Some Basic Properties of Length Rate Quotient

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Abstract. Length Rate Quotient (LRQ) is the first algorithm of interleaved shaping – a novel concept proposed to provide per-flow shaping for a flow aggregate without per-flow queuing. This concept has been adopted by Time-Sensitive Networking (TSN) and Deterministic Networking (DetNet). In this paper, we investigate basic properties of LRQ interleaved shapers. One is the so-called "shaping-for-free" property, which is, when an LRQ interleaved shaper is appended to a FIFO system, it does not increase the worst-case delay of the system. The other basic properties include conformance, output characterization, a sufficient and necessary condition for bounded delay, Guaranteed Rate characterization, and delay and backlog bounds for LRQ interleaved shapers as stand-alone elements. The derived properties of LRQ shed new insights on understanding interleaved shaping, which may be further exploited to achieve bounded delay in TSN / DetNet networks.

Keywords: Interleaved Shaping · Length Rate Quotient (LRQ) · Time Sensitive Networking (TSN) · Deterministic Networking (DetNet) · Asynchronous Traffic Shaping · Interleaved Shaper · Interleaved Regulator

1 Introduction

Interleaved shaping is a novel concept for traffic shaping, originally proposed in [1]. Conceptually, its idea is to perform per-flow traffic shaping within a flow aggregate using only one FIFO queue. An appealing property of interleaved shaping is the so-called "shaping-for-free" property: When an interleaved shaper is appended to a FIFO system and shapes flows to their initial traffic constraints, it does not increase the worst-case delay of the system. Based on this property, an approach to achieve bounded worst-case end-to-end (e2e) delay in the network is investigated in [1]. The approach includes a specific way to allocate shaping and scheduling queues in switches and re-shaping flows to their initial traffic constraints using the corresponding interleaved shaping algorithms.

The concept of interleaved shaping, together with the approach of allocating queues and reshaping traffic [1], has been adopted and extended by IEEE Time-Sensitive Networking (TSN) [2] and IETF Deterministic Networking (DetNet) [3] to deliver bounded e2e latency. The concept is called Asynchronous Traffic Shaping (ATS) in the former [4] while Interleaved Regulation in the latter [5].

In [1], two algorithms for interleaved shaping are introduced, which are Length Rate Quotient (LRQ) and Token Bucket Emulation (TBE). While LRQ is for traffic constraints where the gap between consecutive packets satisfies a length rate quotient condition, TBE is for the well-known token bucket (TB) or leaky bucket (TB) traffic constraints. In [6], more types of traffic constraints are investigated under a unified traffic constraint concept called "Pi-regularity" and the resultant interleaved shapers are called Interleaved Regulators (IRs). The "shaping-for-free" property is also proved for IRs in [6].

Surprisingly, other than the "shaping-for-free" property, few other properties of interleaved shapers have been reported. To bridge the gap, this paper is intended. Specifically, we focus on *LRQ, the first interleaved shaping algorithm.* In addition to "shaping-for-free", a set of basic properties, not previously investigated, are proved in this paper, which include conformance, output characterization, a sufficient and necessary condition to ensure the existence of bounded delay, Guaranteed Rate [7][8] service characterization, and delay and backlog bounds for LRQ interleaved shapers as standalone elements.

The rest is organized as follows. In the next section, the LRQ interleaved shaping algorithm and its modeling are first introduced, followed by some other preliminaries. They include traffic and server models used in the investigation. In Section 3, various basic properties of LRQ are proved. Concluding remarks are given in Sec. 4

2 The LRQ Algorithm and Preliminaries

2.1 The LRQ Interleaved Shaping Algorithm

The LRQ algorithm Length Rate Quotient (LRQ) is the first algorithm of interleaved shaping [1]. Consider an LRQ shaper, whose FIFO queue is shared by an aggregate of flows. The LRQ shaper performs per-flow *interleaved shaping* on the aggregate, according to the algorithm shown in Algorithm 1 [1].

Algorithm 1 Pseudo code of the LRQ algorithm

Initialization: $\forall f : [f].eligibility\_time = 0$
Shaping:

while (true) {
  wait until $q.size > 0$;
  $p := q.head(); l := p.length; f := p.flow\_index$;
  $E^f := [f].eligibility\_time$;

  wait until $t^{now} \geq E^f$; **output** p;

  $E^f := t^{now} + \frac{l}{r^f}$;
  $[f].eligibility\_time := E^f$;
}

The LRQ algorithm shown in Algorithm 1 takes the original form in [1]. As is clear from Algorithm 1, there is only one FIFO queue q where per-flow shaping is conducted. In Algorithm 1, q denotes the queue of the shaper, where packets join in the order of their arrival times. After reaching the head of the queue, the head packet p is checked for its eligibility of output from the queue, which depends on the flow f that it belongs to. Time stamp E^f stores the eligible time of flow f for its next packet. At the output time d of packet p, the time stamp E^f is updated to equal the present / output time $d(= t_{now})$ plus the quotient (l/r^f), where l is the length of p. In this way, the next packet of flow f after this packet p is at least delayed until the time t_{now} reaches E^f.

A model for LRQ To model the LRQ algorithm, let j denote the packet number of p in Algorithm 1, i.e., p is the j-th packet of the aggregate flow g coming out of the queue q in Line 3. In addition, let a^j and d^j denote the arrival time and output / departure time of the packet. Furthermore, let $f_{(j)}$ denote the flow where the packet is from, $i_{(j)}$ its packet number in this flow $f_{(j)}$, and $E^{f,i}$ the eligibility time of packet $p^{f,i}$, i.e., the i-th packet of flow f.

Line 6 tells that under the condition implied by Line 2, LRQ outputs the packet immediately when the present time t_{now} reaches the eligibility time of the packet $E^{f_{(j)},i_{(j)}}$. In other words, the output time equals the eligibility time, i.e., $d^j = E^{f_{(j)},i_{(j)}}$. The condition of Line 2 is that the packet must have already arrived, i.e. $d^j \geq a^j$. In addition, the loop, particularly the two highlighted lines, Lines 2 and 6 imply the FIFO order is preserved when outputting packets, or in other words, $d^j \geq d^{j-1}$. Combining these, we have

$$d^j = \max\{a^j, d^{j-1}, E^{f_{(j)},i_{(j)}}\} \tag{1}$$

with *the initialization condition* $E^{f,1} = 0$ for $\forall f$, and $d^0 = 0$ since the queue is initially empty, where the eligibility time function E^f is updated according to Lines 8 and 9 as:

$$E^{f_{(j)},i_{(j)}+1} = d^j + l^j / r^{f_{(j)}}. \tag{2}$$

Remark on model difference The concept of interleaved shaping has been extended to consider other shaping constraints, such as token-bucket constraint [1] [4] and "Pi-regularity" constraint [6], and has been adopted by IEEE TSN [4] and IETF DetNet [3]. In these standards as well as in the modeling work [6], the interleaved shaping algorithms directly take (1) as the form, where the eligibility time function (2) is adapted according to the targeted shaping constraint. Specifically, the corresponding time functions of d and E are respectively called *GroupEligibilityTime* and individual flow' *schedulerEligibilityTime* in the IEEE Standard 802.1Qcr [4].

In the modeling work [6], the introduced Π^f function is indeed the function E^f (2) here. For interleaved LRQ, the Pi-function has the following expression:

$$\begin{aligned} \Pi^{f,i}_{LRQ}(r^f) &= d^{f,i-1} + l^{f,i-1}/r^f \quad \text{for} \quad i \geq 2 \\ \Pi^{f,1}_{LRQ}(r^f) &= -\infty \quad\quad\quad \text{for} \quad i = 1 \end{aligned} \tag{3}$$

As a highlight, the initial condition for the Π^f function is different from that for E^f. While it is $E^{f,1} = 0$ in the initial LRQ algorithm [1] and the model (1) above, it is $\Pi^{f,1}_{LRQ} = -\infty$ in [6]. Also in [6], this initial condition is discussed to be necessary for its proposed "Pi-regularity" traffic constraint model.

2.2 Flow and Server Models

Notation Consider a FIFO system. Let $\mathcal{F}$ denote the set of flows. For each flow f, let $p^{f,i}$ denote the i-th packet in the sequence, where $i \in \mathcal{N}^+ \equiv \{1, 2, \dots\}$, and $l^{f,max}$ its maximum packet length. For every packet $p^{f,i}$, we denote by $a^{f,i}$ its arrival time to the system, $d^{f,i}$ its departure time from the system, and $l^{f,i}$ its length. The maximum packet length of the system is denoted by l^{max}. In addition, define $A^f(t) \equiv \sum_i \{l^{f,i} | a^{f,i} < t\}$, i.e., the cumulative amount of traffic from flow f which has arrived up to time t, $A^f(s,t) \equiv A^f(t) - A^f(s)$, and $A^f(0) = 0$. For the departure process of the flow from the system, $D^f(t)$ is similarly defined. Sometimes, reference time functions are used to characterize how the flow f is treated by the system. Specifically, we use E^f and F^f to respectively refer to the times when the packets have reached the head of the queue and become eligible for receiving service, and the times when packets are expected to depart. They define reference eligible time $E^{f,i}$ and expected finish time $F^{f,i}$ for each packet $p^{f,i}$ of the flow f. When the concern is on a single flow, the upper script ^{f} may be removed for representation simplicity.

Flow models For flows, two specific traffic models are considered. One is the g-regularity model [9], also known as the *max-plus arrival curve* model [10][11]:

Definition 1. *A flow is said to be g-regular for some non-negative non-decreasing function $g(\cdot)$ iff for all $i \geq j \geq 0$, there holds $a^i \geq a^j + g(L(i) - L(j))$ or equivalently, $\forall i \geq 0$,*

$$a^i \geq \sup_{0 \leq k \leq i} \{a^k + g(L(i) - L(k))\} \equiv a \overline{\otimes} g^{(i)} \tag{4}$$

where $L(i) \equiv \sum_{k=0}^{i-1} l^k$ with $g(0) = 0$ and $L(0) = 0$, and $\overline{\otimes}$ is called the max-plus convolution operator.

In the case $g(x) = \frac{x}{r}$ with a constant rate r, which is equivalent to $a^{i+1} \geq a^i + \frac{l^i}{r}$, $\forall i \geq 1$, we also say the flow is ***LRQ*(*r*)-constrained**.

Another traffic model that will be used is the *(min-plus) arrival curve* model.

Definition 2. *A flow is said to have a (min-plus) arrival curve α, which is a non-negative non-decreasing function, iff the traffic of the flow is constrained by [12], $\forall s, t \geq 0$, $A(s, s+t) \leq \alpha(t)$ or equivalently, $\forall t \geq 0$,*

$$A(t) \leq \inf_{0 \leq s \leq t} \{A(s) + \alpha(t-s)\} \equiv A \otimes \alpha(t) \tag{5}$$

where define $\alpha(0) = 0$ and $\otimes$ is the min-plus convolution operator.

A special type of arrival curve has the form: $\alpha(t) = \rho \cdot t + \sigma$. In this case, we will also say that the flow is leaky-bucket or token-bucket (σ, ρ)**-constrained**. The (σ, ρ) model was first introduced by Cruz in his seminal work [13]. It can be verified that if a flow is $LRQ(r)$-constrained, it is also (σ, ρ)-constrained with $\sigma = l^{max}$ and $\rho = r$, i.e., having a (min-plus) arrival curve $\alpha(t) = rt + l^{max}$.

As shown by the two definitions, while the g-regularity or max-plus arrival curve model characterizes a flow based on the arrival time a^i, the (min-plus) arrival curve model does so based on the cumulative traffic amount function $A(t)$. In the literature, e.g., [9][11], the relationship between the min-plus and max-plus arrival curves has been investigated. Particularly, it has been shown [11] that they can be converted to and are dual of each other.

Server models For server modeling, define two reference time functions $E(\cdot)$ and $F(\cdot)$ iteratively as: $\forall i \geq 1$

$$E^i(r) = \max\{a^i, E^{i-1} + \frac{l^{i-1}}{r}\} \tag{6}$$

$$F^i(r) = \max\{a^i, F^{i-1}\} + \frac{l^i}{r} \tag{7}$$

with $E^0 = 0$, $F^0 = 0$, and $l^0 = 0$ where r denotes the reference service rate. Later E will also be referred to as the eligibility time or *virtual start time (VST)* function, and F the *virtual finish time (VFT)* function. The following relationship between functions E and F can be easily verified, e.g., see [14]: $\forall i \geq 1$,

$$F^i = E^i + \frac{l^i}{r^f} \tag{8}$$

Definition 3. *A system is said to be a Guaranteed Rate (GR) server with guaranteed rate r and error term e to a flow, written as $GR(r, e)$, iff it guarantees that for any packet p^i of the flow, its departure time satisfies [7][8]:*

$$d^i \leq F^i(r) + e \tag{9}$$

or equivalently

$$d^i \leq a \overline{\otimes} g^{(i)} + \frac{l^i}{r} \tag{10}$$

with $g^{(x)} = \frac{x}{r} + e$, where $\overline{\otimes}$ is the max-plus convolution operator.

It has been shown that a wide range of scheduling algorithms, including priority, weighted fair queueing and its various variations, round robin and its variations, hierarchical fair queueing, Earliest Due Date (EDD) and rate-controlled scheduling disciplines (RCSDs), can be modeled using GR [7][8]. For this reason and to simplify the representation, instead of presenting results for schedulers implementing specific scheduling algorithms, we use the GR model to represent them. A summary of the corresponding GR parameters of various scheduling algorithms can be found, e.g., in [14].

Considering the relationship (8), a server model may similarly be defined based on E, which is called the Start-Time (ST) server model, written as $ST(r, \tau)$, iff for any packet p^i of the flow, the system guarantees its departure time [14]:

$$d^i \leq E^i(r) + \tau \tag{11}$$

or equivalently

$$d^i \leq a \overline{\otimes} g^{(i)} \tag{12}$$

with $g(x) = \frac{x}{r} + \tau$ and $\tau = e + l^{max}/r$.

As indicated by the max-plus convolution operator used in (10) and (12), these models are server models for the max-plus branch of network calculus [9]. In the min-plus branch of network calculus, the *(min-plus) service curve* model is well-known. The latency-rate type (min-plus) service curve is defined as follows.

Definition 4. *A system is said to offer to a flow a latency-rate service curve* $\beta(t) = r(t - \tau)^+$ *iff for all* $t \geq 0$ *[12],*

$$D(t) \geq A \otimes \beta(t) \tag{13}$$

where $(x)^+ \equiv \max\{x, 0\}$.

In [12][14], the relationship between the GR model, the ST model, the latency-rate server model and the (min-plus) latency-rate service curve has been investigated. Particularly, it is shown [14] that the latency-rate server model is equivalent to the start-time (ST) server model. With the relation (8), it can be verified that if a system is a $GR(r, e)$ server to a flow, it is also a $ST(r, e + \frac{l^{max}}{r})$ server and provides a latency-rate service curve β to the flow [14][12]:

$$\beta(t) = r[t - (e + \frac{l^{max}}{r})]^+ \tag{14}$$

Conversely, if the system is an $ST(r, \tau)$ or latency-rate server with the same parameters to the flow, it is also a $GR(r, \tau - \frac{l^{min}}{r})$ to the flow [14].

Delay and backlog bounds With the flow and server models introduced above, the following delay and backlog bounds can be found or proved from literature results, e.g.,[8][12].

Proposition 1. *Consider a flow served by a system. The flow has an arrival curve* α*, and the system is a* $GR(r, e)$ *server to the flow. If* $\lim_{t\to\infty} \frac{\alpha(t)}{t} \leq r$*, the delay of any packet* i*, i.e.,* $d^i - a^i$*, is upper-bounded by,* $\forall i \geq 1$*,*

$$d^i - a^i \leq \frac{\sup_{t\geq 0}[\alpha(t) - rt]}{r} + e$$

and the backlog of the system at any time, i.e., $D(t) - A(t)$*, is upper-bounded by,* $\forall t \geq 0$*,*

$$D(t) - A(t) \leq \sup_{t\geq 0}[\alpha(t) - r(t - e - \frac{l^{max}}{r})^+]$$

As a special case, the flow is (σ, ρ)-constrained, i.e. $\alpha(t) = \rho t + \sigma$. If $\rho \leq r$, the bounds in Proposition 1 can be written more explicitly as, $\forall i \geq 1$,

$$d^i - a^i \leq \frac{\sigma}{r} + e \tag{15}$$

for delay and $\forall t \geq 0$,

$$D(t) - A(t) \leq \sigma + \rho \cdot (e + \frac{l^{max}}{r}) \tag{16}$$

for backlog.

In the TSN / DetNet literature, the delay and backlog bounds are derived commonly based on the assumption that the flow has a (min-plus) arrival curve and the server has a latency-rate (min-plus) service curve [15], except in the initial interleaved shaping paper [1] that adopts a timing analysis technique directly on the reference time functions similar to our analysis in this paper. It has also been noticed that the delay bounds from the service curve analysis are more pessimistic than from the timing based analysis [15]. This difference is also seen here as discussed in the following.

Specifically, service curve-based analysis can result in a delay bound that is $\frac{l^{max}}{r}$ larger than the bound from GR-based analysis shown in Proposition 1. The difference is due to the extra term $\frac{l^{max}}{r}$ in the service curve characterization as shown in (14). By exploiting an advanced property of network calculus (NC), which is "the last packetizer can be ignored for delay computation" (see e.g. [12]), the packetizer delay can be deducted from the service curve based delay bound. However, considering that the delay bound must hold for all packets, only $\frac{l^{min}}{r}$ may thus be extracted. Consequently, the "improved" service curve based delay bound becomes:

$$\frac{\sigma}{r} + e + \frac{l^{max}}{r} - \frac{l^{min}}{r}.$$

Then its difference from GR-based analysis can be reduced to

$$\frac{l^{max}}{r} - \frac{l^{min}}{r}.$$

As a remark, the discussion on the delay bound difference is only based on the server models themselves. When delay bound analysis is conducted on a specific scheduling discipline, the GR-based analysis may benefit additionally, e.g., an example of this can be found in [16].

3 Basic Properties of Interleaved LRQ Shapers

3.1 The "Shaping-for-Free" Property

As introduced in Section 2, functions (1) and (2) capture the essence of the LRQ algorithm. In addition, by adapting (2), interleaved shaping of flows with other

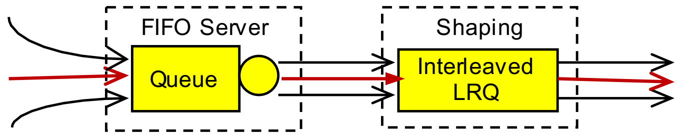

Fig. 1. The shaping-for-free property setup

traffic constraints can be implemented, for which, a systematic investigation has been conducted in [6]. Applying (2) to (1), we can rewrite and obtain the following model for LRQ: $\forall j \geq 1$,

$$d^j = \max\{a^j, d^{j-1}, d^{f_{(j)}, i_{(j)}-1} + \frac{l^{f_{(j)}, i_{(j)}-1}}{r^{f_{(j)}}}\} \tag{17}$$

with **the initial condition:** $d^{f,0} = 0$ **and** $l^{f,0} = 0$ **for** $\forall f$**, which is equivalent to the initial condition** $E^{f,0} = 0$ for (1), since the three involved parameters d, l and r in (2) are non-negative in nature and r is non-zero.

In the literature, "shaping-for-free" is a well known property of per-flow shapers. Specifically, if a shaper is greedy and the initial traffic constraint of the flow is used as the shaping curve, the worst-case delay of the flow in a system composed of the shaper and a server is not increased in comparison with a system of the server only, in spite of the order of the shaper and the server in the combined system [9] [12].

Figure 1 illustrates a typical setup when studying the shaping-for-free property, where interleaved shaping is performed after the FIFO system. In Theorem 1, we extend the study and prove that interleaved shaping does not affect the worst-case delay no matter if interleaved shaping is introduced before or after the FIFO system.

Theorem 1. *Consider a set of flows $\mathcal{F}$, where every flow $f (\in \mathcal{F})$ is LRQ(r^f)-regulated, i.e., $a^{f,i} \geq a^{f,i-1} + \frac{l^{f,i-1}}{r^f}$. These flows pass through a system composed of a FIFO server and an interleaved LRQ shaper with rate r^f for f, $\forall f \in \mathcal{F}$. No matter about the order of the server and the shaper, a delay upper bound for the FIFO server is also a delay upper bound for the composite system.*

Proof. The property has two parts: (I) the LRQ shaper is before the FIFO server; (II) the FIFO server is followed by the LRQ shaper as illustrated in Figure 1.

For part (I), the proof needs Lemma 1 and Lemma 2, which are introduced in Section 3.2. Specifically, with the former, the regulator introduces no delay. With the latter, the output from the regulator, i.e., the input to the server, is regulated with the same traffic constraint and hence the same delay bound remains.

For part (II), the proof is as follows. Let $\hat{a}$ denote the departure from the server and hence the arrival to the regulator. Suppose Δ is a delay bound for all packets through the FIFO server, i.e., $\hat{a}^j \leq a^j + \Delta$ for $\forall j \geq 1$. We prove by strong induction that for the composite system shown in Figure 1, Δ is also a delay bound, i.e. $d^j - a^j \leq \Delta$ for $\forall j \geq 1$, where a^j and d^j respectively denote the arrival and departure times of the j-th packet through the composite system.

For the base step, consider both the 1st and the 2nd packets. By definition and the initial condition $d^{f,0} = 0$ and $l^{f,0} = 0$ for $\forall f$ as discussed above, for the 1st packet, it is obtained immediately $d^1 = \hat{a}^1 \leq a^1 + \Delta$. For the 2nd packet, by the LRQ model (17), $d^2 = \max\{\hat{a}^2, d^1, d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} \leq \max\{a^2 + \Delta, a^1 + \Delta, d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\}$. There are two cases. (i) The 2nd packet is from a different flow, which is the first packet of that flow. In this case, $d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}} = 0$ by definition, and hence $d^2 \leq a^2 + \Delta$ since $a^2 \geq a^1$. (ii) The 2nd packet is from the same flow. Then, $d^2 \leq \max\{a^2 + \Delta, a^1 + \Delta, d^1 + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} \leq \max\{a^2 + \Delta, a^1 + \Delta, a^1 + \Delta + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} = \max\{a^2, a^1 + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} + \Delta \leq a^2 + \Delta$. This completes the base step.

For the induction, assume the theorem holds for all packets till $j-1$ with $j > 2$, which implies (i) $d^{j-1} \leq a^{j-1} + \Delta$. The induction assumption also implies (ii) $d^{f_{(j)},i_{(j)}-1} \leq a^{f_{(j)},i_{(j)}-1} + \Delta$. Applying these to (17), together with $\hat{a}^j \leq a^j + \Delta$, gives:

$$
\begin{aligned}
d^j &= \max\{\hat{a}^j, d^{j-1}, d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} \\
&\leq \max\{a^j + \Delta, a^{j-1} + \Delta, a^{f_{(j)},i_{(j)}-1} + \Delta + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} \\
&= \max\{a^j, a^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} + \Delta \\
&= a^j + \Delta
\end{aligned}
$$

where the last step is due to the *LRQ* traffic constraint. Note that in the induction step, we have implicitly assumed that packet j is not the first packet of flow $f_{(j)}$ to apply (ii). In the case that j is the first packet, by definition and the initial condition, we also have $d^j = \max\{\hat{a}^j, d^{j-1}, 0\} \leq \max\{a^j + \Delta, a^{j-1} + \Delta\} \leq a^j + \Delta$, where we have applied the induction assumption (i). This completes the proof. □

Remark 1. Under interleaved shaping, the shaping-for-free property, corresponding to Part II of Theorem 1, is first investigated in [1], but only implicitly. In [6], a generalized treatment is provided, where the property is proved for a wide range of traffic constraints, including both Chang's g-regularity and (min-plus) arrival curve constraints. However, it is worth highlighting that **the investigation in [6] requires a necessary initialization condition** (3), which is different from that used by LRQ, Algorithm 1. Theorem 1 has bridged the gap.

Remark 2. The shaping-for-free property investigated in [1] and [6] assumes that the interleaved shaper is immediately placed after the FIFO server as illustrated by Figure 1, Theorem 1 extends this and additionally proves that placing the shaper before the FIFO server does not increase worst-case delay either.

3.2 Basic Properties of a Standalone LRQ Interleaved Shaper

In this subsection, a number of basic properties of a standalone LRQ interleaved shaper, which have not been previously reported, are proved.

Lemma 1. *(Conformance) Consider an interleaved LRQ shaper with a set of input flows $\mathcal{F}$, where for every flow $f \in \mathcal{F}$, rate r^f is applied. If at the input, every flow $f \in \mathcal{F}$ is LRQ(r^f)-regulated, then the shaper introduces no delay, i.e., for every packet p^j, there holds $d^j = a^j$.*

Proof. The proof is similar to that for the second part of Theorem 1. We prove by (strong) induction. For the base case, consider the 1st packet and the 2nd packet. By definition and the initial condition, it is obtained immediately $d^1 = a^1$. For the 2nd packet, $d^2 = \max\{a^2, d^1, d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} = \max\{a^2, a^1, d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} = \max\{a^2, d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\}$. There are two cases. (i) The 2nd packet is from a different flow. In this case, $d^{f_{(2)},i_{(2)}-1} + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}} = 0$ by definition, and hence $d^2 = a^2$. (ii) The 2nd packet is from the same flow. Then, $d^2 = \max\{a^2, d^1 + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} = \max\{a^2, a^1 + \frac{l^{f_{(2)},i_{(2)}-1}}{r^{f_{(2)}}}\} = a^2$. This proves the base case.

For the induction, assume the theorem holds for all packets till $j-1$, which implies $d^{j-1} = a^{j-1}$ and $d^{f_{(j)},i_{(j)}-1} = a^{f_{(j)},i_{(j)}-1}$. Applying these to (17) gives:

$$\begin{aligned} d^j &= \max\{a^j, d^{j-1}, d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} \\ &= \max\{a^j, a^{j-1}, a^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} \\ &= \max\{a^j, a^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} \\ &= a^j \end{aligned}$$

which completes the proof. □

An implication of Lemma 1 is that at any time, there is at most one packet in the LRQ system from each flow. This information may be used for conformance check. For instance, from each flow, at most one packet is allowed and additional non-conformant packets are dropped. This way can prevent delaying other flows' packets if one flow is non-conformant to its $LRQ(r^f)$-constraint.

Lemma 2. *(Output Characterization) Consider an interleaved LRQ shaper with a set of flows $\mathcal{F}$, where for every flow $f \in \mathcal{F}$, rate r^f is applied. Regardless of the traffic constraint for each flow at the input, the output of the flow f is constrained by LRQ(r^f), i.e., $\forall i \geq 1$,*

$$d^{f,i} \geq d^{f,i-1} + \frac{l^{f,i-1}}{r^f}.$$

Proof. The output characterization result follows from (17), since the right hand side of (17) is not smaller than $d^{f,i-1} + \frac{l^{f,i-1}}{r^f}$ for any packet $p^{f,i}$ of the flow f. $\square$

Having proved Lemma 1 and Lemma 2, we now focus on the worst-case delay. Unfortunately, its analysis is notoriously challenging. In the rest of this section, we approach it step by step. First, the following result provides a sufficient and necessary condition for an LRQ interleaved shaper to have bounded delay.

Lemma 3. *(Sufficient and Necessary Condition) For an LRQ interleaved shaper with rates* $\{r^f\}$ *for its flow set* $\mathcal{F}$*, the delay for any packet is upper-bounded, if and only if there exists a non-negative constant* $\Delta(<\infty)$ *such that,* $\forall j \geq 1$*,*

$$d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}} - a^j \leq \Delta \tag{18}$$

and if the condition is satisfied, Δ *is also an upper-bound on the delay.*

Proof. For proving (18) is a necessary condition, let's first assume the condition does not hold and then prove the conclusion does not hold consequently. Specifically, the assumption is that for some j, $d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}} - a^j$ is not bounded. Since by definition $d^j \geq d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}$ and hence $d^j - a^j \geq d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}} - a^j$, so for this j, $d^j - a^j$ is not bounded. This completes the necessary condition part.

For the sufficient condition part, we prove by induction that if (18) holds for $\forall j \geq 1$, we also have $d^j - a^j \leq \Delta$ for $\forall i \geq 1$, and hence it is a delay upper-bound. For the base case, $j = 1$. By definition, we have $d^1 = a^1$, and hence $d^1 - a^1 = 0 \leq \Delta$. For the induction case, let's assume Δ is an upper bound for $j-1, (\forall j > 1)$ and then prove it is also an upper bound for j. With the definition of d^j, we have for its delay:

$$\begin{aligned} d^j - a^j &= \max\{a^j, d^{j-1}, d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\} - a^j \\ &= \max\{0, d^{j-1} - a^j, d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}} - a^j\} \\ &\leq \max\{0, d^{j-1} - a^{j-1}, \Delta\} \leq \max\{0, \Delta, \Delta\} = \Delta \end{aligned}$$

which completes the proof. $\square$

Note that, in Lemma 3, the condition does not assume how each flow is regulated at the input. If the flow is $LRQ(r^f)$-regulated at the input, applying this traffic condition together with $d^{f_{(j)},i_{(j)}-1} = a^{f_{(j)},i_{(j)}-1}$ from Lemma 1 gives $d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}} - a^j \leq 0$. In other words, the sufficient and necessary condition is satisfied with $\Delta = 0$. This also confirms Lemma 1 .

When the flow is not $LRQ(r^f)$-regulated, the condition constant Δ is not as easily found. Additional approaches are needed to help find delay bounds. For this, in Lemma 4, we relate the departure time with a generalized version of the virtual start time and virtual finish time functions defined in (6) and (7). Specifically, their generalized counterparts are: $\forall j \geq 1$,

$$\tilde{E}^j = \max\{a^j, \tilde{E}^{j-1} + \frac{l^{j-1}}{r^{(j-1)}}\} \tag{19}$$

$$\tilde{F}^j = \max\{a^j, \tilde{F}^{j-1}\} + \frac{l^j}{r^{(j)}} \tag{20}$$

with $\tilde{E}^0 = \tilde{E}^0 = l^0 = 0$ and $r^0 = \infty$, where, for ease of expression, we use $r^{(j)}$ to denote the rate of the flow that packet j belongs to, i.e., $r^{(j)} \equiv r^{f_{(j)}}$.

The difference between (19) and (6), and the difference between (20) and (7), are that while the rate in the function for each packet is the same in the latter, it may differ from packet to packet in the former. These generalized virtual start time and virtual finish time functions (19) and (20) are similarly defined in the generalized Guaranteed Rate server model [8].

Lemma 4. *(GR Characterization) Consider an interleaved LRQ shaper with a set of input flows $\mathcal{F}$, where for every flow $f \in \mathcal{F}$, rate r^f is applied. The departure time of any packet p^j is bounded by: for $\forall j \geq 1$*

$$d^j \leq \tilde{E}^j = \tilde{F}^j - \frac{l^j}{r^{(j)}} \tag{21}$$

where $\tilde{E}^j$ and $\tilde{F}^j$ are defined in (19) and (20) respectively.

Proof. The definitions of $\tilde{E}^j$ and $\tilde{E}^j$ imply the following relationship between them: $\forall j \geq 1$,

$$\tilde{F}^j = \tilde{E}^j + \frac{l^j}{r^{(j)}} \tag{22}$$

which can be verified with induction. For the base step, it holds because $\tilde{F}^1 = a^1 + \frac{l^1}{r^{(1)}}$ and $\tilde{E}^1 = a^1$. For the induction step, under the induction assumption $\tilde{F}^{j-1} = \tilde{E}^{j-1} + \frac{l^{j-1}}{r^{(j-1)}}$, it also holds.

With the fact $\tilde{E}^{j-1} \leq \tilde{E}^{j-1} + \frac{l^{j-1}}{r^{(j-1)}}$, $\tilde{E}^j$ can also be written as:

$$\tilde{E}^j = max\{a^j, \tilde{E}^{j-1}, \tilde{E}^{j-1} + \frac{l^{j-1}}{r^{(j-1)}}\}$$

Compare $\tilde{E}^j$ and d^j that is copied below

$$d^j = \max\{a^j, d^{j-1}, d^{f_{(j)}, i_{(j)}-1} + \frac{l^{f_{(j)}, i_{(j)}-1}}{r^{f_{(j)}}}\}$$

We prove (21) by induction. For the base case $j = 1$, since $d^1 = a^1$, $\tilde{E}^j = a^1$ and the initial condition, (21) holds, i.e., $d^1 \leq \tilde{E}^1$. For the induction step, we suppose (21) holds for all packets $1, \ldots, j-1$, and consider packet j. There are two cases. (i) Packet p^{j-1} and packet p^j belong to the same flow. In this case, $\frac{l^{j-1}}{r^{(j-1)}} = \frac{l^{f_{(j)}, i_{(j)}-1}}{r^{f_{(j)}}}$ and hence $d^j \leq \tilde{E}^j$ under the induction assumption.

(ii) Packet p^{j-1} belongs to a different flow. In this case, since packet p^{j-1} is the immediate previous packet of p^j, due to FIFO, packet $p^{f_{(j)},i_{(j)}-1}$ must be an earlier packet than p^{j-1}, implying $d^{f_{(j)},i_{(j)}-1} \leq d^{j-1}$. Let $j^*(< j-1)$ denote the packet number of $p^{f_{(j)},i_{(j)}-1}$ in the aggregate. For $\tilde{E}^{j-1}$, by applying the definition of $\tilde{E}$ iteratively, we have

$$\begin{aligned}\tilde{E}^{j-1} = \max\{&a^{j-1}, a^{j-2} + \frac{l^{j-2}}{r^{(j-2)}}, \ldots, \\ &\tilde{E}^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}} + \sum_{k=j^*+1}^{j-2} \frac{l^k}{r^{(k)}}\} \\ \geq\ & d^{f_{(j)},i_{(j)}-1} + \frac{l^{f_{(j)},i_{(j)}-1}}{r^{f_{(j)}}}\end{aligned}$$

where for the last step, the induction assumption $d^{f_{(j)},i_{(j)}-1} \leq \tilde{E}^{f_{(j)},i_{(j)}-1}$ has also been applied. With the above and the induction assumption $d^{j-1} \leq \tilde{E}^{j-1}$, the three terms in $\tilde{E}^j$ are all not smaller than the corresponding ones in d^j. Hence $d^j \leq \tilde{E}^j$ also holds for the second case. Combining both cases, the induction step is proved, i.e. (21) holds for j. □

With Lemma 4, the following corollary is immediately from the definition of the generalized GR server model, the corresponding delay bound analysis [8] and Proposition 1.

Corollary 1. *The LRG regulator is (i) a generalized GR server with guaranteed rate* $r = \min r^f$ *and error term* $e = -\min \frac{l^{f,min}}{r^f}$ *and (ii) provides a service curve* $\alpha(t) = \min_f r^f t$*. (iii) If every flow is* (σ^f, ρ^f)*-constrained and* $\sum_f \rho^f \leq r$*, then the delay of any packet* p^j *is bounded by,* $\forall j \geq 1$*,*

$$d^j - a^j \leq \frac{\sum_f \sigma^f}{\min_f r^f} - \min_f \frac{l^{f,min}}{r^f} \tag{23}$$

and (iv) the backlog of the system at any time t *is bounded by:* $\forall t \geq 0$*,*

$$D(t) - A(t) \leq \sum_f \sigma^f + l^{max} \tag{24}$$

While it is encouraging to have the delay bound (23) for LRQ interleaved shapers as the first step, the condition $\sum_f \rho^f \leq \min_f r^f$ and the term $\min_f r^f$ in (23) make the bound conservative. We improve in the follow result.

Theorem 2. *Consider an interleaved LRQ shaper with rates* $\{r^f\}$ *for its flow set* $\mathcal{F}$*. If every flow* $f(\in \mathcal{F})$ *is* (σ^f, ρ^f)*-constrained, and* $\sum_f \frac{\rho^f}{r^f} \leq 1$*, the delay of any packet* p^j *is bounded by,* $\forall j \geq 1$*,*

$$d^j - a^j \leq \sum_f \frac{\sigma^f}{r^f} - \frac{l^{f,j}}{r^f} \tag{25}$$

which implies the following delay bound for all packets:

$$\sup_{j\geq 1}[d^j - a^j] \leq \sum_f \frac{\sigma^f}{r^f} - \min_f \frac{l^{f,min}}{r^f}$$

Proof. For any packet p^j, there exists a packet p^{j_0} whose arrival starts the "virtual busy" period that packet p^j is in, where for all packets that arrive in $[a^{j_0}, \tilde{F}^j]$ there holds $a^k \leq \tilde{F}^{k-1}$, $\forall k = j_0 + 1, \ldots, j$. Alternatively, the start of the period is by the latest packet with $a^{j_0} > \tilde{F}^{j_0 - 1}$.

Consider a virtual reference FIFO system which has the same input sequence a^j and its output is $\tilde{F}^j$. Then this period is a busy period in the virtual reference system. Note that such a "virtual busy" period always exists, since in one extreme case, p^{j_0} is the first packet for which $a^1 > \tilde{F}^0 = 0$ always holds, and in another extreme case, the period is started by the packet p^j itself and in this case, $j_0 = j$.

Applying $a^k \leq \tilde{F}^{k-1}$ to the definition of $\tilde{F}^j$ gives:

$$\tilde{F}^j = t^0 + \sum_{k=j_0}^{j} \frac{l^k}{r^{(k)}} = t^0 + \sum_{m=1}^{N} \frac{W_m(t^0, \tilde{F}^j)}{r^m} \tag{26}$$

where $W_m(t^0, d^j) = \sum_{k=j_0}^{j} l^k I_{p^k \in f}$ denotes the total amount of service (in accumulated packet lengths) from flow f, served in $[t^0, \tilde{F}^j]$, where the indicator function $I_{p^k \in f}$ has the value 1 when the condition $\{p^k \in f\}$, i.e. packet p^k is from flow f, is true.

Because of FIFO and that the virtual system is empty at t^0_-, $W_m(t^0, d^{g,j})$ is hence limited by the amount of traffic that arrives in $[t^0, a^{f_n,i}]$: $W_m(t^0, \tilde{F}^j) \leq A_m(t^0, a^{g,j})$.

We then have,

$$\tilde{F}^j \leq t^0 + \sum_f \frac{A^f(t^0, a^j)}{r_m} \tag{27}$$

Under the condition that $\sum_f \frac{\rho^f}{r^f} \leq 1$, we obtain:

$$\tilde{F}^j - a^j \leq \sum_f \frac{A^f(t^0, a^j)}{r^f} + t_0 - a^j \leq \sum_f \frac{\rho^f(a^j - t^0) + \sigma^f}{r^f} - (a^j - t_0) \leq \sum_f \frac{\sigma^f}{r^f}$$

with which, the delay bound is obtained together with Lemma 4, specifically (21). □

4 Conclusion

Though being the first algorithm of interleaved shaping, the properties of LRQ were previously little studied. As a step towards filling the gap, a set of properties for LRQ have been derived in this paper. These properties include the shaping-for-free property that has been proved without altering the initialization condition introduced in the original LRQ algorithm. In addition, a set of basic properties of a standalone LRQ interleaved shaper, which were not previously investigated, have also been derived, which include conformance, output characterization, a sufficient and necessary condition for bounded delay, GR characterization, and delay and backlog bounds. These results provide new insights on understanding interleaved shaping, which may be further exploited to deliver bounded delays in TSN / DetNet networks [16].

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