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On the accuracy of the finite volume approximations to nonlocal conservation laws

Aekta Aggarwal[1] · Helge Holden[2] · Ganesh Vaidya[2]

Abstract

In this article, we discuss the error analysis for a certain class of monotone finite volume schemes approximating nonlocal scalar conservation laws, modeling traffic flow and crowd dynamics, without any additional assumptions on monotonicity or linearity of the kernel μ or the flux f. We first prove a novel Kuznetsov-type lemma for this class of PDEs and thereby show that the finite volume approximations converge to the entropy solution at the rate of $\sqrt{\Delta t}$ in $L^1(\mathbb{R})$. To the best of our knowledge, this is the first proof of any type of convergence rate for this class of conservation laws. We also present numerical experiments to illustrate this result.

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1 Introduction

The celebrated Lighthill–Whitham–Richards (LWR) model [30, 32] given by

$$u_t + (u\nu(u))_x = 0, \qquad (t, x) \in (0, \infty) \times \mathbb{R}, \qquad (1.1)$$

with u being the mean traffic density and ν the mean traffic speed, is one of the widely used models in traffic flow modeling. However, being a non-linear hyperbolic

✉ Helge Holden
helge.holden@ntnu.no

Aekta Aggarwal
aektaaggarwal@iimidr.ac.in

Ganesh Vaidya
ganesh.k.vaidya@ntnu.no

1   Operations Management and Quantitative Techniques Area, Indian Institute of Management Indore, Prabandh Shikhar, Rau-Pithampur Road, Indore, Madhya Pradesh 453556, India

2   Department of Mathematical Sciences, NTNU — Norwegian University of Science and Technology, 7491 Trondheim, Norway

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conservation law, it can have solutions with discontinuities and infinite accelerations, adversely impacting its capability to capture physical traffic phenomena effectively. Thus, over last decade, the parallel class of conservation laws with *nonlocal* parts in the flux is gaining particular interest in the modeling as well as mathematical community, where a convolution is introduced in the flux to produce Lipschitz-continuous velocities ensuring bounded accelerations. The two most popular strategies are to evaluate the traffic speed, from either averaging the traffic density, or averaging over the velocity, leading to the following two conservation laws,

$$u_t + \Big(uv(u * \mu)\Big)_x = 0, \tag{1.2}$$

$$u_t + \Big(u(v(u) * \mu)\Big)_x = 0, \tag{1.3}$$

where $v, \mu \in (C^2 \cap W^{2,\infty})(\mathbb{R})$, and

$$(\mu * u)(t, x) = \int_{\mathbb{R}} \mu(x - \xi)\, u(t, \xi)\, \mathrm{d}\xi\,.$$

Such conservation laws have been studied in the recent literature, see [2, 7, 11, 13, 15, 18, 19, 25, 27], and the references therein. From the point of view of modeling, this *nonlocal* nature is particularly suitable in describing the behavior of traffic, where each vehicle moves according to its evaluation of the density and its variations within its horizon.

The article studies a very general class of these nonlocal conservation laws, namely

$$\partial_t u + \partial_x (f(u)v(\mu * \beta(u))) = 0, \qquad (t, x) \in Q_T := (0, T) \times \mathbb{R}, \tag{1.4}$$
$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.5}$$

where

(H1) $f \in \mathrm{Lip}(\mathbb{R})$ with $f(0) = 0$,
(H2) $\beta, v \in (C^2 \cap \mathrm{BV} \cap W^{2,\infty})(\mathbb{R})$, with $\beta(0) = 0$,
(H3) $\mu \in (C^2 \cap \mathrm{BV} \cap W^{2,\infty})(\mathbb{R})$,

with f being non-linear (in contrast to (1.2) or (1.3)). They serve as working models for a variety of real life applications, for example, sedimentation models [6], crowd dynamics models [13–15], vehicular traffic [7, 14], biological applications in structured population dynamics [31], supply chain models [14], granular material dynamics [4], as well as conveyor belt dynamics [21].

The wellposedness of this class has been of interest in the last few years. The local counterpart of (1.4)–(1.5) enjoys a rich literature, with [28] as one of the pioneering papers to fix the wellposedness of the entropy solutions for such PDEs. Similar to its local counterpart, since f can be possibly nonlinear, there can be multiple weak solutions of IVP (1.4)–(1.5). Hence, an additional entropy condition is required to single out a unique solution.

Definition 1.1 A function $u \in C([0, T]; L^1(\mathbb{R})) \cap L^\infty(\overline{Q_T})$ is an entropy solution of the IVP (1.4)–(1.5), if for every $k \in \mathbb{R}$, and for all non-negative $\phi \in C_c^\infty([0, T) \times \mathbb{R})$,

$$
\int_{Q_T} |u(t, x) - k| \phi_t \, dt \, dx + \int_{Q_T} \text{sgn}(u(t, x) - k) \mathcal{U}(t, x)(f(u) - f(k)) \phi_x \, dt \, dx
$$

$$
- \int_{Q_T} f(k) \text{sgn}(u(t, x) - k) \mathcal{U}_x(t, x) \phi \, dt \, dx + \int_{\mathbb{R}} |u_0(x) - k| \phi(0, x) \, dx \geq 0,
$$

$$(1.6)$$

where $\mathcal{U}(t, x) = \nu(\mu * \beta(u(t)))(x))$.

With some appropriate modifications, the proof of [6] can be adapted to prove that any two entropy solutions satisfying Definition 1.1 are equal, while existence of these solutions has been proven in [1, 5] for non linear f and in [9] for linear f, via the convergence of finite volume approximations. These articles dealing with existence of solutions establish that the schemes converge to the entropy solution $u \in C([0, T]; L^1(\mathbb{R}))$.

What remains unexplored is to analyze the rate of convergence, i.e., how fast the error $||u^\Delta(T, \cdot) - u(T, \cdot)||_{L^1(\mathbb{R})}$ made by the numerical solution u^Δ in approximating the exact solution u goes to zero as the mesh size Δx goes to zero. That is the precise aim of this article. In other words, we look for an (optimal) α satisfying

$$
||u^\Delta(T, \cdot) - u(T, \cdot)||_{L^1(\mathbb{R})} \leq C \Delta x^\alpha, \tag{1.7}
$$

with C being an appropriate positive constant. To achieve this, we first prove a Kuznetsov-type lemma using the entropy formulation (1.6). We further estimate the relative entropy functional involving the solution u and numerical approximation u^Δ to obtain (1.7) with an optimal $\alpha = 1/2$, same as the one obtained in [29, 33] for local fluxes (homogeneous). To the best of our knowledge, this is the first result in this direction for such nonlocal conservation laws. It is to be noted that the results of the article hold under no additional assumptions on monotonicity/linearity of the kernel μ or the flux f or ν.

The paper is organized as follows. In Sect. 2, we discuss the wellposedness of (1.4)–(1.5) via convergence of a general class of monotone finite volume approximations. In Sect. 3, we prove the Kuznetsov-type lemma for (1.4)–(1.5) and obtain the rate of convergence as 1/2. In Sect. 4, we also briefly comment on the extensions to higher dimensions. In Sect. 5, we present some numerical experiments which illustrate the theory.

2 Finite volume approximations and wellposedness

We now introduce the notations to be used in the article:

i. $|u|_{L_t^\infty \mathrm{BV}_x} := \sup\limits_{t \in [0,T]} \mathrm{TV}(u(t, \cdot))$.

ii. $|u|_{\mathrm{Lip}_t L_x^1} := \sup\limits_{0 \le t_1 < t_2 \le T} \dfrac{\|u(t_1, \cdot) - u(t_2, \cdot)\|_{L^1(\mathbb{R})}}{|t_1 - t_2|}$.

iii. $K := \{u : \overline{Q}_T \to \mathbb{R} : \|u\|_{L^\infty(\overline{Q}_T)} + |u|_{L_t^\infty \mathrm{BV}_x} < \infty\}$.

iv. $\gamma(u, \sigma) := \sup\limits_{\substack{|t_1 - t_2| \le \sigma \\ 0 \le t_1 < t_2 \le T}} \|u(t_1, \cdot) - u(t_2, \cdot)\|_{L^1(\mathbb{R})}$.

2.1 Uniqueness of the entropy solution

Any two entropy solutions of the IVP (1.4)–(1.5) are equal. More precisely, we have the following result:

Theorem 2.1 (Uniqueness) *Let $u, v \in C([0, T]; L^1(\mathbb{R})) \cap (L_t^\infty \mathrm{BV}_x)(\overline{Q}_T)$ be two entropy solutions of the IVP (1.4)–(1.5) corresponding to the initial data u_0 and v_0 respectively. Then, there exists a constant*

$$\mathcal{M} = \mathcal{M}(f, \mu, \eta, \nu, \beta, \|u\|_{L^1(Q_T)}, \|v\|_{L^1(Q_T)}, |u|_{L_t^\infty \mathrm{BV}_x}, |v|_{L_t^\infty \mathrm{BV}_x}, T) > 0$$

such that

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})}(1 + \mathcal{M}t \exp(\mathcal{M}t)), \quad t \in [0, T].$$

In particular, if $u_0 = v_0$, then $u = v$ a.e. in $\overline{Q}_T$.

Proof Note that the nonlocal coefficient of the flux function considered in this article is more general than the ones in all the previous results on the uniqueness of nonlinear-nonlocal conservation laws. However, the continuous dependence estimates for the entropy solution of the conservation laws (local) derived in [24, Thm. 1.3] can still be invoked to prove the desired weighted contraction estimates as in [6, Thm. 4.1]. Alternatively, this can also be seen as a consequence of the Kuznetsov-type estimates derived in the sequel (see Lemma 3.2), by sending $\varepsilon, \varepsilon_0 \to 0$. □

2.2 Existence via numerical approximations

For $\Delta x, \Delta t > 0$, and $\lambda := \Delta t / \Delta x$, consider equidistant spatial grid points $x_i := i \Delta x$ for $i \in \mathbb{Z}$ and temporal grid points $t^n := n \Delta t$ for non-negative integers $n \le N$, such that $T = N \Delta t$. Let $\chi_i(x)$ denote the indicator function of $C_i := [x_{i-1/2}, x_{i+1/2})$, where $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$ and let $\chi^n(t)$ denote the indicator function of $C^n := [t^n, t^{n+1})$. We approximate the initial data according to:

$$u_0^\Delta(x) := \sum_{i \in \mathbb{Z}} \chi_i(x) u_i^0 \quad \text{where } u_i^0 = \int_{C_i} u_0(x) \, \mathrm{d}x \text{ for } i \in \mathbb{Z}. \tag{2.1}$$

We define a piecewise constant approximate solution u^Δ to (1.4) by

$$u^\Delta(t, x) = u_i^n \quad \text{for} \quad (t, x) \in C^n \times C_i, \quad n \in \mathbb{N}, i \in \mathbb{Z},$$

through the following marching formula:

$$
\begin{aligned}
u_i^{n+1} &= H(v(c_{i-1/2}^n), v(c_{i+1/2}^n), u_{i-1}^n, u_i^n, u_{i+1}^n) \\
&:= u_i^n - \lambda \big[\mathcal{F}(v(c_{i+1/2}^n), u_i^n, u_{i+1}^n) - \mathcal{F}(v(c_{i-1/2}^n), u_{i-1}^n, u_i^n) \big] \\
&:= u_i^n - \lambda \big[\mathcal{F}_{i+1/2}^n(u_i^n, u_{i+1}^n) - \mathcal{F}_{i-1/2}^n(u_{i-1}^n, u_i^n) \big],
\end{aligned}
\tag{2.2}
$$

where the convolution term $\mu * \beta(u)$ is computed through a standard quadrature formula using the same space mesh, i.e.,

$$c_{i+1/2}^n := \Delta x \sum_{p \in \mathbb{Z}} \mu_{i+1/2-p} \beta(u_{p+1/2}^n) \approx \int_{\mathbb{R}} \mu(x_{i+1} - y) \beta(u^\Delta(t^n)(y)) \, dy, \tag{2.3}$$

with $u_{p+1/2}^n$ being any convex combination of u_p^n and u_{p+1}^n, and $\mu_{i+1/2} = \mu(x_{i+1/2})$. Further, $\mathcal{F}(v(c_{i+1/2}^n), u_i^n, u_{i+1}^n)$ denotes the numerical approximation of the flux $f(u)v(\mu * \beta(u))$ at the interface $x = x_{i+1/2}$ for $i \in \mathbb{Z}$, with H being increasing in the last three arguments. The approximations generated by the scheme, namely $u_i^n \approx u(t^n, x_i)$ are extended to a function defined on Q_T via

$$u^\Delta(t, x) = \sum_{n=0}^{N-1} \sum_{i \in \mathbb{Z}} \chi_i(x) \chi^n(t) u_i^n. \tag{2.4}$$

In general, $\mathcal{F}$ can be defined as an appropriate nonlocal extension of any monotone numerical flux, meant for local conservation laws. Here, we present examples of celebrated Lax–Friedrichs flux and Godunov flux.

1. **Lax–Friedrichs type flux:** For any $\theta \in \left(0, \frac{2}{3}\right)$, define

$$\mathcal{F}_{\mathrm{LF}}(a, b, c) = \frac{a}{2}\Big(f(b) + f(c)\Big) - \theta \frac{(c - b)}{2\lambda},$$

where Δt is chosen in order to satisfy the CFL condition

$$\lambda \le \frac{\min(1, 4 - 6\theta, 6\theta)}{1 + 6 |f|_{\mathrm{Lip}(\mathbb{R})} \|v\|_{L^\infty(\mathbb{R})}}. \tag{2.5}$$

2. **Godunov type flux:**

$$\mathcal{F}_{\mathrm{Godunov}}(a, b, c) = a F_{\mathrm{Godunov}}(b, c),$$

where the function F_{Godunov} is the Godunov flux for the corresponding local conservation law $u_t + f(u)_x = 0$, and Δt is chosen in order to satisfy the CFL condition

$$\lambda \, |f|_{\text{Lip}(\mathbb{R})} \, \|\nu\|_{L^\infty(\mathbb{R})} \leq \frac{1}{6}.$$

Theorem 2.2 (Existence) *Assume that (**H1**)–(**H3**) hold. For initial data $u_0 \in (L^1 \cap BV)(\mathbb{R})$ and $\Delta x > 0$, there exist constants $\mathcal{L}_i$, $i = 1, 2, 3$, independent of Δt such that the sequence of approximations u^n defined by* (2.2) *satisfies the following for all $i \in \mathbb{Z}$, $n \in \mathbb{N}$ and $0 \leq n \leq N$:*

1. L^∞ estimate:

$$\left\| u^n \right\|_{L^\infty} \leq \exp(\mathcal{L}_1 T) \left\| u^0 \right\|_{L^\infty}, \tag{2.6}$$

where $\mathcal{L}_1 = \mathcal{L}_1 \left(|\mu|_{\text{Lip}(\mathbb{R})}, \|u_0\|_{L^1(\mathbb{R})}, |f|_{\text{Lip}(\mathbb{R})}, |\beta|_{\text{Lip}(\mathbb{R})}, |\nu|_{\text{Lip}(\mathbb{R})} \right).$

2. L^1 estimate:

$$\left\| u^n \right\|_{L^1} \leq \left\| u^0 \right\|_{L^1}. \tag{2.7}$$

3. BV estimate:

$$\text{TV}(u^n) \leq \exp(\mathcal{L}_2 T)(\text{TV}(u^0) + \mathcal{L}_2), \tag{2.8}$$

where $\mathcal{L}_2 = \mathcal{L}_2(\|\mu\|_{W^{2,\infty}(\mathbb{R})}, \|\nu\|_{W^{2,\infty}(\mathbb{R})}, |f|_{\text{Lip}(\mathbb{R})}, \|u_0\|_{L^1(\mathbb{R})}).$

4. Time continuity:

$$\Delta x \sum_{i \in \mathbb{Z}} \left| u_i^m - u_i^n \right| \leq \mathcal{L}_3 |m - n| \Delta t, \quad m, n \in \mathbb{N} \cup \{0\}, \tag{2.9}$$

where $\mathcal{L}_3 = \mathcal{L}_3(\|u_0\|_{L^1(\mathbb{R})}, |f|_{\text{Lip}(\mathbb{R})}, |\mu|_{\text{Lip}(\mathbb{R})}, \|\nu\|_{W^{1,\infty}(\mathbb{R})}, \text{TV}(u^n)).$

5. Discrete entropy inequality: For any $k \in \mathbb{R}$ we have

$$\left| u_i^{n+1} - k \right| - \left| u_i^n - k \right| + \lambda \left(\mathcal{G}_{i+1/2}^n(u_i^n, u_{i+1}^n, k) - \mathcal{G}_{i-1/2}^n(u_{i-1}^n, u_i^n, k) \right) \tag{2.10}$$

$$+ \lambda \text{sgn}(u_i^{n+1} - k) f(k)(\nu(c_{i+\frac{1}{2}}^n) - \nu(c_{i-\frac{1}{2}}^n)) \leq 0, \tag{2.11}$$

where $\mathcal{G}_{i+1/2}^n(a, b, k) = \mathcal{F}_{i+1/2}^n(a \vee k, b \vee k) - \mathcal{F}_{i+1/2}^n(a \wedge k, b \wedge k)$ for all $i \in \mathbb{Z}$, $n \in \mathbb{N}$.

Furthermore, the finite volume approximations converge to the unique entropy solution u of the IVP (1.4)–(1.5).

Proof The proof follows by invoking the monotonicity of the scheme and writing it in the incremental form. The details can be worked out exactly on the similar lines of [1, Lem. 2.4–2.7] and [5, Lem. 2.2–2.8] with proper modification in the estimations on the nonlocal coefficient. □

The above theorem implies that the entropy solution satisfies the following regularity estimates.

Corollary 2.3 (Regularity of the entropy solution) *Assume that (**H1**)–(**H3**) hold. For $0 < t \leq T$ and $u_0 \in (L^1 \cap BV)(\mathbb{R})$, the entropy solution u of the IVP* (1.4)–(1.5) *satisfies the following:*

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \exp(\mathcal{L}_1 T)\|u_0\|_{L^\infty(\mathbb{R})},$$
$$\|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})},$$
$$\mathrm{TV}(u(t, \cdot)) \leq \exp(\mathcal{L}_2 T)(\mathrm{TV}(u_0) + \mathcal{L}_2),$$
$$\|u(t_2, \cdot) - u(t_1, \cdot)\|_{L^1(\mathbb{R})} \leq \mathcal{L}_3 |t_2 - t_1|, \text{ where } 0 \leq t_1, t_2 \leq T.$$

Remark 2.4 In addition, if $u_i^0 \geq 0$, the numerical approximations defined by (2.2) satisfy $u_i^n \geq 0$, see [5, Lem. 2.2] for details of the proof. This further implies that $\|u^n\|_{L^1} = \|u^0\|_{L^1}$, since the scheme is conservative. Consequently, if initial data is non-negative, so is the entropy solution.

3 Error estimate

We define $\Phi : \overline{Q}_T^2 \to \mathbb{R}$ by

$$\Phi(t, x, s, y) := \Phi^{\varepsilon, \varepsilon_0}(t, x, s, y) = \omega_\varepsilon(x - y)\omega_{\varepsilon_0}(t - s),$$

where $\omega_a(x) = \frac{1}{a}\omega\left(\frac{x}{a}\right), a > 0$ and ω is a standard symmetric mollifier with $\mathrm{supp}(\omega) \subseteq [-1, 1]$. Furthermore, we assume that $\int_{\mathbb{R}} \omega_a(x)\,\mathrm{d}x = 1$ and $\int_{\mathbb{R}} \left|\omega_a'(x)\right|\,\mathrm{d}x = \frac{1}{a}$. Now, it is straight forward to see that Φ is symmetric and $\Phi_x = \omega_\varepsilon'(x - y)\omega_{\varepsilon_0}(t - s) = -\Phi_y$, $\Phi_t = \omega_\varepsilon(x - y)\omega_{\varepsilon_0}'(t - s) = -\Phi_s$. Further, define the following functions.

Definition 3.1

$$G(a, b) := \mathrm{sgn}(a - b)(f(a) - f(b)),$$
$$\Lambda_T(u, \phi, k) := \int_{Q_T} \left(|u - k|\phi_t + \mathcal{U}(t, x)G(u, k)\phi_x - \mathrm{sgn}(u - k)f(k)\mathcal{U}_x(t, x)\phi\right)\mathrm{d}t\,\mathrm{d}x$$
$$- \int_{\mathbb{R}} |u(T, x) - k|\phi(T, x)\,\mathrm{d}x + \int_{\mathbb{R}} |u_0(x) - k|\phi(0, x)\,\mathrm{d}x,$$
$$\text{where } \phi \in C_c^\infty(\overline{Q}_T),$$
$$\Lambda_{\varepsilon, \varepsilon_0}(u, v) := \int_{Q_T} \Lambda_T(u, \Phi(\cdot, \cdot, s, y), v(s, y))\,\mathrm{d}y\,\mathrm{d}s.$$

We now state and prove the Kuznetsov-type lemma for nonlocal conservation laws.

Lemma 3.2 *Let u be the entropy solution of* (1.4)–(1.5) *and $v \in K$. Then,*

$$\|u(T, \cdot) - v(T, \cdot)\|_{L^1(\mathbb{R})} \leq \mathcal{K}\left(-\Lambda_{\varepsilon, \varepsilon_0}(v, u) + \|u_0 - v_0\|_{L^1(\mathbb{R})} + \gamma(v, \varepsilon_0) + \varepsilon + \varepsilon_0\right),$$

where $\mathcal{K}$ is constant that depends on

$$\mathcal{K} = \mathcal{K}(f, \mu, \eta, \nu, \beta, \|u\|_{L^1(Q_T)}, \|v\|_{L^1(Q_T)}, |u|_{L_t^\infty \text{BV}_x}, |v|_{L_t^\infty \text{BV}_x}, |u|_{L_t^\infty \text{BV}_x}, T)$$

and is independent of $\varepsilon, \varepsilon_0$.

Proof Consider the sum $\Lambda_{\varepsilon,\varepsilon_0}(u, v) + \Lambda_{\varepsilon,\varepsilon_0}(v, u)$:

$$\Lambda_{\varepsilon,\varepsilon_0}(u, v) + \Lambda_{\varepsilon,\varepsilon_0}(v, u)$$
$$= \int_{Q_T^2} \left(|u(t, x) - v(s, y)| \Phi_t + \mathcal{U}(t, x) G(u, v) \Phi_x \right) dt \, dx \, dy \, ds$$
$$- \int_{Q_T^2} f(v) \text{sgn}(u(t, x) - v(s, y)) \mathcal{U}_x(t, x) \Phi \, dt \, dx \, dy \, ds$$
$$- \int_{Q_T} \int_{\mathbb{R}} |u(T, x) - v(s, y)| \Phi(T, x, s, y) \, dx \, dy \, ds$$
$$+ \int_{Q_T} \int_{\mathbb{R}} |u_0(x) - v(s, y)| \Phi(0, x, s, y) \, dx \, dy \, ds$$
$$+ \int_{Q_T^2} \left(|u(t, x) - v(s, y)| \Phi_s + \mathcal{V}(s, y) G(u, v) \Phi_y \right) dt \, dx \, dy \, ds$$
$$- \int_{Q_T^2} f(u) \text{sgn}(v(s, y) - u(t, x)) \mathcal{V}_y(s, y) \Phi \, dt \, dx \, dy \, ds$$
$$- \int_{Q_T} \int_{\mathbb{R}} |v(T, y) - u(t, x)| \Phi(t, x, T, y) \, dx \, dy \, ds$$
$$+ \int_{Q_T} \int_{\mathbb{R}} |v_0(y) - u(t, x)| \Phi(t, x, 0, y) \, dx \, dy \, ds,$$

where $\mathcal{V}(s, y) = v(\mu * \beta(v(s))(y))$. Since $\Phi_s = -\Phi_t$, $\Phi_y = -\Phi_x$, we have

$$\Lambda_{\varepsilon,\varepsilon_0}(u, v) + \Lambda_{\varepsilon,\varepsilon_0}(v, u)$$
$$= \int_{Q_T^2} G(u, v) \Phi_x (\mathcal{U}(t, x) - \mathcal{V}(s, y)) \, dt \, dx \, dy \, ds$$
$$- \int_{Q_T^2} \text{sgn}(u(t, x) - v(s, y)) \left(f(v) \mathcal{U}_x(t, x) - f(u) \mathcal{V}_y(s, y) \right) \Phi \, dt \, dx \, dy \, ds$$
$$- \int_{Q_T} \int_{\mathbb{R}} |u(T, x) - v(s, y)| \Phi(T, x, s, y) \, dx \, dy \, ds$$
$$+ \int_{Q_T} \int_{\mathbb{R}} |u_0(x) - v(s, y)| \Phi(0, x, s, y) \, dx \, dy \, ds$$
$$- \int_{Q_T} \int_{\mathbb{R}} |v(T, y) - u(t, x)| \Phi(t, x, T, y) \, dy \, dt \, dx$$
$$+ \int_{Q_T} \int_{\mathbb{R}} |v_0(y) - u(t, x)| \Phi(t, x, 0, y) \, dy \, dt \, dx.$$

In other words, we have

$$\Lambda_{\varepsilon, \varepsilon_0}(u, v) = -\Lambda_{\varepsilon, \varepsilon_0}(v, u) + I_{\Phi'} + I_{\Phi} + I_0 - I_T,$$

with

$$I_{\Phi'} = \int_{Q_T^2} G(u, v)\Phi_x(\mathcal{U}(t, x) - \mathcal{V}(s, y)) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \,,$$

$$I_{\Phi} = -\int_{Q_T^2} \mathrm{sgn}(u(t, x) - v(s, y))\Big(f(v)\mathcal{U}_x(t, x) - f(u)\mathcal{V}_y(s, y)\Big)\Phi \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}s \,,$$

$$I_T = \int_{Q_T} \int_{\mathbb{R}} \Big(|u(T, x) - v(t, y)| + |v(T, y) - u(t, x)|\Big)\Phi(t, x, T, y) \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}x \,,$$

$$I_0 = \int_{Q_T} \int_{\mathbb{R}} \Big(|u_0(x) - v(t, y)| + |v_0(y) - u(t, x)|\Big)\Phi(t, x, 0, y) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \,.$$

Since u is the entropy solution of (1.4)–(1.5), we have that $\Lambda_{\varepsilon, \varepsilon_0}(u, v) \geq 0$, and hence,

$$I_T \leq -\Lambda_{\varepsilon, \varepsilon_0}(v, u) + I_{\Phi'} + I_{\Phi} + I_0. \tag{3.1}$$

The terms I_0 and I_T appear in the local case as well so they can be estimated on the similar lines of [20, 23] to get:

$$I_T \geq \|u(T, \cdot) - v(T, \cdot)\|_{L^1(\mathbb{R})} - \mathcal{K}_1(\varepsilon + \varepsilon_0 + \gamma(v, \varepsilon_0)), \tag{3.2}$$

$$I_0 \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} + \mathcal{K}_1(\varepsilon + \varepsilon_0 + \gamma(v, \varepsilon_0)), \tag{3.3}$$

where $\mathcal{K}_1 = \mathcal{K}_1(|u|_{L_t^\infty \mathrm{BV}_x}, |v|_{L_t^\infty \mathrm{BV}_x})$. Now, we estimate the other terms one by one. Using integration by parts $I_{\Phi'}$ can be written as,

$$I_{\Phi'} = -\int_{Q_T^2} \Phi\big[G_x(u, v)(\mathcal{U}(t, x) - \mathcal{V}(s, y)) + G(u, v)\mathcal{U}_x(t, x)\big] \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \,.$$

Consequently,

$$
\begin{aligned}
I_{\Phi'} &+ I_{\Phi} \\
&= -\int_{Q_T^2} \Phi(G_x(u, v)(\mathcal{U}(t, x) - \mathcal{V}(s, y)) \\
&\qquad + \mathrm{sgn}(u - v)(f(u) - f(v))\mathcal{U}_x(t, x)) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \\
&\quad - \int_{Q_T^2} \mathrm{sgn}(u(t, x) - v(s, y))\Big(f(v)\mathcal{U}_x(t, x) - f(u)\mathcal{V}_y(s, y)\Big)\Phi \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}s \\
&= -\int_{Q_T^2} \Phi(G_x(u, v)\big[(\mathcal{U}(t, x) - \mathcal{V}(s, y)) \\
&\qquad + \mathrm{sgn}(u - v)f(u)(\mathcal{U}_x(t, x) - \mathcal{V}_y(s, y))\big] \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s
\end{aligned}
$$

$$:= I_{\mathcal{U}} + I_{\mathcal{U}_x}.$$

Consider the term

$$I_{\mathcal{U}} = \int_{Q_T^2} \Phi\big[G_x(u, v)(\mathcal{V}(s, y) - \mathcal{U}(t, x))\big] \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}s \,.$$

Since $|G_x(u, v)| \le |f|_{\mathrm{Lip}(\mathbb{R})} \, |u_x|$ (in the sense of measures, see [8, Lem. A2.1] for details), we have,

$$I_{\mathcal{U}} \le |f|_{\mathrm{Lip}(\mathbb{R})} \int_{Q_T^2} \Phi \, |u_x| \, |\mathcal{V}(s, y) - \mathcal{U}(t, x)| \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}s \,.$$

Note that the term

$$
\begin{aligned}
&|\mathcal{V}(s, y) - \mathcal{U}(t, x)| \\
&\le |\mathcal{V}(s, y) - \mathcal{V}(s, x)| + |\mathcal{V}(s, x) - \mathcal{U}(t, x)| \\
&= |v(\mu * \beta(v(s)))(y) - v(\mu * \beta(v(s)))(x)| + |v(\mu * \beta(v(s)))(x) - v(\mu * \beta(u(t)))(x)| \\
&\le |v|_{\mathrm{Lip}(\mathbb{R})} \, |(\mu * \beta(v(s)))(y) - (\mu * \beta(v(s)))(x)| \\
&\quad + |v|_{\mathrm{Lip}(\mathbb{R})} \, |(\mu * \beta(v(s)))(x) - (\mu * \beta(u(t)))(x)| \\
&= |v|_{\mathrm{Lip}(\mathbb{R})} \left(\left| \int_{\mathbb{R}} \beta(v(s, z))(\mu(y - z) - \mu(x - z)) \, \mathrm{d}z \right| \right. \\
&\quad \left. + \left| \int_{\mathbb{R}} (\beta(v(s, z)) - \beta(u(t, z)))\mu(x - z) \, \mathrm{d}z \right| \right) \\
&= |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, |\mu|_{\mathrm{Lip}(\mathbb{R})} \, \|v(s, \cdot)\|_{L^1(\mathbb{R})} \, |y - x| \\
&\quad + |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \|v(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

Consequently we get:

$$
\begin{aligned}
I_{\mathcal{U}} &\le |f|_{\mathrm{Lip}(\mathbb{R})} \, |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \int_{Q_T^2} \Phi \, |u_x| \left(|\mu|_{\mathrm{Lip}(\mathbb{R})} \, \|v(s, \cdot)\|_{L^1(\mathbb{R})} \, |y - x| \right. \\
&\quad \left. + \|\mu\|_{L^\infty(\mathbb{R})} \|v(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})} \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}s \\
&\le |f|_{\mathrm{Lip}(\mathbb{R})} \, |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, |\mu|_{\mathrm{Lip}(\mathbb{R})} \int_{Q_T^2} \Phi \, |u_x| \, \|v(s, \cdot)\|_{L^1(\mathbb{R})} \, |y - x| \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}s \\
&\quad + |f|_{\mathrm{Lip}(\mathbb{R})} \, |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \int_{Q_T^2} \Phi \, |u_x| \, \|u(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}s \\
&\quad + |f|_{\mathrm{Lip}(\mathbb{R})} \, |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \int_{Q_T^2} \Phi \, |u_x| \, \|v(s, \cdot) - u(s, \cdot)\|_{L^1(\mathbb{R})} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \, \mathrm{d}s \\
&:= I_{\mathcal{U}}^1 + I_{\mathcal{U}}^2 + I_{\mathcal{U}}^3,
\end{aligned}
$$

where $I_{\mathcal{U}}^1, I_{\mathcal{U}}^2$, and $I_{\mathcal{U}}^3$ satisfy the following estimates.

$$I_{\mathcal{U}}^1 = |f|_{\mathrm{Lip}(\mathbb{R})} \, |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, |\mu|_{\mathrm{Lip}(\mathbb{R})}$$

$$\times \int_{Q_T^2} \omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)\,|u_x|\,\|v(s,\cdot)\|_{L^1(\mathbb{R})}\,|y-x|\,dx\,dy\,dt\,ds$$

$$\leq |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|\mu|_{\mathrm{Lip}(\mathbb{R})}$$

$$\times \int_{Q_T^2} \omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)\,|u_x|\,\|v(s,\cdot)\|_{L^1(\mathbb{R})}\,\varepsilon\,dx\,dy\,dt\,ds$$

$$\leq |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|\mu|_{\mathrm{Lip}(\mathbb{R})}\,|u|_{L_t^\infty \mathrm{BV}_x}\,\|v\|_{L^1(Q_T)}\varepsilon,$$

$$I_{\mathcal{U}}^2 = |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,\|\mu\|_{L^\infty(\mathbb{R})}$$

$$\times \int_{Q_T^2} \omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)\,|u_x|\,\|u(s,\cdot)-u(t,\cdot)\|_{L^1(\mathbb{R})}\,dx\,dy\,dt\,ds$$

$$\leq |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|u|_{\mathrm{Lip}_t\,L_x^1}\|\mu\|_{L^\infty(\mathbb{R})}$$

$$\times \int_{Q_T^2} \omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)\,|u_x|\,|t-s|\,dx\,dy\,dt\,ds$$

$$\leq |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|u|_{L_t^\infty \mathrm{BV}_x}\,|u|_{\mathrm{Lip}_t\,L_x^1}\|\mu\|_{L^\infty(\mathbb{R})}\int_0^T\int_0^T \omega_{\varepsilon_0}(t-s)\varepsilon_0\,dt\,ds$$

$$= |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,\|\mu\|_{L^\infty(\mathbb{R})}\,|u|_{L_t^\infty \mathrm{BV}_x}\,|u|_{\mathrm{Lip}_t\,L_x^1}T\varepsilon_0,$$

$$I_{\mathcal{U}}^3 = |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|u|_{L_t^\infty \mathrm{BV}_x}\,\|\mu\|_{L^\infty(\mathbb{R})}$$

$$\times \int_0^T\int_0^T \omega_{\varepsilon_0}(t-s)\|v(s,\cdot)-u(s,\cdot)\|_{L^1(\mathbb{R})}\,dt\,ds$$

$$\leq |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|u|_{L_t^\infty \mathrm{BV}_x}\,\|\mu\|_{L^\infty(\mathbb{R})}\int_0^T \|v(s,\cdot)-u(s,\cdot)\|_{L^1(\mathbb{R})}\,ds\,.$$

Collectively, we have

$$I_{\mathcal{U}} \leq |f|_{\mathrm{Lip}(\mathbb{R})}\,|v|_{\mathrm{Lip}(\mathbb{R})}\,|\beta|_{\mathrm{Lip}(\mathbb{R})}\,|u|_{L_t^\infty \mathrm{BV}_x}$$

$$\times \Big(|\mu|_{\mathrm{Lip}(\mathbb{R})}\,\|v\|_{L^1(Q_T)}\varepsilon + \|\mu\|_{L^\infty(\mathbb{R})}(|u|_{L_t^\infty \mathrm{BV}_x}\,T\varepsilon_0$$

$$+ \int_0^T \|v(s,\cdot)-u(s,\cdot)\|_{L^1(\mathbb{R})}\,ds)\Big)$$

$$\leq \mathcal{K}_2(\varepsilon+\varepsilon_0) + \mathcal{K}_3 \int_0^T \|v(s,\cdot)-u(s,\cdot)\|_{L^1(\mathbb{R})}\,ds\,, \tag{3.4}$$

for some appropriate constants $\mathcal{K}_2$ and $\mathcal{K}_3$. Now, we consider,

$$I_{\mathcal{U}_x} = \int_{Q_T^2} \Phi\,\mathrm{sgn}(u-v)f(u)(\mathcal{U}_x(t,x)-\mathcal{V}_y(s,y))\,dt\,dx\,dy\,ds$$

$$\leq |f|_{\mathrm{Lip}(\mathbb{R})}\int_{Q_T^2} \omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)\,|u|\,\big|\mathcal{U}_x(t,x)-\mathcal{V}_y(s,y)\big|\,dt\,dx\,dy\,ds\,.$$

Note that

$$|\mathcal{V}_y(s,y)-\mathcal{U}_x(t,x)| \leq \big|\mathcal{V}_y(s,y)-\mathcal{V}_x(s,x)\big| + |\mathcal{V}_x(s,x)-\mathcal{U}_x(t,x)|\,.$$

Now, adding and subtracting $v'((\mu * \beta(v(s)))(y))(\mu' * \beta(v(s)))(x)$ to $\big|\mathcal{V}_y(s, y) - \mathcal{V}_x(s, x)\big|$, we get

$$\big|\mathcal{V}_y(s, y) - \mathcal{V}_x(s, x)\big| \leq \big|v'((\mu * \beta(v))(s, y))((\mu' * \beta(v))(s, y) - (\mu' * \beta(v))(s, x))\big|$$
$$+ \big|(v'((\mu * \beta(v(s)))(y)) - v'((\mu * \beta(v(s)))(x)))(\mu' * \beta(v(s)))(x)\big|.$$

Furthermore,

$$|\mu' * \beta(v(s))(y) - \mu' * \beta(v(s))(x)| \leq |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu'|_{\mathrm{Lip}(\mathbb{R})} \|v(s, \cdot)\|_{L^1(\mathbb{R})} |y - x|,$$
$$|v'(\mu * \beta(v(s)))(y) - v'(\mu * \beta(v(s)))(x)| \leq |v'|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|v(s, \cdot)\|_{L^1(\mathbb{R})} |y - x|,$$

which implies that

$$\big|\mathcal{V}_y(s, y) - \mathcal{V}_x(s, x)\big|$$
$$\leq |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu'|_{\mathrm{Lip}(\mathbb{R})} \|v(s, \cdot)\|_{L^1(\mathbb{R})} |y - x|$$
$$+ |v'|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|v(s, \cdot)\|_{L^1(\mathbb{R})} |y - x| |\mu|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})}$$
$$\times \|v(s, \cdot)\|_{L^1(\mathbb{R})}.$$

Adding and subtracting $v'((\mu * \beta((s)))(x))(\mu' * \beta(u(t)))(x)$ to $|\mathcal{V}_x(s, x) - \mathcal{U}_x(t, x)|$, we ge

$$|\mathcal{V}_x(s, x) - \mathcal{U}_x(t, x)| \leq \big|v'((\mu * \beta(v(s)))(x))((\mu' * \beta(v(s)))(x) - (\mu' * \beta(u(t)))(x))\big|$$
$$+ \big|(v'((\mu * \beta(v(s)))(x)) - v'((\mu * \beta(u(t)))(x)))(\mu' * \beta(u(t)))(x)\big|.$$

Moreover,

$$|(\mu' * \beta(v))(s, x) - (\mu' * \beta(u))(t, x)| \leq |\mu|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} \|v(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})},$$
$$|v'(\mu * \beta(v))(s, x) - v'(\mu * \beta(u))(t, x)| \leq |v'|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} \|\mu\|_{L^\infty(\mathbb{R})}$$
$$\times \|v(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})}.$$

Collecting all terms, we have

$$|\mathcal{V}_x(s, x) - \mathcal{U}_x(t, x)|$$
$$\leq \|v'\|_{L^\infty(\mathbb{R})} \|\mu'\|_{L^\infty(\mathbb{R})} \|v(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})}$$
$$+ |v'|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} \|\mu\|_{L^\infty(\mathbb{R})} \|v(s, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})}$$
$$\times \|v(s, \cdot)\|_{L^1(\mathbb{R})}.$$

Now, it can be observed that $I_{\mathcal{U}_x}$ can be handled like (3.4), leading to the following estimate:

$$I_{\mathcal{U}_x} \leq \mathcal{K}_4(\varepsilon + \varepsilon_0) + \mathcal{K}_5 \int_0^T \|v(s, \cdot) - u(s, \cdot)\|_{L^1(\mathbb{R})} \, \mathrm{d}s, \qquad (3.5)$$

for some appropriate constants $\mathcal{K}_4$ and $\mathcal{K}_5$. Substituting the above estimates in (3.1), we get

$$\|u(T, \cdot) - v(T, \cdot)\|_{L^1(\mathbb{R})} \leq -\Lambda_{\varepsilon,\varepsilon_0}(v, u) + \|u_0 - v_0\|_{L^1(\mathbb{R})} + \mathcal{K}_6(\varepsilon + \varepsilon_0 + \gamma(v, \varepsilon_0))$$
$$+ \mathcal{K}_7 \int_0^T \|v(s, \cdot) - u(s, \cdot)\|_{L^1(\mathbb{R})} \, ds \, .$$

Now, the result follows by Gronwall's inequality. $\qquad\square$

The remaining of this section is dedicated to estimating the relative entropy functional $\Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u)$ for which we follow the following notations:
For $i \in \mathbb{Z}, n \in \mathbb{N}, k \in \mathbb{R}, (t, x) \in Q_T$, define

1. $\eta_i^n(k) := |u_i^n - k|$,
2. $C_i^n := C^n \times C_i$,
3. $p_i^n(k) := G(u_i^n, k) = \operatorname{sgn}(u_i^n - k)(f(u_i^n) - f(k))$,
4. $\mathcal{U}^\Delta(t, x) := \nu(\mu * \beta(u^\Delta(t)))(x)$.

Lemma 3.3 *The relative entropy functional $\Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u)$ satisfies:*

$$-\Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u) \leq C\left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\varepsilon_0}\right),$$

where C is a constant independent of $\Delta x, \Delta t$.

Proof Let $\sum_{i,n}$ denote the double summation $\sum_{i \in \mathbb{Z}} \sum_{n=0}^{N-1}$. For the piecewise constant function u^Δ (cf. (2.2)–(2.4)) the relative entropy can be written as

$$
\begin{aligned}
&- \Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u) \\
&= - \int_{Q_T} \sum_{i,n} \int_{C_i^n} \eta_i^n(u(s, y)) \Phi_t(s, y, t, x) \, dt \, dx \, ds \, dy \\
&\quad - \int_{Q_T} \sum_{i,n} \int_{C_i^n} p_i^n(u(s, y)) \mathcal{U}^\Delta(t, x) \Phi_x(s, y, t, x) \, dt \, dx \, ds \, dy \\
&\quad + \int_{Q_T} \sum_{i,n} \int_{C_i^n} \operatorname{sgn}(u_i^n - u(s, y)) f(u(s, y)) \partial_x \mathcal{U}^\Delta(t, x) \Phi(s, y, t, x) \, dt \, dx \, ds \, dy \\
&\quad - \int_{Q_T} \sum_i \int_{C_i} \eta_i^0(u(s, y)) \Phi(s, y, 0, x) \, dx \, ds \, dy \\
&\quad + \int_{Q_T} \sum_i \int_{C_i} \eta_i^N(u(s, y)) \Phi(s, y, T, x) \, dx \, ds \, dy \, .
\end{aligned}
$$

Applying the fundamental theorem of calculus, followed by summation by parts, we get

$$-\Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u)$$

$$= \int_{Q_T} \sum_{i,n} \left(\eta_i^{n+1}(u(s,y)) - \eta_i^n(u(s,y)) \right) \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \, ds \, dy$$

$$- \int_{Q_T} \sum_{i,n} \int_{C_i^n} p_i^n(u(s,y)) \mathcal{U}^\Delta(t,x) \Phi_x(s,y,t,x) \, dt \, dx \, ds \, dy$$

$$+ \int_{Q_T} \sum_{i,n} \int_{C_i^n} \operatorname{sgn}(u_i^n - u(s,y)) f(u(s,y)) \mathcal{U}_x^\Delta(t,x) \Phi(s,y,t,x) \, dt \, dx \, ds \, dy$$

$$:= \lambda_1 + \lambda_2 + \lambda_3. \tag{3.6}$$

We have

$$\lambda_2 = - \int_{Q_T} \sum_{i,n} \int_{C_i^n} p_i^n(u(s,y)) v(c_{i+1/2}^n) \Phi_x(s,y,t,x) \, dt \, dx \, ds \, dy$$

$$- \int_{Q_T} \sum_{i,n} \int_{C_i^n} p_i^n(u(s,y)) \Phi_x(s,y,t,x) \left(\mathcal{U}^\Delta(t,x) - v(c_{i+1/2}^n) \right) \, dt \, dx \, ds \, dy$$

$$:= \lambda_2' + \mathcal{E}_2. \tag{3.7}$$

For $(n,i) \in \mathbb{N} \times \mathbb{Z}$ and $(t,x) \in C_i^n$ observe that

$$\left| \mathcal{U}^\Delta(t,x) - v(c_{i+1/2}^n) \right|$$
$$\leq \left| \mathcal{U}^\Delta(t,x) - \mathcal{U}^\Delta(t^n,x) \right| + \left| \mathcal{U}^\Delta(t^n,x) - \mathcal{U}^\Delta(t^n, x_{i+1/2}) \right|$$
$$+ \left| \mathcal{U}^\Delta(t^n, x_{i+1/2}) - v(c_{i+1/2}^n) \right|.$$

Also for $t \in C^n$, using (2.9), we have

$$\left| \mathcal{U}^\Delta(t^n,x) - \mathcal{U}^\Delta(t,x) \right| \leq |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} \|\mu\|_{L^\infty(\mathbb{R})} \mathcal{L}_3 \Delta t. \tag{3.8}$$

Now using the Lipschitz continuity of $\mathcal{U}^\Delta$ in the space variable, we have

$$\left| \mathcal{U}^\Delta(t^n,x) - \mathcal{U}^\Delta(t^n, x_{i+1/2}) \right| \leq \left\| u_0^\Delta \right\|_{L^1(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \Delta x.$$

Furthermore,

$$\left| \mathcal{U}^\Delta(t^n, x_{i+1/2}) - v(c_{i+1/2}^n) \right| \tag{3.9}$$

$$\leq |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} \left| \sum_{p \in \mathbb{Z}} \int_{C_p} \mu(x_{i+1/2} - y) u_p^n \, dy - \Delta x \sum_{p \in \mathbb{Z}} \mu(x_{i+1/2} - x_p) u_p^n \right|$$

$$= |v|_{\text{Lip}(\mathbb{R})} \, |\beta|_{\text{Lip}(\mathbb{R})} \left| \sum_{p \in \mathbb{Z}} u_p^n \left(\int_{C_p} \mu(x_{i+1/2} - y) \, dy - \Delta x \mu(x_{i+1/2} - x_p) \right) \right|$$

$$= |v|_{\text{Lip}(\mathbb{R})} \, |\beta|_{\text{Lip}(\mathbb{R})} \, \left\| u_0^\Delta \right\|_{L^1(\mathbb{R})} \, |\mu|_{\text{Lip}(\mathbb{R})} \, \Delta x := C_0 \Delta x. \tag{3.10}$$

Finally, combining all the above estimates we get,

$$
\begin{aligned}
|\mathcal{E}_2| &\le \int_{Q_T} \sum_{i,n} C_0 \Delta x \, |f(u_i^n) - f(u(s,y))| \int_{C_i^n} |\Phi_x(s,y,t,x)| \, dt \, dx \, ds \, dy \\
&\le \int_{Q_T} \sum_{i,n} C_0 \Delta x (|f(u_i^n)| + |f(u(s,y))|) \int_{C_i^n} |\Phi_x(s,y,t,x)| \, dt \, dx \, ds \, dy \\
&= \sum_{i,n} \int_{C_i^n} C_0 \Delta x (|f(u_i^n)| \int_{Q_T} |\Phi_x(s,y,t,x)| \, dt \, dx \, ds \, dy \\
&\quad + \int_{Q_T^2} C_0 \Delta x \, |f(u(s,y))| \, |\Phi_x(s,y,t,x)| \, dt \, dx \, ds \, dy \\
&\le C_0 \, |f|_{\text{Lip}(\mathbb{R})} \, \left\| u^\Delta \right\|_{L^1(Q_T)} \frac{\Delta x}{\varepsilon} + C_0 \, |f|_{\text{Lip}(\mathbb{R})} \, \|u\|_{L^1(Q_T)} \frac{\Delta x}{\varepsilon}.
\end{aligned}
$$

Thus, we have

$$\lambda_2 = \lambda_2' + \mathcal{O}\left(\frac{\Delta x}{\varepsilon} \right). \tag{3.11}$$

Now, we consider

$$
\begin{aligned}
\lambda_3 &= \int_{Q_T} \sum_{i,n} \text{sgn}(u_i^n - u(s,y)) f(u(s,y)) (v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n)) \\
&\qquad \times \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, dt \, ds \, dy \\
&\quad + \int_{Q_T} \sum_{i,n} \text{sgn}(u_i^n - u(s,y)) f(u(s,y)) \\
&\qquad \times \int_{C_i^n} \mathcal{U}_x^\Delta(t,x) \left(\Phi(s,y,t,x) - \Phi(s,y,t,x_{i+1/2}) \right) dx \, dt \, dy \, ds \\
&\quad + \int_{Q_T} \sum_{i,n} \text{sgn}(u_i^n - u(s,y)) f(u(s,y)) \\
&\qquad \times \int_{C_i^n} \left(\mathcal{U}_x^\Delta(t,x) - \frac{1}{\Delta x} \left(v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n) \right) \right) \Phi(s,y,t,x_{i+1/2}) \, dx \, dt \, dy \, ds \\
&:= \lambda_3' + \mathcal{E}_{31} + \mathcal{E}_{32}.
\end{aligned} \tag{3.12}
$$

The error terms can be estimated as below,

$$
\begin{aligned}
|\mathcal{E}_{31}| &\le \int_{Q_T} \sum_{i,n} |f(u(s,y))| \int_{C_i^n} \left| \mathcal{U}_x^\Delta(t,x) \right| \left| \Phi(s,y,t,x) - \Phi(s,y,t,x_{i+1/2}) \right| dt \, dx \, ds \, dy \\
&\le |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|u_0\|_{L^1(\mathbb{R})} \\
&\quad \times \int_{Q_T} \sum_{i,n} |f(u(s,y))| \int_{C_i^n} \left| \Phi(s,y,t,x) - \Phi(s,y,t,x_{i+1/2}) \right| dx \, dt \, dy \, ds \\
&\le |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|u_0\|_{L^1(\mathbb{R})} \\
&\quad \times \int_{Q_T} |f(u(s,y))| \sum_i \int_{C_i} \left| \omega_\varepsilon(y-x) - \omega_\varepsilon(y - x_{i+1/2}) \right| dx \, ds \, dy \\
&\le |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|u_0\|_{L^1(\mathbb{R})} |w^\varepsilon|_{BV(\mathbb{R})} \Delta x \int_{Q_T} |f(u(s,y))| \, ds \, dy \\
&\le |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|u_0\|_{L^1(\mathbb{R})} \|f(u)\|_{L^1(Q_T)} \frac{\Delta x}{\varepsilon},
\end{aligned}
$$

using Theorem 2.2, equation (2.7), and

$$
\begin{aligned}
|\mathcal{U}_x^\Delta(t,x)| &= |v'((\beta(u^\Delta) * \mu)(t,x))(\mu' * \beta(u^\Delta))(t,x)| \\
&\le |v|_{\mathrm{Lip}(\mathbb{R})} |\beta|_{\mathrm{Lip}(\mathbb{R})} |\mu|_{\mathrm{Lip}(\mathbb{R})} \|u_0\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

Furthermore, we have, by applying fundamental theorem of calculus and rearrange the terms,

$$
\begin{aligned}
|\mathcal{E}_{32}| &= \left| \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u_i^n - u(s,y)) f(u(s,y)) \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \right. \\
&\quad \left. \times \int_{C_i} \left(\mathcal{U}_x^\Delta(t,x) - \frac{1}{\Delta x}(v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n)) \right) dx \, dt \, dy \, ds \right| \\
&= \left| \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u_i^n - u(s,y)) f(u(s,y)) \right. \\
&\quad \times \left(\mathcal{U}^\Delta(t^n, x_{i+1/2}) - \mathcal{U}^\Delta(t^n, x_{i-1/2}) - v(c_{i+\frac{1}{2}}^n) + v(c_{i-\frac{1}{2}}^n) \right) \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, ds \, dt \, dy \\
&\quad + \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u_i^n - u(s,y)) f(u(s,y)) \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \\
&\quad \left. \times \left(\mathcal{U}^\Delta(t, x_{i+1/2}) - \mathcal{U}^\Delta(t^n, x_{i+1/2}) - \mathcal{U}^\Delta(t, x_{i-1/2}) + \mathcal{U}^\Delta(t^n, x_{i-1/2}) \right) dt \, ds \, dy \right|.
\end{aligned}
$$

Apply summation by parts in i to get,

$$
\begin{aligned}
|\mathcal{E}_{32}| &= \left| \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u_i^n - u(s,y)) f(u(s,y)) \right. \\
&\quad \left. \times \left(\mathcal{U}^\Delta(t^n, x_{i+1/2}) - v(c_{i+\frac{1}{2}}^n) \right) \int_{C^n} (\Phi(s,y,t,x_{i+3/2}) - \Phi(s,y,t,x_{i+1/2})) \, dt \, ds \, dy \right|
\end{aligned}
$$

$$+ \left| \int_{Q_T} \sum_{i,n} \operatorname{sgn}(u_i^n - u(s,y)) f(u(s,y)) \right.$$

$$\left. \times \int_{C^n} (\mathcal{U}^\Delta(t, x_{i+1/2}) - \mathcal{U}^\Delta(t^n, x_{i+1/2}))(\Phi(s,y,t,x_{i+1/2}) - \Phi(s,y,t,x_{i-1/2})) \, dt \, ds \, dy \right|$$

$$:= \mathcal{E}_{321} + \mathcal{E}_{322}.$$

Now, using (3.10), we have

$$|\mathcal{E}_{321}| \leq \int_{Q_T} \left(\sum_n |f(u(s,y))| \, |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|u^\Delta\|_{L^1(Q_T)} \, |\mu|_{\operatorname{Lip}(\mathbb{R})} \, \Delta x \right.$$

$$\left. \times \int_{C^n} \sum_i \left| \Phi(s,y,t,x_{i+1/2}) - \Phi(s,y,t,x_{i-1/2}) \right| \right) dt \, ds \, dy$$

$$\leq \int_{Q_T} |f(u(s,y))| \, |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|u^\Delta\|_{L^1(Q_T)} \, |\mu|_{\operatorname{Lip}(\mathbb{R})} \, \Delta x |\omega_\varepsilon|_{BV(\mathbb{R})} \, ds \, dy$$

$$\leq \int_{Q_T} |f(u(s,y))| \, |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|u^\Delta\|_{L^1(Q_T)} \, |\mu|_{\operatorname{Lip}(\mathbb{R})} \, \Delta x \frac{1}{\varepsilon} \, ds \, dy$$

$$= |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|u^\Delta\|_{L^1(Q_T)} \, |\mu|_{\operatorname{Lip}(\mathbb{R})} \, |f|_{\operatorname{Lip}(\mathbb{R})} \, \|u\|_{L^1(Q_T)} \frac{\Delta x}{\varepsilon}.$$

Using (2.9), for $t \in C^n$, we have

$$\left| \mathcal{U}^\Delta(t, x_{i+1/2}) - \mathcal{U}^\Delta(t^n, x_{i+1/2}) \right|$$

$$\leq |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})}$$

$$\times \int_{\mathbb{R}} \left| u^\Delta(t, x_{i+1/2} - y)\mu(y) - u^\Delta(t^n, x_{i+1/2} - y)\mu(y) \right| dy$$

$$= |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \left\| u^\Delta(t, \cdot) - u^\Delta(t^n, \cdot) \right\|_{L^1(\mathbb{R})}$$

$$\leq |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \mathcal{L}_3 \Delta t.$$

Consequently,

$$|\mathcal{E}_{322}| \leq \int_{Q_T} \sum_n |f(u(s,y))| \, |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \mathcal{L}_3 \Delta t$$

$$\times \int_{C^n} \sum_i \left| \Phi(s,y,t,x_{i+1/2}) - \Phi(s,y,t,x_{i-1/2}) \right| dt \, ds \, dy$$

$$\leq \int_{Q_T} |f(u(s,y))| \, |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \mathcal{L}_3 \Delta t |w^\varepsilon|_{BV(\mathbb{R})}$$

$$\times \int_0^T |\omega_{\varepsilon_0}(s-t)| \, dt \, ds \, dy$$

$$\leq \int_{Q_T} |f(u(s,y))| \, |v|_{\operatorname{Lip}(\mathbb{R})} \, |\beta|_{\operatorname{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \mathcal{L}_3 \Delta t \frac{1}{\varepsilon} \, ds \, dy$$

$$= |v|_{\text{Lip}(\mathbb{R})} \, |\beta|_{\text{Lip}(\mathbb{R})} \, \|\mu\|_{L^\infty(\mathbb{R})} \mathcal{L}_3 \, |f|_{\text{Lip}(\mathbb{R})} \, \|u\|_{L^1(Q_T)} \frac{\Delta t}{\varepsilon}.$$

Finally, combining the above estimates, we get:

$$\lambda_3 = \lambda_3' + \mathcal{O}\left(\frac{\Delta x}{\varepsilon}\right). \tag{3.13}$$

Thus, so far we have proved

$$-\Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u) = \lambda_1 + \lambda_2' + \lambda_3' + \mathcal{O}\left(\frac{\Delta x}{\varepsilon}\right). \tag{3.14}$$

Recall λ_1, cf. (3.6),

$$\lambda_1 = \int_{Q_T} \sum_{i,n} \left(\eta_i^{n+1}(u(s,y)) - \eta_i^n(u(s,y))\right) \int_{C_i} \Phi(s,y,t^{n+1},x)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y$$

$$\leq -\lambda \int_{Q_T} \sum_{i,n} \left(\mathcal{G}_{i+1/2}^n(u_i^n, u_{i+1}^n, u(s,y)) - \mathcal{G}_{i-1/2}^n(u_{i-1}^n, u_i^n, u(s,y))\right)$$

$$\times \int_{C_i} \Phi(s,y,t^{n+1},x)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y$$

$$- \lambda \int_{Q_T} \sum_{i,n} \operatorname{sgn}(u_i^{n+1} - u(s,y)) f(u(s,y))(v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n))$$

$$\times \int_{C_i} \Phi(s,y,t^{n+1},x)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y$$

$$:= A_2 + A_3,$$

by applying the discrete entropy inequality (2.11).

Furthermore, we can rewrite λ_2', cf. (3.7), as follows

$$\lambda_2' = -\int_{Q_T} \sum_{i,n} \int_{C_i^n} p_i^n(u(s,y))v(c_{i+1/2}^n)\Phi_x(s,y,t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y$$

$$= \int_{Q_T} \sum_{i,n} \int_{C^n} \left(p_i^n(u(s,y))v(c_{i+1/2}^n) - p_{i-1}^n(u(s,y))v(c_{i-1/2}^n)\right)$$

$$\times \Phi(s,y,t,x_{i+1/2})\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}y$$

by using the fundamental theorem of calculus, followed by summation by parts.

Claim 1 $A_2 + \lambda_2' = \mathcal{O}\left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\varepsilon_0}\right).$

Adding and subtracting

$$\lambda \int_{Q_T} \sum_{i,n} \left(p_i^n(u(s,y))v(c_{i+1/2}^n) - p_{i-1}^n(u(s,y))v(c_{i-1/2}^n) \right) \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx\,dy\,ds$$

we have

$$
\begin{aligned}
A_2 &+ \lambda_2' \\
&= -\lambda \int_{Q_T} \left(\mathcal{G}_{i+1/2}^n(u_i^n, u_{i+1}^n, u(s,y)) - \mathcal{G}_{i-1/2}^n(u_{i-1}^n, u_i^n, u(s,y)) \right) \\
&\qquad \times \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx\,dy\,ds \\
&\quad + \int_{Q_T} \sum_{i,n} \left(p_i^n(u(s,y))v(c_{i+1/2}^n) - p_{i-1}^n(u(s,y))v(c_{i-1/2}^n) \right) \\
&\qquad \times \left(\int_{C^n} \Phi(s,y,t,x_{i+\frac{1}{2}})\,dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx \right) dy\,ds \\
&\quad + \lambda \int_{Q_T} \sum_{i,n} \left(p_i^n(u(s,y))v(c_{i+1/2}^n) - p_{i-1}^n(u(s,y))v(c_{i-1/2}^n) \right) \\
&\qquad \times \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx\,dy\,ds \\
&= \lambda \int_{Q_T} \sum_{i,n} \left(p_i^n(u(s,y))v(c_{i+1/2}^n) - \mathcal{G}_{i+1/2}^n(u_i^n, u_{i+1}^n, u(s,y)) \right) \\
&\qquad \times \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx\,dy\,ds \\
&\quad - \lambda \int_{Q_T} \sum_{i,n} \left(p_{i-1}^n(u(s,y))v(c_{i-1/2}^n) - \mathcal{G}_{i-1/2}^n(u_{i-1}^n, u_i^n, u(s,y)) \right) \\
&\qquad \times \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx\,dy\,ds \\
&\quad + \int_{Q_T} \sum_{i,n} \left(p_i^n(u(s,y))v(c_{i+1/2}^n) - p_{i-1}^n(u(s,y))v(c_{i-1/2}^n) \right) \\
&\qquad \times \left(\int_{C^n} \Phi(s,y,t,x_{i+\frac{1}{2}})\,dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x)\,dx \right) dy\,ds .
\end{aligned}
$$

$$(3.15)$$

Apply summation by parts to get

$$A_2 + \lambda_2' = \lambda \int_{Q_T} \sum_{i,n} \left(\mathcal{G}_{i+1/2}^n(u_i^n, u_{i+1}^n, u(s, y)) - p_i^n(u(s, y))v(c_{i+1/2}^n) \right)$$

$$\times \left(\int_{C_{i+1}} \Phi(s, y, t^{n+1}, x)\, dx - \int_{C_i} \Phi(s, y, t^{n+1}, x)\, dx \right) dy\, ds$$

$$+ \int_{Q_T} \sum_{i,n} \left(p_i^n(u(s, y))v(c_{i+1/2}^n) - p_{i-1}^n(u(s, y))v(c_{i-1/2}^n) \right)$$

$$\times \left(\int_{C^n} \Phi(s, y, t, x_{i+\frac{1}{2}})\, dt - \lambda \int_{C_i} \Phi(s, y, t^{n+1}, x)\, dx \right) dy\, ds .$$

Note that from the definition of the numerical entropy flux $G_{i+1/2}^n$ (see Thm. 2.2(5)), it follows that $v(c_{i+1/2}^n)p_i^n(k) = G_{i+1/2}^n(u_i^n, u_i^n, k)$. Now, invoking the Lipschitz continuity of $G_{i+1/2}^n$ in its second argument, we get

$$\left| \mathcal{G}_{i+1/2}^n(u_i^n, u_{i+1}^n, u(s, y)) - v(c_{i+1/2}^n)p_i^n(u(s, y)) \right| \le \mathcal{C}_1 \left| u_{i+1}^n - u_i^n \right|, \qquad (3.16)$$

for $\mathcal{C}_1 = \mathcal{C}_1(|f|_{\mathrm{Lip}(\mathbb{R})}, \|v\|_{L^\infty(\mathbb{R})})$. Furthermore,

$$\left| v(c_{i+1/2}^n)p_i^n(u(s, y)) - v(c_{i-1/2}^n)p_{i-1}^n(u(s, y)) \right|$$

$$\le \left| v(c_{i+1/2}^n) \right| \left| p_i^n(u(s, y)) - p_{i-1}^n(u(s, y)) \right| + \left| p_{i-1}^n(u(s, y)) \right| \left| v(c_{i+1/2}^n) - v(c_{i-1/2}^n) \right|$$

$$\le \mathcal{C}_2 \left(\left| v(c_{i+1/2}^n) - v(c_{i-1/2}^n) \right| + \left| u_{i+1}^n - u_i^n \right| \right), \qquad (3.17)$$

where the last inequality follows from the Lipschitz continuity of the function $u \mapsto \mathrm{sgn}(u - k)(f(u) - f(k))$ with $\mathcal{C}_2 = \mathcal{C}_2(|f|_{\mathrm{Lip}(\mathbb{R})}, \|v\|_{L^\infty(\mathbb{R})}, \|u^\Delta\|_{L^\infty(\overline{Q}_T)})$. Using (3.16)–(3.17), we have

$$A_2 + \lambda_2' \le \mathcal{C}_1 \lambda \int_{Q_T} \sum_{i,n} \left| u_{i+1}^n - u_i^n \right|$$

$$\times \left| \int_{C_{i+1}} \Phi(s, y, t^{n+1}, x)\, dx - \int_{C_i} \Phi(s, y, t^{n+1}, x)\, dx \right| dy\, ds$$

$$+ \mathcal{C}_2 \int_{Q_T} \sum_{i,n} \left[\left| v(c_{i+1/2}^n) - v(c_{i-1/2}^n) \right| + \left| u_{i+1}^n - u_i^n \right| \right]$$

$$\times \left| \int_{C^n} \Phi(s, y, t, x_{i+\frac{1}{2}})\, dt - \lambda \int_{C_i} \Phi(s, y, t^{n+1}, x)\, dx \right| dy\, ds .$$

Now, since μ has bounded variation, the claim follows because of the following estimates,

$$\sum_i \left| v(c^n_{i+1/2}) - v(c^n_{i-1/2}) \right| \leq |v|_{\mathrm{Lip}(\mathbb{R})} \sum_{i,p} \Delta x \left| \beta(u^n_{p+1/2})(\mu_{i+1/2-p} - \mu_{i-1/2-p}) \right|$$

$$\leq |v|_{\mathrm{Lip}(\mathbb{R})} \sum_{i,p} \Delta x \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \left| u^n_{p+1/2} \right| \int_{C_{i-p}} |\mu'(x)| \, \mathrm{d}x$$

$$= |v|_{\mathrm{Lip}(\mathbb{R})} \sum_{p} \Delta x \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \left| u^n_{p+1/2} \right| \int_{\mathbb{R}} |\mu'(x)| \, \mathrm{d}x$$

$$\leq |v|_{\mathrm{Lip}(\mathbb{R})} \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, \|u_0\|_{L^1(\mathbb{R})} \, |\mu|_{BV(\mathbb{R})} := C_3, \quad (3.18)$$

which is true because of (2.7) and the estimates

$$\int_{Q_T} \left(\int_{C^n} \Phi(s, y, t, x_{i+\frac{1}{2}}) \, \mathrm{d}t - \lambda \int_{C_i} \Phi(s, y, t^{n+1}, x) \, \mathrm{d}x \right) \mathrm{d}y \, \mathrm{d}s = \mathcal{O}\left(\frac{\Delta x^2}{\varepsilon} + \frac{\Delta t^2}{\varepsilon_0} \right),$$

(3.19)

$$\int_{Q_T} \left(\int_{C_{i+1}} \Phi(s, y, t^{n+1}, x) \, \mathrm{d}x - \int_{C_i} \Phi(s, y, t^{n+1}, x) \, \mathrm{d}x \right) \mathrm{d}y \, \mathrm{d}s = \mathcal{O}\left(\frac{\Delta x^2}{\varepsilon} \right),$$

(3.20)

the proofs of which can be found in [23, Ex. 3.17].

Claim 2 $A_3 + \lambda'_3 = \mathcal{O}\left(\frac{\Delta x}{\varepsilon} \right).$

We find

$$A_3 + \lambda'_3$$

$$= -\lambda \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u^{n+1}_i - u(s, y)) f(u(s, y)) (v(c^n_{i+\frac{1}{2}}) - v(c^n_{i-\frac{1}{2}}))$$

$$\times \int_{C_i} \Phi(s, y, t^{n+1}, x) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}y$$

$$+ \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u^n_i - u(s, y)) f(u(s, y)) (v(c^n_{i+\frac{1}{2}}) - v(c^n_{i-\frac{1}{2}}))$$

$$\times \int_{C^n} \Phi(s, y, t, x_{i+1/2}) \, \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}y$$

$$= -\lambda \int_{Q_T} \sum_{i,n} \mathrm{sgn}(u^{n+1}_i - u(s, y)) f(u(s, y)) (v(c^n_{i+\frac{1}{2}}) - v(c^n_{i-\frac{1}{2}}))$$

$$\times \int_{C_i} \Phi(s, y, t^{n+1}, x) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}y$$

$$+\lambda \int_{Q_T} \sum_{i,n} \operatorname{sgn}(u_i^n - u(s,y)) f(u(s,y)) (v(c_{i+\frac{1}{2}}^n)$$

$$-v(c_{i-\frac{1}{2}}^n)) \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \, ds \, dy$$

$$-\lambda \int_{Q_T} \sum_{i,n} \operatorname{sgn}(u_i^n - u(s,y)) f(u(s,y)) (v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n))$$

$$\times \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \, ds \, dy$$

$$+\int_{Q_T} \sum_{i,n} \operatorname{sgn}(u_i^n - u(s,y)) f(u(s,y)) (v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n))$$

$$\times \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, dt \, ds \, dy$$

$$= \int_{Q_T} \sum_{i,n} \operatorname{sgn}(u_i^n - u(s,y)) f(u(s,y)) (v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n))$$

$$\times \left(\int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, dt) - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right) ds \, dy$$

$$-\lambda \int_{Q_T} \sum_{i,n} \left(\operatorname{sgn}(u_i^{n+1} - u(s,y)) - \operatorname{sgn}(u_i^n - u(s,y)) \right)$$

$$\times f(u(s,y)) (v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n)) \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \, ds \, dy$$

$$:= \tilde{\mathcal{E}}_1 + \tilde{\mathcal{E}}_2.$$

The terms $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ can be estimated as follows:

$$\left| \tilde{\mathcal{E}}_1 \right| \leq \int_{Q_T} \sum_{i,n} |f(u(s,y))| |v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n)|$$

$$\times \left| \int_{C^n} \Phi(s,y,t,x_{i+1/2}) \, dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right| ds \, dy$$

$$\leq \| f(u) \|_{L^\infty(Q_T)} \sum_{i,n} \left| v(c_{i+1/2}^n) - v(c_{i-1/2}^n) \right|$$

$$\times \int_{Q_T} \left| \int_{C^n} \Phi(s,y,t,x_{i+\frac{1}{2}}) \, dt - \lambda \int_{C_i} \Phi(s,y,t^{n+1},x) \, dx \right| dy \, ds$$

$$\leq T \| f(u) \|_{L^\infty(Q_T)} \mathcal{C}_3 \left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\varepsilon_0} \right),$$

using (3.18)–(3.19). Furthermore,

$$
\tilde{\mathcal{E}}_2 = -\lambda \int_{Q_T} \sum_i \sum_{n=1}^{N} \mathrm{sgn}(u_i^n - u(s, y)) f(u(s, y)) \left(v \left(c_{i+\frac{1}{2}}^{n-1} \right) - v \left(c_{i-\frac{1}{2}}^{n-1} \right) \right)
$$
$$
\times \int_{C_i} \Phi(s, y, t^n, x) \, dx \, ds \, dy
$$
$$
+ \lambda \int_{Q_T} \sum_i \sum_{n=1}^{N} \mathrm{sgn}(u_i^n - u(s, y)) f(u(s, y)) \left(v \left(c_{i+\frac{1}{2}}^{n} \right) - v \left(c_{i-\frac{1}{2}}^{n} \right) \right)
$$
$$
\times \int_{C_i} \Phi \left(s, y, t^{n+1}, x \right) dx \, ds \, dy
$$
$$
+ \lambda \int_{Q_T} \sum_i \mathrm{sgn}(u_i^0 - u(s, y)) f(u(s, y)) \left(v \left(c_{i+\frac{1}{2}}^{0} \right) - v \left(c_{i-\frac{1}{2}}^{0} \right) \right)
$$
$$
\times \int_{C_i} \Phi \left(s, y, t^1, x \right) dx \, ds \, dy
$$
$$
- \lambda \int_{Q_T} \sum_i \mathrm{sgn} \left(u_i^N - u(s, y) \right) f(u(s, y))(v(c_{i+\frac{1}{2}}^{N}) - v \left(c_{i-\frac{1}{2}}^{N} \right))
$$
$$
\times \int_{C_i} \Phi(s, y, t^{N+1}, x) \, dx \, ds \, dy \, .
$$

Adding and subtracting the term

$$
-\lambda \int_{Q_T} \sum_i \sum_{n=1}^{N} \mathrm{sgn} \left(u_i^n - u(s, y) \right) f(u(s, y)) \left(v \left(c_{i+\frac{1}{2}}^{n-1} \right) - v \left(c_{i-\frac{1}{2}}^{n-1} \right) \right)
$$
$$
\times \int_{C_i} \Phi(s, y, t^{n+1}, x) \, dx \, ds \, dy \, ,
$$

we have

$$
\tilde{\mathcal{E}}_2 = -\lambda \int_{Q_T} \sum_i \sum_{n=1}^{N} \mathrm{sgn}(u_i^n - u(s, y)) f(u(s, y)) \left(v \left(c_{i+\frac{1}{2}}^{n-1} \right) - v \left(c_{i-\frac{1}{2}}^{n-1} \right) \right)
$$
$$
\times \int_{C_i} (\Phi(s, y, t^n, x) - \Phi(s, y, t^{n+1}, x)) \, dx \, ds \, dy
$$
$$
+ \lambda \int_{Q_T} \sum_i \sum_{n=1}^{N} \mathrm{sgn}(u_i^n - u(s, y)) f(u(s, y))
$$

$$\times \left[v\left(c_{i+\frac{1}{2}}^n\right) - v\left(c_{i-\frac{1}{2}}^n\right) - v\left(c_{i+\frac{1}{2}}^{n-1}\right) + v\left(c_{i-\frac{1}{2}}^{n-1}\right) \right]$$

$$\times \int_{C_i} \Phi(s, y, t^{n+1}, x)\, dx\, ds\, dy$$

$$+ \lambda \int_{Q_T} \sum_i \operatorname{sgn}(u_i^0 - u(s, y)) f(u(s, y)) \left(v\left(c_{i+\frac{1}{2}}^0\right) - v\left(c_{i-\frac{1}{2}}^0\right) \right)$$

$$\times \int_{C_i} \Phi(s, y, t^1, x)\, dx\, ds\, dy$$

$$- \lambda \int_{Q_T} \sum_i \operatorname{sgn}\left(u_i^N - u(s, y)\right) f(u(s, y)) \left(v\left(c_{i+\frac{1}{2}}^N\right) - v\left(c_{i-\frac{1}{2}}^N\right) \right)$$

$$\times \int_{C_i} \Phi(s, y, t^{n+1}, x)\, dx\, ds\, dy$$

$$:= \tilde{\mathcal{E}}_{21} + \tilde{\mathcal{E}}_{22} + \tilde{\mathcal{E}}_{23} + \tilde{\mathcal{E}}_{24}.$$

Now, let us estimate $\tilde{\mathcal{E}}_{21}$.

$$\left|\tilde{\mathcal{E}}_{21}\right| = \left| -\lambda \int_{Q_T} \sum_i \sum_{n=1}^N \operatorname{sgn}(u_i^n - u(s, y)) f(u(s, y)) \left(v\left(c_{i+\frac{1}{2}}^{n-1}\right) - v\left(c_{i-\frac{1}{2}}^{n-1}\right) \right) \right.$$

$$\times \int_{C_i} \left. \left(\Phi(s, y, t^n, x) - \Phi\left(s, y, t^{n+1}, x\right)\right) dx\, ds\, dy \right|$$

$$\leq \lambda \|f(u)\|_{L^\infty(Q_T)}$$

$$\times \int_{Q_T} \left(\sum_i \sum_{n=1}^N \left| v(c_{i+\frac{1}{2}}^{n-1}) - v\left(c_{i-\frac{1}{2}}^{n-1}\right) \right| \right.$$

$$\times \int_{C_i} \left. \left|\Phi(s, y, t^n, x) - \Phi(s, y, t^n + \Delta t, x)\right| dx \right) ds\, dy$$

$$\leq \Delta x\, \lambda \|f(u)\|_{L^\infty(Q_T)}$$

$$\times \int_0^T \left(\sum_i \sum_{n=1}^N \left| v\left(c_{i+\frac{1}{2}}^{n-1}\right) - v\left(c_{i-\frac{1}{2}}^{n-1}\right) \right| \left|\omega_{\varepsilon_0}(s - t^n) - \omega_{\varepsilon_0}(s - t^n - \Delta t)\right| \right) ds$$

$$\leq \Delta t\, \mathcal{C}_3 \|f(u)\|_{L^\infty(Q_T)} \sum_{n=1}^N \int_0^T \left(\left|\omega_{\varepsilon_0}(s - t^n) - \omega_{\varepsilon_0}(s - t^n - \Delta t)\right| \right) ds$$

$$\leq \Delta t\, \mathcal{C}_3 \|f(u)\|_{L^\infty(Q_T)} \sum_{n=1}^N \left|\omega_{\varepsilon_0}\right|_{BV(\mathbb{R})} \Delta t$$

$$= \mathcal{C}_4 \frac{\Delta t}{\varepsilon_0}.$$

Now, let us estimate $\tilde{\mathcal{E}}_{22}$.

$$\left|\tilde{\mathcal{E}}_{22}\right| = \left| \lambda \int_{Q_T} \sum_i \sum_{n=1}^N \mathrm{sgn}(u_i^n - u(s, y)) f(u(s, y)) \right.$$
$$\times \left[v(c_{i+\frac{1}{2}}^n) - v(c_{i-\frac{1}{2}}^n) - v(c_{i+\frac{1}{2}}^{n-1}) + v(c_{i-\frac{1}{2}}^{n-1}) \right]$$
$$\left. \times \int_{C_i} \Phi(s, y, t^{n+1}, x) \, dx \, ds \, dy \right|$$

$$\leq \lambda \, |v|_{\mathrm{Lip}(\mathbb{R})} \, \|f(u)\|_{L^\infty(Q_T)} \int_{Q_T} \sum_i \sum_{n=1}^N \left| c_{i+\frac{1}{2}}^n - c_{i-\frac{1}{2}}^n - c_{i+\frac{1}{2}}^{n-1} + c_{i-\frac{1}{2}}^{n-1} \right|$$
$$\times \int_{C_i} |\Phi(s, y, t^{n+1}, x)| \, dx \, ds \, dy$$

$$= \Delta t \, |v|_{\mathrm{Lip}(\mathbb{R})} \, \|f(u)\|_{L^\infty(Q_T)} \sum_i \sum_{n=1}^N \left| c_{i+\frac{1}{2}}^n - c_{i-\frac{1}{2}}^n - c_{i+\frac{1}{2}}^{n-1} + c_{i-\frac{1}{2}}^{n-1} \right|.$$

Recall, cf. (2.3), that

$$c_{i+1/2}^n = \Delta x \sum_p \beta(u_{p+1/2}^n)\mu_{i+1/2-p},$$

which implies

$$\sum_i \left| (c_{i+1/2}^n - c_{i-1/2}^n) - (c_{i+1/2}^{n-1} - c_{i-1/2}^{n-1}) \right|$$

$$= \Delta x \sum_i \left| \left(\sum_p \beta(u_{p+1/2}^n)\mu_{i+1/2-p} - \sum_p \beta(u_{p+1/2}^n)\mu_{i-1/2-p} \right) \right.$$
$$\left. - \left(\sum_p \beta(u_{p+1/2}^{n-1})\mu_{i+1/2-p} - \sum_p \beta(u_{p+1/2}^{n-1})\mu_{i-1/2-p} \right) \right|$$

$$= \Delta x \sum_i \left| \sum_p \beta(u_{p+1/2}^n)(\mu_{i+1/2-p} - \mu_{i-1/2-p}) \right.$$
$$\left. - \sum_p \beta(u_{p+1/2}^{n-1})(\mu_{i+1/2-p} - \mu_{i-1/2-p}) \right|$$

$$\leq \Delta x \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \sum_{i,p} \left| u_{p+1/2}^n - u_{p+1/2}^{n-1} \right| \left| \mu_{i+1/2-p} - \mu_{i-1/2-p} \right|$$

$$\leq \Delta x \, |\beta|_{\mathrm{Lip}(\mathbb{R})} \, |\mu|_{BV(\mathbb{R})} \sum_p \left| u_{p+1/2}^n - u_{p+1/2}^{n-1} \right|$$

$$\leq C_5 \Delta t,$$

where $\mathcal{C}_5 = |\beta|_{\text{Lip}(\mathbb{R})}\, |\mu|_{BV(\mathbb{R})}\mathcal{L}_3$, by applying (2.9). Therefore,

$$\tilde{\mathcal{E}}_{22} \leq \mathcal{C}_6 \Delta t,$$

where $\mathcal{C}_6 = |\nu|_{\text{Lip}(\mathbb{R})}\, \|f(u)\|_{L^\infty(Q_T)}\mathcal{C}_5$. Finally, estimates on the remaining boundary terms $\tilde{\mathcal{E}}_{23}$ and $\tilde{\mathcal{E}}_{24}$, easily follow from (3.18). Specifically,

$$\tilde{\mathcal{E}}_{23} = \lambda \int_{Q_T} \sum_i \text{sgn}(u_i^0 - u(s, y)) f(u(s, y)) \left(\nu\left(c_{i+\frac{1}{2}}^0\right) - \nu\left(c_{i-\frac{1}{2}}^0\right) \right)$$
$$\times \int_{C_i} \Phi(s, y, t^1, x)\, dx\, ds\, dy$$
$$\leq \mathcal{C}_7 \Delta t,$$

where $\mathcal{C}_7 = \mathcal{C}_3 \|f(u)\|_{L^\infty(Q_T)}$. Similarly, $\tilde{\mathcal{E}}_{24} \leq \mathcal{C}_7 \Delta t\, \|f(u)\|_{L^\infty(Q_T)}$. Substituting the assertions of Claim 1 and Claim 2 in (3.14) we get

$$-\Lambda_{\varepsilon,\varepsilon_0}(u^\Delta, u) \leq \mathcal{C}\left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\varepsilon_0}\right),$$

where $\mathcal{C} = \mathcal{C}(T, \|\nu\|_{W^{2,\infty}(\mathbb{R})}, |\beta|_{\text{Lip}(\mathbb{R})}, |f|_{\text{Lip}(\mathbb{R})}, |\mu|_{BV(\mathbb{R})}, \|u_0\|_{L^1(\mathbb{R})}, |u_0|_{BV(\mathbb{R})}, \|\mu\|_{W^{2,\infty}(\mathbb{R})})$. $\qquad\square$

Now, we state and prove the main result of this paper.

Theorem 3.4 (Rate of Convergence) *Let u be the entropy solution of (1.4)–(1.5) and u^Δ be the numerical solution given by (2.2). Then we have the following convergence rate:*

$$\left\|u^\Delta(T, \cdot) - u(T, \cdot)\right\|_{L^1(\mathbb{R})} = \mathcal{O}(\sqrt{\Delta t}).$$

Proof The CFL condition implies that $\Delta x = \mathcal{O}(\Delta t)$. Furthermore, the initial approximation (2.1) implies $\left\|u_0^\Delta - u_0\right\|_{L^1(\mathbb{R})} = \mathcal{O}(\Delta t)$. Now, the desired error estimate follows from Lemma 3.3 and Lemma 3.2 setting $\varepsilon = \varepsilon_0 = \sqrt{\Delta t}$, as $\gamma(u^\Delta, \sqrt{\Delta t}) = \mathcal{O}(\sqrt{\Delta t})$. $\qquad\square$

4 Extension to multi dimensions

We consider the case of two space dimensions and denote the space variables by $(x, y) \in \mathbb{R}^2$, and consider the following PDE:

$$\partial_t u + \partial_x(f^1(u)\nu^1(\beta^1(u) * \mu^1)) + \partial_y(f^2(u)\nu^2(\beta^2(u) * \mu^2)) = 0. \qquad (4.1)$$

Further, for numerical scheme, fix a rectangular grid with sizes Δx and Δy in $\mathbb{R}^2$ and choose a time step Δt. For later use, we also introduce the usual notation

$$(t^n, x_i, y_j) = (n\Delta t, i\Delta x, j\Delta y), \quad n \in \mathbb{N}, \ i, j \in \mathbb{Z}, \quad \lambda_x = \frac{\Delta t}{\Delta x}, \ \lambda_y = \frac{\Delta t}{\Delta y}.$$

Throughout, we fix initial data $u_0 \in (L^\infty \cap \mathrm{BV})(\mathbb{R}^2; \mathbb{R})$ and introduce

$$u_{ij}^0 = \frac{1}{\Delta x \, \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} u_0(x, y) \, \mathrm{d}y \, \mathrm{d}x \quad \text{for } i, j \in \mathbb{Z}.$$

We define a piecewise constant approximate solution u^Δ by

$$u^\Delta(t, x, y) = u_{ij}^n \chi_{[t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2}) \times [y_{j-1/2}, y_{j+1/2})}(t, x, y), \quad n \in \mathbb{N}, \ i, j \in \mathbb{Z},$$

where χ_A denotes the indicator function of a set A, through the following marching formula based on dimensional splitting, (see [16, Sec. 3] and [22, Sec. 5] for details):

$$\begin{aligned}
u_{ij}^{n+1/2} &= u_{ij}^n - \lambda_x \big[\mathcal{F}_{i+1/2,j}^{x,n}(u_{ij}^n, u_{i+1,j}^n) - \mathcal{F}_{i-1/2,j}^{x,n}(u_{i-1,j}^n, u_{ij}^n)\big], \\
u_{ij}^{n+1} &= u_{ij}^{n+1/2} - \lambda_y \big[\mathcal{F}_{i,j+1/2}^{y,n}(u_{ij}^{n+1/2}, u_{i,j+1}^{n+1/2}) - \mathcal{F}_{i,j-1/2}^{y,n}(u_{i,j-1}^{n+1/2}, u_{ij}^{n+1/2})\big],
\end{aligned} \tag{4.2}$$

where $\mathcal{F}_{i+1/2,j}^{x,n}$ and $\mathcal{F}_{i,j+1/2}^{y,n}$ denote the numerical approximations of the fluxes $f^1(u)v^1(\mu^1 * \beta^1(u))$ and $f^2(u)v^2(\mu^2 * \beta^2(u))$ at the interfaces $(x_{i+1/2}, y_j)$ and $(x_i, y_{j+1/2})$, respectively, for $i, j \in \mathbb{Z}$. The convolution terms are computed through quadrature formula, i.e.,

$$\begin{aligned}
c_{i+1/2,j}^{x,n} &= \Delta x \, \Delta y \sum_{l,p \in \mathbb{Z}} \mu_{i+1/2-l,j-p}^1 \beta^1(u_{l+1/2,p}^n), \\
c_{i,j+1/2}^{y,n} &= \Delta x \, \Delta y \sum_{l,p \in \mathbb{Z}} \mu_{i+1/2-l,j-p}^2 \beta^2(u_{l+1/2,p}^n),
\end{aligned} \tag{4.3}$$

where, $u_{l+1/2,p}^n$ is any convex combination of $u_{l,p}^n$ and $u_{l+1,p}^n$, with $\mu_{i+1/2,j}^1 = \mu^1(x_{i+1/2}, y_j)$ and $\mu_{i+1/2,j}^2 = \mu^2(x_{i+1/2}, y_j)$. Throughout, we require that Δt is chosen in order to satisfy the CFL conditions

$$\lambda_x \leq \frac{\min(1, 4 - 6\theta_x, 6\theta_x)}{1 + 6\left|f^1\right|_{\text{Lip}(\mathbb{R})} \left\|\nu^1\right\|_{L^\infty(\mathbb{R})}},$$

$$\lambda_y \leq \frac{\min(1, 4 - 6\theta_y, 6\theta_y)}{1 + 6\left|f^2\right|_{\text{Lip}(\mathbb{R})} \left\|\nu^2\right\|_{L^\infty(\mathbb{R})}}, \qquad \theta_x, \theta_y \in \left(0, \frac{2}{3}\right), \qquad (4.4)$$

and

$$\lambda_x \left|f^1\right|_{\text{Lip}(\mathbb{R})} \left\|\nu^1\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}, \quad \lambda_y \left|f^2\right|_{\text{Lip}(\mathbb{R})} \left\|\nu^2\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2},$$

with numerical fluxes $\mathcal{F}^x$ or $\mathcal{F}^y$ chosen as Lax–Friedrichs flux and Godunov flux, respectively. Extension to other monotone fluxes and for higher dimensions is similar. The numerical scheme can now be shown to converge to entropy solution, see, for example, [1]. The Kuznetsov Lemma and the theorem on error estimate presented in Sect. 3 can now be extended to several space dimensions using dimension splitting arguments (see [23, Sec. 4.3]) with appropriate modifications throughout the proof.

5 Numerical experiments

We now present some numerical experiments to illustrate the theory presented in the previous section. We show the results for the Lax–Friedrichs scheme. The results obtained by Godunov scheme are similar, and are not shown here. Throughout the section, $\theta = \theta_x = \theta_y$ is chosen to be 0.3333, and λ and $\lambda_x = \lambda_y$ are chosen to be 0.1286 and 0.2857, respectively, so as to satisfy the CFL condition (2.5) and (4.4), respectively, in one and two dimensions, for any grid size Δx or Δy used in this section.

5.1 One dimension

We employ the nonlocal version of the standard LWR model (1.1), i.e., IVP (1.4)–(1.5), with

$$\mu(x) = L(-x(\eta + x))^3 \mathbb{1}_{(-\eta, 0)}(x),$$

where L is such that $\int_{\mathbb{R}} \mu(x)\, \mathrm{d}x = 1$.

Further, $\beta(r) = r$ and $\nu(r) = 1 - r$, $f(u) = u$. This PDE fits the hypothesis of the article. Further, the domain of integration is chosen to be the interval $[-1.5, 1.5]$ with $t \in [0, 0.5]$, and

$$u_0(x) = 0.25\mathbb{1}_{(-0.9, 0.1)}(x) + 0.5\mathbb{1}_{(0.1, 0.3)}(x). \qquad (5.1)$$

Figure 1 displays the numerical approximations of (1.4), (5.1) generated by the numerical scheme (2.2), with decreasing grid size Δx, starting with $\Delta x = 0.00625$, and $\eta = 0.0625$. It can be seen that the numerical scheme is able to capture both shocks and
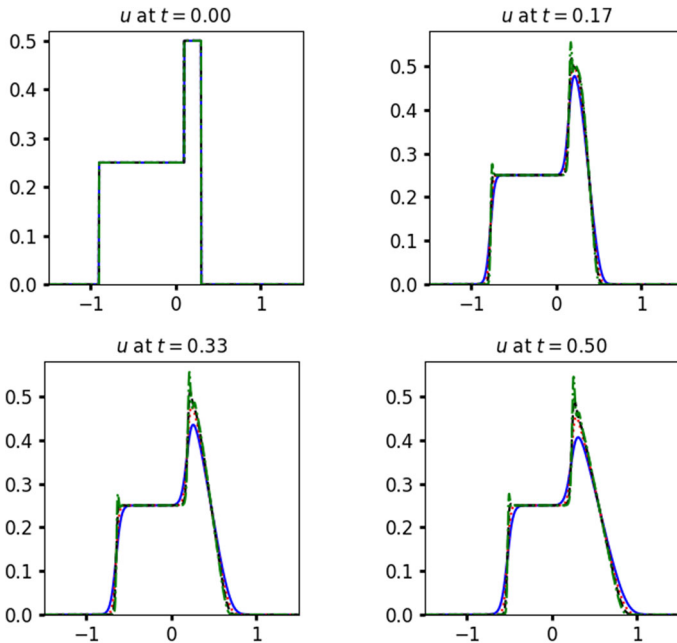
Fig. 1 Solution to the nonlocal conservation law (1.4), (5.1) on the domain $[-1.5, 1.5]$ at times $t = 0.00,\ 0.017,\ 0.33,\ 0.5$, with decreasing mesh size $\Delta x = 0.00625$(——), $\Delta x = 0.003125$(⋯⋯), $\Delta x = 0.0015625$(- - -) and $\Delta x = 0.00078125$(-·-·)

rarefactions well. To compute the observed convergence rate of the scheme (2.2) we obtain numerical approximations to (1.4), (5.1) with decreasing grid sizes Δx, starting with $\Delta x = 0.00625$. The observed convergence rate α is then calculated at time $T = 0.5$ by computing the L^1 distance between the numerical solutions $u_{\Delta x}(T, \cdot)$ and $u_{\Delta x/2}(T, \cdot)$ obtained for the grid size Δx and $1/2\Delta x$, for each grid size Δx. Let $e_{\Delta x}(T) = \left\| u_{\Delta x}(T, \cdot) - u_{\Delta x/2}(T, \cdot) \right\|_{L^1(\mathbb{R})}$. The observed convergence rate α is given by $\log_2(e_{\Delta x}(T)/e_{\Delta x/2}(T))$. The results recorded in Fig. 2 show that $\alpha > 0.5$. The present numerical integration resonates well with the theoretical convergence rate obtained in Theorem 3.4 in this article. It can be seen that the density u goes beyond the maximal initial density 0.5, violating the maximum principle, a phenomenon observed in non-local conservation laws with linear local part, see [6, Ex. 1]. It is to be noted that, in general, for nonlocal conservation laws with linear local part, entropy solutions do not satisfy the invariant region principle, i.e., the density may cross the value 1 even for $0 \leq u_0 \leq 1$ (see [6, Ex. 1]). On the other hand, for fluxes such as $f(u) = u(1-u)$, invariant region principle holds but not necessarily the maximum principle (see [6, Ex. 2]). The numerical results for such fluxes can be computed analogously and we do not present them here.

Figure 3 illustrates the nonlocal to local limit, see [10, 12, 26] and references therein, namely that the entropy solutions of the nonlocal conservation laws converge to the entropy solution of the corresponding local conservation law as the radius of the kernel goes to zero.

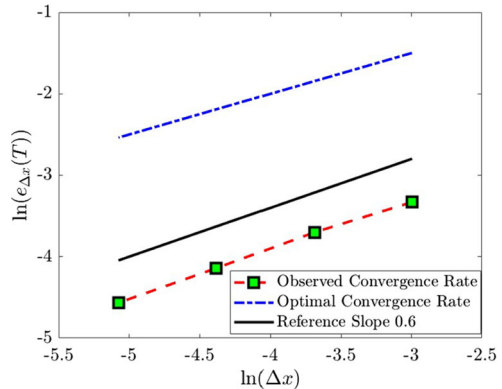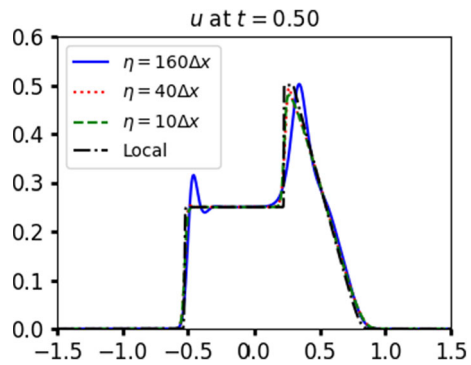| $\Delta x/0.00625$	$e_{\Delta x}(T)$	α
1	0.0358	0.5354
1/2	0.0247	0.6355
1/4	0.0159	0.6124
1/8	0.0104	

Fig. 2 Convergence rate α for the numerical scheme (2.2) for the approximate solutions to the problem (1.4), (5.1) on the domain $[-1.5, 1.5]$ at time $T = 0.5$

Fig. 3 Domain $[-1.5, 1.5]$, $T = 0.5$: Solution to the local conservation law (1.1), (5.1) ($-\cdot-\cdot$); Solution to the nonlocal conservation law (1.4), (5.1) with decreasing convolution radii

5.2 Two dimensions

To illustrate our results in two dimensions, we employ the model introduced in [1], modeling crowd dynamics in two dimensions, which fits in the framework of the article. Assume that a group of pedestrians in a square room $[-4, 4]^2$, can be described through the density $u = u(t, x, y)$ that satisfies the nonlocal conservation law

$$\partial_t u + \nabla \cdot (u\,(1-u)\,(1 - u * \mu)) = 0, \qquad (5.2)$$

where the smooth, non-negative and compactly supported function μ models the way in which each individual averages the density around her/his position to adjust her/his speed.

We choose:

$$\tilde{\mu}(x, y) = (0.16 - x^2 - y^2)^3 \, \chi_{\{(x,y):\, x^2+y^2 \leq 0.16\}}(x, y),$$

$$\mu(x, y) = \left(\iint_{\mathbb{R}^2} \tilde{\mu} \, dx \, dy \right)^{-1} \tilde{\mu}(x, y) \, dx \, dy, \qquad (5.3)$$
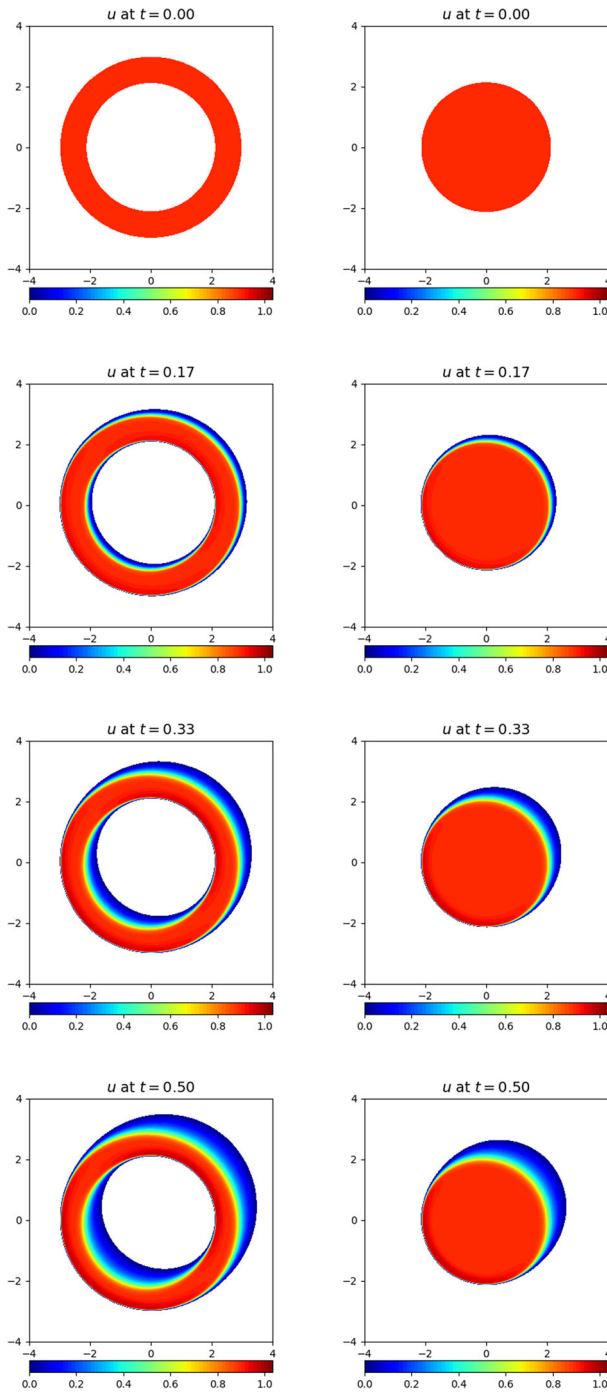
Fig. 4 Solution to (5.2), (5.4) (Left) and (5.2), (5.5) (Right) at times $t = 0.00,\ 0.017,\ 0.33,\ 0.5$ with space mesh $\Delta x = \Delta y = 0.00625$
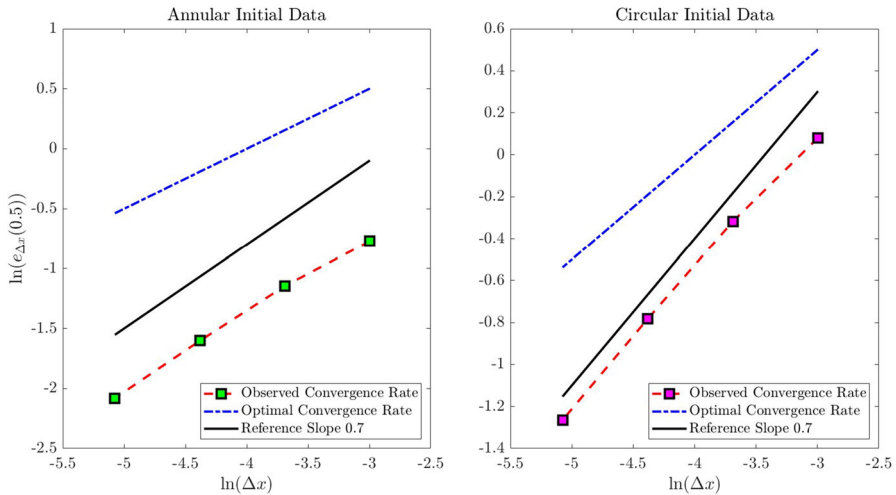
Fig. 5 Convergence rate α for the numerical scheme (2.2) on the domain $[-4, \ 4]^2$ at time $T = 0.5$ for the approximate solutions to the problem (5.2), (5.4) and (5.2), (5.5)

so that $\iint_{\mathbb{R}^2} \mu(x, y) \, dx \, dy = 1$.

As initial data, we consider two cases:

1. Annular initial data, where the crowd is concentrated in an annulus:

$$u_0(x, y) = \chi_{\{(x,y): \ 4 \leq x^2 + y^2 \leq 9\}}(x, y). \tag{5.4}$$

2. Circular initial data, where the crowd is concentrated in a circle:

$$u_0(x, y) = \chi_{\{(x,y): \ x^2 + y^2 \leq 4\}}(x, y). \tag{5.5}$$

The system (5.2) fits into the setting of (4.2) with

$$f^k(u) = u(1 - u), \ \beta^k(u) = u, \ v^k(u) = 1 - u, \ \mu^k = \mu, \quad k = 1, 2.$$

The numerical integrations of (5.2), (5.4) and (5.2), (5.5) are obtained by the the algorithm described in Sect. 4 and are shown in Fig. 4. The figures depict that the density does not cross the maximal density 1 and the numerical simulations are able to capture the physical properties well. To compute the convergence rate of the scheme (4.2) with Lax–Friedrichs flux, we apply the algorithm to problem (5.2), (5.4) and (5.2), (5.5) on the domain $[-4, \ 4]^2$ on the time interval $[0, 0.5]$ with different grid sizes with $\lambda_x = \lambda_y = 0.2857$. The convergence rate α is then calculated at time $T = 0.5$ by computing the L^1 distance between the numerical solutions $u_{\Delta x}(T, \ \cdot)$ and $u_{\Delta x/2}(T, \ \cdot)$ obtained for the grid size Δx and $\Delta x/2$, for each grid size Δx. The results recorded in Table 1 and Fig. 5 show that the observed convergence rates lie strictly between 0.5 and 1. The present numerical integration resonate well with theoretical convergence rate obtained in Theorem 3.4 in this article.

Table 1 Convergence rate α for the numerical scheme (2.2) on the domain $[-4, \, 4]^2$ at time $T = 0.5$ for the approximate solutions to the problem (5.2), (5.4) and (5.2), (5.5)

Initial Data $\Delta x/0.05$	$e_{\Delta x}(T)$		α	
	Annular	Circular	Annular	Circular
1	0.9314	0.3989	0.5406	0.5425
1/2	0.6403	0.2677	0.6580	0.6704
1/4	0.4057	0.1682	0.6901	0.6954
1/8	0.2515	0.1039		

6 Conclusions

In this article, we have established the convergence rate estimates for scalar nonlocal nonlinear conservation laws, modeling traffic and crowd dynamics. Also, our analysis is not case specific, and covers most of the kernels (irrespective of its monotonicity) considered in traffic modelling such as forward, backward and central kernels.

The rate is shown to be 1/2 which is consistent with its local counterparts. It is interesting to see that the obtained convergence rate of 1/2 is independent of the radius of the convolution matrix and monotonicity of the kernel, with constant in the estimate depending on the radius of the kernel η. For $\nu(u) = 1 - u$, $\beta(u) = u$, the nonlocal conservation laws boil down to the local conservation law and hence the convergence rate 1/2 is optimal due to [33].

The extensions of these results to a general coupled system of nonlocal conservation laws, for the convergent finite volume schemes proposed in [1] and for nonlocal conservation laws with discontinuous flux (see [3]), are not straightforward and are works in progress. Furthermore, using the Kuznetsov-type lemma proved in this article, the rate at which the solutions of the nonlocal FTL (see [11, 17]) models converge to its continuum limit, can be explored, which we aim to address in our upcoming article.

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