Explicit Backstepping Kernel Solutions for Leak Detection in Pipe Flow Networks Containing Loops

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***Abstract*— A recursive procedure to obtain explicit expressions to a set of observer backstepping kernel equations for an interconnection (cascade) of $N+1$ systems of 2×2 linear hyperbolic PDEs, $N>0$ an integer, for use in leak detection in pipe flow networks containing loops is developed. The kernel equations, consisting of two sets each of $N+1$ pairs of Goursat PDEs defined over a triangular domain, and $\frac{N(N+1)}{2}$ pairs of Goursat PDEs defined over a square domain, interconnected to each other in an overarching triangular structure, is separated into $2(N+1)$ systems consisting of $k+1$ pairs of PDEs over a triangular domain interconnected with $(N-\frac{k}{2})(k+1)$ pairs of PDEs over a square domain, $k\in\{0,1,\dots,N\}$. Under the assumption that the mean friction factor of the network may be used in place of individual friction factors for each pipe, it is shown that the solution to each of the simplified kernel equation systems is expressed explicitly in terms of modified Bessel functions of the first kind, and may be constructed recursively. A numerical example is provided to illustrate the results.**

I. INTRODUCTION

A. Background

Leakage from networks of pipes transporting fluids is a common problem both in industrial [1] and municipal infrastructure [2] settings. A range of methods have been developed over the years to address this critical problem, ranging from hardware-based methods relying on accurate but expensive infrastructure, fibre-optic cables [3] being a notable example, to be installed along the pipes, to software-based methods relying on processing limited data, obtained from cheap, non-invasive sensors, via sophisticated signal processing techniques.

One class of software-based techniques for automatically detecting and locating leaks is that of observer-based leak detection, where the leak detection algorithm is based on a state observer designed for a mathematical model of the pipe network one is performing leak detection for. Some early contributions that considered leak detection within such a setting include [4], [5], [6]. A common characteristic of much of the early work in this area is the use of a so-called early lumping approach, where the mathematical model describing the transient behaviour of the fluid in the pipelines, being distributed in nature, is discretized (lumped) before algorithm design is performed.

More recently, late-lumping approaches for designing observer-based leak detection systems have been considered. A notable example is [7], where an adaptive observer-based leak detection method for a single pipe is designed via the infinite-dimensional backstepping approach, initially developed for application to 2×2 systems of linear hyperbolic PDEs in [8]. This leak detection method was extended to the case of leak detection in pipes interconnected via a single branching point in [9], and to pipes connected to each other via a loop-shaped network in [10]. To implement the leak detection systems designed in [7], [9], [10], observer gains need to be calculated by solving a system of hyperbolic PDEs known as the *kernel equations*. In general the kernel equations obtained from backstepping designs need to be approximated numerically, but in certain cases closed-form solutions may be found. The observer gains used in the leak detection system from [7] are expressed explicitly by applying results from [11], while in [12] closed-form solutions to the kernel equations from [9] are found.

The aim of this paper is to complement these results by proposing a recursive procedure to construct explicit solutions for kernel equations for use in calculating observer gains for the leak detection system from [10]. We define next the precise problem statement.

B. Problem statement

We consider here the leak detection problem from [10], namely that of automatically detecting, estimating the size of, and locating leaks in a pipe flow network of $N+1$ pipes interconnected in a ring topology. The dynamics of the pressure p_i and volumetric flow q_i within each pipe $i\in\{0,1,\dots,N\}$ of length l_i and cross-sectional area A_i is assumed, for $z_i\in(0,l_i)$ and $t>0$, to be governed by (dropping the index i in z_i for brevity)

$$\partial_t p_i(z,t)+\frac{\beta}{A_i}\partial_z q_i(z,t)=-\frac{\beta}{A_i}d_i(z)\chi_i \tag{1a}$$

$$\partial_t q_i(z,t)+\frac{A_i}{\rho}\partial_z p_i(z,t)=-\frac{F_i}{\rho}q_i(z,t)-A_i g\sin(\phi_i(z)) -\frac{\eta_i}{A_i}d_i(z)\chi_i \tag{1b}$$

where β is the fluid bulk modulus, A_i is the pipe cross-sectional area of pipe i, ρ is the fluid density, F_i is the friction factor of pipe i, ϕ_i is the inclination angle of pipe i at position z_i, and g is the acceleration of gravity. Possible leaks are modelled by the total amount χ_i leaking from each pipe i together with the normalized leak distribution d_i along the pipeline. The factor η_i is proportional to the mean volumetric flow rate $q_{i,0}$ in pipe i, with the constant of proportionality determined by the shape, size and direction of the leak (see [13]). To model the interconnections between

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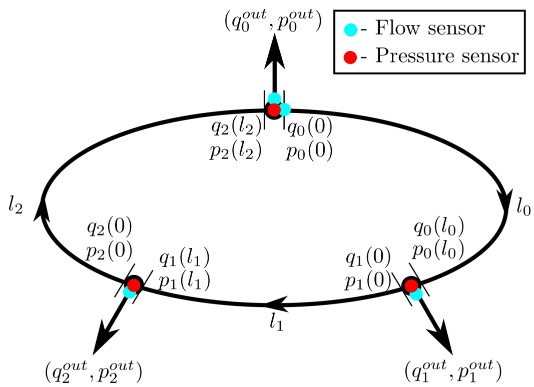

Fig. 1: Loop-shaped water distribution network for $N = 2$. (From [10])

the $N+1$ pipes, the boundary conditions

$$p_i(l_i,t) = p_{\mu_N(i)}(0,t) = p^{out}_{\mu_N(i)}(t) \tag{2a}$$

$$q_i(l_i,t) = q_{\mu_N(i)}(0,t) + q^{out}_{\mu_N(i)}(t) \tag{2b}$$

for $i \in \{0,1,\dots,n\}$, where

$$\mu_N(i) := i+1(\mod N+1), \tag{3}$$

are imposed. The signals p_i^{out}, q_i^{out} represent respectively the pressure and net outflow from the junction situated by the outlet of pipe i. Assume p_i^{out}, q_i^{out} together with the auxiliary flow sensor signal

$$q_{0,0}(t) := q_0(0,t) \tag{4}$$

are measured (see Figure 1). It is then shown in Lemma 3.2 of [10] that the ring-shaped pipe network system (1)–(2) is via an invertible change of coordinates mapped into an interconnection (cascade) of 2×2 linear hyperbolic PDE systems in (u_i, v_i) coordinates, defined over $x \in (0,1)$ and $t > 0$. Given the boundary output signal

$$y(t) := v_0(0,t), \tag{5}$$

which is constructed from a linear combination of $q_{0,0}$ and p_0^{out}, an adaptive observer for the plant in (u_i, v_i) coordinates is designed in [10] to produce estimates $\hat{u}_i, \hat{v}_i, \hat{\kappa}$ of the unknown states u_i, v_i and unknown parameter κ characterising leaks in the network. The observer reads

$$\partial_t \hat{u}_i(x,t) = -\lambda_i \partial_x \hat{u}_i(x,t) + c_{i1}(x)\hat{v}_i(x,t) + P_i^+(x)(y(t) - \hat{v}_0(0,t)) \tag{6a}$$

$$\partial_t \hat{v}_i(x,t) = \lambda_i \partial_x \hat{v}_i(x,t) + c_{i2}(x)\hat{u}_i(x,t) + P_i^-(x)(y(t) - \hat{v}_0(0,t)) \tag{6b}$$

$$\hat{u}_0(0,t) = \varphi_0 y(t) + \sigma_0(t) \tag{6c}$$

$$\hat{v}_{i-1}(1,t) = \vartheta_{i-1}\hat{u}_{i-1}(1,t) + \tau_{i,i-1}\hat{v}_i(0,t) + \sigma_{i,i-1}(t) \tag{6d}$$

$$\hat{u}_i(0,t) = \varphi_i \hat{v}_i(0,t) + \tau_{i-1,i}\hat{u}_{i-1}(1,t) + \sigma_{i-1,i}(t) \tag{6e}$$

$$\hat{v}_N(1,t) = \vartheta_N \hat{u}_N(1,t) + \sigma_N(t) + \hat{\kappa}(t) \tag{6f}$$

$$\dot{\hat{\kappa}}(t) = L(y(t) - \hat{v}_0(0,t)) \tag{6g}$$

where (6a)–(6b) are valid for $i \in \{0,1,\dots,N\}$, while (6d)–(6e) are valid for $i \in \{1,\dots,N\}$. The other system parameters used in (6) are defined as

$$c_{i,1}(x) := -\lambda_i \gamma_i e^{2\gamma_i x}, \qquad c_{i,2}(x) := -\lambda_i \gamma_i e^{-2\gamma_i x} \tag{7a}$$

$$\lambda_i := \frac{1}{l_i}\sqrt{\frac{\beta}{\rho}}, \qquad \gamma_i := \frac{l_i F_i}{2\sqrt{\beta\rho}} \tag{7b}$$

$$\varphi_0 := 1, \qquad \varphi_i := \frac{A_{i-1} - A_i}{A_i + A_{i-1}} \tag{7c}$$

$$\vartheta_{i-1} := e^{-2\gamma_{i-1}} \frac{A_i - A_{i-1}}{A_i + A_{i-1}}, \qquad \vartheta_n := -e^{-2\gamma_n} \tag{7d}$$

$$\tau_{i,i-1} := 2e^{-\gamma_{i-1}} \frac{A_{i-1}}{A_i + A_{i-1}}, \qquad \tau_{i-1,i} := 2e^{-\gamma_{i-1}} \frac{A_i}{A_i + A_{i-1}} \tag{7e}$$

while the boundary input signals $\sigma_{i,i-1}, \sigma_{i-1,i}, \sigma_N$ are known signals of time computed from the junction pressure and flow measurements (see [10] for their respective definitions). In (6), $P_i^+, P_i^- : [0,1] \mapsto \mathbb{R}$, $L \in \mathbb{R}$ are output injection gains, or observer gains, that need to be carefully selected to provide convergent estimates.

To find suitable expressions for these gains, the problem is reformulated as finding P_i^+, P_i^- such that the dynamics of the estimation errors $\tilde{u}_i := u_i - \hat{u}_i$, $\tilde{v}_i := v_i - \hat{v}_i$, $\tilde{\kappa} := \kappa - \hat{\kappa}$ converge to the origin in some sense. In [10] this is done via the backstepping methodology, where a suitable target system in $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\kappa})$ is designed and which by Lemma 4.4 from [10] has a globally exponentially stable origin. The expressions for P_i^+, P_i^- are then uniquely found to be those for which the diffeomorphism

$$\tilde{\kappa}(t) = \tilde{\kappa}(t) \tag{8a}$$

$$\begin{aligned}\tilde{u}_i(x,t) = \tilde{\alpha}_i(x,t) &- \int_0^x \Big(K_{i,i}^{\alpha\alpha}(x,\xi)\tilde{\alpha}_i(\xi,t) \\ &+ K_{i,i}^{\alpha\beta}(x,\xi)\tilde{\beta}_i(\xi,t)\Big)d\xi - \sum_{j=0}^{i-1}\int_0^1 \Big(K_{j,i}^{\alpha\alpha}(x,\xi)\tilde{\alpha}_j(\xi,t) \\ &+ K_{j,i}^{\alpha\beta}(x,\xi)\tilde{\beta}_j(\xi,t)\Big)d\xi\end{aligned} \tag{8b}$$

$$\begin{aligned}\tilde{v}_i(x,t) = \tilde{\beta}_i(x,t) &- \int_0^x \Big(K_{i,i}^{\beta\alpha}(x,\xi)\tilde{\alpha}_i(\xi,t) \\ &+ K_{i,i}^{\beta\beta}(x,\xi)\tilde{\beta}_i(\xi,t)\Big)d\xi - \sum_{j=0}^{i-1}\int_0^1 \Big(K_{j,i}^{\beta\alpha}(x,\xi)\tilde{\alpha}_j(\xi,t) \\ &+ K_{j,i}^{\beta\beta}(x,\xi)\tilde{\beta}_j(\xi,t)\Big)d\xi\end{aligned} \tag{8c}$$

exists between the dynamics of $(\tilde{u}_i, \tilde{v}_i, \tilde{\kappa})$ and $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\kappa})$, and are given by

$$\begin{aligned}P_i^+(x) = -\lambda_0 K_{0,i}^{\alpha\beta}(x,0) &- Lg_i^{\alpha}\left(-1 + \int_0^x K_{i,i}^{\alpha\alpha}(x,\xi)d\xi\right) \\ &- Lg_i^{\beta}\int_0^x K_{i,i}^{\alpha\beta}(x,\xi)d\xi - \sum_{j=0}^{i-1} Lg_j^{\alpha}\int_0^1 K_{j,i}^{\alpha\alpha}(x,\xi)d\xi \\ &- \sum_{j=0}^{i-1} Lg_j^{\beta}\int_0^1 K_{j,i}^{\alpha\beta}(x,\xi)d\xi\end{aligned} \tag{9a}$$

$$\begin{aligned}P_i^-(x) = -\lambda_0 K_{0,i}^{\beta\beta}(x,0) &- Lg_i^{\alpha}\int_0^x K_{i,i}^{\beta\alpha}(x,\xi)d\xi \\ &- Lg_i^{\beta}\left(-1 + \int_0^x K_{i,i}^{\beta\beta}(x,\xi)d\xi\right) \\ &- \sum_{j=0}^{i-1}\left(Lg_j^{\alpha}\int_0^1 K_{j,i}^{\beta\alpha}(x,\xi)d\xi + Lg_j^{\beta}\int_0^1 K_{j,i}^{\beta\beta}(x,\xi)d\xi\right),\end{aligned} \tag{9b}$$

where L is chosen so that $g_0^{\beta} L > 0$ (see Lemma 4.4 from [10] for the definition of g_0^{β}).

Furthermore, the integral kernels $K_{j,i}^{\alpha\alpha}$, $K_{j,i}^{\alpha\beta}$, $K_{j,i}^{\beta\alpha}$, $K_{j,i}^{\beta\beta}$, for $i \in \{0,1,\dots,N\}$, $j \in \{0,1,\dots,i\}$, defining the backstepping transformation (8) are found to satisfy

$$-\lambda_i \partial_\xi K_{i,i}^{\alpha\beta}(x,\xi) = -\lambda_i \partial_x K_{i,i}^{\alpha\beta}(x,\xi) + c_{i,1}(x) K_{i,i}^{\beta\beta}(x,\xi) \tag{10a}$$

$$-\lambda_i \partial_\xi K_{i,i}^{\beta\beta}(x,\xi) = \lambda_i \partial_x K_{i,i}^{\beta\beta}(x,\xi) + c_{i,2}(x) K_{i,i}^{\alpha\beta}(x,\xi) \tag{10b}$$

$$K_{i,i}^{\alpha\beta}(x,x) = \frac{-c_{i,1}(x)}{2\lambda_i} \tag{10c}$$

$$K_{i,i}^{\beta\beta}(1,\xi) = \vartheta_i K_{i,i}^{\alpha\beta}(1,\xi) + \tau_{i+1,i} K_{i,i+1}^{\beta\beta}(0,\xi) \tag{10d}$$

and

$$\lambda_i \partial_\xi K_{i,i}^{\alpha\alpha}(x,\xi) = -\lambda_i \partial_x K_{i,i}^{\alpha\alpha}(x,\xi) + c_{i,1}(x) K_{i,i}^{\beta\alpha}(x,\xi) \tag{11a}$$

$$\lambda_i \partial_\xi K_{i,i}^{\beta\alpha}(x,\xi) = \lambda_i \partial_x K_{i,i}^{\beta\alpha}(x,\xi) + c_{i,2}(x) K_{i,i}^{\alpha\alpha}(x,\xi) \tag{11b}$$

$$K_{i,i}^{\beta\alpha}(x,x) = \frac{c_{i,2}(x)}{2\lambda_i} \tag{11c}$$

$$K_{i,i}^{\alpha\alpha}(1,\xi) = \frac{\varphi_{i+1}}{\varphi_{i+1}\vartheta_i - \tau_{i,i+1}\tau_{i+1,i}} K_{i,i}^{\beta\alpha}(1,\xi) - \frac{\tau_{i+1,i}}{\varphi_{i+1}\vartheta_i - \tau_{i,i+1}\tau_{i+1,i}} K_{i,i+1}^{\alpha\alpha}(0,\xi) \tag{11d}$$

for $j = i$ (where for $i = N$ we have $\tau_{N+1,N} = 0$) and coupled over the triangular domain $\mathscr{T} := \{(x,\xi) \mid 0 \le \xi \le x \le 1\}$. For $0 \le j < i$ they are given by

$$-\lambda_j \partial_\xi K_{j,i}^{\alpha\beta}(x,\xi) = -\lambda_i \partial_x K_{j,i}^{\alpha\beta}(x,\xi) + c_{i,1}(x) K_{j,i}^{\beta\beta}(x,\xi) \tag{12a}$$

$$-\lambda_j \partial_\xi K_{j,i}^{\beta\beta}(x,\xi) = \lambda_i \partial_x K_{j,i}^{\beta\beta}(x,\xi) + c_{i,2}(x) K_{j,i}^{\alpha\beta}(x,\xi) \tag{12b}$$

$$K_{j,i}^{\alpha\beta}(x,1) = \frac{\lambda_{j+1}}{\lambda_j \tau_{j+1,j}} (K_{j+1,i}^{\alpha\beta}(x,0) - \varphi_{j+1} K_{j+1,i}^{\alpha\alpha}(x,0)) \tag{12c}$$

$$K_{j,i}^{\beta\beta}(x,1) = \frac{\lambda_{j+1}}{\lambda_j \tau_{j+1,j}} (K_{j+1,i}^{\beta\beta}(x,0) - \varphi_{j+1} K_{j+1,i}^{\beta\alpha}(x,0)) \tag{12d}$$

$$K_{j,i}^{\alpha\beta}(0,\xi) = \varphi_i K_{j,i}^{\beta\beta}(0,\xi) + \tau_{i-1,i} K_{j,i-1}^{\alpha\beta}(1,\xi) \tag{12e}$$

$$K_{j,i}^{\beta\beta}(1,\xi) = \vartheta_i K_{j,i}^{\alpha\beta}(1,\xi) + \tau_{i+1,i} K_{j,i+1}^{\beta\beta}(0,\xi) \tag{12f}$$

and

$$\lambda_j \partial_\xi K_{j,i}^{\alpha\alpha}(x,\xi) = -\lambda_i \partial_x K_{j,i}^{\alpha\alpha}(x,\xi) + c_{i,1}(x) K_{j,i}^{\beta\alpha}(x,\xi) \tag{13a}$$

$$\lambda_j \partial_\xi K_{j,i}^{\beta\alpha}(x,\xi) = \lambda_i \partial_x K_{j,i}^{\beta\alpha}(x,\xi) + c_{i,2}(x) K_{j,i}^{\alpha\alpha}(x,\xi) \tag{13b}$$

$$K_{j,i}^{\alpha\alpha}(x,1) = \frac{\lambda_{j+1}}{\lambda_j \tau_{j+1,j}} (\vartheta_j K_{j+1,i}^{\alpha\beta}(x,0) + (\tau_{j,j+1}\tau_{j+1,j} - \vartheta_j \varphi_{j+1}) K_{j+1,i}^{\alpha\alpha}(x,0)) \tag{13c}$$

$$K_{j,i}^{\beta\alpha}(x,1) = \frac{\lambda_{j+1}}{\lambda_j \tau_{j+1,j}} (\vartheta_j K_{j+1,i}^{\beta\beta}(x,0) + (\tau_{j,j+1}\tau_{j+1,j} - \vartheta_j \varphi_{j+1}) K_{j+1,i}^{\beta\alpha}(x,0)) \tag{13d}$$

$$K_{j,i}^{\beta\alpha}(0,\xi) = \frac{\vartheta_{i-1}}{\varphi_i \vartheta_{i-1} - \tau_{i-1,i}\tau_{i,i-1}} K_{j,i}^{\alpha\alpha}(0,\xi) - \frac{\tau_{i-1,i}}{\varphi_i \vartheta_{i-1} - \tau_{i-1,i}\tau_{i,i-1}} K_{j,i-1}^{\beta\alpha}(1,\xi) \tag{13e}$$

$$K_{j,i}^{\alpha\alpha}(1,\xi) = \frac{\varphi_{i+1}}{\varphi_{i+1}\vartheta_i - \tau_{i,i+1}\tau_{i+1,i}} K_{j,i}^{\beta\alpha}(1,\xi) - \frac{\tau_{i+1,i}}{\varphi_{i+1}\vartheta_i - \tau_{i,i+1}\tau_{i+1,i}} K_{j,i+1}^{\alpha\alpha}(0,\xi), \tag{13f}$$

coupled over the square domain $\mathscr{S} := \{(x,\xi) \mid 0 \le x, \xi \le 1\}$.

The contribution of this paper is to make progress towards providing explicit expressions for the observer gains in (6), which were only implicitly defined in [10] by the kernel equations. We show that after fixing

$$F_i \equiv F \tag{14}$$

in (7b) for all i, explicit solutions for (10)–(13) may be obtained, which also, of course, implies well-posedness of (10)–(13) under this assumption.

Remark 1.1: Assumption (14) holds when the cross-sectional area of the pipes and the mean flow through the pipes are equal, which admittedly is quite restrictive in a practical setting. For cases when the cross-sectional areas of the pipes are different, we propose using the mean friction factor $\bar{F}$ for the network as the constant friction factor F in the explicit kernel expressions developed here.

Next in Section II the kernel equations (10)–(13) are decomposed into $2(N+1)$ independent systems of equations, before they are solved explicitly in Section III under the assumption (14). The results are subsequently demonstrated on a numerical example in Section IV, before concluding remarks are offered in Section V.

II. Simplification of kernel equations

A. Separating kernels into independent systems

To solve (10)–(13) we propose a structure for the kernels as a linear combination of multiple "sub-kernels" satisfying structurally simpler PDE systems. Consider that the expressions for $K_{j,i}^{\cdot\cdot}$ are recursively constructed from

$$K_{j,i}^{\alpha\beta}(x,\xi) = \frac{\gamma_i}{2} \bar{K}_{j,i,j}^{\alpha\beta}(x,\xi) + \pi_j^1 \begin{bmatrix} K_{j+1,i}^{\alpha\alpha}(x, \frac{\lambda_{j+1}}{\lambda_j}(1-\xi)) \\ K_{j+1,i}^{\alpha\beta}(x, \frac{\lambda_{j+1}}{\lambda_j}(\xi-1)) \end{bmatrix} \tag{15a}$$

$$K_{j,i}^{\beta\beta}(x,\xi) = -\frac{\gamma_i}{2} \bar{K}_{j,i,j}^{\beta\beta}(x,\xi) + \pi_j^1 \begin{bmatrix} K_{j+1,i}^{\beta\alpha}(x, \frac{\lambda_{j+1}}{\lambda_j}(1-\xi)) \\ K_{j+1,i}^{\beta\beta}(x, \frac{\lambda_{j+1}}{\lambda_j}(\xi-1)) \end{bmatrix} \tag{15b}$$

$$K_{j,i}^{\alpha\alpha}(x,\xi) = -\frac{\gamma_i}{2} \bar{K}_{j,i,j}^{\alpha\alpha}(x,\xi) + \pi_j^0 \begin{bmatrix} K_{j+1,i}^{\alpha\alpha}(x, \frac{\lambda_{j+1}}{\lambda_j}(\xi-1)) \\ K_{j+1,i}^{\alpha\beta}(x, \frac{\lambda_{j+1}}{\lambda_j}(1-\xi)) \end{bmatrix} \tag{15c}$$

$$K_{j,i}^{\beta\alpha}(x,\xi) = -\frac{\gamma_i}{2} \bar{K}_{j,i,j}^{\beta\alpha}(x,\xi) + \pi_j^0 \begin{bmatrix} K_{j+1,i}^{\beta\alpha}(x, \frac{\lambda_{j+1}}{\lambda_j}(\xi-1)) \\ K_{j+1,i}^{\beta\beta}(x, \frac{\lambda_{j+1}}{\lambda_j}(1-\xi)) \end{bmatrix} \tag{15d}$$

starting from $K_{N+1,N}^{\cdot\cdot} \equiv 0$, in terms of closed-form expressions $\bar{K}_{j,i,k}^{\cdot\cdot}$ and π_j^0, π_j^1 constant row vectors, all to be found.

For $j = i$, $0 \le i \le k \le N$, consider that $\bar{K}_{i,i,k}^{\alpha\beta}$, $\bar{K}_{i,i,k}^{\beta\beta}$ satisfy

$$(-\partial_\xi + \partial_x) \bar{K}_{i,i,k}^{\alpha\beta}(x,\xi) = \gamma_i e^{2\gamma_i x} \bar{K}_{i,i,k}^{\beta\beta}(x,\xi) \tag{16a}$$

$$(-\partial_\xi - \partial_x) \bar{K}_{i,i,k}^{\beta\beta}(x,\xi) = \gamma_i e^{-2\gamma_i x} \bar{K}_{i,i,k}^{\alpha\beta}(x,\xi) \tag{16b}$$

$$\bar{K}_{i,i,k}^{\alpha\beta}(x,x) = \delta_{i,k} e^{2\gamma_i x} \tag{16c}$$

$$\bar{K}^{\beta\beta}_{i,i,k}(1,\xi) = -\vartheta_i \bar{K}^{\alpha\beta}_{i,i,k}(1,\xi) + \bar{\tau}_{i+1,i}\bar{K}^{\beta\beta}_{i,i+1,k}(0,\xi) \tag{16d}$$

and $\bar{K}^{\alpha\alpha}_{i,i,k}$, $\bar{K}^{\beta\alpha}_{i,i,k}$ satisfy

$$(-\partial_\xi - \partial_x)\bar{K}^{\alpha\alpha}_{i,i,k}(x,\xi) = \gamma_i e^{2\gamma_i x}\bar{K}^{\beta\alpha}_{i,i,k}(x,\xi) \tag{17a}$$

$$(-\partial_\xi + \partial_x)\bar{K}^{\beta\alpha}_{i,i,k}(x,\xi) = \gamma_i e^{-2\gamma_i x}\bar{K}^{\alpha\alpha}_{i,i,k}(x,\xi) \tag{17b}$$

$$\bar{K}^{\beta\alpha}_{i,i,k}(x,x) = \delta_{i,k}e^{-2\gamma_i x} \tag{17c}$$

$$\bar{K}^{\alpha\alpha}_{i,i,k}(1,\xi) = \frac{1}{D_{i+1}}(\varphi_{i+1}\bar{K}^{\beta\alpha}_{i,i,k}(1,\xi) - \bar{\tau}_{i+1,i}\bar{K}^{\alpha\alpha}_{i,i+1,k}(0,\xi)), \tag{17d}$$

over $\mathscr{T}$, where we have defined

$$\delta_{j,k} := \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k \end{cases}, \tag{18}$$

$$\bar{\tau}_{j,i} := \frac{\gamma_j}{\gamma_i}\tau_{j,i}, \tag{19}$$

$$D_i := \det\Theta_i, \qquad \Theta_i := \begin{bmatrix} \varphi_i & \tau_{i-1,i} \\ \tau_{i,i-1} & \vartheta_{i-1} \end{bmatrix}. \tag{20}$$

Likewise, consider that for $0 \leq j < i \leq N$, $j \leq k \leq N$, $\bar{K}^{\alpha\beta}_{j,i,k}$, $\bar{K}^{\beta\beta}_{j,i,k}$ satisfy

$$\left(-\frac{\lambda_j}{\lambda_i}\partial_\xi + \partial_x\right)\bar{K}^{\alpha\beta}_{j,i,k}(x,\xi) = \gamma_i e^{2\gamma_i x}\bar{K}^{\beta\beta}_{j,i,k}(x,\xi) \tag{21a}$$

$$\left(-\frac{\lambda_j}{\lambda_i}\partial_\xi - \partial_x\right)\bar{K}^{\beta\beta}_{j,i,k}(x,\xi) = \gamma_i e^{-2\gamma_i x}\bar{K}^{\alpha\beta}_{j,i,k}(x,\xi) \tag{21b}$$

$$\bar{K}^{\alpha\beta}_{j,i,k}(x,1) = (1-\delta_{j,k})\bar{K}^{\alpha\beta}_{j+1,i,k}(x,0) \tag{21c}$$

$$\bar{K}^{\beta\beta}_{j,i,k}(x,1) = (1-\delta_{j,k})\bar{K}^{\beta\beta}_{j+1,i,k}(x,0) \tag{21d}$$

$$\bar{K}^{\alpha\beta}_{j,i,k}(0,\xi) = -\varphi_i\bar{K}^{\beta\beta}_{j,i,k}(0,\xi) + \bar{\tau}_{i-1,i}\bar{K}^{\alpha\beta}_{j,i-1,k}(1,\xi) \tag{21e}$$

$$\bar{K}^{\beta\beta}_{j,i,k}(1,\xi) = -\vartheta_i\bar{K}^{\alpha\beta}_{j,i,k}(1,\xi) + \bar{\tau}_{i+1,i}\bar{K}^{\beta\beta}_{j,i+1,k}(0,\xi) \tag{21f}$$

and $\bar{K}^{\alpha\alpha}_{j,i,k}$, $\bar{K}^{\beta\alpha}_{j,i,k}$ satisfy

$$\left(-\frac{\lambda_j}{\lambda_i}\partial_\xi - \partial_x\right)\bar{K}^{\alpha\alpha}_{j,i,k}(x,\xi) = \gamma_i e^{2\gamma_i x}\bar{K}^{\beta\alpha}_{j,i,k}(x,\xi) \tag{22a}$$

$$\left(-\frac{\lambda_j}{\lambda_i}\partial_\xi + \partial_x\right)\bar{K}^{\beta\alpha}_{j,i,k}(x,\xi) = \gamma_i e^{-2\gamma_i x}\bar{K}^{\alpha\alpha}_{j,i,k}(x,\xi) \tag{22b}$$

$$\bar{K}^{\alpha\alpha}_{j,i,k}(x,1) = (1-\delta_{j,k})\bar{K}^{\alpha\alpha}_{j+1,i,k}(x,0) \tag{22c}$$

$$\bar{K}^{\beta\alpha}_{j,i,k}(x,1) = (1-\delta_{j,k})\bar{K}^{\beta\alpha}_{j+1,i,k}(x,0) \tag{22d}$$

$$\bar{K}^{\beta\alpha}_{j,i,k}(0,\xi) = \frac{1}{D_i}(\vartheta_{i-1}\bar{K}^{\alpha\alpha}_{j,i,k}(0,\xi) - \bar{\tau}_{i-1,i}\bar{K}^{\beta\alpha}_{j,i-1,k}(1,\xi)) \tag{22e}$$

$$\bar{K}^{\alpha\alpha}_{j,i,k}(1,\xi) = \frac{1}{D_{i+1}}(\varphi_{i+1}\bar{K}^{\beta\alpha}_{j,i,k}(1,\xi) - \bar{\tau}_{i+1,i}\bar{K}^{\alpha\alpha}_{j,i+1,k}(0,\xi)), \tag{22f}$$

over $\mathscr{S}$. We have the following result.

Lemma 2.1: Denote by π^0_j, π^1_j the row vectors of the 2×2 matrix $\Pi_j = [(\pi^0_j)^\top \ (\pi^1_j)^\top]^\top$ defined as

$$\Pi_j := \frac{\lambda_{j+1}}{\lambda_j\tau_{j+1,j}}\begin{bmatrix} -D_{j+1} & \vartheta_l \\ -\varphi_{j+1} & 1 \end{bmatrix}. \tag{23}$$

Then the system of kernel equations (10)–(13) is related to (16)–(22) via (15).

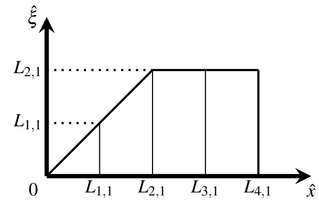

Fig. 2: Example of trapezoidal domain $\hat{\mathscr{T}}_k$ for $k=1$, $N=3$.

Proof: Noting that (12e)–(12f) and (13e)–(13f) can respectively be written in matrix form as

$$\begin{bmatrix} K^{\alpha\beta}_{j,i}(0,\xi) \\ K^{\beta\beta}_{j,i-1}(1,\xi) \end{bmatrix} = \Theta_i \begin{bmatrix} K^{\beta\beta}_{j,i}(0,\xi) \\ K^{\alpha\beta}_{j,i-1}(1,\xi) \end{bmatrix}$$

$$\begin{bmatrix} K^{\beta\alpha}_{j,i}(0,\xi) \\ K^{\alpha\alpha}_{j,i-1}(1,\xi) \end{bmatrix} = \Theta_i^{-1} \begin{bmatrix} K^{\alpha\alpha}_{j,i}(0,\xi) \\ K^{\beta\alpha}_{j,i-1}(1,\xi) \end{bmatrix},$$

and using the definition (23) of Π_j, the result is verified by using (15) to substitute the dynamics and boundary conditions of (16)–(22) into (10)–(13). ■

B. Global scaled coordinates

We consider in the following (16), (21), only, due to considerations of conciseness and space restrictions. The solutions to (17), (22) may be found in an equivalent manner.

For every $k \in \{0,1,\ldots,N\}$, define the trapezoidal domain $\hat{\mathscr{T}}_k := \{(\hat{x},\hat{\xi}) \mid 0 \leq \hat{\xi} \leq \hat{x} \leq L_{N+1,k}, \hat{\xi} \leq L_{k+1,k}\}$, where

$$L_{i,k} := \lambda_k\gamma_k\sum_{l=0}^{i-1}\frac{1}{\lambda_l}. \tag{24}$$

An example of such a domain is shown in Figure 2. Since the coupled system of equations (16), (21) form an independent system for each $k \in \{0,1,\ldots,N\}$, we would like to rewrite (16), (21), for each k, as a single system over $\hat{\mathscr{T}}_k$.

Introduce therefore $\hat{\hat{K}}^{\cdot\cdot}_k$ over $\hat{\mathscr{T}}_k$, where for $L_{i,k} < \hat{x} < L_{i+1,k}$, $L_{j,k} < \hat{\xi} < L_{j+1,k}$ it is defined in terms of $\bar{K}^{\cdot\cdot}_{j,i,k}$, over $(x,\xi) \in \mathscr{T}$ when $j = i$ and $(x,\xi) \in \mathscr{S}$ when $j < i$, as

$$\hat{\hat{K}}^{\alpha\beta}_k\left(L_{i,k} + \frac{\lambda_k\gamma_k}{\lambda_i}x, L_{j,k} + \frac{\lambda_k\gamma_k}{\lambda_j}\xi\right) := \bar{K}^{\alpha\beta}_{j,i,k}(x,\xi) \tag{25a}$$

$$\hat{\hat{K}}^{\beta\beta}_k\left(L_{i,k} + \frac{\lambda_k\gamma_k}{\lambda_i}x, L_{j,k} + \frac{\lambda_k\gamma_k}{\lambda_j}\xi\right) := \bar{K}^{\beta\beta}_{j,i,k}(x,\xi). \tag{25b}$$

Furthermore, introducing $\hat{K}^{\cdot\cdot}_k$ as

$$\hat{K}^{\alpha\beta}_k(x,\xi) := e^{-\frac{\gamma_i\lambda_i}{\gamma_k\lambda_k}(x-L_{i,k}) - \frac{\gamma_j\lambda_j}{\gamma_k\lambda_k}(\xi-L_{k,k})}\hat{\hat{K}}^{\alpha\beta}_k(x,\xi) \tag{26a}$$

$$\hat{K}^{\beta\beta}_k(x,\xi) := e^{\frac{\gamma_i\lambda_i}{\gamma_k\lambda_k}(x-L_{i,k}) - \frac{\gamma_j\lambda_j}{\gamma_k\lambda_k}(\xi-L_{k,k})}\hat{\hat{K}}^{\beta\beta}_k(x,\xi) \tag{26b}$$

we find by combining (16), (21) with (25)–(26) that $\hat{K}^{\alpha\beta}_k, \hat{K}^{\beta\beta}_k$ satisfy

$$(-\partial_\xi + \partial_x)\hat{K}^{\alpha\beta}_k(x,\xi) = \mu_k(x,\xi)\hat{K}^{\alpha\beta}_k(x,\xi) + \nu_k(x,\xi)\hat{K}^{\beta\beta}_k(x,\xi) \tag{27a}$$

$$(-\partial_\xi - \partial_x)\hat{K}^{\beta\beta}_k(x,\xi) = \mu_k(x,\xi)\hat{K}^{\beta\beta}_k(x,\xi) + \nu_k(x,\xi)\hat{K}^{\alpha\beta}_k(x,\xi) \tag{27b}$$

$$\hat{K}_k^{\alpha\beta}(x,x) = \begin{cases} 0 \text{ if } 0 < x < L_{k,k} \\ 1 \text{ if } L_{k,k} < x < L_{k+1,k} \end{cases} \quad (27c)$$

$$\hat{K}_k^{\alpha\beta}(x, L_{k+1,k}) = 0 \text{ if } L_{k+1,k} < x < L_{N+1,k} \quad (27d)$$

$$\hat{K}_k^{\beta\beta}(x, L_{k+1,k}) = 0 \text{ if } L_{k+1,k} < x < L_{N+1,k} \quad (27e)$$

$$\hat{K}_k^{\alpha\beta}(L_{i,k}^+, \xi) = -\varphi_i \hat{K}_k^{\beta\beta}(L_{i,k}^+, \xi) + e^{\gamma_{i-1}} \bar{\tau}_{i-1,i} \hat{K}_k^{\alpha\beta}(L_{i,k}^-, \xi) \quad (27f)$$

$$\hat{K}_k^{\beta\beta}(L_{i,k}^-, \xi) = -e^{2\gamma_{i-1}} \vartheta_{i-1} \hat{K}_k^{\alpha\beta}(L_{i,k}^-, \xi) + e^{\gamma_{i-1}} \bar{\tau}_{i,i-1} \hat{K}_k^{\beta\beta}(L_{i,k}^+, \xi), \quad (27g)$$

over $\hat{\mathscr{T}}_k$ and $L_{i,k}^-$ and $L_{i,k}^+$ are given by

$$L_{i,k}^- := L_{i,k} - \varepsilon, \qquad L_{i,k}^+ := L_{i,k} + \varepsilon \quad (28)$$

for $\varepsilon > 0$ an infinitesimally small real number[1], while ν_k, μ_k are piecewise constant functions defined for $L_{i,k} < x < L_{i+1,k}$ as

$$\nu_k(x) := \frac{\lambda_k \gamma_k}{\lambda_i \gamma_i}, \qquad \mu_k(x,\xi) := \frac{\lambda_j \gamma_j - \lambda_i \gamma_i}{\lambda_k \gamma_k}. \quad (29)$$

III. SOLVING THE KERNELS

A. Initial part of solution

We solve in this section (27) under the assumption of (14). Using the definitions of λ_i, γ_i from (7b) we then have that

$$\nu_k(x) \equiv 1, \qquad \mu_k(x,\xi) \equiv 0,$$

for all $(x,\xi) \in \hat{\mathscr{T}}_k$, and the set of equations (27) may be written as

$$(-\partial_\xi + \partial_x)\hat{K}_k^{\alpha\beta}(x,\xi) = \hat{K}_k^{\beta\beta}(x,\xi) \quad (30a)$$

$$(-\partial_\xi - \partial_x)\hat{K}_k^{\beta\beta}(x,\xi) = \hat{K}_k^{\alpha\beta}(x,\xi) \quad (30b)$$

$$\hat{K}_k^{\alpha\beta}(x,x) = \begin{cases} 0 \text{ if } 0 < x < L_{k,k} \\ 1 \text{ if } L_{k,k} < x < L_{k+1,k} \end{cases} \quad (30c)$$

$$\hat{K}_k^{\alpha\beta}(x, L_{k+1,k}) = 0 \text{ if } L_{k+1,k} < x < L_{N+1,k} \quad (30d)$$

$$\hat{K}_k^{\beta\beta}(x, L_{k+1,k}) = 0 \text{ if } L_{k+1,k} < x < L_{N+1,k} \quad (30e)$$

$$\hat{K}_k^{\alpha\beta}(L_{i,k}^+, \xi) = -\varphi_i \hat{K}_k^{\beta\beta}(L_{i,k}^+, \xi) + e^{\gamma_{i-1}} \bar{\tau}_{i-1,i} \hat{K}_k^{\alpha\beta}(L_{i,k}^-, \xi) \quad (30f)$$

$$\hat{K}_k^{\beta\beta}(L_{i,k}^-, \xi) = -e^{2\gamma_{i-1}} \vartheta_{i-1} \hat{K}_k^{\alpha\beta}(L_{i,k}^-, \xi) + e^{\gamma_{i-1}} \bar{\tau}_{i,i-1} \hat{K}_k^{\beta\beta}(L_{i,k}^+, \xi). \quad (30g)$$

The equations (30) represent a scalar Goursat problem that may be solved explicitly. Differently from the previous case of a single scalar 2×2 linear hyperbolic Goursat problem from [11], we have in the problem (30) discontinuous boundary information along the diagonal boundary condition (30c), as well as vertical interfaces of reflection and transmission coefficients (30f)–(30g). Overcoming these challenges is the main contribution of this section.

Firstly, due to the horizontal boundary conditions (30d)–(30e), together with the direction of the characteristics due to (30a)–(30b), we have that

$$\hat{K}_k^{\alpha\beta}(x,\xi) \equiv \hat{K}_k^{\beta\beta}(x,\xi) \equiv 0$$

$$\text{for } \xi \geq 2L_{k+1,k} - x, \; L_{k+1,k} < x < L_{N+1,k}. \quad (31)$$

Hence, the boundary conditions (30d)–(30e) may be replaced by the boundary condition

$$\hat{K}_k^{\beta\beta}(x, 2L_{k+1,k} - x) = 0 \text{ for } L_{k+1,k} < x < L_{N+1,k}, \; \xi > 0. \quad (32)$$

To solve for (30) over the remaining part of the domain, we consider first the region immediately below the point $(x,\xi) = (L_{k+1,k}, L_{k+1,k})$. Letting

$$\mathscr{D}_{i,k} := \min\{L_{i,k} - L_{i-1,k}, L_{i+1,k} - L_{i,k}\}, \quad (33)$$

we consider the sub-problem of solving (30) for $(x,\xi) \in \mathscr{T}_{k+1,k} := \{L_{k+1,k} - \mathscr{D}_{k+1,k} \leq \xi \leq x \leq L_{k+1,k}\} \cup \{L_{k+1,k} \leq 2L_{k+1,k} - \xi \leq x \leq L_{k+1,k} + \mathscr{D}_{k+1,k}\}$. Define the constants

$$R_{i,j} := \begin{cases} \frac{A_j - A_i}{A_j + A_i}, \text{ if } i < N+1 \\ 1 \text{ for } i = N+1 \end{cases}, \quad T_{i,j} := 2\frac{\gamma_j}{\gamma_i}\frac{A_i}{A_j + A_i} \quad (34)$$

and the functions

$$F_{0,k}(x,\xi) := \mathscr{I}_0(\sqrt{(2L_{k+1,k} - x - \xi)(x - \xi)}) \quad (35a)$$

$$F_{1,k}(x,\xi) := \sqrt{\frac{x-\xi}{2L_{k+1,k} - x - \xi}} \times \mathscr{I}_1(\sqrt{(2L_{k+1,k} - x - \xi)(x - \xi)}) \quad (35b)$$

$$F_{-1,k}(x,\xi) := \sqrt{\frac{2L_{k+1,k} - x - \xi}{x-\xi}} \times \mathscr{I}_1(\sqrt{(2L_{k+1,k} - x - \xi)(x - \xi)}), \quad (35c)$$

where $\mathscr{I}_m$ denotes the m^{th} order modified Bessel function of the first kind. We have the following result.

Lemma 3.1: The solution to (30) over $\mathscr{T}_{k+1,k}$ may be written explicitly as

$$\hat{K}_k^{\alpha\beta}(x,\xi) = \begin{cases} F_{0,k}(x,\xi) + R_{k+1,k}F_{1,k}(x,\xi) \\ \qquad \text{if } L_{k+1,k} - \mathscr{D}_{k+1,k} < x < L_{k+1,k}, \\ T_{k+1,k}F_{0,k}(x,\xi) \\ \qquad \text{if } L_{k+1,k} < x < L_{k+1,k} + \mathscr{D}_{k+1,k}, \end{cases} \quad (36a)$$

$$\hat{K}_k^{\beta\beta}(x,\xi) = \begin{cases} F_{-1,k}(x,\xi) + R_{k+1,k}F_{0,k}(x,\xi) \\ \qquad \text{if } L_{k+1,k} - \mathscr{D}_{k+1,k} < x < L_{k+1,k}, \\ T_{k+1,k}F_{-1,k}(x,\xi) \\ \qquad \text{if } L_{k+1,k} < x < L_{k+1,k} + \mathscr{D}_{k+1,k}. \end{cases} \quad (36b)$$

Proof: Let $k < N$. The proof for the case of $k = N$ is almost identical and hence omitted. Mirroring $\hat{K}_k^{\alpha\beta}, \hat{K}_k^{\beta\beta}$ for $L_{k+1,k} < x < L_{k+1,k} + \mathscr{D}_{k+1,k}$ across the line $x = L_{k+1,k}$ by defining

$$\check{K}_k^{\alpha\alpha}(x,\xi) := \hat{K}_k^{\alpha\alpha}(L_{k+1,k} - x, \xi),$$

$$\check{K}_k^{\beta\alpha}(x,\xi) := \hat{K}_k^{\beta\alpha}(L_{k+1,k} - x, \xi),$$

the scalar Goursat problem (30) over $\mathscr{T}_{k+1,k}$ may be rewritten as a 2×2 matrix Goursat problem for $[\check{K}_k^{\alpha\beta} \; \hat{K}_k^{\beta\beta}]^\top$, $[\check{K}_k^{\beta\beta} \; \hat{K}_k^{\alpha\beta}]^\top$ over $(x,\xi) \in \{L_{k+1,k} - \mathscr{D}_{k+1,k} \leq \xi \leq x \leq L_{k+1,k}\}$. Performing similar steps to [12], the solution to $[\check{K}_k^{\alpha\beta} \; \hat{K}_k^{\beta\beta}]^\top$ is found to satisfy

$$\begin{bmatrix} \check{K}_k^{\alpha\beta}(x,\xi) \\ \hat{K}_k^{\beta\beta}(x,\xi) \end{bmatrix} = \Bigg(I \sum_{n=0}^{\infty} \frac{(\frac{2L_{k+1,k} - x - \xi}{2})^{n+1} (\frac{x-\xi}{2})^n}{n!(n+1)!} + \hat{\Theta}_{k+1} \sum_{n=0}^{\infty} \frac{(\frac{2L_{k+1,k} - x - \xi}{2})^n (\frac{x-\xi}{2})^n}{n!n!}$$

[1]The terms $L_{i,k}^-$, $L_{i,k}^+$ represent a number immediately smaller than or larger than $L_{i,k}$, respectively, so that $\hat{K}_k^{\cdot\cdot}(L_{i,k}^-, \hat{\xi}) \equiv \bar{K}_{j,i-1,k}^{\cdot\cdot}(1,\xi)$ and $\hat{K}_k^{\cdot\cdot}(L_{i,k}^+, \hat{\xi}) \equiv \bar{K}_{j,i,k}^{\cdot\cdot}(0,\xi)$.

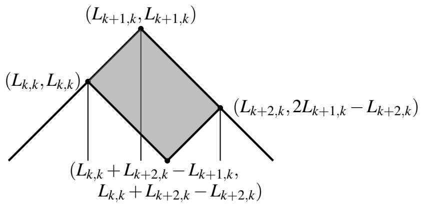

Fig. 3: Diagonally oriented rectangle for which (36) holds, highlighted by shaded region. Shown for case of $L_{k+2,k} - L_{k+1,k} > L_{k+1,k} - L_{k,k}$.

$$+\sum_{m=0}^{\infty} \hat{\Theta}_{k+1}^{m+2} \sum_{n=0}^{\infty} \frac{(\frac{2L_{k+1,k}-x-\xi}{2})^n (\frac{x-\xi}{2})^{n+m+1}}{n!(n+m+1)!}$$
$$- \sum_{m=0}^{\infty} \hat{\Theta}_{k+1}^{m} \sum_{n=0}^{\infty} \frac{(\frac{2L_{k+1,k}-x-\xi}{2})^n (\frac{x-\xi}{2})^{n+m+1}}{n!(n+m+1)!} \Bigg) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (37)$$

where we have defined

$$\hat{\Theta}_{k+1} := \begin{bmatrix} -\varphi_{k+1} & e^{\gamma_k} \bar{\tau}_{k,k+1} \\ e^{\gamma_k} \bar{\tau}_{k+1,k} & -e^{2\gamma_k} \vartheta_k \end{bmatrix} \quad (38)$$

and I denotes the 2×2 identity matrix. Applying the definitions of the reflection and transmission coefficients from (7c)–(7e) for $k < N$ in (38), we see that $\hat{\Theta}_{k+1}$ is involutary, so the last two double sums in (37) cancel. Using then the identity ([14])

$$\sum_{n=0}^{\infty} \frac{a^{n+m} b^n}{(n+m)!n!} = \sqrt{\frac{a^m}{b^m}} \mathscr{I}_m(2\sqrt{ab}),$$

we may write

$$\begin{bmatrix} \check{K}_k^{\alpha\beta}(x,\xi) \\ \hat{K}_k^{\beta\beta}(x,\xi) \end{bmatrix} = \Bigg(\sqrt{\frac{2L_{k+1,k} - x - \xi}{x - \xi}}$$
$$\times \mathscr{I}_1(\sqrt{(2L_{k+1,k} - x - \xi)(x - \xi)})$$
$$+ \hat{\Theta}_{k+1} \mathscr{I}_0(\sqrt{(2L_{k+1,k} - x - \xi)(x - \xi)}) \Bigg) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (39)$$

Writing (39) out component-wise and mirroring $\check{K}_k^{\alpha\beta}$ back across the line $x = L_{k+1,k}$, we obtain (36a) for $L_{k+1,k} < x < L_{k+1,k} + \mathscr{D}_{k+1,k}$ and (36b) for $L_{k+1,k} - \mathscr{D}_{k+1,k} < x < L_{k+1,k}$. Performing similar steps for $[\check{K}_k^{\beta\beta} \ \ \hat{K}_k^{\alpha\beta}]^\top$ yields (36a) for $L_{k+1,k} - \mathscr{D}_{k+1,k} < x < L_{k+1,k}$ and (36b) for $L_{k+1,k} < x < L_{k+1,k} + \mathscr{D}_{k+1,k}$. ■

B. Recursive construction of complete solution

To propagate the solutions to (30) beyond that given within $\mathscr{T}_{k+1,k}$ by (36), firstly by continuity the solutions stay as (36) within the diagonally oriented rectangle with corners in $(x,\xi) = (L_{k,k}, L_{k,k}), (L_{k+1,k}, L_{k+1,k}), (L_{k,k} + L_{k+2,k} - L_{k+1,k}, L_{k,k} + L_{k+1,k} - L_{k+2,k}), (L_{k+2,k}, 2L_{k+1,k} - L_{k+2,k})$ (see Figure 3). Below the point $(L_{k,k}, L_{k,k})$, we have the vertical boundary conditions (30f)–(30g) for $i = k$, and for $x < L_{k,k}$ we have that $\hat{K}_k^{\alpha\beta}(x,x) = 0$. Likewise, below the point $(L_{k+2,k}, 2L_{k+1,k} - L_{k+2,k})$ we have the vertical boundary conditions (30f)–(30g) for $i = k+2$.

To deal with these discontinuities, consider that the solutions (36) consist of two components for $L_{k+1,k} - \mathscr{D}_{k+1,k} < x < L_{k+1,k}$, namely one originating from the boundary data $\hat{K}^{\alpha\beta}(x,x) = 1$, and the other originating from a reflection along the vertical interface due to (30f)–(30g). On the other hand, for $L_{k+1,k} < x < L_{k+1,k} + \mathscr{D}_{k+1,k}$ the solutions are seen to consist of a single component, namely a transmission of the term that originated in the boundary data $\hat{K}^{\alpha\beta}(x,x) = 1$ through (30f)–(30g).

We suggest therefore that the solution to (30) for $(x,\xi) \in \mathscr{T}_{k+2,k} := \{L_{k+2,k} - \mathscr{D}_{k+2,k} \leq \xi - 2(L_{k+1,k} - L_{k+2,k}) \leq x \leq L_{k+2,k}\} \cup \{L_{k+2,k} \leq 2L_{k+2,k} - \xi \leq x \leq L_{k+2,k} + \mathscr{D}_{k+2,k}\}$ to be given by

$$\hat{K}_k^{\alpha\beta}(x,\xi) = \begin{cases} T_{k+1,k}\Big(F_{0,k}(x,\xi) \\ \quad +R_{k+2,k+1} F_{1,k}(x - 2(L_{k+2,k} - L_{k+1,k}), \xi)\Big) \\ \quad \text{if } L_{k+2,k} - \mathscr{D}_{k+2,k} < x < L_{k+2,k} \\ T_{k+1,k} T_{k+2,k+1} F_{0,k}(x,\xi) \\ \quad \text{if } L_{k+2,k} < x < L_{k+2,k} + \mathscr{D}_{k+2,k}, \end{cases} \quad (40a)$$

$$\hat{K}_k^{\beta\beta}(x,\xi) = \begin{cases} T_{k+1,k}\Big(F_{-1,k}(x,\xi) \\ \quad +R_{k+2,k+1} F_{0,k}(x - 2(L_{k+2,k} - L_{k+1,k}), \xi)\Big) \\ \quad \text{if } L_{k+2,k} - \mathscr{D}_{k+2,k} < x < L_{k+2,k} \\ T_{k+1,k} T_{k+2,k+1} F_{-1,k}(x,\xi) \\ \quad \text{if } L_{k+2,k} < x < L_{k+2,k} + \mathscr{D}_{k+2,k}. \end{cases} \quad (40b)$$

It is straightforward to verify by substitution that (40) satisfy the kernel equations (30) within $\mathscr{T}_{k+2,k}$. Similar expressions may be developed for $(x,\xi) \in \mathscr{T}_{k,k} := \{L_{k,k} - \mathscr{D}_{k,k} \leq \xi \leq x \leq L_{k,k}\} \cup \{L_{k,k} \leq 2L_{k,k} - \xi \leq x \leq L_{k,k} + \mathscr{D}_{k,k}\}$. The structure of the solutions (36), (40) suggests that the solution to (30) over the entire trapezoidal domain $\hat{\mathscr{T}}_k$ may be constructed recursively via reflections and transmissions of the initial, unreflected solution component from (36), for $x < L_{k+1,k}$, throughout the solution domain. To do this, we suggest the routine given by Algorithm 1.

Algorithm 1 Recursive construction of $\hat{K}_k^{\alpha\beta}, \hat{K}_k^{\beta\beta}$

▷ *Initialize solution domain*
for $(x,\xi) \in \hat{\mathscr{T}}_k$ **do**
  if $L_{k,k} < x < L_{k+1,k}$ & $2L_{k,k} - x < \xi < x$ **then**
    $\hat{K}_k^{\alpha\beta}(x,\xi) \leftarrow F_{0,k}(x,\xi)$, $\hat{K}_k^{\beta\beta}(x,\xi) \leftarrow F_{-1,k}(x,\xi)$
  else
    $\hat{K}_k^{\alpha\beta}(x,\xi) \leftarrow 0$, $\hat{K}_k^{\beta\beta}(x,\xi) \leftarrow 0$
  end if
end for

▷ *Solve for* $\hat{K}_k^{\alpha\beta}$
$I_\xi \leftarrow (\max(0, L_{k+1,k} - 2\mathscr{D}_{k+1,k}),\ L_{k+1,k})$
if $k < N$ **then**
  TRANSMITRIGHT($\hat{K}_k^{\alpha\beta}$, $F_{0,k}$, $k+1$, I_ξ, 0)
end if
REFLECTLEFT($\hat{K}_k^{\alpha\beta}$, $F_{0,k}$, $k+1$, I_ξ, 0)

▷ *Solve for* $\hat{K}_k^{\beta\beta}$
if $k < N$ **then**
  TRANSMITRIGHT($\hat{K}_k^{\beta\beta}$, $F_{-1,k}$, $k+1$, I_ξ,0)
end if
REFLECTLEFT($\hat{K}_k^{\beta\beta}$, $F_{-1,k}$, $k+1$, I_ξ, 0)

The procedures TRANSMITRIGHT, REFLECTLEFT (along with TRANSMITLEFT, REFLECTRIGHT that are called recursively) used by Algorithm 1 are documented in the Appendix. We have the following result.

Theorem 3.2: Algorithm 1 halts after a finite number of steps and produces the explicit solutions to (30) over $\hat{\mathscr{T}}_k$.

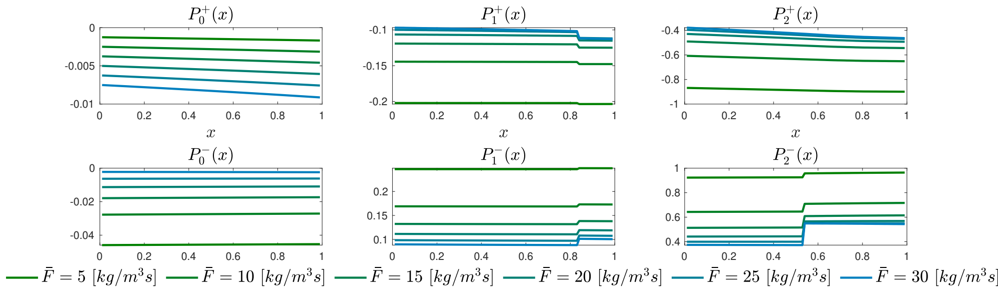

Fig. 4: Observer gains computed for a range of mean friction factors.

Proof: We prove first that the algorithm halts after a finite number of steps. It can be seen that the upper and lower bounds of the open intervals $I_\xi(L_{i+1,k})$, $I_\xi(L_{i-1,k})$ passed in the recursive calls are decreasing with each call while remaining greater than or equal to zero. Hence, after a finite number of steps we have that $I_\xi(L_{i+1,k}), I_\xi(L_{i-1,k}) = \emptyset$, where $\emptyset$ denotes the empty set, and the recursion returns.

Next we prove that the explicit solutions to (30) are returned after Algorithm 1 halts. Using the definitions (35) of respectively $F_{0,k}, F_{-1,k}$, it is verified that the initial solution produced by Algorithm 1 for $L_{k,k} < x < L_{k+1,k}$, $2L_{k,k} - x < \xi < x$ satisfies (30a)–(30b). Seeing that REFLECTLEFT is inevitably the first reflection function to be called for both $\hat{K}^{\alpha\beta}, \hat{K}^{\beta\beta}$, the solution to $\hat{K}^{\alpha\beta}$ consists of $F_{0,k}, F_{1,k}$ functions, only, while $\hat{K}^{\beta\beta}$ only consists of $F_{-1,k}, F_{0,k}$ functions. Since the recursive calls are made in the same order for both $\hat{K}^{\alpha\beta}, \hat{K}^{\beta\beta}$, the equations (30a)–(30b) are satisfied throughout all of $\mathscr{T}_k$.

For the boundary conditions, because $F_{0,k}(x,x) = 1$, (30c) is satisfied for $L_{k,k} < x < L_{k+1,k}$. Due to REFLECTLEFT we see that $\hat{K}^{\alpha\beta}(x,x) \propto F_{1,k}(x,x) = 0$ for $x < L_{k,k}$ and hence the remainder of (30c) is satisfied. Along the line $(x, L_{k+1,k})$ for $x > L_{k+1,k}$, $\hat{K}^{\alpha\beta}, \hat{K}^{\beta\beta}$ are both initialized to 0, and since this region is not altered by the recursive calls, the boundary conditions (30d)–(30e) are satisfied. Lastly, considering that the variable Δ that is passed through successive recursive calls made by the subroutines of Algorithm 1 (see the Appendix) tracks the relative distance in the x-direction from $L_{k+1,k}$, using (35) it is seen that the terms produced by the recursive calls to the transmit and reflect functions together with the term produced by the previous function calling these, for both $\hat{K}^{\alpha\beta}$ and $\hat{K}^{\beta\beta}$, satisfy the boundary conditions (30f)–(30g) at $x = L_{i+1,k}$ for the case of TRANSMITRIGHT, REFLECTRIGHT, and $x = L_{i-1,k}$ for TRANSMITLEFT, REFLECTLEFT. ■

Having produced explicit solutions to (30) with Algorithm 1, the explicit solutions $\bar{K}_{i,j,k}^{\alpha\beta}, \bar{K}_{i,j,k}^{\beta\beta}$ to (16)–(21) may be obtained by reversing the variable changes done via (25), (26). Repeating equivalent steps as done in Sections II-B, III for (17)–(22), explicit solutions $\hat{K}_k^{\alpha\alpha}, \hat{K}_k^{\beta\alpha}$ from which explicit solutions $\bar{K}_{i,j,k}^{\alpha\alpha}, \bar{K}_{i,j,k}^{\beta\alpha}$ may be found. Substituting these then into (15) yields explicit solutions for the kernel equations (10)–(13), from which observer gains (9) may be computed.

IV. NUMERICAL EXAMPLE

We consider here a numerical example of observer gains (9) computed from the closed-form kernel solutions developed in Sections II–III. We use the same system parameters as used in Section VI of [10], apart from the friction factors, for which a mean value throughout the network is used for each operating point. In [10] a ring-shaped water distribution network such as the one shown in Figure 1 with $N = 2$ pipes is considered. Hence, a total of 6 observer gains, namely $P_0^+, P_0^-, P_1^+, P_1^-, P_2^+, P_2^-$ need to be computed from (9) for each operating point. The average water consumption level considered in [10] results in a mean friction factor of $\bar{F} \approx 20$ $[kg/m^3s]$ throughout the network.

To emphasize the use of the observer gains in a gain-scheduling setting, we consider the observer gains computed for a range of mean friction factors, representing variations in the mean water consumption in the network, a quantity that in practice slowly varies throughout the day and year [15] based on cyclical variations in water demand patterns. Specifically, we compute here the observer gains (9), with $i \in \{0,1,2\}$, for the mean friction factors given by $\bar{F} = \{5, 10, 15, 20, 25, 30\}$ $[kg/m^3s]$. Using the mean friction factors throughout the network for each operating point, the resultant observer gains calculated from the explicit solutions for $K_{j,i}^{\alpha\alpha}, K_{j,i}^{\beta\alpha}, K_{j,i}^{\alpha\beta}, K_{j,i}^{\beta\beta}$ are plotted in Figure 4.

It is seen from Figure 4 that the observer gains vary in magnitude as the mean friction in the network changes. In general it is seen that for lower mean friction factor $\bar{F}$, the constant offset terms Lg_i^α, Lg_i^β for respectively P_i^+, P_i^- dominate the gains, with the gains gradually shifting away from these values as the mean friction factor increases. Additionally, for $P_1^\pm$, $P_2^\pm$ there is a clear discontinuity within the domain of each of the respective gains. For $P_1^\pm$, the location of the discontinuity is, for the parameters used here, calculated to be at $x = \frac{\lambda_1}{\lambda_0} = 0.833$, being the point where the characteristic line originating from the point $(L_{1,k}, L_{1,k})$ intersects the $\hat{x}$-axis between $\hat{x} = L_{1k}$ and $\hat{x} = L_{2k}$ ($\hat{x}$ being in the global coordinates from Section II-B). Likewise, for $P_2^\pm$ the location of the discontinuity is seen, for the particular parameters used here, to be at $x = 2 - \lambda_2(\frac{1}{\lambda_0} + \frac{1}{\lambda_1}) = 0.533$, which corresponds to the point where the characteristic line originating from $(L_{2,k}, L_{2,k})$ intersects the $\hat{x}$-axis between $\hat{x} =$

$L_{2,k}$ and $\hat{x} = L_{3,k}$, after reflecting from the vertical boundary located along $(L_{3,2}, \hat{\xi})$.

V. Conclusion

A method for finding the explicit solutions to the kernel equations (10)–(13), under the assumption (14), for the computation of observer gains (9) has been developed. The explicit solution is found from the recursive procedure given in Algorithm 1 together with coordinate changes (15)–(25), (26), and is expressed in terms of modified Bessel functions of the first kind. The main practical value of having explicit expressions for the kernel equation solutions is in a gain-scheduling setting, where the observer gains need to be updated by recomputing the kernel equations as underlying parameters change. Since Algorithm 1 only needs to be run once for each value of k for a given network configuration to obtain the explicit kernel solutions, the kernel equations can in this case be updated as the mean network friction changes by simply varying the mean network friction parameter in the closed-form expressions for the solution, and sampling the solution at the desired grid points.

One direction for future work building on the contribution of this paper is to study how well using the mean friction factor of the entire network in computing the observer gains via the explicit expressions found here compares to numerically approximating the kernels with different friction factors for each pipe in the network. Also, future work should address whether explicit solutions to the kernel equations (10)–(13) may be found without the restriction (14), and additionally whether the kernel solution method developed here extends itself to finding explicit kernel equation solutions associated to more complex topologies than a single branching point as considered in [12] or a loop-shaped network as considered here.

References

[1] M. A. Adegboye, W.-K. Fung, and A. Karnik, "Recent advances in pipeline monitoring and oil leakage detection technologies: Principles and approaches," *Sensors*, vol. 19, no. 11, p. 2548, 2019.

[2] S. El-Zahab and T. Zayed, "Leak detection in water distribution networks: an introductory overview," *Smart Water*, vol. 4, no. 1, pp. 1–23, 2019.

[3] F. Tanimola and D. Hill, "Distributed fibre optic sensors for pipeline protection," *Journal of Natural Gas Science and Engineering*, vol. 1, no. 4-5, pp. 134–143, 2009.

[4] L. Billmann and R. Isermann, "Leak detection methods for pipelines," *Automatica*, vol. 23, no. 3, pp. 381–385, 1987.

[5] C. Verde, "Minimal order nonlinear observer for leak detection," *J. Dyn. Sys., Meas., Control*, vol. 126, no. 3, pp. 467–472, 2004.

[6] G. Besançon, D. Georges, O. Begovich, C. Verde, and C. Aldana, "Direct observer design for leak detection and estimation in pipelines," in *2007 European Control Conference (ECC)*. IEEE, 2007, pp. 5666–5670.

[7] O. M. Aamo, "Leak detection, size estimation and localization in pipe flows," *IEEE Transactions on Automatic Control*, vol. 61, no. 1, pp. 246–251, 2015.

[8] R. Vazquez, M. Krstic, and J.-M. Coron, "Backstepping boundary stabilization and state estimation of a 2× 2 linear hyperbolic system," in *2011 50th IEEE conference on decision and control and european control conference*. IEEE, 2011, pp. 4937–4942.

[9] H. Anfinsen and O. M. Aamo, "Leak detection, size estimation and localization in branched pipe flows," *Automatica*, p. 110213, 2022.

[10] N. C. A. Wilhelmsen and O. M. Aamo, "Leak detection, size estimation and localization in water distribution networks containing loops," in *61st IEEE Conference on Decision and Control (CDC)*, 2022, pp. 5429–5436.

[11] R. Vazquez and M. Krstic, "Marcum q-functions and explicit kernels for stabilization of 2× 2 linear hyperbolic systems with constant coefficients," *Systems & Control Letters*, vol. 68, pp. 33–42, 2014.

[12] N. C. A. Wilhelmsen and O. M. Aamo, "Explicit backstepping kernel solutions for leak detection in branched pipe flows," *IEEE Control Systems Letters*, vol. 7, pp. 913–918, 2022.

[13] R. A. Bajura, "A model for flow distribution in manifolds," *Journal of Engineering for Power*, vol. 93, no. 1, pp. 7–12, 1971.

[14] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. US Government printing office, 1964, vol. 55.

[15] D. R. Maidment and S.-P. Miaou, "Daily water use in nine cities," *Water Resources Research*, vol. 22, no. 6, pp. 845–851, 1986.

Appendix

The subroutines used in Algorithm 1 are documented here. The symbol $\emptyset$ denotes the empty set.

procedure TransmitRight(K, dK, i, $(\xi_l,\ \xi_u)$, Δ)
  $I_x \leftarrow (L_{i,k},\ L_{i+1,k})$
  $I_\xi(x) \leftarrow (\max(0, \xi_l - x + L_{i,k}),\ \max(0, \xi_u - x + L_{i,k}))$
  for $x \in I_x$, $\xi \in I_\xi(x)$ **do**
    $K(x,\xi) \leftarrow K(x,\xi) + T_{i,i-1} dK(x,\xi)$
  end for
  if $I_\xi(L_{i+1,k}) \neq \emptyset$ **then**
    $\Delta \leftarrow \Delta - 2(L_{i+1,k} - L_{i,k})$
    if $i < N$ **then**
      TransmitRight(K, $T_{i,i-1}dK$, $i+1$, $I_\xi(L_{i+1,k})$, Δ)
    end if
    ReflectLeft(K, $T_{i,i-1}dK$, $i+1$, $I_\xi(L_{i+1,k})$, Δ)
  end if
end procedure

procedure TransmitLeft(K, dK, i, $(\xi_l,\ \xi_u)$, Δ)
  $I_x \leftarrow (L_{i-1,k},\ L_{i,k})$
  $I_\xi(x) \leftarrow (\max(0, \xi_l + x - L_{i,k}),\ \max(0, \xi_u + x - L_{i,k}))$
  for $x \in I_x$, $\xi \in I_\xi(x)$ **do**
    $K(x,\xi) \leftarrow K(x,\xi) + T_{i-1,i} dK(x,\xi)$
  end for
  if $I_\xi(L_{i-1,k}) \neq \emptyset$ **then**
    $\Delta \leftarrow \Delta + 2(L_{i,k} - L_{i-1,k})$
    TransmitLeft(K, $T_{i-1,i}dK$, $i-1$, $I_\xi(L_{i-1,k})$, Δ)
    ReflectRight(K, $T_{i-1,i}dK$, $i-1$, $I_\xi(L_{i-1,k})$, Δ)
  end if
end procedure

procedure ReflectRight(K, dK, i, $(\xi_l,\ \xi_u)$, Δ)
  $I_x \leftarrow (L_{i,k},\ L_{i+1,k})$
  $I_\xi(x) \leftarrow (\max(0, \xi_l - x + L_{i,k}),\ \max(0, \xi_u - x + L_{i,k}))$
  for $x \in I_x$, $\xi \in I_\xi(x)$ **do**
    $dM(x,\xi) \leftarrow (-\partial_\xi + \partial_x) dK(x+\Delta, \xi)$
    $K(x,\xi) \leftarrow K(x,\xi) + R_{i-1,i} dM(x,\xi)$
  end for
  if $I_\xi(L_{i+1,k}) \neq \emptyset$ **then**
    $\Delta \leftarrow \Delta - 2(L_{i+1,k} - L_{i,k})$
    if $i < N$ **then**
      TransmitRight(K, $R_{i-1,i}dM$, $i+1$, $I_\xi(L_{i+1,k})$, Δ)
    end if
    ReflectLeft(K, $R_{i-1,i}dM$, $i+1$, $I_\xi(L_{i+1,k})$, Δ)
  end if
end procedure

procedure ReflectLeft(K, dK, i, $(\xi_l,\ \xi_u)$, Δ)
  $I_x \leftarrow (L_{i-1,k},\ L_{i,k})$
  $I_\xi(x) \leftarrow (\max(0, \xi_l + x - L_{i,k}),\ \max(0, \xi_u + x - L_{i,k}))$
  for $x \in I_x$, $\xi \in I_\xi(x)$ **do**
    $dM(x,\xi) \leftarrow (-\partial_\xi - \partial_x) dK(x+\Delta, \xi)$
    $K(x,\xi) \leftarrow K(x,\xi) + R_{i,i-1} dM(x,\xi)$
  end for
  if $I_\xi(L_{i-1,k}) \neq \emptyset$ **then**
    $\Delta \leftarrow \Delta + 2(L_{i,k} - L_{i-1,k})$
    TransmitLeft(K, $R_{i,i-1}dM$, $i-1$, $I_\xi(L_{i-1,k})$, Δ)
    ReflectRight(K, $R_{i,i-1}dM$, $i-1$, $I_\xi(L_{i-1,k})$, Δ)
  end if
end procedure