Mathematics and engineering: Interplay between praxeologies

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Introduction

There are several on-going initiatives aiming at strengthening the connection between mathematics and applications in engineering, both within study programmes and between study programmes and work life. The CDIO (Conceive, Design, Implement, Operate http://www.cdio.org) approach has formulated some general principles for engineering education, such as the principle of *contextual learning*: "Concepts … are presented in the context of their use", and "[e]xamples include believable situations that students recognize as being important to their current or possible future lives" (Crawley et al., 2014, pp. 32-33). In the CDIO approach, a deep working knowledge and conceptual understanding are emphasised (Crawley et al., 2014, p. 13). This may be interpreted in the way that to use mathematics in engineering contexts requires understanding of mathematics at the level of *studied reflection*: "[t]o be able to use mathematics to solve problems" and "[t]o understand how mathematics applies to other situations" (Booth, 2004, p. 25). In the literature, there is evidence that students often do not see the relevance of the mathematics they are expected to learn (Flegg et al., 2012), that they find it challenging to apply mathematics they have learnt when they need it in engineering courses (Carvalho & Oliviera, 2018), that mathematics in engineering courses is often invisible (González-Martín, 2021; González-Martín & Hernandez-Gomes, 2017), and that connections between mathematics and engineering are often lacking (Faulkner et al., 2019).

This paper is based on a collaborative project between mathematics and electrical engineering. The students are in their first year of the Master of Technology (MT) programme Electronic Systems Design and Innovation. I will present an example showing that engineering problems may require (rather advanced) mathematical knowledge to be solved. However, also deep knowledge from the engineering field is necessary to model the problem in mathematical terms. Following the Anthropological Theory of the Didactic (ATD), I see mathematics and engineering as two institutions, each with their own praxeology, Π_M (mathematics) and Π_E (engineering). The analysis shows that to answer the generating question, arising in Π_E, essential elements from both Π_M and Π_E are required. I will write $\Pi_i = [P_i/L_i] = [T_i, \tau_i, \theta_i, \Theta_i]$, $i = E$ or M, according to standard notation in ATD (Bosch & Gascón, 2014). The interplay between praxeologies in the same project is further elaborated in Rønning (2022). Other authors have also discussed interplay between praxeologies, e.g., Peters et al. (2017), in the extended praxeological ATD-model.

To investigate whether the context-based teaching affects the students' perceived relevance of mathematics, a survey was distributed in the spring of 2022 both to the students within the project and to all other first-year MT students. Some results from this survey are shown at the end.

The example

The fundamental example is the oscillator circuit shown in Figure 1, and the generating question Q is to determine the output voltage y. This circuit is an extension of the simpler circuit shown in Figure 2 which contains an amplifier, described by the linear relation $z = Gy,$ where G is a positive number and y is the voltage. The circuit in Figure 1 was used as an example both in the mathematics course and in an electronics course running in parallel. In the electronics course, the students built the circuit from physical components (τ_E), and they could observe and measure its behaviour (τ_E). However, to compute the output, mathematical concepts and techniques (τ_M) were necessary. Furthermore, to explain why the mathematical techniques worked, a mathematical technology (θ_M) was required. This can be seen as an interplay between the praxeologies Π_M and Π_E where the mathematical understanding is lifted to the level of studied reflection (Booth, 2004). I will now discuss the two circuits more in detail to see how the praxeologies interact. The circuit in Figure 2 can be modelled with the differential equation (1) (Lundheim, 2021).

$$(1)\ y'' + (1 - G)\frac{R}{L}y' + \frac{1}{LC}y = 0.$$

This differential equation can be solved using analytic methods, and the solution can be written

$$y(t) = e^{-\delta t}(A\sin(\omega t) + B\cos(\omega t)),$$

where $\delta = (1 - G)R/2L$ and $\omega = \sqrt{1/LC - \delta^2}$. Modelling the circuit requires knowledge from Π_E, and solving the differential equation (1) requires knowledge from Π_M. It follows that when $G = 1$, harmonic oscillations are obtained. When $G < 1$ ($\delta > 0$), $|y(t)| \to 0$, hence, the oscillations will die out. When $G > 1$ ($\delta < 0$), $|y(t)| \to \infty$, the oscillations will blow up, and the system will be unstable.

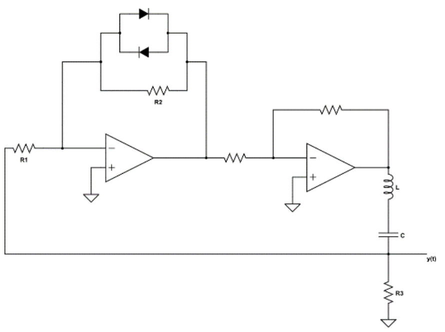

Figure 1. The oscillator

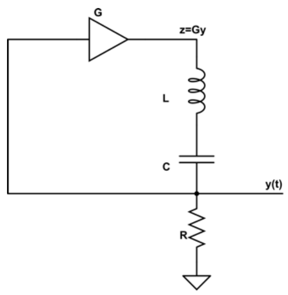

Figure 2. A simple circuit

From an engineering point of view, stable oscillations are desirable, so one would like to set the constant G in the amplifier equal to 1, which would be mathematically easy. However, the engineer knows (Π_E) that keeping the number G exactly equal to 1 is, for physical reasons, impossible in practice. Hence, it is necessary to modify the circuit, which is how the circuit in Figure 1 is created. Without going into details, I will only say that this modification results in a non-linear model. This modification requires knowledge both from the praxis block and the logos block of Π_E. Mathematically, the modification means that the linear function $z = Gy$ is replaced with $z = g(y)$, where g is the function, whose inverse is given in (3). Using properties of the elements of the circuit (Lundheim, 2021), the equation for the circuit in Figure 1 becomes

$$(2)\ \ y'' + (1 - g'(y))\frac{R}{L}y' + \frac{1}{LC}y = 0,$$

where the inverse of the function g can be written as

$$(3)\ \ g^{-1}(y) = 2R_1 I_0 \sinh\left(\frac{y}{V_0}\right) + \frac{1}{R_2}y.$$

The constants in the expressions come from the specifications of the components in the circuit. To formulate the equation (2) and the expression for g^{-1} in (3) knowledge both from Π_E and Π_M is necessary. As a result of the modification, the mathematical problem has changed to the non-linear differential equation (2) instead of the linear equation (1). This challenges both the praxis block and the logos block of Π_M: Does the equation have a solution (L_M) and if so, how can it be solved (P_M)? The equation (2) is a special case of Lienard's equation, and Lienard's theorem gives conditions, which are part of L_M, for this equation to have a stable limit cycle (see e.g., Lins et al., 1977, pp. 335-336). It can be solved (P_M) using e.g., the symplectic Euler method (Hairer & Wanner, 2015).

Discussion

Traditionally, students in their first year will encounter only analytic methods for solving second order differential equations with constant coefficients, like equation (1). Numerical methods are usually at this stage restricted to simple methods (e.g., Euler's method) for solving the first order initial value problem $y' = f(x, y), y(0) = y_0$. Solving systems is also at this stage usually restricted to linear systems. Seen from a mathematical point of view, these choices are natural since they give simple, elegant solutions and they show how various parts of mathematics are useful, such as complex numbers, or eigenvalues and eigenvectors of matrices. However, from an engineering point of view, these methods have limited value since they can only be applied to situations which are not so often found in real engineering applications, or in engineering courses, as the example with the oscillator in Figure 1 shows. The possible discrepancy between the classical methods (τ_M) from Π_M and the relevant applications (T_E) in Π_E raises the question of *relevance* of mathematics for engineering. The table below shows answers to two of the statements presented in the survey administered to the students. The percentages in boldface show the results from the students within the project (n = 45) and those in normal font in parenthesis show the results from the rest of the students (n = 494). These results indicate that working with mathematics in context may increase the perceived relevance of mathematics.

	Completely agree	Partly agree	Partly disagree	Completely disagree
In my work with other courses (i.e., not mathematics courses), I have seen the importance of learning mathematics.	**85 %** (37 %)	**13 %** (44 %)	**0 %** (14 %)	**0 %** (5 %)
I don't think the mathematics I have learned is very relevant for my study programme.	**2 %** (5 %)	**2 %** (25 %)	**18 %** (44 %)	**78 %** (26 %)

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