{A}^k, \Gamma^k, D, \theta)$ is an infinite rank flat Higgs bundle.

Proof. Since the total degree of the Kodaira–Spencer tensor is zero, $\theta^2 = 0$ is always true. Moreover

$$D^{1,0}\theta + \theta D^{1,0} = 0$$

follows from $[V_j, V_k] \equiv 0$, which is true for every relative Kähler fibration. Assume further that ω is a Monge-Ampère form, then we have

$$[V_j, \overline{V_k}] \equiv 0$$

by Proposition 1.5, which gives

$$D^{0,1}\theta + \theta D^{0,1} = 0 \ \text{ i.e. } \theta \text{ is holomorphic,}$$

and (by (3.23) and (3.21))

$$\nabla^2 = \sum (dt^j \wedge d\bar{t}^k) \otimes \Theta^E(V_j, \overline{V_k}).$$

Thus $\nabla^2 = 0$ if one further assumes that $\Theta^E \equiv 0$. □

Theorem 3.16. *A relative Kähler fibration $p : (\mathcal{X}, \omega) \to \mathcal{B}$ is a Monge-Ampère fibration if and only if the following associated infinite rank Higgs bundle*

$$(\mathcal{A}, \Gamma, D, \theta)$$

is Higgs-flat, where each fiber $\mathcal{A}_t$ denotes the space of smooth differential forms on X_t.

Proof. By taking E to be a trivial bundle, then the bundle $\mathcal{A}$ is precisely $\oplus_{k=0}^{2n} \mathcal{A}^k$. Thus if $p : (\mathcal{X}, \omega) \to \mathcal{B}$ is a Monge-Ampère fibration, then Proposition 3.15 implies that $\mathcal{A}$ is Higgs flat. On the other hand, since

$$\nabla^2 = \sum (dt^j \wedge d\bar{t}^k) \otimes [d, \delta_{[V_j, \overline{V_k}]}],$$

we know that if $\mathcal{A}$ is Higgs flat, then $\nabla^2 \equiv 0$ gives

$$[d, \delta_{[V_j, \overline{V_k}]}] u \equiv 0$$

on fibers for all smooth form u on $\mathcal{X}$. Take u to be an arbitrary smooth function, we get

$$[d, \delta_{[V_j, \overline{V_k}]}] u = [V_j, \overline{V_k}] u = 0,$$

which implies $[V_j, \overline{V_k}] \equiv 0$. Thus ω is a Monge-Ampère form by Proposition 1.5. The proof is complete. □

4. Examples of Monge-Ampère fibrations

In this section, we will introduce some examples of Monge-Ampère fibrations, which are also the motivations for studying such kinds of fibrations.

4.1. **Family of elliptic curves.** For each t in the upper half plane $\mathbb{H} := \{t \in \mathbb{C} : \operatorname{Im} t > 0\}$, consider the the following elliptic curve (one dimensional torus)

$$X_t := \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}).$$

There is a canonical diffeomorphism from each X_t to a fixed elliptic curve, say X_i. In fact, the $\mathbb{R}$-linear quasi-conformal mapping $f^t : \mathbb{C} \to \mathbb{C}$ defined by

$$f^t(1) = 1, \quad f^t(t) = i, \tag{4.1}$$

naturally induces a map, still denoted by f^t, from X_t to X_i. A direct computation gives

$$f^t(\zeta) = z = \frac{i - \bar{t}}{t - \bar{t}}\,\zeta + \frac{t - i}{t - \bar{t}}\,\bar{\zeta}.$$

Now $\{f^t\}_{t \in \mathbb{H}}$ defines a smooth trivialization of $\mathcal{X} := \{X_t\}_{t \in \mathbb{H}} \simeq (\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2$ as follows

$$f : \mathcal{X} \to \mathbb{H} \times X_i, \quad f(t, \zeta) := (t, f^t(\zeta)).$$

The natural Kähler form $i\, dz \wedge d\bar{z}$ on $\mathbb{C}$ induces a Kähler form on X_i, thus a relative Kähler form, say ω_i on $\mathbb{H} \times X_i$. Consider its pull back, say $\omega := f^*\omega_i$, on $\mathcal{X}$, we have

Proposition 4.1. *ω is a Monge-Ampère form on the following canonical fibration*

$$p : \mathcal{X} \to \mathbb{H}, \quad p(X_t) := t.$$

Proof. Notice that $(i\, dz \wedge d\bar{z})^2 = 0$ gives $\omega^2 = 0$. Moreover, ω can be written as the following form:

$$\omega = i\alpha \wedge \bar{\alpha},$$

where

$$\alpha := f^* dz = \frac{i - \bar{t}}{t - \bar{t}}\, d\zeta + \frac{t - i}{t - \bar{t}}\, d\bar{\zeta} + \frac{(i - \bar{t})(\bar{\zeta} - \zeta)}{(t - \bar{t})^2}\, dt + \frac{(t - i)(\bar{\zeta} - \zeta)}{(t - \bar{t})^2}\, d\bar{t},$$

we get

$$\omega = \frac{i}{\operatorname{Im} t}\left(d\zeta \wedge d\bar{\zeta} + A\, d\zeta \wedge d\bar{t} + \bar{A}\, dt \wedge d\bar{\zeta} + |A|^2 dt \wedge d\bar{t}\right), \quad A := \frac{\zeta - \bar{\zeta}}{\bar{t} - t}.$$

Thus ω is of degree-$(1,1)$ and positive on each fiber. Hence ω is a Monge-Ampère form. □

Remark 4.2. The above fibration possesses a natural $SL_2(\mathbb{Z})$ action

$$SL_2(\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (t, \zeta) \mapsto \left(\frac{at + b}{ct + d}, \frac{\zeta}{ct + d}\right),$$

which preserves ω. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, then each Γ quotient of the upper half-plane $\mathbb{H}$ can be compactified, thus the regular part induces a Monge-Ampère fibration over a quasi-projective manifold. Similarly, one can also construct the Monge-Ampère family of Abelian varieties, see Remark 4.4 for another approach.

4.2. **Finite dimensional Higgs bundles.** Denote by $\mathfrak{gl}_n(\mathbb{C})$ the space of n by n complex matrices. Consider the following bounded symmetric domain of the third type

$$\mathrm{BSD}_{\mathrm{III}} := \{B \in \mathfrak{gl}_n(\mathbb{C}) : B = B^T,\ B\bar{B}^T < 1\},$$

where B^T denotes the transpose of B and $B\bar{B}^T < 1$ means all eigenvalues of $B\bar{B}^T$ are less than one. One may define a canonical *holomorphic motion* of $\mathbb{C}^n$:

$$F : \mathrm{BSD}_{\mathrm{III}} \times \mathbb{C}^n \to \mathrm{BSD}_{\mathrm{III}} \times \mathbb{C}^n; \quad F(B,z) = (B,\zeta), \quad \zeta := z + B\bar{z}, \tag{4.2}$$

where we think of z as a column vector and $B\bar{z}$ denotes the matrix multiplication. The natural metric $i\partial\overline{\partial}|z|^2$ on $\mathbb{C}^n$ defines a relative Kähler metric, still write it as $i\partial\overline{\partial}|z|^2$, on $\mathrm{BSD}_{\mathrm{III}} \times \mathbb{C}^n$. Then one can check that

$$\Omega := (F^{-1})^*(i\partial\overline{\partial}|z|^2)$$

is of degree $(1,1)$ with respect to the (B,ζ) coordinate on $\mathrm{BSD}_{\mathrm{III}} \times \mathbb{C}^n$.

Theorem 4.3. *Put $\mathcal{X} := \mathrm{BSD}_{\mathrm{III}} \times \mathbb{C}^n$, then the natural projection*

$$p : (B,\zeta) \to B,$$

defines a (non-proper) Monge-Ampère fibration $p : (\mathcal{X},\Omega) \to \mathrm{BSD}_{\mathrm{III}}$.

Proof. Notice that it is positive on the central fiber and symplectic on each fiber, thus Ω is relative Kähler. Moreover, $(i\partial\overline{\partial}|z|^2)^n = 0$ implies that $\Omega^n = 0$. Thus Ω is a Monge-Ampère form. $\square$

Remark 4.4. Fix an abelian variety $\mathbb{C}^n/\mathbb{Z}^{2n}$, the map F in (4.2) induces a natural $\mathbb{Z}^{2n}$ action on $\mathcal{X}$, which gives a Monge-Ampère family of Abelian varieties $\mathcal{X}/\mathbb{Z}^{2n} \to \mathrm{BSD}_{\mathrm{III}}$.

4.2.1. *Higgs bundles over* $\mathrm{BSD}_{\mathrm{III}}$. For each $t \in \mathrm{BSD}_{\mathrm{III}}$, let us denote by $\mathcal{A}_t^k$ the space of *translation invariant* k-forms on $p^{-1}(t) = \mathbb{C}^n$. Then we have the following finite rank vector bundle

$$\mathcal{A}^k := \{\mathcal{A}_t^k\}_{t\in\mathrm{BSD}_{\mathrm{III}}}.$$

Notice that our holomorphic motion F in (4.2) defines a flat connection

$$\nabla := \sum dt^j \otimes L_{V_j} + \sum d\bar{t}^j \otimes L_{\bar{V}_j}, \quad V_j := F_*\left(\frac{\partial}{\partial t^j}\right), \tag{4.3}$$

on $\mathcal{A}^k$ (since F is linear on fibers, the above connection is well defined on the space of invariant forms; flatness follows from $[\frac{\partial}{\partial t^j}, \frac{\partial}{\partial t^k}] = [\frac{\partial}{\partial t^j}, \frac{\partial}{\partial \bar{t}^k}] = 0$). Denote by $\mathcal{A}^{p,q} := \{\mathcal{A}_t^{p,q}\}_{t\in\mathrm{BSD}_{\mathrm{III}}}$ each (p,q) component of $\mathcal{A}^k$, i.e. each $\mathcal{A}_t^{p,q}$ is the space of translation invariant (p,q)-forms on $p^{-1}(t)$. By the Cartan formula for the Lie derivative, we have

$$L_{V_j} = [d,\delta_{V_j}] = [\partial,\delta_{V_j}] + [\overline{\partial},\delta_{V_j}], \tag{4.4}$$

thus only $[\partial,\delta_{V_j}]$ preserve the bidegree, from which we know the induced connection on each $\mathcal{A}^{p,q}$ can be written as

$$D = \sum dt^j \otimes D_{\partial/\partial t^j} + \sum d\bar{t}^j \otimes D_{\partial/\partial \bar{t}^k}, \quad D_{\partial/\partial t^j} := [\partial,\delta_{V_j}],\ D_{\partial/\partial \bar{t}^k} := [\overline{\partial},\delta_{\bar{V}_k}],$$

Moreover, we have

$$\nabla - D = \theta + \bar{\theta}, \quad \theta := \sum dt^j \otimes [\overline{\partial},\delta_{V_j}].$$

We call θ the *Higgs field* on $\mathcal{A}^k$. We also need the following lemma, which is a special case of Theorem 5.6 in [41].

Lemma 4.5. *D defines a Chern connection on each $\mathcal{A}^{p,q}$ with respect to the metric defined by Ω, moreover $[\overline{\partial}, \delta_{V_j}]^* = [\partial, \delta_{\bar{V}_j}]$.*

Proof. To show that the $(0,1)$-part of D is integrable, it is enough to prove

$$[[\overline{\partial}, \delta_{\bar{V}_j}], [\overline{\partial}, \delta_{\bar{V}_k}]] = 0,$$

which follows from $[L_{\bar{V}_j}, L_{\bar{V}_k}] = L_{[\bar{V}_j, \bar{V}_k]} = 0$. Now it suffices to check that D preserves the metric and $[\overline{\partial}, \delta_{V_j}]^* = [\partial, \delta_{\bar{V}_j}]$. The idea is to use the primitive decomposition and the fact that ∇ commutes with $\Omega\wedge$. Details can be found in [41]. $\square$

Theorem 4.6. *The above lemma implies that each $(\mathcal{A}^k, \theta, D)$ is a flat Hermitian Higgs bundle.*

4.2.2. *Curvature properties of the space of complex structures.* Let (V, ω) be a $2n$ dimensional real vector space V with a symplectic form ω. Denote by $\mathcal{J}(V, \omega)$ the space of ω-compatible complex structures on V. For each $J \in \mathrm{BSD_{III}}$ and $p + q = k$, denote by $\wedge_J^{p,q}$ the space of J-(p, q)-forms in $\wedge^k(\mathbb{C} \otimes V^*)$. It is known that $\mathcal{J}(V, \omega)$ is isomorphic to $\mathrm{BSD_{III}}$, and the Higgs bundle $\mathcal{A}^k$ has the following description

$$\mathcal{A}^k \simeq \mathcal{H}^k := \oplus_{p+q=k} \mathcal{H}^{p,q}, \quad \mathcal{H}^{p,q} := \{\wedge_J^{p,q}\}_{J \in \mathcal{J}(V,\omega)}.$$

Thus as in [38] one may define the associated *Lu's Hodge metric*, say $\omega_{\mathcal{WP},k}$, on $\mathcal{J}(V, \omega)$. One may verify that all $\omega_{\mathcal{WP},k}$ are equal up to positive constants, i.e.

$$\omega_{\mathcal{WP},k} = c(k, n)\omega_{\mathcal{WP},1},$$

where $c(k, n)$ depends only on k and n. In fact, $\omega_{\mathcal{WP},1}$ is just the generalized Weil-Petersson metric in Definition 1.12 (up to a factor). Hence $\omega_{\mathcal{WP},1}$ is Kähler on $\mathcal{J}(V, \omega)$ with non-positive holomorphic bisectional curvature; moreover, its holomorphic sectional curvature is bounded above by $-2/n$.

4.3. **Geodesics.**

4.3.1. *Kähler metric geodesics.* Let (X, ω) be a fixed n-dimensional compact Kähler manifold. Consider the following Mabuchi space of Kähler potentials

$$\mathcal{K} := \{\phi \in C^\infty(X, \mathbb{R}) : \omega + i\partial\overline{\partial}\phi > 0\}$$

on X. Fix ϕ_0, ϕ_1 in $\mathcal{K}$, if there exists a smooth function ϕ on a neighborhood of the closure of

$$\mathcal{X} := \mathbb{H}_{0,1} \times X, \quad \mathbb{H}_{0,1} := \{\tau \in \mathbb{C} : 0 < \mathrm{Re}\,\tau < 1\},$$

such that $\phi(0, x) = \phi_0(x)$, $\phi(1, x) = \phi_1(x)$, ϕ does not depend on the imaginary part of τ and

$$(\omega + i\partial\overline{\partial}\phi)^{n+1} \equiv 0 \text{ on } \mathcal{X}, \quad \phi(t, \cdot) \in \mathcal{K},$$

then we say that $\{\phi(t, \cdot)\}_{t\in[0,1]}$ is a smooth geodesic in $\mathcal{K}$ connecting ϕ_0, ϕ_1. Associated with a smooth geodesic, the following trivial fibration

$$p : (\mathcal{X}, \omega + i\partial\overline{\partial}\phi) \to \mathbb{H}_{0,1}$$

is a Monge-Ampère fibration.

4.3.2. *Convex function geodesics.* If ϕ is a smooth, strictly convex function on $\mathbb{R}^n$, then we know that its gradient map

$$\nabla\phi : x \mapsto (\phi_{x_1}(x), \cdots, \phi_{x_n}(x)), \quad \phi_{x_j} := \partial\phi/\partial x_j,$$

defines a diffeomorphism from $\mathbb{R}^n$ onto an open set

$$A_\phi := \nabla\phi(\mathbb{R}^n)$$

in $\mathbb{R}^n$. Moreover, one can check that A_ϕ is convex in $\mathbb{R}^n$.

Definition 4.7. Let A be a bounded open convex set in $\mathbb{R}$. A smooth, strictly convex function ϕ on $\mathbb{R}^n$ is said to be of type A if $A_\phi = A$. We call denote by $\mathcal{C}_A$ the space of type A functions.

Note that $\mathcal{C}_A$ is not empty. In fact, if ψ is a smooth, strictly convex function on A that tends to infinity at the boundary of A, then its Legendre transform

$$\psi^*(x) := \sup_{y\in A} x \cdot y - \psi(y), \quad \forall\ x \in \mathbb{R},$$

lies in $\mathcal{C}_A$. $A_{\phi+\psi} = A_\phi + A_\psi$ implies that $\mathcal{C}_A$ is a convex set.

The Legendre transform of $\phi \in \mathcal{C}_A$ is defined by

$$\phi^*(y) := \sup_{x\in\mathbb{R}} x \cdot y - \phi(x), \quad \forall\ y \in A.$$

We know that ϕ^* is smooth and strictly convex on A. Moreover, if $\phi_0, \phi_1 \in \mathcal{C}_A$, then

$$\phi : (t, x) \mapsto (t\phi_1^* + (1-t)\phi_0^*)^*(x) \tag{4.5}$$

satisfies

$$MA(\phi) = 0$$

on $[0, 1] \times \mathbb{R}^n$, where $MA(\phi)$ denotes the determinant of the full Hessian of ϕ.

Definition 4.8. We call ϕ defined in (4.5) the geodesic between $\phi_0, \phi_1 \in \mathcal{C}_A$.

Let $\mathcal{X} := [0, 1] \times \mathbb{R}^n \times \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$ be the natural complexification of $[0, 1] \times \mathbb{R}^n$. Think of ϕ as a function on $\mathcal{X}$, then

$$p : (\mathcal{X}, i\partial\overline{\partial}\phi) \to \mathcal{B}, \quad \mathcal{B} := [0, 1] \times \mathbb{R} \subset \mathbb{C},$$

is a (non-proper) Monge-Ampère fibration.

4.3.3. *Hermitian form geodesics.* Denote by $\mathcal{H}$ the space of Hermitian forms on $\mathbb{C}^n$. Let $\{e_j\}$ be the canonical basis of $\mathbb{C}^n$ then a Hermitian form, say $\omega \in \mathcal{H}$, can be written as

$$\omega = i \sum_{j,k=1}^{n} a_{j\bar{k}}\, e_j^* \wedge \overline{e_k^*},$$

where $A := (a_{j\bar{k}})$ satisfies

$$a_{j\bar{k}} = \overline{a_{k\bar{j}}}$$

and $\sum a_{j\bar{k}}\xi^j\bar{\xi}^k > 0$ if $\xi \neq 0$. Thus we can identify ω with a Hermitian matrix A. Now let

$$\mathbb{A} := \{A_t\}_{t\in[0,1]}$$

be a smooth family (smooth on a neighborhood of $[0,1]$) of Hermitian matrices. We know that $\mathbb{A}$ defines a smooth metric on the trivial bundle

$$p : \mathcal{X} \to \mathcal{B}, \quad \mathcal{X} := [0,1] \times \mathbb{R} \times \mathbb{C}^n, \quad \mathcal{B} := [0,1] \times \mathbb{R} \subset \mathbb{C},$$

with Chern curvature

$$\Theta_{tt}(\mathbb{A})e_j = \sum (a_{j\bar{k},t} a^{\bar{k}l})_t e_l = \sum (a_{j\bar{k},tt} a^{\bar{k}l} - a_{j\bar{k},t} a_{p\bar{q},t} a^{\bar{k}p} a^{\bar{q}l}) e_l,$$

where $(a^{\bar{k}l})$ denotes the inverse matrix of $(a_{j\bar{k}})$ and $f_{,t}$ denotes the derivative of f with respect to t. Think of

$$\phi(t,z) := \sum a_{j\bar{k}}(t) z^j \bar{z}^k$$

as a function on $\mathcal{X}$. Then $i\partial\overline{\partial}\phi$ defines a relative Kähler form on $\mathcal{X}$. A direct computation gives

Proposition 4.9. $\Theta_{tt}(\mathbb{A}) \equiv 0$ *if and only if* $(i\partial\overline{\partial}\phi)^{n+1} \equiv 0$.

Now we know that if $\mathbb{A}$ is flat, then

$$p : (\mathcal{X}, i\partial\overline{\partial}\phi) \to \mathcal{B}$$

is a (non-proper) Monge-Ampère fibration.

Definition 4.10. We say that $\mathbb{A}$ is the geodesic between A_0 and A_1 if $\Theta_{tt}(\mathbb{A}) \equiv 0$.

References

[1] L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. Math. (1961), 171-191. pages 2

[2] L. V. Ahlfors, *Curvature properties of Teichmüller's space*, Journal d'Analyse Mathématique **9** (1961), 161-176. pages 2

[3] T. Aikou, *Projective flatness of complex Finsler metrics*, Publ. Math. Debrecen **63** (2003), no. 3, 343-362. pages 19

[4] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. Math. **169**, (2009), 531-560. pages 18

[5] B. Berndtsson. Positivity of direct image bundles and convexity on the space of Kähler metrics. J. Diff. Geom., **81**(3), (2009), 457-482. pages 19

[6] B. Berndtsson, *Strict and non strict positivity of direct image bundles*, Math. Z. **269** (3-4), (2011), 1201-1218. pages 18

[7] B. Berndtsson, *Long geodesics in the space of Kähler metrics*, Anal Math (2022). https://doi.org/10.1007/s10476-022-0140-z pages 3

[8] B. Berndtsson, M. Păun, X. Wang, *Algebraic fiber spaces and curvature of higher direct images*, Journal of the Institute of Mathematics of Jussieu, 2020, 1-56. doi:10.1017/S147474802000050X. pages 2, 5, 20, 21

[9] R. Bott, L. W. Tu, Differential Forms in Algebraic Topology (Springer, 1982). pages 16

[10] D. Burns, *Curvatures of Monge-Ampère foliations and parabolic manifolds*, Ann. Math. **115** (1982), 349-373. pages 2, 3

[11] H. Feng, K. Liu, X. Wan, *Chern forms of holomorphic Finsler vector bundles and some applications*, Inter. J. Math. Vol. **27**, No. 4, (2016) 1650030. pages 15

[12] S. Finski, *On Monge-Ampère volumes of direct images*, International Mathematics Research Notices, 2021, rnab058. https://doi.org/10.1093/imrn/rnab058 pages 19

[13] W. Fulton, Intersection Theory, Second Edition, Springer, 1998. pages 17

[14] A. Fujiki, G, Schumacher, *The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics.* Publications of the Research Institute for Mathematical Sciences, **26** (1990), 101-183. pages 8, 9

[15] D. Huybrechts, Complex geometry. An introduction. Universitext. Springer-Verlag, Berlin, 2005. xii+309 pp. pages 19

[16] S. Kobayashi, Differential geometry of complex vector bundles, Iwanami-Princeton Univ. Press, 1987. pages 14, 17, 18, 19

[17] N. Koiso, *Einstein metrics and complex structures*, Invent. Math. **73** (1983), no. 1, 71-106. pages 2

[18] S. Kobayashi, T. Ochiai, *On complex manifolds with positive tangent bundles*, J. Math. Soc. Japan, Vol. **22**, No. 4, (1970), 499-525. pages 17

[19] K. Liu, X. Sun, S. T. Yau, *Good Geometry on the Curve Moduli*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 699-724. pages 2

[20] K. Liu, X. Sun, X. Yang, S.-T. Yau, *Curvatures of Moduli Space of Curves and Applications*, Asian Journal of Mathematics **21**, (2017) no. 5, 841-54. pages 2

[21] Z. Lu, *On the geometry of classifying spaces and horizontal slices*, Amer. J. Math. **121** (1999), no. 1, 177-198. pages 2

[22] Z. Lu, X. Sun, *Weil-Petersson geometry on moduli space of polarized Calabi-Yau manifolds*, J. Inst. Math. Jussieu **3** (2004), no. 2, 185-229. pages 2

[23] P. Naumann, *Curvature of higher direct images*, Ann. Fac. Sci. Toulouse Math. (6) **30** (2021), no. 1, 171-201. pages 2

[24] A. Nannicini, *Weil–Petersson metric in the space of compact polarized Kähler Einstein manifolds with* $c_1 = 0$, Manuscripta Math. **54** (1986), 405–438. pages 2

[25] G. Schumacher, *On the geometry of moduli spaces.* Manuscripta Math. **50** (1985), 229–267. pages 2

[26] G. Schumacher, *Harmonic maps of the moduli space of compact Riemann surfaces*, Math. Ann. 275 (1986), no. 3, 455-466. pages 2

[27] G. Schumacher, *The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds*, Complex analysis and geometry, 339-354, Univ. Ser. Math., Plenum, New York, 1993. pages 2, 9, 10, 12

[28] G. Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. **190** (2012), 1–56. pages 2, 9, 10

[29] B. Shiffman, A. Sommese, Vanishing theorems on complex manifolds, Birkhäuser, Boston, Basel, Stuttgart, 1985. pages 17

[30] Y.-T. Siu, *Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class*, in Complex Analysis, Papers in Honour of Wilhelm Stall (P.-M. Wong and A. Howard, eds.) (Vieweg, Braunschweig, 1986). pages 2

[31] N. K. Smolentsev, *Curvature of the space of associated metrics on a symplectic manifold*, Siberian Mathematical Journal **33** (1992), 111–117. pages 3

[32] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric.* Mathematical aspects of string theory (San Diego, Calif., 1986), 629–646, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987. 32G13 (32G15 53C25 58D99) pages 2

[33] A. N. Todorov, *The Weil-Petersson geometry of the moduli space of* $SU(n \geq 3)$ *(Calabi-Yau) manifolds*, I. Comm. Math. Phys. **126** (1989), no. 2, 325-346. pages 2

[34] A. J. Tromba, *On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil-Petersson metric*, Manuscripta Math. **56** (1986), no. 4, 475-497. pages 2

[35] X. Wan, X. Wang, *Poisson-Kähler fibration I: curvature of base manifold*, 2019, arXiv:1908.03955v2. pages 13, 19, 20

[36] X. Wan, G. Zhang, *The asymptotic of curvature of direct image bundle associated with higher powers of a relatively ample line bundle*, Geom. Dedicata 214 (2021), 489-517. pages 9, 10

[37] C.-L. Wang, *Curvature properties of the Calabi-Yau moduli*, Doc. Math. **8** (2003), 577-590. pages 2

[38] X. Wang, *Curvature restrictions on a manifold with a flat Higgs bundle*, arXiv: 1608.00777. pages 25

[39] X. Wang, *A curvature formula associated to a family of pseudoconvex domains*, Ann. Inst. Fourier (Grenoble) **67** (2017), no. 1, 269-313. pages 6, 21

[40] X. Wang, *Curvature of higher direct image sheaves and its application on negative-curvature criterion for the Weil-Petersson metric*, arXiv: 1607.03265. pages 8

[41] X. Wang, *Notes on variation of Lefschetz star operator and T-Hodge theory*, arXiv:1708.07332. pages 21, 25

[42] R. Wells, Raymond, Differential analysis on complex manifolds. Third edition. With a new appendix by Oscar Garcia-Prada. Graduate Texts in Mathematics, **65**. Springer, New York, 2008. pages 16

[43] S. Wolpert, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1986), no. 1, 119-145. pages 2

[44] Y. Wu, *The Riemannian sectional curvature operator of the Weil-Petersson metric and its application*, J. Differential Geom. **96** (2014), no. 3, 507-530. pages 2

Xueyuan Wan: Mathematical Science Research Center, Chongqing University of Technology, Chongqing 400054, China
E-mail address: `xwan@cqut.edu.cn`

Xu Wang: Department of Mathematical Sciences, Norwegian University of Science and Technology, No-7491 Trondheim, Norway.
E-mail address: `xu.wang@ntnu.no`