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Athanasios Kouroupis

Aspects of Dirichlet series: Composition operators, Bohr's theorem and universality

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Thesis for the Degree of Philosophiae Doctor
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Athanasios Kouroupis
Trondheim, 2024

Dedicated to the memory of my friend, M. Manos.

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Introduction

1. Dirichlet series

In the present thesis, we are interested in questions related to the classical theory of Dirichlet series and the theory of Hardy spaces of Dirichlet series. A Dirichlet series is a series of functions of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad s = \sigma + it. \tag{1}$$

One of the most important and well studied examples is the Riemann zeta function, which can be written as a Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\operatorname{Re} s > 1$.

For what follows we will assume that a Dirichlet series will converge in at least one point $s = \sigma + it$ in the complex plane. Such a series converges in half-planes and we will be using the notation $\mathbb{C}_\theta$ to denote the half-plane of complex numbers with $\operatorname{Re} s > \theta$, $\theta \in \mathbb{R}$. We define the abscissa of pointwise, absolute, uniform, and square convergence, [58]:

$$\sigma_c(f) = \inf\left\{\sigma \in \mathbb{R} : \, f(s) = \sum_{n\geq 1} \tfrac{a_n}{n^\sigma} \ \text{ converges}\right\},$$

$$\sigma_a(f) = \inf\left\{\sigma \in \mathbb{R} : \, f(s) = \sum_{n\geq 1} \tfrac{|a_n|}{n^\sigma} \ \text{ converges}\right\},$$

$$\sigma_u(f) = \inf\left\{\sigma \in \mathbb{R} : \, f(s) = \sum_{n\geq 1} \tfrac{a_n}{n^s} \ \text{ converges uniformly in } \mathbb{C}_\sigma\right\},$$

$$\sigma_2(f) = \inf\left\{\sigma \in \mathbb{R} : \, f(s) = \sum_{n\geq 1} \tfrac{|a_n|^2}{n^{2\sigma}} \ \text{ converges}\right\}.$$

A result of key importance for the classical theory of Dirichlet series is the following theorem due to Bohr.

THEOREM 1. *Let f be a Dirichlet series. If there is a real number θ and a bounded set Ω such that f has an analytic continuation to $\mathbb{C}_\theta$ that maps $\mathbb{C}_\theta$ to Ω, then $\sigma_{\mathrm{u}}(f) \leq \theta$.*

Queffélec and Seip [52] (see also [51, Theorem 8.4.1]) showed that the assumption that Ω is a bounded set may be replaced with the weaker assumption that Ω is a half-plane. In Article 5 we extend this result in an optimal way to hyperbolic domains, that is, to domains Ω that omit two distinct points in the complex plane.

1.1. Spaces of Dirichlet series. The Hardy space $\mathcal{H}^2$ of Dirichlet series, which was first systematically studied by H. Hedenmalm, P. Lindqvist, and K. Seip [35], is defined as

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|^2_{\mathcal{H}^2} = \sum_{n\geq 1} |a_n|^2 < \infty \right\}.$$

It is a space of holomorphic function in the half-plane $\mathbb{C}_{\frac{1}{2}}$ and equipped with the ℓ^2-inner product of the coefficients it is a Hilbert space.

$$\langle f, g\rangle_{\mathcal{H}^2} = \sum_{n\geq 1} a_n \overline{b_n}, \tag{2}$$

where $f(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$, $g(s) = \sum\limits_{n\geq 1} \frac{b_n}{n^s}$.

For a point $s_0 \in \mathbb{C}_{\frac{1}{2}}$, the point evaluation linear functional $L_{s_0} : \mathcal{H}^2 \mapsto \mathbb{C}$, $f \mapsto f(s_0)$ is bounded. The reproducing kernels, K_{s_0} of $\mathcal{H}^2$ at points $s_0 \in \mathbb{C}_{\frac{1}{2}}$ are translations of the Riemann zeta function

$$K_{s_0}(s) = \zeta(s + \overline{s_0}) = \sum_{n\geq 1} \frac{1}{n^{s+\overline{s_0}}},$$

$$\langle f(s), \zeta(s + \overline{s_0})\rangle_{\mathcal{H}^2} = f(s_0), \qquad f \in \mathcal{H}^2.$$

The role of the Riemann zeta function as reproducing kernels illustrates its importance in the theory of $\mathcal{H}^2$ and portends connections with the field of analytic number theory.

1.2. Hardy spaces on the infinite polytorus and the Bohr-lift. We will make a short presentation of those topics and we refer the interested reader to [26, 35, 51]. In the one dimensional case the Hardy space H^p, $1 \leq p \leq \infty$ consists of all functions in $L^p(\mathbb{T}, dm)$ with vanishing negative Fourier coefficients. We will have a brief discussion on those space in Section 2.

The infinite polytorus $\mathbb{T}^\infty$ is the countable infinite Cartesian product of copies of the unit circle $\mathbb{T}$,

$$\mathbb{T}^\infty = \{\chi = (\chi_1, \chi_2, \dots) : \chi_j \in \mathbb{T},\, j \geq 1\}.$$

As a compact abelian group it possess a unique Haar measure m_∞, see for example [18, 54]. We can identify the measure m_∞ with the countable infinite product measure $m \times m \times \cdots$, where m is the normalized Lebesgue measure of the unit circle.

The Fourier coefficient of a function $g \in L^1(\mathbb{T}^\infty)$ at a sequence $a = (a_1, a_2, \dots) \in \mathbb{Z}_0^\infty$ is defined as

$$\widehat{g}(a) = \int\limits_{\mathbb{T}^\infty} g(z) z^{-a}\, dm_\infty(z),$$

where $\mathbb{Z}_0^\infty$ is the set of all compactly supported sequences with integer terms and

$$z^a = z_1^{a_1} \cdot z_2^{a_2} \cdot \dots,$$

is the multi-index notation. Similarly, we will denote by $\mathbb{N}_0^\infty$ the set of all compactly supported sequences of non-negative integers.

In a similar manner to the unit circle, the Hardy space $H^p(\mathbb{T}^\infty)$, $1 \leq p \leq \infty$ is defined as the subspace of $L^p(\mathbb{T}^\infty)$, which contains all the functions with vanishing Fourier coefficients at sequences in $\mathbb{Z}_0^\infty \setminus \mathbb{N}_0^\infty$.

We will denote by $\{p_n\}_{n\geq 1}$ the increasing sequence of prime numbers. The fundamental theorem of arithmetics implies that $\mathbb{T}^\infty$ is isomorphic to the group of characters of $((\mathbb{Q})_+, \cdot)$. Given a point $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, the corresponding character $\chi : (\mathbb{Q})_+ \to \mathbb{T}$ is the completely multiplicative function on $\mathbb{N}$ such that $\chi(p_j) = \chi_j$, extended to $(\mathbb{Q})_+$ through the relation $\chi(n^{-1}) = \overline{\chi(n)}$. From now on we identify a point $\chi = (\chi_1, \dots) \in \mathbb{T}^\infty$ with the corresponding character $\chi(n)$.

Suppose $f(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$ and $\chi(n)$ is a character. The vertical limit function f_χ is defined as

$$f_\chi(s) = \sum_{n\geq 1} \frac{a_n}{n^s} \chi(n).$$

Kronecker's theorem [15] justifies the name, since for any $\epsilon > 0$, there exists a sequence of real numbers $\{t_j\}_{j\geq 1}$ such that $f(s + it_j) \to f_\chi(s)$ uniformly on $\mathbb{C}_{\sigma_u(f)+\epsilon}$.

Starting with a Dirichlet polynomial $f(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$ and mapping its prime term to a new variable:

$$p_i^{-s} \mapsto \chi_i, \qquad i \in \mathbb{N},$$

we define the Bohr-lift of f as

$$B(f) := \sum_{n\geq 1} a_n \chi(n). \tag{3}$$

The Bohr-lift is an isometric isomorphism between $\mathcal{H}^p$ [5, 16, 51] and $H^p(\mathbb{T}^\infty)$, $p \in (0, \infty]$. By $\mathcal{H}^\infty$ we denote the space of all bounded Dirichlet series in $\mathbb{C}_0$, equipped with the uniform norm. For $0 < p < \infty$, the Hardy space $\mathcal{H}^p$ of

Dirichlet series is defined as the completion of Dirichlet polynomials under the Besicovitch norm (or quasi-norm if $0<p<1$)

$$\|P\|_{\mathcal{H}^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt \right)^{\frac{1}{p}}.$$

For every $f \in \mathcal{H}^2$ the series $B(f) := \sum_{n\geq 1} a_n \chi(n)$ converges for almost every character $\chi \in \mathbb{T}^\infty$, [36]. Thus, for every $f \in \mathcal{H}^2$ and for almost every $\chi \in \mathbb{T}^\infty$, we have that $\sigma_c(f_\chi) \leq 0$, which is actually a result due to H. Helson [38], see also [5] for an easy proof using the Rademacher–Menchov theorem.

1.3. The ergodic theorem. It is known [51, Section 2.2] that given a sequence $\{a_n\}_{n\geq 1}$ of $\mathbb{Q}$-linear independent real numbers, then the Kronecker flow $\{T_t\}_{t\in\mathbb{R}}$ is ergodic, where

$$T_t(\chi_1, \chi_2, \dots) = (e^{-ita_1}\chi_1, e^{-ita_2}\chi_2, \dots). \tag{4}$$

By Birkhoff–Khinchin ergodic theorem, we obtain the following.

THEOREM 2 ([24, 51]). *If $g \in L^1(\mathbb{T}^\infty)$, then for almost every $\chi_0 \in \mathbb{T}^\infty$,*

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} g\left(T_t\chi_0\right) \, dt = \int_{\mathbb{T}^\infty} g(\chi) \, dm_\infty(\chi). \tag{5}$$

If g is continuous, then (5) holds for every $\chi_0 \in \mathbb{T}^\infty$.

Consequently, for every $f \in \mathcal{H}^p$, $0<p<\infty$ and for almost every character $\chi_0(n)$

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |f_{\chi_0}(it)|^p \, dt = \int_{\mathbb{T}^\infty} |B(f)|^p \, dm_\infty(\chi). \tag{6}$$

2. Composition operators on Hardy spaces of the unit disk

We will make a very short presentation of the topic and we refer the reader to [25, 57]. The classical Hardy spaces H^p, $p>0$ consist of all analytic functions in the unit disk $\mathbb{D}$ with finite p-integral means

$$H^p = \left\{ f \text{ holomorphic in } \mathbb{D} : \|f\|_{H^p}^p := \lim_{r\to 1^-} \int_{o}^{2\pi} \left|f(re^{i\theta})\right|^p \frac{d\theta}{2\pi} < +\infty \right\}.$$

The H^∞ space contains all bounded holomorphic function in the unit disk and it is equipped with the uniform norm.

For $p = 2$, by Parseval's formula we have the following equivalent definition for the H^2 space:

$$H^2 = \left\{ f(z) = \sum_{n\geq 0} a_n z^n : \|f\|_{H^2}^2 := \sum_{n\geq 0} |a_n|^2 < +\infty \right\}.$$

Every analytic self-map of the unit disk, ϕ, induces a bounded composition operator

$$C_\phi : H^p \mapsto H^p, \qquad f \mapsto f \circ \phi \qquad \text{on} \qquad H^p.$$

This result is a consequence of subharmonicity known as the Littlewood subordination principle [46].

J. Shapiro in his seminal paper [56] characterized the compact composition operator C_ϕ in terms of the Nevanlinna counting function

$$N_\phi(z) = \sum_{z_i \in \phi^{-1}(\{z\})} \log \frac{1}{|z_i|}, \qquad z \neq \phi(0).$$

We recall that an operator is called compact if it maps every bounded set to a relatively compact one.

The composition operator C_ϕ is compact on H^p, $p \in (0, \infty)$ if and only if

$$\lim_{|z|\to 1^-} \frac{N_\phi(z)}{\log \frac{1}{|z|}} = 0. \tag{7}$$

As the above characterization demonstrates compactness of the composition operator C_ϕ on H^p is independent of $p > 0$. This is actually a special case of the fact that the Hardy spaces H^p share the same Carleson measures [23], the measure in our case would be the pull-back measure induced by the symbol ϕ, see for example [25]. A positive Borel measure μ in the unit disk is a Carleson measure for H^p if there exists a constant $C > 0$ such that, for all $f \in H^p$,

$$\int_{\mathbb{D}} |f(w)|^p d\mu(w) \leq C \|f\|_{H^p}^p .$$

In Article 3 we will consider Carleson measures on spaces of Dirichlet series.

From the previous discussion if we are interested in compactness of composition operators on Hardy spaces it is sufficient to only consider the case $p = 2$. Two identities of key importance for our point of view that also illustrate why the above characterization of compactness is natural are the Littlewood–Paley and the Stanton's formulae for the norm of a function $f \in H^2$ and its image $C_\phi(f)$, respectively.

$$\|f\|_{H^2}^2 = |f(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z). \tag{8}$$

$$\|C_\phi(f)\|^2_{H^2} = |f\circ\phi(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 N_\phi(z)\,dA(z), \tag{9}$$

where $dA(z) = dx\,dy$, $z = x+iy$, is the area measure.

3. Composition operators on spaces of Dirichlet series

Gordon and Hedenmalm [32] determined the class $\mathfrak{G}$ of symbols which generate bounded composition operators on the Hardy space $\mathcal{H}^2$.

$$C_\psi : \mathcal{H}^2 \mapsto \mathcal{H}^2, \qquad f \mapsto f\circ\psi.$$

The Gordon–Hedenmalm class $\mathfrak{G}$ consists of all functions $\psi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer, called the characteristic of ψ, and φ is a Dirichlet series with the following mapping properties:

(1) If $c_0 = 0$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_{\frac{1}{2}}$.
(2) If $c_0 \geq 1$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_0$ or $\varphi \equiv i\tau$ for some $\tau \in \mathbb{R}$.

We will use the notation $\mathfrak{G}_0$ and $\mathfrak{G}_{\geq 1}$ for the subclasses of symbols that satisfy (1) and (2), respectively.

For $p \geq 1$ [5, 51] the condition $\psi \in \mathfrak{G}$ is necessary for the operator C_ϕ to be bounded on $\mathcal{H}^p$ and for the special cases $\psi \in \mathfrak{G}_{\geq 1}$ or $p = 2n$, $n \in \mathbb{N}$ it is also sufficient. Also, for $1 \leq p < 2$, there exists a Dirichlet series symbol $\varphi \in \mathfrak{G}_0$, such that C_φ is not bounded on $\mathcal{H}^p$. The characterization of bounded composition operators with Dirichlet series symbols on $\mathcal{H}^p$, $p \notin 2\mathbb{N}$ remains an open question. In Article 3 we give geometric conditions that suffice boundedness and compactness of C_φ on $\mathcal{H}^p$.

3.1. Compactness. Compact composition operators induced by Dirichlet series symbols $C_\varphi\colon \mathcal{H}^2 \to \mathcal{H}^2$ were characterized by O. F. Brevig and K-M. Perfekt in [21]. As in the disk case, this characterization involves a counting function.

$$M_\varphi(w) = \lim_{\sigma\to 0^+}\lim_{T\to\infty}\frac{\pi}{T}\sum_{\substack{s\in\varphi^{-1}(\{w\})\\ |\operatorname{Im} s|<T\\ \sigma<\operatorname{Re} s<\infty}} \operatorname{Re} s, \qquad w \in \mathbb{C}_{\frac{1}{2}}\setminus\{\varphi(+\infty)\}.$$

It turns out that C_φ is compact if and only if

$$\lim_{\operatorname{Re} w\to\frac{1}{2}^+} \frac{M_\varphi(w)}{\operatorname{Re} w - \frac{1}{2}} = 0. \tag{10}$$

Suppose $\varphi \in \mathfrak{G}_0$ and f a Dirichlet series in $\mathcal{H}^2$. The analogues of the Littlewood–Paley and Stanton's formulae are respectively:

(11)

$$\|f\|_{\mathcal{H}^2}^2 = |f(+\infty)|^2 + \lim_{\sigma_0\to 0^+}\lim_{T\to\infty}\frac{2}{T}\int\limits_{\sigma_0}^{\infty}\int\limits_{-T}^{T}|f'(\sigma+it)|^2\,\sigma\,dt\,d\sigma, \qquad \sigma_u(f)\leq 0.$$

(12) $$\|C_\varphi(f)\|_{\mathcal{H}^2}^2 = |f(\varphi(+\infty))|^2 + \frac{2}{\pi}\int\limits_{\mathbb{C}_{\frac{1}{2}}}|f'(w)|^2\,M_\varphi(w)\,dA(w),$$

for the proof of those formulas we refer to [21]. By $f(+\infty)$ we denote the first coefficient a_1 of the Dirichlet series $f(s)=\sum\limits_{n\geq 1}\frac{a_n}{n^s}$.

We apply the polarization identity in (12), yielding to

(13) $$\langle C_\varphi(f), C_\varphi(g)\rangle = f(\varphi(+\infty))\overline{g(\varphi(+\infty))} + \frac{2}{\pi}\int\limits_{\mathbb{C}_{\frac{1}{2}}} f'(w)\overline{g'(w)}M_\varphi(w)\,dA(w).$$

The above identity will play a crucial role when we investigate the existence of composition operators in the Schatten classes.

Two key properties of the counting function $M_\varphi(w)$ proved in [21] are the submean value property and a Littlewood type inequality. Those are respectively:

(14) $$M_\varphi(w)\leq\frac{1}{|D(w,r)|}\int\limits_{D(w,r)}M_\varphi(z)\,dA(z),$$

for every disk $D(w,r)\subset\mathbb{C}_{\frac{1}{2}}$ that does not contain $\varphi(+\infty)$, and

(15) $$M_\varphi(w)\leq\log\left|\frac{\varphi(+\infty)+\overline{w}-1}{\varphi(+\infty)-w}\right|, \qquad w\in\mathbb{C}_{\frac{1}{2}}\backslash\{\varphi(+\infty)\}.$$

For $\psi\in\mathfrak{G}_{\geq 1}$ the characterization of compact composition operators remains open. F. Bayart [8] gave the following sufficient condition for the composition operator C_ψ to be compact

(16) $$\lim_{\operatorname{Re} w\to 0^+}\frac{\mathcal{N}_\psi(w)}{\operatorname{Re} w}=0,$$

where the Nevanlinna-type counting function $\mathcal{N}_\psi$ is defined as

$$\mathcal{N}_\psi(w)=\sum_{\substack{s\in\psi^{-1}(\{w\})\\ \operatorname{Re} s>0}}\operatorname{Re} s.$$

For $\psi(s)=c_0 s+\varphi(s)\in\mathfrak{G}$, we set

$$\psi_\chi(s)=c_0 s+\varphi_\chi(s),$$

then for every $\chi \in \mathbb{T}^\infty$ we have that

$$(C_\psi(f))_\chi = f_{\chi^{c_0}} \circ \psi_\chi. \tag{17}$$

The symbol ψ has boundary values $\psi_\chi(it) = \lim_{\sigma\to 0^+} \psi_\chi(\sigma + it)$ for almost every $\chi \in \mathbb{T}^\infty$ and for almost every $t \in \mathbb{R}$, see [20].

The Littlewood–Paley type formula [6] that is convenient to use when we work on composition operators with symbols $\psi \in \mathfrak{G}_{\geq 1}$ is

$$\|f\|^2_{\mathcal{H}^2} = |f(+\infty)|^2 + \frac{2}{T} \int_{\mathbb{T}^\infty} \int_0^\infty \int_{-T}^{T} \left|f'_\chi(\sigma + it)\right|^2 \sigma \, dt \, d\sigma \, dm_\infty(\chi), \tag{18}$$

where $f \in \mathcal{H}^2$ and $T > 0$.

Suppose $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$, by a non-injective change of variables we get the following Stanton's type formula:

$$\|C_\psi(f)\|^2 = |f(+\infty)|^2 + \frac{2}{\pi} \int_{\mathbb{C}_0} \int_{\mathbb{T}^\infty} \left|f'_{\chi^{c_0}}(w)\right|^2 \mathcal{N}_{\psi_\chi}(w, T) \, dm_\infty(\chi) \, dA(w), \tag{19}$$

where the (restricted) mean counting function $\mathcal{N}_{\psi_\chi}(w, T)$ is defined as

$$\mathcal{N}_{\psi_\chi}(w, T) = \frac{\pi}{T} \sum_{\substack{s \in \psi_\chi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \operatorname{Re} s > 0}} \operatorname{Re} s.$$

In Article 2 having as a starting point the above expressions for the norms in $\mathcal{H}^2$, we give a necessary condition for composition operators C_ψ, $\psi \in \mathfrak{G}_{\geq 1}$ to be compact on $\mathcal{H}^2$ answering in this way a question posed by F. Bayart in [8].

3.2. Weighted Hilbert spaces of Dirichlet series. For $a \leq 1$ we define the weighted Hilbert space $\mathcal{D}_a$ of Dirichlet series as

$$\mathcal{D}_a = \left\{ f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} : \|f\|_a^2 = |a_1|^2 + \sum_{n \geq 2} |a_n|^2 \log(n)^a < \infty \right\}.$$

The condition that $\psi \in \mathfrak{G}$ is necessary for a composition operator $C_\psi : \mathcal{D}_a \to \mathcal{D}_a$ to be bounded. In the Bergman case $a < 0$, this is also known to be sufficient [3, 4].

Working in a similar manner with the Hardy case $\mathcal{H}^2$ applying Carlson's theorem [35, Lemma 3.2] one deduces the following Littlewood–Paley type formula:

$$\|f\|_a^2 = |f(+\infty)|^2 + \frac{2^{1-a}}{\Gamma(2-a)} \lim_{\sigma_0 \to 0^+} \lim_{T \to \infty} \frac{1}{T} \int_{\sigma_0}^\infty \int_{-T}^{T} \left|f'(\sigma + it)\right|^2 \sigma^{1-a} \, dt \, d\sigma, \tag{20}$$

where $f \in \mathcal{D}_a$ and $\sigma_u(f) \leq 0$.

The space $\mathcal{D}_a$ is analogous to the weighted Hilbert space D_α, consisting of those holomorphic functions g on the unit disk such that

$$\|g\|_{D_\alpha}^2 = |g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2(1-|z|^2)^\alpha \, dA(z) < \infty. \tag{21}$$

By the results of [41, 50, 56], a holomorphic self-map of the unit disk $\phi\colon \mathbb{D} \to \mathbb{D}$ induces a compact composition operator on D_α, $\alpha > 0$, if and only if

$$\lim_{|z|\to 1^-} \frac{N_{\phi,\alpha}(z)}{(1-|z|^2)^\alpha} = 0, \tag{22}$$

where for $\alpha = 1$, $N_{\phi,1}$ is the classical Nevanlinna counting function N_ϕ and for $\alpha \neq 1$, $N_{\phi,\alpha}$ is the generalized Nevanlinna counting function

$$N_{\phi,\alpha}(z) = \sum_{z_i \in \phi^{-1}(\{z\})} (1-|z_i|^2)^\alpha.$$

For $\varphi \in \mathfrak{G}_0$, making a non-injective change of variables in (20) yields that

$$\|C_\varphi(f)\|_a^2 = |f(\varphi(+\infty))|^2 + \frac{2^{1-a}}{\pi\Gamma(2-a)} \lim_{\sigma\to 0^+} \lim_{T\to\infty} \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 M_{\varphi,1-a}(w,\sigma,T) \, dA(w),$$

where

$$M_{\varphi,a}(w,\sigma,T) = \frac{\pi}{T} \sum_{\substack{s\in\varphi^{-1}(\{w\}) \\ |\operatorname{Im} s|<T \\ \sigma<\operatorname{Re} s<\infty}} (\operatorname{Re} s)^a, \qquad w \neq \varphi(+\infty).$$

We introduce the weighted mean counting functions

$$M_{\varphi,a}(w,\sigma) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{s\in\varphi^{-1}(\{w\}) \\ |\operatorname{Im} s|<T \\ \sigma<\operatorname{Re} s<\infty}} (\operatorname{Re} s)^a, \qquad w \neq \varphi(+\infty),$$

and

$$M_{\varphi,a}(w) = \lim_{\sigma\to 0^+} M_{\varphi,a}(w,\sigma),$$

if these limits exist.

In Article 1 we work on composition operators on such weighted Hilbert spaces of Dirichlet series. We prove the existence of the above weighted mean counting functions and the associated Stanton's formulae, we give a characterization of compact composition operators C_φ, $\varphi \in \mathfrak{G}_0$ on Bergman spaces of Dirichlet series in terms of those counting functions, and we investigate when analogous conditions are sufficient or necessary.

3.3. Schatten classes. A compact operator T acting on a separable Hilbert space H can be written as

$$T(x) = \sum_{n\geq 1} s_n \langle x, e_n \rangle h_n, \qquad x \in H, \tag{23}$$

where $\{s_n\}_{n\geq 1}$ is the sequence of singular values and $\{e_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ are orthonormal sequences. In case T is self-adjoint, then $e_n = \pm h_n$ for all $n \geq 1$.

For $p > 0$ the S_p Schatten class of compact operators T on H is defined as

$$S_p = S_p(H) = \left\{ T \text{ compact on } H : \|T\|_{S_p}^p := \sum_{n\geq 1} s_n^p < \infty \right\}.$$

Equivalently (see [40]), for $p \geq 1$, a bounded linear operator T belongs to S_p if and only if there exists a positive constant C such that

$$\sum_{n\geq 1} |\langle Te_n, e_n \rangle|^p \leq C,$$

for every orthonormal basis (e_n). Furthermore, if T is self-adjoint, then

$$\|T\|_{S_p}^p = \sup \sum_{n\geq 1} |\langle Te_n, e_n \rangle|^p,$$

where the supremum is being taken over all orthonormal bases of H.

A non-trivial result is that S_p is a Banach spaces for $p \geq 1$, [42, Chapter 1.]. For a positive operator T on H we define the power T^p, $p > 0$, as

$$T^p(x) = \sum_{n\geq 1} s_n^p \langle x, e_n \rangle e_n, \qquad x \in H.$$

We observe that $T \in S_p$, if and only if $T^p \in S_1$. If T is not assumed to be positive, we can still use that $T \in S_p$, if and only if $|T|^p = (T^*T)^{p/2} \in S_1$ if and only if $T^*T \in S_{p/2}$.

D. H. Luecking and K. Zhu [47] proved that a composition operator C_ϕ on the Hardy space $H^2(\mathbb{D})$ belongs to the Schatten class S_p, $p > 0$ if and only if

$$\int_{\mathbb{D}} \frac{(N_\phi(z))^{\frac{p}{2}}}{(1 - |z|^2)^{\frac{p}{2}+2}} \, dA(z) < +\infty. \tag{24}$$

In Article 3 we study when the analogous condition is necessary and sufficient for a composition operator on Hilbert spaces of Dirichlet series to exists in the Schatten classes.

4. Conformally invariant quantities in the complex plane

At first sight, geometric function theory and potential theory may look irrelevant to the theory of Dirichlet series. Although, conformal invariant quantities are crucial to the development of our results.

4.1. Hyperbolic metric. Let us recall the classical Schwarz–Pick for the unit disk: Every holomorphic self-map of the unit disk $\phi : \mathbb{D} \to \mathbb{D}$ satisfies the following contractive inequality

$$\frac{|\phi'(z)|}{1-|\phi(z)|^2} \leq \frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{25}$$

Equality holds in (25) for a single point $z_0 \in \mathbb{D}$, and consequently for every point in the disk, if and only if ϕ is a holomorphic automorphism of the unit disk.

Definition. The hyperbolic metric and distance in the unit disk are defined respectively as

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2},$$

$$d_{\mathbb{D}}(z,w) = \inf_{\gamma} \int_{\gamma} \lambda_{\mathbb{D}}(\zeta)\,|d\zeta|,$$

where the infimum is taken over all piecewise smooth curves γ in $\mathbb{D}$ that join z and w.

The Schwarz–Pick lemma has the following reformulation in terms of the quantities defined above:

Theorem 3. *Let ϕ be a holomorphic self-map of the unit disk. Then it is a contraction of the hyperbolic distance*

$$\lambda_{\mathbb{D}}(\phi(z))|\phi'(z)| \leq \lambda_{\mathbb{D}}(z), \tag{26}$$

and

$$d_{\mathbb{D}}(\phi(z),\phi(w))) \leq d_{\mathbb{D}}(z,w), \tag{27}$$

where z, $w \in \mathbb{D}$. If equality holds in (26) for one point, or in (27) for a pair of distinct points, then ϕ is a holomorphic automorphism of the unit disk, and thus an isometry.

Using the conformal invariance of the hyperbolic distance, one can prove that

$$d_{\mathbb{D}}(z,w) = \log \frac{1+\left|\frac{w-z}{1-\overline{w}z}\right|}{1-\left|\frac{w-z}{1-\overline{w}z}\right|} = 2\operatorname{arctanh}\left|\frac{w-z}{1-\overline{w}z}\right|.$$

Via the Riemann mapping theorem we can define hyperbolic metrics and distances to any simply connected proper subdomain Ω of the complex plane.

Actually the domains where we can define such metrics are called hyperbolic, which are domains that omit two distinct points in the complex plane, see for example [13]. Let f be a Riemann map from Ω onto the unit disk. Then

$$\lambda_\Omega(z) = \lambda_{\mathbb{D}}(f(z))|f'(z)|,$$

$$d_\Omega(z,w) = d_{\mathbb{D}}(f(z), f(w)) = \inf_\gamma \int_\gamma \lambda_\Omega(\zeta)\,|d\zeta|,$$

where the infimum is taken over all piecewise smooth curves γ in Ω that join z and w. The quantities λ_Ω and d_Ω are independent of the choice of the Riemann map. In the case of the right-half plane, considering the Riemann map $f(z) = \frac{z-w}{z+\overline{w}}$ we obtain that

$$\lambda_{\mathbb{C}_0}(z) = \frac{1}{\operatorname{Re} z},$$

and

$$d_{\mathbb{C}_0}(z,w) = \log \frac{1 + \left|\frac{z-w}{z+\overline{w}}\right|}{1 - \left|\frac{z-w}{z+\overline{w}}\right|} = \log \frac{(|z+\overline{w}| + |z-w|)^2}{4 \operatorname{Re} z \operatorname{Re} w},$$

where z, $w \in \mathbb{C}_0$.

The following Schwarz–Pick lemma for simply connected domains is a direct consequence of the definition and the classical Schwarz–Pick lemma, see [13].

THEOREM 4. *Suppose that Ω_1 and Ω_2 are simply connected proper subdomains of the complex plane and that $f : \Omega_1 \to \Omega_2$ is a holomorphic function. Then, for every z, $w \in \Omega_1$,*

$$\lambda_{\Omega_2}(f(z))|f'(z)| \leq \lambda_{\Omega_1}(z), \tag{28}$$

and

$$d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z,w). \tag{29}$$

Furthermore, equality holds in (28) for one point, or in (29) for a pair of distinct points, if and only if f is a biconformal map from Ω_1 onto Ω_2.

In Article 1 we apply the results of this subsection proving a Schwarz–type lemma for Dirichlet series. Our methodology leads to a new proof of the characterization of bounded composition operators on Bergman spaces of Dirichlet series. Furthermore, to prove an optimal extension of Bohr's theorem in Article 5 we used a Schwarz–type lemma for hyperbolic domains. This lemma is the so called Schottsky's theorem, see [1]:

THEOREM 5. *Let $D(c,r)$ be the open disc with center c and radius $r>0$. If f is analytic and different from 0 and 1 in $D(c,r)$, then*

$$|f(s)| \leq \exp\left(\frac{r+|s-c|}{r-|s-c|}\left(7+\max(0,\log|f(c)|)\right)\right), \tag{30}$$

for all $s \in D(c,r)$.

4.2. Green's function. A Green's function [53] for a domain $\Omega \subset \mathbb{C}$ is a function $g_\Omega : \Omega \times \Omega \to (-\infty, +\infty]$ such that, for all $w \in \Omega$, $g(\cdot, w)$ is harmonic in $\Omega \backslash \{w\}$, $g_\Omega(z,w) \to 0$ n.e as $z \to \partial\Omega$ and $g_\Omega(\cdot, w) + \log|\cdot - w|$ is harmonic in a neighborhood of w. If a domain admits a Green's function then it is necessarily unique. For instance, the Green's function on the disk $g_\mathbb{D} : \mathbb{D} \times \mathbb{D} \mapsto (0, +\infty]$ has the form

$$g_\mathbb{D}(z,w) = \log\left|\frac{1-z\overline{w}}{z-w}\right|.$$

By conformal invariance we can easily define Green's function on every simply connected subdomain of the complex plane, for example

$$g_{\mathbb{C}_0}(z,w) = \log\left|\frac{z+\overline{w}}{z-w}\right|, \qquad z,\, w \in \mathbb{C}_0.$$

The class of domains D possessing a Green's function g_D is much larger than the simply connected domains, see [53, Chapter 4]. Lindelöf's principle for Green's function (see for instance [14]) states that if f is a holomorphic function mapping D_1 to D_2, where both of those domains possess Green's function, then for $z_0 \in D_1$ and $w \in D_2 \setminus \{f(z_0)\}$

$$\sum_{z \in f^{-1}(\{w\})} g_{D_1}(z, z_0) \leq g_{D_2}(w, f(z_0)). \tag{31}$$

In Article 3 we apply Lindelöf's principle for Green's function to establish sufficient conditions for composition operators on $\mathcal{H}^p$ spaces to be bounded, compact or exist in the S_p classes in terms of the geometric properties of their symbols.

5. Beurling primes

When we defined the correspondence (Bohr-lift) between Dirichlet series in the Hardy spaces and functions in the Hardy spaces of the infinite polytorus we only used the fact that monomials of the form p_n^{-s} behave in some sense like independent variables z_n, $n \in \mathbb{N}$. Can we work with similar sequences of "primes", which have the same behaviour but are in some sense more easy to control?

Let us start doing that considering an increasing sequence $q = \{q_n\}_{n\geq 1}$ of real numbers and imitating the construction of the natural numbers through the fundamental theorem of arithmetic.

We want the sequence q to behave like the increasing sequence of primes, thus we assume that $\{\log q_n\}_{n\geq 1}$ is linearly independent over $\mathbb{Q}$ and that

$$1 < q_1 < q_2 < \cdots < q_n \to \infty.$$

We will denote by $\mathbb{N}_q = \{\nu_n\}_{n\geq 1}$ the set of numbers that can be written as finite products with factors from q, ordered in an increasing manner.

$$\nu_n = q^a := q_1^{a_1} \cdot q_2^{a_2} \cdot \ldots .$$

This representation of the number ν_n is unique since we assumed $\mathbb{Q}$-linear independence of the sequence $\{\log q_n\}_{n\geq 1}$.

The numbers q_n are called Beurling primes, and the numbers ν_n are Beurling integers. This sequence of Beurling integers corresponds to a class of generalized Dirichlet series of the form

$$f(s) = \sum_{n\geq 1} a_n \nu_n^{-s}.$$

This is actually a special case of generalized Dirichlet series

$$f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s},$$

where $\lambda = \{\lambda_n\}_{n\geq 1}$ is an increasing and unbounded sequence of positive numbers called the frequencies of the Dirichlet series. These series have similar convergent properties as in the classical case $\lambda_n = \log n$ and the abscissas σ_c, σ_u, and σ_a of pointwise, uniform, and absolute convergence are defined in the same way, see for example [34].

Our first motivation is to consider Hardy spaces of generalized Dirichlet series, which are at least in a functional analytic point of view similar to the $\mathcal{H}^p$ spaces, meaning isometric isomorphic. In this way when we work on problems that do not depend on the sequence of primes we will have the opportunity to transfer our notions to a space where we may have better control, for example of the reproducing kernels.

5.1. Hardy spaces of generalized Dirichlet series and Helson's conjecture. Henry Helson [37] was the first who worked on spaces of generalized Dirichlet series. For a Beurling system $q = \{q_n\}_{n\geq 1}$ and $\mathbb{N}_q = \{\nu_n\}_{n\geq 1}$, the corresponding Hardy space $\mathcal{H}_q^2$ of generalized Dirichlet series is defined as

$$\mathcal{H}_q^2 = \left\{ f(s) = \sum_{n\geq 1} a_n \nu_n^{-s} \; : \; \|f\|_{\mathcal{H}_q^2}^2 = \sum_{n\geq 1} |a_n|^2 < \infty \right\}.$$

For the other values of $p \in [1, \infty)$ as in the classical case, we define $\mathcal{H}_q^p$ as the completion of Dirichlet polynomials under the Besicovitch norm

$$\|P\|_{\mathcal{H}_q^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt \right)^{\frac{1}{p}}.$$

For $p = \infty$, the space $\mathcal{H}_q^\infty$ will consists of bounded and convergent Dirichlet series $\sum_{n\geq 1} a_n \nu_n^{-s}$ in the right half-plane $\mathbb{C}_0$ equipped with the uniform norm.

As we have already mentioned, a result of key importance to the theory of Hardy spaces of Dirichlet series is the correspondence, the Bohr lift, between $\mathcal{H}_q^p$ and the Hardy space $H^p(\mathbb{T}^\infty)$ of the infinite polytorus. For $p \in (0, \infty)$, this holds also for the spaces $\mathcal{H}_q^p$, [**27, 38**]. The situation when $p = \infty$ is not the same [**55**]. In order to have that the Bohr-lift is an isometric isomorphism between $H^\infty(\mathbb{T}^\infty)$ and $\mathcal{H}_q^\infty$ we will require for the sequence of Beurling natural numbers to satisfied the following condition:

There exist positive constants c_1 and c_2 such that

$$\nu_{n+1} - \nu_n \geq c_1 \nu_{n+1}^{-c_2}. \tag{32}$$

This condition is known as Bohr's condition. Bohr was interested in frequencies $\{\lambda_n\}_{n\geq 1}$ for which the analogue of the Theorem 1 hold's. Landau in [**45**] and recently Bayart in [**7**] gave weaker conditions on the sequence of frequencies that suffice the validity of Bohr's theorem for generalized Dirichlet series of the form $\sum_{n\geq 1} a_n e^{-\lambda_n s}$.

As in the classical case, if $f \in \mathcal{H}_q^2$, then $f_\chi(s)$ converges in $\mathbb{C}_0$ for almost every $\chi \in \mathbb{T}^\infty$. This follows from the Carleson theorem for the infinite polytorus [**36**]. Helson in [**38**] proved the same result in a more general setting with the extra assumption that the frequencies satisfy Bohr's condition. In the same article Helson posed a conjecture, which was a way (if correct) to disprove Riemann's hypothesis. In order to state the conjecture we first need to recall what an outer function is.

Definition. Let $\mathbb{N}_q$ be a Beurling system as above. A function $f \in \mathcal{H}_q^2$ is said to be outer (or cyclic) if $\left\{ fg : g \in \mathcal{H}_q^\infty \right\}$ is dense in $\mathcal{H}_q^2$.

Note that an outer function $f \in H^2$ has no zeros in the unit disk, that is a consequence of the fact that reproducing kernels in the classical Hardy spaces are well defined in the whole disk. That is exactly what the conjecture states for our setting:

Conjecture. *If $\mathbb{N}_q$ is a Beurling system that satisfies Bohr's condition and f is outer in $\mathcal{H}_q^2$, then f_χ never has any zeros in its half-plane of convergence.*

In the classical setting where q is the ordinary sequence of primes, under the Riemann hypothesis Helson's conjecture fails. Let $f = 1/\zeta(s + 1/2 + \varepsilon)$, under our assumption the function f is a convergent Dirichlet series in $\mathbb{C}_0$, see for example [59]. One can also prove that f, f^2, $1/f$, $1/f^2 \in \mathcal{H}^2$. Therefore, there are polynomials p_n which converge to $1/f$ in $\mathcal{H}^4$, so that

$$\|1 - p_n f\|_{\mathcal{H}^2} \leq \|f\|_{\mathcal{H}^4} \|1/f - p_n\|_{\mathcal{H}^4} \to 0, \qquad n \to \infty.$$

Thus $1/\zeta(s + 1/2 + \varepsilon)$ is outer. On the other hand, it has a zero at $s = 1/2 - \varepsilon$.

In Article 4 we disprove Helson's conjecture finding a Beurling system which satisfies the analogue of Riemann hypothesis and Bohr's condition. Furthermore, Beurling primes were also used in Article 2 to extend our notions to a larger space of Dirichlet series and establish a necessary condition for composition operators of the form C_ψ, $\psi \in \mathfrak{G}_{\geq 1}$ to be compact on $\mathcal{H}^2$.

For an overview of the theory of Beurling primes from a number theoretical point of view we refer to [28, 39].

6. Universality

Universality refers to the phenomenon where an object, via a countable process, yields approximations to all members of some collection of interest. In our case this object will be a Dirichlet series, the collection will be a set of holomorphic functions and the approximation will happen through the vertical translations of this Dirichlet series.

Let Ω be a vertical strip of the form $\{\sigma_0 < \operatorname{Re} s < \sigma_1\}$ and f be a Dirichlet series with an analytic continuation in Ω. The Dirichlet series f is called universal in Ω if for every compact set $K \subset \Omega$ with connected complement, for every non-vanishing holomorphic function g, which is continuous in K and holomorphic in the interior of K and for every $\varepsilon > 0$, the set

$$D = \{t \geq 0 : \|f(s + it) - g(s)\|_K \leq \varepsilon,\, s \in K\}$$

has positive lower density $\underline{dens}(D) := \liminf_{T\to\infty} \frac{1}{T} \int_0^T 1_D(t)\, dt$, where $\|\cdot\|_K$ denotes the supremum norm on K. Furthermore, we say that f is strongly universal if the restriction that g is non-vanishing can be eased.

Essentially, f is strongly universal or universal, if the vertical translations $f(\cdot + it)$ can approximate every or every non-vanishing holomorphic function in K, respectively. One of the most important results of this field, a key stone for all the further developments, is the universality of the Riemann zeta function due to Voronin [2, 60].

Theorem 6. *The Riemann zeta function ζ is universal in the critical strip $\{\frac{1}{2} < \operatorname{Re} s < 1\}$.*

The theory of universal Dirichlet series is connected to the properties of the series in the domains where it is difficult to investigate their behaviour. An example that illustrates the importance of universality is the following reformulation of the Riemann hypothesis due to B. Bagchi [2]:

THEOREM 7. *The Riemann hypothesis holds if and only if for every compact subset K of the critical strip with connected complement and every $\varepsilon > 0$, there exists a subset D of the positive real numbers, such that:*

$$\sup_K |\zeta(s+it) - \zeta(s)| \leq \varepsilon, \qquad t \in D, \qquad \underline{dens}(D) > 0. \tag{33}$$

After the seminal work of Voronin, universality attracted the attention of a lot of mathematicians. We refer the interested reader to the survey article of K. Matsumoto [48] for further developments on the subject.

In Article 6 we set out to find examples of convergent Dirichlet series that are (strongly) universal and to establish criteria for generalized Dirichlet series $\sum\limits_{n\geq 1} a_n e^{-\lambda_n s}$ to be (strongly) universal. It is worth mentioning that at least in our knowledge there are not many examples of universal objects that we can actually "write down" (without an analytic continuation). The same phenomenon also happens in other fields related to universality. For instance, it is still an open question to find an example of a universal Taylor series, [33, 49].

One of the starting points of our work was the following question posed by F. Bayart in [9]:

QUESTION. Is the alternating prime zeta function $\sum\limits_{n\geq 1} (-1)^n p_n^{-s}$ strongly universal in $\{\frac{1}{2} < \mathrm{Re} < 1\}$?

Now, we will make a short sketch on how one can study generalized Dirichlet series $f(s) = \sum\limits_{n\geq 1} a_n e^{-\lambda_n s}$ in terms of their universal properties. Normally, when we try to prove that such a series is universal we split the proof into two parts. The first part has to do with the density of the subspace generated by the terms $a_n e^{-\lambda_n s}$ of the Dirichlet series in a certain Hilbert space of holomorphic functions. We start mentioning the following Hilbertian density criterion, [2, 12, 51]:

THEOREM 8. *Let $\{x_n\}_{n\geq 1}$ be a sequence of vectors of a complex Hilbert space H such that:*

(i) For every $x^* \in H \setminus \{0\}$, we have

$$\sum_{n\geq 1} |\langle x^*, x_n\rangle| = \infty.$$

(ii)

$$\sum_{n\geq 1} \|x_n\|^2 < \infty.$$

Then, for any positive integer N_0*, the set* $\left\{ \sum_{n=N_0}^{N} z_n x_n : |z_n| = 1, N \in \mathbb{N} \right\}$ *is dense in* H*.*

Definition. We say that a Dirichlet series $f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ belongs to $\mathcal{D}_{\text{dens}}$ if for all $\alpha, \beta > 0$, there exist $C > 0$ and $x_0 \geq 1$ such that, for all $x \geq x_0$,

$$\sum_{\lambda_n \in \left[x, x+\frac{\alpha}{x^2}\right]} |a_n| \geq C e^{(\sigma_a(f)-\beta)x}.$$

Under some mild conditions, the existence of $f \in \mathcal{D}_{\text{dens}}$ implies (i) of Theorem 8, [9]. The Hilbert space in our case will be a Bergman space $A^2(U)$, where U is a smooth Jordan subdomain of Ω.

$$A^2(U) = \left\{ f \ \text{ holomorphic in } \ U : \|f\|^2_{A^2(U)} := \int_U |f(z)|^2 \, dx dy < \infty \right\}.$$

The second part of the proof is related to how well the partial sums of the series $f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ approximate the function itself.

Definition. We say that a Dirichlet series $f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ belongs to $\mathcal{D}_{\text{w.a.}}(\sigma_0)$, where $\sigma_0 \geq \sigma_2(f)$, if it satisfies the following conditions:

(i) It has an analytic extension to $\{\operatorname{Re} s > \sigma_0, \operatorname{Im} s > 0\} \cup \mathbb{C}_{\sigma_c(D)}$.

(ii) It is of finite order, that is: For all $\sigma_1 > \sigma_0$, there exist t_0, $B > 0$ such that, for all $s = \sigma + it$ with $\sigma \geq \sigma_1$ and $t \geq t_0$, $|f(\sigma + it)| \leq t^B$.

(iii) For all $\sigma_2 > \sigma_1 > \sigma_0$,

$$\sup_{\sigma \in [\sigma_1, \sigma_2]} \sup_{T>0} \frac{1}{T} \int_1^T |f(\sigma + it)|^2 dt < +\infty.$$

(iv) The sequence (λ_n) is $\mathbb{Q}$-linearly independent.

For a probabilistic point of view of the value distribution for members of the class $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ we refer to [29, 30].

It is proven in [9] that membership in those two classes of Dirichlet series suffices strong universality.

Theorem 9. *Suppose that the Dirichlet series* $f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ *exists in* $\mathcal{D}_{\text{w.a.}}(\sigma_0) \cap \mathcal{D}_{\text{dens}}$*. Then it is strongly universal in the strip* $\{\sigma_0 < \operatorname{Re} s < \sigma_a(f)\}$*.*

The main difficulty that we face trying to answer problems like Question 6 has to do with the order and the square moments of the corresponding Dirichlet series, see (ii) and (iii) of Definition 6.

Classical and also recent results indicate that such problems, related to the order of Dirichlet series, require techniques from the intersection of harmonic analysis and analytic number theory, see for example how the method of non-stationary phase or decoupling can be used to estimate exponential sums, [**12**, **17**]. For series of the form $f(s) = \sum_{n\geq 1} a(n)e^{-\lambda(n)s}$ where the frequencies $\lambda(n)$ and the terms $a(n)$ are not regular enough, meaning non-smooth and not of bounded variation respectively, those methods are not applicable. Although, we expect a proper decay at least for the alternating prime zeta function, see Article 6.

In Article 6 we study in addition to the alternating prime zeta function, generalized Dirichlet series of the form $f(s) = \sum_{n\geq 1} Q(n)(\log(n))^{\kappa}[P(n)]^{-s}$, where P and Q are polynomials. We provide a sufficient condition for such a Dirichlet series f to be strongly universal. Our result can be applied for example to give a new proof of the fact that the Hurwitz zeta functions $\sum_{n\geq 1}(n+\alpha)^{-s}$, where α is transcendental, is strongly universal in $\{\frac{1}{2} < \operatorname{Re} s < 1\}$, see also [**31**, **48**].

7. Overview of the thesis

Article 1 [**44**]**.** We study composition operators with Dirichlet series symbols on weighted Hilbert spaces of Dirichlet series $\mathcal{D}_a$. For this purpose we demonstrate the existence of the associated weighted mean counting functions $M_{\varphi,a}$, and provide a corresponding change of variables formula for the composition operator. This leads to natural necessary conditions for the boundedness and compactness. For Bergman-type spaces, we are able to show that the compactness condition is also sufficient, by employing a Schwarz-type lemma for Dirichlet series. Combining this Schwarz-type lemma and the techniques of [**32**] we give a new proof of the characterization of bounded composition operators induced by symbols with characteristic zero on $\mathcal{D}_a$, $a \leq 0$.

Article 2 [**43**]**.** O. F. Brevig and K–M. Perfekt [**21**] characterized compact composition operators on $\mathcal{H}^2$ with symbols in $\mathfrak{G}_0$. For symbols $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$, F. Bayart [**8**] proved that condition (16) suffices compactness of the composition operator C_ψ on $\mathcal{H}^2$. Our main result is that the following $L^1(\mathbb{T}^\infty)$ analogue of condition (16) is necessary for such composition operators to be compact on $\mathcal{H}^2$:

$$\lim_{\operatorname{Re} w\to 0} \int_{\mathbb{T}^\infty} \frac{\mathcal{N}_{\psi_\chi}(w)}{\operatorname{Re} w}\, dm_\infty(\chi) = 0, \tag{34}$$

recall that the Nevanlinna-type counting function $\mathcal{N}_\psi$ is defined as

$$\mathcal{N}_\psi(w) = \sum_{\substack{s \in \psi^{-1}(\{w\}) \\ \operatorname{Re} s > 0}} \operatorname{Re} s.$$

To prove this result we extend our notions to a Hardy space $\mathcal{H}^2_\Lambda$ of generalized Dirichlet series, induced in a natural way by a sequence of Beurling primes.

Article 3 [**11**]**.** We give necessary and sufficient conditions, in the same spirit with the Schatten class conditions for the disk setting (24), for a composition operator with Dirichlet series symbol to belong to the Schatten classes S_p of the Hardy space $\mathcal{H}^2$ of Dirichlet series. For $p \geq 2$, these conditions lead to a characterization for the subclass of symbols with bounded imaginary parts. Finally, we establish a comparison-type principle for composition operators exploiting the connections between the mean counting function M_φ and the counting function appearing in Lindelöf's principle (31). Applying our techniques in conjunction with classical geometric function theory methods, we prove the analogue of the polygonal compactness theorem for $\mathcal{H}^2$ and we give examples of bounded composition operators with Dirichlet series symbols on $\mathcal{H}^p$, $p > 0$.

Article 4 [**22**]**.** Given a sequence of frequencies $\{\lambda_n\}_{n\geq 1}$, a corresponding generalized Dirichlet series is of the form $f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$. We are interested in multiplicatively generated systems, where each number e^{λ_n} arises as a finite product of some given numbers $\{q_n\}_{n\geq 1}$, $1 < q_n \to \infty$, referred to as Beurling primes. In the classical case, where $\lambda_n = \log n$, Bohr's theorem holds. We prove, under very mild conditions, that given a sequence of Beurling primes, a small perturbation yields another sequence of primes such that the corresponding Beurling integers satisfy Bohr's condition, and therefore the theorem. Applying our technique in conjunction with a probabilistic method, we find a system of Beurling primes for which both Bohr's theorem and the Riemann hypothesis are valid. This provides a counterexample to a conjecture of H. Helson.

Article 5 [**19**]**.** The following extension of Bohr's theorem is established: If a somewhere convergent Dirichlet series f has an analytic continuation to the half-plane $\mathbb{C}_\theta = \{s = \sigma + it : \sigma > \theta\}$ that maps $\mathbb{C}_\theta$ to $\mathbb{C} \setminus \{\alpha, \beta\}$ for complex numbers $\alpha \neq \beta$, then f converges uniformly in $\mathbb{C}_{\theta+\varepsilon}$ for any $\varepsilon > 0$. The extension is optimal in the sense that the assertion no longer holds should $\mathbb{C} \setminus \{\alpha, \beta\}$ be replaced with $\mathbb{C} \setminus \{\alpha\}$.

Article 6. [10] We establish sufficient conditions for a Dirichlet series induced by general frequencies to be universal with respect to vertical translations. Applying our methodology we give examples of universal Dirichlet series such as the alternating prime zeta function $\sum_{n\geq 1}(-1)^n p_n^{-s}$, partially answering in this way Question 6.

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Part 1

Composition operators

Article 1: Composition operators on weighted Hilbert spaces of Dirichlet series

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COMPOSITION OPERATORS ON WEIGHTED HILBERT SPACES OF DIRICHLET SERIES

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Abstract. We study composition operators of characteristic zero on weighted Hilbert spaces of Dirichlet series. For this purpose we demonstrate the existence of weighted mean counting functions associated with the Dirichlet series symbol, and provide a corresponding change of variables formula for the composition operator. This leads to natural necessary conditions for the boundedness and compactness. For Bergman-type spaces, we are able to show that the compactness condition is also sufficient, by employing a Schwarz-type lemma for Dirichlet series.

1. Introduction

For $a \leq 1$ we define the weighted Hilbert space $\mathcal{D}_a$ of Dirichlet series as

$$\mathcal{D}_a = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|_a^2 = |a_1|^2 + \sum_{n\geq 2} |a_n|^2 \log(n)^a < \infty \right\}.$$

The space $\mathcal{D}_0$ coincides with the Hardy space $\mathcal{H}^2$ of Dirichlet series with square summable coefficients, which was systematically studied in an influential article of Hedenmalm, Lindqvist, and Seip [13]. For $a < 0$ we refer to $\mathcal{D}_a$ as a Bergman space and for $a > 0$ as a Dirichlet space, see [18].

By the Cauchy–Schwarz inequality, $\mathcal{D}_a$ is a space of analytic functions in the half-plane $\mathbb{C}_{\frac{1}{2}}$, where $\mathbb{C}_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}$. Therefore, if $\psi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ is an analytic function, the composition operator $C_\psi(f) = f \circ \psi$ defines an analytic function in $\mathbb{C}_{\frac{1}{2}}$ for every $f \in \mathcal{D}_a$. Gordon and Hedenmalm [12] determined the class $\mathfrak{G}$ of symbols which generate bounded composition operators on the Hardy space $\mathcal{H}^2$. The Gordon–Hedenmalm class $\mathfrak{G}$ consists of all functions $\psi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer, called the characteristic of ψ, and φ is a Dirichlet series such that:

(i) If $c_0 = 0$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_{\frac{1}{2}}$.
(ii) If $c_0 \geq 1$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_0$ or $\varphi \equiv i\tau$ for some $\tau \in \mathbb{R}$.

We will use the notation $\mathfrak{G}_0$ and $\mathfrak{G}_{\geq 1}$ for the subclasses of symbols that satisfy (i) and (ii), respectively. In either case, the mapping properties of φ and Bohr's theorem imply that the Dirichlet series φ necessarily has abscissa of uniform convergence $\sigma_u(\varphi) \leq 0$, see [23, Theorem 8.4.1].

By what is essentially the original argument of Gordon and Hedenmalm, the condition that $\psi \in \mathfrak{G}$ is necessary for a composition operator $C_\psi : \mathcal{D}_a \to \mathcal{D}_a$ to be bounded. In the Bergman case $a < 0$, this is also known to be sufficient [2, 3]. When $\psi \in \mathfrak{G}_0$, the proof of boundedness of $C_\psi : \mathcal{D}_a \to \mathcal{D}_a$, $a < 0$, due to Bailleul and Brevig [3], has a rather serendipitous flavor. In Section 3 we will supply a more systematic proof based on a Schwarz lemma for Dirichlet series, Lemma 3.4.

Beyond this, we will focus on composition operators induced by symbols $\varphi \in \mathfrak{G}_0$. The compact operators $C_\varphi \colon \mathcal{H}^2 \to \mathcal{H}^2$ were characterized only very recently in [10], in terms of the behavior of the mean counting function

$$M_{\varphi,1}(w) = \lim_{\sigma \to 0^+} \lim_{T \to \infty} \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} \operatorname{Re} s, \qquad w \neq \varphi(+\infty).$$

The main purpose of this article is to explore analogous tools and results in the weighted setting.

From Carlson's theorem [13, Lemma 3.2] one deduces the following formula of Littlewood–Paley type,

$$\|f\|_a^2 = |f(+\infty)|^2 + \frac{2^{1-a}}{\Gamma(2-a)} \lim_{\sigma_0 \to 0^+} \lim_{T \to \infty} \frac{1}{T} \int_{\sigma_0}^{\infty} \int_{-T}^{T} |f'(\sigma + it)|^2 \sigma^{1-a} \, dt \, d\sigma, \tag{1}$$

valid for $f \in \mathcal{D}_a$ such that $\sigma_u(f) \leq 0$. From this point of view, the space $\mathcal{D}_a$ is analogous to the weighted Hilbert space D_α, consisting of those holomorphic functions g on the unit disk such that

$$\|g\|_{D_\alpha}^2 = |g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < \infty, \tag{2}$$

where $\alpha = 1 - a \geq 0$ and $dA(z) = dx\,dy$, $z = x + iy$. By the results of [16, 20, 26], a holomorphic self-map of the unit disk $\phi \colon \mathbb{D} \to \mathbb{D}$ induces a compact composition operator on D_α, $\alpha > 0$, if and only if

$$\lim_{|z| \to 1^-} \frac{N_{\phi,\alpha}(z)}{(1 - |z|^2)^\alpha} = 0, \tag{3}$$

where for $\alpha = 1$, $N_{\phi,1}$ is the classical Nevanlinna counting function

$$N_\phi(z) = N_{\phi,1}(z) = \sum_{z_i \in \phi^{-1}(\{z\})} \log \frac{1}{|z_i|}, \qquad z \neq \phi(0),$$

and for $\alpha \neq 1$, $N_{\phi,\alpha}$ is the generalized Nevanlinna counting function

$$N_{\phi,\alpha}(z) = \sum_{z_i \in \phi^{-1}(\{z\})} (1 - |z_i|^2)^\alpha.$$

A key step in the disk setting is to introduce a non-injective change of variables in (2), resulting in what is known as a Stanton formula. In our setting, for $\varphi \in \mathfrak{G}_0$, making the change of variables in (1) yields that

$$\|C_\varphi(f)\|_a^2 = |f(\varphi(+\infty))|^2 + \frac{2^{1-a}}{\pi\Gamma(2-a)} \lim_{\sigma\to 0^+} \lim_{T\to\infty} \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 M_{\varphi,1-a}(w,\sigma,T)\, dA(w),$$

where

$$M_{\varphi,a}(w,\sigma,T) = \frac{\pi}{T} \sum_{\substack{s\in\varphi^{-1}(\{w\}) \\ |\operatorname{Im} s|<T \\ \sigma<\operatorname{Re} s<\infty}} (\operatorname{Re} s)^a, \qquad w \neq \varphi(+\infty).$$

For a Dirichlet series φ with abscissa of uniform convergence $\sigma_u(\varphi) \leq 0$, we therefore introduce the weighted mean counting functions

$$M_{\varphi,a}(w,\sigma) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{s\in\varphi^{-1}(\{w\}) \\ |\operatorname{Im} s|<T \\ \sigma<\operatorname{Re} s<\infty}} (\operatorname{Re} s)^a, \qquad w \neq \varphi(+\infty),$$

and

$$M_{\varphi,a}(w) = \lim_{\sigma\to 0^+} M_{\varphi,a}(w,\sigma),$$

if these limits exist.

Jessen and Tornehave [15, Theorem 31] studied the unweighted counting function $M_{\varphi,0}(w,\sigma)$ in the context of Lagrange's mean motion problem. They proved that the counting function exists for $\sigma > 0$ and $w \neq \varphi(+\infty)$, and that it satisfies

$$M_{\varphi,0}(w,\sigma) = -\mathcal{J}'_{\varphi-w}(\sigma^+),$$

where $\mathcal{J}'_{\phi-w}(\sigma^+)$ is the right-derivative of the Jessen function,

$$\mathcal{J}_{\varphi-w}(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \log|\varphi(\sigma+it) - w|\, dt. \tag{4}$$

On the basis of this and Littlewood's lemma, it was demonstrated in [10] that the weighted mean counting function $M_{\varphi,1}(w,\sigma)$ also exists for $\sigma > 0$ and $w \neq$

$\varphi(+\infty)$. Additionally, if φ belongs to the Nevanlinna class of Dirichlet series $\mathcal{N}_u$, that is, $\sigma_u(\varphi) \le 0$ and

$$\limsup_{\sigma\to 0^+} \frac{1}{2T}\int_{-T}^{T} \log^+ |\varphi(\sigma+it)|dt < +\infty,$$

then

$$M_{\varphi,1}(w) = \lim_{\sigma\to 0^+} \mathcal{J}_{\varphi-w}(\sigma) - \log|\varphi(+\infty)-w| < +\infty.$$

In Section 4 we will investigate the existence of the weighted mean counting functions $M_{\varphi,a}$.

Theorem 1.1. *For $a \in \mathbb{R}$, let φ be a Dirichlet series such that $\sigma_u(\varphi) \le 0$ and $\varphi(+\infty) \neq w$. Then the counting function $M_{\varphi,a}(w,\sigma)$ exists and is right-continuous on $\sigma > 0$. Furthermore,*

$$M_{\varphi,a}(w,\sigma) = M_{\varphi,0}(w,\sigma)\sigma^a + a\int_{\sigma}^{\infty} t^{a-1}M_{\varphi,0}(w,t)\,dt. \tag{5}$$

For $\sigma_\infty > 0$ sufficiently large, depending on φ and w, we also have that

(6)

$$M_{\varphi,a}(w,\sigma) - M_{\varphi,0}(w,\sigma)\sigma^a = a\sigma^{a-1}\mathcal{J}_{\varphi-w}(\sigma) - a\sigma_\infty^{a-1}\log|\varphi(+\infty)-w| - a(1-a)\int_{\sigma}^{\sigma_\infty} t^{a-2}\mathcal{J}_{\varphi-w}(t)dt.$$

In Theorem 4.8 we will furthermore obtain the integral representation

$$M_{\varphi,a}(w) = \int_{\mathbb{T}^\infty} M_{\varphi_\chi,a}(w,0,1)\,dm_\infty(\chi)$$

of the weighted mean counting function, where dm_∞ denotes the Haar measure on the infinite polytorus $\mathbb{T}^\infty$, and φ_χ denotes the Dirichlet series φ twisted by the character $\chi \in \mathbb{T}^\infty$, see Section 2. In the case that $a \ge 1$, we are from this formula able to deduce that

$$M_{\varphi,a}(w) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{s\in\varphi_\chi^{-1}(\{w\}) \\ |\operatorname{Im} s|<T \\ \operatorname{Re} s>0}} (\operatorname{Re} s)^a$$

for almost every $\chi \in \mathbb{T}^\infty$. That is, it is almost surely possible to interchange the T- and σ-limits in the definition of $M_{\varphi,a}(w)$. When $a = 1$, this partially resolves [10, Problem 1].

In Section 5 we then prove the analogue of the Stanton formula.

Theorem 1.2. *Suppose that $\varphi \in \mathfrak{G}_0$ and that $a \leq 1$. Then, for every $f \in \mathcal{D}_a$,*

$$\|C_\varphi(f)\|_a^2 = |f(\varphi(+\infty))|^2 + \frac{2^{1-a}}{\Gamma(2-a)\pi} \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 \, M_{\varphi,1-a}(w) dA(w). \tag{7}$$

If $a \leq 0$, then $M_{\varphi,1-a}(w)$ exists and is finite for every $w \in \mathbb{C}_{\frac{1}{2}} \setminus \{\varphi(+\infty)\}$.

Remark. For $a \geq 1/2$, the mean counting function $M_{\varphi,1-a}(w)$ can be infinite everywhere, see Example 4.6. In particular, both sides of (7) can be infinite. When $0 < a < 1/2$, we do not know if $M_{\varphi,1-a}(w)$ is finite for every $\varphi \in \mathfrak{G}_0$ and $\varphi(+\infty) \neq w$.

We use Theorem 1.2 to characterize the compact composition operators in the Bergman setting.

Theorem 1.3. *Let $\varphi \in \mathfrak{G}_0$. Then the induced composition operator C_φ is compact on the Bergman space $\mathcal{D}_{-a}$, $a > 0$, if and only if*

$$\lim_{\operatorname{Re} w \to \frac{1}{2}^+} \frac{M_{\varphi,1+a}(w)}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a}} = 0. \tag{8}$$

In addition to the change of variable formula, our Schwarz-type Lemma 3.4, is essential to proving the sufficiency of (8). In this context, we note that Bayart [5] recently showed that the condition $\lim_{\operatorname{Re} s \to 0^+} \frac{\operatorname{Re} \varphi(s) - \frac{1}{2}}{\operatorname{Re} s} = \infty$ is sufficient, but not necessary, for the operator $C_\varphi \colon \mathcal{D}_{-a} \to \mathcal{D}_{-a}$ to be compact.

Finally, we consider the Dirichlet-type spaces $\mathcal{D}_a$ for $0 < a < 1$. We prove that the analogue of (8) remains necessary for the composition operator to be compact, and we give an analogous necessary condition for boundedness. In Example 5.7 we observe that this condition is not sufficient for the operator to be bounded, at least not when $a \geq 1/2$.

Theorem 1.4. *Suppose that $0 < a < 1$ and let $\varphi \in \mathfrak{G}_0$. If the operator C_φ is bounded on the Dirichlet space $\mathcal{D}_a$, then for every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that*

$$\frac{M_{\varphi,1-a}(w)}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1-a}} < C(\delta), \qquad w \in \mathbb{C}_{\frac{1}{2}} \setminus D(\varphi(+\infty), \delta). \tag{9}$$

If $C_\varphi : \mathcal{D}_a \to \mathcal{D}_a$ is compact, then

$$\lim_{\operatorname{Re} w \to \frac{1}{2}^+} \frac{M_{\varphi,1-a}(w)}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1-a}} = 0. \tag{10}$$

In the special case where the symbol φ has bounded imaginary parts and the associated counting function is locally integrable, we can also prove that (9) is sufficient for the composition operator C_φ to be bounded, and that (10) is sufficient for a bounded composition operator C_φ to be compact.

Notation. Throughout the article, we will employ the convention that C denotes a positive constant which may vary from line to line. When we wish to clarify that the constant depends on some parameter P, we will write that $C = C(P)$. Furthermore, if $A = A(P)$ and $B = B(P)$ are two quantities depending on P, we write $A \approx B$ to signify that there are constants $c_1, c_2 > 0$ such that $c_1 B \leq A \leq c_2 B$ for all relevant choices of P.

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2. Background material

2.1. **The infinite polytorus and vertical limits.** The infinite polytorus is defined as the (countable) infinite Cartesian product of copies of the unit circle $\mathbb{T}$,

$$\mathbb{T}^\infty = \{\chi = (\chi_1, \chi_2, \dots) : \chi_j \in \mathbb{T},\ j \geq 1\}.$$

It is a compact abelian group with respect to coordinate-wise multiplication. We can identify the Haar measure m_∞ of the infinite polytorus with the countable infinite product measure $m \times m \times \cdots$, where m is the normalized Lebesgue measure of the unit circle.

By the unique prime factorization, $\mathbb{T}^\infty$ is isomorphic to the group of characters of $(\mathbb{Q}_+, \cdot)$. Given a point $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, the coresponding character $\chi : \mathbb{Q}_+ \to \mathbb{T}$ is the completely multiplicative function on $\mathbb{N}$ such that $\chi(p_j) = \chi_j$, where $\{p_j\}_{j \geq 1}$ is the increasing sequence of primes, extended to $\mathbb{Q}_+$ through the relation $\chi(n^{-1}) = \overline{\chi(n)}$.

Suppose $f(s) = \sum\limits_{n \geq 1} \frac{a_n}{n^s}$ is a Dirichlet series and $\chi(n)$ is a character. The vertical limit function f_χ is defined as

$$f_\chi(s) = \sum_{n \geq 1} \frac{a_n \chi(n)}{n^s}.$$

The name comes from Kronecker's theorem [7]; for any $\epsilon > 0$, there exists a sequence of real numbers $\{t_j\}_{j \geq 1}$ such that $f(s + t_j) \to f_\chi(s)$ uniformly on $\mathbb{C}_{\sigma_u(f)+\epsilon}$.

If $f \in \mathcal{D}_a$, then the abscissa of convergence satisfies $\sigma_c(f_\chi) \leq 0$ for almost every $\chi \in \mathbb{T}^\infty$. This is a consequence of the Rademacher–Menchov theorem [30, Ch. XIII], following an argument of [4]. Finally, we note that if $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}$, and we set

$$\psi_\chi(s) = c_0 s + \varphi_\chi(s),$$

then for every $\chi \in \mathbb{T}^\infty$ we have that

$$(C_\psi(f))_\chi = f_{\chi^{c_0}} \circ \psi_\chi. \tag{11}$$

2.2. **The hyperbolic metric and distance.** The classical Schwarz–Pick lemma states that for every holomorphic self-map of the unit disk $\phi : \mathbb{D} \to \mathbb{D}$ and for any $z \in \mathbb{D}$,

$$\frac{|\phi'(z)|}{1-|\phi(z)|^2} \leq \frac{1}{1-|z|^2}. \tag{12}$$

Equality holds in (12) for one point $z_0 \in \mathbb{D}$, and consequently for all points, if and only if ϕ is a holomorphic automorphism of the unit disk. The hyperbolic metric and distance in the unit disk are defined respectively as

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$$

and

$$d_{\mathbb{D}}(z,w) = \inf_{\gamma} \int_{\gamma} \lambda_{\mathbb{D}}(\zeta)\, |d\zeta|,$$

where the infimum is taken over all piecewise smooth curves γ in $\mathbb{D}$ that join z and w. The Schwarz–Pick lemma implies that every holomorphic self-map of the unit disk is a contraction of the hyperbolic distance,

$$\lambda_{\mathbb{D}}(\phi(z))|\phi'(z)| \leq \lambda_{\mathbb{D}}(z), \tag{13}$$

and

$$d_{\mathbb{D}}(\phi(z), \phi(w)) \leq d_{\mathbb{D}}(z,w), \tag{14}$$

where z, $w \in \mathbb{D}$.

If equality holds in (13) for one point, or in (14) for a pair of distinct points, then ϕ is a holomorphic automorphism of the unit disk, and thus an isometry. Using the conformal invariance of the hyperbolic distance, one can prove that

$$d_{\mathbb{D}}(z,w) = \log \frac{1 + \left|\frac{w-z}{1-\overline{w}z}\right|}{1 - \left|\frac{w-z}{1-\overline{w}z}\right|} = 2 \operatorname{arctanh} \left|\frac{w-z}{1-\overline{w}z}\right|.$$

The Riemann mapping theorem allows us to transfer these notions to any simply connected proper subdomain Ω of the complex plane. More precisely, let f be a Riemann map from Ω onto the unit disk. Then

$$\lambda_{\Omega}(z) = \lambda_{\mathbb{D}}(f(z))|f'(z)|,$$

and

$$d_{\Omega}(z,w) = d_{\mathbb{D}}(f(z), f(w)) = \inf_{\gamma} \int_{\gamma} \lambda_{\Omega}(\zeta)\, |d\zeta|,$$

where the infimum is taken over all piecewise smooth curves γ in Ω that join z and w. By the Schwarz lemma it is easy to prove that λ_{Ω} and d_{Ω} are independent of the choice of the Riemann map. In the case of the right-half plane, considering the Riemann map $f(z) = \frac{z-w}{z+\overline{w}}$ we obtain that

$$\lambda_{\mathbb{C}_0}(z) = \frac{1}{\operatorname{Re} z},$$

and

$$d_{\mathbb{C}_0}(z,w) = \log \frac{1 + \left|\frac{z-w}{z+\overline{w}}\right|}{1 - \left|\frac{z-w}{z+\overline{w}}\right|} = \log \frac{\left(|z+\overline{w}| + |z-w|\right)^2}{4 \operatorname{Re} z \operatorname{Re} w},$$

where $z,\, w \in \mathbb{C}_0$.

The following Schwarz–Pick lemma for simply connected domains is a direct consequence of the definition and the ordinary Schwarz–Pick lemma.

Theorem 2.1 ([6]). *Suppose that Ω_1 and Ω_2 are simply connected proper subdomains of the complex plane and that $f : \Omega_1 \to \Omega_2$ is a holomorphic function. Then, for every $z,\, w \in \Omega_1$,*

$$\lambda_{\Omega_2}(f(z))|f'(z)| \leq \lambda_{\Omega_1}(z), \tag{15}$$

and

$$d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z,w). \tag{16}$$

Furthermore, equality holds in (15) for one point, or in (16) for a pair of distinct points, if and only if f is a biconformal map from Ω_1 onto Ω_2.

3. Bounded composition operators on Bergman spaces of Dirichlet series

Consider the maps $T_\beta(z) = \beta\frac{1-z}{1+z}$, $\beta > 0$, and $S_\theta(z) = z + \theta$, $\theta > 0$, taking the unit disk $\mathbb{D}$ onto $\mathbb{C}_0$ and the half-plane $\mathbb{C}_0$ onto $\mathbb{C}_\theta$, respectively. Following [12], the space $H^2_i(\mathbb{C}_\theta, \beta)$ consists of those holomorphic functions on $\mathbb{C}_\theta$ such that $f \circ S_\theta \circ T_\beta \in H^2(\mathbb{D})$, with norm

$$\|f\|^2_{H^2_i(\mathbb{C}_\theta,\beta)} := \|f \circ S_\theta \circ T_\beta\|^2_{H^2(\mathbb{D})} = \frac{\beta}{\pi} \int_{-\infty}^{+\infty} |f(\theta + it)|^2 \frac{dt}{\beta^2 + t^2}.$$

We recall the following two lemmas.

Lemma 3.1 ([12, 22]). *Let $f \in \mathcal{H}^2$ be such that $\sigma_u(f) \leq 0$. Then*

$$\lim_{\beta\to\infty} \|f\|_{H^2_i(\mathbb{C}_0,\beta)} = \|f\|_0\,.$$

Lemma 3.2 ([9]). *For $\beta > 0$ and $f \in \mathcal{H}^2$,*

$$\|f\|^2_{H^2_i(\mathbb{C}_{\frac{1}{2}},\beta)} \leq \max\left\{\frac{2}{\beta}, \zeta(1+\beta)\right\} \|f\|^2_0\,.$$

The Cauchy–Schwarz inequality shows that point evaluations in $\mathbb{C}_{\frac{1}{2}}$ are bounded on $\mathcal{D}_a$. We record the following statement for easy reference.

Lemma 3.3. *Let $a \le 1$ and $\delta > 0$. Then there exists a constant $C = C(a, \delta)$ such that for every $s \in \mathbb{C}_{\frac{1}{2}+\delta}$ and $f \in \mathcal{D}_a$,*

$$|f'(s)| \le C \|f\|_a |2^{-s}|.$$

Our next goal is to establish a kind of Schwarz lemma for Dirichlet series. Note that the Schwarz–Pick lemma for the hyperbolic distance implies that

$$\liminf_{|z|\to 1^-} \frac{1 - |\phi(z)|}{1 - |z|} > 0$$

for any holomorphic self-map ϕ of $\mathbb{D}$, see [8, Lemma 1.4.5]. The corresponding inequality does not hold for all self-maps of the right half-plane. However, for a Dirichlet series $\varphi \in \mathfrak{G}_0$ we will prove that

$$\liminf_{\operatorname{Re} s \to 0^+} \frac{\operatorname{Re}\varphi(s) - \frac{1}{2}}{\operatorname{Re} s} = \delta > 0. \tag{17}$$

This implies a quantitative version of [12, Prop. 4.2]. Namely, that for sufficiently small $\epsilon > 0$,

$$\varphi(\mathbb{C}_\epsilon) \subset \mathbb{C}_{\frac{1}{2}+\epsilon\delta}.$$

The key idea in proving (17) is to exploit the vertical translations of $\varphi \in \mathfrak{G}_0$ to restrict the limit to a half-strip, where the quantity in (17) can be shown to be uniformly bounded from below by virtue of Theorem 2.1.

Lemma 3.4. *For every $\varphi \in \mathfrak{G}_0$ there exists a constant $C = C(\varphi) > 0$ such that*

$$\operatorname{Re} s \le C\left((\operatorname{Re} s)^2 + 1\right)\left(\operatorname{Re}\varphi(s) - \frac{1}{2}\right), \qquad s \in \mathbb{C}_0.$$

Proof. We consider the vertical translations $\varphi(s + it) = \varphi_{\chi_t}(s)$, where $\chi_t = n^{-it}$, $t \in \mathbb{R}$. Observing that $\varphi_{\chi_t} \in \mathfrak{G}_0$, we have

$$\begin{aligned}
\log \frac{\lambda_{\mathbb{C}_0}(z)}{\lambda_{\mathbb{C}_0}(\varphi_{\chi_t}(z) - \frac{1}{2})} &= \log \frac{\operatorname{Re}\varphi_{\chi_t}(z) - \frac{1}{2}}{\operatorname{Re} z} \\
&= \log \frac{\left(|z+1| + |z-1|\right)^2}{4\operatorname{Re} z} - \log \frac{\left(|\varphi_{\chi_t}(z) + \frac{1}{2}| + |\varphi_{\chi_t}(z) - \frac{3}{2}|\right)^2}{4(\operatorname{Re}\varphi_{\chi_t}(z) - \frac{1}{2})} \\
&\quad + 2\log \frac{|\varphi_{\chi_t}(z) + \frac{1}{2}| + |\varphi_{\chi_t}(z) - \frac{3}{2}|}{|z+1| + |z-1|} \\
&\ge d_{\mathbb{C}_0}(z, 1) - d_{\mathbb{C}_0}(\varphi_{\chi_t}(z) - \frac{1}{2}, 1) + 2\log \frac{2}{|z+1| + |z-1|}.
\end{aligned}$$

By the Schwarz–Pick lemma and the triangle inequality for the hyperbolic distance, we find from here that

$$\log\left(\frac{\operatorname{Re}\varphi_{\chi_t}(z)-\frac{1}{2}}{\operatorname{Re} z}\right) \geq -d_{\mathbb{C}_0}(\varphi_{\chi_t}(1)-\frac{1}{2},1)+2\log\left(\frac{2}{|z+1|+|z-1|}\right). \tag{18}$$

The crucial step is to note that the quantity $d_{\mathbb{C}_0}(\varphi_{\chi_t}(1)-\frac{1}{2},1)$ is uniformly bounded, since φ maps the line $\operatorname{Re} z=1$ into a compact subset of $\mathbb{C}_{1/2}$. Given $s\in\mathbb{C}_0$, we can therefore choose $z=\operatorname{Re} s$ and $t=\operatorname{Im} s$ to obtain that

$$\operatorname{Re} s \leq C\left[(|z+1|+|z-1|)^2\right]\left(\operatorname{Re}\varphi(s)-\frac{1}{2}\right),$$

which is the desired inequality. □

We next recall Littlewood's subordination principle, which implies that any holomorphic self-map of the unit disk generates a bounded composition operator on the Hardy space $H^2(\mathbb{D})$.

Lemma 3.5 ([17, 27])**.** *Suppose ϕ is a holomorphic self-map of the unit disk $\mathbb{D}$. Then, for every $f\in H^2(\mathbb{D})$,*

$$\|f\circ\phi\|_{H^2(\mathbb{D})} \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}\,\|f\|_{H^2(\mathbb{D})}.$$

We also borrow the following lemma from [12].

Lemma 3.6 ([12])**.** *Let $a\leq 1$ and let $\{p_j\}_{j\geq1}$ be the increasing sequence of primes. Then, the function $f(s)=\sum_{j\geq1} a_{p_j}p_j^{-s}$, with $a_{p_j}=\frac{1}{\sqrt{p_j}\log(p_j)^{1+\frac{a}{2}}}$, satisfies the following:*

(i) $f\in\mathcal{D}_a$ and $\sigma_c(f)=\frac{1}{2}$.
(ii) $\sigma_c(f_\chi)=0$, for almost every $\chi\in\mathbb{T}^\infty$.

As promised in the introduction, we now provide a proof of the characterization of the bounded composition operators on the Bergman spaces $\mathcal{D}_a$, $a\leq 0$, which is new for Dirichlet series symbols. To do so, we will combine the original argument of Gordon and Hedenmalm [12] with the Schwarz lemma for Dirichlet series.

Theorem 3.7 ([2, 3])**.** *For $a>0$, the class $\mathfrak{G}$ determines all bounded composition operators on the Bergman space of Dirichlet series $\mathcal{D}_{-a}$.*

Proof. It was essentially already proven in [12] that it is necessary that $\psi\in\mathfrak{G}$ in order for $C_\psi\colon\mathcal{D}_a\to\mathcal{D}_a$ to be bounded. Indeed, by [23, Theorem 8.3.1], $P\circ\psi$ is a Dirichlet series for every polynomial P if and only if the symbol $\psi:\mathbb{C}_{\frac{1}{2}}\to\mathbb{C}_{\frac{1}{2}}$ has the form $\psi(s)=c_0s+\varphi(s)$, where c_0 is a non-negative integer and φ is a

Dirichlet series. The mapping properties of ψ are deduced from the composition rule (11) and Lemma 3.6, noting that $\varphi(\mathbb{C}_0) = \varphi_\chi(\mathbb{C}_0)$ for every $\chi \in \mathbb{T}^\infty$.

Conversely, suppose $\psi \in \mathfrak{G}$. Let μ denote the probability measure $d\mu(\sigma) = \frac{2^a}{\Gamma(a)}\sigma^{a-1}e^{-2\sigma}\,d\sigma$ on $(0,\infty)$, observing that

$$\|f\|_{-a}^2 \approx \int_{\mathbb{T}^\infty}\int_0^\infty\int_0^1 |f_\chi(\sigma+it)|^2\,dt\,d\mu(\sigma)\,dm_\infty(\chi) = \int_0^\infty \|f_\sigma\|_0^2\,d\mu(\sigma),$$

where $f_\sigma = f(\cdot+\sigma)$. First we will consider the case when $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$. In this case the analogue of the Schwarz lemma is trivial: $\operatorname{Re} s \leq \operatorname{Re}\psi(s)$. Fix $\sigma > 0$, for a Dirichlet polynomial f and a positive number $\beta > 0$, we define the functions

$$F_\beta = f \circ S_\sigma \circ T_\eta$$

and

$$g_\beta = T_\eta^{-1} \circ S_\sigma^{-1} \circ \psi_\sigma \circ T_\beta,$$

where $\eta = c_0(\beta+\sigma) - \sigma$. Note that $g_\beta(0) \to 0$ as $\beta \to \infty$. By Lemma 3.1 and Lemma 3.5, we have

$$\begin{aligned}
\|C_{\psi_\sigma}(f)\|_0 &= \lim_{\beta\to\infty} \|f \circ \psi_\sigma\|_{H_i^2(\mathbb{C}_0,\beta)} \\
&= \lim_{\beta\to\infty} \|F_\beta \circ g_\beta\|_{H^2(\mathbb{D})} \leq \lim_{\beta\to\infty} \sqrt{\frac{1+|g_\beta(0)|}{1-|g_\beta(0)|}}\,\|F_\beta\|_{H^2(\mathbb{D})} \\
&= \lim_{\eta\to\infty} \|f \circ S_\sigma \circ T_\eta\|_{H^2(\mathbb{D})} = \|f_\sigma\|_0\,.
\end{aligned}$$

Therefore

$$\|C_\psi(f)\|_{-a}^2 \approx \int_0^\infty \|C_{\psi_\sigma}(f)\|_0^2\,d\mu(\sigma) \leq \int_0^\infty \|f_\sigma\|_0^2\,d\mu(\sigma) \approx \|f\|_{-a}^2\,,$$

which demonstrates that the composition operator is bounded in this case.

Suppose next that $\varphi \in \mathfrak{G}_0$. Considering a vertical translation of the argument f by $\operatorname{Im}\varphi(+\infty)$, there is no loss of generality in assuming that $\varphi(+\infty) > 1/2$. By Lemma 3.4 there exists a constant $\lambda = \lambda(\varphi) > 0$ such that

$$\lambda \operatorname{Re} s \leq \operatorname{Re}\varphi(s) - \frac{1}{2}, \qquad 0 < \operatorname{Re} s < 1.$$

In this case, for a Dirichlet polynomial f and a positive numbers $\beta > 0$, we define the functions

$$F = f \circ S_{\lambda\sigma+\frac{1}{2}} \circ T_\eta$$

and

$$g_\beta = T_\eta^{-1} \circ S_{\lambda\sigma+\frac{1}{2}}^{-1} \circ \varphi_\sigma \circ T_\beta,$$

where $\eta = \varphi(+\infty) - \lambda\sigma - \frac{1}{2}$ and $0 < \sigma < \delta := \frac{\varphi(+\infty) - \frac{1}{2}}{2\lambda}$. Then we again have that $\lim_{\beta\to\infty} g_\beta(0) = 0$, and

$$\begin{aligned} \|C_{\varphi_\sigma}(f)\|_0 &= \lim_{\beta\to\infty} \|f \circ \varphi_\sigma\|_{H_i^2(\mathbb{C}_0,\beta)} = \lim_{\beta\to\infty} \|F \circ g_\beta\|_{H^2(\mathbb{D})} \\ &\le \lim_{\beta\to\infty} \sqrt{\frac{1+|g_\beta(0)|}{1-|g_\beta(0)|}}\, \|F\|_{H^2(\mathbb{D})} = \|f_{\lambda\sigma}\|^2_{H_i^2(\mathbb{C}_{\frac{1}{2}},\eta)} . \end{aligned}$$

By Lemma 3.2 we conclude that there is a constant such that

$$\|C_{\varphi_\sigma}(f)\|_0 \le C\, \|f_{\lambda\sigma}\|_0 , \qquad 0 < \sigma < \delta.$$

For $\sigma \ge \delta$, we simply note that $\varphi_\sigma(\mathbb{C}_0) \subset \mathbb{C}_{1/2+\varepsilon}$ for some $\varepsilon > 0$, and therefore by the Cauchy–Schwarz inequality that

$$\sup_{s\in\mathbb{C}_0} |f(\varphi_\sigma(s))| \le C\|f\|_{-a}.$$

Hence $\|C_{\varphi_\sigma}(f)\|_0 \le C\|f\|_{-a}$, as can be seen for example from Carlson's theorem, see [13, Lemma 3.2]. We conclude that

$$\|C_\varphi(f)\|^2_{-a} \approx \int_0^\infty \|C_{\varphi_\sigma}(f)\|_0^2 \, d\mu(\sigma) \le C \int_0^\delta \|f_{\lambda\sigma}\|_0^2 \, d\mu(\sigma) + C\, \|f\|^2_{-a} \le C\, \|f\|^2_{-a} .$$

□

Remark. Using the same argument one can prove that Theorem 3.7 holds for all Bergman-like spaces of Dirichlet series [18],

$$\mathcal{D}_\mu = \left\{ f(s) = \sum_{n\ge1} \frac{a_n}{n^s} : \|f\|_\mu^2 = \sum_{n\ge1} |a_n|^2 w_\mu(n) < \infty \right\},$$

assuming that the coefficients are of the form

$$w_\mu(n) = \int_0^\infty \frac{d\mu(\sigma)}{n^{2\sigma}},$$

where μ is a probability measure on $(0,\infty)$ with $0 \in \operatorname{supp}(\mu)$ and satisfying

$$\int_0^\infty \frac{d\mu(\sigma)}{n^{2\lambda\sigma}} \le C(\lambda) \int_0^\infty \frac{d\mu(\sigma)}{n^{2\sigma}}, \qquad 0 < \lambda < 1. \tag{19}$$

Every symbol $\psi \in \mathfrak{G}_{\ge 1}$ induces a contraction C_ψ on $\mathcal{D}_\mu$, even without the condition (19).

4. Weighted mean counting functions

In this section, we will investigate the properties of the weighted counting function $M_{\varphi,a}(w,\sigma)$, $\sigma > 0$, where φ is a Dirichlet series with abscissa of convergence $\sigma_u(\varphi) \le 0$. Firstly, we will prove the existence of this function, generalizing [10, Theorem 6.2]. Monotonicity then ensures the existence of the limit function $M_{\varphi,a}(w)$ (finitely or infinitely). Secondly, following the ideas of Aleman [1] and Shapiro [26] from the disk case, we will give a weak version of the submean value property for the weighted counting function $M_{\varphi,a}(w)$, $a > 0$. Note that the (strong) submean value property of $M_{\varphi,1}(w)$ was proven in [10, Lemma 6.5].

4.1. **Existence.** In [10], the existence of $M_{\varphi,1}(w,\sigma)$ was established through Littlewood's lemma [28, Sec. 9.9], which is a rectangular version of Jensen's formula [26, Sec. 10.2]. We will replace Littlewood's lemma with the following theorem, which allows us to count the zeros of a non-zero holomorphic function in an arbitrary domain.

Theorem 4.1 ([25])**.** *Let $u \not\equiv -\infty$ be a subharmonic function on a domain Ω in $\mathbb{C}$. Then, there exists a unique Radon measure Δu on Ω such that for every compactly supported function $v \in C^\infty(\Omega)$, it holds that*

$$\int_\Omega v\Delta u = \int_\Omega u\Delta v\, dA.$$

In the special case that $u = \log|f|$, where $f \not\equiv 0$ is a holomorphic function on the domain Ω, the measure $\frac{1}{2\pi}\Delta u$ is the sum of Dirac masses at the zeros of f, counting multiplicity.

The almost periodicity of the Dirichlet series φ in $\mathbb{C}_{\sigma_0}$, $\sigma_0 > 0$, implies an argument principle for the unweighted counting function $M_{\varphi,0}$, see [15].

Lemma 4.2. *Suppose that φ is a Dirichlet series with abscissa of uniform convergence $\sigma_u(\varphi) \le 0$. If $\varphi(+\infty) \neq 0$, $\{\operatorname{Re} s = \sigma_0\}$ is a zero-free line for the function φ and $\{T_j\}_{j\ge 1}$ is an increasing sequence of positive real numbers, relatively dense in $[0,+\infty)$, such that*

$$|\varphi(\sigma + iT_j)| \ge \delta > 0, \qquad \sigma \ge \sigma_0,$$

then

$$M_{\varphi,0}(0,\sigma_0) = -\lim_{j\to\infty} \frac{1}{2T_j} \int_{-T_j}^{T_j} \frac{\varphi'(\sigma_0 + it)}{\varphi(\sigma_0 + it)} dt.$$

Proof. Let $\sigma_\infty > 0$ be such that the equation $\varphi(s) = 0$ has no solution for $\operatorname{Re} s \ge \sigma_\infty$. We will denote by R_j the rectangle with vertices at $\sigma_0 \pm iT_j$, $\sigma_\infty \pm iT_j$.

By the argument principle, we then have that

$$\begin{aligned} M_{\varphi,0}(0,\sigma_0,T_j) &= \frac{1}{2iT_j}\int\limits_{\partial R_j}\frac{\varphi'(\zeta)}{\varphi(\zeta)}d\zeta \\ &= -\frac{1}{2T_j}\int\limits_{-T_j}^{T_j}\frac{\varphi'(\sigma_0+it)}{\varphi(\sigma_0+it)}dt + \frac{i}{2T_j}\int\limits_{\sigma_0}^{\sigma_\infty}\frac{\varphi'(\sigma+iT_j)}{\varphi(\sigma+iT_j)}d\sigma \\ &- \frac{i}{2T_j}\int\limits_{\sigma_0}^{\sigma_\infty}\frac{\varphi'(\sigma-iT_j)}{\varphi(\sigma-iT_j)}d\sigma + \frac{1}{2T_j}\int\limits_{-T_j}^{T_j}\frac{\varphi'(\sigma_\infty+it)}{\varphi(\sigma_\infty+it)}dt. \end{aligned}$$

We observe that the first coefficient of the Dirichlet series $f = \frac{\varphi'}{\varphi}$ satisfies $f(+\infty) = 0$. Thus, letting $T_j \to \infty$ and then $\sigma_\infty \to \infty$ follows that

$$M_{\varphi,0}(0,\sigma_0) = \lim_{j\to\infty} M_{\varphi,0}(0,\sigma,T_j) = -\lim_{j\to\infty}\frac{1}{2T_j}\int\limits_{-T_j}^{T_j}\frac{\varphi'(\sigma_0+it)}{\varphi(\sigma_0+it)}dt. \qquad \square$$

We begin by proving a special case of Theorem 1.1.

Theorem 4.3. *Let φ be a Dirichlet series such that $\sigma_u(\varphi) \leq 0$, and let $w \neq \varphi(+\infty)$ be such that $\{\operatorname{Re} s = \sigma_0\}$ is a zero free line for the function $\varphi - w$. Then, for every $a \in \mathbb{R}$, the counting function $M_{\varphi,a}(w,\sigma_0)$ exists and satisfies*

$$M_{\varphi,a}(w,\sigma_0) = M_{\varphi,0}(w,\sigma_0)\sigma_0^a + a\int\limits_{\sigma_0}^{\infty} t^{a-1}M_{\varphi,0}(w,t)\,dt. \tag{20}$$

Furthermore, for sufficiently large $\sigma_\infty > 0$,

(21)

$$\begin{aligned} &M_{\varphi,a}(w,\sigma_0) - M_{\varphi,0}(w,\sigma_0)\sigma_0^a = \\ &\qquad a\sigma_0^{a-1}\mathcal{J}_{\varphi-w}(\sigma_0) - a\sigma_\infty^{a-1}\log|\varphi(+\infty)-w| - a(1-a)\int\limits_{\sigma_0}^{\sigma_\infty} t^{a-2}\mathcal{J}_{\varphi-w}(t)dt, \end{aligned}$$

where $\mathcal{J}_{\varphi-w}$ is the Jessen function (4).

Proof. Without loss of generality we assume that $w = 0$. By almost periodicity there exists an increasing sequence $\{T_j\}_{j\geq 1}$ of positive real numbers, relatively dense in $[0,+\infty)$, such that for every $\sigma \geq \sigma_0$,

$$|\varphi(\sigma \pm iT_j)| \geq \delta > 0.$$

Let $\sigma_\infty > 0$ be so large that $\varphi \neq 0$ in $\mathbb{C}_{\frac{\sigma_\infty}{2}}$. Then $\Delta\log|\varphi| = 0$ near the boundary of the rectangle R_j with vertices at $\sigma_0 \pm iT_j$, $\sigma_\infty \pm iT_j$.

By a C^∞ version of Urysohn's lemma [11, Theorem 8.18] there exists a function $\psi \in C_c^\infty(R_j)$ such that $\psi(s) = (\operatorname{Re} s)^a$ for $s \in \operatorname{supp}(\Delta \log|\varphi|)$. Theorem 4.1 implies that

$$\int\limits_{R_j} (\operatorname{Re} z)^a \, \Delta \log|\varphi(z)| = \int\limits_{R_j} \psi(z) \Delta \log|\varphi(z)|$$

$$= 2\pi \sum_{\substack{s\in\varphi^{-1}(\{0\}) \\ s\in R_j}} (\operatorname{Re} s)^a = 2T_j M_{\varphi,a}(0,\sigma_0,T_j).$$

On the other hand, Green's theorem implies that

$$\int\limits_{R_j} (\operatorname{Re} z)^a \, \Delta \log|\varphi(z)| + a(1-a) \int\limits_{R_j} (\operatorname{Re} z)^{a-2} \log|\varphi(z)| dA(z)$$

$$= \oint\limits_{\partial R_j} (\operatorname{Re}\zeta)^a \left(-\frac{d\log|\varphi(\zeta)|}{dy}, \frac{d\log|\varphi(\zeta)|}{dx} \right) \cdot d\zeta$$

$$- \oint\limits_{\partial R_j} \log|\varphi(\zeta)| \left(-\frac{d\,(\operatorname{Re}\zeta)^a}{dy}, \frac{(\operatorname{Re}\zeta)^a}{dx} \right) \cdot d\zeta,$$

where $\zeta = x + iy$. For the first line integral on the right-hand side, we have that

$$\oint\limits_{\partial R_j} (\operatorname{Re}\zeta)^a \left(-\frac{d\log|\phi(\zeta)|}{dy}, \frac{d\log|\phi(\zeta)|}{dx} \right) \cdot d\zeta =$$

$$-\sigma_0^a \operatorname{Re} \int\limits_{-T_j}^{T_j} \frac{\phi'(\sigma_0+it)}{\phi(\sigma_0+it)}\, dt + \sigma_\infty^a \operatorname{Re} \int\limits_{-T_j}^{T_j} \frac{\phi'(\sigma_\infty+it)}{\phi(\sigma_\infty+it)}\, dt \pm \operatorname{Re} \int\limits_{\sigma_0}^{\sigma_\infty} \sigma^a \frac{i\phi'(\sigma \pm iT_j)}{\phi(\sigma\pm iT_j)}\, d\sigma.$$

From Lemma 4.2, dividing through by $2T_j$ and letting $j \to \infty$, we obtain that

$$\lim_{j\to\infty} \frac{1}{2T_j} \oint\limits_{\partial R_j} (\operatorname{Re}\zeta)^a \left(-\frac{d\log|\phi(\zeta)|}{dy}, \frac{d\log|\phi(\zeta)|}{dx} \right) \cdot d\zeta = M_{\phi,0}(0,\sigma_0)\sigma_0^a.$$

Writing out the second line integral,

$$\oint\limits_{\partial R_j} \log|\phi(\zeta)| \left(-\frac{d\,(\operatorname{Re}\zeta)^a}{dy}, \frac{(\operatorname{Re}\zeta)^a}{dx} \right) \cdot d\zeta =$$

$$- a\sigma_0^{a-1} \int\limits_{-T_j}^{T_j} \log|\phi(\sigma_0+it)| dt + a\sigma_\infty^{a-1} \int\limits_{-T_j}^{T_j} \log|\phi(\sigma_\infty+it)| dt,$$

we have that

$$\lim_{j\to\infty} \frac{1}{2T_j} \oint_{\partial R_j} \log|\phi(\zeta)| \left(-\frac{d\,(\operatorname{Re}\zeta)^a}{dy}, \frac{(\operatorname{Re}\zeta)^a}{dx} \right) \cdot d\zeta$$

$$= -a\sigma_0^{a-1} J_\phi(\sigma_0) + a\sigma_\infty^{a-1} J_\phi(\sigma_\infty),$$

where $J_\phi(\sigma_\infty) = \log|\phi(+\infty)|$ by [15, Theorem 31].

We apply Fubini's theorem to the area integral,

$$\frac{1}{2T_j} \int_{R_j} (\operatorname{Re} z)^{a-2} \log|\phi(z)| dA(z) = \int_{\sigma_0}^{\sigma_\infty} \sigma^{a-2} \frac{1}{2T_j} \int_{-T_j}^{T_j} \log|\phi(\sigma+it)| dt d\sigma.$$

Since the sequence of functions $\left\{ \frac{1}{T_j} \int_{-T_j}^{T_j} \log|\varphi(\sigma+it)| dt \right\}_{j\geq 1}$ is uniformly bounded on $[\sigma_0, +\infty)$ by [15, Theorem 5], the dominated convergence theorem implies that

$$a(1-a) \lim_{j\to\infty} \frac{1}{2T_j} \int_{R_j} (\operatorname{Re} z)^{a-2} \log|\varphi(z)| dA(z) = a(1-a) \int_{\sigma_0}^{\sigma_\infty} \sigma^{a-2} \mathcal{J}_\varphi(\sigma) d\sigma.$$

We conclude that

$$\lim_{j\to\infty} M_{\varphi,a}(0,\sigma_0,T_j) - M_{\varphi,0}(0,\sigma_0)\sigma_0^a =$$

$$a\sigma_0^{a-1} \mathcal{J}_\varphi(\sigma_0) - a\sigma_\infty^{a-1} \log|\varphi(+\infty)| - a(1-a) \int_{\sigma_0}^{\sigma_\infty} \sigma^{a-2} \mathcal{J}_\varphi(\sigma) d\sigma.$$

This finishes the proof of (21), since almost periodicity and the argument principle show that the number of zeros of φ on any rectangle $(\sigma_0, \sigma_\infty) \times (T-d, T+d)$ is uniformly bounded, where $d = \sup_{j\geq 1}(T_{j+1} - T_j)$, see for example [15, Theorem 3].

Finally, the Jessen function $\mathcal{J}_\varphi(\sigma)$ is convex and, as a consequence, absolutely continuous on every closed sub-interval of the positive semi-axis. Thus, we can integrate by parts, yielding that

$$M_{\varphi,a}(0,\sigma_0) = M_{\varphi,0}(0,\sigma_0)\sigma_0^a + a \int_{\sigma_0}^{\infty} t^{a-1} M_{\varphi,0}(0,t) dt,$$

which is (20). □

Before proving Theorem 1.1, we extract the following technical lemma from the work of [10, Lemma 2.4].

Lemma 4.4. *Let φ be a Dirichlet series such that $\sigma_u(\varphi) \le 0$ and $\varphi(+\infty) \ne 0$. Then for every $\sigma_0 > 0$, and for $T > 0$ sufficiently large, there exists a constant $C(\sigma_0, \varphi) > 0$ such that,*

$$\frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} 1 \le C, \qquad \sigma \ge \sigma_0. \tag{22}$$

Proof. Let Θ denote the unique conformal map from the unit disk to the half-strip

$$S_1 = \{s : \operatorname{Re} s > 0, \, |\operatorname{Im} s| < 1\}$$

with $\Theta(0) = 1$ and $\Theta'(0) > 0$. We observe that

$$\Theta^{-1}(s) = \frac{\sinh(\frac{s\pi}{2}) - \sinh(\frac{\pi}{2})}{\sinh(\frac{s\pi}{2}) + \sinh(\frac{\pi}{2})},$$

and that there exist absolute constants δ_1, $\delta_2 > 0$ such that

$$\delta_1 < \left| \left(\Theta^{-1}\right)'(s) \right| < \delta_2$$

whenever $|\operatorname{Im} s| \le \frac{1}{2}$ and $0 \le \operatorname{Re} s \le \frac{1}{2}$. The Koebe quarter theorem [21, Corollary 1.4] implies that for every $s \in S_1$,

$$\frac{1 - |\Theta^{-1}(s)|^2}{4 \left| (\Theta^{-1})'(s) \right|} \le \operatorname{dist}(s, \partial S_1) \le \frac{1 - |\Theta^{-1}(s)|^2}{\left| (\Theta^{-1})'(s) \right|}.$$

Thus, there exists an absolute constant C_0 such that

$$\pi \operatorname{Re} s \le C_0 \log \left| \frac{1}{\Theta^{-1}(s)} \right|, \tag{23}$$

when $|\operatorname{Im} s| \le \frac{1}{2}$ and $0 \le \operatorname{Re} s \le \frac{1}{2}$.

For $T > 0$ we will denote by S_T the half-strip TS_1 and by $\Theta_T : \mathbb{D} \to S_T$ the map $\Theta_T = T\Theta$. We consider the function $\varphi_{\frac{\sigma_0}{2}}(s) = \frac{\varphi(s + \frac{\sigma_0}{2})}{M}$, where $M = \sup_{\mathbb{C}_{\frac{\sigma_0}{2}}} |\varphi(s)|$. Then, for T so large that the equation $\varphi(s) = 0$ has no solutions for $\operatorname{Re} s > T$ and for $\sigma \ge \sigma_0$, we have by (23) that

$$\begin{aligned} \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} 1 &\le C \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} \left(\operatorname{Re} s - \frac{\sigma_0}{2} \right) \le C \frac{\pi}{T} \sum_{\substack{s \in \varphi_{\frac{\sigma_0}{2}}^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \operatorname{Re} s > 0}} \operatorname{Re} s \\ &\le C \sum_{\substack{s \in \varphi_{\frac{\sigma_0}{2}}^{-1}(\{0\}) \\ |\operatorname{Im} s| < 2T \\ \operatorname{Re} s > 0}} \log \left| \frac{1}{\Theta_{2T}^{-1}(s)} \right| = C N_{\psi_{2T}}(0), \end{aligned}$$

where $\psi_{2T} = \varphi_{\frac{\sigma_0}{2}} \circ \Theta_{2T}$ and $N_{\psi_{2T}}$ is the classical Nevanlinna counting function. Thus, the Littlewood inequality [26],

$$N_{\psi_{2T}}(0) \leq \log \left| \frac{1}{\psi_{2T}(0)} \right|,$$

implies that

$$\frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} 1 \leq C \log \left| \frac{M}{\varphi(\frac{\sigma_0}{2} + 2T)} \right| \qquad \square$$

We can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality we can assume that $w = 0$. Let $\omega(\sigma) = \sigma^a$. Then,

$$\int_{\sigma_0}^{\infty} \omega'(\sigma) M_{\varphi,0}(0,\sigma) d\sigma = \int_{\sigma_0}^{\sigma_\infty} \omega'(\sigma) \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} 1 \, d\sigma,$$

where $\sigma_\infty > 0$ is such that the equation $\varphi(s) = 0$ has no solutions in $\mathbb{C}_{\sigma_\infty}$. By the dominated convergence theorem, which applies in light of (22), and then by Fubini's theorem, we obtain that

$$\begin{aligned}
\int_{\sigma_0}^{\infty} \omega'(\sigma) M_{\varphi,0}(0,\sigma)\, d\sigma &= \lim_{T\to\infty} \frac{\pi}{T} \int_{\sigma_0}^{\infty} \omega'(\sigma) \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} 1 \, d\sigma \\
&= \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{0\}) \\ |\operatorname{Im} s| < T \\ \sigma_0 < \operatorname{Re} s < \infty}} \int_{\sigma_0}^{\operatorname{Re} s} \omega'(\sigma)\, d\sigma \\
&= M_{\varphi,a}(0,\sigma_0) - \sigma_0^a M_{\varphi,0}(0,\sigma_0).
\end{aligned}$$

This proves the existence of the function $M_{\varphi,a}(0,\sigma_0)$ and (5). The right continuity of $M_{\varphi,a}(0,\sigma)$ is now a consequence of the right continuity of $M_{\varphi,0}(0,\sigma)$, see [10, Lemma 5.1]. Integrating by parts as in the proof of Theorem 4.3, we also obtain (6). $\square$

Strictly speaking, this argument is independent of Theorem 4.3. However, we find the proof of Theorem 4.3 to be illuminating and interesting in its own right. Note that the proof for Theorem 1.1 can also be applied to the more general counting function induced by a twice continuously differentiable weight $\omega(s) = \omega(\operatorname{Re} s)$ on $(0, \infty)$,

$$M_{\varphi,\omega}(w, \sigma, T) = \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} \omega(s), \qquad w \neq \varphi(+\infty).$$

By monotonicity we deduce the following.

Corollary 4.5. *Let φ be a Dirichlet series such that $\sigma_u(\varphi) \leq 0$ and $\varphi(+\infty) \neq w$. Then, for every $a \in \mathbb{R}$ the counting function $M_{\varphi,a}(w) = \lim_{\sigma \to 0^+} M_{\varphi,a}(w, \sigma)$ exists, finitely or infinitely.*

The limit is not finite in general for $a \geq 0$, as we now exemplify.

Example 4.6. Applying the transference principle [24] to the example constructed by Zorboska in [29], we obtain a Dirichlet series φ such that the a-weighted counting function is finite if and only if $a > \frac{1}{2}$. More precisely, we consider the Dirichlet series $\varphi(s) = g(2^{-s})$, where $g(z) = e^{-\frac{1+z}{1-z}}$, $z \in \mathbb{D}$. We observe that φ is a periodic function (with period $ip = \frac{2\pi i}{\log(2)}$) and abscissa of uniform convergence $\sigma_u(\varphi) \leq 0$.

Let $w \in g(\mathbb{D}) \setminus \{g(0)\} = \varphi(\mathbb{C}_0) \setminus \{\varphi(+\infty)\}$. The periodicity implies that

(24)
$$\left[\frac{2T}{p}\right] \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ 0 \leq \operatorname{Im} s < p \\ \sigma < \operatorname{Re} s < \infty}} (\operatorname{Re} s)^a \leq \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} (\operatorname{Re} s)^a \leq \left(\left[\frac{2T}{p}\right] + 1\right) \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ 0 \leq \operatorname{Im} s < p \\ \sigma < \operatorname{Re} s < \infty}} (\operatorname{Re} s)^a,$$

where $[x]$ is the integer part of the real number x. Note that

$$(\log 2)^a \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| < p \\ \operatorname{Re} s > 0}} (\operatorname{Re} s)^a = \sum_{\substack{z \in g^{-1}(\{w\}) \\ |z| < 1}} \left(\log \frac{1}{|z|}\right)^a.$$

Writing $w = e^{-b} e^{i\theta}$, where $b > 0$ and $\theta \in [0, 2\pi)$, so that

$$g^{-1}(\{w\}) = \left\{ z_n = \frac{1 - b + i(\theta + 2\pi n)}{i(\theta + 2\pi n) - b - 1} : n \in \mathbb{Z} \right\}.$$

We thus have

$$\sum_{\substack{s\in\varphi^{-1}(\{w\})\\ 0\leq \operatorname{Im} s<p\\ \operatorname{Re} s>0}} (\operatorname{Re} s)^a \approx \sum_{n\in\mathbb{Z}} \left(1-|z_n|^2\right)^a$$

$$= \sum_{n\in\mathbb{Z}} \left(\frac{4b}{(1+b)^2+(\theta+2n\pi)^2}\right)^a.$$

This shows that $M_{\varphi,a}(w)=\infty$ for all $w\in\mathbb{D}$ and $a\leq\frac{1}{2}$.

4.2. **Weighted mean counting functions as integrals.** The purpose of this subsection is to replace the limiting processes in the definition of $M_{\varphi,a}$ with integration. For $a\geq 1$, this allows us to show that it is almost always possible to directly take $\sigma=0$ in the definition of the weighted mean counting function.

Lemma 4.7. *Let $a\in\mathbb{R}$ and let φ be a Dirichlet series such that $\sigma_u(\varphi)\leq 0$ and $\varphi(+\infty)\neq w$. Then, for every $\sigma>0$, the weighted mean counting function is invariant under vertical limits, that is,*

$$M_{\varphi,a}(w,\sigma)=M_{\varphi_\chi,a}(w,\sigma), \qquad \chi\in\mathbb{T}^\infty.$$

Proof. The statement holds for the Jessen function, see [14, Satz A] or [10, Lemma 4.1],

$$\mathcal{J}_{\varphi-w}(\sigma)=\mathcal{J}_{\varphi_\chi-w}(\sigma), \qquad \chi\in\mathbb{T}^\infty.$$

Thus, for the unweighted counting function, we have that

$$M_{\varphi_\chi,0}(w,\sigma)=-\mathcal{J}'_{\varphi_\chi-w}(\sigma^+)=-\mathcal{J}'_{\varphi-w}(\sigma^+)=M_{\varphi,0}(w,\sigma).$$

By Theorem 1.1 it follows that every weighted mean counting function $M_{\varphi_\chi,a}(w,\sigma)$ is invariant under vertical limits. □

Of course we may let $\sigma\to 0^+$ to obtain that $M_{\varphi,a}(w)=M_{\varphi_\chi,a}(w)$ for every $\chi\in\mathbb{T}^\infty$.

Theorem 4.8. *Let φ be a Dirichlet series such that $\sigma_u(\varphi)\leq 0$ and $\varphi(+\infty)\neq w$. Then, for every $a\in\mathbb{R}$ the weighted mean counting function can be written as*

$$M_{\varphi,a}(w)=\int_{\mathbb{T}^\infty} M_{\varphi_\chi,a}(w,0,1)\,dm_\infty(\chi). \tag{25}$$

Proof. For fixed $\sigma>0$, almost periodicity and Hurwitz's theorem imply that $M_{\varphi_\chi,a}(w,\sigma,1)$ is uniformly bounded in $\chi\in\mathbb{T}^\infty$, cf. [15, Theorem 3]. Thus, we can apply the Birkhoff–Khinchin theorem: for almost every character $\chi'\in\mathbb{T}^\infty$, it holds that

$$\int_{\mathbb{T}^\infty} M_{\varphi_\chi,a}(w,\sigma,1)\,dm_\infty(\chi) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} M_{\varphi_{n^{-it}\chi'},a}(w,\sigma,1)\,dt$$

$$= \lim_{T\to\infty}\frac{\pi}{2T}\int_{-T}^{T}\sum_{\substack{s\in\varphi^{-1}_{n^{-it}\chi'}(\{w\})\\ |\operatorname{Im} s|<1\\ \operatorname{Re} s>\sigma}} (\operatorname{Re} s)^a\,dt = \lim_{T\to\infty}\frac{\pi}{2T}\int_{-T}^{T}\sum_{\substack{s\in\varphi^{-1}_{\chi'}(\{w\})\\ -1+t<\operatorname{Im} s<1+t\\ \operatorname{Re} s>\sigma}} (\operatorname{Re} s)^a\,dt.$$

Interchanging the order of integration and summation yields that

$$\int_{\mathbb{T}^\infty} M_{\varphi_\chi,a}(w,\sigma,1)\,dm_\infty(\chi) = \lim_{T\to\infty}\frac{\pi}{2T}\sum_{\substack{s\in\varphi^{-1}_{\chi'}(\{w\})\\ |\operatorname{Im} s|<1+T\\ \operatorname{Re} s>\sigma}} (\operatorname{Re} s)^a\int_{\operatorname{Im} s-1}^{\operatorname{Im} s+1} dt = M_{\varphi_{\chi'},a}(w,\sigma).$$

Applying Lemma 4.7 and then letting $\sigma\to 0^+$ with the monotone convergence theorem, we obtain (25). □

From this argument we are also able to give a partial solution to [10, Problem 1].

Theorem 4.9. *Let φ be a Dirichlet series such that $\sigma_u(\varphi)\le 0$ and $\varphi(+\infty)\neq w$. Then, for $a\ge 1$ and almost every $\chi\in\mathbb{T}^\infty$,*

$$M_{\varphi,a}(w) = \lim_{T\to\infty}\frac{\pi}{T}\sum_{\substack{s\in\varphi^{-1}_{\chi}(\{w\})\\ |\operatorname{Im} s|<T\\ \operatorname{Re} s>0}} (\operatorname{Re} s)^a\,. \tag{26}$$

Proof. By [10, Lemma 2.4], applied in conjunction with Lemma 3.4 for $a>1$, we have that $M_{\varphi_\chi,a}(w,0,T_0)\in L^\infty(\mathbb{T}^\infty)$ for all sufficiently large $T_0>0$. Applying the Birkhoff–Khinchin theorem as in the proof of Theorem 4.8, it holds for almost every $\chi'\in\mathbb{T}^\infty$ that

$$\int_{\mathbb{T}^\infty} M_{\varphi_\chi,a}(w,0,T_0)\,dm_\infty(\chi) = \lim_{T\to\infty}\frac{\pi}{T}\sum_{\substack{s\in\varphi^{-1}_{\chi'}(\{w\})\\ |\operatorname{Im} s|<T\\ \operatorname{Re} s>0}} (\operatorname{Re} s)^a\,.$$

However, exactly as in Theorem 4.8, we also have that

$$M_{\varphi_{\chi'},a}(w) = M_{\varphi,a}(w) = \int_{\mathbb{T}^\infty} M_{\varphi_\chi,a}(w,0,T_0)\,dm_\infty(\chi). \qquad \square$$

4.3. **The submean value property.** Let Ω be an open subset of $\mathbb{C}$. We say that a function $u : \Omega \to [-\infty, \infty)$ satisfies the submean value property if for every disk $\overline{D(w,r)} \subset \Omega$

$$u(w) \leq \frac{1}{|D(w,r)|} \int\limits_{D(w,r)} u(z) dA(z),$$

where $|D(w,r)| = \pi r^2$ is the area of the disk.

Shapiro [26, Section 4] proved that for every holomorphic self-map of the unit disk ϕ, the Nevanlinna counting function N_ϕ satisfies the submean value property in $\mathbb{D} \setminus \{\phi(0)\}$. Kellay and Lefevre [16, Lemma 2.3] proved that for $\alpha \in (0,1)$, the generalized Nevanlinna counting function $N_{\phi,\alpha}$ satisfies the submean value property in $\mathbb{D} \setminus \{\phi(0)\}$. In fact, this result follows directly from the submean value property of the classical Nevanlinna counting function and the following formula due to Aleman.

Theorem 4.10 ([1])**.** *Let $0 < \alpha < 1$ and $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic and non-constant. Then*

$$N_{\phi,\alpha}(w) = -\frac{1}{2} \int\limits_{\mathbb{D}} \Delta\omega_a(z) N_{\phi\circ\tau_z}(w)\, dA(z), \qquad w \in \mathbb{D},$$

where $\omega_a(z) = \left(1 - |z|^2\right)^\alpha$ and $\tau_z(w) = \frac{z-w}{1-\bar{z}w}$.

In the Hardy space case the mean counting function $M_{\varphi,1}$ satisfies the submean value property [10, Lemma 6.5], for every Dirichlet series φ that belongs to the Nevanlinna class. For periodic symbols $\varphi(s) = g(2^{-s})$, where g is a holomorphic self-map of the unit disk, we also know that $M_{\varphi,a}(w)$ satisfies the submean value property for all $a \in (0,1)$, by an application of Theorem 4.10.

This subsection is devoted to proving the following.

Theorem 4.11. *Let φ be a Dirichlet series with $\sigma_u(\varphi) \leq 0$. Then, for every positive $a > 0$, there exists a constant $C = C(a) > 0$ such that*

$$M_{\varphi,a}(w) \leq \frac{C}{|D(w,r)|} \int\limits_{D(w,r)} M_{\varphi,a}(z)\, dA(z), \tag{27}$$

for every disk $D(w,r)$ that does not contain $\varphi(+\infty)$.

For $a = 0$ the (unweighted) counting function does not satisfy (27), as can be seen from the following example.

Example 4.12. Let $\varphi_\rho(s) = g_\rho(2^{-s})$, where g_ρ is a Riemann map from the unit disk onto the domain

$$\Omega_\rho = D(0,r) \bigcup \{x + iy : x \in [0, 2r),\ y \in (-r/\rho, r/\rho)\},$$

where $r < \frac{1}{3}$ is fixed. Then for $w = \frac{119r}{60}$, we have that

$$\lim_{\rho\to\infty} \frac{1}{N_{g_\rho,0}(w)} \int\limits_{D(w,1-2r)} N_{g_\rho,0}(z)\,dA(z) = 0,$$

and thus by (24), that

$$\lim_{\rho\to\infty} \frac{1}{M_{\varphi_\rho,0}(w)} \int\limits_{D(w,1-2r)} M_{\varphi_\rho,0}(z)\,dA(z) = 0.$$

Therefore Theorem 4.11 could not be true for $a = 0$.

First we need the following lemma.

Lemma 4.13. *Suppose that Ω is a bounded subdomain of $\mathbb{C}$ and let $\phi : \mathbb{D} \to \Omega$ be holomorphic. Then the generalized Nevanlinna counting function $N_{\phi,\alpha}(w)$ satisfies the submean value property for $0 < \alpha < 1$.*

Proof. Let $R > 0$ be such that $\overline{\Omega} \subset D(0, R)$. Then, the function $\Phi = \frac{\phi}{R}$ is a holomorphic self-map of the unit disk. We observe that for every $w \in \Omega \setminus \{\phi(0)\}$,

$$N_{\phi,\alpha}(w) = N_{\Phi,\alpha}\left(\frac{w}{R}\right).$$

Let $\overline{D(w,r)} \subset \Omega \setminus \{\phi(0)\}$. Then, by the submean value property of $N_{\Phi,\alpha}$, we have that

$$N_{\phi,\alpha}(w) \le \frac{R^2}{|D(w,r)|} \int\limits_{D\left(\frac{w}{R},\frac{r}{R}\right)} N_{\Phi,\alpha}(z)\,dA(z) = \frac{1}{|D(w,r)|} \int\limits_{D(w,r)} N_{\phi,\alpha}(z)\,dA(z).$$

□

Proof of Theorem 4.11. First we consider the case $a \in (0, 1]$. In the notation of the proof of Lemma 4.4, let $\Theta_{\sigma,2T}(z) = \Theta_{2T}(z) + \sigma = 2T\Theta(z) + \sigma$ be the Riemann map from the unit disk onto the half-strip

$$S_{\sigma,2T} = \{z : \operatorname{Re} z > \sigma,\ |\operatorname{Im} z| < 2T\},$$

with $\Theta_{\sigma,2T}(0) = 2T + \sigma$ and $\Theta'_{\sigma,2T}(0) > 0$. We observe that

$$\Theta^{-1}_{\sigma,2T}(s) = \Theta^{-1}_{2T}(s-\sigma) = \Theta^{-1}\left(\frac{s-\sigma}{2T}\right).$$

By the Koebe quarter theorem, working as in Lemma 4.4,

$$1 - \left|\Theta^{-1}_{\sigma,2T}(s)\right|^2 \approx \frac{\operatorname{Re} s - \sigma}{2T},$$

whenever $\sigma < \operatorname{Re} s < T$ and $|\operatorname{Im} s| < T$.

For $T > 0$ so large that $\varphi(s) \notin D(w,r)$ for all $\operatorname{Re} s \geq T$, we have that

$$\frac{\pi}{T^a} \sum_{\substack{s\in\varphi^{-1}(\{z\}) \\ |\operatorname{Im} s|<T \\ 2\sigma<\operatorname{Re} s<\infty}} (\operatorname{Re} s)^a \leq 2^a \frac{\pi}{T^a} \sum_{\substack{s\in\varphi^{-1}(\{z\}) \\ |\operatorname{Im} s|<T \\ \sigma<\operatorname{Re} s<T}} (\operatorname{Re} s - \sigma)^a$$

$$\leq C \sum_{\substack{s\in\varphi^{-1}(\{z\}) \\ |\operatorname{Im} s|<T \\ \operatorname{Re} s>\sigma}} \left(1 - \left|\Theta^{-1}_{\sigma,2T}(s)\right|^2\right)^a \leq C N_{\varphi\circ\Theta_{\sigma,2T},a}(z),$$

for all $z \in D(w,r)$.

Conversely, again by the Koebe quarter theorem, there exists an absolute constant $C > 0$ such that

$$1 - \left|\Theta^{-1}(s)\right|^2 \leq C \operatorname{Re} s,$$

for $0 < \operatorname{Re} s < \frac{1}{2}$ and $|\operatorname{Im} s| < 1$. That is,

$$1 - \left|\Theta^{-1}_{\sigma,2T}(s)\right|^2 \leq C \frac{\operatorname{Re} s - \sigma}{2T},$$

whenever $\sigma < \operatorname{Re} s < T$ and $|\operatorname{Im} s| < 2T$. Thus, for $z \in D(w,r)$

$$N_{\varphi\circ\Theta_{\sigma,2T},a}(z) \leq C \sum_{\substack{s\in\varphi^{-1}(\{z\}) \\ |\operatorname{Im} s|<2T \\ \operatorname{Re} s>\sigma}} \left(1 - \left|\Theta^{-1}_{\sigma,2T}(s)\right|^2\right)^a \leq C \frac{\pi}{T^a} \sum_{\substack{s\in\varphi^{-1}(\{z\}) \\ |\operatorname{Im} s|<2T \\ \operatorname{Re} s>\sigma}} (\operatorname{Re} s)^a .$$

In summary, we have shown that for all sufficiently large $T > 0$ and for all $z \in D(w,r)$,

$$M_{\varphi,a}(z, 2\sigma, T) \leq C_1 T^{a-1} N_{\varphi\circ\Theta_{\sigma,2T},a}(z) \leq C_2 M_{\varphi,a}(z, \sigma, 2T).$$

Since $N_{\varphi\circ\Theta_{\sigma,2T},a}$ satisfies the submean value property by Lemma 4.13, we conclude that

$$M_{\varphi,a}(w, 2\sigma, T) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} M_{\varphi,a}(z, \sigma, 2T)\, dA(z).$$

By Theorem 1.1 and (22) we can apply the dominated convergence theorem to let $T \to \infty$, and then let $\sigma \to 0^+$ with the monotone convergence theorem, to obtain the desired property,

$$M_{\varphi,a}(w) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} M_{\varphi,a}(z)\, dA(z),$$

for a constant $C > 0$ that depends only on $a \in (0,1]$.

We assume now that $a > 1$. By Tonelli's theorem we have, for every $T > 0$, $\sigma > 0$, and $w \neq \varphi(+\infty)$, that

$$(a-1)\int_{\sigma}^{\infty} t^{a-2} M_{\varphi,1}(w,t,T)\,dt = (a-1)\frac{\pi}{T}\sum_{\substack{s\in\varphi^{-1}(\{w\})\\ |\operatorname{Im} s|<T\\ \sigma<\operatorname{Re} s<\infty}} \operatorname{Re} s \int_{\sigma}^{\operatorname{Re} s} t^{a-2}dt$$
$$= M_{\varphi,a}(w,\sigma,T) - \sigma^{a-1} M_{\varphi,1}(w,\sigma,T).$$

Applying the submean value property for $a = 1$, we thus find that

$$M_{\varphi,a}(w,2\sigma,T) \leq \frac{C}{|D(w,r)|}\int_{D(w,r)} M_{\varphi,a}(z,\sigma,2T)\,dA(z).$$

This concludes the proof, by letting $T \to \infty$ and then $\sigma \to 0^+$ in the same way as before. □

5. Composition Operators

5.1. **Reproducing kernels.** To obtain necessary conditions for a composition operator to be compact, we will make use of reproducing kernels. The reproducing kernel $k_{s,a}$ of $\mathcal{D}_a$ at a point $s \in \mathbb{C}_{\frac{1}{2}}$ is given by the equation

$$k_{s,a}(w) = 1 + \sum_{n\geq 2} \frac{1}{(\log(n))^a}\frac{1}{n^{\overline{s}+w}}, \qquad w \in \mathbb{C}_{1/2}.$$

For fixed $a < 1$, we have that

$$\|k_{s,a}\|_a^2 = 1 + \sum_{n\geq 2}\frac{1}{n^{2\operatorname{Re} s}(\log(n))^a} \approx \frac{1}{(2\operatorname{Re} s - 1)^{1-a}},$$

as $\operatorname{Re} s \to \frac{1}{2}^+$. We will also require slightly more detailed information about the behavior of the reproducing kernel, cf. [19, Lemma 3.1].

Lemma 5.1. *Let* $a \leq 1$ *and* $J_a(w) = \sum_{n\geq 1}\frac{(\log(n))^{1-a}}{n^w}$ *for* $\operatorname{Re} w > 1$. *Then there exists a holomorphic function* E_a *on* $\mathbb{C}_0$ *such that*

$$J_a(w) = \frac{\Gamma(2-a)}{(w-1)^{2-a}} + E_a(w), \qquad w \in \mathbb{C}_1.$$

Proof. We consider the summatory function

$$A(x) = \sum_{n\leq x} (\log(n))^{1-a}, \qquad x \geq 1.$$

Summation by parts yields that

$$J_a(w) = w \sum_{n\geq 1} A(n) \int_n^{n+1} x^{-w-1} dx = w \int_1^{\infty} A(x) x^{-w-1} dx. \tag{28}$$

We observe that

$$A(n) \leq \int_1^{n+1} (\log(t))^{1-a}\, dt \leq A(n+1).$$

Thus, with $g_a(x) := A(x) - \int_1^x (\log(t))^{1-a}\, dt$, we have that

$$\begin{aligned} |g_a(x)| &\leq \left| A(\lfloor x \rfloor) - \int_1^{\lfloor x \rfloor} (\log(t))^{1-a}\, dt \right| + \int_{\lfloor x \rfloor}^{x} (\log(t))^{1-a}\, dt \\ &\leq 2\,(\log(x+1))^{1-a}. \end{aligned}$$

Therefore the function $E_a(w) := w \int_1^{\infty} g_a(x) x^{-w-1} dx$ is holomorphic on $\mathbb{C}_0$, so that (28) can be written in the following form,

$$J_a(w) = w \int_1^{\infty} \int_1^{x} (\log(t))^{1-a}\, x^{-w-1} dt dx + E_a(w).$$

We can compute the integral by a change of variables,

$$\begin{aligned} w \int_1^{\infty} \int_1^{x} (\log(t))^{1-a}\, x^{-w-1} dt dx &= w \int_0^{\infty} e^{-wr} \int_0^{r} u^{1-a} e^{u} du dr \\ &= \int_0^{\infty} e^{-(w-1)u} u^{1-a} du = \frac{1}{(w-1)^{2-a}} \oint_{\Lambda_w} z^{1-a} e^{-z} dz, \end{aligned}$$

where $\Lambda_w = \{t(w-1) \,:\, t \geq 0\}$. Applying Cauchy's theorem to shift the path of integration to the positive real axis, we see that

$$w \int_1^{\infty} \int_1^{x} (\log(t))^{1-a}\, x^{-w-1} dt dx = \frac{\Gamma(2-a)}{(w-1)^{2-a}}.$$

□

5.2. **The Stanton formula.** The proof of the analogue of the Stanton formula for the weighted spaces $\mathcal{D}_a$, $a \leq 1$, relies on the work of [10] and a generalized version of the dominated convergence theorem.

Proof of Theorem 1.2. As explained on [10, p. 10], it is a consequence of Bohr's theorem that for any $f \in \mathcal{D}_a$, the abscissa of uniform convergence satisfies $\sigma_u(f \circ \varphi) \leq 0$. By making a non-injective change of variables in the Littlewood–Paley formula (1), we thus have that

$$\|C_\varphi(f)\|_a^2 = |f(\varphi(+\infty))|^2 + \frac{2^{1-a}}{\Gamma(2-a)\pi} \lim_{\sigma_0 \to 0^+} \lim_{T \to +\infty} \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 M_{\varphi,1-a}(w,\sigma_0,T) dA(w).$$

We extract the following equation from the proof of [10, Theorem 1.3],

$$\lim_{T \to +\infty} \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 \left(M_{\varphi,1}(w,\sigma_0,T) - M_{\varphi,1}(w,\sigma_1,T)\right) dA(w) = \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 \left(M_{\varphi,1}(w,\sigma_0) - M_{\varphi,1}(w,\sigma_1)\right) dA(w).$$

Since both $M_{\varphi,1-a}(w,\sigma,T)$ and $M_{\varphi,1}(w,\sigma,T)$ converge pointwise for $\sigma > 0$, and additionally, since for $\sigma_0 < \operatorname{Re} s < \sigma_1$ there is a constant $C > 0$ such that $(\operatorname{Re} s)^{1-a} \leq C \operatorname{Re} s$, the generalized dominated convergence theorem [11, Sec. 2, Ex. 20] thus yields that

$$\lim_{T \to +\infty} \frac{\pi}{T} \int_{-T}^{T} \int_{\sigma_0}^{\sigma_1} |(f \circ \varphi(\sigma + it))'|^2 \sigma^{1-a} d\sigma dt = \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 \left(M_{\varphi,1-a}(w,\sigma_0) - M_{\varphi,1-a}(w,\sigma_1)\right) dA(w).$$

By the monotone convergence theorem, letting $\sigma_0 \to 0^+$ and then $\sigma_1 \to +\infty$, we conclude that

$$\|C_\varphi(f)\|_a^2 = |f(\varphi(+\infty))|^2 + \frac{2^{1-a}}{\Gamma(2-a)\pi} \int_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 M_{\varphi,1-a}(w) dA(w). \qquad \square$$

When $a \leq 0$, Theorem 1.2, the boundedness of the composition operator $C_\varphi \colon \mathcal{D}_a \to \mathcal{D}_a$, and Theorem 4.11 allow us to deduce that the weighted counting function is finite.

Corollary 5.2. *Suppose that $\varphi \in \mathfrak{G}_0$ and that $a \leq 0$. Then $M_{\varphi,1-a}(w)$ is finite for every $w \neq \varphi(+\infty)$.*

5.3. **Compact composition operators on Bergman spaces of Dirichlet series.** To prove the sufficiency part of characterization we will make use of the following Littlewood-type inequality from [10, Theorem 1.1].

Theorem 5.3 ([10]). *For every $\varphi \in \mathfrak{G}_0$ the mean counting function $M_{\varphi,1}$ satisfies the estimate*

$$M_{\varphi,1}(w) \leq \log\left|\frac{\overline{w} + \varphi(+\infty) - 1}{w - \varphi(+\infty)}\right|,$$

where $w \in \mathbb{C}_{\frac{1}{2}} \setminus \{\varphi(+\infty)\}$.

The Schwarz lemma for Dirichlet series, Lemma 3.4, allows us to extend this estimate to the Bergman case.

Proposition 5.4. *Let $\varphi \in \mathfrak{G}_0$ and $a \geq 0$. Then for every $\delta > 0$ there exists a constant $C(\varphi, \delta, a) > 0$ such that*

$$M_{\varphi,1+a}(w) \leq C(\varphi, \delta, a)\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a} \frac{\operatorname{Re}\varphi(+\infty) - \frac{1}{2}}{|w - \varphi(+\infty)|^2}$$

for every $w \notin D(\varphi(+\infty), \delta)$.

Proof. Theorem 5.3 and the inequality $\log x \leq \frac{1}{2}(x^2 - 1)$, $x > 0$, together with a trivial computation, shows that for every $w \neq \varphi(+\infty)$,

$$M_{\varphi,1}(w) \leq 2\left(\operatorname{Re} w - \frac{1}{2}\right)\frac{\operatorname{Re}\varphi(+\infty) - \frac{1}{2}}{|w - \varphi(+\infty)|^2}.$$

Given $\delta > 0$, let $\sigma > 0$ be such that $\varphi(s) \in D(\varphi(+\infty), \delta)$ for every $\operatorname{Re} s > \sigma$. By Lemma 3.4, for $w \notin D(\varphi(+\infty), \delta)$, we thus have that

$$\begin{aligned}
M_{\varphi,1+a}(w) &= \lim_{\sigma_0 \to 0^+} \lim_{T \to +\infty} \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \sigma_0 < \operatorname{Re} s \leq \sigma}} (\operatorname{Re} s)^{a+1} \\
&\leq C(\varphi)\left(1 + \sigma^2\right)^a \left(\operatorname{Re} w - \frac{1}{2}\right)^a M_{\varphi,1}(w) \\
&\leq C(\varphi, \delta, a)\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a} \frac{\operatorname{Re}\varphi(+\infty) - \frac{1}{2}}{|w - \varphi(+\infty)|^2}.
\end{aligned}$$

□

Remark. By combining a similar argument with [10, Lem. 2.4], we find that

$$M_{\varphi,1+a}(w) = O\left(\left(\log\frac{1}{|w - \varphi(+\infty)|}\right)^{1+2a}\right)$$

as $w \to \varphi(+\infty)$. We do not expect the exponent in this inequality to be the best possible.

Before proceeding with the proof of Theorem 1.3, we require two simple lemmas, the first of which has the same proof as [10, Lemma 7.3].

Lemma 5.5. *Let $\nu \in \mathbb{C}_{\frac{1}{2}}$, $a \geq 0$, and $\delta > 0$. Then there exists a constant $C = C(\delta, a) > 0$ such that*

$$\int\limits_{\frac{1}{2}<\operatorname{Re} w<\theta} |f'(w)|^2 \frac{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a}}{|w-\nu|^{1+\delta}} dA(w) \leq \frac{C}{\left(\operatorname{Re}\nu - \theta\right)^{1+\delta}} \|f\|_{-a}^2$$

for every $f \in \mathcal{D}_{-a}$ and $\frac{1}{2} < \theta < \operatorname{Re}\nu$.

Lemma 5.6. *Let $\varphi \in \mathfrak{G}_0$ and $\{f_n\}_{n\geq 1}$ be a sequence in $\mathcal{D}_{-a}$, $a \geq 0$, that converges weakly to 0. Then, for every $\theta > \frac{1}{2}$,*

(29)
$$\lim_{n\to+\infty} \left(|f_n(\varphi(+\infty))|^2 + \frac{2^{1+a}}{\Gamma(2+a)\pi} \int\limits_{\operatorname{Re} w \geq \theta} |f_n'(w)|^2 M_{\varphi,1+a}(w) dA(w) \right) = 0.$$

Proof. The point evaluation at $\nu = \varphi(+\infty)$ is bounded, and thus

$$\lim_{n\to\infty} f_n(\varphi(+\infty)) = \lim_{n\to\infty} \langle f_n, k_{\nu,-a}\rangle_{\mathcal{D}_{-a}} = 0.$$

Next, applying Montel's theorem for $\mathcal{H}^\infty$ [4, Lemma 18], it is easy to see that the sequence of the derivatives $\{f_n'\}_{n\geq 1}$ converges uniformly to 0 in $\mathbb{C}_\theta$. The dominated convergence theorem, which can be applied in light of Lemma 3.3, thus gives us (29). □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We first assume that the operator C_φ is compact on $\mathcal{D}_{-a}$. Suppose $\{s_n\}_{n\geq 1} \subset \mathbb{C}_{\frac{1}{2}}$ is an arbitrary sequence such that $\operatorname{Re} s_n \to \frac{1}{2}$. We observe that the induced sequence of normalized reproducing kernels $\{K_{s_n,-a}\}_{n\geq 1}$ converges weakly to 0, as $n \to \infty$, and therefore

$$\lim_{n\to+\infty} \|C_\varphi(K_{s_n,-a})\|_{-a} = 0. \tag{30}$$

Without loss of generality we can assume that for every $n \geq 1$,

$$\operatorname{Re} s_n < \frac{2\operatorname{Re}\varphi(+\infty) + \frac{1}{2}}{3},$$

so that the disks $D(s_n, r_n)$, $r_n = \frac{\operatorname{Re} s_n - \frac{1}{2}}{2}$, do not contain $\varphi(+\infty)$.

By Lemma 5.1 there exists a constant $C = C(a) > 0$ such that

$$\left|K'_{s_n,a}(w)\right|^2 \geq C\left(\operatorname{Re} s_n - \frac{1}{2}\right)^{-a-3}$$

whenever $w \in D(s_n, r_n)$. Therefore

$$\begin{aligned}\|C_\varphi(K_{s_n,a})\|^2 &\geq C(a)\int_{\mathbb{C}_{\frac{1}{2}}} |K'_{s_n,a}(w)|^2 M_{\varphi,1+a}(w)dA(w)\\ &\geq C(a)\left(\operatorname{Re} s_n - \frac{1}{2}\right)^{-a-3}\int_{D(s_n,r_n)} M_{\varphi,1+a}(w)dA(w).\end{aligned}$$

The submean value property of Theorem 4.11 therefore yields that

$$M_{\varphi,a+1}(s_n)\left(\operatorname{Re} s_n - \frac{1}{2}\right)^{-a-1} \leq C(a)\left\|C_\varphi(K_{s_n,a})\right\|^2.$$

We conclude, by (30), that

$$\lim_{\operatorname{Re} w\to\frac{1}{2}} \frac{M_{\varphi,1+a}(w)}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a}} = 0.$$

Conversely, we suppose that (8) holds and argue as in the proof of [10, Theorem 1.4]. Let $\{f_n\}_{n\geq 1}$ be a sequence in $\mathcal{D}_{-a}$ that converges weakly to 0, such that $\|f_n\|_{-a} \leq 1$ for all $n \geq 1$. Fix $\delta \in (0,1)$. By Proposition 5.4 and (8), there exists for every $\epsilon > 0$ a θ, $\frac{1}{2} < \theta < \frac{\frac{1}{2}+\operatorname{Re}\varphi(+\infty)}{2}$, such that

$$M_{\varphi,1+a}(w) \leq \epsilon\frac{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a}}{|w-\varphi(+\infty)|^{1+\delta}}$$

for all $\frac{1}{2} < \operatorname{Re} w < \theta$. This and Lemma 5.5 gives us that

$$\begin{aligned}\int_{\frac{1}{2}<\operatorname{Re} w<\theta} |f_n'(w)|^2 M_{\varphi,1+a}(w)dA(w) &\leq \epsilon\int_{\frac{1}{2}<\operatorname{Re} w<\theta} |f_n'(w)|^2\frac{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a}}{|w-\varphi(+\infty)|^{1+\delta}}dA(w)\\ &\leq \epsilon C(\varphi,\delta,a).\end{aligned}$$

Combined with Lemma 5.6 and Theorem 1.2 we see that $\|C_\varphi(f_n)\|_{-a} \to 0$. Thus C_φ is compact on $\mathcal{D}_{-a}$. □

Remark. We can extract an alternative proof for the boundedness of the operator $C_\varphi\colon \mathcal{D}_{-a} \mapsto \mathcal{D}_{-a}$ from the second half of the proof of Theorem 1.3.

5.4. **Composition operators on Dirichlet-type spaces.** The proof of Theorem 1.4 is completely analogous to the proof of necessity in Theorem 1.3. We leave the details to the reader. The following example illustrates that the necessary condition (9) of Theorem 1.4 is not sufficient for the composition operator to be bounded on the Dirichlet space $\mathcal{D}_a$, $a \geq \frac{1}{2}$.

Example 5.7. For $\nu \in \mathbb{C}_{\frac{1}{2}}$, the function $g_\nu(z) = \frac{(\overline{\nu}-1)z+\nu}{1-z}$ maps the unit disk $\mathbb{D}$ onto $\mathbb{C}_{\frac{1}{2}}$. Consider the Dirichlet series $\varphi(s) = g_\nu(2^{-s})$. For $1/2 \leq a \leq 1$, by (24), we have that

$$M_{\varphi,1-a}(w) = \log(2)^{2-a} \log \left| \frac{1}{g_\nu^{-1}(w)} \right|^{1-a} = \log(2)^{2-a} \left(\log \left| \frac{w+\overline{\nu}-1}{w-\nu} \right| \right)^{1-a}.$$

Thus, φ satisfies (9). However, for sufficiently small $\varepsilon > 0$,

$$\left\| C_\varphi(2^{-s}) \right\|_a^2 \geq C(a) \int_{\frac{1}{2}}^{\frac{1}{2}+\epsilon} 2^{-2\sigma} \int_{-\infty}^{\infty} \left(\frac{(\sigma - \frac{1}{2})(\operatorname{Re}\nu - \frac{1}{2})}{|\sigma + \overline{\nu} - 1 + it|^2} \right)^{1-a} dt d\sigma = \infty.$$

We finish the article by noting that when $\operatorname{Im}\varphi$ is bounded, it is simple to establish the converse to Theorem 1.4. Note that if C_φ is bounded on $\mathcal{D}_a$, $0 < a \leq 1$, then Theorem 1.2 also implies that $M_{\varphi,1-a}$ is locally integrable at $w = \varphi(+\infty)$.

Theorem 5.8. *Let $0 < a < 1$, and suppose that $\varphi \in \mathfrak{G}_0$ has bounded imaginary part. If the counting function $M_{\varphi,1-a}$ is locally integrable and satisfies (9), then C_φ is bounded on $\mathcal{D}_a$. In addition, if we assume that φ satisfies (10), then C_φ is compact on $\mathcal{D}_a$.*

Proof. We present the proof of the first part of the theorem only. Let $\delta > 0$ be small. By the hypothesis of local integrability and Lemma 3.3, it holds that

$$\int_{D(\varphi(+\infty),\delta)} |f'(w)|^2 M_{\varphi,1-a}(w) dA(w) \leq C \left\| f \right\|_a^2.$$

Let $T = \sup|\operatorname{Im}\varphi|$. The local embedding theorem for the Hardy space $\mathcal{H}^2 = \mathcal{D}_0$ [13, Theorem 4.11] says that

$$\int_{-T}^{T} |g(1/2+it)|^2 \, dt \leq C\|g\|_0, \qquad g \in \mathcal{D}_0.$$

Applying this with (9), we find that

$$\begin{aligned}\int_{\mathbb{C}_{\frac{1}{2}}\setminus D(\varphi(+\infty),\delta)} |f'(w)|^2 M_{\varphi,1-a}(w)dA(w) &\leq C\int_0^\infty\int_{-T}^{T}\left|f'\left(\frac{1}{2}+\sigma+it\right)\right|^2 dt\,\sigma^{1-a}d\sigma\\ &\leq C\int_0^\infty\sum_{n\geq 2}\frac{|a_n|^2\log(n)^2}{n^{2\sigma}}\sigma^{1-a}d\sigma\\ &\leq C\,\|f\|_a^2.\end{aligned}$$

In light of Theorem 1.2, this shows that $C_\varphi\colon \mathcal{D}_a\to\mathcal{D}_a$ is bounded. □

From the proof it is clear that Theorem 5.8 also holds under milder decay assumptions on $M_{\varphi,1-a}(w)$ as $|\operatorname{Im} w|\to\infty$.

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Article 2: Composition operators and generalized primes

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COMPOSITION OPERATORS AND GENERALIZED PRIMES

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Abstract. We study composition operators on the Hardy space $\mathcal{H}^2$ of Dirichlet series with square summable coefficients. Our main result is a necessary condition, in terms of a Nevanlinna-type counting function, for a certain class of composition operators to be compact on $\mathcal{H}^2$. To do that we extend our notions to a Hardy space $\mathcal{H}^2_\Lambda$ of generalized Dirichlet series, induced in a natural way by a sequence of Beurling's primes.

1. Introduction

We consider the increasing sequence $\{p_n\}_{n\geq 1}$ of primes and an arbitrary increasing sequence $\{q_n\}_{n\geq 1}$ satisfying the following:

(i) The set $\{\log(p_n)\}_{n\geq 1} \cup \{\log(q_n)\}_{n\geq 1}$ is $\mathbb{Q}$-linear independent.
(ii) $\{q_n\}_{n\geq 1}$ is increasing, unbounded and each term is greater than 1.

For our purposes we will say that a real number $q > 1$ is a generalized prime if the set $\{\log(p_n)\}_{n\geq 1} \cup \{\log(q)\}$ is $\mathbb{Q}$-linear independent.

We will denote by $\mathbb{N}_{p,q} = \{\lambda_n\}_{n\geq 1}$ the increasing sequence of numbers that can be written as a (unique) finite product of terms of the set $\{p_n\}_{n\geq 0} \cup \{q_n\}_{n\geq 1}$, i.e.

$$\lambda = p^a q^b := p_1^{a_1} \cdot p_2^{a_2} \cdot \ \ldots \ \cdot q_1^{b_1} \cdot q_2^{b_2} \cdot \ldots \ .$$

A Dirichlet series is a function g of the form

$$g(s) = \sum_{n\geq 1} \frac{a_n}{n^s}, \qquad s = \sigma + it.$$

The set of numbers $\mathbb{N}_{p,q} = \{\lambda_n\}_{n\geq 1}$ corresponds to generalized Dirichlet series, meaning function of the form

$$f(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^s}, \qquad s = \sigma + it.$$

It is well-known that if a generalized Dirichlet series converges at a point $s_0 = \sigma_0 + it_0$, then it converges for every $s \in \mathbb{C}_{\sigma_0}$, where by $\mathbb{C}_\theta$ we denote the half-plane $\{z : \operatorname{Re} z \geq \theta\}$, $\theta \in \mathbb{R}$.

The first to introduce such systems of numbers was Beurling [6]. Studying general Beurling's systems gives us a better understanding of the exceptional system of the classical primes. We refer the interested reader to [12, 18, 13, 25] for results related to number theory, like the prime number theorem and the Riemann hypothesis. Our point of view is more operator theoretical, a system of Beurling's primes naturally induces a Hardy space of generalized Dirichlet series, with frequencies $\Lambda = \{\log \lambda_n\}_{n\geq 1}$ [17]. The idea behind using such systems is that the behavior of certain operators does not depend on the choice of primes.

The space $\mathcal{H}^2_\Lambda$ of generalized Dirichlet series with square summable coefficients is defined as

$$\mathcal{H}^2_\Lambda = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^s} : \|f\|^2_{\mathcal{H}^2_\Lambda} = \sum_{n\geq 1} |a_n|^2 < +\infty \right\}.$$

The Hardy space $\mathcal{H}^2$ [15] is the subspace of $\mathcal{H}^2_\Lambda$ containing all Dirichlet series,

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|^2_{\mathcal{H}^2} = \sum_{n\geq 1} |a_n|^2 < +\infty \right\}.$$

Gordon and Hedenmalm [14] determined the class $\mathfrak{G}$ of analytic functions $\psi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ that induce bounded composition operators $C_\psi(f) = f \circ \psi$ on $\mathcal{H}^2$. The class of symbols $\mathfrak{G}$ consists of all $\psi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer and φ is a Dirichlet series such that:

(i) If $c_0 = 0$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_{\frac{1}{2}}$.
(ii) If $c_0 \geq 1$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_0$ or $\varphi \equiv i\tau$ for some $\tau \in \mathbb{R}$.

Furthermore, a symbol $\psi \in \mathfrak{G}$ with $c_0 \geq 1$ induces a norm-one composition operator. We will use the notation $\mathfrak{G}_0$ and $\mathfrak{G}_{\geq 1}$ for the subclasses that satisfy (i) and (ii), respectively.

Defining the space $\mathcal{H}^2_\Lambda$, in some sense we added infinitely many prime-like numbers on the structure of $\mathcal{H}^2$. Our first aim is to prove that this does not have an effect on the behavior, meaning boundedness and compactness of a composition operator with symbol $\psi \in \mathfrak{G}_{\geq 1}$.

Theorem 1.1. *A symbol $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$, induces a bounded operator C_ψ on $\mathcal{H}^2_\Lambda$ with norm $\|C_\psi\| = 1$.*

Theorem 1.2. *Suppose $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$ and that C_ψ is a compact operator on $\mathcal{H}^2$. Then, C_ψ is compact on $\mathcal{H}^2_\Lambda$.*

In Section 4, we work on the compactness of composition operators on the Hardy space $\mathcal{H}^2$. O. F. Brevig and K–M. Perfekt [9] characterized compact composition operators on $\mathcal{H}^2$ with symbols in $\mathfrak{G}_0$. For symbols $\psi(s) = c_0 s + \varphi(s) \in$

$\mathfrak{G}_{\geq 1}$, F. Bayart [5] gave the following sufficient condition for the composition operator C_ψ to be compact

$$\lim_{\operatorname{Re} w \to 0^+} \frac{\mathcal{N}_\psi(w)}{\operatorname{Re} w} = 0, \tag{1}$$

where the Nevanlinna-type counting function $\mathcal{N}_\psi$ is defined as

$$\mathcal{N}_\psi(w) = \sum_{\substack{s \in \psi^{-1}(\{w\}) \\ \operatorname{Re} s > 0}} \operatorname{Re} s.$$

Conversely, Bailleul [1] for finitely valent symbols, where ϕ is supported on a finite set of primes and Brevig and Perfekt [8] under the assumption that ϕ is supported on single prime, proved that (1) is also necessary for the composition operator C_ψ to be compact. We say that a Dirichlet series ϕ is supported on a set of primes $\mathbb{P}$ if

$$\phi(s) = \sum_{\substack{p \mid n \\ p \in \mathbb{P}}} \frac{a_n}{n^s}.$$

Our next result is a necessary condition without any additional assumption on the symbol $\psi \in \mathfrak{G}_{\geq 1}$. Specifically, we replace pointwise convergence in (1) with $L^1(\mathbb{T}^\infty)$ convergence. This answers a question posed by F. Bayart [5, Question 3.6].

Theorem 1.3. *Suppose a symbol $\psi \in \mathfrak{G}_{\geq 1}$ induces a compact composition operator C_ψ on $\mathcal{H}^2$. Then*

$$\lim_{\operatorname{Re} w \to 0} \int_{\mathbb{T}^\infty} \frac{\mathcal{N}_{\psi_\chi}(w)}{\operatorname{Re} w} \, dm_\infty(\chi) = 0. \tag{2}$$

The classical technique for proving such necessary conditions for compactness goes through the submean value property of the associated counting function and the behavior of the reproducing kernels near the boundary, see for example [24]. In Section 4.1 we prove the weak submean value property for the average counting function $\int_{\mathbb{T}^\infty} N_{\psi_\chi}(w) \, dm_\infty(\chi)$, Theorem 4.6. Using geometric function theory results, related to the distortion and the boundary behavior of conformal maps, we will be able to transfer our notions to the disk setting. The weak submean value property will then follow by classical results due to Shapiro [24].

The main difficult in our setting is that reproducing kernels, $k_w(s) = \zeta(\overline{w} + s)$, on $\mathcal{H}^2$ are well defined only for points $w \in \mathbb{C}_{\frac{1}{2}}$.

F. Bayart have found an Example 5.2, of a non-compact and bounded composition operator with symbol in $\mathfrak{G}_{\geq 1}$, that satisfies (2). Theorem 1.3 gives us the $L^1(\mathbb{T}^\infty)$ convergence of the quantity $\mathcal{N}_\psi(w)(\operatorname{Re} w)^{-1} \to 0$. It may be a step closer, but the characterization of compact composition operators, with symbols in $\mathfrak{G}_{\geq 1}$, remains an open problem [4, 5].

For our purposes it was enough to study composition operators C_ψ with symbols in the class $\mathfrak{G}_{\geq 1}$. It would be interesting to have a characterization of the symbols that induce bounded composition operators on $\mathcal{H}^2_\Lambda$.

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Notation. Throughout the article, we will be using the convention that C denotes a positive constant which may vary from line to line. We will write that $C = C(\Omega)$ to indicate that the constant depends on a parameter Ω.

2. Background material

2.1. Composition operators in the disk setting.

The classical Hardy spaces H^2 consists of all holomorphic functions in the unit disk with square summable Taylor coefficients

$$H^2 = \left\{ f(s) = \sum_{n\geq 0} a_n z^n : \|f\|^2_{H^2} = \sum_{n\geq 0} |a_n|^2 < +\infty \right\}.$$

By the Littlewood subordination principle [20], every holomorphic self-map of the unit disk, ϕ, induces a bounded composition operator on H^2. J. Shapiro in his seminal paper [24] characterized the compact composition operator C_ϕ in terms of the Nevanlinna counting function

$$N_\phi(z) = \sum_{z_i \in \phi^{-1}(\{z\})} \log \frac{1}{|z_i|}, \qquad z \neq \phi(0).$$

The composition operator C_ϕ is compact on H^2 if and only if

$$\lim_{|z|\to 1^-} \frac{N_\phi(z)}{\log \frac{1}{|z|}} = 0. \tag{3}$$

In order to prove the above theorem, J. Shapiro makes use of the Littlewood–Paley and the Stanton formulae for the norm of a function $f \in H^2$ and its image

$C_\phi(f)$, respectively.

$$\|f\|_{H^2}^2 = |f(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z). \tag{4}$$

$$\|C_\phi(f)\|_{H^2}^2 = |f \circ \phi(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 N_\phi(z) \, dA(z), \tag{5}$$

where $dA(z) = dx\,dy$, $z = x + iy$, is the area measure.

2.2. **The infinite polytorus and vertical limits.** The infinite polytorus $\mathbb{T}^\infty$ is the countable infinite Cartesian product of copies of the unit circle $\mathbb{T}$,

$$\mathbb{T}^\infty = \{\chi = (\chi_1, \chi_2, \dots) : \chi_j \in \mathbb{T},\, j \geq 1\}.$$

As a compact abelian group with respect to coordinate-wise multiplication it posses a unique Haar measure m_∞ [23]. We can identify the measure m_∞ with the countable infinite product measure $m \times m \times \cdots$, where m is the normalized Lebesgue measure of the unit circle.

The $\mathbb{Q}$-linear independence of the set $\{\log(p_n)\}_{n\geq 1} \cup \{\log(q_n)\}_{n\geq 1}$, implies that $\mathbb{T}^\infty$ is isomorphic to the group of characters of $((\mathbb{Q}_{p,q})_+, \cdot)$, where $(\mathbb{Q}_{p,q})_+$ are the fractions of $(\mathbb{N}_{p,q}, \cdot)$. Given a point $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, the corresponding character $\chi : (\mathbb{Q}_{p,q})_+ \to \mathbb{T}$ is the completely multiplicative function on $\mathbb{N}_{p,q}$ such that $\chi(p_j) = \chi_{2j}$, $\chi(q_j) = \chi_{2j-1}$, extended to $(\mathbb{Q}_{p,q})_+$ through the relation $\chi(\lambda_n^{-1}) = \overline{\chi(\lambda_n)}$. From now on we identify a point $\chi = (\chi_1, \dots) \in \mathbb{T}^\infty$ with the corresponding character $\chi(\lambda_n)$.

Suppose $f(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^s}$ and $\chi(\lambda_n)$ is a character. The vertical limit function f_χ is defined as

$$f_\chi(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^s} \chi(\lambda_n).$$

Kronecker's theorem [7] justifies the name, since for any $\epsilon > 0$, there exists a sequence of real numbers $\{t_j\}_{j\geq 1}$ such that $f(s + it_j) \to f_\chi(s)$ uniformly on $\mathbb{C}_{\sigma_u(f)+\epsilon}$. The abscissae of convergence are defined likewise with the theory of

Dirichlet series.

$$\sigma_c(f) = \inf\left\{\sigma \in \mathbb{R} : f(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^\sigma} \quad \text{converges}\right\},$$

$$\sigma_a(f) = \inf\left\{\sigma \in \mathbb{R} : f(s) = \sum_{n\geq 1} \frac{|a_n|}{\lambda_n^\sigma} \quad \text{converges}\right\},$$

$$\sigma_u(f) = \inf\left\{\sigma \in \mathbb{R} : f(s) = \sum_{n\geq 1} \frac{a_n}{\lambda_n^s} \quad \text{converges uniformly in} \quad \mathbb{C}_\sigma\right\}.$$

For a symbol $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}$ we set

$$\psi_\chi(s) = c_0 s + \varphi_\chi(s),$$

and we observe that for every $\chi \in \mathbb{T}^\infty$ and $f \in \mathcal{H}^2_\Lambda$,

$$(C_\psi(f))_\chi = f_{\chi^{c_0}} \circ \psi_\chi. \tag{6}$$

Note that for a Dirichlet series $f \in \mathcal{H}^2 \subset \mathcal{H}^2_\Lambda$ the vertical limit function has the form

$$f_\chi(s) = \sum_{n\geq 1} \frac{a_n \chi(n)}{n^s},$$

where the character $\chi(n)$ exists in the dual group of $(\mathbb{Q}_+, \cdot)$, which is also isomorphic to $\mathbb{T}^\infty$.

2.3. **Hardy spaces on the infinite polytorus and the Bohr-lift.** We will make a short presentation of those topics and we refer to [11, 15, 22] for further information. For our purposes it would be enough to define only the spaces $H^2(\mathbb{T}^\infty)$ and $H^\infty(\mathbb{T}^\infty)$, but for expository reasons we will consider $1 \leq p \leq \infty$. Let us first recall what happens in one dimension. The Hardy space H^p, $1 \leq p \leq \infty$ consists of all functions in $L^p(\mathbb{T}, dm)$ with vanishing negative Fourier coefficients. The Fourier coefficient of a function $g \in L^1(\mathbb{T}^\infty)$ at a sequence $a = (a_1, a_2, \dots) \in \mathbb{Z}_0^\infty$ is defined as

$$\widehat{g}(a) = \int_{\mathbb{T}^\infty} g(z) z^{-a} \, dm_\infty(z),$$

where $\mathbb{Z}_0^\infty$ is the set of all compactly supported sequences with integer terms and

$$z^a = z_1^{a_1} \cdot z_2^{a_2} \cdot \dots,$$

is the multi-index notation. Similarly, we will denote by $\mathbb{N}_0^\infty$ the set of all compactly supported sequences of non-negative integers.

In a similar manner to the unit circle, the Hardy space $H^p(\mathbb{T}^\infty)$, $1 \le p \le \infty$ is defined as the subspace of $L^p(\mathbb{T}^\infty)$, which contains all the functions with vanishing Fourier coefficients at sequences in $\mathbb{Z}_0^\infty \setminus \mathbb{N}_0^\infty$.

By the definition of $\mathbb{N}_{p,q} = \{\lambda_n\}_{n\ge1}$, for every $n \in \mathbb{N}$ there exist two unique sequences in $\mathbb{N}_0^\infty$, $\gamma_p(\lambda_n)$ and $\gamma_q(\lambda_n)$, such that

$$\lambda_n = p^{\gamma_p(\lambda_n)} q^{\gamma_q(\lambda_n)}.$$

Starting with a generalized Dirichlet polynomial $f(s) = \sum\limits_{n\ge1} \frac{a_n}{\lambda_n^s}$ and mapping its prime term to a new variable, in the following way

$$p_i^{-s} \mapsto \chi_{2j}, \qquad q_i^{-s} \mapsto \chi_{2j-1}, \qquad i \in \mathbb{N},$$

we define the Bohr-lift of f as

$$B(f) := \sum_{n\ge1} a_n \chi(\lambda_n). \tag{7}$$

The Bohr-lift is an isometric isomorphism between $\mathcal{H}_\Lambda^2$ and $H^2(\mathbb{T}^\infty)$. It is also, a norm preserving homeomorphism from $\mathcal{H}^\infty$ into $H^\infty(\mathbb{T}^\infty)$, see for example [15]. By $\mathcal{H}^\infty$ we denote the space of all bounded Dirichlet series in $\mathbb{C}_0$, equipped with the uniform norm.

By Carleson theorem for $H^2(\mathbb{T}^\infty)$ [16, Theorem 1.5], for every $f \in \mathcal{H}_\Lambda^2$ the series $B(f) := \sum\limits_{n\ge1} a_n\chi(\lambda_n)$ converges for almost every character $\chi \in \mathbb{T}^\infty$.

Thus, for every $f \in \mathcal{H}_\Lambda^2$ and for almost every $\chi \in \mathbb{T}^\infty$, we have that

$$\sigma_c(f_\chi) \le 0.$$

2.4. **The ergodic theorem.** It is known [22, Section 2.2] that given a sequence $\{a_n\}_{n\ge1}$ of $\mathbb{Q}$-linear independent real numbers, then the Kronecker flow $\{T_t\}_{t\in\mathbb{R}}$ is ergodic, where

$$T_t(\chi_1, \chi_2, \dots) = (e^{-ita_1}\chi_1, e^{-ita_2}\chi_2, \dots). \tag{8}$$

By Birkhoff–Khinchin ergodic theorem, we obtain the following.

Theorem 2.1 ([10, 22]). *If $g \in L^1(\mathbb{T}^\infty)$, then for almost every $\chi_0 \in \mathbb{T}^\infty$,*

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} g\left(T_t\chi_0\right) dt = \int_{\mathbb{T}^\infty} g(\chi)\, dm_\infty(\chi). \tag{9}$$

If g is continuous, then (9) holds for every $\chi_0 \in \mathbb{T}^\infty$.

Consequently, for every $f \in \mathcal{H}_\Lambda^2$ and for almost every character $\chi_0(\lambda_n)$

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left|f_{\chi_0}(it)\right|^2 dt = \int_{\mathbb{T}^\infty} \left|B(f)\right|^2 dm_\infty\left(\chi\right). \tag{10}$$

2.5. **The Littlewood–Paley and Stanton's formulae.** As in [4, Lemma 2], for every $f \in \mathcal{H}^2_\Lambda$ and $T > 0$, we have the following Littlewood–Paley formula

$$\|f\|^2_{\mathcal{H}^2_\Lambda} = |f(+\infty)|^2 + \frac{2}{T} \int_{\mathbb{T}^\infty} \int_0^\infty \int_{-T}^{T} \left|f'_\chi(\sigma + it)\right|^2 \sigma \, dt \, d\sigma \, dm_\infty(\chi). \tag{11}$$

Suppose $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$, by a non-injective change of variables [24],

$$\|C_\psi(f)\|^2 = |f(+\infty)|^2 + \frac{2}{\pi} \int_{\mathbb{C}_0} \int_{\mathbb{T}^\infty} \left|f'_{\chi^{c_0}}(w)\right|^2 \mathcal{N}_{\psi_\chi}(w, T) \, dm_\infty(\chi) \, dA(w), \tag{12}$$

where the counting function $\mathcal{N}_{\psi_\chi}(w, T)$ is defined as

$$\mathcal{N}_{\psi_\chi}(w, T) = \frac{\pi}{T} \sum_{\substack{s \in \psi_\chi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \operatorname{Re} s > 0}} \operatorname{Re} s.$$

3. When do composition operators change adding primes?

In this section we will study the behavior of a composition operator C_ψ, $\psi \in \mathfrak{G}_{\geq 1}$ on the space $\mathcal{H}^2_\Lambda$. Our approach has been inspired by results in [8, Section 3].

Let $\mathbb{N}_q = \{b_k\}_{k \geq 1}$ be the increasing sequence of numbers that can be written as a finite product of terms of the set $\{q_n\}_{n \geq 1}$. We observe that $b_i^{-s}\mathcal{H}^2 \perp b_j^{-s}\mathcal{H}^2$, when $b_i \neq b_j$. Thus, $\mathcal{H}^2_\Lambda$ has the following orthogonal decomposition

$$\mathcal{H}^2_\Lambda = \bigoplus_{k \geq 1} b_k^{-s} \mathcal{H}^2. \tag{13}$$

Proposition 3.1. *Let $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$ and $k \in \mathbb{N}$. Then, the composition operator C_ψ maps $b_k^{-s}\mathcal{H}^2$ into $b_k^{-c_0 s}\mathcal{H}^2$ and its restriction $C_{\psi,k}$ to $b_k^{-s}\mathcal{H}^2$ has norm $\|C_\psi\| = 1$.*

Proof. First, we observe that

$$C_\psi(b_k^{-s} n^{-s}) = b_k^{-sc_0} b_k^{-\varphi(s)} C_\psi(n^{-s}) \in b_k^{-sc_0}\mathcal{H}^2.$$

The Bohr–lift respects multiplication [15], that is

$$B(mf) = B(m)\, B(f), \qquad m \in \mathcal{H}^\infty, \qquad f \in \mathcal{H}^2.$$

For every Dirichlet polynomial f, we have that

$$\left\|C_\psi(b_k^{-s} f)\right\|^2_{\mathcal{H}^2_\Lambda} = \left\|b_k^{-sc_0} b_k^{-\varphi(s)} C_\psi(f)\right\|^2_{\mathcal{H}^2_\Lambda} = \int_{\mathbb{T}^\infty} \left|B\left(b_k^{-\varphi(s)}\right) B\left(C_\psi(f)\right)\right|^2 dm_\infty(\chi).$$

As we have already discussed, the Bohr–lift is norm preserving between $\mathcal{H}^\infty$ and $H^\infty(\mathbb{T}^\infty)$. Therefore

$$\left\| B\left(b_k^{-\varphi(s)}\right)\right\|_{H^\infty(\mathbb{T}^\infty)} = \left\|b_k^{-\varphi(s)}\right\|_{\mathcal{H}^\infty} \leq 1.$$

Thus

$$\left\|C_\psi(b_k^{-s} f)\right\|_{\mathcal{H}^2_\Lambda} \leq \|C_\psi(f)\|_{\mathcal{H}^2} \leq \left\|b_k^{-s} f\right\|_{\mathcal{H}^2_\Lambda}. \qquad \square$$

Corollary 3.2. *Let $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$. Then, the induced composition operator on $\mathcal{H}^2_\Lambda$ has the following orthogonal decomposition*

$$C_\psi = \bigoplus_{k\geq 0} C_{\psi,k}. \tag{14}$$

Proof of Theorem 1.1. The proof follows directly from the Corollary 3.2. $\square$

Proof of Theorem 1.2. By (14), it is sufficient to prove the following:

(i) $\|C_{\psi,k}\| \to 0$.
(ii) $C_{\psi,k}$ is compact for every $k \geq 0$.

First we will prove (i). By Theorem 2.1, for every Dirichlet polynomial $f \in \mathcal{H}^2$ and for almost every $\chi_0 \in \mathbb{T}^\infty$, we have that

$$\begin{aligned}\left\|C_{\psi,k}(b_k^{-s} f)\right\|^2_{\mathcal{H}^2_\Lambda} &= \int_{\mathbb{T}^\infty} \left|B\left(b_k^{-\psi} C_\psi(f)\right)(\chi)\right|^2 dm_\infty(\chi) \\ &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} b_k^{-2\,\mathrm{Re}\left(\psi_{\chi_0}(it)\right)} \left|B\left(C_\psi(f)\right)(\lambda_n^{-it}\chi_0)\right|^2 dt.\end{aligned}$$

The symbol ψ has boundary values $\psi_\chi(it) = \lim_{\sigma\to 0^+} \psi_\chi(\sigma+it)$ for almost every $t \in \mathbb{R}$ and for almost every $\chi \in \mathbb{T}^\infty$. Furthermore, the vertical limit ψ_χ is in the class $\mathfrak{G}_{\geq 1}$, see [3, 8]. Thus

$$\begin{aligned}\left\|C_{\psi,k}(b_k^{-s} f)\right\|^2_{\mathcal{H}^2_\Lambda} &\leq \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \lfloor b_k\rfloor^{-2\,\mathrm{Re}\left(\psi_{\chi_0}(it)\right)} \left|B\left(C_\psi(f)\right)(\lambda_n^{-it}\chi_0)\right|^2 dt \\ &= \left\|C_\psi(\lfloor b_k\rfloor^{-s} f)\right\|^2_{\mathcal{H}^2}, \end{aligned} \tag{15}$$

where $\lfloor\cdot\rfloor$ is the floor function. We assume that (i) fails, without loss of generality there exist $\delta > 0$ and a sequence of Dirichlet polynomials $\{f_k\}_{k\geq 1}$ in the unit ball of $\mathcal{H}^2$ such that

$$\left\|C_\psi(\lfloor b_k\rfloor^{-s} f_k)\right\|_{\mathcal{H}^2} \geq \left\|C_\psi(b_k^{-s} f_k)\right\|_{\mathcal{H}^2_\Lambda} > \delta, \qquad k \in \mathbb{N}. \tag{16}$$

The sequence $\{\lfloor b_k \rfloor^{-s} f_k\}_{k\geq 1}$ converges weakly to 0 in $\mathcal{H}^2$ and as consequence

$$\lim_{n\to+\infty} \left\| C_\psi(\lfloor b_k \rfloor^{-s} f_k) \right\|_{\mathcal{H}^2} = 0.$$

This contradicts with (16). Therefore,

$$\|C_{\psi,k}\| \to 0.$$

For (ii), we consider an arbitrary sequence $\{b_k^{-s} g_j\}_{j\geq 1}$, which converges weakly to 0 and we observe that $\{g_j\}_{j\geq 1}$ is also weakly convergent to 0 in $\mathcal{H}^2$. This implies that

$$\left\| C_\psi(b_k^{-s} g_j) \right\|_{\mathcal{H}^2_\Lambda} \leq \|C_\psi(g_j)\|_{\mathcal{H}^2} \to 0.$$

Thus, $C_{\psi,k}$ is compact for every $k \geq 1$. □

4. Symbols that do not depend on a prime

4.1. **Submean value property.** Let Ω be an open subset of $\mathbb{C}$. We say that a function $u : \Omega \to [-\infty, \infty)$ satisfies the submean value property if for every disk $\overline{D(w,r)} \subset \Omega$

$$u(w) \leq \frac{1}{|D(w,r)|} \int_{D(w,r)} u(z)\, dA(z),$$

where $|D(w,r)| = \pi r^2$ is the area of the disk.

Shapiro [24, Section 4] proved that for every holomorphic self-map of the unit disk ϕ, the Nevanlinna counting function N_ϕ satisfies the submean value property in $\mathbb{D} \setminus \{\phi(0)\}$.

The aim of this subsection is to prove the weak submean value property Theorem 4.6 for the average $\int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi)$, where $\psi \in \mathfrak{G}_{\geq 1}$.

$$\int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(z)\, dm_\infty(\chi)\, dA(z). \tag{17}$$

Our argument will rely on a technique which has been developed in [9, 19] and allows us to transfer our notions in the disk setting.

We consider the unique conformal map F from the unit disk onto the rectangle

$$R = \{z : |\operatorname{Im} z| < 2,\ 0 < \operatorname{Re} z < 2\},$$

with $F(0) = 1$ and $F'(0) > 0$.

Lemma 4.1. *Suppose s is a point with $0 < \operatorname{Re} s < 1$ and $|\operatorname{Im} s| < 2$. Then*

$$1 - |F^{-1}(s)|^2 \leq C \operatorname{Re} s. \tag{18}$$

Furthermore, if $0 < \operatorname{Re} s < 1$ and $|\operatorname{Im} s| < 1$. Then

$$1 - |F^{-1}(s)|^2 \geq C \operatorname{Re} s. \tag{19}$$

Proof. By the Koebe quarter theorem [21, Corollary 1.4], for every $s \in R$, we have that

$$\frac{1-|F^{-1}(s)|^2}{4\left|\left(F^{-1}\right)'(s)\right|} \leq \operatorname{dist}(s, \partial R) \leq \frac{1-|F^{-1}(s)|^2}{\left|\left(F^{-1}\right)'(s)\right|}. \tag{20}$$

By the Caratheodory [21, Theorem 2.6] and the Kellogg-Warschawski theorems [21, Theorem 3.9], there exist absolute constants δ_1, $\delta_2 > 0$ such that for $0 < \operatorname{Re} s < 1$ and $|\operatorname{Im} s| < 1$

$$0 < \delta_1 < \left|\left(F^{-1}\right)'\right| < \delta_2 < \infty.$$

This and (20) imply (19).

Again, by the Kellogg-Warschawski theorem there exists $r > 0$ such that $\left|\left(F^{-1}\right)'(s)\right|$ is bounded in $\overline{R} \cap D\left(F^{-1}(\pm 2i), r\right)$. Now, (18) follows by the Koebe quarter theorem working as above. □

Lemma 4.2 ([19])**.** *Let Ω be a bounded subdomain of $\mathbb{C}$ and $\phi : \mathbb{D} \to \Omega$ be holomorphic. Then, the classical Nevanlinna counting function $N_\phi(w)$ satisfies the submean value property.*

Lemma 4.3. *Let $\psi \in \mathfrak{G}_{\geq 1}$. Then, there exists an absolute constant $C > 0$ such that*

$$\mathcal{N}_\psi(w, 1) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} \mathcal{N}_\psi(z, 2)\, dA(z), \tag{21}$$

for every disk $D(w,r) \subset \mathbb{C}_0 \setminus \mathbb{C}_{\frac{1}{2}}$.

Proof. Let $F_\sigma(z) = F(z) + \sigma$ be the Riemann map from the unit disk onto the rectangle

$$R_\sigma = \{z : \sigma < \operatorname{Re} z < 2 + \sigma,\ |\operatorname{Im} z| < 2\},$$

with $F_\sigma(0) = 1 + \sigma$ and $F'_\sigma(0) > 0$.

By Lemma 4.1

$$1 - |F_\sigma^{-1}(s)|^2 \leq C_1(\operatorname{Re} s - \sigma), \tag{22}$$

whenever $\sigma < \operatorname{Re} s < 1$, $|\operatorname{Im} s| < 2$ and

$$1 - |F_\sigma^{-1}(s)|^2 \geq C_2(\operatorname{Re} s - \sigma), \tag{23}$$

whenever $\sigma < \operatorname{Re} s < 1$, $|\operatorname{Im} s| < 1$.

We observe that $\operatorname{Re} s \leq \operatorname{Re} \psi(s)$ and that

$$1 - \left|F_\sigma^{-1}(s)\right|^2 \sim \log \frac{1}{\left|F_\sigma^{-1}(s)\right|}, \qquad s \in R_\sigma \cap \mathbb{C}_0 \setminus \mathbb{C}_{\frac{1}{2}}.$$

By (23), for $z \in D(w,r) \subset \mathbb{C}_0 \setminus \mathbb{C}_{\frac{1}{2}}$

$$\begin{aligned}\mathcal{N}_\psi(z,1,2\sigma) &:= \pi \sum_{\substack{s\in\psi^{-1}(\{z\})\\ |\operatorname{Im} s|<1\\ \operatorname{Re} s>2\sigma}} \operatorname{Re} s = \pi \sum_{\substack{s\in\psi^{-1}(\{z\})\\ |\operatorname{Im} s|<1\\ 2\sigma<\operatorname{Re} s<\frac{1}{2}}} \operatorname{Re} s \\ &\leq 2\pi \sum_{\substack{s\in\psi^{-1}(\{z\})\\ |\operatorname{Im} s|<1\\ \sigma<\operatorname{Re} s<\frac{1}{2}}} (\operatorname{Re} s - \sigma) \leq C \sum_{\substack{s\in\psi^{-1}(\{z\})\\ |\operatorname{Im} s|<2\\ \sigma<\operatorname{Re} s<\frac{1}{2}}} \left(1 - \left|F_\sigma^{-1}(s)\right|^2\right) \leq C N_{\psi\circ F_\sigma}(z).\end{aligned}$$

By (22), for $z \in D(w,r)$, we have that

$$N_{\psi\circ F_\sigma}(z) \leq C \sum_{\substack{s\in\psi^{-1}(\{z\})\\ |\operatorname{Im} s|<2\\ \sigma<\operatorname{Re} s<\frac{1}{2}}} \left(1 - \left|F_\sigma^{-1}(s)\right|^2\right) \leq C\frac{\pi}{2} \sum_{\substack{s\in\psi^{-1}(\{z\})\\ |\operatorname{Im} s|<2\\ \sigma<\operatorname{Re} s<\frac{1}{2}}} \operatorname{Re} s = C\mathcal{N}_\psi(w,2,\sigma).$$

By Lemma 4.2 the function $N_{\psi\circ F_\sigma}$ satisfies the submean value property and

$$\mathcal{N}_\psi(z,1,2\sigma) \leq C_1 N_{\psi\circ F_\sigma}(z) \leq C_2 \mathcal{N}_\psi(z,2,\sigma).$$

Therefore

$$\mathcal{N}_\psi(w,1,2\sigma) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} \mathcal{N}_\psi(z,2,\sigma)\, dA(z). \tag{24}$$

We can apply the monotone convergence theorem to let $\sigma \to 0^+$, yielding that

$$\mathcal{N}_\psi(w,1) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} \mathcal{N}_\psi(z,2)\, dA(z),$$

for an absolute constant $C > 0$. □

The following theorem will allow us to apply Theorem 2.1 for the counting function $\mathcal{N}_{\psi_\chi}(w)$.

Theorem 4.4. [4] *Let $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$. Then, for every $w \in \mathbb{C}_0$*

$$\mathcal{N}_\psi(w) \leq \frac{\operatorname{Re} w}{c_0}. \tag{25}$$

The following lemma will be of key importance for the proof of the weak submean value property, Theorem 4.6, and Theorem 4.7. Despite its technical and maybe serendipitous look, the idea behind Lemma 4.5 may be useful. See, for example the interchange of limits problem [9, Problem 1] and the partial solution of it [19, Theorem 4.9].

Lemma 4.5. *Let $\psi \in \mathfrak{G}_{\geq 1}$, $T > 0$ and $w \in \mathbb{C}_0$. Then*

$$\int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi) = \frac{1}{2\pi c_0} \int_{\mathbb{T}^\infty} \int_{-\infty}^{+\infty} \mathcal{N}_{\psi_\chi}(w+it, T)\, dt\, dm_\infty(\chi). \tag{26}$$

Proof. We observe that $s \in \psi_\chi^{-1}(\{w+it\})$ if and only if $s - \frac{it}{c_0} \in \psi_{\chi\chi_t}^{-1}(\{w\})$, where $\chi_t(n) = n^{-\frac{it}{c_0}}$ and $t \in \mathbb{R}$. Therefore

$$\int_{\mathbb{T}^\infty} \int_{-\infty}^{+\infty} \mathcal{N}_{\psi_\chi}(w+it, T)\, dt\, dm_\infty(\chi) = \frac{\pi}{T} \int_{\mathbb{T}^\infty} \int_{-\infty}^{+\infty} \sum_{\substack{s \in \psi_{\chi\chi_t}^{-1}(\{w\}) \\ -T - \frac{t}{c_0} < \operatorname{Im} s < T - \frac{t}{c_0} \\ \operatorname{Re} s > 0}} \operatorname{Re} s\, dt\, dm_\infty(\chi).$$

The Haar measure m_∞ is rotation invariant. This and Tonelli's theorem imply that

$$\begin{aligned}
\int_{\mathbb{T}^\infty} \int_{-\infty}^{+\infty} \mathcal{N}_{\psi_\chi}(w+it, T)\, dt\, dm_\infty(\chi) &= \frac{\pi}{T} \int_{\mathbb{T}^\infty} \int_{-\infty}^{+\infty} \sum_{\substack{s \in \psi_{\chi}^{-1}(\{w\}) \\ -T - \frac{t}{c_0} < \operatorname{Im} s < T - \frac{t}{c_0} \\ \operatorname{Re} s > 0}} \operatorname{Re} s\, dt\, dm_\infty(\chi) \\
&= \frac{\pi}{T} \int_{\mathbb{T}^\infty} \sum_{\substack{s \in \psi_{\chi}^{-1}(\{w\}) \\ \operatorname{Re} s > 0}} \operatorname{Re} s \int_{c_0(-\operatorname{Im} s - T)}^{c_0(T - \operatorname{Im} s)} dt\, dm_\infty(\chi) \\
&= 2c_0\pi \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi) \\
&= 2c_0\pi \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(\operatorname{Re} w)\, dm_\infty(\chi).
\end{aligned}$$

□

Theorem 4.6. *Let $\psi \in \mathfrak{G}_{\geq 1}$. Then, there exists an absolute constant $C > 0$ such that*

$$\int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(z)\, dm_\infty(\chi)\, dA(z), \tag{27}$$

for every disk $D(w,r) \subset \mathbb{C}_0 \setminus \mathbb{C}_{\frac{1}{2}}$.

Proof. By Lemma 4.3

$$\mathcal{N}_{\psi_\chi}(w+it, 1) \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} \mathcal{N}_{\psi_\chi}(z+it, 2)\, dA(z). \tag{28}$$

The proof follows by Lemma 4.5 integrating (28) with respect to $\chi \in \mathbb{T}^\infty$ and then $t \in \mathbb{R}$. □

4.2. **Necessity when omitting a prime.** This subsection is devoted to the following weaker version of Theorem 1.3.

Theorem 4.7. *Suppose $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}_{\geq 1}$ with $\varphi(s) = \sum_{p \nmid n} \frac{a_n}{n^s}$, where p is a prime number. If the induced composition operator is compact on $\mathcal{H}^2$, then*

$$\lim_{\operatorname{Re} w \to 0} \int_{\mathbb{T}^\infty} \frac{\mathcal{N}_{\psi_\chi}(w)}{\operatorname{Re} w} \, dm_\infty(\chi) = 0.$$

To prove the theorem we will use a variant of the classical technique, which gives necessary conditions for compactness. But, it is worth mentioning the ideas behind the steps of the proof. First, we considered a symbol that does not depend on a prime. We did that in order to separate the derivative of the reproducing kernel and the counting function under the integral sign on the infinite polytorus, (30). Then, using Lemma 4.5, the average counting function arises, (31). In the last step of the proof, we make use of the translation invariance of that average, to start with an integral on the real line and then introduce a proper disk to gain an additional power of $\operatorname{Re} s_n$ and derive the necessary inequality, (33).

Proof. Without loss of generality we assume that $p = 2$. Let $\{s_n\}_{n \geq 1} \subset \mathbb{C}_0$ be an arbitrary sequence such that $\operatorname{Re} s_n \to 0^+$. We observe that the induced sequence $\{K_{s_n,2}\}_{n \geq 1}$ of normalized reproducing kernels associated to the prime 2, defined as

$$K_{s_n,2}(s) = \sqrt{1 - 4^{-\operatorname{Re} s_n}} \sum_{j \geq 0} \frac{1}{2^{j(\overline{s_n} + s)}}$$

converges weakly to 0, as $n \to \infty$. Therefore

$$\lim_{n \to +\infty} \|C_\psi(K_{s_n,2})\| = 0. \tag{29}$$

Stanton's formula (12) yields to the following

$$\begin{aligned} \|C_\psi(K_{s_n,2})\|^2 &\geq C \int_{\mathbb{C}_0} \int_{\mathbb{T}^\infty} \left|(K_{s_n,2})'_{\chi^{c_0}}(w)\right|^2 \mathcal{N}_{\psi_\chi}(w,1) \, dm_\infty(\chi) \, dA(w) \\ &\geq C \int_{\mathbb{C}_0} \int_{\mathbb{T}} \left|(K_{s_n,2})'_{\chi_1^{c_0}}(w)\right|^2 \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w,1) \, dm_\infty(\chi) \, dA(w). \end{aligned} \tag{30}$$

By Parseval's formula and Lemma 4.5

$$\|C_\psi(K_{s_n,2})\|^2 \geq C(1-4^{-\operatorname{Re} s_n}) \int_{\mathbb{C}_0} \sum_{j\geq 1} j^2 4^{-j(\operatorname{Re} s_n+\operatorname{Re} s)} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w,1)\, dm_\infty(\chi)$$

$$\geq C(1-4^{-\operatorname{Re} s_n}) \int_0^{+\infty} \sum_{j\geq 1} j^2 4^{-j(\operatorname{Re} s_n+\sigma)} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(\sigma)\, dm_\infty(\chi)\, d\sigma$$

$$\geq C \int_{\frac{\operatorname{Re} s_n}{2}}^{\frac{3\operatorname{Re} s_n}{2}} \frac{1-4^{-\operatorname{Re} s_n}}{\left(1-4^{-(\operatorname{Re} s_n+\sigma)}\right)^3} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(\sigma)\, dm_\infty(\chi)\, d\sigma.$$

For sufficiently large $n \in \mathbb{N}$, we have that for every $t \in \mathbb{R}$

$$\|C_\psi(K_{s_n,2})\|^2 \geq C\,(\operatorname{Re} s_n)^{-2} \int_{\frac{\operatorname{Re} s_n}{2}}^{\frac{3\operatorname{Re} s_n}{2}} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(\sigma)\, dm_\infty(\chi)\, d\sigma \tag{31}$$

$$= C\,(\operatorname{Re} s_n)^{-2} \int_{\frac{\operatorname{Re} s_n}{2}}^{\frac{3\operatorname{Re} s_n}{2}} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(\sigma+it)\, dm_\infty(\chi)\, d\sigma.$$

Thus

$$\|C_\psi(K_{s_n,2})\|^2$$

$$\geq C\,(\operatorname{Re} s_n)^{-3} \int_{\frac{\operatorname{Re} s_n}{2}}^{\frac{3\operatorname{Re} s_n}{2}} \int_{-\sqrt{(\frac{\operatorname{Re} s_n}{2})^2-(\sigma-\operatorname{Re} s_n)^2}}^{\sqrt{(\frac{\operatorname{Re} s_n}{2})^2-(\sigma-\operatorname{Re} s_n)^2}} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(\sigma+it)\, dm_\infty(\chi)\, dt\, d\sigma$$

$$\geq C(\operatorname{Re} s_n)^{-1} \frac{1}{\left|D(\operatorname{Re} s_n, \frac{\operatorname{Re} s_n}{2})\right|} \int_{D(\operatorname{Re} s_n, \frac{\operatorname{Re} s_n}{2})} \int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi)\, dA(w). \tag{32}$$

By Theorem 4.6,

$$\|C_\psi(K_{s_n,2})\|^2 \geq C \frac{\int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(s_n)\, dm_\infty(\chi)}{\operatorname{Re} s_n}. \tag{33}$$

The proof now follows from the equation (29),

$$\lim_{\operatorname{Re} w\to 0} \frac{\int_{\mathbb{T}^\infty} \mathcal{N}_{\psi_\chi}(w)\, dm_\infty(\chi)}{\operatorname{Re} w} = 0. \qquad \square$$

5. Proof of Theorem 1.3

The proof of Theorem 1.3 follows from Theorem 4.7 and Theorem 1.2. More specifically, let C_ψ be a compact composition operator with symbol $\psi \in \mathfrak{G}_{\geq 1}$. Then, by Theorem 1.2 C_ψ is compact on $\mathcal{H}^2_\Lambda$. Theorem 4.7 remains true if we substitute $\mathcal{H}^2$ with $\mathcal{H}^2_\Lambda$. The symbol $\psi \in \mathfrak{G}_{\geq 1}$ does not depend on the generalized prime q_1 and thus

$$\lim_{\operatorname{Re} w \to 0} \int_{\mathbb{T}^\infty} \frac{\mathcal{N}_{\psi_\chi}(w)}{\operatorname{Re} w} \, dm_\infty(\chi) = 0.$$

Note that in order to prove Theorem 1.3 it would be sufficient to add just one generalized prime, for example $q = \pi$.

Now we present the counterexample of F. Bayart. We will make use of the following characterization of compact composition operators with symbols $\psi(s) = c_0 s + \psi(s) \in \mathfrak{G}_{\geq 1}$, where ϕ is a Dirichlet polynomial.

Theorem 5.1 ([2])**.** *Let $\psi(s) = c_0 s + \phi(s) \in \mathfrak{G}_{\geq 1}$, where ϕ is a Dirichlet polynomial. Then, the induced composition operator C_ψ is compact on $\mathcal{H}^2$ if and only if the symbol has restricted range.*

We say that a symbol $\psi \in \mathfrak{G}$ has unrestricted range if

$$\inf_{s \in \mathbb{C}_0} \operatorname{Re} \phi(s) = \begin{cases} \frac{1}{2} & \text{if } c_0 = 0, \\ 0 & \text{if } c_0 \geq 1. \end{cases} \tag{34}$$

It is worth mentioning that a symbol with restricted range always induces a compact composition operator on $\mathcal{H}^2$, [3, Theorem 20, Theorem 21].

Example 5.2. We consider the symbol $\psi(s) = 1 + s - 2^{-s}$. The composition operator C_ψ is not compact on $\mathcal{H}^2$, since $\psi(s) = 1 + s - 2^{-s}$ has unrestricted range. The vertical translations of it have the following form

$$\psi_z(s) = 1 + s - z2^{-s}, \qquad z \in \mathbb{T}.$$

The function $h(z) := \inf_{\operatorname{Re} s > 0} |\psi_z(s)|$ is continuous on z and vanishes only for $z = 1$. For $\varepsilon > 0$ sufficiently small there exists a constant $C(\varepsilon) > 0$ such that

$$h(z) \geq 2\varepsilon, \qquad |z - 1| > C(\varepsilon)$$

and $C(\varepsilon) \to 0^+$, as $\varepsilon \to 0^+$. Applying Theorem 4.4, we have that

$$\int_{\mathbb{T}} \frac{N_{\psi_z}(\varepsilon)}{\varepsilon} \, dz = \int_{1-C(\varepsilon)}^{1+C(\varepsilon)} \frac{N_{\psi_z}(\varepsilon)}{\varepsilon} \, dz \leq 2C(\varepsilon).$$

Thus

$$\lim_{\operatorname{Re} w \to 0} \int_{\mathbb{T}} \frac{N_{\psi_z}(w)}{\operatorname{Re} w} \, dz = 0.$$

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Article 3: Schatten class composition operators on the Hardy space of Dirichlet series and a comparison-type principle

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SCHATTEN CLASS COMPOSITION OPERATORS ON THE HARDY SPACE OF DIRICHLET SERIES AND A COMPARISON-TYPE PRINCIPLE

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Abstract. We give necessary and sufficient conditions for a composition operator with Dirichlet series symbol to belong to the Schatten classes S_p of the Hardy space $\mathcal{H}^2$ of Dirichlet series. For $p \geq 2$, these conditions lead to a characterization for the subclass of symbols with bounded imaginary parts. Finally, we establish a comparison-type principle for composition operators. Applying our techniques in conjunction with classical geometric function theory methods, we prove the analogue of the polygonal compactness theorem for $\mathcal{H}^2$ and we give examples of bounded composition operators with Dirichlet series symbols on $\mathcal{H}^p$, $p > 0$.

1. Introduction

The Hardy space $\mathcal{H}^2$ of Dirichlet series, which was first systematically studied by H. Hedenmalm, P. Lindqvist, and K. Seip [13], is defined as

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|^2_{\mathcal{H}^2} = \sum_{n\geq 1} |a_n|^2 < \infty \right\}.$$

Gordon and Hedenmalm [12] determined the class $\mathfrak{G}$ of symbols which generate bounded composition operators on the Hardy space $\mathcal{H}^2$. The Gordon–Hedenmalm class $\mathfrak{G}$ consists of all functions $\psi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer, called the characteristic of ψ, and φ is a Dirichlet series such that:

(i) If $c_0 = 0$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_{\frac{1}{2}}$.
(ii) If $c_0 \geq 1$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_0$ or $\varphi \equiv i\tau$ for some $\tau \in \mathbb{R}$.

We denote by $\mathbb{C}_\theta$, $\theta \in \mathbb{R}$ the half-plane $\{s : \operatorname{Re} s > \theta\}$. We will also use the notation $\mathfrak{G}_0$ and $\mathfrak{G}_{\geq 1}$ for the subclasses of symbols that satisfy i and ii, respectively.

In this paper, we are mostly interested in the case $\psi = \varphi \in \mathfrak{G}_0$. In that context, the compact operators $C_\varphi \colon \mathcal{H}^2 \to \mathcal{H}^2$ were characterized only very recently in

[9], in terms of the behavior of the mean counting function

$$M_\varphi(w) = \lim_{\sigma\to 0^+}\lim_{T\to\infty}\frac{\pi}{T}\sum_{\substack{s\in\varphi^{-1}(\{w\})\\ |\operatorname{Im} s|<T\\ \sigma<\operatorname{Re} s<\infty}}\operatorname{Re} s, \qquad w\in\mathbb{C}_{\frac12}\backslash\{\varphi(+\infty)\}.$$

It turns out that C_φ is compact if and only if

$$\lim_{\operatorname{Re} w\to\frac12^+}\frac{M_\varphi(w)}{\operatorname{Re} w-\frac12}=0. \tag{1}$$

The next step would be to characterize symbols $\varphi\in\mathfrak{G}_0$ such that C_φ belongs to the Schatten class S_p, $p>0$. In the disk setting D. H. Luecking and K. Zhu [17] proved that a composition operator C_ϕ on the Hardy space $H^2(\mathbb{D})$ belongs to the Schatten class S_p, $p>0$ if and only if

$$\int_{\mathbb{D}}\frac{(N_\phi(z))^{\frac p2}}{(1-|z|^2)^{\frac p2+2}}\,dA(z)<+\infty, \tag{2}$$

where $dA(z)=dx\,dy$, $z=x+iy$ is the area measure, ϕ is a holomorphic self-map of the unit disk and N_ϕ is the associated Nevanlinna counting function [24].

Our first main result is that the analogue characterization holds in the Dirichlet series setting provided the symbol has bounded imaginary part.

Theorem 1.1. *Suppose that the symbol $\varphi\in\mathfrak{G}_0$ has bounded imaginary part and that $p\geq 1$. Then, the composition operator C_φ belongs to the class S_{2p} if and only if φ satisfies the condition*

$$\int_{\mathbb{C}_{\frac12}}\frac{(M_\varphi(w))^p}{\left(\operatorname{Re} w-\frac12\right)^{p+2}}\,dA(w)<+\infty. \tag{3}$$

For $p>0$ the above condition remains necessary and if $p\geq 2$, then it is necessary for all symbols in $\mathfrak{G}_0$.

When $p=1$, namely if we want to know if C_φ is Hilbert-Schmidt, things are easier and Hilbert-Schmidt composition operators with symbols φ in $\mathfrak{G}_0$ have already been characterized in [9]. This is equivalent to saying that

$$\int_{\mathbb{C}_{\frac12}}\zeta''(2\operatorname{Re}(w))M_\varphi(w)dA(w)<+\infty.$$

We generalize this characterization for $C_\varphi\in S_{2m}$, $m\in\mathbb{N}$.

Theorem 1.2. *Let $\varphi \in \mathfrak{G}_0$ and $m \in \mathbb{N}$. Then C_φ belongs to S_{2m} if and only if*

(4)
$$\int_{\mathbb{C}_{\frac{1}{2}}} \cdots \int_{\mathbb{C}_{\frac{1}{2}}} \zeta''(\overline{w_1}+w_2)\cdots\zeta''(\overline{w_{m-1}}+w_m)\zeta''(\overline{w_m}+w_1)\prod_{k=1}^{m} M_\varphi(w_k)\,dA(w_k) < \infty.$$

Our next result is a comparison-type principle. Using the Lindelöf principle for Green's functions, we will be able to establish geometric conditions on the symbols that imply that the associated composition operator is compact or belongs to S_p. To our knowledge this is the first example of a technique that gives geometric conditions that apply to all symbols $\varphi \in \mathfrak{G}_0$. To exemplify this, we focus on symbols whose range is contained in angular sectors.

Theorem 1.3. *Let $\varphi \in \mathfrak{G}_0$ and assume that the range of the symbol lies inside an angular sector of the form $\varphi(\mathbb{C}_0) \subset \left\{ s \in \mathbb{C}_{\frac{1}{2}} : \left|\arg(s) - \frac{1}{2}\right| < \frac{\pi}{2\alpha} \right\}$, for some $\alpha > 1$. Then C_φ is compact. If we further assume that $\alpha \leq 2$, then $C_\varphi \in S_{2p}$ for any $p > 1/(\alpha - 1)$.*

We can strengthen the previous result proving that if the range of the symbol meets the boundary inside a finite union of angular sectors, then the induced composition operator is compact.

This geometric method also applies to continuity and compactness of composition operators acting on the other Hardy spaces of Dirichlet series $\mathcal{H}^p$, $p \neq 2$. Recall that for $0 < p < \infty$, the Hardy space $\mathcal{H}^p$ of Dirichlet series is defined as the completion of Dirichlet polynomials under the Besicovitch norm (or quasi-norm if $0 < p < 1$)

$$\|P\|_{\mathcal{H}^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt \right)^{\frac{1}{p}}.$$

The characterization of bounded composition operators with Dirichlet series symbols on $\mathcal{H}^p$, $p \notin 2\mathbb{N}$ is an open and challenging question. The condition $\varphi \in \mathfrak{G}_0$ is necessary but not sufficient, [21] and there is no known sufficient conditions which may be applied to a large class of symbols whose range touches the boundary of $\mathbb{C}_0$. We provide such a sufficient condition under the assumption that the range of the symbol is contained in an angular sector.

Theorem 1.4. *Let $k \in \mathbb{N}$ and $p \in (0, 2k]$. If the symbol $\varphi \in \mathfrak{G}_0$ maps the right half-plane into an angular sector of the form $\Omega = \left\{ s \in \mathbb{C}_{\frac{1}{2}} : \left|\arg\left(s - \frac{1}{2}\right)\right| < \frac{p\pi}{4k} \right\}$, then C_φ is bounded on $\mathcal{H}^p$. Furthermore, if $\max(1, p) < q \leq 2k$, then the composition operator is compact on $\mathcal{H}^q$.*

In the last section we briefly discuss the case of Bergman spaces of Dirichlet series as well as some results on Carleson measures.

Notation. Throughout the article, we will be using the convention that C denotes a positive constant which may vary from line to line. We will write that $C = C(x)$ to indicate that the constant depends on a parameter x. If f, g are two real functions defined on the same set Ω, we will write $f \ll g$ if there exists $C > 0$ such that for all $x \in \Omega$, $f(x) \leq Cg(x)$ and $f \sim g$ if $f \ll g$ and $g \ll f$.

2. Background material

2.1. **Schatten classes.** A compact operator T acting on a separable Hilbert space H can be written as

$$T(x) = \sum_{n\geq 1} s_n \langle x, e_n\rangle h_n, \qquad x \in H, \tag{5}$$

where $\{s_n\}_{n\geq 1}$ is the sequence of singular values and $\{e_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ are orthonormal sequences. In case T is self-adjoint, then $e_n = \pm h_n$ for all $n \geq 1$.

For $p > 0$ the S_p Schatten class of compact operators T on H is defined as

$$S_p = S_p(H) = \left\{ T \in \mathfrak{K}(H) : \|T\|_{S_p}^p := \sum_{n\geq 1} s_n^p < \infty \right\}.$$

Equivalently (see [14]), for $p \geq 1$, a bounded linear operator $T \in \mathfrak{L}(H)$ belongs to S_p if and only if there exists a positive constant C such that

$$\sum_n |\langle Te_n, e_n\rangle|^p \leq C$$

for every orthonormal basis (e_n). Furthermore, if T is self-adjoint,

$$\|T\|_{S_p}^p = \sup \sum_n |\langle Te_n, e_n\rangle|^p$$

the supremum being taken over all orthonormal basis of H.

For a compact and positive operator T on H we define the power T^p, $p > 0$, as

$$T^p(x) = \sum_{n\geq 1} s_n^p \langle x, e_n\rangle e_n, \qquad x \in H.$$

When $p = n \in \mathbb{N}$, the operator T^n is the n-th iteration of T. We observe that $T \in S_p$ if and only if $T^p \in S_1$. It T is not assumed to be positive, we can still use that $T \in S_p$ iff $|T|^p = (T^*T)^{p/2} \in S_1$ iff $T^*T \in S_{p/2}$.

For a unit vector $x \in H$ and a positive operator T, applying Hölder's inequality in (5) we obtain the following inequality

$$\langle T^p(x), x\rangle \geq (\langle T(x), x\rangle)^p, \qquad p \geq 1. \tag{6}$$

For $0 < p \leq 1$ the inequality is reversed.

2.2. **The infinite polytorus and vertical limits.** The infinite polytorus $\mathbb{T}^\infty$ is defined as the (countable) infinite Cartesian product of copies of the unit circle $\mathbb{T}$,

$$\mathbb{T}^\infty = \{\chi = (\chi_1, \chi_2, \dots) : \chi_j \in \mathbb{T},\, j \geq 1\}.$$

It is a compact abelian group with respect to coordinatewise multiplication. We can identify the Haar measure m_∞ of the infinite polytorus with the countable infinite product measure $m \times m \times \cdots$, where m is the normalized Lebesgue measure of the unit circle.

$\mathbb{T}^\infty$ is isomorphic to the group of characters of $(\mathbb{Q}_+, \cdot)$. Given a point $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, the corresponding character $\chi : \mathbb{Q}_+ \to \mathbb{T}$ is the completely multiplicative function on $\mathbb{N}$ such that $\chi(p_j) = \chi_j$, where $\{p_j\}_{j\geq 1}$ is the increasing sequence of primes, extended to $\mathbb{Q}_+$ through the relation $\chi(n^{-1}) = \overline{\chi(n)}$.

Suppose $f(s) = \sum\limits_{n\geq 1} \frac{a_n}{n^s}$ is a Dirichlet series and χ is a character. The vertical limit function f_χ is defined as

$$f_\chi(s) = \sum_{n\geq 1} \frac{a_n \chi(n)}{n^s}.$$

By Kronecker's theorem [6], for any $\epsilon > 0$, there exists a sequence of real numbers $\{t_j\}_{j\geq 1}$ such that $f(s+it_j) \to f_\chi(s)$ uniformly on $\mathbb{C}_{\sigma_u(f)+\epsilon}$, where $\sigma_u(f)$ denotes the abscissa of uniform convergence of f.

If $f \in \mathcal{H}^2$, then for almost every character $\chi \in \mathbb{T}^\infty$ the vertical limit function f_χ converges in the right half-plane and has boundary values $f_\chi(it) = \lim\limits_{\sigma\to 0^+} f_\chi(\sigma + it)$ for almost every $t \in \mathbb{R}$, [2]. For $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}$, we set

$$\psi_\chi(s) = c_0 s + \varphi_\chi(s),$$

then for every $\chi \in \mathbb{T}^\infty$ we have that

$$(C_\psi(f))_\chi = f_{\chi^{c_0}} \circ \psi_\chi. \tag{7}$$

The symbol ψ has boundary values $\psi_\chi(it) = \lim\limits_{\sigma\to 0^+} \psi_\chi(\sigma + it)$ for almost every $\chi \in \mathbb{T}^\infty$ and for almost every $t \in \mathbb{R}$.

2.3. **Composition operators on $\mathcal{H}^2$.** O. F. Brevig and K–M. Perfekt [9, Theorem 1.3.] proved the following analogue of Stanton's formula for the Hardy spaces of Dirichlet series:

$$\|C_\varphi(f)\|^2_{\mathcal{H}^2} = |f(\varphi(+\infty))|^2 + \frac{2}{\pi} \int\limits_{\mathbb{C}_{\frac{1}{2}}} |f'(w)|^2 M_\varphi(w)\, dA(w), \tag{8}$$

where $\varphi \in \mathfrak{G}_0$ and $f \in \mathcal{H}^2$. By $f(+\infty)$ we denote the first coefficient a_1 of the Dirichlet series $f(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$. We apply the polarization identity in (8) yielding to

$$\langle C_\varphi(f), C_\varphi(g)\rangle = f(\varphi(+\infty))\overline{g(\varphi(+\infty))} + \frac{2}{\pi}\int_{\mathbb{C}_{\frac{1}{2}}} f'(w)\overline{g'(w)}M_\varphi(w)\,dA(w). \tag{9}$$

We will make use of two properties of the counting function $M_\varphi(w)$ proved in [9], the submean value property and a Littlewood type inequality. Those respectively are

$$M_\varphi(w) \leq \frac{1}{|D(w,r)|}\int_{D(w,r)} M_\varphi(z)\,dA(z), \tag{10}$$

for every disk $D(w,r) \subset \mathbb{C}_{\frac{1}{2}}$ that does not contain $\varphi(+\infty)$, and

$$M_\varphi(w) \leq \log\left|\frac{\varphi(+\infty)+\overline{w}-1}{\varphi(+\infty)-w}\right|, \qquad w \in \mathbb{C}_{\frac{1}{2}}\backslash\{\varphi(+\infty)\}. \tag{11}$$

In Subsection 4 we will prove a weaker version of the Littlewood inequality (11) but sufficient for our purpose. The standard technique to prove such inequalities goes through regularity results for conformal maps [9, 15]. We shall use the following consequence of (11) (see [9, Lemma 2.3]): for $\sigma_\infty > \operatorname{Re}(\varphi(+\infty))$, there exists $C > 0$ such that, for all $w \in \mathbb{C}_{\sigma_\infty}$,

$$M_\varphi(w) \leq C\frac{\operatorname{Re}(w)-\frac{1}{2}}{\big(1+|\operatorname{Im}(w)|\big)^2}. \tag{12}$$

2.4. **Carleson measures.** Let H be a Hilbert space of holomorphic functions on a domain Ω. A Borel measure μ in Ω is called a Carleson measure for H if there exists a constant $C > 0$ such that, for all $f \in H$,

$$\int_\Omega |f(w)|^2 d\mu(w) \leq C\,\|f\|_H^2\,.$$

We will denote by $C(\mu, H)$ or simply by $C(\mu)$ the infimum of such constants. For instance, Carleson measures on the Hardy space $H^2(\mathbb{C}_{\frac{1}{2}})$, that consist of holomorphic function f in $\mathbb{C}_{\frac{1}{2}}$ equipped with norm

$$\|f\|^2_{H^2(\mathbb{C}_{\frac{1}{2}})} := \sup_{\sigma>\frac{1}{2}}\int_{\mathbb{R}} |f(\sigma+it)|^2\,dt < \infty,$$

are characterized as follows.

Theorem 2.1 ([10]). *A Borel measure μ on $\mathbb{C}_{\frac{1}{2}}$ is a Carleson measure for $H^2(\mathbb{C}_{\frac{1}{2}})$ if and only if there exists a constant $D > 0$ such that for every square Q with one side I on the line $\{\operatorname{Re} s = \frac{1}{2}\}$*

$$\mu(Q) \leq D\,|I|\,. \tag{13}$$

Moreover, there exist two absolute constants $a, b > 0$ such that, for all Borel measures μ on $\mathbb{C}_{\frac{1}{2}}$, denoting by $D(\mu)$ the infimum of the constants D verifying (13), then $aD(\mu) \leq C(\mu) \leq bD(\mu)$.

2.5. **Weighted Hilbert spaces of Dirichlet series.** Our main strategy (inspired by [17]) to obtain the membership of C_φ to S_{2p} is to derive it from the membership to S_p of an associated Toeplitz operator defined on another space of Dirichlet series. We now introduce this class of spaces. For $a \leq 1$ we define the weighted Hilbert space $\mathcal{D}_a$ of Dirichlet series as

$$\mathcal{D}_a = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|_a^2 = |a_1|^2 + \sum_{n\geq 2} |a_n|^2 (\log n)^a < \infty \right\}.$$

The reproducing kernel $k_{w,-a}$, $a \geq 0$ of $(\mathcal{D}_{-a})_0$ (space mod constants) at a point $w \in \mathbb{C}_{\frac{1}{2}}$ is given by

$$k_{w,-a}(s) = \sum_{n>1} \frac{(\log n)^a}{n^{s+\overline{w}}} = \frac{\Gamma(1+a)}{(\overline{w}+s-1)^{1+a}} + E_a(s+\overline{w}), \qquad s \in \mathbb{C}_{\frac{1}{2}}, \tag{14}$$

where $E_a(\cdot)$ is a holomorphic function on $\mathbb{C}_0$, [15, Lemma 5.1]. Observe that

$$\|k_{w,-a}\|_{-a}^2 = k_{w,-a}(w) \sim_{\operatorname{Re}(w)\to\frac{1}{2}} \frac{1}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^{a+1}}. \tag{15}$$

Note that for $a = -2$, we have $k_{w,-a}(s) = \zeta''(s+\overline{w})$ and $\zeta''(w) \sim_{\operatorname{Re}(w)\to\frac{1}{2}} \big(\operatorname{Re}(w) - 1/2\big)^{-3}$. Recall also that for any orthonormal basis (f_n) of $(\mathcal{D}_{-a})_0$, for any $w \in \mathbb{C}_{\frac{1}{2}}$,

$$\sum_n |f_n(w)|^2 = k_{w,-a}(w). \tag{16}$$

The local embedding theorem, [13], states that there exists an absolute constant $C > 0$ such that for every $f \in \mathcal{H}^2$

$$\frac{1}{2T}\int_{-T}^{T} \left| f\left(\frac{1}{2} + it\right)\right|^2 dt \leq C\,\|f\|_{\mathcal{H}^2}^2\,, \qquad T > 0. \tag{17}$$

A direct application of (17) is that for every $f(s)=\sum_{n\geq 2}\frac{a_n}{n^s}\in(\mathcal{D}_{-a})_0$, $a>0$, we have the following embedding

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\int_{\frac12}^{\infty}|f(\sigma+it)|^2(\sigma-\frac12)^{a-1}\,d\sigma\,dt &\leq C\sum_{n\geq 2}|a_n|^2\int_{\frac12}^{\infty}n^{1-2\sigma}(\sigma-\frac12)^{a-1}\,d\sigma\,dt\\ &\leq C\frac{\Gamma(a)}{2^a}\sum_{n\geq 2}|a_n|^2(\log n)^{-a}\\ &\leq C\frac{\Gamma(a)}{2^a}\|f\|_{-a}^2, \qquad (18)\end{aligned}$$

where C is the constant appearing in (17). In particular, if B is a subset of $\mathbb{C}_{\frac12}$ with bounded imaginary part, then $\mathbf{1}_B\left(\mathrm{Re}(\cdot)-\frac12\right)^{a-1}dA$ is a Carleson measure for $(\mathcal{D}_{-a})_0$. More generally, if $\kappa:[0,+\infty)\to[0,+\infty)$ is integrable, bounded and decreasing then $\kappa(|\mathrm{Im}(\cdot)|)\left(\mathrm{Re}(\cdot)-\frac12)\right)^{a-1}dA$ is a Carleson measure for $(\mathcal{D}_{-a})_0$.

The differentiation operator $D(f)=f'$ is an isometry between $\mathcal{H}^2_0$ and $(\mathcal{D}_{-2})_0$. By (9) the composition operator C_φ belongs to $S_{2p}(\mathcal{H}^2)$, $p>0$ if and only if the operator $\left(D\circ C_\varphi\circ D^{-1}\right)^*D\circ C_\varphi\circ D^{-1}$ exists in $S_p((\mathcal{D}_{-2})_0)$ if and only if the operator $T_\varphi:(\mathcal{D}_{-2})_0\to(\mathcal{D}_{-2})_0$ defined as

$$\langle T_\varphi(f),g\rangle=\int_{\mathbb{C}_{\frac12}}f(w)\overline{g(w)}M_\varphi(w)\,dA(w) \qquad (19)$$

belongs to $S_p((\mathcal{D}_{-2})_0)$.

3. Composition operators belonging to Schatten classes

3.1. Schatten class and Carleson measures. We shall divide the proof of Theorem 1.1 into several parts. We first handle the case $p\geq 1$ in a more general context by giving a necessary and a sufficient condition for C_φ to belong to S_{2p}. Both conditions involve M_φ and Carleson measures. At this stage, we do not assume anything on the imaginary part of φ.

Theorem 3.1. *Let $p\geq 1$ and $\varphi\in\mathfrak{G}_0$.*

a) Assume that $C_\varphi\in S_{2p}$ and let μ be a Carleson measure for $(\mathcal{D}_{-2})_0$. Then

$$\int_{\mathbb{C}_{\frac12}}\frac{(M_\varphi(w))^p\zeta''(2\,\mathrm{Re}(w))}{\left(\mathrm{Re}(w)-\frac12\right)^p}d\mu(w)<+\infty.$$

b) Assume that there exists $\rho : \varphi(\mathbb{C}_0) \to (0,+\infty)$ such that ρdA is a Carleson measure for $(\mathcal{D}_{-2})_0$ and that

$$\int_{\varphi(\mathbb{C}_0)} \frac{(M_\varphi(w))^p \zeta''(2\operatorname{Re}(w))}{\rho(w)^{p-1}} dA(w) < +\infty. \tag{20}$$

Then $C_\varphi \in S_{2p}$.

Proof. We start by proving a). Let I_μ be the inclusion operator from $(\mathcal{D}_{-2})_0$ into $L^2(\mathbb{C}_{\frac{1}{2}}, \mu)$ which is bounded since μ is Carleson. Moreover, assuming $C_\varphi \in S_{2p}$ or, equivalently, $T_\varphi \in S_p$, we get by the ideal property of Schatten classes that $I_\mu \circ T_\varphi^{p/2} \in S_2$. Let (f_n) be any orthonormal sequence of $(\mathcal{D}_{-2})_0$. One can write

$$\begin{aligned}
\infty > \|I_\mu \circ T_\varphi^{p/2}\|_{S_2}^2 &= \sum_{n\geq 1} \|T_\varphi^{p/2}(f_n)\|^2_{L^2(\mu)} \\
&= \sum_{n\geq 1} \int_{\mathbb{C}_{\frac{1}{2}}} \left|\langle T_\varphi^{\frac{p}{2}}(f_n), k_{w,-2}\rangle\right|^2 d\mu(w) \\
&= \int_{\mathbb{C}_{\frac{1}{2}}} \left\|T_\varphi^{\frac{p}{2}}(k_{w,-2})\right\|^2_{(\mathcal{D}_{-2})_0} d\mu(w) \\
&= \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^p(k_{w,-2}), k_{w,-2}\rangle\, d\mu(w) \\
&\geq \int_{\mathbb{C}_{\frac{1}{2}}} \left(\langle T_\varphi(K_{w,-2}), K_{w,-2}\rangle\right)^p \|k_{w,-2}\|^2\, d\mu(w)
\end{aligned}$$

by (6), where $K_{w,-2}$ is the normalized reproducing kernel of $(\mathcal{D}_{-2})_0$ at w. Observe that the exchange of integral and sum is justified by Tonelli's theorem. Fix $\sigma_\infty > \operatorname{Re}\varphi(+\infty)$. By (19),

$$\begin{aligned}
\|I_\mu \circ T_\varphi^{p/2}\|_{S_2}^2 &\geq \int_{\mathbb{C}_{\frac{1}{2}}} \left(\int_{\mathbb{C}_{\frac{1}{2}}} |K_{w,-2}(z)|^2 M_\varphi(z) dA(z)\right)^p \|k_{w,-2}\|^2\, d\mu(w) \\
&\geq \int_{\mathbb{C}_{\frac{1}{2}} \setminus \mathbb{C}_{\sigma_\infty}} \left(\int_{D\left(w, \frac{1}{2}\left(\operatorname{Re}(w) - \frac{1}{2}\right)\right)} |K_{w,-2}(z)|^2 M_\varphi(z) dA(z)\right)^p \|k_{w,-2}\|^2\, d\mu(w).
\end{aligned}$$

By (14) one can estimate the behaviour of $K_{w,-2}(z)$ in the disc $D\left(w, \frac{\operatorname{Re} w - \frac{1}{2}}{2}\right)$, whenever $\operatorname{Re} w \leq \sigma_\infty$ to obtain

$$\int_{D\left(w,\frac{1}{2}\left(\operatorname{Re}(w)-\frac{1}{2}\right)\right)} |K_{w,-2}(z)|^2 M_\varphi(z) dA(z) \gg \int_{D\left(w,\frac{1}{2}\left(\operatorname{Re}(w)-\frac{1}{2}\right)\right)} \frac{M_\varphi(z)}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^3} dA(z)$$
$$\gg \frac{M_\varphi(w)}{\operatorname{Re}(w)-\frac{1}{2}}$$

where the last inequality follows from the submean value property of the mean counting function (10). Taking into account the value of $\|k_{w,-2}\|$ we get

$$\int_{\mathbb{C}_{\frac{1}{2}} \setminus \mathbb{C}_{\sigma_\infty}} \frac{(M_\varphi(w))^p \zeta''(2\operatorname{Re}(w))}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^p} d\mu(w) < +\infty.$$

Finally, (12) yields

$$\int_{\mathbb{C}_{\sigma_\infty}} \frac{(M_\varphi(w))^p \zeta''(2\operatorname{Re}(w))}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^p} d\mu(w)$$
$$\ll \int_{\mathbb{C}_{\sigma_\infty}} \zeta''(2\operatorname{Re}(w)) d\mu(w) \ll \int_{\mathbb{C}_{\sigma_\infty}} \left|2^{-w}\right|^2 d\mu(w) < +\infty.$$

Conversely, assume that (20) holds and let q be the conjugate exponent of p. For $p = 1$ the validity of (20) follows from the Hilbert-Schmidt characterization. Thus, we will also assume that $p > 1$. Let (f_n) be any orthonormal basis of $(\mathcal{D}_{-2})_0$. Then

$$\sum_{n\geq 1} \langle T_\varphi(f_n), f_n\rangle^p = \sum_{n\geq 1} \left(\int_{\mathbb{C}_{\frac{1}{2}}} |f_n(w)|^2 M_\varphi(w) dA(w) \right)^p$$
$$= \sum_{n\geq 1} \left(\int_{\varphi(\mathbb{C}_0)} \frac{|f_n(w)|^{2/p} M_\varphi(w)}{\rho(w)^{1/q}} |f_n(w)|^{2/q} \rho(w)^{1/q} dA(w) \right)^p$$
$$\leq \sum_{n\geq 1} \left(\int_{\varphi(\mathbb{C}_0)} \frac{|f_n(w)|^2 M_\varphi(w)^p}{\rho(w)^{p/q}} dA(w) \right) \left(\int_{\varphi(\mathbb{C}_0)} |f_n(w)|^2 \rho(w) dA(w) \right)^{p/q}.$$

Since ρdA is a Carleson measure and since $\sum_n |f_n(w)|^2 = k_{w,-2}(w)$ for any orthonormal basis of $(\mathcal{D}_{-2})_0$, we get

$$\sum_{n\geq 1}\langle T_\varphi(f_n), f_n\rangle^p \ll \int\limits_{\varphi(\mathbb{C}_0)} \frac{(M_\varphi(w))^p k_{w,-2}(w)}{\rho(w)^{p-1}} dA(w)$$
$$\ll \int\limits_{\varphi(\mathbb{C}_0)} \frac{(M_\varphi(w))^p \zeta''(2\operatorname{Re}(w))}{\rho(w)^{p-1}} dA(w).$$

Hence, T_φ belongs to S_p. □

In view of the above theorem, the ideal case would be to choose a function $\rho : \varphi(\mathbb{C}_0) \to (0, +\infty)$ such that ρdA is a Carleson measure for $(\mathcal{D}_{-2})_0$ and

$$\frac{1}{\rho(w)^{p-1}} = \frac{\rho(w)}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^p},\ w \in \varphi(\mathbb{C}_0).$$

This yields to $\rho(w) = \operatorname{Re}(w) - \frac{1}{2}$. Now if φ has bounded imaginary part, then the embedding inequality implies that $\mathbf{1}_{\varphi(\mathbb{C}_0)}\left(\operatorname{Re}(w) - \frac{1}{2}\right) dA$ is a Carleson measure for $(\mathcal{D}_{-2})_0$. This gives the way to the case $p > 1$ of Theorem 1.1.

Corollary 3.2. *Let $p \geq 1$ and $\varphi \in \mathfrak{G}_0$ with bounded imaginary part. Then C_φ belongs to S_{2p} if and only if*

$$\int\limits_{\mathbb{C}_{\frac{1}{2}}} \frac{(M_\varphi(w))^p}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{p+2}} dA(w) < +\infty.$$

Proof. Our discussion actually shows that, under the assumptions of the corollary, $C_\varphi \in S_{2p}$ if and only if

$$\int\limits_{\mathbb{C}_{\frac{1}{2}}} \frac{(M_\varphi(w))^p}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{p-1}} \zeta''(2\operatorname{Re}(w))\, dA(w) < +\infty.$$

It remains to show that this is equivalent to (3). Let $\sigma_\infty = 2\operatorname{Re}\varphi(+\infty)$. Then for $w \in \mathbb{C}_{\frac{1}{2}} \backslash \mathbb{C}_{\sigma_\infty}$,

$$\frac{1}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^3} \ll \zeta''(2\operatorname{Re}(w)) \ll \frac{1}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^3}.$$

We may conclude if we are able to prove that for any $\varphi \in \mathfrak{G}_0$,

$$\int\limits_{\mathbb{C}_{\sigma_\infty}} \frac{M_\varphi(w)^p}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^{p-1}} \zeta''(2\operatorname{Re}(w)) dA(w) < +\infty,$$

and

$$\int_{\mathbb{C}_{\sigma_\infty}} \frac{M_\varphi(w)^p}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^{p+2}} dA(w) < +\infty.$$

Both of these properties follow from (12). ☐

When φ does not have bounded imaginary part, there are still interesting Carleson measures for $(\mathcal{D}_{-2})_0$, for instance $\left(\operatorname{Re}(w)-\frac{1}{2}\right)/(1+|\operatorname{Im}(w)|)^a dA$ for any $a>1$. This yields to the following result.

Corollary 3.3. *Let $p>1$, let $\varphi\in\mathfrak{G}_0$ and let $a>1$.*

a) If C_φ belongs to S_{2p}, then

$$\int_{\mathbb{C}_{\frac{1}{2}}} \frac{M_\varphi(w)^p}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^{p+2}(1+|\operatorname{Im}(w)|)^a} dA(w) < +\infty.$$

b) Assume that

$$\int_{\mathbb{C}_{\frac{1}{2}}} \frac{M_\varphi(w)^p}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^{p+2}}(1+|\operatorname{Im}(w)|)^{a(p-1)} dA(w) < +\infty.$$

Then $C_\varphi\in S_{2p}$.

Proof. This follows from Theorem 3.1 with $\rho(w)=\left(\operatorname{Re}(w)-\frac{1}{2}\right)/(1+|\operatorname{Im}(w)|)^a$ and $d\mu=\rho dA$. Again we can replace everywhere $\zeta''(2\operatorname{Re}(w))$ by $\left(\operatorname{Re}(w)-\frac{1}{2}\right)^{-3}$ since for a),

$$\int_{\mathbb{C}_{\sigma_\infty}} \frac{M_\varphi(w)^p}{\left(\operatorname{Re}(w)-\frac{1}{2}\right)^{p+2}(1+|\operatorname{Im}(w)|)^a} dA(w) < +\infty$$

and for b), $\zeta''(2\operatorname{Re}(w)) \ll \left(\operatorname{Re}(w)-\frac{1}{2}\right)^{-3}$ is valid throughout $\mathbb{C}_{\frac{1}{2}}$. ☐

We now prove that (3) remains necessary for $p\geq 2$ without any assumption on φ.

Theorem 3.4. *Let $p\geq 2$ and $\varphi\in\mathfrak{G}_0$. Assume that $C_\varphi\in S_{2p}$. Then*

$$\int_{\mathbb{C}_{\frac{1}{2}}} \frac{(M_\varphi(w))^p}{\left(\operatorname{Re} w-\frac{1}{2}\right)^{p+2}} dA(w) < +\infty.$$

Proof. For the positive operator T_φ belonging to S_p, denoting by (f_n) an orthonormal basis of eigenvectors of T_φ,

$$\begin{aligned}\infty > \left\|T_\varphi^p\right\|_{S_1} &= \sum_{n\geq 1}\langle T_\varphi^p(f_n), f_n\rangle \\ &= \sum_{n\geq 1}\int_{\mathbb{C}_{\frac{1}{2}}} T_\varphi^{p-1}(f_n)(w)\overline{f_n(w)}M_\varphi(w)\,dA(w) \\ &= \sum_{n\geq 1}\int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^{p-1}(f_n), k_{w,-2}\rangle\overline{f_n(w)}M_\varphi(w)\,dA(w).\end{aligned}$$

The quantity under the integral sign is nonnegative since

$$\langle T_\varphi^{p-1}(f_n), k_{w,-2}\rangle\overline{f_n(w)} = s_n^{p-1}\langle f_n, k_{w,-2}\rangle\overline{f_n(w)} = s_n^{p-1}|f_n(w)|^2.$$

An application of Tonelli's theorem yields

$$\begin{aligned}\infty > \left\|T_\varphi^p\right\|_{S_1} &= \int_{\mathbb{C}_{\frac{1}{2}}} \sum_{n\geq 1}\langle f_n, T_\varphi^{p-1}(k_{w,-2})\rangle\overline{f_n(w)}M_\varphi(w)\,dA(w) \\ &= \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^{p-1}(k_{w,-2}), k_{w,-2}\rangle M_\varphi(w)\,dA(w) \\ &= \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^{p-1}(K_{w,-2}), K_{w,-2}\rangle \left\|k_{w,-2}\right\|^2 M_\varphi(w)\,dA(w).\end{aligned}$$

We now use (6)

$$\begin{aligned}\infty > \left\|T_\varphi^p\right\|_{S_1} &\geq \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi(K_{w,-2}), K_{w,-2}\rangle^{p-1} \left\|k_{w,-2}\right\|^2 M_\varphi(w)\,dA(w) \\ &\geq \int_{\mathbb{C}_{\frac{1}{2}}} \left(\int_{\mathbb{C}_{\frac{1}{2}}} |K_{w,-2}(z)|^2 M_\varphi(z)\,dA(z)\right)^{p-1} \left\|k_{w,-2}\right\|^2 M_\varphi(w)\,dA(w).\end{aligned}$$

We conclude as above using the submean value property of the counting function (10) to deduce that (3) holds true. □

We end up the proof of Theorem 1.1 by considering the case $p \in (0, 1)$.

Theorem 3.5. *Let $p \in (0,1)$ and $\varphi \in \mathfrak{G}_0$. Assume that φ has bounded imaginary part and $C_\varphi \in S_{2p}$. Then*

$$\int_{\mathbb{C}_{\frac12}} \frac{(M_\varphi(w))^p}{(\operatorname{Re} w - \frac12)^{p+2}}\, dA(w) < +\infty.$$

Proof. We still denote by (f_n) an orthonormal basis of eigenvectors of T_φ. We now write

$$\begin{aligned}\|T_\varphi\|_{S_p}^p &= \sum_{n\geq 1} (\langle T_\varphi(f_n), f_n\rangle)^p \\ &= \sum_{n\geq 1}\left(\int_{\mathbb{C}_{\frac12}} |f_n(w)|^2 M_\varphi(w) dA(w)\right)^p \\ &= \sum_{n\geq 1}\left(\int_{\varphi(\mathbb{C}_0)} \frac{M_\varphi(w)}{\operatorname{Re}(w)-\frac12}|f_n(w)|^2\left(\operatorname{Re}(w)-\frac12\right)dA(w)\right)^p.\end{aligned}$$

Now by (18) the measures $\mathbf{1}_{\varphi(\mathbb{C}_0)}|f_n(\cdot)|^2\left(\operatorname{Re}(\cdot)-\frac12\right)dA$ are finite measures on $\mathbb{C}_{\frac12}$ with uniformly bounded mass. It follows from Hölder inequality and (16) that

$$\begin{aligned}\|T_\varphi\|_{S_p} &\gg \int_{\mathbb{C}_{\frac12}} \sum_{n\geq 1} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-\frac12)^p}|f_n(w)|^2\left(\operatorname{Re}(w)-\frac12\right)dA(w) \\ &\gg \int_{\mathbb{C}_{\frac12}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-\frac12)^{p-1}}\zeta''(2\operatorname{Re}(w))dA(w).\end{aligned}$$

□

3.2. **The case of even integers.** We now prove the $2m$-Schatten class characterization (4).

Proof of Theorem 1.2. We first prove that (4) implies that C_φ is compact. If this was the case, then we could find $\delta > 0$ and a sequence $(w(k)) \subset \mathbb{C}_{\frac12}$ with real part going to $1/2$ such that for every $\varepsilon \in (0,1)$ the rectangles

$$\begin{aligned}R_k = &\left(\frac{\operatorname{Re} w(k) - \frac12}{2}, \frac{3\left(\operatorname{Re} w(k)-\frac12\right)}{2}\right) \\ &\times \left(\operatorname{Im} w(k) - \varepsilon\left(\operatorname{Re} w(k)-\frac12\right), \operatorname{Im} w(k) + \varepsilon\left(\operatorname{Re} w(k)-\frac12\right)\right)\end{aligned}$$

are pairwise disjoint and for all $k \geq 1$,

$$\frac{M_\varphi(w(k))}{\mathrm{Re}(w(k)) - \frac{1}{2}} \geq \delta.$$

Let $A_k = \prod_{j=1}^m R_k$. We recall that $\zeta''(s)$ has a pole of order 3 at $s = 1$, thus we can choose $\varepsilon > 0$ close to zero such that

$$\mathrm{Re}\left(\zeta''\left(\overline{w_1} + w_2\right) \cdots \zeta''\left(\overline{w_{m-1}} + w_m\right) \zeta''\left(\overline{w_m} + w_1\right)\right) \gg \left(\mathrm{Re}\, w(k) - \frac{1}{2}\right)^{-3m},$$

for every $w = (w_1, \dots, w_m) \in A_k$. Using the mean-value property of the counting function as well as the estimate above, we obtain that

$$\int_{A_k} \zeta''\left(\overline{w_1} + w_2\right) \cdots \zeta''\left(\overline{w_{m-1}} + w_m\right) \zeta''\left(\overline{w_m} + w_1\right) \prod_{j=1}^m M_\varphi(w_j)\, dA(w_j) \gg \prod_{j=1}^m \frac{M_\varphi(w_k)}{\mathrm{Re}(w(k)) - \frac{1}{2}} \geq \delta^m.$$

Since the sets A_k are pairwise disjoint, this would contradict (4).

Hence, for both implications of Theorem 1.2, we may assume that C_φ hence T_φ is compact. Let us consider the canonical decomposition of T_φ,

$$T_\varphi(f) = \sum_{n\geq 1} s_n \langle f, f_n \rangle f_n.$$

We know that $C_\varphi \in S_{2m}$ if and only if $T_\varphi^m \in S_1$ if and only if

$$\sum_{n\geq 1} \langle T_\varphi^m(f_n), f_n \rangle < \infty.$$

We observe that

$$\sum_{n\geq 1} \langle T_\varphi^m(f_n), f_n \rangle = \sum_{n\geq 1} \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^{m-1}(f_n), k_{w_1,-2} \rangle \overline{f_n(w_1)} M_\varphi(w_1)\, dA(w_1).$$

Arguing as in the proof of Theorem 3.4, we may use Tonelli's theorem to get

$$\sum_{n\geq 1} \langle T_\varphi^m(f_n), f_n \rangle = \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^{m-1}(k_{w_1,-2}), k_{w_1,-2} \rangle M_\varphi(w_1)\, dA(w_1)$$

$$= \int_{\mathbb{C}_{\frac{1}{2}}} \int_{\mathbb{C}_{\frac{1}{2}}} \langle T_\varphi^{m-2}(k_{w_1,-2}), k_{w_2,-2} \rangle \zeta''(w_1 + \overline{w_2}) M_\varphi(w_2) M_\varphi(w_1)\, dA(w_2)\, dA(w_1).$$

By induction one obtains

$$\|T_\varphi^m\|_{S_1} = \int\limits_{\mathbb{C}_{\frac{1}{2}}} \cdots \int\limits_{\mathbb{C}_{\frac{1}{2}}} \zeta''\left(\overline{w_1}+w_2\right)\cdots\zeta''\left(\overline{w_{m-1}}+w_m\right)\zeta''\left(\overline{w_m}+w_1\right)\prod_{k=1}^m M_\varphi(w_k)\,dA(w_k).$$

□

We now intend to give a similar characterization involving the boundary values of $\varphi \in \mathfrak{G}_0$. For every $\chi \in \mathbb{T}^\infty$, φ_χ belongs to $\mathfrak{G}_0$ and for almost every χ, the generalized boundary value $\varphi(\chi) = \lim_{\sigma\to 0^+} \varphi_\chi(\sigma)$ does exist (see [8, Section 2] or [9, Corollary 3.3]). Of course, $\mathrm{Re}(\varphi(\chi)) \geq 1/2$ for almost every $\chi \in \mathbb{T}^\infty$. We first show that when C_φ is compact, this inequality is strict for almost every $\chi \in \mathbb{T}^\infty$.

Theorem 3.6. *Let $\varphi \in \mathfrak{G}_0$ such that C_φ induces a compact operator on $\mathcal{H}^2$. Then $\mathrm{Re}(\varphi(\chi)) > 1/2$ for almost every $\chi \in \mathbb{T}^\infty$.*

Proof. The norm of the image of a function $f \in \mathcal{H}^2$ under C_φ can be written as

$$\|C_\varphi(f)\|^2_{\mathcal{H}^2} = \int\limits_{\mathbb{T}^\infty} |f\circ\varphi(\chi)|^2\, dm_\infty(\chi) = \int\limits_{\overline{\mathbb{C}_{\frac{1}{2}}}} |f(w)|^2\, d\mu_\varphi(w),$$

where μ_φ if the push-forward measure of m_∞ by $\varphi(\chi)$, see [8]. Since C_φ is compact and the reproducing kernel $\zeta(\cdot+\bar{w})$ of $\mathcal{H}^2$ at w satisfies

$$\zeta(s+\bar{w}) = \frac{1}{\bar{w}+s-1} + O(1),$$

we can argue like in the proof of [19, Theorem 3] to deduce that

$$\mu_\varphi(Q) = o(|I|), \qquad |I| \to 0, \tag{21}$$

where Q is a (Carleson) square in $\mathbb{C}_{\frac{1}{2}}$ with one side I on the vertical line $\left\{\mathrm{Re}\, s = \frac{1}{2}\right\}$. This means that μ_φ is a vanishing Carleson measure for $H^2(\mathbb{C}_{\frac{1}{2}})$ and this implies that $\mu_\varphi\big|_{\left\{\mathrm{Re}\, s=\frac{1}{2}\right\}}$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}$. Following a standard argument, see for example [11, Chapter 3], we will prove that $\mu_\varphi\big|_{\left\{\mathrm{Re}\, s=\frac{1}{2}\right\}}$ is equal to 0. By the Lebesgue-Radon-Nikodym theorem there exists a positive function $f \in L^1(\mathbb{R})$ such that

$$d\mu_\varphi\big|_{\left\{\mathrm{Re}\, s=\frac{1}{2}\right\}} = f(t)\,dt.$$

The set $E = \{\chi : \mathrm{Re}\,\varphi(\chi) > \frac{1}{2}\}$ is of full measure if and only if $f \equiv 0$. Let us assume that there exists $\varepsilon > 0$ such that $\left|f^{-1}\left((\varepsilon,\infty)\right)\right| > 0$. Let $F \subset f^{-1}((\varepsilon,+\infty))$

with positive and finite measure and let $\delta > 0$ be such that

$$\mu_\varphi(Q) \le \frac{\varepsilon}{4}|I|$$

for every Carleson square in $\mathbb{C}_{\frac{1}{2}}$ with length $|I| \le \delta$. We can cover F by a sequence of intervals (I_n) such that $|I_n| \le \delta$ and

$$\sum_n |I_n| \le 2|F|.$$

Now,

$$\varepsilon|F| \le \mu_\varphi\Big|_{\{\mathrm{Re}\, s=\frac{1}{2}\}}(F) \le \frac{\varepsilon}{4}\sum_n |I_n| \le \frac{\varepsilon}{2}|F|$$

which is a contradiction with $|F| > 0$. Thus $\mathrm{Re}(\varphi(\chi)) > 1/2$ for a.e. $\chi \in \mathbb{T}^\infty$. □

We are now ready to give an analogue of Theorem 1.2 involving the symbol directly.

Theorem 3.7. *Suppose that the symbol $\varphi \in \mathfrak{G}_0$ induces a compact operator and let $m \in \mathbb{N}$. Then, C_φ belongs to S_{2m} if and only if*

$$\text{(22)} \quad \infty > \int\limits_{\mathbb{T}^\infty} \cdots \int\limits_{\mathbb{T}^\infty} \zeta(\overline{\varphi(\chi_1)}+\varphi(\chi_2)) \cdots \zeta(\overline{\varphi(\chi_{m-1})}+\varphi(\chi_m))\zeta(\overline{\varphi(\chi_m)}+\varphi(\chi_1)) \prod_{k=1}^{m} dm_\infty(\chi_k).$$

Proof. Let $T = C_\varphi^* \circ C_\varphi$ and let us consider its canonical decomposition

$$T(f) = \sum_{n\ge1} s_n \langle f, f_n\rangle f_n.$$

We know that $C_\varphi \in S_{2m} \iff T^m \in S_1$ and that

$$\langle T(f), g\rangle = \int\limits_{\mathbb{T}^\infty} f(\varphi(\chi))\overline{g(\varphi(\chi))}dm_\infty(\chi).$$

Then

$$\begin{aligned}\sum_{n\ge1}\langle T^m(f_n), f_n\rangle &= \sum_{n\ge1}\langle T(f_n), T^{m-1}(f_n)\rangle \\ &= \sum_{n\ge1}\int\limits_{\mathbb{T}^\infty} f_n(\varphi(\chi_1))\overline{\langle T^{m-1}(f_n), \zeta(\cdot + \overline{\varphi(\chi_1)})\rangle}dm_\infty(\chi_1).\end{aligned}$$

As in the proof of Theorem 1.2 the quantity inside the integral is nonnegative which allows us to use Tonelli's theorem. Hence

$$\sum_{n\geq 1}\langle T^m(f_n),f_n\rangle = \int_{\mathbb{T}^\infty}\langle T^{m-1}(k_{\varphi(\chi_1),0}),k_{\varphi(\chi_1),0}\rangle dm_\infty(\chi_1)$$
$$= \int_{\mathbb{T}^\infty}\int_{\mathbb{T}^\infty}\langle T^{m-2}(k_{\varphi(\chi_1),0}),k_{\varphi(\chi_2),0}\rangle \zeta(\varphi(\chi_1)+\overline{\varphi(\chi_2)})dm_\infty(\chi_1)dm_\infty(\chi_2).$$

By induction one finally obtains

$$\|T^m\|_{S_1} =$$
$$\int_{\mathbb{T}^\infty}\cdots\int_{\mathbb{T}^\infty}\zeta(\overline{\varphi(\chi_1)}+\varphi(\chi_2))\cdots\zeta(\overline{\varphi(\chi_{m-1})}+\varphi(\chi_m))\zeta(\overline{\varphi(\chi_m)}+\varphi(\chi_1))\prod_{k=1}^m dm_\infty(\chi_k).$$

□

4. A comparison-type principle

4.1. **Lindelöf principle and Littlewood inequality.** In this section we will use Lindelöf principle for Green's function to give a simple proof of a non-contractive Littlewood inequality (11). Similar techniques have been used in the disk setting, [5].

We recall (see for instance [22]) that a Green's function for a domain $\Omega\subset\mathbb{C}$ is a function $g_\Omega:\Omega\times\Omega\to(-\infty,+\infty]$ such that, for all $w\in\Omega$, $g(\cdot,w)$ is harmonic in $\Omega\backslash\{w\}$, $g_\Omega(z,w)\to 0$ n.e as $z\to\partial\Omega$ and $g_\Omega(\cdot,w)+\log|\cdot-w|$ is harmonic in a neighbourhood of w. If a domain admits a Green's function then it is necessarily unique. For instance, the Green's function on the disk $g_{\mathbb{D}}:\mathbb{D}\times\mathbb{D}\mapsto(0,+\infty]$ has the form

$$g_{\mathbb{D}}(z,w)=\log\left|\frac{1-z\overline{w}}{z-w}\right|.$$

By conformal invariance we can easily define Green's function on every simply connected subdomain of the complex plane, for example

$$g_{\mathbb{C}_0}(z,w)=\log\left|\frac{z+\overline{w}}{z-w}\right|,\qquad z,\,w\in\mathbb{C}_0.$$

The class of domains D possessing a Green's function g_D is much larger than the simply connected domains, see [22, Chapter 4]. Lindelöf principle for Green's function (see for instance [4]) states that if f is a holomorphic function mapping D_1 to D_2, where both of those domains possess Green's function, then for $z_0\in D_1$ and $w\in D_2\setminus\{f(z_0)\}$

$$\sum_{z\in f^{-1}(\{w\})} g_{D_1}(z,z_0)\leq g_{D_2}(w,f(z_0)). \tag{23}$$

Let us first show how to deduce, up to a multiplicative constant, Littlewood inequality (11) and also a corresponding inequality for a symbol in $\mathfrak{G}_{\geq 1}$ (such an inequality was used in [3] to obtain a sufficient condition for composition operators with symbols in $\mathfrak{G}_{\geq 1}$ to be compact on $\mathcal{H}^2$). Recall that for $\psi \in \mathfrak{G}_{\geq 1}$, its restricted Nevanlinna counting function is defined by

$$N_\psi(w) = \sum_{\substack{s\in\psi_\chi^{-1}(\{w\})\\ |\operatorname{Im} s|\leq 1}} \operatorname{Re} s.$$

Theorem 4.1. *a) Let $\varphi \in \mathfrak{G}_0$. Then for all $w \in \mathbb{C}_{\frac12}\backslash\{\varphi(+\infty)\}$,*

$$M_\varphi(w) \leq \pi \log\left|\frac{\varphi(+\infty)+\overline{w}-1}{\varphi(+\infty)-w}\right|.$$

b) Let $\psi \in \mathfrak{G}_{\geq 1}$. There exists $C>0$ such that, for all $\chi \in \mathbb{T}^\infty$, for all $w \in \mathbb{C}_0$ with $\operatorname{Re} w \leq c_0$,

$$N_{\psi_\chi}(w) \leq C\frac{\operatorname{Re} w}{1+(\operatorname{Im} w)^2}.$$

Proof. a) Let $\varphi \in \mathfrak{G}_0$ and $w \in \mathbb{C}_{\frac12}\backslash\{\varphi(+\infty)\}$. For $T>0$ sufficiently large, $w \neq \varphi(T)$ so that Lindelöf principle implies

$$\sum_{s\in\varphi^{-1}(\{w\})} \log\left|\frac{T+\overline{s}}{T-s}\right| \leq \log\left|\frac{\varphi(T)+\overline{w}-1}{\varphi(T)-w}\right|.$$

On the other hand, using the elementary inequality $\log(x) \geq \frac12(1-x^{-2})$ valid for $x>0$,

$$\sum_{s\in\varphi^{-1}(\{w\})} \log\left|\frac{T+\overline{s}}{T-s}\right| \geq \sum_{s\in\varphi^{-1}(\{w\})} \frac{2T\operatorname{Re} s}{|T+s|^2}.$$

Observe that $\{\operatorname{Re}(s) : s \in \varphi^{-1}(\{w\})\}$ is bounded. Therefore, for all $\varepsilon \in (0,1)$, we can choose T large enough so that

$$\sum_{s\in\varphi^{-1}(\{w\})} \log\left|\frac{T+\overline{s}}{T-s}\right| \geq \frac{(1-\varepsilon)}{T} \sum_{\substack{s\in\varphi^{-1}(\{w\})\\ |\operatorname{Im} s|\leq T}} \operatorname{Re} s.$$

We can conclude by letting T to $+\infty$ and ε to 0.

Regarding b), let $\chi \in \mathbb{T}^\infty$ and $w \in \mathbb{C}_0$ with $\operatorname{Re} w \leq c_0$. Lindelöf principle says that

$$\sum_{s\in\psi_\chi^{-1}(\{w\})} \log\left|\frac{\overline{s}+2}{s-2}\right| \leq \log\left|\frac{\overline{w}+\psi_\chi(2)}{w-\psi_\chi(2)}\right|.$$

Now, when $\psi_\chi(s)=w$, then $0<\operatorname{Re}s=\frac{\operatorname{Re}w-\operatorname{Re}\varphi(s)}{c_0}\le 1$ since $\operatorname{Re}w\le c_0$. We apply again the inequality $\log(x)\ge\frac{1}{2}(1-x^{-2})$, $x>0$, yielding to

$$N_{\psi_\chi}(w)\le C\sum_{s\in\psi_\chi^{-1}(\{w\})}\log\left|\frac{\overline{s}+2}{s-2}\right|.$$

Finally it was shown in [3] that

$$\log\left|\frac{\overline{w}+\psi_\chi(2)}{w-\psi_\chi(2)}\right|\le C\frac{\operatorname{Re}w}{1+(\operatorname{Im}w)^2}$$

where C does not depend neither on $\chi\in\mathbb{T}^\infty$ nor on w with $\operatorname{Re}w\le c_0$. □

4.2. **A comparison-type principle and a polygonal compactness theorem.** We shall now apply the idea of the previous subsection when $\varphi\in\mathfrak{G}_0$ maps $\mathbb{C}_0$ into a subdomain D of $\mathbb{C}_{\frac{1}{2}}$. Lindelöf principle helps us to find better estimates on M_φ. Indeed, provided D admits a Green's function, the proof of Theorem 4.1 shows that

$$M_\varphi(w)\ll g_D(w,\varphi(+\infty)),\ w\in\mathbb{C}_{\frac{1}{2}}\backslash\{\varphi(+\infty)\}. \tag{24}$$

We deduce the following comparison principle. Under similar conditions a norm-comparison principle appeared in [8].

Theorem 4.2. *Let $\varphi\in\mathfrak{G}_0$ be such that $\varphi(\mathbb{C}_0)\subset D\subset\mathbb{C}_{\frac{1}{2}}$ where D is a simply connected domain. Let R_D be the Riemann map from $\mathbb{D}$ onto D such that $R_D(0)=\varphi(+\infty)$ and let $\varphi_D=R_D(2^{-s})$. Assume that C_{φ_D} is compact. Then C_φ is compact. Moreover, if $D\subset\{|\operatorname{Im}(s)|\le C\}$ for some $C>0$ and if C_{φ_D} belongs to S_{2p}, $p\ge 1$, then C_φ belongs to S_{2p}.*

Proof. Let $p=\frac{2\pi}{\log(2)}$. By the ip-periodicity of φ_D, we have that for all $T>0$

$$\left\lfloor\frac{2T}{p}\right\rfloor\sum_{\substack{s\in\varphi_D^{-1}(\{w\})\\0\le\operatorname{Im}s<p\\\sigma<\operatorname{Re}s<\infty}}\operatorname{Re}s\le\sum_{\substack{s\in\varphi_D^{-1}(\{w\})\\|\operatorname{Im}s|<T\\\sigma<\operatorname{Re}s<\infty}}\operatorname{Re}s\le\left(\left\lfloor\frac{2T}{p}\right\rfloor+1\right)\sum_{\substack{s\in\varphi_D^{-1}(\{w\})\\0\le\operatorname{Im}s<p\\\sigma<\operatorname{Re}s<\infty}}\operatorname{Re}s,$$

where $\lfloor x\rfloor$ is the integer part of the real number x, [15, Example 4.6]. Thus

$$\begin{aligned}M_{\varphi_D}(w)&\sim\sum_{\substack{0\le\operatorname{Im}s<p\\s\in\varphi_D^{-1}(\{w\})}}\operatorname{Re}s\\&\sim\sum_{\substack{z\in\mathbb{D}\\z\in R_D^{-1}(\{w\})}}\log\left(\frac{1}{|z|}\right)\end{aligned}$$

since

$$\log 2 \sum_{\substack{s\in\varphi_D^{-1}(\{w\})\\ |\operatorname{Im} s|<p\\ \operatorname{Re} s>0}} \operatorname{Re} s = \sum_{\substack{z\in R_D^{-1}(\{w\})\\ |z|<1}} \log\frac{1}{|z|}.$$

By the conformal invariance of the Green's function

$$M_{\varphi_D}(w) \sim g_D(w, \varphi_D(+\infty)).$$

Hence our assumption on C_{φ_D} gives an estimate on M_{φ_D} which transfers to M_φ thanks to (24) which itself gives the corresponding result on C_φ. Observe that in both cases, we use the *characterization* of compactness or membership to S_{2p}. □

Remark. In Theorem 4.2, we can only assume that D admits a Green's function and use for R_D a universal covering map of D.

The most interesting case occurs when $\varphi(\mathbb{C}_0)$ is mapped into an angular sector contained in $\mathbb{C}_{\frac{1}{2}}$. This leads to Theorem 1.3 that we now prove.

Proof of Theorem 1.3. Green's function of the domain

$$\Omega = \left\{ s\in\mathbb{C}_{\frac{1}{2}} : \left|\arg(s)-\frac{1}{2}\right| < \frac{\pi}{2\alpha}\right\}$$

is

$$g_\Omega(z,w) = \log\left|\frac{\left(z-\frac{1}{2}\right)^\alpha + \overline{\left(w-\frac{1}{2}\right)^\alpha}}{\left(z-\frac{1}{2}\right)^\alpha - \left(w-\frac{1}{2}\right)^\alpha}\right|.$$

By (24) and [9, Lemma 2.3], for $w\in\varphi(\mathbb{C}_0)\subset\Omega$,

$$\begin{aligned} M_\varphi(w) &\ll \log\left|\frac{\left(w-\frac{1}{2}\right)^\alpha + \overline{\left(\varphi(+\infty)-\frac{1}{2}\right)^\alpha}}{\left(w-\frac{1}{2}\right)^\alpha - \left(\varphi(+\infty)-\frac{1}{2}\right)^\alpha}\right| \\ &\ll \operatorname{Re}\left(w-\frac{1}{2}\right)^\alpha \\ &\ll \left(\operatorname{Re}(w)-\frac{1}{2}\right)^\alpha \end{aligned}$$

provided $|w-\varphi(+\infty)| > \delta$ for some fixed $\delta > 0$. The proof of compactness follows from the characterization (1), the proof of the Schatten class part follows from Corollary 3.3. Indeed, letting $\sigma_\infty = 2\operatorname{Re}\varphi(+\infty)$, let $T > 0$ be such that

$|\operatorname{Im}(w)| \leq T$ for all $w \in \varphi(\mathbb{C}_0) \cap (\mathbb{C}_{\frac{1}{2}} \backslash \mathbb{C}_{\sigma_\infty})$. Then

$$\int\limits_{\mathbb{C}_{\frac{1}{2}} \backslash \mathbb{C}_{\sigma_\infty}} \frac{(M_\varphi(w))^p}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^{p+2}} (1 + |\operatorname{Im}(w)|)^{2(p-1)} dA(w) \ll$$

$$\int\limits_{\frac{1}{2}}^{\sigma_\infty} \int\limits_{-T}^{T} \frac{(1 + |\operatorname{Im}(w)|)^{2(p-1)}}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^{p+2-p\alpha}} dt d\sigma < +\infty, \qquad w = \sigma + it,$$

since $p > 1/(\alpha - 1)$. Moreover,

$$\int\limits_{\mathbb{C}_{\sigma_\infty}} \frac{(M_\varphi(w))^p}{\left(\operatorname{Re}(w) - \frac{1}{2}\right)^{p+2}} (1 + |\operatorname{Im}(w)|)^{2(p-1)} dA(w) \ll$$

$$\int\limits_{\mathbb{C}_{\sigma_\infty}} \frac{dA(w)}{(1 + |\operatorname{Im}(w)|)^2 \left(\operatorname{Re}(w) - \frac{1}{2}\right)^2} < +\infty$$

by (12). □

Using properties of conformal maps, we can extend the compactness part of Theorem 1.3 to slightly more general domains. This is the analogue of the polygonal compactness theorem, [25], in our setting.

Theorem 4.3. *Let $\varphi \in \mathfrak{G}_0$ be such that for some $\delta > 0$, the set $\varphi(\mathbb{C}_0) \bigcap \{\frac{1}{2} < \operatorname{Re} s \leq \frac{1}{2} + \delta\}$ is contained in a finite union of angular sectors of the form $\bigcup_{j=1}^{d} \left\{\left|\arg\left(z - \frac{1}{2} - i\tau_j\right)\right| < \alpha_j\right\}$ with $\tau_j \in \mathbb{R}$ and $\alpha_j \in (0, \pi/2)$. Then C_φ is compact on $\mathcal{H}^2$.*

Proof. Let us consider the Riemann map $f = R_D$, where

$$D = \mathbb{C}_{\frac{1}{2}+\delta} \bigcup \bigcup_{j=1}^{d} \left\{ \left| \arg\left(z - \frac{1}{2} - i\tau_j\right) \right| < \alpha_j \right\}$$

and let $\{w_n\}_{n\geq 1}$ be an arbitrary sequence such that $\operatorname{Re} w_n \to \frac{1}{2}^+$. Since $M_\varphi(w_n) = 0$ if $w_n \notin D$, we can assume that the sequence converges to a corner boundary point $f(e^{i\theta_0}) \neq \infty$. Then, by (24) and Koebe's quarter theorem, see [20, Corollary 1.4],

$$M_\varphi(w_n) \ll g_D(w_n, f(0)) \ll 1 - \left|f^{-1}(w_n)\right|^2 \ll$$

$$\operatorname{dist}(w_n, \partial D) \left|f'\left(f^{-1}(w_n)\right)\right|^{-1} \leq \left(\operatorname{Re} w_n - \frac{1}{2}\right) \left|f'\left(f^{-1}(w_n)\right)\right|^{-1}.$$

It is sufficient to prove that $\left|f'\left(f^{-1}(w_n)\right)\right| \to \infty$, as $n \to \infty$. By the Kellogg-Warschawski theorem [20, Theorem 3.9] and the Carathéodory extension theorem [20, Chapter 2]

$$\left|f'\left(f^{-1}(w_n)\right)\right| \gg \left|f^{-1}(w_n) - e^{i\theta_0}\right|^{\alpha-1} \to \infty,$$

where $\alpha = \max\{a_j : 1 \le j \le d\}$. □

Remark. Our techniques apply also for symbols $\psi = c_0 s + \varphi \in \mathfrak{G}_{\ge 1}$. Although, the range of such a symbol cannot meet the imaginary axis in an angular sector or more generally inside a domain D where $\operatorname{Im} w$ is bounded for $w \in D \bigcap (\mathbb{C}_0 \setminus \mathbb{C}_\varepsilon)$, $\varepsilon > 0$. If that was the case then we would be able to find a point $s(\varepsilon) \in \mathbb{C}_0 \setminus \mathbb{C}_{\frac{\varepsilon}{4c_0}}$ such that $\operatorname{Re}\varphi\left(s(\varepsilon)\right) \le \varepsilon/4$. The Dirichlet series φ converges uniformly in $\mathbb{C}_{\operatorname{Re} s(\varepsilon)/2}$, see [7]. By almost periodicity we can find an increasing unbounded sequence of positive numbers $\{T_n\}_{n\ge1}$ such that

$$\operatorname{Re}\varphi(s(\varepsilon) + iT_n)) \le \frac{\varepsilon}{2}$$

so that $\operatorname{Re}\psi(s(\varepsilon) + iT_n) \le \frac{3\varepsilon}{4}$. We observe that $|\operatorname{Im}\psi(s(\varepsilon) + iT_n)| \to +\infty$, this contradicts our assumption.

4.3. **On the boundedness on $\mathcal{H}^p$.** We conclude this section by the proof of Theorem 1.4. We will use Hilbertian methods to prove that our assumption implies that C_φ is bounded as an operator from H to $\mathcal{H}^2$, where H is a Hilbert space of Dirichlet series containing $\mathcal{H}^p$. To do this, we need another class of Bergman spaces of Dirichlet series, the spaces $\mathcal{A}_a$, $\alpha \ge 1$. They are defined as

$$\mathcal{A}_\alpha = \left\{ f(s) = \sum_{n\ge1} a_n n^{-s} : \|f\|^2_{\mathcal{A}_\alpha} = \sum_{n\ge1} \frac{|a_n|^2}{d_\alpha(n)} < \infty \right\},$$

where by $d_\alpha(n)$ we denote the coefficients of the Dirichlet series $\left(\zeta(s)\right)^\alpha$, $s \in \mathbb{C}_1$. In particular, $d_2(\cdot)$ is the divisor counting function. The space $\mathcal{A}_\alpha$ is a reproducing kernel Hilbert space, the reproducing kernel at a point $s_0 \in \mathbb{C}_{\frac{1}{2}}$ being the function $\left(\zeta(\overline{s_0} + \cdot)\right)^\alpha$. The analogue of the embedding theorem for $\mathcal{A}_\alpha$ reads:

Lemma 4.4 ([18])**.** *For every $f \in \mathcal{A}_\alpha$ and every interval $I \subset \mathbb{R}$ there exists a constant $C = C(|I|)$ such that*

$$\int_{\frac{1}{2}}^{1} \int_I |f'(\sigma + it)|^2 \left(\sigma - \frac{1}{2}\right)^\alpha dt\, d\sigma \le C \|f\|^2_{\mathcal{A}_\alpha}. \tag{25}$$

Proof of Theorem 1.4. Let us set $\alpha = 2k/p$. Working in a similar manner to Theorem 1.3, there exist $\varepsilon > 0$, $C > 0$ such that $\operatorname{Re} w \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ implies

$$M_\varphi(w) \leq C \left(\operatorname{Re} w - \frac{1}{2}\right)^\alpha .$$

Let $T > 0$ be such that $\varphi(\mathbb{C}_0) \cap (\mathbb{C}_{\frac{1}{2}} \backslash \mathbb{C}_{\frac{1}{2}+\varepsilon}) \subset \left[\frac{1}{2}, \frac{1}{2} + \varepsilon\right] \times [-T, T]$. By (8), for $f \in \mathcal{H}^p \subset \mathcal{H}^{2k}$,

$$\begin{aligned}
\|C_\varphi(f)\|_{\mathcal{H}^{2k}}^{2k} &= \left\|C_\varphi(f^k)\right\|_{\mathcal{H}^2}^2 \\
&= \left|f^k(\varphi(+\infty))\right|^2 + \frac{2}{\pi} \int_{\mathbb{C}_{\frac{1}{2}}} \left|(f^k)'(w)\right|^2 M_\varphi(w)\, dA(w) \\
&\ll \int_{\frac{1}{2}}^{\frac{1}{2}+\varepsilon} \int_{-T}^{T} \left|(f^k)'(\sigma + it)\right|^2 \left(\sigma - \frac{1}{2}\right)^\alpha dt\, d\sigma \\
&\qquad + \left|f^k(\varphi(+\infty))\right|^2 + \frac{2}{\pi} \int_{\mathbb{C}_{\frac{1}{2}+\varepsilon}} \left|(f^k)'(w)\right|^2 M_\varphi(w)\, dA(w).
\end{aligned}$$

Let us write $f^k = \sum_{j \geq 1} a_j j^{-s}$. By the Cauchy-Schwarz inequality, for all $w \in \mathbb{C}_{\frac{1}{2}+\varepsilon}$,

$$\begin{aligned}
\left|(f^k)'(w)\right| &\leq \sum_{j \geq 2} \frac{|a_j| \log j}{j^{\operatorname{Re} w}} \\
&\leq \left(\sum_{j \geq 2} \frac{|a_j|^2}{d_\alpha(j)}\right)^{1/2} \left(\sum_{j \geq 2} \frac{d_\alpha(j) \log^2 j}{j^{2 \operatorname{Re} w}}\right)^{1/2} \\
&\leq C(\varepsilon)|2^{-w}|\|f^k\|_{\mathcal{A}_\alpha}.
\end{aligned}$$

Note that $(\zeta^\alpha(s))'' = \sum\limits_{j \geq 2} \frac{d_\alpha(j) \log^2 j}{j^s}$ converges absolutely for $\operatorname{Re} s > 1 + \varepsilon$, $\varepsilon > 0$.

By the local embedding theorem (25), the boundedness of pointwise evaluation at $\varphi(+\infty)$ and the continuity of C_φ on $\mathcal{H}^2$, applied to 2^{-s}, we get

$$\|C_\varphi(f)\|_{\mathcal{H}^{2k}}^{2k} \ll \|f^k\|_{\mathcal{A}_\alpha}^2 .$$

Now, the inclusion operator $i : \mathcal{H}^{p/k} \to \mathcal{A}_\alpha$ is contractive, [16]. Therefore

$$\|C_\varphi(f)\|_{\mathcal{H}^p} \leq \|C_\varphi(f)\|_{\mathcal{H}^{2k}} \ll \|f^k\|_{\mathcal{A}_\alpha}^{1/k} \leq \|f^k\|_{\mathcal{H}^{p/k}}^{1/k} = \|f\|_{\mathcal{H}^p} .$$

Let us turn to compactness. Let $\{f_n\}_{n\geq 1}$ be a sequence of $\mathcal{H}^q$ converging weakly to 0. We set $g_n = f_n^k$ and observe that (g_n) converges pointwise to 0 on $\mathbb{C}_{\frac{1}{2}}$ and that the Dirichlet coefficients $\widehat{g}_n(j)$ converge to 0 for each $j \geq 1$.

We work as above but we now set $\alpha = 2k/q$ and consider $\delta \in (0, \varepsilon)$. Then

$$\begin{aligned}\|C_\varphi(f_n)\|_{\mathcal{H}^q}^{2k} &\leq \|C_\varphi(f_n)\|_{\mathcal{H}^{2k}}^{2k} = \|C_\varphi(g_n)\|_{\mathcal{H}^2}^2 \\ &\leq |g_n(\varphi(+\infty)|^2 + \delta^{\frac{1}{p}-\frac{1}{q}} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \int_{-T}^{T} |g_n'(\sigma+it)|^2 \left(\sigma - \frac{1}{2}\right)^\alpha dt\, d\sigma \\ &\quad + \frac{2}{\pi} \int_{\mathbb{C}_{\frac{1}{2}+\delta}} |g_n'(w)|^2\, M_\varphi(w)\, dA(w).\end{aligned}$$

The first term goes to zero as n tends to $+\infty$ and the second term is as small as we want for every n if we adjust δ small enough. Therefore it remains to show that, for a fixed $\delta > 0$, the last terms tends to 0 as n tends to $+\infty$. Now, for all $n \geq 1$ and all $w \in \mathbb{C}_{\frac{1}{2}+\delta}$,

$$\begin{aligned}|g_n'(w)| &\leq \sum_{j\geq 2} \frac{|\widehat{g_n}(j)| \log j}{j^{\operatorname{Re} w}} \\ &\leq \left(\sum_{j\geq 2} \frac{d_\alpha(j)\log^2 j}{j^{2\operatorname{Re} w - \frac{\delta}{2}}}\right)^{1/2} \left(\sum_{j\geq 2} \frac{|\widehat{g_n}(j)|^2}{d_\alpha(j) j^{\frac{\delta}{2}}}\right)^{1/2} \\ &\ll |2^{-w}| \left(\sum_{j=2}^{N} \frac{|\widehat{g_n}(j)|^2}{d_\alpha(j) j^{\frac{\delta}{2}}} + \frac{1}{N^{\frac{\delta}{2}}} \|g_n\|_{\mathcal{A}_\alpha}^2\right)^{1/2}.\end{aligned}$$

Since (g_n) is bounded in $\mathcal{A}_\alpha$, for any $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that, $|g_n'(w)| \leq \eta |2^{-w}|$. We now argue as above to conclude that $(C_\varphi(f_n))$ tends to 0 in $\mathcal{H}^q$. □

Remark. We choose to work with symbols with range into angular sectors for the sake of simplicity. It will be interesting to know if our techniques can be applied to give other examples of geometric conditions related to the behavior of composition operators on Hardy spaces of Dirichlet series.

5. Further discussion

5.1. **Bergman spaces.** We focused on the Hardy space $\mathcal{H}^2$, but we can extend our results to Bergman spaces of Dirichlet series $\mathcal{D}_{-a}$, $a \geq 0$. The class $\mathfrak{G}$ determines again the bounded composition operators on $\mathcal{D}_{-a}$, $a \geq 0$. For Dirichlet series symbols $\varphi \in \mathfrak{G}_0$, the compact composition operators C_φ have been characterized, [15], in terms of the weighted counting function

$$M_{\varphi,1+a}(w) = \lim_{\sigma\to 0^+} \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{s\in\varphi^{-1}(\{w\}) \\ |\operatorname{Im} s|<T \\ \sigma<\operatorname{Re} s<\infty}} (\operatorname{Re} s)^{1+a}, \qquad w \neq \varphi(+\infty),$$

and similarly with the Hardy space case, C_φ is compact on $\mathcal{D}_{-a}$, $a \geq 0$ if and only if

$$\lim_{\operatorname{Re} w\to \frac{1}{2}^+} \frac{M_{\varphi,1+a}(w)}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{1+a}} = 0.$$

Theorem 5.1. *Let $\varphi \in \mathfrak{G}_0$ and $p \geq 4$. A necessary condition for the composition operator C_φ to belong to the class S_p is the following:*

$$\int_{\mathbb{C}_{\frac{1}{2}}} \frac{(M_{\varphi,1+a}(w))^{\frac{p}{2}}}{\left(\operatorname{Re} w - \frac{1}{2}\right)^{(a+1)\frac{p}{2}+2}} \, dA(w) < +\infty. \tag{26}$$

If we further assume that φ has bounded imaginary part, then C_φ belongs to the class S_p, $p \geq 2$ if and only if φ satisfies (26), and for $p > 0$ the condition remains necessary.

To prove Theorem 5.1 one can argue in a similar manner with the Hardy space $\mathcal{H}^2$, using the analogue key ingredients, those are: The change of variables formula [15, Theorem 1.2], the Littlewood-type inequality [15, Proposition 5.4], the weak submean value property [15, Theorem 4.11] and the behavior of reproducing kernels (14).

5.2. **Carleson measures.** E. Saksman and J–F. Olsen [19] proved that if μ is a Carleson measure for $\mathcal{H}^2$, then it is a Carleson measure for $H^2(\mathbb{C}_{\frac{1}{2}})$. The converse is also true with the extra assumption that μ has compact support.

A direct consequence of the local embedding theorem is that a sufficient condition for a measure μ in $\left\{\frac{1}{2} < \operatorname{Re} s < \sigma_\infty\right\}$ to be Carleson for $\mathcal{H}^2$ is

$$\{C(\mu_n, H^2(\mathbb{C}_{\frac{1}{2}}))\}_{n\in\mathbb{Z}} \in \ell^1,$$

where μ_n is the restriction of μ on the half-strip $\{s \in \mathbb{C}_{\frac{1}{2}} : n \le \operatorname{Im} s < n+1\}$. Indeed,

$$\begin{aligned}\int_{\mathbb{C}_{\frac{1}{2}}} |f(w)|^2 d\mu(w) &\ll \sum_{n\in\mathbb{Z}} \int_{\mathbb{C}_{\frac{1}{2}}} \left| \frac{f(w)}{w-in} \right|^2 d\mu_n(w) \\ &\ll \sum_{n\in\mathbb{Z}} C(\mu_n) \left\| \frac{f(\cdot)}{\cdot - in} \right\|_{H^2(\mathbb{C}_{\frac{1}{2}})} \\ &\ll \sum_{n\in\mathbb{Z}} C(\mu_n) \, \|f\|^2_{\mathcal{H}^2} \\ &\ll \|f\|^2_{\mathcal{H}^2} .\end{aligned}$$

An example of such a measure is the restriction of $\frac{M_\varphi(w)}{\operatorname{Re} w - \frac{1}{2}} dA(w)$ to $\{\frac{1}{2} < \operatorname{Re} s < \frac{1+\operatorname{Re}\varphi(+\infty)}{2}\}$. The above condition is not necessary, as we will exemplify now.

We consider the sequence $\{s_n\}_{n\ge 1}$, where

$$s_n = \frac{1}{2} + \left(\frac{1}{2}\right)^n + i\left(n + \frac{1}{2}\right).$$

As we will prove in a moment the measure $d\mu(w) = \sum_{n\ge1} (\operatorname{Re} s_n - \frac{1}{2})\delta_{s_n}(w)$ is a Carleson measure for $\mathcal{H}^2$, where $\delta_{s_n}(w)$ is a Dirac mass at s_n . The restriction μ_n, $n \ge 1$ has the form

$$d\mu_n(w) = \left(\operatorname{Re} s_n - \frac{1}{2}\right) \delta_{s_n}(w).$$

Let Q_n, $n \ge 1$ be the square with center at the point s_n and one side I_n on the line $\{\operatorname{Re} s = \frac{1}{2}\}$. Then,

$$\mu_n(Q_n) = \frac{|I_n|}{2},$$

consequently $\{C(\mu_n)\}_{n\in\mathbb{Z}} \notin \ell^1$. It remains to prove that $d\mu(w) = \sum_{n\ge1} (\operatorname{Re} s_n - \frac{1}{2})\delta_{s_n}(w)$ is a Carleson measure for $\mathcal{H}^2$. Actually, this is true for every sequence $\{s_n\}_{n\ge1}$ in $\mathbb{C}_{\frac{1}{2}}$ such that

$$\operatorname{Re} s_{n+1} - \frac{1}{2} \le a\left(\operatorname{Re} s_n - \frac{1}{2}\right), \qquad n \in \mathbb{N}, \tag{27}$$

for some $a \in (0,1)$. We follow an argument of [23, Section 4], see also [1]. It is sufficient to prove that the matrix $A = \left[\frac{\zeta(s_i + \overline{s_j})}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}}\right]_{i,j\ge1}$ defines a bounded operator on ℓ^2. We will prove that for every $j \in \mathbb{N}$

$$\sum_{i\geq 1} \frac{|\zeta(s_i+\overline{s_j})|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \leq C, \tag{28}$$

the result will then follow from Schur's test [26, Section 3.3].

The Riemann zeta function has a simple pole at 1. Therefore, (27) yields the existence of $i_0 \geq 1$ and of $b \in (0,1)$ such that, for all $i \geq i_0$,

$$\frac{\zeta(2\operatorname{Re} s_i)}{\zeta(2\operatorname{Re} s_{i+1})} \leq b,$$

where $b \in (0,1)$. We only need to prove (28) for $j \geq i_0$. On the one hand,

$$\sum_{1\leq i\leq i_0} \frac{|\zeta\left(s_i+\overline{s_j}\right)|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \leq i_0 \frac{|\zeta\left(\operatorname{Re} s_{i_0}+\operatorname{Re} s_j\right)|}{\sqrt{\zeta(2\operatorname{Re} s_1)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \leq C.$$

On the other hand,

$$\begin{aligned}\sum_{i\geq i_0} \frac{|\zeta\left(s_i+\overline{s_j}\right)|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} &\leq \sum_{i\geq i_0} \frac{|\zeta(\frac{1}{2}+\max\{\operatorname{Re} s_i,\operatorname{Re} s_j\})|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}}\\ &\ll \sum_{i_0\leq i\leq j}\sqrt{\frac{\zeta(2\operatorname{Re} s_i)}{\zeta(2\operatorname{Re} s_j)}}+\sum_{i\geq j\geq i_0}\sqrt{\frac{\zeta(2\operatorname{Re} s_j)}{\zeta(2\operatorname{Re} s_i)}}\\ &\ll \sum_{i_0\leq i\leq j} b^{\frac{j-i}{2}}+\sum_{i\geq j\geq i_0} b^{\frac{i-j}{2}}\\ &\leq C.\end{aligned}$$

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Part 2

Bohr's theorem and Beurling primes

Article 4: A note on Bohr's theorem for Beurling integer systems

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A NOTE ON BOHR'S THEOREM FOR BEURLING INTEGER SYSTEMS

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Abstract. Given a sequence of frequencies $\{\lambda_n\}_{n\geq 1}$, a corresponding generalized Dirichlet series is of the form $f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$. We are interested in multiplicatively generated systems, where each number e^{λ_n} arises as a finite product of some given numbers $\{q_n\}_{n\geq 1}$, $1 < q_n \to \infty$, referred to as Beurling primes. In the classical case, where $\lambda_n = \log n$, Bohr's theorem holds: if f converges somewhere and has an analytic extension which is bounded in a half-plane $\{\operatorname{Re} s > \theta\}$, then it actually converges uniformly in every half-plane $\{\operatorname{Re} s > \theta + \varepsilon\}$, $\varepsilon > 0$. We prove, under very mild conditions, that given a sequence of Beurling primes, a small perturbation yields another sequence of primes such that the corresponding Beurling integers satisfy Bohr's condition, and therefore the theorem. Applying our technique in conjunction with a probabilistic method, we find a system of Beurling primes for which both Bohr's theorem and the Riemann hypothesis are valid. This provides a counterexample to a conjecture of H. Helson concerning outer functions in Hardy spaces of generalized Dirichlet series.

1. Introduction

For an increasing sequence of positive frequencies $\lambda = \{\lambda_n\}_{n\geq 1}$, and a generalized Dirichlet series

$$f(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s},$$

the abscissas σ_c, σ_u, and σ_a of point-wise, uniform, and absolute convergence are defined as in the classical theory of Dirichlet series [12]. In this article we wish to find sets of frequencies such that the analogue of a theorem of Bohr [4] holds: if $\sigma_c(f) < \infty$ and f has a bounded analytic extension to a half-plane $\{\operatorname{Re} s > \theta\}$, then $\sigma_u(f) \leq \theta$. The problem of finding frequencies for which the abscissas of bounded and uniform convergence always coincide, which originated with Bohr and Landau [18], has recently been revisited [2, 20] with the context of Hardy spaces of Dirichlet series in mind. Indeed, Bohr's theorem is essentially a necessity for a satisfactory Hardy space theory, see [19, Ch. 6].

An important class of frequencies were introduced by Beurling [3]. Given an arbitrary increasing sequence $q = \{q_n\}_{n\geq 1}$, $1 < q_n \to \infty$, such that $\{\log q_n\}_{n\geq 1}$ is linearly independent over $\mathbb{Q}$, we will denote by $\mathbb{N}_q = \{\nu_n\}_{n\geq 1}$ the set of numbers that can be written (uniquely) as finite products with factors from q, ordered in an increasing manner. The numbers q_n are known as Beurling primes, and the numbers ν_n are Beurling integers. The corresponding generalized Dirichlet series are of the form

$$f(s) = \sum_{n\geq 1} a_n \nu_n^{-s}.$$

There are a number of criteria to guarantee the validity of Bohr's theorem for frequencies $\{\lambda_n\}_{n\geq 1}$. Bohr's original condition asks for the existence of $c_1, c_2 > 0$ such that

$$\lambda_{n+1} - \lambda_n \geq c_1 e^{-c_2 \lambda_{n+1}}, \qquad n \in \mathbb{N}. \tag{1}$$

Landau relaxed the condition somewhat: for every $\delta > 0$ there should be a $c > 0$ such that

$$\lambda_{n+1} - \lambda_n \geq c e^{-e^{\delta \lambda_{n+1}}}, \qquad n \in \mathbb{N}. \tag{2}$$

Landau's condition was recently relaxed further by Bayart [2]: for every $\delta > 0$ there should be a $C > 0$ such that for every $n \geq 1$ it holds that

$$\inf_{m>n} \left(\log\left(\frac{\lambda_m + \lambda_n}{\lambda_m - \lambda_n} \right) + (m - n) \right) \leq C e^{\delta \lambda_n}. \tag{3}$$

For frequencies of Beurling type, $\lambda_n = \log \nu_n$, these conditions have natural reformulations. For example, Bohr's condition (1) is equivalent to the existence of $c_1, c_2 > 0$ such that

$$\nu_{n+1} - \nu_n \geq c_1 \nu_{n+1}^{-c_2}. \tag{4}$$

Conditions (1)-(4) are usually very difficult to check for any given Beurling system, since they involve the distances between the corresponding Beurling integers. Furthermore, they often fail. This is especially true if one wants to retain properties of the ordinary integers, such as the asympotic behaviour of the counting function $N_q(x) = \sum_{\nu_n \leq x} 1$, see e.g. [11] for the subtleties that arise already when dealing with a finite sequence $q = (q_1, \ldots, q_N)$ of Beurling primes.

One motivation for considering Beurling integers is to investigate the properties of the q-zeta function

$$\zeta_q(s) = \sum_{n\geq 1} \nu_n^{-s} = \prod_{n\geq 1} \frac{1}{1 - q_n^{-s}},$$

and their interplay with the counting functions

$$N_q(x) = \sum_{\nu_n \leq x} 1, \qquad \pi_q(x) = \sum_{q_n \leq x} 1.$$

As an example, Beurling [3] himself showed that the condition

$$N_q(x) = ax + O(\frac{x}{(\log x)^\gamma}), \qquad \text{for some } \gamma > \frac{3}{2}, \tag{5}$$

implies the analogue of the prime number theorem,

$$\pi_q(x) := \sum_{q_n \le x} 1 \sim \frac{x}{\log x}. \tag{6}$$

We refer to [10] for a comprehensive overview of further developments.

In Section 2 we begin with a preparatory result which is interesting in its own right. It states that starting with the classical set of primes numbers we can add almost any finite sequence of Beurling primes while retaining the validity of Bohr's theorem.

Theorem 1.1. *Let $\{p_n\}_{n\ge1}$ be the sequence of ordinary prime numbers and let $N \ge 1$. Then Bohr's condition (1) holds for the Beurling integers generated by the Beurling primes*

$$q = \{p_n\}_{n\ge1} \bigcup \{q_j\}_{j=1}^N,$$

for almost every choice $(q_1, \dots, q_N) \in (1,\infty)^N$.

Sequences of Beurling primes of the type considered in Theorem 1.1 previously appeared in [17].

Our next result requires more careful analysis.

Theorem 1.2. *Let $q = \{q_n\}_{n\ge1}$ be an increasing sequence of Beurling primes such that $q_1 > 1$ and $\sigma_c(\zeta_q) < \infty$. Then, for every $A > 0$ there exists a sequence of Beurling primes $\tilde{q} = \{\tilde{q}_n\}_{n\ge1}$ for which Bohr's condition (1) holds and*

$$|q_n - \tilde{q}_n| \le q_n^{-A}, \qquad n \in \mathbb{N}.$$

Combining our techniques with a probabilistic method from [6], which refined previous work of Diamond, Montgomery, Vorhauer [9] and Zhang [23], we are able to construct a system of Beurling primes that satisfies Bohr's theorem as well as the analogue of the Riemann hypothesis. Specifically, we say that a Beurling system satisfies RH if the corresponding Beurling zeta function $\zeta_q(s)$ has an analytic non-vanishing extension to $\operatorname{Re} s > 1/2$, except for a simple pole at $s = 1$, of zero order in any substrip $\operatorname{Re} s > \sigma_0$, $\sigma_0 > 1/2$. This latter property means that for every $\varepsilon > 0$ and $\sigma > \sigma_0$,

$$\zeta_q(\sigma + it) = O_\varepsilon(|t|^\varepsilon), \qquad |t| \to \infty.$$

Theorem 1.3. *There exists a system of Beurling primes $q = \{q_n\}_{n\ge1}$ such that:*

(*i*) *The system satisfies RH.*

(*ii*) *The prime counting function* $\pi_q(x)$ *satisfies*

$$\pi_q(x) = \mathrm{li}(x) + O(1),$$

where $\mathrm{li}(x) = \int_1^x (1 - u^{-1}) (\log u)^{-1} \, du$.

(*iii*) *The integer counting function* $N_q(x)$ *satisfies*

$$N_q(x) = ax + O_\varepsilon(x^{1/2+\varepsilon}), \quad \textit{for all } \varepsilon > 0,$$

for some $a > 0$.

(*iv*) *The corresponding Beurling integer system satisfies Bohr's condition.*

Note that there are examples of Beurling systems such that ζ_q has a non-vanishing meromorphic extension to $\mathrm{Re}\, s > 1/2$ of infinite order in any substrip $\mathrm{Re}\, s > \sigma_0$, $1/2 < \sigma_0 < 1$, see [5]. On the other hand, if ζ_q does not vanish and has finite order in every such substrip, then it is actually of zero order, and thus the corresponding Beurling system satisfies RH. Therefore *(ii)* and *(iii)* actually imply *(i)*. The latter two statements can both be deduced from [16, Theorem 2.3].

The proofs of our results investigate how well "irrational numbers" may be approximated by fractions of Beurling integers. We will comment further on this kind of Diophantine approximation problems in Section 3. In Section 3 we will also return to the original idea behind our work. There has been an interest in studying Hardy spaces of generalized Dirichlet series since the 60s [8, 13, 14]. However, to our knowledge, except for examples that are very closely related to the ordinary integers, there has not been any discussion of the existence of Beurling primes q satisfying the prime number theorem such that Bohr's theorem holds true for the corresponding Hardy space $\mathcal{H}^\infty_q$. This is in spite of the fact that Bohr's theorem is crucial for a meaningful theory of the Hardy spaces $\mathcal{H}^p_q$, $1 \le p \le \infty$.

Since other aspects of the function theory of Hardy spaces do not depend on the choice q of Beurling primes, the motivation for Theorem 1.3 was to find a canonical Beurling system $\mathbb{N}_q$ which allows us to assume the Riemann hypothesis in the $\mathcal{H}^p_q$-theory. As a specific function theoretic application, we construct an outer function, or synonymously, a cyclic function, $f \in \mathcal{H}^2_q$ which has a zero in its half-plane of convergence,

$$f(s) = \frac{1}{\zeta_q(s + 1/2 + \varepsilon)}, \qquad 0 < \varepsilon < 1/2. \tag{7}$$

The existence of such an f constitutes a counterexample to a conjecture posed by Helson [15].

Notation. Throughout the article, we will be using the convention that C denotes a positive constant which may vary from line to line. We will write that $C = C(\Omega)$ when the constant depends on the parameter Ω.

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2. Proof of the main results

Lemma 2.1. *Suppose that $\{q_n\}_{n\geq 1}$ is a Beurling system such that $d_n := \nu_{n+1} - \nu_n \gg \nu_{n+1}^{-C}$. Then, for every $\varepsilon > 0$ and for almost every $q' > 1$, the Beurling system $\{q_n\}_{n\geq 1} \cup \{q'\}$ has a distance function satisfying*

$$d_n' = \nu_{n+1}' - \nu_n' \gg \nu_{n+1}^{-C'}, \qquad n \in \mathbb{N}, \tag{8}$$

where $C'(q',\, q) = \max\,(C,\, 2\sigma_c(\zeta_q) - 1 + \varepsilon)$.

Proof. Let $x_0 > 1$. First we will prove that the set $\mathcal{M}$ of all numbers $q' \geq x_0$ such that there exist infinitely many triples $(j, n, m) \in \mathbb{N}^3$ with

$$\left|(q')^j - \frac{\nu_n}{\nu_m}\right| \leq \nu_n^{-C_0}\nu_m^{-C_0}, \qquad C_0 = \sigma(\zeta_q) + \varepsilon,$$

has measure zero. Since

$$\left|q' - \left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}}\right| \leq x_0^{1-j}\left|(q')^j - \frac{\nu_n}{\nu_m}\right|,$$

we have that $\mathcal{M} \subset \limsup_{m,n,j} \Omega_{m,n,j}$, where

$$\Omega_{m,n,j} = \left[\left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}} - x_0^{1-j}\nu_n^{-C_0}\nu_m^{-C_0},\, \left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}} + x_0^{1-j}\nu_n^{-C_0}\nu_m^{-C_0}\right],\ j, n, m \geq 1.$$

The Borel–Cantelli lemma thus shows that $|\mathcal{M}| = 0$, since

$$\sum_{m\geq 1}\sum_{n\geq 1}\sum_{j\geq 1} |\Omega_{m,n,j}| \leq \frac{2x_0}{x_0 - 1}\zeta_q\,(C_0)^2 < \infty.$$

Fix a number $q' \in [x_0, \infty) \setminus \mathcal{M}$ such that $\log q'$ is not in the (countable) set $\operatorname{span}_{\mathbb{Q}}\{\log q_n\}$. Note that the set of such numbers has full measure in $[x_0, \infty)$, and that $x_0 > 1$ is arbitrary. By construction, there are finitely many triples (j, n, m) such that

$$\left|(q')^j - \frac{\nu_n}{\nu_m}\right| \leq \nu_n^{-C_0}\nu_m^{-C_0}. \tag{9}$$

For these exceptional triples, the left-hand side is at least positive, since $\log q' \notin \operatorname{span}_{\mathbb{Q}}\{\log q_n\}$. Therefore

$$\left|(q')^j - \frac{\nu_n}{\nu_m}\right| \gg \nu_n^{-C_0}\nu_m^{-C_0}$$

for *all* $(j,n,m) \in \mathbb{N}^3$.

Now we consider two arbitrary consecutive Beurling integers generated by the prime system $\{q_n\}_{n\geq 1} \cup \{q'\}$,

$$\nu'_{n+1} = (q')^a \nu_m, \qquad \nu'_n = (q')^b \nu_l.$$

If $a = b$, then $l = m - 1$ and

$$\nu'_{n+1} - \nu'_n \gg \nu_m^{-C} \geq \left(\nu'_{n+1}\right)^{-C},$$

by the hypothesis on the distances d_n for the original Beurling system. Otherwise, if, say, $b < a$, then

$$\left|\nu'_{n+1} - \nu'_n\right| = (q')^b \nu_m \left|(q')^{a-b} - \frac{\nu_l}{\nu_m}\right| \gg \nu_l^{-C_0}\nu_m^{-C_0+1}(q')^b \gg \left(\nu'_{n+1}\right)^{-C'}.$$

where $C' = 2\sigma_c(\zeta_q) - 1 + \varepsilon$. □

Proof of Theorem 1.1. The proof is a direct consequence of Lemma 2.1. □

In order to prove Bohr's theorem for more general Beurling systems, we need to control the constant in the distance estimate (8), which comes from the exceptional triples satisfying (9).

Proof of Theorem 1.2. Fix a small $\varepsilon > 0$ and $x_0 \in (1 + \varepsilon/2, 1 + \varepsilon)$. Consider first any Beurling system $\mathbb{N}_\rho = \{\nu_n\}_{n\geq 1}$ generated by Beurling primes such that $\rho_1 > 1 + \varepsilon$ and $\sigma_c(\zeta_\rho) < \infty$. For a number $\sigma_\infty > \max(2, A)$ to be chosen in a moment, let

$$\mathcal{N} = \bigcup_{m\geq 2}\bigcup_{n\geq 2}\bigcup_{j\geq 1}\Omega_{m,n,j},$$

where $\Omega_{m,n,j}$ is defined as in the proof of Lemma 2.1,

$$\Omega_{m,n,j} = \left[\left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}} - x_0^{1-j}\nu_n^{-\sigma_\infty}\nu_m^{-\sigma_\infty}, \left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}} + x_0^{1-j}\nu_n^{-\sigma_\infty}\nu_m^{-\sigma_\infty}\right].$$

Then $|\mathcal{N}| \leq C(\varepsilon)\left(\zeta_\rho(\sigma_\infty) - 1\right)^2$. Furthermore, for $x > 2$, let

$$I_x = [x - x^{-\frac{\sigma_\infty}{2}}, x + x^{-\frac{\sigma_\infty}{2}}].$$

Note that if σ_∞ is sufficiently large, $\sigma_\infty \geq C(\varepsilon)$, then

$$\left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}} + x_0^{1-j}\nu_n^{-\sigma_\infty}\nu_m^{-\sigma_\infty} \leq \frac{4}{3}\nu_n \qquad \text{and} \qquad x - x^{-\frac{\sigma_\infty}{2}} \geq \frac{2}{3}x.$$

Thus, $I_x \cap \Omega_{m,n,j} \neq \emptyset$ only if $\nu_n \geq x/2$. Therefore

$$|I_x \cap \mathcal{N}| \leq \sum_{\substack{m\geq 2\\ j\geq 1\\ \nu_n \geq \frac{x}{2}}} |\Omega_{m,n,j}| \leq C(\varepsilon)\left(\zeta_\rho(\sigma_\infty)-1\right) \sum_{\nu_n\geq\frac{x}{2}} \nu_n^{-\sigma_\infty}$$
$$\leq C(\varepsilon)\left(\zeta_\rho(\sigma_\infty)-1\right)\zeta_\rho\left(\frac{\sigma_\infty}{4}\right) x^{-\frac{3\sigma_\infty}{4}}.$$

We will construct a sequence of Beurling systems such that

$$(10) \qquad \left(\zeta_\rho(\sigma_\infty)-1\right)\zeta_\rho\left(\frac{\sigma_\infty}{4}\right) \leq 1$$

for the number $\sigma_\infty > 0$, still to be chosen later. Therefore

$$(11) \qquad |I_x \cap \mathcal{N}| \leq C(\varepsilon) x^{-\frac{\sigma_\infty}{4}} |I_x|,$$

We conclude that whenever x is sufficiently large, $I_x \not\subset \mathcal{N}$.

To include triples where ν_n or ν_m equals one in our considerations, we increase the power σ_∞. The inequality

$$(12) \qquad \left| x^j - \frac{\nu_n}{\nu_m} \right| \leq \nu_n^{-3\sigma_\infty} \nu_m^{-3\sigma_\infty}$$

implies, whenever $x \geq x_0$, that

$$\left| x - \left(\frac{\nu_n}{\nu_m}\right)^{\frac{1}{j}} \right| \leq x_0^{1-j} \left| x^j - \frac{\nu_n^2 \nu_m}{\nu_m^2 \nu_n} \right| \leq x_0^{1-j} \left(\nu_m^2 \nu_n\right)^{-\sigma_\infty} \left(\nu_n^2 \nu_m\right)^{-\sigma_\infty}.$$

Therefore $\mathcal{M} \subset \mathcal{N}$, where $\mathcal{M}$ this time denotes the set of all $x \geq x_0$ for which there exists an exceptional triple $(j,n,m) \in \mathbb{N}^3$ such that (12) holds.

Now let q be a sequence of primes in the statement of Theorem 1.2, assuming that $\varepsilon < q_1 - 1$. As described, we will only be able to effectively apply (11) when x is sufficiently large, say, $x \geq B = B(\varepsilon) = C(\varepsilon)^4 + 2$, where $C(\varepsilon)$ in this instance refers to the same constant as in (11). Let N be such that $\{q_1, \ldots, q_N\} = (1, B) \cap q$. Then, as a corollary of Theorem 1.1, we already know that there exists an increasing finite sequence of primes $\{\tilde{q}_1, \ldots \tilde{q}_N\}$, $\tilde{q}_1 > 1$, such that $|q_j - \tilde{q}_j| \leq q_j^{-A}$, $j = 1, \ldots, N$, and such that Bohr's condition holds for $\{\nu_n^{(N)}\}_{n\geq 1} = \mathbb{N}_{\{\tilde{q}_1,\ldots\tilde{q}_N\}}$. Further, we choose σ_∞ so large that

$$\left|\nu_{n+1}^{(N)} - \nu_n^{(N)}\right| \geq \left(\nu_{n+1}^{(N)}\right)^{-6\sigma_\infty}, \qquad n \in \mathbb{N},$$

and

$$(13) \qquad \left(\zeta_{q'}(\sigma_\infty)-1\right)\zeta_{q'}\left(\frac{\sigma_\infty}{4}\right) \leq 1, \quad q' = \{\tilde{q}_1, \ldots \tilde{q}_N, q_{N+1}-1, q_{N+2}-1, q_{N+3}-1, \ldots\}.$$

This is made possible by the hypothesis that $\sigma_c(\zeta_q) < \infty$, since

$$\zeta_{q'}(\sigma) \leq \prod_{j\geq 1} \frac{1}{1-(q'_j)^{-\sigma}} \leq \zeta_q\left(\frac{\sigma}{C}\right), \qquad \sigma > 0, \qquad C \geq \sup_{n\geq 1} \frac{\log(q_n)}{\log(q'_n)}.$$

From here we proceed by induction. Suppose that $\tilde{q}_1, \dots \tilde{q}_k$ have been chosen, where $k \geq N$, with corresponding Beurling integers $\{\nu_n^{(k)}\}_{n\geq 1} = \mathbb{N}_{\{\tilde{q}_n\}_{n=1}^k}$ satisfying that

$$\left|\nu_{n+1}^{(k)} - \nu_n^{(k)}\right| \geq \left(\nu_{n+1}^{(k)}\right)^{-6\sigma_\infty}.$$

We apply the preceding discussion to the Beurling primes $\rho = \{\tilde{q}_1, \dots \tilde{q}_k\}$ and $x = q_{k+1}$, concluding that there exists a number $\tilde{q}_{k+1} \in I_{q_{k+1}}$ such that

$$\left|\tilde{q}_{k+1}^j - \frac{\nu_n^{(k)}}{\nu_m^{(k)}}\right| \geq \left(\nu_n^{(k)}\right)^{-3\sigma_\infty} \left(\nu_m^{(k)}\right)^{-3\sigma_\infty}, \qquad (j,n,m) \in \mathbb{N}^3.$$

By the same argument as in the last paragraph of the proof of Theorem 1.1 the Beurling system $\{\nu_n^{(k+1)}\}_{n\geq 1} = \mathbb{N}_{\{\tilde{q}_n\}_{n=1}^{k+1}}$, then satisfies that

$$\left|\nu_{n+1}^{(k+1)} - \nu_n^{(k+1)}\right| \geq \left(\nu_{n+1}^{(k+1)}\right)^{-6\sigma_\infty}, \qquad n \in \mathbb{N}.$$

At each step of the construction, (13) ensures that (10) holds. We hence obtain a sequence $\tilde{q} = \{\tilde{q}_n\}_{n\geq 1}$, satisfying that $|\tilde{q}_n - q_n| \leq q_n^{-\frac{\sigma_\infty}{2}}$ as well as Bohr's condition (1), specifically,

$$|\tilde{\nu}_{n+1} - \tilde{\nu}_n| \geq (\tilde{\nu}_{n+1})^{-6\sigma_\infty}, \qquad n \in \mathbb{N}.$$

where $\{\tilde{\nu}_n\}_{n\geq 1} = \mathbb{N}_{\tilde{q}}$. □

To prove Theorem 1.3, we shall combine the proof of Theorem 1.2 with the probabilistic construction of [6, Theorem 1.2]. Let

$$F(x) = \mathrm{li}(x) = \int_1^x \frac{1-u^{-1}}{\log u}\, du$$

and set $x_n = F^{-1}(n)$. We select the nth Beurling prime q_n randomly from the interval $[x_n, x_{n+1}]$ according to the probability measure $d\,\mathrm{li}(x)|_{[x_n,x_{n+1}]}$. That is, we consider a sequence of independent random variables Q_n, representing the coordinate functions $(q_1, q_2, \dots) \mapsto q_n$, with cumulative distribution function $\int_{x_n}^x d\,\mathrm{li}(u) = \mathrm{li}(x) - n$, $x_n \leq x \leq x_{n+1}$. Formally, the probability space is $X = \prod_{n=1}^\infty [x_n, x_{n+1}]$, and by appealing to Kolmogorov's extension theorem, we can equip

X with a probability measure dP such that

$$P\left(A\times\prod_{n=k+1}^{\infty}[x_n,x_{n+1}]\right)=\int_A d\,\mathrm{li}(u_1)\cdots d\,\mathrm{li}(u_k),\quad \text{if } A\subseteq[x_1,x_2]\times\cdots\times[x_k,x_{k+1}].$$

Proof of Theorem 1.3. Let $A>1$. We will show the existence of a sequence of Beurling primes $q=\{q_n\}_{n\geq 1}$ generating integers $\mathbb{N}_q=\{\nu_n\}_{n\geq 1}$ with the following properties:

(a) The Beurling zeta function $\zeta_q(s)$ can be written as

$$\zeta_q(s)=\frac{se^{Z(s)}}{s-1},$$

where $Z(s)$ is an analytic function in $\{\operatorname{Re}s>1/2\}$ which in every closed half-plane $\{\operatorname{Re}s\geq\sigma_0\}$, $\sigma_0>1/2$, satisfies

$$|Z(s)|\ll_{\sigma_0}\sqrt{\log(|t|+2)},\qquad s=\sigma+it.$$

(b) The Beurling integers satisfy $|\nu_n-\nu_m|\geq(\nu_n\nu_m)^{-A}$ whenever $n\neq m$.

Suppose that ν,μ are relatively prime Beurling integers, $(\nu,\mu)=1$, satisfying $|\nu-\mu|<(\nu\mu)^{-A}$. Let q_k be the largest prime factor of $\nu\mu$. Without loss of generality $q_k\mid\nu$ and $q_k\nmid\mu$, and let j be such that $\nu=q_k^j\nu'$ where $q_k\nmid\nu'$. Then

$$\left|q_k^j-\frac{\mu}{\nu'}\right|<\frac{1}{q_k^{Aj}(\nu')^{1+A}\mu^A},$$

so that

$$\left|q_k-\left(\frac{\mu}{\nu'}\right)^{1/j}\right|<\frac{1}{q_k^{(A+1)j-1}(\nu')^{1+A}\mu^A}\leq\frac{1}{x_k^{(A+1)j-1}(\nu')^{1+A}\mu^A}.$$

This motivates the following definitions. For $k,j\geq 1$ and

$$q_1\in[x_1,x_2],q_2\in[x_2,x_3],\dots,q_{k-1}\in[x_{k-1},x_k],$$

we set

$$\begin{aligned}&\mathcal{M}_{k,j}(q_1,\dots,q_{k-1})\\ &=\bigcup_{\nu,\mu\in\mathbb{N}_{(q_1,\dots,q_{k-1})}}\left[\left(\frac{\mu}{\nu}\right)^{1/j}-\frac{1}{x_k^{(A+1)j-1}\nu^{1+A}\mu^A},\left(\frac{\mu}{\nu}\right)^{1/j}+\frac{1}{x_k^{(A+1)j-1}\nu^{1+A}\mu^A}\right],\end{aligned}$$

and we consider the events

$$B_{k,j}=\big\{(q_1,q_2,\dots):q_k\in\mathcal{M}_{k,j}(q_1,\dots,q_{k-1})\big\}.$$

Denoting $\mathcal{X} = (x_1, x_2, \dots)$, the Lebesgue measure of $\mathcal{M}_{k,j}(q_1, \dots, q_{k-1})$ is bounded by $2\zeta_{\mathcal{X}}(1+A)\zeta_{\mathcal{X}}(A)x_k^{1-(A+1)j}$. Note that $\pi_{\mathcal{X}}(x) = \operatorname{li}(x) + O(1)$, and so $\zeta_{\mathcal{X}}(s)$ has abscissa of convergence 1. Hence, for the probability of $B_{k,j}$ we have

$$\begin{aligned} P(B_{k,j}) &= \int_{x_1}^{x_2} d\operatorname{li}(u_1) \int_{x_2}^{x_3} d\operatorname{li}(u_2) \dots \int_{x_{k-1}}^{x_k} d\operatorname{li}(u_{k-1}) \int_{[x_k, x_{k+1}] \cap \mathcal{M}_{k,j}(u_1, \dots, u_{k-1})} d\operatorname{li}(u_k) \\ &\leq \frac{2\zeta_{\mathcal{X}}(1+A)\zeta_{\mathcal{X}}(A)}{x_k^{(A+1)j-1}}. \end{aligned}$$

In particular, $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} P(B_{k,j}) < \infty$.

We also consider the events

$$A_{k,m} = \left\{ (q_1, q_2, \dots) : \Big| \sum_{n=1}^{k} q_n^{-im} - \int_{x_1}^{x_k} u^{-im} d\operatorname{li}(u) \Big| \geq 8 \sqrt{\frac{x_k}{\log x_k}} \big(\sqrt{\log x_k} + \sqrt{\log m} \big) \right\}.$$

In the proof of [6, Theorem 1.2] it was shown that also $\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} P(A_{k,m}) < \infty$. Hence, by the Borel–Cantelli lemma we have with probability 1 that the sequence $(q_1, q_2, \dots)$ is contained in only finitely many of the sets $A_{k,m}$ and $B_{k,j}$. Take any such sequence q.

Note that by construction, $\pi_q(x) = \operatorname{li}(x) + O(1)$. As q is contained in only finitely many $A_{k,m}$, we obtain that

$$\begin{aligned} &\left| \sum_{q_n \leq x} q_n^{-it} - \int_{x_1}^{x} u^{-it} \, d\operatorname{li}(u) \right| \\ &\qquad \ll \sqrt{\frac{x}{\log(x+1)}} \big(\sqrt{\log(x+1)} + \sqrt{\log(|t|+1)} \big), \quad x \geq 1, \quad t \in \mathbb{R}. \end{aligned} \tag{14}$$

For $x = x_k$ and $t = m \in \mathbb{Z}$ this is clear. If $x \in (x_k, x_{k+1})$, then both terms in the absolute value of (14) change by at most $O(1)$ upon replacing x by x_k. Hence the bound also holds for any $t = m \in \mathbb{Z}$ and arbitrary $x \geq 1$. To obtain the bound for $t \in (m, m+1)$, we write

$$\sum_{q_n \leq x} q_n^{-it} = \int_{x_1}^{x} u^{-i(t-m)} \, d\Big(\sum_{q_n \leq u} q_n^{-im} \Big),$$

$$\int_{x_1}^{x} u^{-it} \, d\operatorname{li}(u) = \int_{x_1}^{x} u^{-i(t-m)} \, d\Big(\int_{x_1}^{u} v^{-im} \, d\operatorname{li}(v) \Big),$$

and integrate by parts.

The bound (14) implies (a). Indeed, setting $\Pi_q(x) = \pi_q(x) + \frac{\pi_q(x^{1/2})}{2} + \cdots$, we have

$$\log \zeta_q(s) = \int_1^\infty x^{-s}\, d\Pi_q(x)$$
$$= \log \frac{s}{s-1} + \int_1^\infty x^{-s}\, d\big(\pi_q(x) - \mathrm{li}(x)\big) + \int_1^\infty x^{-s}\, d\big(\Pi_q(x) - \pi_q(x)\big).$$

Let $\sigma \geq \sigma_0 > 1/2$. In the second integral we integrate by parts and use (14) to see that it is $O_{\sigma_0}\big(\sqrt{\log(|t|+2)}\big)$. The third integral is $O_{\sigma_0}(1)$ for $\sigma \geq \sigma_0 > 1/2$, since $\Pi_q(x) - \pi_q(x)$ is non-decreasing and $\ll \sqrt{x}/\log x$.

Now let $k_1, k_2, \ldots, k_l$ be the exceptional integers of the construction. Then $q \notin B_{k,j}$ for all j and $k \neq k_1, \ldots, k_l$. We simply remove the corresponding Beurling primes from the system: $\tilde{q} = q \setminus \{q_{k_1}, \ldots, q_{k_l}\}$. As $\zeta_{\tilde{q}}(s) = \zeta(s)(1 - q_{k_1}^{-s}) \cdots (1 - q_{k_l}^{-s})$, (a) remains valid for $\zeta_{\tilde{q}}(s)$. Finally, every two distinct Beurling integers $\nu_n \neq \nu_m$ from $\mathbb{N}_{\tilde{q}}$ satisfy $|\nu_n - \nu_m| \geq (\nu_n \nu_m)^{-A}$. For if this were not the case, then, by the argument at the beginning of the proof, the largest prime factor q_k of $\nu_n \nu_m/(\nu_n, \nu_m)^2$ would be contained in some $\mathcal{M}_{k,j}(q_1, \ldots, q_{k-1})$, which is impossible by construction.

Every point in Theorem 1.3 has now been proven, except for *(iii)*. However, since $\zeta_{\tilde{q}}(s)$ is of zero order in $\{\mathrm{Re}\, s > \sigma_0\}$ for every $\sigma_0 > 1/2$ by (a), this follows from a standard application of Perron inversion, see [16, 23]. □

3. Further discussion

Diophantine approximation and Beurling integers. Using the Borel–Cantelli theorem to study the irrationality of real numbers is a standard technique of Diophantine approximation. The irrationality measure $\mu(x)$ of a real number $x \in \mathbb{R}$ is defined as the infimum of the set

$$R_x = \left\{ r > 0 : \left| x - \frac{m}{n} \right| < \frac{1}{n^r} \text{ for at most finitely many pairs } (m,n) \in \mathbb{N} \times \mathbb{N} \right\}.$$

For a Beurling system $\mathbb{N}_q = \{\nu_n\}_{n \geq 1}$, we may also introduce the irrationality measure $\mu_q(x)$ of a real number $x \in \mathbb{R}$ as the infimum of the set

$$R_x = \left\{ r > 0 : \left| x - \frac{\nu_m}{\nu_n} \right| < \frac{1}{\nu_n^r} \text{ for at most finitely many pairs } (m,n) \in \mathbb{N} \times \mathbb{N} \right\}.$$

Then, by slightly modifying the proof of Lemma 2.1, we obtain the following proposition.

Proposition 3.1. *Let $q = \{q_n\}_{n\geq 1}$ be a sequence of Beurling primes with $\sigma_c(\zeta_q) < \infty$. Then, for almost every $x \in \mathbb{R}$, it holds that*

$$\mu_q(x) \leq 2\sigma_c(\zeta_q).$$

In the classical case, Dirichlet's approximation theorem therefore implies that $\mu(x) = 2$ for almost every $x \in \mathbb{R}$. We also recall Roth's theorem [7], which states that $\mu(x) = 2$ for every algebraic irrational number. It would be very interesting to develop corresponding results in the context of Beurling integers.

Hardy spaces of Dirichlet series and a conjecture of Helson. For a sequence q of Beurling primes, we introduce the Hardy space $\mathcal{H}_q^2$ as

$$\mathcal{H}_q^2 = \left\{ f(s) = \sum_{n\geq 1} a_n \nu_n^{-s} \; : \; \|f\|_{\mathcal{H}_q^2}^2 = \sum_{n\geq 1} |a_n|^2 < \infty \right\}.$$

More generally, for $1 \leq p < \infty$, we define $\mathcal{H}_q^p$ as the completion of polynomials (finite sums $\sum a_n \nu_n^{-s}$) under the Besicovitch norm

$$\|P\|_{\mathcal{H}_q^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt \right)^{\frac{1}{p}}.$$

The function theory of these spaces originated with Helson [14], and was, in the distuingished case where q is the sequence of ordinary primes, continued in very influential papers of Bayart [1] and Hedenmalm, Lindqvist, and Seip [13]. More generally, there is a developing theory of Hardy spaces of Dirichlet series $\sum a_n e^{-\lambda_n s}$ whose frequencies are related to other groups than $\mathbb{T}^\infty$, but we shall restrict our attention to frequencies given by Beurling primes. A cornerstone of the theory is that there is a natural multiplicative linear isometric isomorphism between $\mathcal{H}_q^p$ and the Hardy space $H^p(\mathbb{T}^\infty)$ of the infinite torus [8, 15]. However, more is needed in order to identify $H^\infty(\mathbb{T}^\infty)$ with $\mathcal{H}_q^\infty$, the space of Dirichlet series $\sum a_n \nu_n^{-s}$ which converge to a bounded function in $\mathbb{C}_0 = \{\operatorname{Re} s > 0\}$. In fact, Bohr's condition is typically used in order to establish this isomorphism [20].

In identifying $\mathcal{H}_q^p$ with $H^p(\mathbb{T}^\infty)$ one is naturally led to consider twisted Dirichlet series

$$f_\chi(s) = \sum_{n\geq 1} a_n \chi(\nu_n) \nu_n^{-s},$$

where a point $\chi \in \mathbb{T}^\infty$ is interpreted as the completely multiplicative character $\chi \colon \mathbb{N}_q \to \mathbb{T}$ such that $\chi(q_n) = \chi_n$. Helson [15] proved that if $f \in \mathcal{H}_q^2$ and the associated frequencies satisfy Bohr's condition, then $f_\chi(s)$ converges in $\mathbb{C}_0$ for almost every $\chi \in \mathbb{T}^\infty$. Helson went on to make a conjecture, which we state only in the special case that the frequencies correspond to a Beurling system. Recall that $f \in \mathcal{H}_q^2$ is said to be outer (or cyclic) if $\{fg : g \in \mathcal{H}_q^\infty\}$ is dense in $\mathcal{H}_q^2$.

Conjecture. If $\mathbb{N}_q$ is a Beurling system that satisfies Bohr's condition and f is outer in $\mathcal{H}^2_q$, then f_χ never has any zeros in its half-plane of convergence.

Suppose now that the Beurling primes q are chosen as in Theorem 1.3, so that we have the Riemann hypothesis at our disposal, and consider the Dirichlet series

$$f(s) = \frac{1}{\zeta_q(s + 1/2 + \varepsilon)},$$

for some $0 < \varepsilon < 1/2$. Through a routine calculation with coefficients, one checks that $f, f^2, 1/f, 1/f^2 \in \mathcal{H}^2_q$. Therefore, there are polynomials p_n which converge to $1/f$ in $\mathcal{H}^4_q$, so that

$$\|1 - p_n f\|_{\mathcal{H}^2_q} \leq \|f\|_{\mathcal{H}^4_q} \|1/f - p_n\|_{\mathcal{H}^4_q} \to 0, \qquad n \to \infty.$$

Thus, f is outer. On the other hand, it has a zero at $s = 1/2 - \varepsilon$. To disprove Helson's conjecture it only remains to prove that f converges in $\mathbb{C}_0$.

Proposition 3.2. *The reciprocal $1/\zeta_q$ of the Beurling zeta function converges in $\{\operatorname{Re} s > 1/2\}$.*

Proof. The reciprocal $1/\zeta_q$ is of zero order in $\{\operatorname{Re} s > \sigma_0\}$, for every $\sigma_0 > 1/2$. This is clear from the proof of Theorem 1.3, but it is also well known that this can be deduced from *(ii)* and *(iii)* using the Borel–Carathéodory and the Hadamard three circles theorems,

$$\log \zeta_q(\sigma + it) = O\left((\log |t|)^\alpha\right), \qquad \alpha \in (0, 1), \tag{15}$$

uniformly for $\frac{1}{2} < \sigma_0 \leq \sigma \leq 1$. See for example [16, Theorem 2.3] or [22, Theorem 14.2]. Since we have Bohr's condition, the standard argument [21, Section 9.44] with Perron's formula then shows that $1/\zeta_q$ is convergent in the half-plane where it is analytic with zero order. □

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Article 5: An extension of Bohr's theorem

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AN EXTENSION OF BOHR'S THEOREM

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Abstract. The following extension of Bohr's theorem is established: If a somewhere convergent Dirichlet series f has an analytic continuation to the half-plane $\mathbb{C}_\theta = \{s = \sigma + it \,:\, \sigma > \theta\}$ that maps $\mathbb{C}_\theta$ to $\mathbb{C} \setminus \{\alpha, \beta\}$ for complex numbers $\alpha \neq \beta$, then f converges uniformly in $\mathbb{C}_{\theta+\varepsilon}$ for any $\varepsilon > 0$. The extension is optimal in the sense that the assertion no longer holds should $\mathbb{C} \setminus \{\alpha, \beta\}$ be replaced with $\mathbb{C} \setminus \{\alpha\}$.

1. Introduction

Let $\mathfrak{D}$ denote the class of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

that converge in at least one point $s = \sigma + it$ in the complex plane. Associated to each Dirichlet series f in $\mathfrak{D}$ is a number $\sigma_{\mathrm{c}}(f)$, called the *abscissa of convergence*, with the property that f converges if $\sigma > \sigma_{\mathrm{c}}(f)$ and f does not converge if $\sigma < \sigma_{\mathrm{c}}(f)$. This note concerns an extension of Bohr's classical theorem on uniform convergence of Dirichlet series [3]. We therefore define the *abscissa of uniform convergence* $\sigma_{\mathrm{u}}(f)$ as the infimum of the real numbers θ such that f converges uniformly in the half-plane $\mathbb{C}_\theta$. Here and in what follows, we set

$$\mathbb{C}_\theta = \{s = \sigma + it \,:\, \sigma > \theta\}.$$

Our starting point reads as follows.

Bohr's theorem. *Let f be in $\mathfrak{D}$. If there is a real number θ and a bounded set Ω such that f has an analytic continuation to $\mathbb{C}_\theta$ that maps $\mathbb{C}_\theta$ to Ω, then $\sigma_{\mathrm{u}}(f) \leq \theta$.*

Queffélec and Seip [10] (see also [9, Theorem 8.4.1]) showed that the assumption that Ω is a bounded set may be replaced with the weaker assumption that Ω is a half-plane. This extension of Bohr's theorem was applied obtain the canonical formulation of the Gordon–Hedenmalm characterization of composition

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operators [6], which has proven to be essential for further developments (see e.g. [5, Section 6]).

The purpose of the present note is to delineate precisely the limits to how far Bohr's theorem may be extended in terms of the mapping properties of f in the half-plane $\mathbb{C}_\theta$. We will achieve this by establishing the following results.

Theorem 1. *Let f be in $\mathfrak{D}$. If there is a real number θ and complex numbers $\alpha \neq \beta$ such that f has an analytic continuation to $\mathbb{C}_\theta$ that maps $\mathbb{C}_\theta$ to $\mathbb{C} \setminus \{\alpha, \beta\}$, then $\sigma_{\mathrm{u}}(f) \leq \theta$.*

Theorem 2. *There is a Dirichlet series f with $\sigma_{\mathrm{c}}(f) \leq 1/2$, $\sigma_{\mathrm{u}}(f) = 1$, and*

$$f(\mathbb{C}_\theta) = \mathbb{C} \setminus \{0\}$$

for any $1/2 \leq \theta \leq 1$.

It must be stressed that both results are fairly direct consequences of well-known techniques and results. The proof of Theorem 1 uses Schottsky's theorem similarly to how it is used by Titchmarsh in the introduction to [12, Chapter XI], while Theorem 2 is deduced from results of Bohr [2] and Helson [8] on the Riemann zeta function.

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2. Proof of Theorem 1 and Theorem 2

We begin with some preparation for the proof of Theorem 1. Let $\mathbb{D}(c, r)$ denote the open disc with center c and radius $r > 0$. If f is analytic and different from 0 and 1 in $\mathbb{D}(c, r)$, then the effective version of Schottsky's theorem due to Ahlfors [1] states that

$$|f(s)| \leq \exp\left(\frac{r + |s-c|}{r - |s-c|}\big(7 + \max(0, \log|f(c)|)\big)\right) \tag{2}$$

for all s in $\mathbb{D}(c, r)$. (We do not actually require the effective version of Schottsky's theorem, but we find it more convenient to work with explicit expressions.)

Proof of Theorem 1. We may assume without loss of generality that $\alpha = 0$ and $\beta = 1$. It is well-known (see e.g. [9, Chapter 4.2]) that $\sigma_{\mathrm{u}}(f) \leq \sigma_{\mathrm{c}}(f) + 1$, so every Dirichlet series in $\mathfrak{D}$ converges uniformly in some half-plane. For $\vartheta > \theta$, we set

$$M(f, \vartheta) = \sup_{t \in \mathbb{R}} |f(\vartheta + it)|.$$

It is plain that $M(f, \vartheta) < \infty$ if $\vartheta > \sigma_{\mathrm{u}}(f)$. We fix $\vartheta > \sigma_{\mathrm{u}}(f)$ and apply (2) with $c = \vartheta + it$, $r = \vartheta - \theta$, and $s = \sigma + it$, to infer that if $\theta < \sigma < \vartheta$, then

$$|f(s)| \leq \exp\left(\frac{2(\vartheta - \theta)}{\sigma - \theta}\big(7 + \max(0, \log|M(f, \vartheta)|)\big)\right). \tag{3}$$

This demonstrates that f is bounded in $\mathbb{C}_{\theta+\varepsilon}$ for any $\varepsilon > 0$, and, consequently, that $\sigma_{\mathrm{u}}(f) \leq \theta$ by Bohr's theorem. □

Ritt [11, Theorem II] established a version of Schottsky's theorem for convergent Dirichlet series. This result provides an upper bound similar to (3) that is valid in all of $\mathbb{C}_\theta$ and that only depends on θ and a_1, under the additional assumption that a_1 is not equal to 0 or 1. Here a_1 denotes the first coefficient in the series (1).

To prepare for the proof of Theorem 2, we consider the vertical translation

$$V_\tau f(s) = f(s + i\tau).$$

The *vertical limit functions* of a Dirichlet series f in $\mathfrak{D}$ are the functions which can be obtained as uniform limits of sequences of vertical translations $(V_{\tau_k} f)_{k\geq 1}$ in $\mathbb{C}_\theta$ for any fixed $\theta > \sigma_{\mathrm{u}}(f)$. Recall from [7, Section 2.3] that the vertical limit functions of the Dirichlet series (1) coincide with the Dirichlet series of the form

$$f_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s},$$

where χ is a completely multiplicative function from the natural numbers to the unit circle. Certain properties of f are preserved under vertical limits. For instance, Bohr's theorem implies that if f is in $\mathfrak{D}$, then $\sigma_{\mathrm{u}}(f) = \sigma_{\mathrm{u}}(f_\chi)$ for any χ. A consequence of Rouché's theorem (see e.g. [4, Lemma 1]) is that $f_\chi(\mathbb{C}_\theta) = f(\mathbb{C}_\theta)$ for any χ and any $\theta \geq \sigma_{\mathrm{u}}(f)$. However, the abscissa of convergence for f and f_χ may in general be different (see [7, 8] or [9, Chapter 8.4]).

Proof of Theorem 2. We begin with the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which satisfies $\sigma_{\mathrm{c}}(\zeta) = \sigma_{\mathrm{u}}(\zeta) = 1$. A result of Bohr [2] (see also [9, Chapter 4.5]) asserts that $\zeta(\mathbb{C}_1) = \mathbb{C} \setminus \{0\}$. By the discussion above, it follows that $\sigma_{\mathrm{u}}(\zeta_\chi) = 1$ and that $\zeta_\chi(\mathbb{C}_1) = \mathbb{C} \setminus \{0\}$ for any χ. Helson [8] established that there are χ such that the Dirichlet series ζ_χ converges and does not vanish in the half-plane $\mathbb{C}_{1/2}$. Choosing $f = \zeta_\chi$ for such a χ, we obtain the stated result. □

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Part 3

Universality

Article 6: On universality of general Dirichlet series

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Submitted

ON UNIVERSALITY OF GENERAL DIRICHLET SERIES

FRÉDÉRIC BAYART AND ATHANASIOS KOUROUPIS

ABSTRACT. In the present work we establish sufficient conditions for a Dirichlet series induced by general frequencies to be universal with respect to vertical translations. Applying our methodology we give examples of universal Dirichlet series such as the alternating prime zeta function $\sum_{n\geq 1}(-1)^n p_n^{-s}$.

1. INTRODUCTION

The study of the universal properties of Dirichet series goes back to 1975 with the seminal work of Voronin on the Riemann zeta function [19]. Voronin's theorem says:

> Let K be a compact subset of $\{1/2 < \mathrm{Re}(s) < 1\}$ with connected complement, let f be a nonvanishing function continuous on K and holomorphic in the interior of K. Then
>
> $$\underline{\mathrm{dens}}\left\{\tau \geq 0 : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$
>
> where $\underline{\mathrm{dens}}(A)$ denotes the lower density of $A \subset \mathbb{R}_+$, that is
>
> $$\underline{dens}(A) = \liminf_{T\to\infty} \int_0^T \mathbf{1}_A(t)\,dt.$$

Let us introduce the following definitions: let $\Omega_1 \subset \Omega \subset \mathbb{C}$ be two domains such that $\Omega_1 + i\tau \subset \Omega_1$ for all $\tau > 0$, and, for all compact sets $K \subset \Omega$, there exists $\tau > 0$ with $K + i\tau \subset \Omega_1$. Let $D : \Omega_1 \to \mathbb{C}$ be holomorphic. We say that D is universal in Ω if, for all compact subsets K of Ω with connected complement, for all nonvanishing functions $f : K \to \Omega$, continuous on K and holomorphic in the interior of K,

$$\underline{\mathrm{dens}}\left\{\tau \geq 0 : \sup_{s\in K} |D(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

We say that f is strongly universal if the restriction that it is non-vanishing can be eased.

Since Voronin's work, the area of universality gained popularity. Many authors studied aspects of (strong) universality for various classes of Dirichlet series $\sum_{n\geq 1} a_n e^{-\lambda_n s}$, where $\{a_n\}_{n\geq 1} \subset \mathbb{C}^{\mathbb{N}}$ and $\{\lambda_n\}_{n\geq 1}$ is an increasing sequence of nonnegative real numbers tending to $+\infty$. The case $\{\lambda_n\}_{n\geq 1} = \{\log n\}_{n\geq 1}$ corresponds to ordinary Dirichlet series. The survey paper [12] provides a thorough examination of the subject up to 2015.

The first author in [4] improving the work of [11] on strong universality of general Dirichel series obtained the following result:

> Let $P \in \mathbb{R}_d[X]$ with $d \geq 1$ and $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$, let $\omega \in \mathbb{R}\backslash 2\pi\mathbb{Z}$ and let $\kappa \in \mathbb{R}$. Assume moreover that the sequence $\{\log P(n)\}_{n\geq 1}$ is $\mathbb{Q}$-linearly independent. Then the Dirichlet series $D(s) = \sum_{n\geq 1} Q(n)(\log n)^{\kappa} e^{i\omega n}(P(n))^{-s}$ is strongly universal in $\{(2d-1)/2d < \mathrm{Re}(s) < 1\}$.

This generalizes the case of the Lerch zeta function (see [10]) when $Q(n) = 1$, $\kappa = 0$ and $P(n) = n + \alpha$ with α transcendental.

One of the objectives of this article is to study potential cases of universal Dirichlet series for which the methods of [4] are not applicable, see for example [4, Question 6.8]. The first one concerns the frequencies $\{\lambda_n\}_{n\geq 1}$. The simplest example of $\mathbb{Q}$-linearly independent frequencies $\{\lambda_n\}_{n\geq 1}$ are probably the sequence $\{\log p_n\}_{n\geq 1}$ of the logarithms of prime numbers. However, this sequence is not regular enough to be handled by the methods of [4]. The main barrier for this problem is that the sequence of primes is not regular enough to estimate the associated exponential sums using the classical techniques from harmonic analysis like the method of non-stationary phase/ Van Der Corput type lemmas 2.5 or decoupling [6]. We give a partial answer to this question.

Theorem 1.1. *Let $D(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ be a Dirichlet series and let $d \in \mathbb{N}$. Assume that*

- $\lambda_n = \log P(p_n)$ *where* $P \in \mathbb{R}_d[X]$ *and* $\lim_{+\infty} P = +\infty$.
- *there exist* $C, \kappa > 0$ *such that for* $n \geq 2$,
$$\frac{n^{d-1}}{C(\log n)^{\kappa}} \leq |a_n| \leq C n^{d-1}(\log(n))^{\kappa}.$$
- $\sigma_c(D) = 0$.
- *The sequence* $\{\lambda_n\}_{n\geq 1}$ *is* $\mathbb{Q}$*-linearly independent.*

Then D is strongly universal in $\left\{1 - \frac{1}{3d} < \mathrm{Re}(s) < 1\right\}$.

Corollary 1.2. *The Dirichlet series $\sum_{n\geq 1} (-1)^n p_n^{-s}$ is strongly universal in the strip $\{\frac{2}{3} < \mathrm{Re}(s) < 1\}$.*

Observe that for the examples coming from [4] or from Theorem 1.1, the Dirichlet series itself converges in its strip of universality. This does not cover the case of the Riemann zeta function or that of the Hurwitz zeta functions $\sum_{n\geq 1}(n+\alpha)^{-s}$, α transcendental, which have a pole at 1 and are known to be universal in $\{\frac{1}{2}<\mathrm{Re}(s)<1\}$. We extend those results to a large class of general Dirichlet series, even in the case 1 is a branching point and not a pole. In what follows we denote by $\mathbb{C}_\sigma$ the half-plane $\{\mathrm{Re}(s)>\sigma\}$ and by $\mathbb{C}_\sigma^+$ its restriction to the complex numbers of positive imaginary part, $\{s\in\mathbb{C}:\ \mathrm{Re}(s)>\sigma,\ \mathrm{Im}(s)>0\}$.

Theorem 1.3. *Let $P\in\mathbb{R}_d[X]$ with $\lim_{+\infty} P=+\infty$. Let $Q\in\mathbb{R}_{d-1}[X]$ and let $\kappa\in\mathbb{R}$. Assume moreover that $\{\log P(n)\}_{n\geq 1}$ is $\mathbb{Q}$-linearly independent. Then the Dirichlet series $D(s)=\sum_{n\geq 1}Q(n)(\log(n))^\kappa[P(n)]^{-s}$ admits a holomorphic continuation to $\mathbb{C}^+_{1-\frac{1}{d}}\cup\mathbb{C}_1$ and even to $\mathbb{C}_{1-\frac{1}{d}}\backslash\{1\}$ if $\kappa\in\mathbb{N}_0$. Moreover, it is strongly universal in the strip $\left\{1-\frac{1}{2d}<\mathrm{Re}(s)<1\right\}$.*

Notation. Throughout the paper, if $f,g:E\to\mathbb{R}$ are two functions defined on the same set E, the notation $f\lesssim g$ will mean that there exists some constant $C>0$ such that $f\leq Cg$ on E.

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2. Preliminaries

2.1. **Abscissas of convergence.** To a Dirichlet series $D=\sum_{n=1}^{+\infty}a_ne^{-\lambda_n s}$ we will associate three abscissas, its abscissa of convergence,

$$\sigma_c(D):=\inf\left\{\mathrm{Re}(s):\ \sum_{n\geq 1}a_ne^{-\lambda_n s}\text{ converges}\right\},$$

its abscissa of absolute convergence

$$\sigma_a(D):=\inf\left\{\sigma\in\mathbb{R}:\ \sum_{n\geq 1}|a_n|e^{-\lambda_n\sigma}\text{ converges}\right\},$$

and also

$$\sigma_2(D):=\inf\left\{\sigma\in\mathbb{R}:\ \sum_{n\geq 1}|a_n|^2e^{-2\lambda_n\sigma}\text{ converges}\right\}.$$

It is well-known that $D=\sum_{n\geq 1}a_ne^{-\lambda_n s}$ converges in the half-plane $\mathbb{C}_{\sigma_c(D)}$ and that it defines a holomorphic function there.

2.2. **How to prove universality.** Let us introduce two definitions from [4].

Definition 2.1. *Let $\sigma_0 \in \mathbb{R}$. We say that a Dirichlet series $D(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ with finite abscissa of convergence belongs to $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ provided*

(1) it extends holomorphically to $\mathbb{C}_{\sigma_0}^+ \cup \mathbb{C}_{\sigma_c(D)}$;
(2) $\sigma_2(D) \leq \sigma_0$;
(3) for all $\sigma_1 > \sigma_0$, there exist t_0, $B > 0$ such that, for all $s = \sigma + it$ with $\sigma \geq \sigma_1$ and $t \geq t_0$, $|D(\sigma + it)| \leq t^B$;
(4) for all $\sigma_2 > \sigma_1 > \sigma_0$,

$$\sup_{\sigma\in[\sigma_1,\sigma_2]} \sup_{T>0} \frac{1}{T}\int_1^T |D(\sigma+it)|^2 dt < +\infty;$$

(5) the sequence $\{\lambda_n\}_{n\geq 1}$ is $\mathbb{Q}$-linearly independent.

Definition 2.2. *We say that a Dirichlet series $D = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ belongs to $\mathcal{D}_{\text{dens}}$ provided for all $\alpha, \beta > 0$, there exist $C > 0$ and $x_0 \geq 1$ such that, for all $x \geq x_0$,*

$$\sum_{\lambda_n \in \left[x, x+\frac{\alpha}{x^2}\right]} |a_n| \geq Ce^{(\sigma_a(D)-\beta)x}.$$

The main interest of introducing these definitions is the following theorem (see [4]).

Theorem 2.3. *Let D be a Dirichlet series and let $\sigma_0 > \sigma_2(D)$. Assume that $D \in \mathcal{D}_{\text{w.a.}}(\sigma_0) \cap \mathcal{D}_{\text{dens}}$. Then D is strongly universal in the strip $\{\sigma_0 < \mathrm{Re}(s) < \sigma_a(D)\}$*

It should be pointed out that Definition 2.1 in [4] mentions the whole half-plane $\mathbb{C}_{\sigma_0}$ and not the quarter half-plane as here. However, this does not change anything for the proofs. The key points are that the half vertical lines $\sigma + it$, $t > 0$, $\sigma > \sigma_0$, are contained in $\mathbb{C}_{\sigma_0}^+$ and that for any compact set K included in the strip $\{\sigma_0 < \mathrm{Re}(s) < \sigma_a(D)\}$, there exists $\tau > 0$ such that $K + i\tau \subset \mathbb{C}_{\sigma_0}^+$.

2.3. **Two lemmas to estimate exponential sums.** We shall need two inequalities which are been widely used in this context. The first one deals with exponential sums and is due to Montgomery and Vaughan (see [13]).

Lemma 2.4. *Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers such that $\sum_{n\geq 1} |a_n|^2 < +\infty$. Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of real numbers and set $\theta_n := \inf_{m\neq n} |\lambda_n - \lambda_m| > 0$ for every n. Then*

$$\int_0^T \left|\sum_{n\geq 1} a_n e^{i\lambda_n t}\right|^2 dt = T\sum_{n\geq 1} |a_n|^2 + O\left(\sum_{n\geq 1} \frac{|a_n|^2}{\theta_n}\right)$$

where the O-constant is absolute.

We also need the following classical inequality for exponential sums, which goes back to J. G. Van der Corput (see [5, Lemma 11.5]).

Lemma 2.5. *Let $a < b$ and let $f, g : [a, b] \to \mathbb{R}$ be two functions of class $\mathcal{C}^2$. Assume that*

- *f' is monotonic with $|f'| < 1/2$;*
- *g is positive, non-increasing and convex.*

Then

$$\sum_{n=a}^{b} g(n)e^{2\pi i f(n)} = \int_{a}^{b} g(u)e^{2\pi i f(u)}du + O(g(a) + |g'(a)|).$$

2.4. **The incomplete Gamma function/ Prym's function.** We will make a short presentation and we refer the interested reader to [1, 16, 17]. For $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(z) > 0$, we define the incomplete Gamma function $\Gamma(a, z)$ by

$$\Gamma(a, z) = \int_{z}^{+\infty} t^{a-1}e^{-t}\, dt.$$

For fixed z, as in the classical case it has a meromorphic extension in $\mathbb{C}$ with simple poles at the nonpositive integers. This can be easily obtained from the recurrence relation:

$$\Gamma(a + 1, z) = a\Gamma(a, z) + z^a e^{-z}.$$

For a fixed value of a, Γ admits a holomorphic extension (its principal branch) to $\mathbb{C}\backslash\mathbb{R}_-$ and even to $\mathbb{C}$ when a is a positive integer. When a is not a nonpositive integer, this follows for instance from the relation

$$\Gamma(a, z) = \Gamma(a)(1 - z^{a-1}\gamma^*(a, z)),$$

where the function γ^* is entire in both a and z. When a is a nonpositive integer, this follows from the corresponding statement for $a = 0$ (in that case, the incomplete Gamma function is also called the exponential integral).

For this principal branch (and for a fixed a), we have the estimation

$$\Gamma(a, z) = e^{-z} \int_{0}^{+\infty} e^{-u}\, (z + u)^{a-1}\, du = O(z^{a-1}e^{-z}), \tag{1}$$

as $|z| \to +\infty$.

2.5. **A remark on** [4]**.** In [4, Theorem 1.6], the theorem on rearrangement universality of Dirichlet series states that a Dirichlet series $\sum_{n\geq 1} a_n e^{-\lambda_n s}$ is rearrangement universal if for any $f \in H(\Omega)$, where Ω is the strip $\sigma_c(D) < \mathrm{Re}(s) < \sigma_a(D)$, there exists a permutation σ of $\mathbb{N}$ such that $\sum_{n\geq 1} a_{\sigma(n)} e^{-\lambda_{\sigma(n)} s}$ converges to f in $H(\Omega)$). This theorem is false. Indeed it would imply that $\sum_{n\geq 1}(-1)^n n^{-s}$ is rearrangement universal. This cannot hold: any rearrangement of $\sum_{n\geq 1}(-1)^n n^{-s}$ will take values in $\mathbb{R}$ for real values of the parameter s. The mistake which is made in [4] lies on the fact that a lemma due to Banaszczyk is only true for some real Fréchet spaces to the complex Fréchet space $H(\Omega)$.

3. Proof of Theorem 1.1

The main difficulty in order to apply Theorem 2.3 is to estimate the square moments of D on vertical lines. We follow a method introduced in [2] where the authors estimate the square moments of the logarithm of the zeta function.

Lemma 3.1. *Let $D = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ be a Dirichlet series with $\sigma_a(D) \leq 1$. Assume that D extends continuously to $\overline{\mathbb{C}_\alpha}$, $0 \leq \alpha < 1$, analytically in $\mathbb{C}_\alpha$, and that D has order B in $\overline{\mathbb{C}_\alpha}$. Let $\sigma_0 \in (\alpha, 1)$ with $\sigma_0 > \sigma_2(D)$ and let us set $A = B/(\sigma_0 - \alpha)$. Then for all $T \geq 1$, for all $\sigma > \sigma_1 > \sigma_0$,*

$$\int_0^T |D(\sigma+it)|^2 dt \lesssim T + \sum_{n\geq 1} \frac{|a_n|^2 \exp(-2\lambda_n \sigma)\exp\left(-2\frac{e^{\lambda_n}}{T^A}\right)}{\min(\lambda_n - \lambda_{n-1}, \lambda_{n+1}-\lambda_n)}.$$

Proof. The inverse Mellin transform (see [14, Appendix 3]) applied to the Γ function says that, for all $x > 0$,

$$e^{-x} = \frac{1}{2\pi i}\int_{2-\sigma_0-i\infty}^{2-\sigma_0+i\infty} x^{-w}\Gamma(w)dw. \tag{2}$$

Let $T \geq 2$ and set $X = T^A$. We apply (2) for $x = \frac{e^{\lambda_n}}{X}$, yielding to

$$\exp\left(-\frac{e^{\lambda_n}}{X}\right) = \frac{1}{2\pi i}\int_{2-\sigma_0-i\infty}^{2-\sigma_0+i\infty} \exp(-\lambda_n w)X^w \Gamma(w)dw.$$

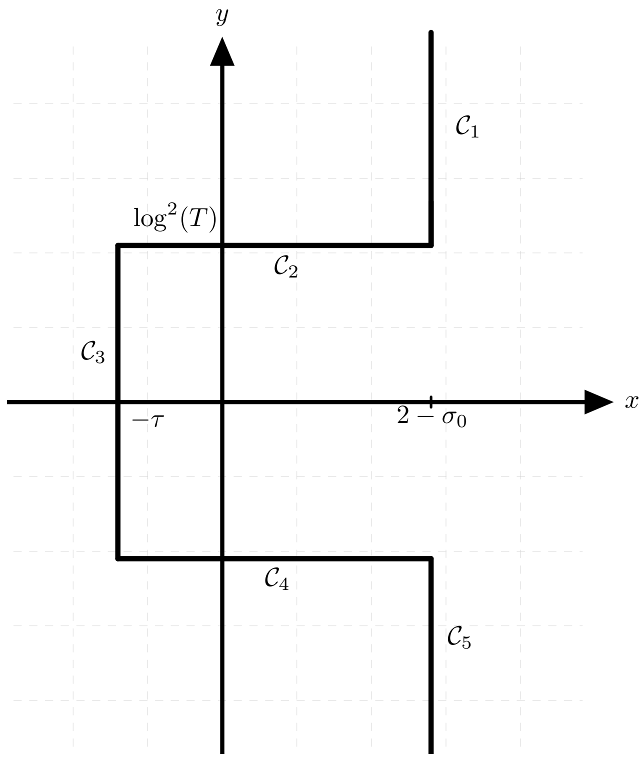
Therefore, for any $\sigma > \sigma_0$ and any $t \in [0,T]$, setting $s = \sigma + it$, for any $n \geq 1$,

$$a_n \exp(-\lambda_n s)\exp\left(-\frac{e^{\lambda_n}}{X}\right) = \frac{1}{2\pi i}\int_{2-\sigma_0-i\infty}^{2-\sigma_0+i\infty} a_n \exp(-\lambda_n(s+w))X^w\Gamma(w)dw.$$

Since $\mathrm{Re}(s+w) > 1$ provided $\mathrm{Re}(w) = 2 - \sigma_0$, we can sum these equalities and interchange summation and integral to get

$$(3) \qquad \sum_{n\geq 1} a_n \exp(-\lambda_n s) \exp\left(\frac{e^{-\lambda_n}}{X}\right) = \frac{1}{2\pi i} \int_{2-\sigma_0-i\infty}^{2-\sigma_0+i\infty} D(s+w) X^w \Gamma(w)\, dw.$$

We set $\tau = \sigma_0 - \alpha$ and we introduce the following contour $\mathcal{C}$, defined as the union of five segments or half-lines, $\mathcal{C}_1, \ldots, \mathcal{C}_5$.



On $\mathcal{C}_1 \cup \mathcal{C}_5$, $|D(s+w)| \lesssim 1$. Moreover, writing $w = u + iv$ with $u = 2 - \sigma_0$, Stirling's formula for the Γ-function (see again [14, Appendix 3]) says that

$$|\Gamma(u+iv)| \lesssim e^{-C|v|}$$

for some $C>0$ (independent of $w\in\mathcal{C}_1\cup\mathcal{C}_5$). Hence,

$$\begin{aligned}\int_{\mathcal{C}_1\cup\mathcal{C}_5}|D(s+w)X^w\Gamma(w)|dw &\lesssim X^{2-\sigma_0}\int_{\log^2(T)}^{+\infty}e^{-Cv}dv\\ &\lesssim 1.\end{aligned}$$

Pick now $w=u+iv\in\mathcal{C}_2\cup\mathcal{C}_3\cup\mathcal{C}_4$. Then

$$\begin{cases}|\operatorname{Im}(s+w)| & \lesssim & (\log T)^2T\\ \operatorname{Re}(s+w) & \geq & \sigma_0-\tau=\alpha.\end{cases}$$

Therefore, $|D(s+w)|\lesssim T^{B_1}$, $B_1<B$. This implies

$$\begin{aligned}\int_{\mathcal{C}_2\cup\mathcal{C}_4}|D(s+w)X^w\Gamma(w)|dw &\lesssim T^BT^{A(2-\sigma_0)}e^{-C\log^2(T)}\\ &\lesssim 1.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{\mathcal{C}_3}|D(s+w)X^w\Gamma(w)|dw &\lesssim T^{-A\tau}T^B\int_{\mathbb{R}}|\Gamma(-\tau+it)|dt\\ &\lesssim 1\end{aligned}$$

by our choice of A and τ. Summing the estimates we obtain

$$\int_{\mathcal{C}}|D(s+w)X^w\Gamma(w)|dw\lesssim 1. \tag{4}$$

Let now $\mathcal{R}$ be the rectangle $\mathcal{C}_2\cup\mathcal{C}_3\cup\mathcal{C}_4\cup\mathcal{C}_6$ with $\mathcal{C}_6=[2-\sigma_0-i\log^2T,2-\sigma_0+i\log^2T]$ so that

$$\int_{\mathcal{C}}=\int_{2-\sigma_0-i\infty}^{2-\sigma_0+i\infty}+\int_{\mathcal{R}}. \tag{5}$$

The function $w\mapsto D(s+w)X^w\Gamma(w)$ has a single pole 0 inside $\mathcal{R}$, with residue $D(s)$. Hence, by (3), (4) and (5),

$$|D(s)|\lesssim 1+\left|\sum_{n\geq 1}a_n\exp(-\lambda_n s)\exp\left(\frac{-e^{\lambda_n}}{X}\right)\right|.$$

We set $b_n = a_n \exp(-\lambda_n\sigma)\exp\left(-\frac{e^{\lambda_n}}{X}\right)$. Taking the square and integrating over $[0,T]$, we find

$$\int_0^T |D(\sigma+it)|^2 dt \lesssim T + \int_0^T \left|\sum_{n\geq 1} b_n \exp(-\lambda_n it)\right|^2 dt.$$

We apply the Montgomery-Vaughan inequality, yielding to the following estimate:

$$\int_0^T |D(\sigma+it)|^2 dt \lesssim T + T\sum_{n\geq 1}|a_n|^2\exp(-2\lambda_n\sigma) + \sum_{n\geq 1}\frac{|b_n|^2}{\min(\lambda_n-\lambda_{n-1},\lambda_{n+1}-\lambda_n)}.$$

Since $\sigma_0 > \sigma_2(D)$, we have proven Lemma 3.1. ☐

To estimate the sum appearing in the last lemma, we shall use the following result.

Lemma 3.2. *Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of frequencies, let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers. Assume that there exist $a \in \mathbb{R}$, b, c, $\beta > 0$ such that*

$$|a_n| \lesssim n^a, \quad \lambda_n \geq b\log n, \quad \min(\lambda_n-\lambda_{n-1},\lambda_{n+1}-\lambda_n) \geq c\exp(-\beta\lambda_n).$$

Then for all $X > 0$ and all $\sigma > \beta/2$,

$$\sum_{n\geq 1}\frac{|a_n|^2\exp(-2\lambda_n\sigma)\exp\left(-2\frac{e^{\lambda_n}}{X}\right)}{\min(\lambda_n-\lambda_{n-1},\lambda_{n+1}-\lambda_n)} \lesssim \max\left(1, X^{\frac{2a+(\beta-2\sigma)b+1}{b}}\right).$$

In particular, if $e^{\lambda_{n+1}} - e^{\lambda_n} \geq 1$ for all n, we may choose $\beta = 1$ in the previous lemma. Indeed, the inequality $\lambda_{n+1} - \lambda_n \geq \exp(-\lambda_n)$ is clear if $\lambda_{n+1} - \lambda_n \geq 1$. Otherwise,

$$\lambda_{n+1} - \lambda_n \gtrsim e^{\lambda_{n+1}-\lambda_n} - 1 = (e^{\lambda_{n+1}} - e^{\lambda_n})e^{-\lambda_n}.$$

Proof. Denote by S the sum. Then

$$\begin{aligned} S &\lesssim \sum_{n\geq 1} n^{2a}\exp\left((\beta-2\sigma)\lambda_n\right)\exp\left(-2\frac{e^{\lambda_n}}{X}\right) \\ &\lesssim \sum_{n\geq 1} n^{2a+(\beta-2\sigma)b}\exp\left(-2\frac{n^b}{X}\right). \end{aligned}$$

We split the sum into two parts. We first sum up to $X^{1/b}$ and we denote by S_1 this sum. Then

$$S_1 \lesssim \sum_{n\leq X^{1/b}} n^{2a+(\beta-2\sigma)b} \lesssim \max\left(1, X^{\frac{2a+(\beta-2\sigma)b+1}{b}}\right).$$

Regarding the second sum, say S_2, we write

$$S_2 = \sum_{n\geq X^{1/b}} n^{2a+(\beta-2\sigma)b}\exp\left(-\frac{2n^b}{X}\right)$$
$$\lesssim \int_{X^{1/b}}^{+\infty} t^{2a+(\beta-2\sigma)b}\exp\left(-\frac{2t^b}{X}\right)dt.$$

We do the change of variables $u = t^b/X$ which yields

$$S_2 \lesssim X^{\frac{2a+(\beta-2\sigma)b+1}{b}} \int_1^{+\infty} u^{\frac{2a+(\beta-2\sigma)b+1-b}{b}}\exp(-2u)du$$
$$\lesssim X^{\frac{2a+(\beta-2\sigma)b+1}{b}}.$$

□

Now we are ready to complete the first part of the proof of Theorem 1.1, that is:

Corollary 3.3. *Let* $D(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ *be a Dirichlet series and let* $d \in \mathbb{N}$. *Assume that*

- $\lambda_n = \log\big(P(p_n)\big)$ *where* $P \in \mathbb{R}_d[X]$ *and* $\lim_{+\infty} P = +\infty$.
- *there exist* $C, \kappa > 0$ *such that for* $n \geq 2$,
$$\frac{n^{d-1}}{C(\log n)^\kappa} \leq |a_n| \leq Cn^{d-1}(\log(n))^\kappa.$$
- $\sigma_c(D) = 0$.

Then D *belongs to* $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ *where* $\sigma_0 = 1 - \frac{1}{3d}$.

Proof. Under the above conditions, $\sigma_a(D) = 1$, $\sigma_c(D) = 0$ and $\sigma_2(D) = 1 - \frac{1}{2d}$. Therefore, conditions (1) to (3) of Definition 2.1 are satisfied. Let us prove (4). Let $\sigma_1 > \sigma_0$ and let $\varepsilon < \sigma_1 - \sigma_0$. We set $\alpha = \varepsilon$, and observe that we can choose $B = 1$ (see [8, Theorem 12]). We apply Lemma 3.1 and then Lemma 3.2 with $a = (d-1) + \varepsilon$, $b = d - \varepsilon$ and $\beta = 1$ to obtain that, for $\sigma \geq \sigma_1$, for $T \geq 2$,

$$\int_0^T |D(\sigma + it)|^2 dt \lesssim T + T^{B(\varepsilon)}$$

with

$$B(\varepsilon) = \frac{2\big((d-1)+\varepsilon\big) + (1-2\sigma_0)(d-\varepsilon) + 1}{(d-\varepsilon)(\sigma_1 - \varepsilon)}.$$

Now,

$$\lim_{\varepsilon\to 0} B(\varepsilon) = \frac{3d - 2d\sigma_0 - 1}{d\sigma_1} = \frac{\sigma_0}{\sigma_1} < 1.$$

Hence, choosing $\varepsilon > 0$ small enough, we have shown that $\int_0^T |D(\sigma+it)|^2 dt \lesssim T$. □

Let us proceed with the second half. First let us state the following lemma proved in [4, Lemma 6.1]:

Lemma 3.4. *Let $P(X) = \sum_{k=0}^{d} b_k X^k$ be a polynomial of degree d, with $b_d > 0$. Then, there exist x_0, $y_0 > 0$ such that P induces a bijection from $[x_0, +\infty]$ to $[y_0, +\infty]$, and*

$$P^{-1}(x) = \frac{x^{1/d}}{(b_d^{1/d})} - \frac{b_{d-1}}{b_d^{(d-1)/d}} + o(1),$$

as $x \to +\infty$.

Proposition 3.5. *Let $D(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ be a Dirichlet series and let $d \in \mathbb{N}$. Assume that*

- *$\lambda_n = \log P(p_n)$ where $P \in \mathbb{R}_d[X]$ and $\lim_{+\infty} P = +\infty$.*
- *there exist C, $\kappa > 0$ such that for $n \geq 2$,*

$$\frac{n^{d-1}}{C(\log n)^\kappa} \leq |a_n| \leq Cn^{d-1}(\log(n))^\kappa.$$

Then D belongs to $\mathcal{D}_{\text{dens}}$.

Proof. Let $\alpha, \beta > 0$. Without loss of generality, we may assume that P is one-to-one on $[\log(p_1), +\infty)$. Then

$$\lambda_n \in \left[x, x + \frac{\alpha}{x^2}\right] \qquad \text{if and only if} \qquad p_n \in \left[P^{-1}(e^x), P^{-1}(e^{x+\frac{\alpha}{x^2}})\right].$$

Using Lemma 3.4, there exist $c_1 > 0$, $c_2 \in \mathbb{R}$ such that for small $\varepsilon > 0$ and for every x sufficiently large:

$$p_n \in \left[c_1 e^{\frac{x}{d}} - c_2 + \varepsilon, c_1 e^{\frac{x}{d} + \frac{\alpha}{dx^2}} - c_2 - \varepsilon\right] \qquad \text{implies that} \qquad \lambda_n \in \left[x, x + \frac{\alpha}{x^2}\right].$$

By Hadamard - De la Vallée Poussin estimate, we also know that

$$\begin{aligned}\Pi(u) &:= \operatorname{card}\{n : \ p_n \leq x\} \\ &= \int_2^u \frac{dt}{\log(t)} + O\left(ue^{-c\sqrt{\log u}}\right).\end{aligned}$$

If $p_n \gtrsim e^{\frac{x}{d}}$, then $n \gtrsim \frac{e^{\frac{x}{d}}}{x}$. Therefore

$$\begin{aligned}
\sum_{\lambda_n \in \left[x, x+\frac{\alpha}{x^2}\right]} |a_n| &\gtrsim \frac{e^{\frac{x(d-1)}{d}}}{x^{d-1+\kappa}} \mathrm{card}\left\{n:\ p_n \in \left[c_1 e^{\frac{x}{d}} - c_2 + \varepsilon, c_1 e^{\frac{x}{d}+\frac{\alpha}{dx^2}} - c_2 - \varepsilon\right]\right\} \\
&\gtrsim \frac{e^{\frac{x(d-1)}{d}}}{x^{d-1+\kappa}} \left(\int_{c_1 e^{\frac{x}{d}} - c_2 + \varepsilon}^{c_1 e^{\frac{x}{d}+\frac{\alpha}{dx^2}} - c_2 - \varepsilon} \frac{dt}{\log(t)} + O\left(e^{\frac{x}{d} - c\sqrt{x}}\right) \right) \\
&\gtrsim \frac{e^{\frac{x(d-1)}{d}}}{x^{d-1+\kappa}} \left(e^{\frac{x}{d}} \frac{e^{\frac{\alpha}{dx^2}} - 1}{x} + O\left(e^{\frac{x}{d} - c'\sqrt{x}}\right) \right) \\
&\gtrsim \frac{e^x}{x^{d+\kappa+2}} \\
&\gtrsim e^{(1-\beta)x}.
\end{aligned}$$

□

Proof of Theorem 1.1. It follows immediately from Theorem 2.3, Corollary 3.3 and Proposition 3.5. □

4. Proof of Theorem 1.3

As above the main difficulty is to estimate the square moments of D. The situation is not as clear as in the previous case since D now will be defined via an analytic continuation. We need to understand how to define this analytic continuation and how close it is to the partial sums of D.

Lemma 4.1. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Then the Dirichlet series $D(s) = \sum_{n\geq 1} Q(n)(\log n)^\kappa (P(n))^{-s}$ admits a holomorphic continuation to $\mathbb{C}^+_{1-\frac{1}{d}} \cup \mathbb{C}_1$ and even to $\mathbb{C}_{1-\frac{1}{d}} \backslash \{1\}$ provided $\kappa \in \mathbb{N}_0$. Moreover, let $\sigma_1 > 1 - \frac{1}{d}$ and $\sigma_2 > 1$.*

(a) There exist t_0, $B > 0$ such that, for all $s = \sigma + it$ with $\sigma \geq \sigma_1$ and $t \geq t_0$,

$$|D(s)| \leq t^B.$$

(b) There exist δ, $\varepsilon > 0$ such that, for all $x > 0$, for all $s = \sigma + it$ with $\sigma \in [\sigma_1, \sigma_2]$ and $1 \leq t \leq \delta x$,

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^\kappa (P(n))^{-s} + O(x^{-\varepsilon}) + O\left(\frac{(\log P(x))^\kappa}{(s-1)P(x)^{s-1}}\right)$$

(here, the O-constants do not depend neither on s nor on x).

Proof. As in the classical case of the Riemann zeta function, see for example [5], our plan is to use the regularity and smoothness of the coefficients and the frequencies of our Dirichlet series D to estimate its order and how close the partial sums approximate D. We will rely again on the principle of non-stationary phase, that is Lemma 2.5. But first we need to deal with some technical difficulties that arise from the "unknown" polynomials P and Q. We start with $s=\sigma+it$, $\sigma>1$ and let $N\geq 1$. We write

$$D(s)=\sum_{n=2}^{N-1}Q(n)(\log n)^{\kappa}(P(n))^{-s}+\sum_{n=N}^{+\infty}Q(n)(\log n)^{\kappa}(P(n))^{-s}$$

and we apply Euler's summation formula (see [5, (11.3)]). Setting

$$\phi(u)=Q(u)(\log u)^{\kappa}(P(u))^{-s}\text{ and }\rho(u)=u-\lfloor u\rfloor-\frac{1}{2},$$

we get

$$D(s)=\sum_{n=2}^{N-1}Q(n)(\log n)^{\kappa}(P(n))^{-s}+\int_N^{+\infty}\phi(u)du+\int_N^{+\infty}\rho(u)\phi'(u)du+\frac{1}{2}\phi(N). \tag{6}$$

These integrals are convergent when $s\in\mathbb{C}_1$. Moreover it is easy to check that there exists $\varepsilon>0$ such that, provided $s=\sigma+it$ with $\sigma\geq\sigma_1>1-\frac{1}{d}$, for any $u>2$,

$$|\phi(u)|\lesssim u^{-\varepsilon}\text{ and }|\phi'(u)|\lesssim|s|u^{-1-\varepsilon}.$$

In particular, the last integral in (6) defines a holomorphic function in $\mathbb{C}_{1-\frac{1}{d}}$ which is $O(|s|N^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. Let us now see how to control the second integral. Up to multiply Q by a constant, we may write it $Q(u)=P'(u)+Q_1(u)$ with $\deg(Q_1)\leq d-2$. As above, the integral $\int_N^{+\infty}Q_1(u)(\log u)^{\kappa}(P(u))^{-s}du$ defines an analytic function in $\mathbb{C}_{1-\frac{1}{d}}$ which is $O(N^{-\varepsilon})$. Therefore we have obtained so far that D may be written in $\mathbb{C}_1$

$$D(s)=\sum_{n=2}^{N-1}Q(n)(\log n)^{\kappa}(P(n))^{-s}+\int_N^{+\infty}\frac{P'(u)(\log u)^{\kappa}}{(P(u))^s}du+R_N(s)$$

where R_N is analytic in $\mathbb{C}_{1-\frac{1}{d}}$ and $|R_N(s)|\lesssim|s|N^{-\varepsilon}$ uniformly for $\sigma\geq\sigma_1$.

We choose N sufficiently large such that P is one-to-one on $[N,+\infty)$. By change of variables we obtain:

$$\int_N^{+\infty}\frac{P'(u)(\log u)^{\kappa}}{(P(u))^s}du=\int_{P(N)}^{+\infty}\frac{(\log P^{-1}(u))^{\kappa}}{u^s}du.$$

By Lemma 3.4 we have the following formula:

$$P^{-1}(u) = a_d u^{1/d}(1+\varepsilon_1(u)) \text{ with } |\varepsilon_1(u)| \lesssim u^{-1/d},$$

where $a_d > 0$. Therefore,

$$(\log P^{-1}(u))^\kappa = \log^\kappa(a_d u^{1/d}) + \varepsilon_2(u)$$

with

$$|\varepsilon_2(u)| \lesssim u^{-1/d}\log^{\kappa-1}(u).$$

As before, the integral $\int_{P(N)}^{+\infty} \varepsilon_2(u)u^{-s}du$ defines an analytic function in $\mathbb{C}_{1-\frac{1}{d}}$ which is $O(P(N)^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. On the other hand, setting $b_d = a_d^d$ and restricting ourselves to $s \in \mathbb{C}_1$, we may write

$$\begin{aligned}
\int_{P(N)}^{+\infty} \frac{\log^\kappa(a_d u^{1/d})}{u^s}du &= \int_{P(N)}^{+\infty} \frac{1}{d^\kappa}\frac{\log^\kappa(b_d u)}{u^s}du \\
&= \frac{b_d^{s-1}}{d^\kappa}\int_{b_d P(N)}^{+\infty} \frac{\log^\kappa(v)}{v^s}dv && (v = b_d u) \\
&= \frac{b_d^{s-1}}{d^\kappa}\int_{\log(b_d P(N))}^{+\infty} y^\kappa e^{(1-s)y}dy && (y = \log v) \\
&= \frac{b_d^{s-1}}{d^\kappa(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N))).
\end{aligned}$$

Hence by analytic continuation we have shown that for $s \in \mathbb{C}_1$, we may write

$$D(s) = \sum_{n=2}^{N-1} Q(n)(\log n)^\kappa (P(n))^{-s} + \widetilde{R_N}(s) + \frac{b_d^{s-1}}{d^\kappa(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N)))$$

where $\widetilde{R_N}(s)$ is holomorphic in $\mathbb{C}_{1-\frac{1}{d}}$ and is $O(|s|N^{-\varepsilon}) + O(P(N)^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. Since we know that $\Gamma(\kappa+1,\cdot)$ admits an analytic continuation to $\mathbb{C}\backslash\mathbb{R}_-$ we can conclude to the analytic continuation of D to $\mathbb{C}^+_{1-\frac{1}{d}} \cup \mathbb{C}_1$. When $\kappa \in \mathbb{N}$, the analytic continuation even holds on $\mathbb{C}_{1-\frac{1}{d}}\backslash\{1\}$. The estimation (a) (which is trivial for $\sigma \geq \sigma_2 > 1$) follows easily for $\sigma \in [\sigma_1, \sigma_2]$ by what we already know on $\widetilde{R_N}$ and by (1).

Let us turn to the proof of (b). Choosing $N \geq x$, we may write

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \sum_{n=x+1}^{N} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \frac{b_d^{s-1}}{d^{\kappa}(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N))) + O(|s|N^{-\varepsilon}) + O(P(N)^{-\varepsilon}).$$

We apply Lemma 2.5 to the second sum with

$$g(u) = Q(u)\log^{\kappa}(u)(P(u))^{-\sigma},\ f(u) = \frac{-t\log(P(u))}{2\pi}.$$

Observe that, for $\sigma \in [\sigma_1, \sigma_2]$ and $u \in [x, N]$, provided $t \leq \delta x$ with δ small enough,

$$|g(u)| \lesssim x^{-\varepsilon},\ |g'(u)| \lesssim \sigma x^{-1-\varepsilon} \leq x^{-\varepsilon},\ |f'(u)| \leq \frac{1}{2}.$$

Hence,

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \int_x^N \frac{Q(u)\log^{\kappa}(u)}{(P(u))^s}du + \frac{b_d^{s-1}}{d^{\kappa}(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N))) + O(x^{-\varepsilon} + |s|N^{-\varepsilon} + P(N)^{-\varepsilon}).$$

We let $N \to +\infty$, yielding to:

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \int_x^{+\infty} \frac{Q(u)\log^{\kappa}(u)}{(P(u))^s}du + O(x^{-\varepsilon}), \qquad s \in \mathbb{C}_1.$$

Now writing $Q(u) = P'(u) + Q_1(u)$ and repeating the argument in the first part of this proof one can obtain the following identity

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \frac{b_d^{s-1}}{d^{\kappa}(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(x))) + \widetilde{R}(s),$$

where $\widetilde{R}(s)$ is holomorphic in $\mathbb{C}_{1-\frac{1}{d}}$ and is $O(x^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. Using one last time (1), we obtain (b) of Lemma 4.1. □

From this and Lemma 2.4, we may deduce the first half of Theorem 1.3.

Proposition 4.2. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Assume moreover that $\{\log P(n)\}_{n\geq 1}$ is $\mathbb{Q}$-linearly independent. Then the Dirichlet series $D(s) = \sum_{n\geq 1} Q(n)(\log n)^{\kappa}(P(n))^{-s}$ belongs to $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ with $\sigma_0 = 1 - \frac{1}{2d}$.*

Proof. It is clear that $\sigma_2(D) = (2d-1)/2d$ and thus it just remains to prove (4). We fix $T \geq 1$ and we first estimate $\int_{T/2}^{T} |D(\sigma+it)|^2 dt$ where $1 - \frac{1}{2d} < \sigma_1 \leq \sigma \leq \sigma_2$. We apply the estimate given by Lemma 4.1 with $x = T/\delta$ so that $O(x^{-\varepsilon}) = O(T^{-\varepsilon})$ and

$$\left| \frac{\log^{\kappa}(P(x))}{(s-1)P(x)^{s-1}} \right| \lesssim \frac{\log^{\kappa} T}{T T^{d(\sigma-1)}} \lesssim T^{-\varepsilon}.$$

Hence, applying Lemma 2.4

$$\begin{aligned}\int_{T/2}^{T} |D(\sigma+it)|^2 dt &\lesssim \int_{T/2}^{T} \left| \sum_{n=2}^{T/\delta} |Q(n)(\log n)^{\kappa}(P(n))^{-s} \right|^2 dt + T^{1-2\varepsilon} \\ &\lesssim T \sum_{n=2}^{T/\delta} |Q(n)|^2 (\log n)^{2\kappa} |P(n)|^{-2\sigma} + \\ &\quad \sum_{n=2}^{T/\delta} \frac{|Q(n)|^2 (\log n)^{2\kappa} |P(n)|^{-2\sigma}}{\log(P(n+1)) - \log(P(n))} + T^{1-2\varepsilon}.\end{aligned}$$

The first sum is dominated by some constant since $\sigma \geq \sigma_1 > \sigma_2(D)$. Regarding the second sum, for $n \in [2, T/\delta]$,

$$\frac{|Q(n)|^2 (\log n)^{2\kappa} |P(n)|^{-2\sigma}}{\log(P(n+1)) - \log(P(n))} \lesssim T n^{2(d-1)-2d\sigma} (\log n)^{2\kappa} \lesssim T n^{2d(1-\sigma_1)-2} (\log n)^{2\kappa},$$

and we get the estimate

$$\sum_{n=2}^{T/\delta} \frac{|Q(n)|^2 (\log n)^{2\kappa} |P(n)|^{-2\sigma}}{\log(P(n+1)) - \log(P(n))} \lesssim T,$$

note that $2d(1-\sigma_1) < 1$. Hence, we have obtained

$$\int_{T/2}^{T} |D(\sigma+it)|^2 dt \lesssim T,$$

for all $T \geq 1$ and all $\sigma \in [\sigma_1, \sigma_2]$, where the involved constant does not depend neither on σ nor on T. Taking $T2^{-j}$ instead of T in the latter formula and summing over j, we get the proposition. □

The second half of the proof of Theorem 1.3 has been proven in [4, Proposition 6.2], for the sake of completeness we repeat the argument below.

Proposition 4.3. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Then the Dirichlet series $D(s) = \sum_{n\geq 1} Q(n)(\log n)^\kappa (P(n))^{-s}$ belongs to $\mathcal{D}_{\text{dens}}$.*

Proof. Let $a, \beta > 0$. There exists a $x_0 > 1$ such that for every $x \geq x_0$ the polynomial P is positive and increasing and Q behaves like its leading term. By the Lemma 3.4 there exist constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that

$$n \in \left[c_1 e^{\frac{x}{d}} - c_2 + \varepsilon, c_1 e^{\frac{x}{d} + \frac{a}{dx^2}} - c_2 - \varepsilon\right] \qquad \text{implies that} \qquad \lambda_n \in \left[x, x + \frac{a}{x^2}\right].$$

Thus

$$\sum_{\lambda_n \in \left[x, x+\frac{a}{x^2}\right]} |Q(n)(\log n)^\kappa| \gtrsim \frac{e^{\frac{x}{d}}}{x^2} e^{\frac{d-1}{d} x(1-\frac{\beta}{2})} \gtrsim e^{(1-\beta)x}.$$

□

Theorem 1.3 now follows from Proposition 4.2, Proposition 4.3 and Theorem 2.3.

5. Random models and further discussion

One of the motivations behind our work is to give concrete examples of convergent universal objects like the alternating prime zeta function $P_-(s) = \sum_{n\geq 1} (-1)^n p_n^{-s}$. As we have already proved P_- is strongly universal in $\{\frac{2}{3} < \text{Re} < 1\}$. But, the universality of this object in the whole critical strip remains an open question.

Question 5.1. Is P_- strongly universal in $\{\frac{1}{2} < \text{Re} \leq \frac{2}{3}\}$?

It is worth mentioning that Theorem 1.1 implies that every series of the form

$$P_\chi(s) = \sum_{n\geq 1} \chi_n p_n^{-s}, \qquad |\chi_n| = 1,$$

with $\sigma_c(P_\chi) \leq 0$, is strongly universal in $\{\frac{2}{3} < \text{Re} < 1\}$.

We expect such P_χ to be strongly universal in the whole critical strip even when we only have $\sigma_c(P_\chi) \leq 1/2$. To justify our claim let us randomize our series. Let $\{X_n\}$ be a sequence of unimodular independent identically distributed Steinhaus or Rademacher (coin tossing) random variables and $P_X(s) = \sum_{n\geq 1} X_n p_n^{-s}$. Kolmogorov's three-series theorem [15, Chapter 5] implies that P_X converges almost surely in $\mathbb{C}_{\frac{1}{2}}$. To obtain that such series are strongly universal almost surely, we need to obtain more information about their order in the critical strip.

Proposition 5.2. *Let* $P_X(s) = \sum_{n\geq 1} X_n p_n^{-s}$*, where* $\{X_n\}$ *is as above. Then,* P_X *is of sub-logarithmic order in the critical strip and as a consequence is strongly universal, almost surely.*

Proof. We consider the corresponding randomized zeta functions

$$\zeta_X(s) = \prod_{n\geq 1} \frac{1}{1 - X_n p_n^{-s}}. \tag{7}$$

It is easy to see that ζ_X converges absolutely for $\operatorname{Re} s > 1 + \varepsilon$, $\varepsilon > 0$. It is also known that ζ_X and the reciprocal $1/\zeta_X$ converge in $\mathbb{C}_{\frac{1}{2}}$, almost surely. For Steinhaus random variables $(X_1, X_2, \dots) \in \mathbb{T} \times \mathbb{T} \times \dots$ this can be obtained from the work of Helson [9] or as an application of Menchoff's theorem [3]. In the case of Rademacher random variables $X_n = r_n(t) = \operatorname{sign}(\sin(2\pi 2^n t))$, $0 < t < 1$ this has been done by Carlson and Wintner [7, 20].

Starting from (7) one can prove that there exists an absolutely convergent Dirichlet series $F(s, X)$ in $\mathbb{C}_{\frac{1}{2}}$ such that:

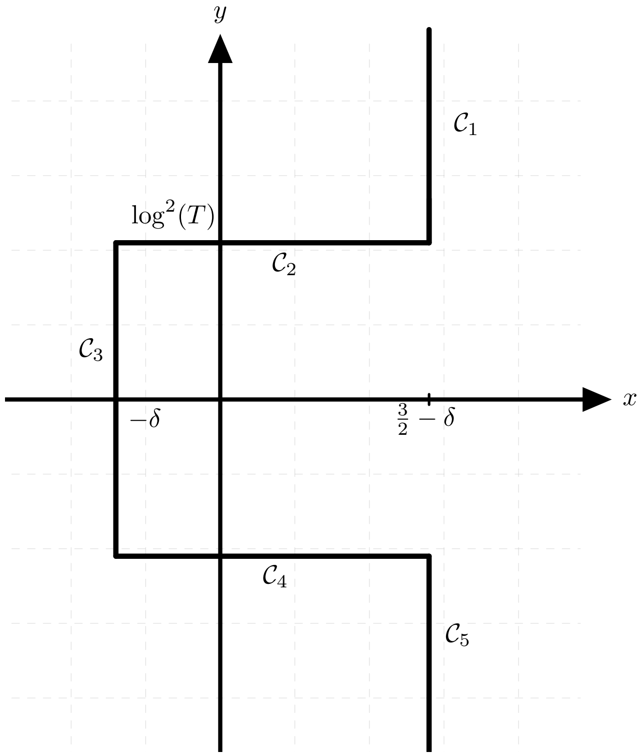
$$\log \zeta_X - P_X = F, \qquad \text{in} \qquad \mathbb{C}_{\frac{1}{2}}.$$

Using the Borel–Carathéodory theorem in a similar manner as in the proof of the implication: The Riemann hypothesis implies the Lindelöf hypothesis, [18, Chapter XIV], we obtain that for all $\varepsilon > 0$

$$|P_X(\sigma + it)| = O\left((\log t)^{2-2\sigma+\varepsilon}\right), \qquad t \to \infty,$$

uniformly for $\sigma \geq \sigma_0 > \frac{1}{2}$.

Fix such X_0, the fact that $P_{X_0} \in \mathcal{D}_{\text{dens}}$ follows immediately from Proposition 3.5. It remains to show that $P_{X_0} \in \mathcal{D}_{\text{w.a.}}(\frac{1}{2})$. We will work as in Lemma 3.1. Let $T > 2$, we set $X = T^{\varepsilon}$, $\tau = \delta$ and $t \in (0, T)$, $\sigma \geq \sigma_0 > \frac{1}{2} + 2\delta$, where ε, $\delta > 0$ are sufficiently small. We consider the contour $\mathcal{C} = \cup_{i=1}^{5} \mathcal{C}_i$, where



We observe that

$$\sum_{n\geq 1} X_n p_n^{-s} e^{-\frac{p_n}{X}} = \frac{1}{2\pi i} \int_{\frac{3}{2}-\delta-i\infty}^{\frac{3}{2}-\delta+i\infty} P_{X_0}(s+w) X^w \Gamma(w)\, dw,$$

$$\int_{\mathcal{C}_i} |P_{X_0}(s+w) X^w \Gamma(w)|\, dw \lesssim 1, \qquad i = 1,2,4,5.$$

To prove that the integral over the line segment $\mathcal{C}_3$ is bounded note that our function P_{X_0} is of zero order uniformly in $\mathbb{C}_{\frac{1}{2}+\delta}$, thus

$$\int_{\mathcal{C}_3} |P_{X_0}(s+w) X^w \Gamma(w)|\, dw \lesssim T^{\frac{\varepsilon}{2}-\varepsilon} \lesssim 1.$$

Applying Cauchy's theorem and the Montgomery-Vaughan inequality as in Lemma 3.1, we obtain

$$\frac{1}{T}\int_0^T |P_{X_0}(\sigma+it)|^2 dt \lesssim 1+\sum_{n\geq 1} p_n^{1-2\sigma} e^{-2\frac{p_n}{X}} \lesssim \sum_{n\geq 1} n^{1-2\sigma} e^{-2\frac{n}{X}} \lesssim 1+T^{-1+\varepsilon(2-(1+\delta))} \lesssim 1.$$

Therefore $P_{X_0} \in \mathcal{D}_{\text{w.a.}}(\frac{1}{2})$. □

Question 5.3. Is it true that if the series P_X converges, then it will be strongly universal in $\{\frac{1}{2} < \text{Re} < 1\}$?

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