Distributional Generating Series in Nonlinear Control Theory

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***Abstract*—In linear system theory, the commutative algebra of integrable functions under convolution does not have a unit unless one includes the Dirac delta or impulse function. In the broader context of nonlinear systems, such generalized functions or distributions have played a more limited role. One exception to this situation is the class of nonlinear input-output systems that have a Chen–Fliess series representation. In this context, a convolutional unit in the form of a distributional generating series was introduced in the literature to characterize the algebraic structures induced by system interconnections. What is missing in this earlier work, however, is a clear description of what a distributional generating series is in an analytic framework. The goal of this paper is to describe this object precisely and show how it interacts with more standard concepts in nonlinear control theory.**

***Keywords*—Chen–Fliess series, generalized functions, nonlinear control systems**

I. Introduction

A long established fact in linear system theory is that the commutative algebra of integrable functions under convolution does not have a unit unless one includes the Dirac delta or impulse function. This object is not a function in the usual sense but instead a generalized function or distribution, that is, a continuous linear functional that behaves like the unit element of convolution [23]. As the impulse response is the primary characterization of a linear time-invariant (LTI) input-output system in the time domain, the impulse function plays an essential role in the theory [22].

In the broader context of nonlinear systems, distributions have played a more limited role since this setting often leads to working with products of distributions, which is analytically a more delicate issue [6]. One exception to this situation, however, is the class of nonlinear input-output systems that have a Chen–Fliess series representation and thus can be represented in terms of a noncommutative formal power series or generating series [12], [13]. This class of systems is closed under composition [10], [11] and induces a noncommutative algebra on the set of formal power series with a locally finite composition product. This product reduces to series convolution in the LTI case and therefore also lacks a unit. Analogous to the situation in linear system theory, a unit for this algebra was introduced in [17] and found to be quite useful in the analysis of system interconnections. As one might expect, this unit is not a normal formal power series but instead a type of generalized or distributional series, that is, a series analogue of the Dirac delta function. It was subsequently used in many works by the authors and others to describe feedback transformation groups on Chen–Fliess series [15], [16]. While its algebraic role in these constructions is straightforward, what is missing is a clear and precise description of what a distributional series is analytically. Thus, the main goal of this paper is to provide this missing mathematical framework.

The basic approach to defining a distributional generating series is to first view a Chen–Fliess series as a linear function of the Chen series representation of the input function [2]–[5], [20]. This allows one to then define the notion of a *test series* as an element of a topological vector space spanned by certain polynomial components of all possible Chen series generated by all inputs having a (not necessarily convergent) Taylor series representation. Distributional generating series are then defined to be continuous linear functionals on the space of test series in a manner completely analogous with the classical notion of distributions acting on test functions, albeit with a much simpler underlying topology since function integration does not play a direct role in this construction. Regular distributional generating series are those functionals that can be written explicitly in terms of actual generating series. The so called Dirac generating series (not to be confused with a similarly named concept in representation theory [1]) is shown to be non-regular. Finally, the issue of how distributional generating series interact with more standard concepts in control theory is established. It is shown that the Dirac series as defined in this analytical context is in fact the unit of the composition product of generating series as expected. Identities concerning the left-shift of the Dirac generating series are derived. The Chen series representation of the Dirac delta function is also computed in this context to establish a certain duality between the Dirac delta function and the Dirac generating series.

The paper is organized as follows. In the next section, a few important preliminaries are reviewed concerning Chen–Fliess series and Chen series. Section III then develops the notion of a distributional generating series. The concept is applied in the next section to demonstrate how it interacts with existing concepts used in control theory. The conclusions of the paper are presented in the final section.

II. Preliminaries

An *alphabet* $X = \{x_0, x_1, \ldots, x_m\}$ is any nonempty finite set of symbols referred to as *letters*. A *word* $\eta = x_{i_1} \cdots x_{i_k}$ is a finite sequence of letters from X. The number of letters in a word η, written as $|\eta|$, is called its *length*. The empty word, $\emptyset$, is taken to have length zero. The collection of all words having length k is denoted by X^k. Define $X^* = \bigcup_{k\geq 0} X^k$, which is a monoid under the concatenation product. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. Often c is written as the formal sum $c = \sum_{\eta\in X^*}(c,\eta)\eta$, where the *coefficient* $(c,\eta)\in\mathbb{R}^\ell$ is the image of $\eta\in X^*$ under c. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. A series c is called *proper* if $\emptyset\notin\mathrm{supp}(c)$. The set of all noncommutative formal power series over the alphabet X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The subset of series with finite support, i.e., polynomials, is represented by $\mathbb{R}^\ell\langle X\rangle$. Each set is an associative $\mathbb{R}$-algebra under the concatenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words

$$(x_i\eta)\sqcup\!\sqcup(x_j\xi) = x_i(\eta\sqcup\!\sqcup(x_j\xi)) + x_j((x_i\eta)\sqcup\!\sqcup\xi),$$

where $x_i, x_j \in X$, $\eta,\xi\in X^*$ and with $\eta\sqcup\!\sqcup\emptyset = \emptyset\sqcup\!\sqcup\eta = \eta$ [12]. For any letter $x_i\in X$, let x_i^{-1} denote the $\mathbb{R}$-linear *left-shift operator* defined by $x_i^{-1}(\eta) = \eta'$ when $\eta = x_i\eta'$ and zero otherwise. Higher order shifts are defined inductively via $(x_i\xi)^{-1}(\cdot) = \xi^{-1}x_i^{-1}(\cdot)$, where $\xi\in X^*$. It acts as a derivation on the shuffle product.

A. Chen–Fliess series

Given any $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ one can associate a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p}\geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1]\to\mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : \ 1\leq i\leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L_\mathfrak{p}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B_\mathfrak{p}^m(R)[t_0, t_1] := \{u\in L_\mathfrak{p}^m[t_0, t_1] : \|u\|_\mathfrak{p}\leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each word $\eta = x_i\bar\eta\in X^*$ the map $E_\eta : L_1^m[t_0, t_1]\to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar\eta}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar\eta}[u](\tau, t_0)\,d\tau,$$

where $x_i\in X$, $\bar\eta\in X^*$, and $u_0 = 1$. The *Chen–Fliess series* corresponding to $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is

$$y(t) = F_c[u](t) = \sum_{\eta\in X^*}(c,\eta)\,E_\eta[u](t, t_0)$$

[12], [13], [21]. If there exist real numbers $K_c, M_c > 0$ such that

$$|(c,\eta)|\leq K_c M_c^{|\eta|}|\eta|!,\quad \forall\eta\in X^*, \tag{1}$$

then F_c constitutes a well defined mapping from $B_\mathfrak{p}^m(R)[t_0, t_0+T]$ into $B_\mathfrak{q}^\ell(S)[t_0, t_0+T]$ for sufficiently small $R, T > 0$ and some $S > 0$, where the numbers $\mathfrak{p},\mathfrak{q}\in[1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}+1/\mathfrak{q} = 1$ [19]. (Here, $|z| := \max_i |z_i|$ when $z\in\mathbb{R}^\ell$.) Any series c satisfying (1) is called *locally convergent*. The set of all locally convergent series is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$, and F_c is referred to as a *Fliess operator*. The following theorem is proved in [24] (see also [14, Theorem 3.4]).

Theorem 2.1: If $c\in\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$ and $u\in B_\mathfrak{p}^m(R)[t_0, t_0+T]$, then $y = F_c[u]$ is differentiable a.e. on $[t_0, t_0+T]$ provided $R, T > 0$ are sufficiently small. In particular,

$$\frac{d}{dt}F_c[u] = \sum_{i=0}^{m} u_i\,F_{x_i^{-1}(c)}[u]. \tag{2}$$

B. Algebras on $\mathbb{R}\langle\langle X\rangle\rangle$ induced by system interconnections

Given Fliess operators F_c and F_d, where $c, d\in\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_cF_d = F_{c\sqcup\!\sqcup d}$, respectively [12]. It is also known that the composition of two Fliess operators F_c and F_d with $c\in\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$ and $d\in\mathbb{R}_{LC}^m\langle\langle X\rangle\rangle$ always yields another Fliess operator with generating series $c\circ d$, where this *composition product* is given by

$$c\circ d = \sum_{\eta\in X^*}(c,\eta)\,\psi_d(\eta)(\mathbf{1})$$

[10], [11]. Here ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to the vector space endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i)\circ\psi_d(\eta)$ with $\psi_d(x_i)(e) = x_0(d_i\sqcup\!\sqcup e)$, $i = 0, 1, \ldots, m$ for any $e\in\mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := \mathbf{1} := 1\emptyset$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Note in particular that this composition product is left linear but in general not right linear unless c is a *linear series*, that is, its support is contained in the language $L = \{x_0^{n_0}x_ix_0^{n_1} : n_0, n_1\geq 0,\ i\neq 0\}$.

In the event that $c\in\mathbb{R}\langle\langle X\rangle\rangle$ is not locally convergent, and thus the Chen–Fliess series may not converge in any sense, it is still possible to define a *formal* Fliess operator using the composition product [14], [20]. Namely, $u\mapsto y = F_c[u]$ is replaced with the always well defined mapping

$$c\circ : \mathbb{R}[[X_0]]\to\mathbb{R}[[X_0]], c_u\mapsto c_y = c\circ c_u, \tag{3}$$

where $X_0 := \{x_0\}$, and $\mathbb{R}[[X_0]]$ is the set of all commutative formal power series on X_0. The following uniqueness theorem will be essential.

Theorem 2.2: [14], [20] Let $c, d\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$. If $c\circ c_u = d\circ c_u$ for all $c_u\in\mathbb{R}^m[[X_0]]$ then $c = d$.

An immediate consequence of this theorem and the left linearity of the composition product is that $c\circ c_u = 0$ for all $c_u\in\mathbb{R}^m[[X_0]]$ if and only if $c = 0$.

C. Topological vector spaces of formal power series

The space $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ of all formal power series is generated as a vector space by the words in X^*. Hence, it can be canonically identified with the product $\prod_{\eta\in X^*}\mathbb{R}^\ell$. This product is a Fréchet space (complete metric space) as a countable product of finite dimensional vector spaces. The resulting topology on $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is called the Fréchet topology, and convergence in this topology corresponds to convergence of all the coefficients as elements in $\mathbb{R}^\ell$. The space $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ contains several interesting subspaces. Consider the following interpretation of (1). Fix $M > 0$ and define

$$\|c\|_{\ell_{\infty,M}} := \sup\left\{\frac{|(c,\eta)|}{M^{|\eta|}\,|\eta|!} : \eta\in X^*\right\}\in[0,\infty] \tag{4}$$

for each $c\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The set of all c with $\|c\|_{\ell_{\infty,M}} < \infty$, written as $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$, is a vector subspace of $\mathbb{R}^\ell\langle\langle X\rangle\rangle$.

The function $\|\cdot\|_{\ell_{\infty,M}}$ is a norm on $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$. The following assignment is an isometry of normed spaces:

$$\ell_{\infty,M}(X^*,\mathbb{R}^\ell) \longrightarrow \ell_\infty(X^*,\mathbb{R}^\ell), c \mapsto \frac{c}{M^{|\eta|}\,|\eta|!},$$

where $\ell_\infty(X^*,\mathbb{R}^\ell) := \{c\colon X^* \to \mathbb{R}^\ell : \sup_\eta |(c,\eta)| < \infty\}$ is the Banach space of all *bounded* functions from X^* to $\mathbb{R}^\ell$. Thus, for each fixed $M > 0$ the space $(\ell_{\infty,M}(X^*,\mathbb{R}^\ell), \|\cdot\|_{\ell_{\infty,M}})$ is a Banach space. A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ belongs to $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$ if and only if the bound (1) holds for some $K \geq 0$ and the fixed number M. In fact, the norm $\|c\|_{\ell_{\infty,M}}$ is the smallest number $K \geq 0$ such that (1) is satisfied. As $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$ is a Banach space, and, in particular, a metric space, the topology of $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$ can be recovered from convergent sequences, where a sequence $(c_j)_j$ in $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$ converges to $c \in \ell_{\infty,M}(X^*,\mathbb{R}^\ell)$ if and only if

$$\lim_{j\to\infty} \|c_j - c\|_{\ell_{\infty,M}} = 0.$$

In this case, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle := \lim_{M\to\infty} \ell_{\infty,M}(X^*,\mathbb{R}^\ell)$ is the locally convex limit of the Banach spaces of bounded functions. It is shown in [18] that the resulting space of all locally bounded series is a Silva space. This space is a subset of $\mathbb{R}\langle\langle X\rangle\rangle$, but its topology, the so called Silva topology, is finer than the Fréchet topology. As shown in [18], the composition product restricted to a product on $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ is continuous with respect to the Silva topology. The composition product on $\mathbb{R}\langle\langle X\rangle\rangle$ is also continuous under the ultrametric topology and the Fréchet topology [7], [18].

D. Chen series

Let $c_u \in \mathbb{R}^m[[X_0]]$. Then the corresponding *Chen series* $P[u]$ is the exponential Lie series satisfying the linear ordinary differential equation (ODE)

$$\frac{d}{dt}P[u] = \left[x_0 + \sum_{i=1}^m x_i u_i\right] P[u], \quad P[u](0) = \mathbf{1}, \tag{5}$$

where

$$u(t) = \sum_{k=0}^\infty (c_u, x_0^k)\frac{t^k}{k!} \tag{6}$$

is viewed as a formal input unless convergence conditions are available. There is no general solution theory beyond ODEs on Banach spaces, even for the linear case. Observe that $\mathbb{R}^m\langle\langle X\rangle\rangle$ is the direct product of copies of the Banach space $\mathbb{R}^m$ each labeled by a word in X^*. One can identify (5) with a system of linear ODEs on these copies of $\mathbb{R}^m$. Note that in general these ODEs depend on components of $P[u]$ labeled by words of shorter length. However, the linear ODEs on $\mathbb{R}^m$ can be solved iteratively by starting with the shortest length words and then inserting the solutions into the ODEs for longer length words. This iterative procedure produces a solution for every component in the collection and forms the unique solution to (5) in the Fréchet space $\mathbb{R}^m\langle\langle X\rangle\rangle$ [9, §6].

A Chen series can also be written as a formal function

$$P[u](t) = \sum_{n=0}^\infty P_{c_u}(n)\frac{t^n}{n!}, \tag{7}$$

where

$$P_{c_u}(n) = \sum_{\eta\in X^*} \eta\,(\eta \circ c_u, x_0^n).$$

The following lemma gives an inductive way to compute $P_{c_u}(n)$.

Lemma 2.1: For any $c_u \in \mathbb{R}^m[[X_0]]$, the sequence $P_{c_u}(n)$, $n \geq 0$ satisfies

$$P_{c_u}(n+1) = x_0 P_{c_u}(n) + \sum_{i=1}^m x_i \sum_{k=0}^n \left[(c_{u_i}, x_0^{n-k}) P_{c_u}(k)\right]\binom{n}{k}$$

with $P_{c_u}(0) = \mathbf{1}$.

Proof: Direct differentiation of (7) gives

$$\frac{dP}{dt} = \sum_{n=0}^\infty P_{c_u}(n+1)\frac{t^n}{n!}. \tag{8}$$

Substituting (7) in (5) yields

$$\begin{aligned}\frac{dP}{dt} &= \sum_{k=0}^\infty \left[x_0 + \sum_{i=1}^m x_i u_i(t)\right] P_{c_u}(k)\frac{t^k}{k!}\\ &= \sum_{k=0}^\infty x_0 P_{c_u}(k)\frac{t^k}{k!} + \sum_{i=1}^m x_i \sum_{k,\ell=0}^\infty \left[(c_{u_i}, x_0^\ell)P_{c_u}(k)\right]\frac{t^{k+\ell}}{k!\ell!}\\ &= \sum_{n=0}^\infty x_0 P_{c_u}(n)\frac{t^n}{n!} + \sum_{i=1}^m x_i \sum_{n=0}^\infty\sum_{k=0}^n \left[(c_{u_i}, x_0^{n-k})P_{c_u}(k)\right]\\ &\quad \binom{n}{k}\frac{t^n}{n!}. \end{aligned}\tag{9}$$

The claim follows directly from comparing (9) against (8). ■

The first few polynomials $P_{c_u}(n)$ when $m = 1$ are:

$$\begin{aligned}
P_{c_u}(0) &= \mathbf{1}\\
P_{c_u}(1) &= x_0 + x_1(c_u,\emptyset)\\
P_{c_u}(2) &= x_0^2 + x_1(c_u,x_0) + (x_0x_1 + x_1x_0)(c_u,\emptyset) +\\
&\quad x_1^2(c_u,\emptyset)^2\\
P_{c_u}(3) &= x_0^3 + x_1(c_u,x_0^2) + (x_0x_1 + 2x_1x_0)(c_u,x_0) +\\
&\quad 3x_1^2(c_u,\emptyset)(c_u,x_0) + (x_0^2x_1 + x_1x_0^2 + x_0x_1x_0)\\
&\quad (c_u,\emptyset) + (x_0x_1^2 + x_1x_0x_1 + x_1^2x_0)(c_u,\emptyset)^2 +\\
&\quad x_1^3(c_u,\emptyset)^3\\
&\ \ \vdots
\end{aligned}$$

Note that every $P_{c_u}(n)$ can be viewed as an element in a finite dimensional $\mathbb{R}$-vector space spanned by monomials in X^* which is independent of c_u but dependent on n.

Since the composition product on $\mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle$ is continuous in both arguments in the Fréchet topology, then $\lim_{k\to\infty} c_u(k) = c_u$ implies that $\lim_{k\to\infty} P_{c_u(k)}(n) = P_{c_u}(n)$.

Observe that for any $c \in \mathbb{R}\langle\langle X\rangle\rangle$

$$\begin{aligned}
c_y &:= c \circ c_u\\
&= \sum_{n=0}^\infty (c\circ c_u, x_0^n)x_0^n\\
&= \sum_{n=0}^\infty \sum_{\eta\in X^*} (c,\eta)(\eta\circ c_u, x_0^n)x_0^n\\
&= \sum_{n=0}^\infty (c, P_{c_u}(n))x_0^n.
\end{aligned}$$

III. Distributional Generating Series

In this section, the notion of a test series is introduced and then applied to define a distributional generating series. The focus henceforth is on the single-input, single-output case. Thus, $X = \{x_0, x_1\}$ and $\ell = 1$. The Dirac generating series is then given as a specific example of a non-regular distributional generating series. This series is shown to be the unit of the algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the composition product.

A. Test series

The linear dependence of $(c_y, x_0^n) = (c \circ c_u, x_0^n) = (c, P_{c_u}(n))$ on $P_{c_u}(n)$ motivates the following definition.

Definition 3.1: Define for each $n \in \mathbb{N}_0$ the finite dimensional $\mathbb{R}$-vector space

$$V_n = \operatorname{span}_{\mathbb{R}}\{P_{c_u}(n) : c_u \in \mathbb{R}[[X_0]]\}.$$

The set of ***test series*** $V := \bigoplus_{n\in\mathbb{N}_0} V_n$ is the locally convex direct sum of the V_n.

Note that since every V_n is finite dimensional, there is only one choice for a locally convex topology. As V is a locally convex direct sum of finite dimensional spaces, the following structural theorem applies.

Lemma 3.1: The space V is a Silva space isomorphic to $\mathbb{R}^{(\mathbb{N})}$, the countable locally convex direct sum of copies of $\mathbb{R}$. Moreover, the summation mapping

$$\Sigma\colon V \to \mathbb{R}_{LC}\langle\langle X\rangle\rangle, (p_n)_{n\in\mathbb{N}_0} \to \sum_{n\in\mathbb{N}_0} p_n$$

is continuous and linear.

Proof: Since V is the direct locally convex sum of the finite dimensional spaces V_n, it is the limit of the inductive system $(E_n := \oplus_{0\leq k\leq n} V_k)_{n\in\mathbb{N}_0}$, where the linking maps $E_n \to E_{n+1}$ are the inclusion of finite dimensional subspaces. In particular, all linking maps are compact operators, hence V is a Silva space [25]. As every V_n is isomorphic to $\mathbb{R}^{d_n} = \oplus_{k=1}^{d_n}\mathbb{R}$ for some natural number d_n, V is a countable direct sum of finite direct sums. Reordering the summands one can thus canonically identify V with a countable direct sum of the reals $V \cong \mathbb{R}^{(\mathbb{N})}$ as a locally convex space.

Now since every polynomial gives rise to a locally bounded series, the summation mapping is well defined (as a sum of finitely many polynomials). As the space V is a locally convex direct sum, it suffices to establish the continuity of Σ on every component V_n of the sum. On these components, the summation mapping simply becomes the inclusion $V_n \to \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. Since every polynomial is already contained in each of the spaces $\ell_{\infty,M}(X^*, \mathbb{R}^\ell)$, the inclusion factors through these Banach spaces. The norm (4) of the Banach space computes the weighted supremum of the series coefficients. Since polynomials only have finitely many nonzero coefficients, it is easy to see that the inclusion is continuous with respect to these norms. ■

Note that as the Silva topology on the space of locally bounded series is finer than the subspace topology induced by the Fréchet topology on $\mathbb{R}\langle\langle X\rangle\rangle$, the summation map is also continuous as a mapping into $\mathbb{R}\langle\langle X\rangle\rangle$ with the Fréchet topology. However, this summation mapping is not injective. For example, it is easy to see that the monomial x_1 is contained both in V_1 and in V_2.

Remark: It is clear from Lemma 3.1 that the strong dual of the space V can be identified as the direct product $\mathbb{R}^{\mathbb{N}} = \prod_{\mathbb{N}} \mathbb{R}$. Spaces of this kind are so called weakly complete spaces [8]. Note that the Fréchet topology on $\mathbb{R}\langle\langle X\rangle\rangle$ also turns the space into a weakly complete space. These spaces and their duals admit an easily accessible duality theory. As a consequence, it follows that (derived here from first principles): Since V is a locally convex direct sum, a linear map $V \to \mathbb{R}$ will be continuous if and only if its restriction to every summand V_n is continuous. However, since every linear map from a finite dimensional vector space is automatically continuous, every linear map on V is automatically continuous.

B. Distributional generating series

The main definition is given first.

Definition 3.2: A ***distributional generating series*** $\mathfrak{c}$ is a (continuous) linear functional

$$\mathfrak{c} : V \to \mathbb{R}, p \mapsto \mathfrak{c}(p).$$

For brevity, such series will often be called distributional series. From the remark above, the continuity of a distributional series is automatic.

Example 3.1: Let $c \in \mathbb{R}\langle\langle X\rangle\rangle$ and define for any $c_u \in \mathbb{R}[[X_0]]$ and $n \in \mathbb{N}_0$

$$\begin{aligned}\mathfrak{c} : P_{c_u}(n) \mapsto (c, P_{c_u}(n)) &:= \sum_{\eta\in X^*} (c,\eta)(P_{c_u}(n),\eta) \\ &= (c \circ c_u, x_0^n).\end{aligned}$$

This uniquely defines a linear mapping on V and constitutes a *regular* distributional series since $\mathfrak{c}$ can be written explicitly in terms of an element $c \in \mathbb{R}\langle\langle X\rangle\rangle$. In light of Theorem 2.2, given any regular $\mathfrak{c}$, the generating series c is uniquely specified. But a systematic way to compute c is an open problem except for the case where c is locally convergent [14], [24]. □

Example 3.2: Consider a linear map on V with the defining property that

$$\boldsymbol{\delta}\colon P_{c_u}(n) \mapsto (c_u, x_0^n)$$

for any $c_u \in \mathbb{R}[[X_0]]$ and $n \in \mathbb{N}_0$. The distributional series $\boldsymbol{\delta}$ will be called the *Dirac generating series*. To see that it is not regular, suppose there existed a $c \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $\boldsymbol{\delta}(p) = (c, p)$ for all $p \in V$. Then necessarily, $\boldsymbol{\delta}(P_{c_u}(n)) = (c_u, x_0^n) = (c, P_{c_u}(n))$ for all $c_u \in \mathbb{R}[[X_0]]$ and $n \in \mathbb{N}_0$. If $n = 1$, then from Lemma 2.1 it follows that

$$(c, P_{c_u}(1)) = (c_u, x_0) = (c, x_0) + (c, x_1)(c_u, \emptyset).$$

Selecting $c_u = x_0$ gives $(c, x_0) = 1$, while if $c_u = x_0^2$ then $(c, x_0) = 0$. Thus, no such c exists. □

As an application, define $(\boldsymbol{\delta} \circ c_u, x_0^n) = \boldsymbol{\delta}(P_{c_u}(n)) = (c_u, x_0^n)$ for all $n \in \mathbb{N}_0$, or equivalently, $\boldsymbol{\delta} \circ c_u = c_u$.

Lemma 3.2: For any $c_u \in \mathbb{R}[[X_0]]$, it follows that

$$c_u \circ \boldsymbol{\delta} = \boldsymbol{\delta} \circ c_u = c_u.$$

Proof: Using the associativity of the composition product, it follows for all $c_v \in \mathbb{R}[[X_0]]$ that

$$(c_u \circ \boldsymbol{\delta}) \circ c_v = c_u \circ (\boldsymbol{\delta} \circ c_v) = c_u \circ c_v.$$

Therefore, applying Theorem 2.2 it follows that $c_u \circ \boldsymbol{\delta} = c_u$. ■

Theorem 3.1: The Dirac series, $\boldsymbol{\delta}$, is the unit of the composition product on $\mathbb{R}\langle\langle X \rangle\rangle$.

Proof: For any $c_u \in \mathbb{R}[[X_0]]$, observe from Lemma 3.2 that

$$(c \circ \boldsymbol{\delta}) \circ c_u = c \circ (\boldsymbol{\delta} \circ c_u) = c \circ c_u.$$

Hence, from Theorem 2.2 it follows that $c \circ \boldsymbol{\delta} = c$. Similarly,

$$(\boldsymbol{\delta} \circ c) \circ c_u = \boldsymbol{\delta} \circ (c \circ c_u) = c \circ c_u.$$

Thus, $\boldsymbol{\delta} \circ c = c$. ■

IV. Applications in Control Theory

It is first confirmed that the composition product on $\mathbb{R}\langle\langle X \rangle\rangle$ is equivalent to function convolution in the context of linear-time invariant systems. Such systems can be represented in terms of a locally convergent Chen–Fliess series F_c, where $c = \sum_{i\geq 0}(c, x_0^i x_1)x_0^i x_1$. In which case, the impulse response is defined to be

$$\begin{aligned} h(t) = F_c[\delta](t) &= \sum_{i=0}^{\infty}(c, x_0^i x_1) E_{x_0^i x_1}[\delta](t, 0^-) \\ &= \sum_{i=0}^{\infty}(c, x_0^i x_1)\frac{t^i}{i!}, \quad t \geq 0^+, \end{aligned} \tag{10}$$

where δ denotes the Dirac delta function. The input-output map $u \mapsto y$ is then also given in terms of the convolution integral

$$\begin{aligned} (h * u)(t) &:= \int_{0^-}^{t} h(t-\tau)u(\tau)\, d\tau \\ &= \int_{0^-}^{t} \sum_{i,j=0}^{\infty} (c, x_0^i x_1)\frac{(t-\tau)^i}{i!}\, (c_u, x_0^j)\frac{\tau^j}{j!}\, d\tau \\ &= \sum_{i,j=0}^{\infty} (c, x_0^i x_1)(c_u, x_0^j)\frac{1}{i!\, j!}\int_{0^-}^{t} (t-\tau)^i \tau^j\, d\tau \\ &= \sum_{i,j=0}^{\infty} (c, x_0^i x_1)(c_u, x_0^j)\frac{1}{i!\, j!}\sum_{k=0}^{i}\binom{i}{k}(-1)^k t^{i-k} \\ &\quad \int_{0^-}^{t} \tau^{k+j}\, d\tau \\ &= \sum_{i,j=0}^{\infty} (c, x_0^i x_1)(c_u, x_0^j)\frac{t^{i+j+1}}{i!\, j!}\left[\sum_{k=0}^{i}\binom{i}{k}(-1)^k \right. \\ &\quad \left. \frac{1}{k+j+1}\right] \\ &= \sum_{i,j=0}^{\infty} (c, x_0^i x_1)(c_u, x_0^j)\frac{t^{i+j+1}}{(i+j+1)!} \\ &= \sum_{k=1}^{\infty}\left[\sum_{j=0}^{k-1}(c, x_0^{k-j-1}x_1)(c_u, x_0^j)\right]\frac{t^k}{k!} \\ &= \sum_{k=1}^{\infty}(c \circ c_u, x_0^k)\frac{t^k}{k!} \\ &= \sum_{k=1}^{\infty}(c_y, x_0^k)\frac{t^k}{k!} \\ &= y(t), \end{aligned}$$

where the identity for integer sequences

$$\sum_{k=0}^{i}\binom{i}{k}(-1)^k\frac{1}{k+j+1} = \frac{i!j!}{(i+j+1)!}, \quad i \geq 0$$

has been used, as well as the fact that the composition product reduces to series convolution in the linear time-invariant case. It is worth noting that the generating series for h in (10) is $c_h = \sum_{i\geq 0}(c, x_0^i x_1)x_0^i \in \mathbb{R}[[X_0]]$, which is not computed in terms of the Dirac generating series as one might expect from (3) since $c \circ \boldsymbol{\delta} = c \neq c_h$.

The next result illustrates a certain *duality* between the Dirac delta function and the Dirac generating series in the context of Chen–Fliess series assuming henceforth that $t_0 = 0^-$.

Theorem 4.1: If $c_u \in \mathbb{R}[[X_0]]$, then

$$F_{c_u}[\delta](t) = F_{\boldsymbol{\delta}}[u](t), \quad t \geq 0^-,$$

where u is given by (6).

Proof: First observe that integrating both sides of (5) and setting $u = \delta$ gives

$$\begin{aligned} P[\delta](t) &= \mathbf{1} + \int_{0^-}^{t} (x_0 + x_1\delta(\tau))P[\delta](\tau)\, d\tau \\ &= \mathbf{1} + x_1 + \int_{0^-}^{t} x_0 P[\delta](\tau)\, d\tau \\ &= \mathbf{1} + x_1 + \int_{0^-}^{t} x_0\left[\mathbf{1} + x_1 + \int_{0^-}^{\tau_1} x_0 P[\delta](\tau_2)\, d\tau_2\right] d\tau_1 \\ &= \mathbf{1} + x_1 + x_0 t + x_0 x_1 t + x_0^2\int_{0^-}^{t}\int_{0^-}^{\tau_1} P[\delta](\tau_2)\, d\tau_2\, d\tau_1 \\ &\vdots \end{aligned}$$

The bilinear structure of (5) provides a Volterra series representation of this series, namely,

$$\begin{aligned} P[\delta](t) &= \mathrm{e}^{x_0 t}\mathbf{1} + \sum_{k=1}^{\infty}\int_{0^-}^{t}\int_{0^-}^{\tau_k}\cdots\int_{0^-}^{\tau_2}\mathrm{e}^{x_0(t-\tau_k)}x_1\mathrm{e}^{x_0(\tau_k-\tau_{k-1})} \\ &\qquad \cdots x_1\mathrm{e}^{x_0\tau_1}\ \delta(\tau_k)\cdots\delta(\tau_1)\, d\tau_1\cdots d\tau_k \\ &= \mathrm{e}^{x_0 t}x_1^*, \end{aligned}$$

where $x_1^* := \sum_{i\geq 0} x_1^i$. Therefore, if $c_u \in \mathbb{R}[[X_0]]$, then

$$\begin{aligned} F_{c_u}[\delta](t) &= (c_u, P[\delta](t)) \\ &= (c_u, \mathrm{e}^{x_0 t}x_1^*) \\ &= \sum_{k=0}^{\infty}(c_u, x_0^k)\frac{t^k}{k!} \\ &= u(t) \end{aligned}$$

$$= F_{\boldsymbol{\delta}}[u](t).$$

■

Consider next how to define distributional series for derivatives of Chen–Fliess series. Observe that $F_{\boldsymbol{\delta}}[u] = u\mathbb{U}$, where $\mathbb{U}$ denotes the unit step function

$$\mathbb{U}(t) = \begin{cases} 1 : t \geq 0 \\ 0 : t < 0. \end{cases}$$

Therefore, in light of (2),

$$\begin{aligned} \frac{d}{dt}F_{\boldsymbol{\delta}}[u] &= \frac{du}{dt}\mathbb{U} + u\delta \\ &= F_{x_0^{-1}(\boldsymbol{\delta})}[u] + uF_{x_1^{-1}(\boldsymbol{\delta})}[u]. \end{aligned}$$

So it makes sense to define

$$\begin{aligned} F_{x_0^{-1}(\boldsymbol{\delta})}[u] &= \frac{du}{dt}\mathbb{U} \\ F_{x_1^{-1}(\boldsymbol{\delta})}[u] &= \delta. \end{aligned}$$

To assign meaning to $x_0^{-1}(\boldsymbol{\delta})$ and $x_1^{-1}(\boldsymbol{\delta})$, it useful to start with the known identify

$$\begin{aligned} &x_0^{-1}(c \circ d) = x_0^{-1}(c) \circ d + d \sqcup\!\sqcup (x_1^{-1}(c) \circ d) \\ &x_1^{-1}(c \circ d) = 0 \end{aligned}$$

for any $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$. Substitute $\boldsymbol{\delta}$ for c and set $d = c_u \in \mathbb{R}[[X_0]]$. Then the only nontrivial equation is

$$x_0^{-1}(\boldsymbol{\delta} \circ c_u) = x_0^{-1}(\boldsymbol{\delta}) \circ c_u + c_u \sqcup\!\sqcup (x_1^{-1}(\boldsymbol{\delta}) \circ c_u).$$

Defining $x_1^{-1}(\boldsymbol{\delta}) = 0$, it would then follow that $x_0^{-1}(\boldsymbol{\delta} \circ c_u) := x_0^{-1}(\boldsymbol{\delta}) \circ c_u$. This defines a new distributional generating series $\boldsymbol{\delta}' := x_0^{-1}(\boldsymbol{\delta})$ uniquely specified on V by

$$\begin{aligned} \boldsymbol{\delta}'(P_{c_u}(n)) &= (x_0^{-1}(\boldsymbol{\delta}), P_{c_u}(n)) \\ &= (x_0^{-1}(\boldsymbol{\delta}) \circ c_u, x_0^n) \\ &= (x_0^{-1}(\boldsymbol{\delta} \circ c_u), x_0^n) \\ &= (c_u, x_0^{n+1}) \\ &= (c_{u'}, x_0^n). \end{aligned}$$

More generally, $\boldsymbol{\delta}^{(k)} := x_0^{-k}(\boldsymbol{\delta})$, $k \in \mathbb{N}_0$, with $\boldsymbol{\delta}^{(k)}(P_{c_u}(n)) = (c_{u^{(k)}}, x_0^n)$. A corollary of Theorem 4.1 then follows directly as given below.

Corollary 4.1: If $c_u \in \mathbb{R}[[X_0]]$, then for $k \in \mathbb{N}_0$

$$F_{c_u}[\delta^{(k)}](t) = F_{\boldsymbol{\delta}^{(k)}}[u](t), \quad t \geq 0^-.$$

V. Conclusions

The notion of a distributional or generalized generating series was introduced in the context Chen–Fliess series to supply the missing analytic framework for such objects that already appear in the literature. A distributional generating series is simply any linear functional that is (automatically) continuous on $\mathbb{R}^{(\mathbb{N})}$ as a locally convex topological vector space. The Dirac generating series and its left-shifts were given as specific examples. Next, it was verified that the Dirac generating series is the unit of the associative algebra of formal power series under the composition product. Finally, a certain duality between this distribution and the classical Dirac delta function was established.

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