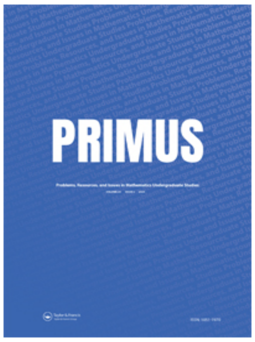

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Learning Eigenvalues and Eigenvectors with Online YouTube Resources: A Journey in the Embodied, Proceptual-symbolic, and Formal Worlds of Mathematics

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ABSTRACT

Linear algebra is one of the service mathematics courses that students in many programs take as part of their undergraduate study. Previous studies indicate that linear algebra courses are often challenging for undergraduate students, leading some to seek help from other resources, including online resources such as YouTube. Through a multiple case study, I focused on identifying the opportunities that YouTube resources could offer students to learn about eigenvalues and eigenvectors, which are among the key concepts in introductory linear algebra. I utilized a combination of APOS (Action-Process-Object-Schema) and Tall's three worlds of mathematics to analyze two highly viewed YouTube videos on this topic. The two analyzed files had different emphases (one more on the embodied world, and the other on the proceptual-symbolic world), catering to students with different levels of understanding of linear algebra. The findings suggest that, overall, these resources could provide students with opportunities to learn about eigen theory in the embodied and proceptual-symbolic worlds, with more focus on action. Lecturers and teacher assistants could benefit from familiarizing themselves with these available resources and considering the possible integration of these resources into their teaching and/or support they provide for students to meet their academic needs and preferences.

KEYWORDS

Eigen theory; APOS; three worlds of mathematics; YouTube; linear algebra

1. INTRODUCTION

Linear algebra is one of the required mathematics courses in many science, technology, engineering and mathematics (STEM) programs worldwide [17,21]. It has numerous applications in mathematics (e.g., in differential equations and Fourier series) and other STEM fields (e.g., molecular modeling in biology and ecology and image processing in engineering) [16]. Linear algebra courses often appear "very intense" to undergraduate students [30, p. 275]. Many students perceive that ideas and definitions are discussed rapidly in such courses and are not well connected to what they learned in high school [30]. Stewart [20] highlighted further:

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> Linear algebra is made out of many languages and representations. Instructors and textbooks often move between these languages and modes fluently, not allowing students time to discuss and interpret their validities as they assume that students will pick up their understandings along the way. (p. 51)

This could potentially lead students to seek assistance in addressing their challenges with linear algebra. Mathematics help-seeking is a self-regulated learning strategy that assists students in overcoming the challenges and uncertainties they encounter while studying mathematics, by drawing on other people and resources (including online resources) [1]. Previous studies have highlighted that many students turn to YouTube for assistance with their mathematics courses at both the school level (e.g., [15]) and university (e.g., [1,12]). As mathematics educators, one could argue that we bear the responsibility of investigating the potential opportunities these online resources could offer for student learning. This exploration could help us in considering their potential integration into our teaching, aiming to meet the academic needs and preferences of our students [12]. This study primarily focuses on the potential opportunities that these YouTube resources could offer students to learn about eigenvalues and eigenvectors. These specific concepts are chosen as they are amongst the "major conceptual ideas" within linear algebra [21, p. 1020], coupled with notable applications (e.g., they "hold the key to the discrete evolution of a dynamical system" [16, p. 310]). However, it is noteworthy that many students struggle with developing a conceptual understanding of them (e.g., [22,30]). The research question sought in this study is: *what opportunities could YouTube resources offer students to learn about eigenvalues and eigenvectors?*

The Framework of Advanced Mathematical Thinking (FAMT)[1] is used to address this research question. Stewart and Thomas [23] have developed this framework, which is a combination of two well-known theoretical frameworks in mathematics education, action-process-object-schema (APOS) theory [10] and Tall's [25–27] theory of three worlds of mathematics, encompassing the embodied, proceptual-symbolic, and formal worlds. I have adopted the FAMT as the analytical framework of this study as it has been shown beneficial to investigate teaching and learning of linear algebra across multiple topics (see [19,21]), including eigenvalues and eigenvectors (e.g., [23,30]). This study contributes to the mathematics education literature in at least two different ways. First, it is one of the first attempts to analyze the content of online video resources (including YouTube videos) available for learning linear algebra. Second, it appears that in past research, the FAMT framework has not been used for such a specific purpose, potentially encouraging its utilization in future research in mathematics education for analyzing online video resources.

[1] It is also called a *Framework for advanced mathematical thinking* and a *framework for mathematical thinking*.

2. THE THEORETICAL BACKGROUNDS

In this section, I provide a brief description of the main tenets of APOS theory [10] and Tall's [25–27] three worlds of mathematics before introducing the FAMT as the main theoretical framework of the study.

2.1. APOS Theory

The APOS theory is a constructivist theory focusing on developing models of how individuals might learn mathematical concepts and use these models to design teaching activities and/or investigate student learning and problem solving [4]. This theory postulates that in constructing mathematical knowledge, various types of reflective abstraction or mental mechanisms (e.g., encapsulation and generalization) are influential and guide individuals in constructing mental structures (i.e., *action*, *process*, *object*, and *schema*). Learning mathematics, in short, according to this theory, begins with

> … manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas. [5, p. 9]

In the following, I describe the action, process, and object in more detail due to their fundamental roles in the FAMT. *Action* is

> an externally directed transformation of a previously conceived Object, or Objects … [E]ach step of the transformation needs to be performed explicitly and guided by external instructions … [E]ach step prompts the next, that is, the steps of the Action cannot yet be imagined and none can be skipped. [4, p. 19]

This type of thinking could be described as "basic or complex," yet it remains "necessary" as *processes* are essentially "interiorized Actions," and *objects* are constructed "because of the application of Actions" [4, p. 20]. As individuals repeat *actions* and reflect on them, they gradually transition from "relying on external cues to having internal control over them" [4, p. 20]. This transition to process could be characterized by the ability "to imagine carrying out the steps without necessarily having to perform each one explicitly and by being able to skip steps, as well as reverse them" [4, p. 20]. When an individual develops awareness "of the process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has encapsulated the process into a cognitive *object*" [11, p. 339].

2.2. Tall's Three Worlds of Mathematics

Tall's [25,26,27] theory of the three worlds of mathematics describes how humans develop mathematical thinking from childhood to adulthood, and even how they become mathematicians [24,29]. This theory postulates that three worlds of mathematical thinking (i.e., embodied, proceptual-symbolic, and formal) exist, where recognition, repetition, and language play significant roles in their development. In short, according to this theory, individuals initially perceive mathematical concepts through recognition and categorization in the embodied world. The second world, the symbolic world, is constructed through the "repetition of sequences of actions to construct mathematical operations either as routine procedures or, through encapsulation, as flexible procepts" [28, p. 134]. Furthermore, language is instrumental in defining concepts, describing and deducing relationships, and in axiomatic mathematics, where set-theoretic language is used to construct formal mathematical theory [26,28]. In what follows, I unpack these three worlds of mathematical thinking.

2.2.1. *The Embodied World*

The embodied world is developed based on humans' sensory experiences and actions as biological beings [25]. In this world, sense-making in mathematics is characterized by "using our human senses to make links between our perceptions and actions" [28, p. 145]. Humans construct mental images by exploring objects' properties that they see and sense in the real world and their "mental world of meaning" [25, p. 30]. This formulation of the embodied world means that this world includes humans' "mental perceptions of real-world objects" (e.g., understanding Euclidean geometry) and "internal conceptions that involve visuo-spatial imagery" (e.g., understanding non-Euclidean geometries) [25, p. 30].

2.2.2. *The Proceptual-symbolic World*

The proceptual-symbolic world, as its name suggests, is the world of symbols used for calculations and manipulations in various fields of mathematics, such as linear algebra and calculus [25]. In this world, we practice doing sequences of actions until we can accurately perform them "with little conscious effort" [27, p. 22]. Furthermore, symbols allow for easy switching from doing processes to thinking about concepts [25]. The sense-making of mathematics changes, and individuals begin to focus "on actions on objects rather than on the objects themselves" [28, p. 141]. This world develops for individuals "in a spectrum of ways from limited procedural learning to flexible proceptual thinking" [27, p. 22], which is "the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object" [13, p. 122].

2.2.3. *The Formal World*

The formal world is constructed based on "*properties*, expressed in terms of formal definitions that are used as axioms to specify mathematical structures" (e.g., group and topological spaces) [25, p. 30]. In this word, we work with axioms that are carefully "formulated to *define* mathematical structures in terms of specified properties" [25, p. 30–31]. Formal proofs are then used to deduce other properties to construct other relevant theorems [25]. Furthermore, we change how we construct meaning "from definitions based on known objects to formal concepts based on set-theoretic definitions" [26, p. 7]. This makes this world more powerful, as it is not bounded by the context in which mathematics is used [28].

2.3. The Framework of Advanced Mathematical Thinking

The APOS theory and Tall's three worlds of mathematics have "natural connections" and can be considered "complementary" [30, p. 277]. This implies that it is possible to develop action, process, and object within each embodied, proceptual-symbolic and formal world [23,30]. Moreover, these two theoretical frameworks can also be thought of "as somewhat orthogonal" [30, p. 277]. When using the FAMT, a common approach involves creating a table to investigate aspects related to the teaching and learning of mathematics. In the table, the action, process, and object are positioned in the left-hand column, while the three worlds are located on the top cells [23]. The level of complexity often increases for many students when moving from left to right (from the embodied to the formal world), from top to bottom (from action to object), or diagonally (from action embodied to object formal) [18].

3. TEACHING AND LEARNING OF EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors are often discussed "through the formal-world linguistic concept definition" in mathematics lectures, without discussing their applications in mathematics [30, p. 280]. Many students tend to think in the proceptual-symbolic word (e.g., focusing on transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$) when learning this topic, as the definitions of these concepts take on a symbolic form [30]. This could lead students to pay less attention to the embodied world when learning eigen theory. Thomas and Stewart [30] highlighted: "the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust" (p. 280). Empirical studies have pointed out this concern and reported that many students in traditional linear algebra courses have limited to no embodied understanding of eigenvalues and eigenvectors [6,30].

Several suggestions have been proposed to improve the teaching and learning of eigen theory. Thomas and Stewart [30] suggested explicitly presenting the procedure for finding eigenvectors and connecting them to conceptual ideas. Beltrán-Meneu et al. [6] reported that designing teaching activities focused on the application of eigen theory in physics could be an approach to improve students' embodied understanding of this topic. Furthermore, several studies highlighted the benefits of incorporating digital dynamic environments, such as GeoGebra [6] and Geometer's sketchpad [7], in designing teaching activities to develop students' understanding of eigen theory. More recently, the integration of a problem-based learning approach into teaching and learning of eigen theory has also been found to be useful [2]. However, as highlighted earlier, it appears that online learning resources available for eigen theory were not the primary focus of past research in linear algebra.

4. METHODOLOGY

The present study is a multiple case study [9] involving two cases. I used the keyword "eigenvectors and eigenvalues" to search on YouTube to identify possible resources for inclusion in the study as cases. Case 1 (https://www.youtube.com/watch?v=PFDu9oVAE-g&t=812s) is the most viewed YouTube video created six years ago, aimed at helping students learn about eigenvectors and eigenvalues. It was developed by *3Blue1Brown* channel, which has 4.97 million subscribers. At the time of completing this paper in February 2023, the video had garnered 3.6 million views. This channel was created by Grant Sanderson, a former Stanford student who received a communication award from the American Mathematical Society (AMS) for his work. The AMS [3] lauded the channel as a "watchable and engaging YouTube channel … about discovery and creativity in mathematics … Through 3Blue1Brown videos and animations, Sanderson presents mathematics both as practically valuable and as an art form, rich with inviting stories and arresting images." Upon watching and conducting an initial analysis of Case 1, it came to my attention that it appears to be well-suited for students with a solid understanding of linear algebra concepts. Consequently, some students with weaker mathematical backgrounds might find this video challenging to comprehend and may look for alternative videos to enhance their understanding of this topic. Then I changed my approach and used a more realistic situation to find relevant resources for this study. I did not use double quotations when searching for a relevant video (only typed in *eigenvectors and eigenvalues*) and did not change the YouTube setting to sort files based on their view counts. Instead, I kept the default format that sorts videos by their relevance. Such a search might be more realistic than the approach taken for identifying Case 1 because, in my view, undergraduate students typically search on YouTube without trying to find the most viewed content, and they probably do not use double quotations when searching on YouTube. With this approach, I came across Case 2 (https://www.youtube.com/watch?v=TQvxWaQnrqI&t=94s), a video file viewed 341,000 times, created three years ago by *Professor Dave Explains* channel with 2.37 million subscribers. It appears after Case 1 in my search.

4.1. Data Analysis

I have developed a rubric inspired by the FAMT and relevant literature that utilized this framework in linear algebra (e.g., [20,30]) to help in analyzing the movements undertaken between the three worlds by the content creators of the YouTube videos. Table 1 presents this rubric in general, and Table 2 exemplifies each cell for eigen theory.

Here, the embodied world for eigen theory is primarily conceptualized in $\mathbb{R}^2$ or $\mathbb{R}^3$ following the pertinent literature in linear algebra (e.g., [20]). Furthermore, it is noteworthy to point out that concerns have been raised regarding the extent to which geometry should be integrated into the teaching and learning of linear algebra. For instance, Gueudet-Chartier [14] emphasized that "geometry must be

Table 1. The general rubric developed for analyzing YouTube videos.

	Embodied	Proceptual-symbolic	Formal
Action	The content creator: – draws (or verbally visualizes) a mathematical object step by step without discussing the rationale behind the procedure; – evaluates the geometric representation of a mathematical object(s) step by step without justifying the rationale behind the evaluation method; and – highlights some properties of a mathematical object(s) by illustrating it geometrically/graphically without justification.	The content creator solves a task by following a method that involves using one or more formulas (step by step), without justifying the undertaken procedure.	The content creator: – provides a formal definition for a mathematical concept; – mentions some properties of a mathematical concept without justification; and – points out two or more mathematical concepts are related without discussing how.
Process	The content creator: – draws (or verbally visualizes) a mathematical object (step by step or might skip some steps) and explains how such a representation is created; – evaluates the geometric representation of a mathematical object(s) (step by step or might skip some steps) and explains the rationale behind the evaluation method; – discusses how the graphical representation of two or more mathematical objects/concepts are related; and – highlights some properties of a mathematical object by illustrating it geometrically/graphically with justification.	The content creator: – solves a task by following a method which involves using one or more formulas (step by step or might skip some steps) and justifies the undertaken procedure; and – discusses how different symbolically presented mathematical relations and formulas are related.	The content creator: – discusses how two or more mathematical concepts are related, mainly by focusing on their formal definitions; and – mentions some properties of a mathematical concept with justification, mainly by focusing on their formal definitions.
Object	The content creator explicitly (or verbally) applies transformations to the geometrical representation of a mathematical object/concept.	The content creator: – discusses the derivation of a mathematical relation or formula; – performs transformations on symbolically presented mathematical relations and formulas to identify new properties of mathematical objects; and – highlights a process as an object/a totality with justification.	The content creator uses the mathematical concept in the process of proving a mathematical theorem.
The content creator discusses the applications of the topic in the real-world and/or other disciplines.			

Table 2. The rubric developed for analyzing YouTube videos related to eigen theory.

	Embodied	Proceptual-symbolic	Formal
Action	The content creator: – draws eigenvectors for a given matrix; – illustrates geometrically/graphically certain matrices do not have eigenvectors and eigenvalues, or only have one eigenvalue; – illustrates geometrically/graphically how the value and sign of eigenvalue impact the direction and magnitude of an eigenvector; – evaluates whether a vector drawn in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an eigenvector for a given matrix; and – calculates a matrix's eigenvalue(s) by analyzing its eigenvector(s) drawn in $\mathbb{R}^2$ or $\mathbb{R}^3$.	The content creator: – calculates eigenvalues and eigenvectors for a given matrix using $AX = \lambda X$; – evaluates whether a vector(s) is an eigenvector of a matrix A using $AX = \lambda X$; and – determines whether a value is an eigenvalue of a matrix A using $AX = \lambda X$.	The content creator: – provides a formal definition for the eigenvalue and eigenvector (e.g., "An **eigenvector** of an $n \times n$ matrix A is a nonzero vector X such that $AX = \lambda X$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution X of $AX = \lambda X$; such an X is called an eigenvector corresponding to λ" [16, p. 299]); – highlights, without justification, that a matrix can have multiple eigenvalues but not more than the number of its rows/columns; – highlights, without justification, that a matrix in $\mathbb{R}$ can have no eigenvalues; and – points out eigen theory is related to linear transformation, determinants, changes of basis, and linear systems of equations without discussing how they are related.

(*continued*).

Table 2. Continued.

	Embodied	Proceptual-symbolic	Formal
Process	The content creator: – considers the given matrix in a task about eigen theory as a linear transformation (on basis vectors), draws the corresponding eigenvectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and provides justifications for both the output and the undertaken method; – justifies geometrically/graphically why certain linear transformations (e.g., rotation by 90°) do not have any eigenvectors in $\mathbb{R}^2$ or $\mathbb{R}^3$; – justifies geometrically/graphically why certain transformations (e.g., $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$) with a single eigenvalue could have multiple forms of eigenvectors; – justifies geometrically/graphically how the value and sign of an eigenvalue impact the direction and magnitude of the corresponding eigenvector(s); and – justifies geometrically/graphically how certain known linear transformations impact the eigenvalues (e.g., when the linear transformation is a rotation, the eigenvalue is 1).	The content creator justifies the *method* undertaken for calculating eigenvalues and eigenvectors, drawing on several procepts such as determinant and linear systems of equations.	The content creator: – justifies (mainly by the definition of eigenvectors and eigenvalues) why a matrix can have multiple eigenvalues but not more than the number of its rows/columns and/or why in $\mathbb{R}$, a matrix can have no eigenvalues; – discusses the relationship between eigenspace and linear span; and – discusses the relationship between eigen theory and linear transformation, determinants, changes of basis, and systems of linear equations.
Object	The content creator discusses that eigenvectors "can be picked up mentally" [23, p. 953] and performs linear transformations on their graphical representations to explore whether they can span the full space. Here the content creator probably illustrates geometrically/graphically what can happen if eigenvectors do not span the full space.	The content creator: – performs operations on $AX = \lambda X$ when solving linear algebra problems/theorems (e.g., multiplying both sides of $AX = \lambda X$ by A in the process of proving $A^k X = \lambda^k X$); and – highlights, with justification, that the resulting object in each side of $AX = \lambda X$ is a vector [30].	The content creator uses eigen theory in proving mathematical theorems (e.g., the diagonalization theorem).

The content creator discusses the applications of eigen theory in various fields, such as in differential equations and electrical engineering (for example, see Section 5.7 in [16]).

used very carefully in linear algebra courses" (p. 500) because, in certain situations, it can create didactical obstacles for students.

The data are analyzed deductively based on the developed framework. I considered several measures to improve the validity and reliability of this qualitative research [8]. These tables underwent multiple iterations of improvement, involving the thorough examination of the YouTube videos on multiple occasions, as well as receiving consultation from a senior lecturer in mathematics education who is well-acquainted with the FAMT and linear algebra research and teaching. I reviewed the video transcripts twice and examined the emerged codes multiple times to ensure there was no *drift* in the code definitions. Furthermore, I provided *a rich, thick description* of the cases and included several excerpts from the YouTube videos in the results section to enhance the credibility of the findings [31].

5. RESULTS

The analysis of the two cases is presented separately at first and then summarized in Table 3.

5.1. Case 1

In the first case, the video begins with a quote by Serge Lang about the potential connections between mathematics and music. Then immediately, the content creator, whom I refer to as Grant, highlights that many students found this topic unintuitive:

> "Eigenvectors and eigenvalues" is one of those topics that a lot of students find particularly unintuitive. Questions like "why are we doing this" and "what does this actually mean" are too often left just floating away in an unanswered sea of computations.

He then points out that this topic is not "particularly complicated or poorly explained"; however, to have an embodied understanding of this topic, one needs to have a "solid visual understanding for many of the topics that precede it." Grant further elaborates that a process or object understanding of four key concepts in linear algebra is essential:

> Most important here is that you know how to think about matrices as linear transformations, but you also need to be comfortable with things like determinants, linear systems of equations and change of basis. Confusion about eigen stuffs usually has more to do with a shaky foundation in one of these topics than it does with eigenvectors and eigenvalues themselves. (*Action formal*)

After this note, Grant moves to the embodied word by demonstrating a linear transformation in two dimensions that moves the basic vectors $\hat{i}$ and $\hat{j}$ to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively. He points out that this transformation can be represented by $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. Grant then encourages viewers to think about what this transformation would do to a particular vector and its span. He illustrates it geometrically/graphically in the video for a vector and then proceeds to introduce eigenvectors in the embodied world:

> Most vectors are going to get knocked off their span during the transformation … it would seem pretty coincidental if the place where the vector landed also happens to be somewhere on that line. But some special vectors do remain on their own span, meaning the effect that the matrix has on such a vector is just to stretch it or squish it, like a scalar.

Grant illustrates this for $\hat{i}$ in two dimensions and notes that any other vectors on the x-axis have the same feature: "What's more, because of the way linear transformations work, any other vector on the x-axis is also just stretched by a factor of 3 and hence remains on its own span." He then points out another eigenvector of this matrix (i.e., $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$) without introducing the term yet. Grant illustrates how the transformation impacts this vector, stretching it by a factor of 2. He points out, "linearity is going to imply that any other vector on the diagonal line spanned by this guy is just going to get stretched out by a factor of 2." He concludes this illustration by highlighting that these are all vectors that do not knock off their span during transformation by $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ (*Process embodied*). Grant now introduces the main terms in this topic:

> As you might have guessed by now, these special vectors are called the "eigenvectors" of the transformation, and each eigenvector has associated with it, what's called an "eigenvalue," which is just the factor by which it stretched or squashed during the transformation. (*Action formal*)

He then illustrates that an eigenvalue could be negative:

> In another example, you could have an eigenvector with eigenvalue $\frac{-1}{2}$, meaning that the vector gets flipped and squished by a factor of 1/2. But the important part here is that it stays on the line that it spans out without getting rotated off of it. (*Action embodied*)

Grant does not stop here and provides another example, this time involving a three-dimensional rotation. During this illustration, he implicitly points out the connection between eigenvalues and rotation: "since rotations never stretch or squish anything, so the length of the vector would remain the same" (*Process embodied*). He concludes the journey in the embodied world by illustrating again how linear transformation presented as a matrix can be interpreted. He emphasizes the significance of finding eigenvalues and eigenvectors:

> With any linear transformation described by a matrix, you could understand what it's doing by reading off the columns of this matrix as the landing spots for basis vectors. (*Process embodied*). But often, a better way to get at the heart of what the linear transformation actually does, less dependent on your particular coordinate system, is to find the eigenvectors and eigenvalues. (*Action formal*)

Grant starts the journey in the proceptual-symbolic world by saying that he does not discuss the computational method for finding eigenvalues and eigenvectors in detail; however, he provides "an overview of the computational ideas that are most important for a conceptual understanding." He introduces eigenvectors and eigenvalues symbolically as follows:

> Symbolically, here's what the idea of an eigenvector looks like, $A\vec{v} = \lambda\vec{v}$. A is the matrix representing some transformation, with $\vec{v}$ as the eigenvector, and λ is a number, namely the corresponding eigenvalue. What this expression is saying is that the matrix-vector product

> $A\vec{v}$ gives the same result as just scaling the eigenvector $\vec{v}$ by some value λ. So, finding the eigenvectors and their eigenvalues of a matrix A comes down to finding the values of $\vec{v}$ and λ that make this expression true. (*Process proceptual-symbolic*)

He then points out that working with this equation might be initially challenging because the left-hand side is a matrix-vector multiplication; In contrast, the right-hand side is a scalar vector multiplication. Grant addresses this issue by rewriting $\lambda\vec{v}$ as a "kind of" matrix-vector multiplication, incorporating the identity matrix into the equation, $A\vec{v} = (\lambda I)\vec{v}$. He elaborates on how this inclusion impacts the right-hand side of the equation. He then subtracts$(\lambda I)\vec{v}$ from both sides and factors out $\vec{v}$, resulting in $(A - \lambda I)\vec{v} = \vec{0}$ (*Process proceptual-symbolic*). Grant then points out we are interested in non-zero solutions for $\vec{v}$, stating: "… this will always be true if $\vec{v}$ itself is the zero vector, but that's boring. What we want is a non-zero eigenvector" (*Action proceptual-symbolic*). He continues by moving to the embodied world for illustration and elaboration:

> And if you watched Chapters 5 and 6, you'll know that the only way it's possible for the product of a matrix with a non-zero vector to become zero is if the transformation associated with that matrix squishes space into a lower dimension. And that squishification corresponds to a zero determinant for the matrix. (*Action embodied*)

To illustrate this, he considers A as $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ and λ as a parameter ranging between 0 and 3. On the screen, Grant calculates the determinant of $A - \lambda I$, which becomes $\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$ for any value of λ within this interval, while presenting this transformation geometrically/graphically. Grant points out:

> Now imagine tweaking λ, turning a knob to change its value. As that value of λ changes, the matrix itself changes, and so the determinant of the matrix changes. The goal here is to find a value of λ that will make this determinant zero, meaning the tweaked transformation squishes space into a lower dimension. In this case, the sweet spot comes when λ equals 1. (*Action embodied* and *Action proceptual-symbolic*)

He concludes this demonstration by emphasizing that if another matrix is chosen, the eigenvalue is not necessarily 1 (*Action formal*). Then, he briefly reviews what has happened in the proceptual-symbolic world: "… and remember, the reason we care about that is because it means

$$
\begin{aligned}
(A - \lambda I)\vec{v} &= \vec{0} \\
A\vec{v} - (\lambda I)\vec{v} &= \vec{0} \\
A\vec{v} &= (\lambda I)\vec{v} \\
A\vec{v} &= \lambda\vec{v}
\end{aligned}
$$

which you can read off as saying that the vector $\vec{v}$ is an eigenvector of A" (*Action proceptual-symbolic*). Grant moves back again to the embodied world and illustrates what having an eigenvalue of 1 means for the vector: "$\vec{v}$ would actually just stay fixed in place" (*Action embodied*). Then Grant moves back to the proceptual-symbolic world and reviews the procedure once again on the screen:

$$\text{“}A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} = (\lambda I)\vec{v}$$
$$A\vec{v} - (\lambda I)\vec{v} = 0$$
$$(A - \lambda I)\vec{v} = 0$$
$$\det(A - \lambda I) = 0\text{”} \quad (\textit{Action proceptual-symbolic}).$$

He then continues by reiterating the importance of constructing processes and objects of relevant concepts in linear algebra in order to develop a solid embodied understanding of the eigen theory:

> This is the kind of thing I mentioned in the introduction. If you didn't have a solid grasp of determinants and why they relate to linear systems of equations having non-zero solutions, an expression like this [det $(A - \lambda I) = 0$] would feel completely out of the blue. (*Action formal*)

Grant revisits the first example (i.e., $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$) and the proceptual-symbolic world by calculating eigenvalues step by step (*Action proceptual-symbolic*). Then only for one of the eigenvalues (i.e., 2), he presents the equation that leads to identification of the corresponding eigenvector (i.e., $\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$) and refrains from performing the computation. This omission could serve as an indication that the procedural aspect of calculating the eigenvector is not the focus of this video: "If you computed this, the way you would any other linear system, you'd see that the solutions are all the vectors on the diagonal line spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$" (*Action proceptual-symbolic*). He illustrates the eigenvectors corresponding to this eigenvalue and continues by reemphasizing the association between the linear transformation and eigenvalue: "… This corresponds to the fact that the unaltered matrix $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ has the effect of stretching all those vectors by a factor of 2" (*Action embodied*).

Grant then proceeds by pointing out that a two-dimensional transformation is not necessarily required to have eigenvectors. He illustrates and justifies this by initially discussing a rotation by 90° in the embodied world (*Process embodied*), then moves to the proceptual-symbolic world, where he demonstrates that the corresponding eigenvalues are $\lambda = i$ and $\lambda = -i$ (*Action proceptual-symbolic*). Grant continues by discussing a shear transformation (i.e., $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$). He first illustrates the shear, its eigenvectors and eigenvalue in the embodied world (*Action embodied*). Subsequently, he substantiates that the eigenvalue is 1 by calculating the eigenvalue in the proceptual-symbolic world (i.e., calculating (det $\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right)$)) (*Action proceptual-symbolic*).

Grant continues by introducing another property of eigenvalues and eigenvectors: "It's also possible to have just one eigenvalue, but with more than a line full of eigenvectors" (*Action formal*). He illustrates this property by moving to the embodied world and demonstrating a transformation that scales every vector by 2 (i.e., $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$) (*Action embodied*).

The final main topic Grant discusses is the concept of eigenbasis. He begins by exemplifying cases of such situations (i.e., $\hat{i}$ scaled by -1 and $\hat{j}$ scaled by 2) (*Action embodied*). Grant shows this transformation using a matrix ($\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$) (*Action proceptual-symbolic*). In the proceptual-symbolic world, he further provides another example involving a 4 by 4 diagonal matrix (*Action proceptual-symbolic*). Grant highlights that both examples represent diagonal matrices and defines this type of matrix (*Action formal*). He then emphasizes that the diagonal entries in diagonal matrices are eigenvalues: "… the way to interpret this is that all the basis vectors are eigenvectors, with the diagonal entries of this matrix being their eigenvalues" (*Action formal*). Continuing in the proceptual-symbolic world, Grant underscores a computational property of diagonal matrices: multiplying diagonal matrices n times by itself corresponds to scaling each basis vector by the n-th power of the corresponding eigenvalue (*Action proceptual-symbolic*). He encourages viewers to multiply a non-diagonal matrix (i.e., $\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$) multiple times to see how useful it is to work with diagonal matrices: "In contrast, try computing the 100th power of a non-diagonal matrix. Really, try it for a moment, it's a nightmare" (*Action proceptual-symbolic*).

Grant proceeds by emphasizing that in many cases, eigenvectors span the full space. In such instances, theses eigenvectors can serve as basis vectors by changing the coordinate system (*Action formal*). He then details the process of expressing a transformation from one coordinate system to another by moving to the proceputal-symbolic world:

> Take the coordinates of the vectors that you want to use as a new basis, which, in this case, means are two eigenvectors, that make those coordinates the columns of a matrix, known as the change of basis matrix. When you sandwich the original transformation putting the change of basis matrix on it's right and the inverse of the change of basis matrix on its left, the result will be a matrix representing that same transformation, but from the perspective of the new basis vectors coordinate system:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{(\textit{Action proceptual-symbolic})}.$$

Grant illustrates it also in the embodied world (*Action embodied*) and underscores its usefulness:

> The whole point of doing this with eigenvectors is that this new matrix is guaranteed to be diagonal with its corresponding eigenvalues down that diagonal. This is because it represents working in a coordinate system where what happens to the basis vectors is that they get scaled during the transformation. (*Action formal*)

He proceeds by providing a definition for the eigenbasis (*Action formal*) and delves deeper into the practicality of this concept, noting that for tasks such as computing the 100th power of a matrix (e.g., $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{100}$) transitioning to an eigenbasis simplifies the computation, followed by conversion back to the original system. Grant explains it without performing the computation (*Action formal*). He then highlights that not all transformations can be approached in this manner: "A shear, for example, doesn't have enough eigenvectors to span the full space. But if you can find an eigenbasis, it makes matrix operations really lovely." (*Action formal*).

Grant concludes the video by setting a challenge for viewers related to computing arbitrary powers of A^n for the given matrix (i.e., $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$) by changing to an eigenbasis.

5.2. Case 2

Here the content creator, let's refer to him as Dave, begins by attempting to motivate viewers/students to learn about eigenvalues and eigenvectors. He mentions several applications of these concepts in physics:

> Eigenvalues and eigenvectors represent an incredibly useful concept in linear algebra, and we can see their application[s] not just in math, but also in physics [...] Eigenvalues can be used to solve systems of linear differential equations, describe natural frequencies of vibrations [...] distinguish states of energy, and much more. (*Highlighting applications of the topic*)

Then, Dave defines eigenvectors and eigenvalues by first introducing matrix A as an $n \times n$ square matrix. He continues by defining eigenvectors as "vectors that have a special relationship with matrix A, such that when you multiply A times the eigenvector $\vec{x}$, you get back that same vector, multiplied by a scalar, λ. These scalars are called eigenvalues... ." On the screen, the symbolic representation (i.e., $A\vec{x} = \lambda\vec{x}$) is also presented (*Action formal*). He continues by pointing out a number of properties of eigenvectors and eigenvalues without justifying them: (a) "eigenvectors must be non-trivial," (b) a matrix can have multiple eigenvalues, but not more than the number of its rows/columns, and (c) "each eigenvalue is associated with a specific eigenvector" (*Action formal*).

Then Dave moves to the proceptual-symbolic world. He introduces a 2 by 2 matrix (i.e., $A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$) and a vector $\left(\text{i.e., } \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for illustration or, in his words, "we can further cement the definition we just mentioned." Dave states the goal of the example is to verify if $\vec{x}$ is an eigenvector for A. He starts by calculating $A\vec{x}$ in detail, and after simplifying the outcomes, he shows that $\vec{x}$ is an eigenvector, and the corresponding eigenvalue is -2 (*Action proceptual-symbolic*).

He continues in proceptual-symbolic world by discussing, in general, how eigenvalues can be calculated. Dave points out: "We know that the eigenvalues and eigenvectors of a square matrix, A, obey the equation $A\vec{x} = \lambda\vec{x}$." He then subtracts $\lambda\vec{x}$ from both sides (i.e., $A\vec{x} - \lambda\vec{x} = \vec{0}$), adds the identity matrix (i.e., $A\vec{x} - \lambda I\vec{x} = \vec{0}$), and points out that such an action does not change the equation. He then factors out the vector and write the equation as $(A - \lambda I)\vec{x} = \vec{0}$ (*Process proceptual-symbolic*). Dave emphasizes again that "we want non-trivial solutions to this equation," and therefore $\vec{x}$ cannot be the zero vector (*Action proceptual-symbolic*). He then turns his attention to $(A - \lambda I)$ and the fact that if it is invertible, both sides of the question can be multiplied by $(A - \lambda I)^{-1}$, leading to

$$(A - \lambda I)^{-1}(A - \lambda I)\vec{x} = (A - \lambda I)^{-1}\vec{0}$$

$$\vec{x} = \vec{0} \quad (\textit{Object proceptual-symbolic}).$$

Dave highlights: “To avoid this situation, $(A - \lambda I)$ can’t be invertible, and therefore, as we recall from learning about inverse matrices, the determinant of this matrix must be zero.” (*Action formal*). He continues by defining the characteristic polynomial (i.e., $|A - \lambda I|$) and the characteristic equation (i.e., $|A - \lambda I| = 0$) (*Action formal*). Then he discusses a concrete example (i.e., $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$) to reinforce student learning, saying, “Let’s try a concrete example so that this will make more sense.” Dave solves this example in detail, step by step (*Action proceptual-symbolic*). Then he continues by discussing how eigenvectors can be calculated, first in general, and then provides a concrete example by returning to the previous matrix to calculate its eigenvectors:

> Once we have found the eigenvalues for a matrix, we can start solving for the eigenvectors… it must be done for each eigenvalue separately. We… actually plug in one of the eigenvalues we found for lambda and get a new matrix… what we are left with is a system of equations that we can solve by any method we’ve learned so far. The most consistent method here is to use row operations to get the matrix into row echelon form.

Up to this point, Dave has described the method step by step (*Action proceptual-symbolic*). However, he then chooses to provide a justification for a part of his approach:

> There will be times when we must “choose” values for the components of the eigenvectors, but it doesn’t really matter what we choose because the solution we find by doing so only represents the form of eigenvectors. Any scalar multiple of the vector we find through this method will also be an eigenvector.

He justifies this by assuming $\vec{x}$ is an eigenvector of A and therefore, $A\vec{x} = \lambda\vec{x}$. Dave then considers the vector $c\vec{x}$ where c is a scalar and justifies that $c\vec{x}$ is also an eigenvector for *A*.

$$A\left(c\vec{x}\right) = cA\vec{x} = c\lambda\vec{x} = \lambda\left(c\vec{x}\right)$$
$$A\left(c\vec{x}\right) = \lambda\left(c\vec{x}\right) \quad (\textit{Object proceptual-symbolic}).$$

He returns to the previous matrix (i.e., $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$) and calculates its eigenvectors in detail, step by step (*Action proceptual-symbolic*). After finding an eigenvector corresponding to the eigenvalues, he highlights again that every scalar multiple of an eigenvector is also an eigenvector:

> So, for the eigenvalue $\lambda = 3$, we get the eigenvector, $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. However, recall that this vector only represents the form eigenvectors take for this eigenvalue. We can have any multiple of this vector, and it will still be an eigenvector. To put it simply, any vector where the second element is twice the first, will be an eigenvector, $\vec{x} = \begin{bmatrix} c \\ 2c \end{bmatrix}$ (*Action proceptual-symbolic*).

Dave then continues by focusing on finding eigenvalues and eigenvectors of a 3 by 3 matrix (i.e., $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix}$) to reinforce learning of this topic. He says: “Now that we’ve gotten our feet wet, let’s go through one more example to really make sure we understand this process.” He provides a full solution, only skipping the explanation of how he subtracts λI from A: “First, to find the eigenvalues, let’s find $A - \lambda I$, which will mean subtracting λ from the terms in the main diagonal of A. $A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 3 & -2-\lambda & 0 \\ 2 & 3 & 4-\lambda \end{bmatrix}$ … ” (*Action proceptual-symbolic*). Dave ends this video by giving a task for viewers to solve. He includes such a practice at the end of his videos on this YouTube channel: “Now that we have sufficiently discussed the important process of finding eigenvalues and eigenvectors, let’s check comprehension. Find the eigenvalues and eigenvectors associated with the following matrix: $A = \begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$.”

5.3. A Cross-case Analysis

I conclude the results section by summarizing the analysis of the two cases in Table 3. These percentages should be interpreted with extreme caution due to variations in the length of episodes (each segment of the videos), as evident in Sections 5.1 and 5.2, ranging from a few seconds to a few minutes. However, Table 3 offers an overview of the content focus. The findings indicate that the embodied world received greater emphasis in Case 1, while the proceptual-symbolic world was more prominent in Case 2. Furthermore, both cases devoted nearly equal attention to the formal world. Across the three worlds, action was the dominant thinking type in both cases. In Case 1, the process was observed in the embodied and proceptual-symbolic worlds, while in Case 2, the object was observed in the proceptual-symbolic world.

Table 3. A summary of the findings across the two cases.

	Embodied		Proceptual-symbolic		Formal	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
Action	9 (21.9%)	0	13 (31.7%)	6 (46.1%)	13 (31.7%)	4 (30.8%)
Process	4 (9.8%)	0	2 (4.9%)	0	0	0
Object	0	0	0	2 (15.4%)	0	0
The application of the topic in the real-world and other disciplines			Case 1		Case 2	
			0		1 (7.7%)	

6. DISCUSSION AND CONCLUSIONS

In this study, conducted through a multiple case study approach, my focus was on exploring the potential opportunities that online YouTube resources could offer to students in enhancing their understanding of eigen theory. This research contributes to the mathematics education literature in several ways. It seems it is the first attempt to analyze available online YouTube resources relating to linear algebra. Secondly, as part of this study, an analytical framework has been developed, drawing from APOS and Tall’s three worlds of mathematics (depicted in Table 1). This framework can serve as a tool for analyzing online YouTube resources on other mathematical topics and has the potential to be adapted for evaluating the content delivered by lecturer/teachers in mathematical lectures/classrooms.

The findings suggest that there exist diverse opportunities for students to learn about eigenvalues and eigenvectors through online YouTube resources. These resources guide the viewers/students across the three worlds of mathematics. However, depending on the focus of the content creator, one world of mathematics could receive more attention. In Case 1, it was the embodied world, while in Case 2, it was the proceptual-symbolic world. In these two cases, one can observe that the process formal and object formal were not discussed. This could be related to the fact that these thinking types are more in focus in linear algebra courses for mathematics major students, rather than the majority of students who take linear algebra as one of their "service mathematics courses" in their STEM programs.

Digital technology has impacted the quality of YouTube resources in the past few years. It was interesting to observe that in Case 1, these technologies were used simultaneously to discuss eigenvalues in the embodied and proceptual-symbolic worlds. Such utilization of technology in YouTube resources could potentially address the difficulties highlighted in the literature, where many students do not or have a limited understanding of eigen theory in the embodied world [6,30].

To conclude, previous research suggests that YouTube resources are widely accepted by students and utilized for learning mathematics at the university level (e.g., [1,12]). The findings indicate that these resources have different focuses, and each is suitable for students/viewers with different competencies in linear algebra. For example, one could argue that the majority of students/viewers aiming to learn about eigen theory can relate to Case 2; however, it seems a strong (embodied) understanding of several key concepts in linear algebra (e.g., change of basis), as mentioned by Grant at the beginning of the video, is necessary for students/viewers to engage with Case 1. Therefore, mathematics lecturers and teacher assistants could utilize or recommend these available resources depending on students' competency and the intended learning outcome.

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