Int. J. Production Economics 263 (2023) 108930

Contents lists available at ScienceDirect

International Journal of Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Preventive investment, malfunctions and liability☆

Peter Kort[a], Maria Lavrutich[b], Cláudia Nunes[c,*], Carlos Oliveira[b,d,e]

[a] CentER, Department of Econometrics and Operations Research, Tilburg University, The Netherlands
[b] Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, 7491 Trondheim, Norway
[c] Department of Mathematics and CEMAT, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
[d] ISEG - School of Economics and Management, Universidade de Lisboa, Rua do Quelhas 6, Lisboa 1200-781, Portugal
[e] REM-Research in Economics and Mathematics, CEMAPRE, Portugal

ARTICLE INFO

Keywords:
Investment decision
Time and capacity optimization
Malfunctions
Liability rule

ABSTRACT

In this paper we study the decision of a firm to undertake a one-time proactive preventive investment to limit the occurrence of future disruptions. The firm is operating in a market with uncertain demand, and its products are subject to a risk of malfunction. Malfunctions lead to direct costs, consisting of, e.g. legal fees, fines and additional costs. But they also make the product less attractive, affecting product demand. Moreover, the firm may be strictly or partially liable for these malfunctions. In order to take into account different levels of liability, we introduce a liability parameter. Our model takes these features into consideration, and we determine the optimal time and size of the preventive investment, depending on the liability rule, that maximize the value of a firm that is already in the market and has the option to invest in a preventive infrastructure. We then determine the liability rule that steers this investment decision in such a way that malfunction damage is minimized.

1. Introduction

In many manufacturing industries, digital transformation and technological progress have led to increased R&D activity and product improvements. At the same time it also brought safety and security concerns to the forefront (Marucheck et al., 2011). This issue is especially important when the consequences of accidents or safety failures are large, the so called *low likelihood but high impact catastrophic events*, which can severely affect firm's profitability.

Knemeyer et al. (2009) emphasize the increasing need for managers to address resilience of their products, systems or supply chains with respect to such failures by undertaking proactive investments, which leads to the following questions: when should these investments be made and how much should the firm invest? These are precisely the questions that we address in this paper. More specifically, we study the decision of a firm to undertake a pro-active preventive investment in order to lower the likelihood of future disruptions where both the timing and consequences of these disruptions are uncertain. Preventive investment is broadly defined as it includes not only R&D investment but also human investment (as receiving preventive training, for instance).

This decision concerns not only the optimal time to invest but also its optimal size. A larger preventive investment reduces accident probability to a larger extent, but it requires a larger investment cost.

In general, such failures/disruptions may negatively affect the project's revenues for different reasons. These include direct negative effects on demand due to loss of consumers, loss of suppliers, loss of reputation and competitive foothold in the market, as well as additional non-anticipated costs associated with law suits and insurance payments. To address this in our model, product failures affect firm's profitability in two ways. *First*, there is a *direct cost* to the firm, having a one-time effect. Upon the occurrence of a malfunction, the firm faces immediate legal or financial repercussions related to the firm's liability for the damages. *Second*, the occurrence of failures reduces demand for the product now and in the future. This is thus a *long-term reduction in demand*. the firm should take into account that a larger preventive investment will reduce accident probability to a larger extent.

The topic of investment in safety is particularly relevant in view of growing market penetration of smart products, which become increasingly autonomous and less reliant on human decision-making (Dawid and Muehlheusser, 2022; Sassone et al., 2016). A prominent example is the industry for automated vehicles (AVs) that have the potential

☆ The authors gratefully acknowledge support from the Portuguese Science Foundation (FCT) through the project CEMAPRE/REM - UIDB/05069/2020, PTDC/EGE-ECO/30535/2017 and CE- MAT - UIDB/04621/2020.

* Corresponding author.
E-mail address: cnunes@math.tecnico.ulisboa.pt (C. Nunes).

https://doi.org/10.1016/j.ijpe.2023.108930
Received 8 September 2022; Received in revised form 22 May 2023; Accepted 26 May 2023
Available online 1 June 2023

to reduce road accidents in the future. However, today the technology is not fully matured. For example, even the established technology providers, such as Google, still experience malfunctions (?). Due to the ability of malfunctions in AVs to cause crashes and potential casualties, more R&D investments are needed to improve their safety.

But the question of safety is also relevant in other areas besides the industry of smart products. One of such examples is the airline industry. In July 2000, an Air France Concorde airplane crashed in Charles de Gaulle airport, just after take-off. After many months of investigation, it was concluded that the accident was caused by a tyre failure. But even after a $17M (2001 values) safety improvement, Concorde flights were ended in 2003. The investment was decided too late and Concorde was unable to overcome its consequences.[1]

More recently, Boeing has faced devastating effects on the projects' revenues as consequence of two major plane crashes (in 2018, in Indonesia, and in 2019, in Ethiopia, both with the Boeing 737 MAX passenger airliner). Both crashes were caused by malfunctioning of the flight-software system. As it later transpired, the disasters were to some degree preventable, had the pilots been trained to regain control of the plane under this type of malfunction.[2]

When one is dealing with such decisions, especially large investments that also may lead to human losses, the question of liability is important. This is especially relevant in connection with smart products like automated vehicles. If an accident occurs with a self-driving car, the big difference with an ordinary car accident is that the behavior of the "driver" is much less influential, if at all, with respect to accident occurrence. Then the question is who is responsible: the car owner or the manufacturer. A similar question arise in airplane crashes, for instance.

The question of liability is far from being trivial, even in terms of its definition. Following Dawid and Muehlheusser (2022), we explicitly introduce a liability parameter, the value of which denotes to what extent either of the involved parties, consumer and firm, are liable. And we consider both strict and partial firm liability. Moreover, we go one step further than Dawid and Muehlheusser (2022), where we determine which liability rule will minimize future malfunction damage. Hence, our liability rule is endogenously determined, as it is the result of an optimization problem. Therefore, it is not intended to describe how it presently occurs in reality, but instead it shows how it should be established to reduce the expected damage from malfunctions.

In a nutshell, our paper presents a full characterization of several decisions that the firm and the regulator need to undertake in order to reduce losses concerning malfunctions. To do so, we first determine, for a given liability level, when the firm should undertake the preventive investment and how large should it be. Second, given this information, which liability rule should the regulator choose to stimulate the firm to act such that expected damages from accidents are minimized.

The main results that we obtain in this paper can be summarized as follows.

- The characterization of the optimal preventive investment depends on which effect is dominant when a malfunction occurs: if the *direct-cost* effect is dominant, the firm will undertake an immediate but relatively small investment; otherwise it is optimal to undertake a large preventive investment at a (possibly) later point in time.
- In case of a large investment, albeit at the (possible) expense of a late undertaking, it is optimal for the regulator to let the consumer be liable. On the other hand, if an immediate investment action is required, let the firm be liable, either fully liable or at least to the extent required to let the firm invest immediately.
- The damage-minimizing liability rule depends on the level of demand uncertainty. If demand is very uncertain, the regulator decides to let the consumer be liable, because then the firm will invest late and a lot. In case of (almost) no uncertainty, the regulator should design a liability rule such that it induces the firm to invest immediately, which requires the firm to be sufficiently liable.

The paper is organized as follows. Section 2 gives an overview of the relevant literature, whereas Section 3 presents the model. Due to the complexity of this framework, a full analytical solution is not available. Therefore, our analysis is partly numerical. The analytical part of our results is presented in Section 4. Based on this analysis we develop numerical results in Section 5. Section 6 concludes the paper.

2. Related literature

The novelty of this paper is that we look at the nexus between timing and size of the investment, and different liability rules in the presence of both market uncertainty (demand and, thus, revenues) and technical risks (uncertain arrival of product failures). Hence our work is related to several strands of the literature.

Concerning optimal preventive and maintenance measures, there exists a large number of papers that study system reliability from an industrial engineering perspective, including maintenance strategies (Wang and Pham, 2006; Garg and Deshmukh, 2006; Rivera-Gómez et al., 2013). The operations management literature has presented a number of different mechanisms to manage disruption risks along the supply chain. These include mitigation strategies, e.g., inventories and sourcing, as well as contingency strategies, e.g. demand management and rerouting (Tomlin, 2006, 2009; Babich et al., 2007; Bakshi and Kleindorfer, 2009; Wang et al., 2010). Within this literature stream, however, the flexibility of the firm with respect to timing the investment decision is typically disregarded either because market uncertainty is ignored or due to the fact that failure mitigation strategies are not associated with a substantial irreversible capital investment.

On the contrary, in our model, we explicitly take into account market uncertainty (as we assume that demand is stochastic, modeled by a geometric Brownian motion), and that the time and impact of the malfunctions are also unknown. Thus we take into consideration a model with a much higher level of stochasticity, and that means a higher level of uncertainty and flexibility. Allowing for demand uncertainty, the decision must be taken based on future expectations about demand growth. Therefore, the problem becomes analytically more challenging but at the same time more realistic. The model is also richer, allowing to study the impact of the demand drift and the volatility parameters on the optimal time and size investment.

This is a novelty in the sense that it extends papers as Kim and Tomlin (2013), which explicitly consider capital investment decisions but uncertainty about the firm's demand and revenue is disregarded; or Jin et al. (2009), where demand is uncertain but random failures are not taken into account.

Closer to our setting in terms of uncertainty is Xu et al. (2020). But they consider continuous control rather than a lump sum capital investment in preventive measures. In our model a firm has the opportunity to undertake a lump sum preventive investment in the presence of both failure risks and revenue uncertainty.

Regarding the liability rule, the closest contribution to our paper is Dawid and Muehlheusser (2022), that study the interplay between product liability, safety investment and the timing of market introduction with regard to the application to smart products, particularly AVs. In their model, the firm chooses the safety stock investment that reduces the accident rate, which is similar to preventive investment in our model. However, we differ from Dawid and Muehlheusser (2022) in two crucial ways. Firstly, in their model, demand is assumed deterministic. Secondly, they consider incremental investments in safety

[1] Westcott, Richard (24 October 2013). "Could Concorde ever fly again? No, says British Airways". BBC News.

[2] https://www.economist.com/books-and-arts/2021/11/27/a-new-book-explains-the-tragic-failure-of-boeings-737-max

stock, whereas we consider one lumpy preventive investment. When setting up an infrastructure designed for preventing malfunctions, it seems more reasonable to indeed model such investments as lumpy, as in many cases they require a one-time large capital expenditure. Moreover, considering just one lumpy investment is not very restrictive, as long as it is reasonable to assume that subsequent investments intend to occur at a (much) later point in time so that their effects are discounted away. The advantage is that the model becomes more tractable, so that we can focus on the interplay between undertaking preventive investments, demand uncertainty, uncertain arrival of malfunctions, and liability rules.

Since we consider a lumpy investment under uncertainty, our paper contributes to the literature of real options, where Dixit and Pindyck (1994) is a seminal work. As we are considering not only the time but also the size of the investment, we refer to Dangl (1999), Huisman and Kort (2015), Huberts et al. (2015), where it is recognized that investment decisions are not only about timing but also about size. Moreover, Kort et al. (2021) develops a real options model of preventive investment to avoid environmental incidents. Unlike our model, however, that paper disregards the issue of liability related to such incidents. Another difference is that in Kort et al. (2021) the effect of a malfunction is multiplicative with respect to the firm's revenues, implying that every new incident has the same relative negative effect on revenue. On the contrary, in our model every repeated product failure leads to a relatively larger negative effect on the revenue, due to the loss of the firm's reputation. Here it is taken into account that reputation loss increases with every new failure: one malfunction can be an unfortunate accident, but with every additional one it becomes more clear that there is a structural problem.

Another methodological novelty of this paper is that it brings to the arena of real options the possibility to optimize not only with respect to time and capacity of the investment, but also with respect to the intensity of the Poisson process, representing in our case the arrival of product failures. In our model undertaking the preventive investment reduces the Poisson arrival probability and thus also the malfunction probability. The idea of controlling the intensity of the point process that affects the outcome of a project is not new. For instance, Araman and Caldentey (2009) study a model with dynamic pricing policies for nonperishable products in the context of a retail operation with uncertain demand. Also Defourny (2018) considers a problem of controlling the intensity of a point process in order to maximize the probability that a target number of arrivals is met exactly by a deadline. But in these papers the objective is different from ours, as well as the model and assumptions used.

Lastly, we contribute to the debate in the legal literature concerning the ability to reduce accident risks by imposing strict liability on AV producers (Hay and Spier, 2005; Shavell, 2020). We provide an extensive analysis in which we determine the liability rule that reduces future accidental risks by providing the maximal incentive for the firms to invest in safety.

3. The model

Consider a firm operating in a market with uncertain demand and, thus, its future revenue is also uncertain. More specifically, we assume that the firm's revenue evolves according to a geometric Brownian motion process

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

in which μ and σ are the constants representing the drift and the volatility parameters, respectively, and $\{W_t, t \geq 0\}$ is the Brownian motion. As usual, (e.g. Dixit and Pindyck, 1994) it also holds that $\mu < r$, where r denotes the discount rate.

In our model, the products produced by the firm are subject to the occurrence of random accidents, or malfunctions. Let $\{Z_i, i \in \mathbb{N}_0\}$ be a sequence of independent and identically distributed random variables, where Z_i stands for the damage due to the ith malfunction. The examples here include the failure of product components, software bugs, or operational failures that can potentially lead to accidents (e.g., airplane accidents, road accidents of automated vehicles, etc.).

The accountability for such malfunctions can either be placed on the firm or the consumers, where our model allows to incorporate different liability rules. In particular, we assume that $\alpha \in [0,1]$ represents the part of the damage for which the firm is liable. This implies that in the case of $\alpha = 0$ the firm does not face any direct costs related to the malfunctions and the burden is borne by the consumers. On the contrary, in the case of $\alpha = 1$ the firm is fully liable. We also take into account partial liability where $0 < \alpha < 1$.

In general, when a malfunction occurs, the firm faces two negative consequences. First, a malfunction results into a direct one time cost to the firm, consisting of, for instance, paying a fine, compensations to buyers/suppliers, costs related to insurance and investigation, or legal fees. It makes sense that this direct cost is related to the damage of this malfunction, and therefore we assume that it is equal to $\alpha K Z_i$, where K is a constant.

Second, every malfunction has a negative effect on current and future demand for the product, as it discourages the consumers to buy the product. This effect will be bigger the more the consumer is held liable for the damage caused by the malfunction. In particular, the part of the demand, and thus of the revenue, that is lost due to malfunction is

$$(1 - \alpha + \epsilon)\frac{Z_i}{1 + Z_i}.$$

Intuitively, consumers may experience negative utility from the occurrence of a malfunction also if they are not liable at all. The parameter $\epsilon > 0$ accounts for this effect. The larger is the damage, the more the consumer is discouraged from buying the product. The fraction $\frac{Z_i}{1+Z_i}$, being increasing in Z_i, captures that the negative demand effect is bigger in case of a larger damage due to the malfunction.

If N_t is the number of malfunctions that have occurred until time t, the total return of the firm until time t, given all the information concerning the occurrence of malfunctions until that time, is given by

$$\int_0^t e^{-rs}\left(X_s\left(1 - (1 - \alpha + \epsilon)\sum_{i=1}^{N_s}\frac{Z_i}{1 + Z_i}\right) - c\right)ds - \alpha K \sum_{i=1}^{N_t} e^{-rT_i} Z_i,$$

where T_i denotes the time of the occurrence of the ith malfunction, and c is a fixed production cost.

In this formulation, every new malfunction has a larger (relative) negative effect on demand (and, thus, on revenue). This is intuitive, as the first malfunction can be accidental. However, every repeated malfunction in the absence of mitigating measures by the firm leads to more serious reputational damage.

The firm holds an option to invest in safety or a preventive infrastructure, where the investment accomplishes that it reduces the probability that malfunctions occur. Upon investment the firm incurs a sunk cost R. The occurrence of malfunctions follows a Poisson process. Before the preventive investment it is denoted by $\left\{N_t^B, t \geq 0\right\}$, with intensity rate λ_B (the subscript B denotes "before"). After the preventive investment is undertaken, the intensity rate of malfunctions is reduced from λ_B to λ_A (the subscript A denotes "after"). In addition, we assume that a larger preventive investment leads to a larger decrease in the malfunctions' arrival rate. In order to emphasize the dependence of λ_A on R, we use the notation $\lambda_A(R)$. We assume that $\lambda_A(R)$ satisfies the following conditions.

Assumption 1. The function $\lambda_A : R_0^+ \to R^+$ is such that $\lambda_A \in C^\infty\left(R_0^+\right)$, $\lambda_A'(R) < 0$, $\lambda_A''(R) > 0$, $\lambda_A(0) = \lambda_B$, and $\lambda_A'''(R) / \left(\lambda_A''(R)\right)^2 > -2/\lambda'(R)$.

The condition $\lambda_A'''(R) / \left(\lambda_A''(R)\right)^2 > -2/\lambda'(R)$ is hard to interpret, but it holds for a very large class of functional forms. Examples here are $\lambda_A(R) = \frac{1}{(a+bR)^k}$ and $\lambda_A(R) = e^{-aR+b}$.

Finally, we let $\{N_t^A, t \geq 0\}$ denote the Poisson process with intensity rate λ_A, where we assume it to be independent of $\{N_t^B, t \geq 0\}$, and both Poisson processes are independent of $\{W_t, t \geq 0\}$.

Considering the situation that the firm is already in the market, we want to find (1) the optimal preventive investment time and (2) the optimal preventive investment size. Hence, given that the current revenue is x and that n malfunctions have already occurred, we arrive at the following optimization problem:

$$\begin{aligned} V_\alpha(x,n) = \sup_{\tau, R\geq 0} E_{x,n}\Bigg[&\int_0^\tau e^{-rt}\left(X_t\left(1-(1-\alpha+\epsilon)\sum_{i=1}^{N_t^B}U_i\right)-c\right)dt \\ &- e^{-r\tau}R - \alpha K\sum_{i=1}^{N_\tau^B} e^{-rT_i^B}Z_i \\ &+\int_\tau^\infty e^{-rt}\left(X_t\left(1-(1-\alpha+\epsilon)\left(\sum_{i=1}^{N_\tau^B}U_i+\sum_{i=N_\tau^B+1}^{N_{t-\tau}^A+N_\tau^B+1}U_i\right)\right)-c\right)dt \\ &-\alpha K\sum_{i=1}^{\infty}e^{-r(\tau+T_i^A)}Z_{N_\tau^B+i}\Bigg] \\ = \sup_{\tau,R\geq 0} J(\tau, x, n|\alpha),& \end{aligned} \tag{1}$$

where $E_{x,n}$ is the conditional expectation, given that $X_0 = x$ and $N_0^B = n$.[3] Moreover, we let $U_i = \frac{Z_i}{1+Z_i}$. Furthermore, T_i^B (resp., T_i^A) denotes the time of the ith malfunction before (resp., after) the timing of the preventive investment.

Based on the optimal preventive investment decision of the firm, which, according to (1), depends on the liability parameter α, the government, or the regulator, can determine which liability rule minimizes the expected discounted total damage as a result of malfunctions. We denote the corresponding level of α by α^*. The insights regarding α^* can be useful for policy makers, regulators and legislators when developing a legal framework related to the liability rules for different industries where malfunctions could eventually occur.

The regulator can determine α^* by solving the following minimization problem:

$$\alpha^*(x) = \arg\min_{\alpha\in[0,1]} E_{x,0}\left[\sum_{i=1}^{N_{\tau^*}^B} e^{-rT_i^B}Z_i + \sum_{i=1}^{\infty} e^{-r(\tau^*+T_i^A)}Z_{N_{\tau^*}^B+i}\right], \tag{2}$$

where τ^* denotes the time at which the firm undertakes the preventive investment. Since a malfunction affects demand and thus revenue, the optimal value, α^*, depends on the firm's current instantaneous revenue, x.

4. Analytical results

The following proposition rewrites the optimal stopping problem (1), where the expected value of U_i is denoted by u.

Proposition 1. *The optimal stopping problem* (1) *can be written as*

$$\begin{aligned} V_\alpha(x,n) =& \frac{x}{r-\mu}\left(1-n(1-\alpha+\epsilon)u - \frac{(1-\alpha+\epsilon)u\lambda_B}{r-\mu}\right)-\frac{c}{r} \\ &+\sup_{\tau,R\geq 0} E_{x,n}\left[e^{-r\tau}g_\alpha(X_\tau;R) - \alpha K E[Z]\frac{\lambda_B}{r}\left(1-\left(\frac{\lambda_B}{r+\lambda_B}\right)^{N_\tau^B}\right)\right], \end{aligned} \tag{3}$$

where

$$g_\alpha(x;R) = \frac{(1-\alpha+\epsilon)ux}{r-\mu}\left(\frac{\lambda_B}{r-\mu}-\frac{\lambda_A(R)}{r-\mu}\right) - R - \alpha K E[Z]\frac{\lambda_A(R)}{r}.$$

[3] Similarly, E_x (resp., E_n) represents the conditional expectation, given that $X_0 = x$ (resp., $N_0^B = n$).

In order to find the optimal investment decision, one needs to consider the function $g_\alpha(X_\tau; R)$ and the final term in $V_\alpha(x,n)$. The first term in g_α indicates the expected reduction in demand losses due to the fact that the malfunction intensity rate decreases from λ_B to $\lambda_A(R)$ at the moment of investment. The second term is the investment expense, and the third term is the expected direct cost due to malfunctions after having undertaken the preventive investment.

Hence, from Proposition 1 we obtain that finding the value function (3), and obtaining the optimal investment decision of the firm, relies on solving the optimization problem

$$\begin{aligned} &v_\alpha(x,n) \\ &= \sup_{\tau,R\geq 0} E_{x,n}\left[e^{-r\tau}g_\alpha(X_\tau;R) - \alpha K E[Z]\frac{\lambda_B}{r}\left(1-\left(\frac{\lambda_B}{r+\lambda_B}\right)^{N_\tau^B}\right)\right]. \end{aligned} \tag{4}$$

We note that the function v_α depends on n because $N_0^B = n$. However, this dependency no longer exists when $\alpha = 0$. Then, in this situation we will use the notation $v_0(x)$. Another implication of $\alpha = 0$ is that all changes in the firm's cash flow as a result of the preventive investment are contained in $g_0(x;R)$. Therefore, $g_0(x;R)$ can be denoted as the net present value of the firm's preventive investment.

Note that, depending on the value of x, the net present value, $g_0(x;R)$, can be negative for all values of R. In this case, the solution is trivial, since the firm will not make a preventive investment. Otherwise, the firm will invest the amount R^*, which in general, for $\alpha \in [0,1]$, is given by

$$R_\alpha^*(x) = \arg\max_{R\geq 0} g_\alpha(x;R). \tag{5}$$

Here, $R_\alpha^*(x)$ is the value of R that maximizes the expected return of the investment, given that the investment decision is undertaken when the instantaneous revenue is given by x. Proposition 2 presents the function $R_\alpha^*(x)$.

Proposition 2. *The optimal size of the preventive investment, R_α^*, when the firm's instantaneous revenue is equal to x, is given by*

$$R_\alpha^*(x) = \begin{cases} 0, & x\in[0,\hat{x}_\alpha] \\ (\lambda_A')^{-1}\left(-\frac{(r-\mu)^2 r}{r(1-\alpha+\epsilon)ux+(r-\mu)^2\alpha K E(Z)}\right), & x\in[\hat{x}_\alpha,\infty] \end{cases},$$

where $\hat{x}_\alpha = -\left(1+\frac{\lambda_A'(0)\alpha K E(Z)}{r}\right)\frac{(r-\mu)^2 r}{(1-\alpha+\epsilon)u\lambda_A'(0)}$.

Proposition 2 states that the revenue needs to be large enough for a preventive investment to be optimal. This makes sense, because the effect of a malfunction is that a fixed part of the demand, and thus also revenue, will be lost, and this loss can only be large when demand itself is large. The next proposition confirms this logic, i.e., given that an investment takes place, its size is increasing with the firm's revenue.

Proposition 3. *For fixed α, the function R_α^* is increasing and concave in x, with*

$$\lim_{x\to+\infty} R_\alpha^*(x) = +\infty,$$

and it is increasing with μ, and constant with σ and n. Moreover, R_α^* decreases in α for values of x larger than $\frac{KE[Z](r-\mu)^2}{r\mu}$, and increases otherwise.

From Proposition 3 we obtain that preventive investment increases with the trend parameter μ. A large preventive investment considerably reduces the probability of malfunction occurrence. This is especially worthwhile when μ is large, because the latter implies that revenue is expected to be large in the long run, while a malfunction would result in losing part of this revenue. On the other hand, Proposition 3 learns that the size of the preventive investment is not influenced by the uncertainty parameter σ. This is at first sight surprising because we know from the literature (Dangl, 1999) that more uncertainty results into a later and larger investment. The point here is that we consider the preventive investment size for a given value of x. Would we also have allowed for a change in the investment threshold, being the minimal value of x at which investing is optimal, then most likely we would have obtained that an increase of σ would have resulted in a larger investment threshold, and since R^*_α is increasing in x (Proposition 3), a larger investment size.

If the value of the liability share α is larger, the firm has to incur a larger part of the direct costs upon malfunction occurrence. On the other hand, since the consumer will then be held less liable, the long term demand effect will be smaller. In this light it is understandable that when direct costs are expected to be large, i.e. $K\mathbb{E}(Z)$ is large, an increase of α triggers the firm to increase its preventive investment size. However, if the direct cost is expected to be small, it is the long term demand effect that is more important for the firm. If α increases, the long term demand effect will be smaller, and this explains why the preventive investment size is decreasing in α when $K\mathbb{E}(Z)$ is small.

We finalize this section presenting an equivalent expression to the minimization problem presented in (2).

Proposition 4. *Given that the current firm's revenue is x and n malfunctions have already occurred, the liability parameter α that minimizes the discounted total damage as a result of accidents is implicitly defined as follows:*

$$\alpha^* = \inf_{\alpha\in(0,1)} \left\{ \lambda_A \left(\frac{x}{x^*}\right)^{\beta_+(r)} - \lambda_B \left(\frac{x}{x^*}\right)^{\beta_+\left(\frac{r\lambda_B}{r+\lambda_B}\right)} \right\}, \tag{6}$$

$$\alpha^*(x,n) = \arg\min_{\alpha\in(0,1)} \left\{ \lambda_A(R^*_\alpha(x))\mathbb{E}_{x,n}\left[e^{-r\tau^*}\right] - \lambda_B\mathbb{E}_{x,n}\left[e^{-r\frac{\lambda_B}{r+\lambda_B}\tau^*}\right] \right\}, \tag{7}$$

where τ^* is the investment time, with λ_A and τ^* depending on α, x and n, and

$$\beta_+(\zeta) = \frac{(\mu-\sigma^2/2)+\sqrt{(\mu-\sigma^2/2)^2+2\sigma^2\zeta}}{\sigma^2}.$$

4.1. Consumers are fully liable

Note that in between Propositions 1 and 2 we explained that the optimization problem (4) becomes simpler in case $\alpha = 0$. This allows us to obtain some additional analytical results for this scenario. Note that consumers being fully liable would for sure apply to ordinary cars. However, in the case of automated vehicles this is less clear: when an automated vehicle is involved in an accident, why should the "driver" be fully blamed for something the car did itself?

In the special case of $\alpha = 0$, the optimization problem simplifies into

$$V_0(x,n) = \frac{x}{r-\mu}\left(1 - n(1+\epsilon)u - \frac{(1+\epsilon)u\lambda_B}{r-\mu}\right) - \frac{c}{r} + v_0(x), \tag{8}$$

in which

$$v_0(x) = \sup_{\tau, R\geq 0} E_x\left[e^{-r\tau} g_0(X_\tau; R)\right],$$

and

$$g_0(x;R) = \frac{(1+\epsilon)ux}{r-\mu}\left(\frac{\lambda_B}{r-\mu} - \frac{\lambda_A(R)}{r-\mu}\right) - R.$$

The following corollary to Proposition 2 specifies the optimal preventive investment size.

Corollary 1. *The optimal size of the preventive investment is given by*

$$R^*_0(x) = \begin{cases} 0, & x\in[0,\hat{x}] \\ (\lambda'_A)^{-1}\left(-\frac{(r-\mu)^2}{(1+\epsilon)ux}\right), & x\in[\hat{x},\infty] \end{cases},$$

where $\hat{x} = -\frac{(r-\mu)^2}{(1+\epsilon)u\lambda'_A(0)}$.

From the next proposition it can be derived when it is optimal for the firm to undertake the preventive investment. In this proposition x^* is the investment threshold, meaning that it is optimal for the firm to invest when the firm's instantaneous revenue is at least as large as x^*. In case the initial instantaneous revenue level falls below the threshold, the firm will invest in size $R^*_0(x^*)$ at the moment the process X_t reaches the level x^* for the first time.

Proposition 5. *The value of the project is given by*

$$V_0(x,n) = \begin{cases} \frac{x}{r-\mu}\left(1-n(1+\epsilon)u - \frac{(1+\epsilon)u\lambda_B}{r-\mu}\right) - \frac{c}{r} + Ax^{\beta_1} & x < x^* \\ \frac{x}{r-\mu}\left(1-n(1+\epsilon)u - \frac{(1+\epsilon)u\lambda_A\left(R^*_0(x)\right)}{r-\mu}\right) - \frac{c}{r} - R^*_0(x) & x\geq x^* \end{cases},$$

where $\beta_1 = \frac{\frac{\sigma^2}{2}-\mu+\sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2+2\sigma^2 r}}{\sigma^2}$, and

$$A = \left(x^*\right)^{-\beta_1} g_0\left(x^*; R^*_0\left(x^*\right)\right),$$

with x^* being the value of the revenue that triggers the preventive investment, implicitly given by

$$\beta_1 g_0\left(x^*; R^*_0\left(x^*\right)\right) - g'_0\left(x^*; R^*_0\left(x^*\right)\right)x^* = 0,$$

where it holds that $x^* > \hat{x}$.

From Corollary 1 we get that the firm will not acquire any preventive equipment as long as $x \leq \hat{x}$. It automatically follows that for a meaningful investment to take place it should hold that $x > \hat{x}$. This explains why for the investment threshold we have that $x^* > \hat{x}$.

5. Numerical results

In this section we provide a numerical illustration of the preventive investment decision, regarding both the optimal investment threshold and its size. We also present results concerning the dependence of the firm's investment decision on the liability rule. Lastly, we determine the liability policy that minimizes the discounted stream of accidental damages.

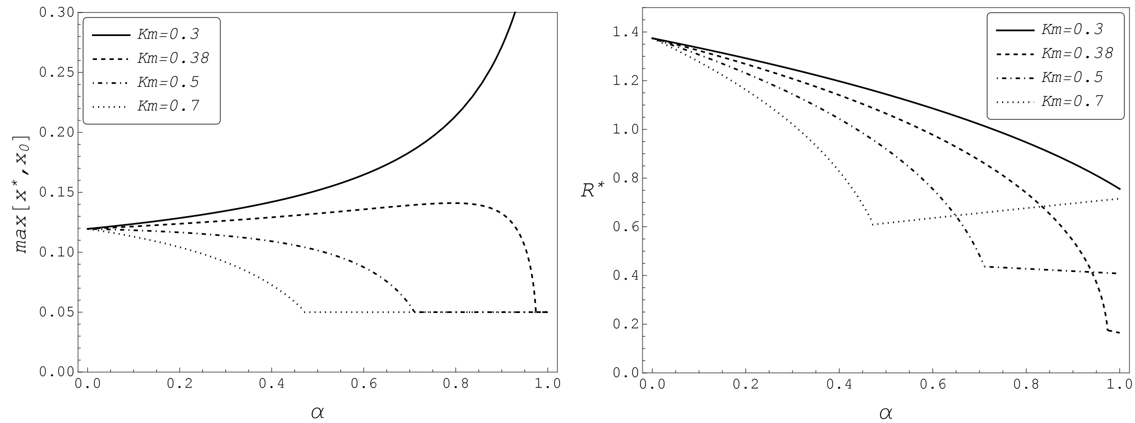
When choosing the values for model parameters, we base ourselves on studies that analyze aviation accidents (Squalli and Saad, 2006; Čokorilo et al., 2010; Akyildirim et al., 2021). As previously referred, one application of our model is safety investments in airline industry. The examples from Concorde and Boeing show that the financial and human costs of aviation accidents are enormous, but fortunately such accidents are rare. Furthermore, sometimes these accidents can be prevented by undertaking proper investments. This motivates the design of models and studies as in this paper.

One of the most important parameters in our model is the (average) intensity of aviation accidents. Akyildirim et al. (2021) collect information about aviation accidents between the period June 1, 1995 and May 31, 2019, concluding that the number of accidents company-by-company in their sample varied between 1 and 8 (Table 3 of Akyildirim et al., 2021). In terms of Poisson intensity, this means a rate between 0,04–0,33(3). In order to satisfy Assumption 1 regarding the functional form for the arrival rates of malfunctions, we assume

$$\lambda_A(R) = e^{-a-bR}, \quad \lambda_B = e^{-a},$$

with $a = 2$ and $b = 1$, which means that λ_B is equal to $e^{-2} = 0.1353$, which is in the range of suggested values. Moreover, setting $b = 1$ means that a preventive investment of size 1 has the effect that the number of malfunctions occur once every 20 years on average, as $\frac{1}{\lambda_A(1)} = e^{2+1} \approx 20$.

Among the studies that investigate the direct costs of aviation accidents is according to Čokorilo et al. (2010) (see Section 7, in particular Table 3). They find that the costs directly related with aviation accidents (from minor to catastrofic) can vary between 0,04



(a) Optimal investment thresholds.

(b) Optimal preventive investment size.

Fig. 1. Optimal investment threshold and optimal preventive investment size as functions of α for different values of Km. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\mu = 0.02$, $\sigma = 0.2$, $a = 2$, $u = 0.2$, and $\epsilon = 0.1$].

Table 1
Values of the parameters regarding the impact of aviation accidents.

Parameters		Values	Real/estimated values
λ_B		$e^{-2} \approx 0.1353$	0.04–0.33
Km		0.5	0.12–0.65
Reputational losses $u(1-\epsilon-\alpha)$	u	0.2	≈ 0.03
	ϵ	0.1	
	α	α	

and 0,21 billion EUR, in 1999 prices, according to the severity of the accident. Using an increasing rate of 5%, as suggested by the authors, we obtain costs (in 2022 prices) between 0,12 and 0,65 billion EUR. In order to focus on more severe accidents when liability issues are particularly important, we choose $KE[Z_i]$ (hereby denoted by Km) to be equal to 0.5 billion EUR. Note that if the company is partially liable, i.e. $\alpha \in (0, 1)$, the expected direct cost equals 0.5α, whereas there are no direct costs in case $\alpha = 0$.

Čokorilo et al. (2010) suggest that the reputation costs of an aviation accident can vary between 0 and 0.921457 billion EUR in 2022 prices (0–0.3 billion EUR, in 1999 prices). These values are also supported by Squalli and Saad (2006), that estimates the revenue losses when there are accidents with serious injuries around 0.3 billion EUR, the maximum value presented by Čokorilo et al. (2010). According to Squalli and Saad (2006), perceptions about accidents resulting in serious injuries significantly decrease enplanement by 3 percent.

In our model, the reputation costs can be interpreted as the reduction in the revenue due to the accident. The revenue is reduced by $u(1-\epsilon-\alpha) \times 100\%$ every time we have an accident. Hence, fixing $u = 0.2$, $\epsilon = 0.1$, and $\alpha = 0.75$, we get the reduction of 3% estimated in Squalli and Saad (2006).

In Table 1 we show the values that we define for these parameters in the baseline case, and the range of values estimated by the above mentioned works.

Finally, we fix the following values for the other parameters:

$$r = 0.05,\ c = 0.01,\ X_0 = 0.05,\ n = 0,\ \mu = 0.02,\ \sigma = 0.2$$

With the values for the baseline case fixed, in the remainder of the section we study the impact of changes of one parameter at a time on the optimal decisions.

We start by determining the liability rule that minimizes (expected) damage from accidents. For that it is crucial to know how the firm's preventive investment decision depends on α. Fig. 1 shows the investment timing, represented by the threshold, and the investment size as function of α, for different values of the direct accident cost $KE\left(Z_i\right) = Km$.

If $\alpha = 0$, consumers are fully liable and the firm faces no direct cost. This explains why the threshold and the investment size are the same for different values of Km. In general it holds that the effect of Km on the firm's investment decision becomes bigger when α increases.

Intuitively, an increase in α has two consequences for the firm. A large α implies that the effect of direct costs is important for the firm, as it is liable to a considerable extent. At the same time, it means that the effect of demand reduction is small, as it also implies limited liability for the consumers The opposite holds when α is small. With this in mind, consider first the small value of Km, i.e. $Km = 0.3$. If α increases, this small direct cost starts playing a bigger role, while at the same time the firm becomes less affected by the demand reduction. Thus, the firm in general is less affected by the occurrence of an accident. Then it feels less need to protect itself against accident occurrence, implying that for $Km = 0.3$ the firm will invest later, the threshold increases, and less when α goes up.

The opposite happens when the direct cost Km is large, i.e. when $Km = 0.7$. Then the firm is more eager to invest preventively when α increases, translating in a decreasing investment threshold. The investment threshold keeps on decreasing until it reaches the initial value $X_0 = 0.05$. Then the firm invests at the initial point in time, which thus happens for $\alpha \geq 0.7$. If the firm invests at a lower threshold, at such a moment in time revenue is lower so then the fraction of the revenue lost upon a malfunction due to demand reduction, is also lower. This explains why the investment size also decreases with α, until α reaches the value 0.7. Then we are in the situation that the firm invests immediately, and we see that from then on the investment size is increasing in α while its liability increases until it becomes fully liable, in order to protect the firm against the large direct cost it can expect.

For the intermediate direct cost level, $Km = 0.38$ or $Km = 0.5$, we observe a combination of the above two developments. The investment threshold first slightly increases with α, and then decreases until it is optimal to invest immediately at $X_0 = 0.05$. The investment size is decreasing all along with α, either because the threshold is decreasing as well, or because the direct cost is not large enough to let the firm undertake a larger preventive investment when the firm's liability increases.

In general, from the perspective of damage minimization, it is desirable that the firm undertakes the preventive investment early and to let the firm invest a lot. It then follows, and Fig. 2 confirms it, that in the case of a small direct cost level, $Km = 0.3$, it is optimal to have

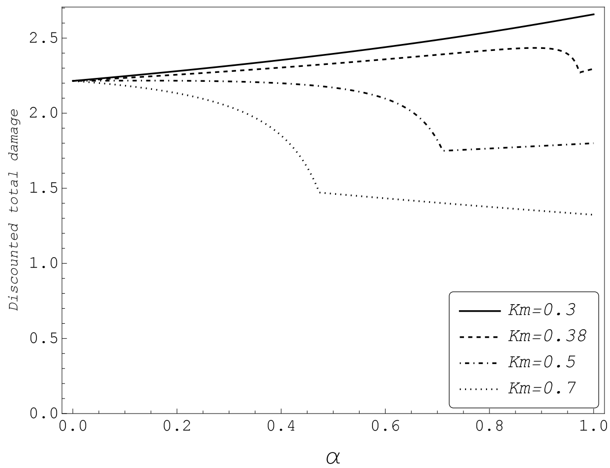


Fig. 2. Discounted total damage as a result of the accidents as a function of α for different values of Km. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $b = 1$, $u = 0.2$, and $\epsilon = 0.1$].

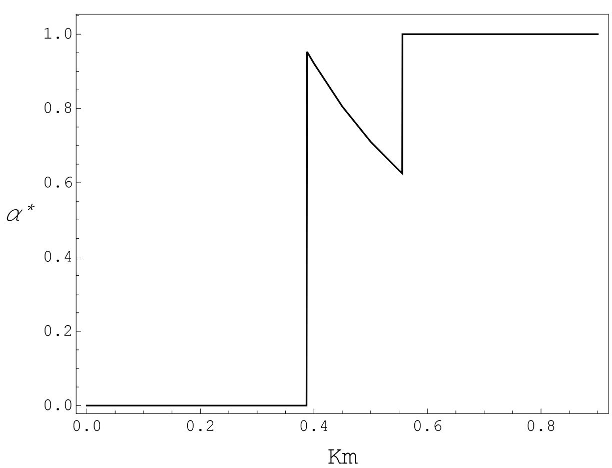


Fig. 3. Optimal liability rule a function of Km. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $b = 1$, $u = 0.2$, and $\epsilon = 0.1$].

$\alpha = 0$, or, in other words, to let the consumer be fully liable. In the opposite case, thus when the direct cost is large, $Km = 0.7$, a fully liable firm is optimally incentivized to preventively invest immediately, and, as long as the firm invests immediately, the size of preventive investment will increase in α. The latter makes it indeed optimal to have α equal to one in this case.

In the intermediate cases, total damage is minimized for the minimal value of α such that the firm will invest immediately. This is because, as long as the firm invests immediately, the firm's investment size is decreasing in α.

From Fig. 2, we can derive the following result.

Result 1. *Three candidate optimal liability rules prevail: full liability for the consumer, full liability for the firm, or choosing the minimal value of α such that the firm will invest immediately.*

The numerical analysis below, where we vary the value of a single parameter while keeping the others constant to see how the firm's investment decision and the damage-minimizing liability rule changes, confirms this result.

Fig. 3 shows then the optimal α, in the sense that this liability rule leads to minimal expected total damage, as a function of the expected direct cost Km.

Indeed, as expected, the consumer should be fully liable when the direct cost is small, whereas the firm should be fully liable when the direct cost of damages for the firm is expected to be large. In the intermediate case the optimal liability rule corresponds to the minimal α under which the firm immediately invests. This α is decreasing in Km because when the direct cost is expected to be larger, the firm is more willing to invest immediately, resulting in a larger α-domain for which this takes place.

In Fig. 4 we have the firm's investment decision in terms of threshold and size as a function of Km. In this and subsequent figures, three curves are shown: the investment decision when the consumer is fully liable ($\alpha = 0$, dashed curve), the firm is fully liable ($\alpha = 1$, dotted curve), and when the liability rule is the optimal one as in Fig. 3 ($\alpha = \alpha^*$, solid curve).

For $\alpha = 0$ the curves are horizontal, because in the case the consumer is fully liable, the expected direct cost does not play a role. They are incurred by the firm only when the firm is (partly) liable, i.e. $\alpha > 0$.

In case the firm is fully liable, $\alpha = 1$, the firm is not so eager to invest preventively when the expected direct cost from a malfunction is small. Then, as Fig. 4a shows, the threshold triggering investment is (very) large. When Km increases, at some point these direct costs get large enough that the firm will invest immediately, and in a larger size as Km increases further. This, i.e. investing preventively immediately in a sufficiently large size, is attractive to limit the number of malfunctions, so the optimal α^* is equal to one for Km large enough.

On the other hand, for Km small it is better not to make the firm liable at all, because it is not incentivized to carry out any preventive investment in the short term. Then it is optimal to make the consumer fully liable, i.e. the curves $\alpha = \alpha^*$ and $\alpha = 0$ are similar there. For intermediate values of Km, both firm and consumer are partly liable. This results in the firm investing immediately, but in a slightly larger size than when the firm is fully liable. This is because, as we know from Fig. 2, for intermediate values of Km, the smallest α for which the firm invests immediately triggers the largest investment size, given that the firm invests immediately. This leads to the following policy implication.

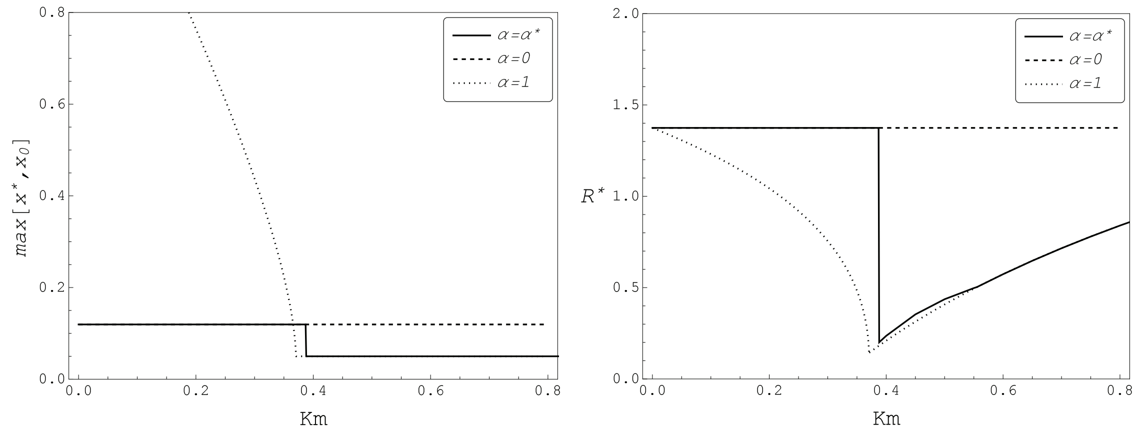
Result 2. *To minimize the expected total malfunction damage it is sufficient to let the firm be liable only when the expected direct cost from a malfunction is large enough.*

Figs. 5 and 6 depict the dependence of firm and regulator decisions on b. The parameter b measures the effectiveness of preventive investment in reducing the malfunction probability.

Intuitively, if b is small the investment size needs to be large in order to have an effect. A large investment is only profitable if it can limit the probability of malfunction occurrence in a situation where malfunction damage is large. Since a malfunction causes the loss of part of the revenue, malfunction damage can only be large if the revenue, X, is large as well.

If b is large, already a small investment is enough to substantially reduce the malfunction probability. This does not require a large revenue, so such an investment could essentially be done immediately. A way to accomplish this is to make the firm partially liable. Partially liability is better than full liability, because, whereas in both cases the firm invests immediately, in the case of partial liability the firm will invest more, as Fig. 1b learns in the case of $Km = 0.5$. As b increases even further, the optimal solution in terms of damage minimization converges to the case of full consumer liability. These observations lead to the following result.

Result 3. *When varying the level of preventive investment efficiency, firms should be held less liable for the damages in two distinct cases. First, when the efficiency of the preventive investment is very large, i.e. for the well established products with known potential defects, for which the firms can efficiently reduce the damages themselves. Second, when preventive investment is, on the contrary, very inefficient, for example, when the malfunction occurred due to new unsolved problems.*



(a) Optimal investment thresholds.

(b) Optimal preventive investment size.

Fig. 4. Optimal investment thresholds and optimal preventive investment size as functions of Km. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $b = 1$, $u = 0.2$, and $\epsilon = 0.1$].

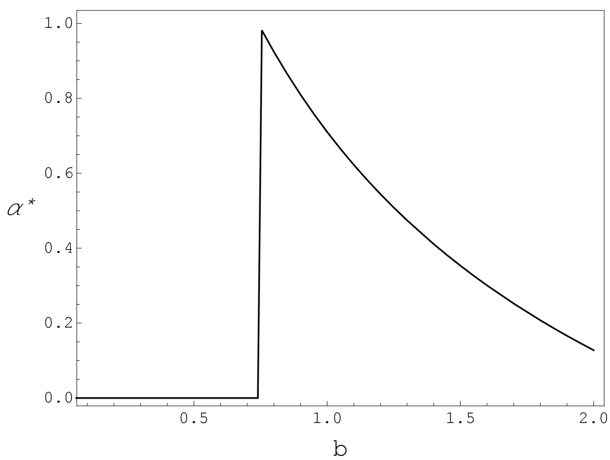


Fig. 5. Optimal liability rule a function of b. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $u = 0.2$, $Km = 0.5$, and $\epsilon = 0.1$].

Next we vary the parameter ϵ, covering the non-liability related demand reduction due to a malfunction. If ϵ becomes larger, a malfunction's effect on demand is more disastrous, so either an earlier or a larger preventive investment is needed. Figs. 7 and 8 present the results in this case.

If the consumer is fully liable, the investment threshold decreases with ϵ, keeping the investment size fixed. If the firm is fully liable, the firm invests immediately, where the size increases with ϵ. We see that to reduce total expected damage, the best thing is to invest immediately, where, as we have seen before, partial liability dominates full firm liability since it will give a larger investment size. When ϵ goes up, a larger α–domain results in immediate investment, and therefore it is possible to have immediate investment for a lower value of α. The policy implications here can be summarized in the following result.

Result 4. *For markets where demand is very sensitive to the occurrence of accidents (for example, when there exists a large number of competitors producing a similar product and it is easy for the consumers to switch), the liability rule should favor the firm. This is because the economic consequences of malfunctions hurt the firm already enough to incentivize preventive investment without needing any additional legal repercussions.*

The influence of the revenue uncertainty parameter σ is analyzed in Figs. 9 and 10. The polar cases of α are clear. If the firm is fully liable, $\alpha = 1$, a malfunction mainly causes some direct cost the firm needs to incur, being independent of revenue uncertainty. Therefore, in this case the firm's preventive investment decision is independent from σ, as Fig. 10 confirms. Then an immediate investment in a relatively small size occurs. If the consumer is fully liable, a malfunction results in a considerable reduction of revenue, which depends a lot on the uncertainty parameter σ. We know from the literature (Dangl, 1999; Huisman and Kort, 2015) that the firm invests later and more in a more uncertain economic environment, and this is what also happens here.

As can be seen in Fig. 9, in order to minimize the expected total damage, under high uncertainty the regulator put full liability on the consumer. This is because such a situation triggers a late and large preventive investment, which is preferable when σ is large. This explains that $\alpha = 0$ when σ is large. In the complementary case immediate investment is preferred where partial liability gives the largest investment size. As σ goes up, there is more incentive to invest later. This implies that a larger α is needed, i.e. a firm should be more liable to let it invest immediately. Thus, in addition to the classical conclusion that firms invest later and more in a more uncertain environment, we also obtain a new result.

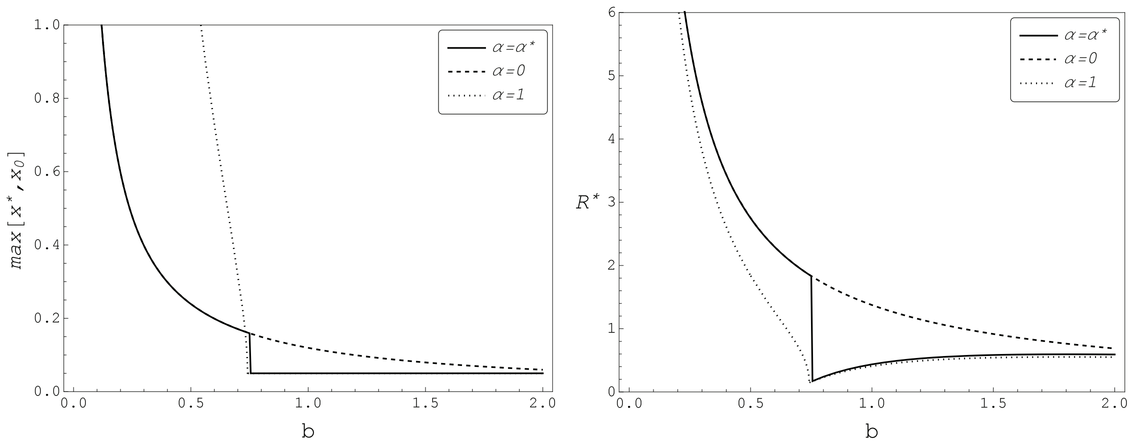
Result 5. *From the point of view of minimizing total malfunction damage, it holds that in highly volatile markets firms should not be held liable for the damage, as this will substantially reduce the preventive investment amount.*

6. Summary of the main results

The characterization of the optimal preventive investment depends on what is the dominant effect when a malfunction occurs: the *direct cost* or the *long term reduction in demand*. The first effect is a one-time effect and deterministic while the latter effect repeats itself over time and is stochastic because it depends on future demand realizations. We find that if the *direct-cost* effect is dominant, the firm will undertake an immediate but relatively small investment.[4] If the *long-term-reduction-in-demand* effect prevails, it is optimal to undertake a large preventive investment at a (possibly) later point in time. The investment is large because the effect is repetitive and it is possibly delayed because it only pays off to undertake such a large investment when the demand level is large enough.

In determining the liability rule that minimizes malfunction damage, one should realize that the regulator can influence the relative dominance of the two just-mentioned effects. The *direct-cost* effect will

[4] Of course, this holds under the condition that the direct accident cost is not too large.



(a) Optimal investment thresholds.

(b) Optimal preventive investment size.

Fig. 6. Optimal investment thresholds and optimal preventive investment size as functions of b. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $u = 0.2$, $Km = 0.5$, and $\epsilon = 0.1$].

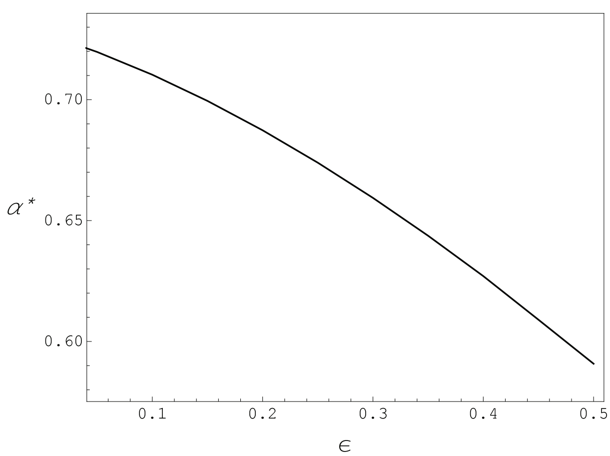


Fig. 7. Optimal liability rule a function of ϵ. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $b = 1$, $u = 0.2$, and $Km = 0.5$].

be dominant if the firm is liable. Demand effects depend on consumer behavior, so making consumers liable for the accidents will enhance the *long-term-reduction-in-demand* effect. Our results show that, if the solution of the malfunction-damage-minimization problem requires a large investment, albeit at the (possible) expense of a late undertaking, it is optimal for the regulator to let the consumer be liable. On the other hand, if an immediate investment action is required, let the firm be liable, either fully liable or at least to the extent required to let the firm invest immediately. Concerning the latter, letting the firm be fully liable is optimal in case of a large *direct-cost* effect, but if the *direct-cost* effect is of intermediate size then choose a firm liability level being just sufficient to trigger an immediate preventive investment.

Another interesting finding is the impact of demand uncertainty on the damage-minimizing liability rule. We know from the literature (for example Dangl, 1999 or Huisman and Kort, 2015) that in an uncertain economic environment it is optimal to undertake a late and large investment. Therefore, if demand is very uncertain, the regulator decides to let the consumer be liable, because then the firm will invest late and a lot. In case of (almost) no uncertainty, the regulator likes to have an immediate investment and also the firm does not have incentives delay investment. Therefore, the regulator can let the firm invest immediately, even if it puts a considerable liability weight on the side of the consumer. Provided that the *direct cost* is not too large, the latter is preferable because consumer liability raises the preventive investment size as explained above. If from there on, uncertainty gradually increases, the firm's incentive to delay the preventive investment becomes larger. So, an immediate investment then requires that the regulator should gradually reduce the liability of the consumer.

7. Conclusion

In this paper, we study the option to undertake a preventive investment by a firm that aims to reduce the frequency of its product's malfunctions. In particular, we consider an active firm selling products on the market, while facing demand uncertainty. The negative implication of product malfunctions is twofold. First, the firm faces some direct costs, related to, e.g., paying legal fees, fines, additional insurance costs. Second, malfunction occurrence makes the product less attractive, resulting in a reduction of product demand now and in the future.

In the current setting, the firm is already in the market, and hence the option that we are addressing concerns solely the investment in a preventive center. An interesting extension of this problem would be considering that the firm has yet to enter the market. This would lead to a sequential investment problem. First, the firm has to enter the market and to do so it has to invest first in production capacity. Second, it has the option to undertake the preventive investment, which can be done either at the moment of market entry or at a later point in time. Then besides considering future damage when designing the liability rule, the effect of this rule on market entry should also be taken into account. This would require our current framework to be extended to a welfare maximization problem that the regulator should solve.

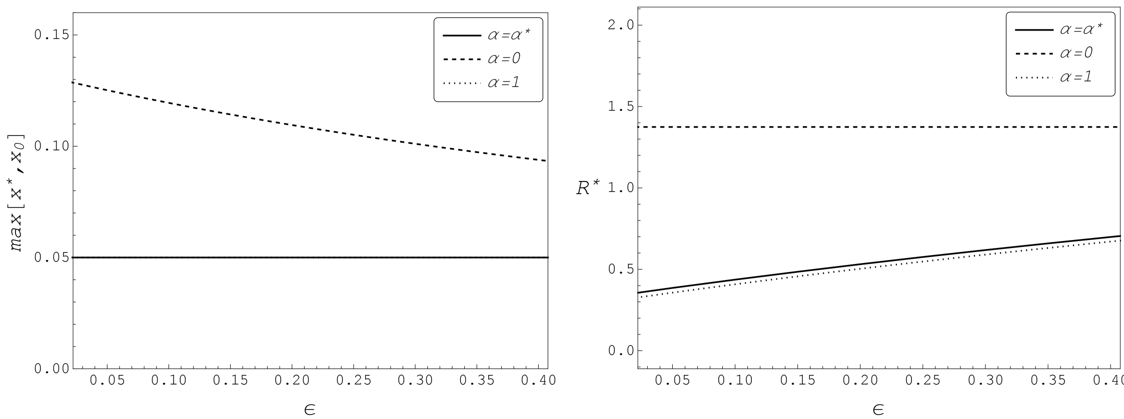
Data availability

No data was used for the research described in the article.

Appendix

Proof of Proposition 1. We start by noticing that

$$\mathbb{E}_{x,n}\left[\int_0^\infty \mathrm{e}^{-rt}\left(X_t\left(1-(1-\alpha+\epsilon)\sum_{i=1}^{N_t^B} U_i\right)\right)dt\right]$$
$$=\int_0^\infty \mathrm{e}^{-rt}\mathbb{E}_x\left[X_t\right]\mathbb{E}_n\left(1-(1-\alpha+\epsilon)\sum_{i=1}^{N_t^B} U_i\right)dt$$
$$=\frac{x}{r-\mu}\left(1-\frac{(1-\alpha+\epsilon)u(n+\lambda_B)}{r-\mu}\right). \tag{9}$$



(a) Optimal investment thresholds.

(b) Optimal preventive investment size.

Fig. 8. Optimal investment thresholds and optimal preventive investment size as functions of ϵ. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\sigma = 0.2$, $\mu = 0.02$, $a = 2$, $b = 1$, $u = 0.2$, and $Km = 0.5$].

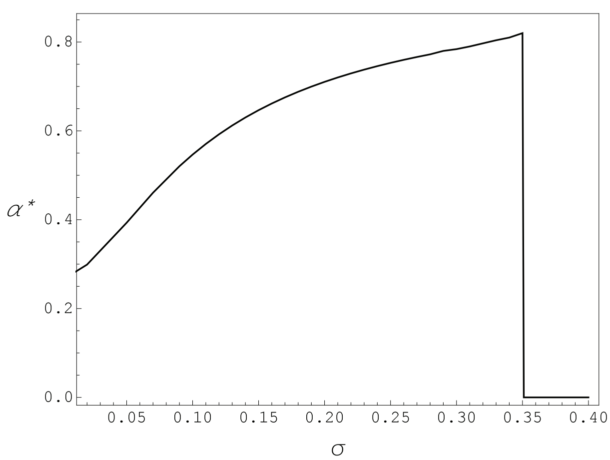


Fig. 9. Optimal liability rule a function of σ. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\mu = 0.02$, $a = 2$, $u = 0.2$, $Km = 0.5$, and $\epsilon = 0.1$].

Moreover, taking into account that T_i^B (resp., T_i^A) is distributed according to an Erlang distribution with parameters i and λ_B (resp., λ_A), and N_τ^B is Poisson distributed, with parameter $\lambda_B \tau$, given τ

$$\mathbb{E}_{x,0}\left[\alpha \sum_{i=n+1}^{N_\tau^B} \mathrm{e}^{-rT_i^B} Z_i\right] = \alpha \mathbb{E}[Z] \frac{\lambda_B}{r} \mathbb{E}_{x,n}\left[1 - \left(\frac{\lambda_B}{r+\lambda_B}\right)^{N_\tau^B}\right] \tag{10}$$

$$\mathbb{E}_{x,n}\left[\alpha \sum_{i=1}^{\infty} \mathrm{e}^{-r(\tau+T_i^A)} Z_i\right] = \alpha \mathbb{E}[Z] \frac{\lambda_A(R)}{r} \mathbb{E}_x[\mathrm{e}^{-r\tau}]. \tag{11}$$

Thus, in light of the previous calculations, we can obtain (3).

Proof of Proposition 2. Since the function g_α is smooth, one may start by finding the critical points of g_α. Indeed, it is a matter of calculations to see that

$$\frac{\partial g_\alpha}{\partial R} = 0 \Leftrightarrow \lambda_A'(R) = -\frac{(r-\mu)^2 r}{r(1-\alpha+\epsilon)ux + (r-\mu)^2 \alpha K E(Z)}. \tag{12}$$

We note that λ_A' is continuous, negative and strictly increasing, which implies that $\lim_{x\to+\infty} \lambda_A'(x) = a$, with $a \in \mathbb{R}_0^-$. Since the function λ_A is decreasing and positive, then $\lim_{x\to+\infty} \lambda_A(x) = b$ with $b \in \mathbb{R}_0^+$. Thus, $a = 0$, because, otherwise, $\lim_{x\to+\infty} \frac{\lambda_A(x)}{ax} = 1$, which is a contradiction. Consequently, the previous equation has no solution when $x \in]0, \hat{x}_\alpha]$, with $\hat{x}_\alpha = -\left(1 + \frac{\lambda_A'(0)\alpha K E(Z)}{r}\right) \frac{(r-\mu)^2}{(1-\alpha_A+\epsilon)u\lambda_A'(0)}$ and has a unique solution when $x \in]\hat{x}_\alpha, +\infty[$. Therefore, noticing that

$$\frac{\partial^2 g_\alpha}{\partial R^2} = -r(1-\alpha+\epsilon)ux + (r-\mu)^2 \alpha K E(Z) \lambda_A'(R) < 0, \tag{13}$$

the optimal R is defined as in Proposition 2.

Proof of Proposition 3. For $x < \hat{x}$ the function R_α^* is constant and equal to 0, therefore we focus our analysis on the case $x \geq \hat{x}$. Since $R_\alpha^*(x)$ satisfies Eq. (12), then we can use implicit differentiation to get

$$(R_\alpha^*)'(x) = \frac{(r-\mu)^2 r^2 (1-\alpha+\epsilon)u}{\left(r(1-\alpha+\epsilon)ux + (r-\mu)^2 \alpha K E(Z)\right)^2 \lambda_A''(R_\alpha^*(x))} > 0,$$

which proves that R_α^* is increasing. Alternatively, we can also rewrite $(R_\alpha^*)'$ as $(R_\alpha^*)'(x) = \frac{A}{(Ax+B)^2 \lambda_A''(R_\alpha^*(x))}$, where $A = \frac{(1-\alpha+\epsilon)u}{(r-\mu)^2}$ and $B = \frac{\alpha K E(Z)}{r}$. Regarding the concavity of the function R_α^*, we note that

$$(R_\alpha^*)''(x) = -A \frac{2A(Ax+B)\lambda_A''(R_\alpha^*(x)) + \lambda_A'''(R_\alpha^*(x))(Ax+B)^2 (R_\alpha^*)'(x)}{(Ax+B)^4 \left(\lambda_A''(R_\alpha^*(x))\right)^2}$$

Taking into account the definition of R_α^* one can see that

$$(R_\alpha^*)''(x) = -A^2 \left(\frac{2}{(Ax+B)^3 \left(\lambda_A''(R_\alpha^*(x))\right)} + \frac{\lambda_A'''(R_\alpha^*(x))}{(Ax+B)^4 \left(\lambda_A''(R_\alpha^*(x))\right)^3}\right) \tag{14}$$

$$= \frac{-A^2}{(Ax+B)^3 \left(\lambda_A''(R_\alpha^*(x))\right)} \left(2 + \frac{\lambda_A'''(R_\alpha^*(x))}{(Ax+B)\left(\lambda_A''(R_\alpha^*(x))\right)^2}\right) \tag{15}$$

Taking into account Assumption 1, we have that

$$2 + \frac{\lambda_A'''(R_\alpha^*(x))}{(Ax+B)\left(\lambda_A''(R_\alpha^*(x))\right)^2} > 2 - \frac{2}{(Ax+B)\lambda_A'(R_\alpha^*(x))} = 0,$$

and, consequently, R_α^* is concave.

To prove that $R_\alpha^*(x) \to +\infty$ as $x \to +\infty$, one can verify that Eq. (12) is true for every $x > \hat{x}_\alpha$. Therefore,

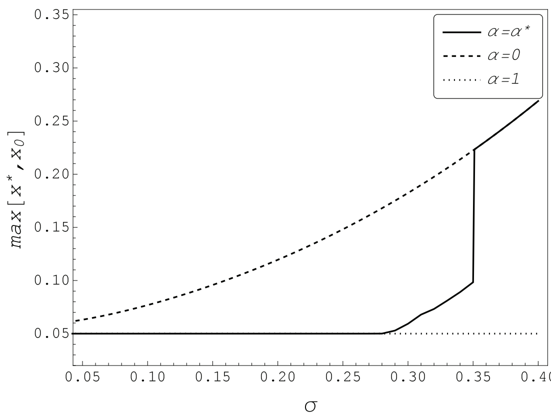
$$\lim_{x\to+\infty} (Ax+B)\lambda_A'(R_\alpha^*(x)) = -1. \tag{16}$$

Taking into account that $\lambda_A'(x) \to 0$ as $x \to +\infty$, we have that

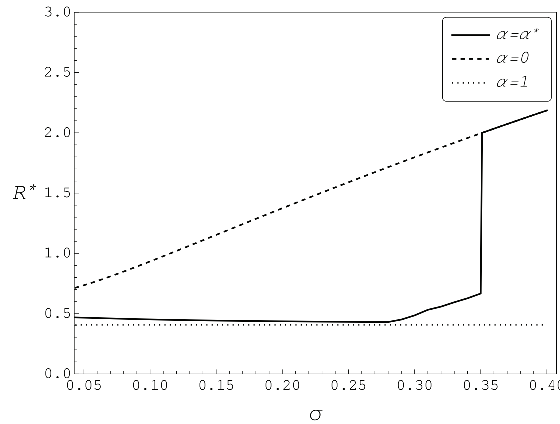
$$\lim_{x\to+\infty} \frac{(1-\alpha+\epsilon)ux}{(r-\mu)^2} \lambda_A'(R_\alpha^*(x)) = -1. \tag{17}$$

Therefore, Eq. (17) may be true only in case $\lim_{x\to+\infty} R_\alpha^*(x) = +\infty$.

Regarding the behavior of R_α^* as a function of α: let $w(\alpha) = -\frac{(r-\mu)^2 r}{r(1-\alpha+\epsilon)ux + (r-\mu)^2 \alpha K E(Z)}$, which is monotone function in α, increasing



(a) Optimal investment thresholds.



(b) Optimal preventive investment size.

Fig. 10. Optimal investment thresholds and optimal preventive investment size as functions of σ. [Parameter values: $X_0 = 0.05$, $n = 0$, $r = 0.05$, $\mu = 0.02$, $a = 2$, $u = 0.2$, $Km = 0.5$, and $\epsilon = 0.1$].

(decreasing) if $KE[Z](r-\mu)^2 > rux$ (otherwise), with

$$w(0) = -\frac{(r-\mu)^2 r}{r(1+\epsilon)ux} < 0$$

$$w(1) = -\frac{(r-\mu)^2 r}{r\epsilon ux + (r-\mu)^2 KE(Z)} < 0$$

Therefore, for $x > \frac{KE[Z](r-\mu)^2}{ru}$, R^* decreases with α, and increases otherwise. Finally, one can easily see that the function R^*_α does not depend on σ and n. Regarding μ, we can use the relationship (12) to compute that

$$\lambda''_A(R^*_\alpha)\frac{\partial R^*_\alpha}{\partial \mu} = \frac{2(1-\alpha+\epsilon)ux}{(r-\mu)^3}\left(\frac{1}{\frac{1-\alpha+\epsilon}{(r-\mu)^2}ux + \frac{\alpha KE(Z)}{r}}\right)^2$$

Taking into account the restriction in the parameters we have, the result follows immediately.

Proof of Proposition 4. Combining the computations in the proof of Proposition 1 and the fact that

$$\mathbb{E}\left[\left(\frac{\lambda_B}{\lambda_B + r}\right)^{N^B_{\tau^*}}\right] = \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\lambda_B}{\lambda_B + r}\right)^{N^B_{\tau^*}} |\tau^*\right]\right] = \mathbb{E}\left[e^{-\frac{\lambda_B \tau^* r}{r+\lambda_B}}\right], \quad (18)$$

the result follows immediately.

Proof of Proposition 5. We start by proving that equation $0 = \beta_1 g_0(x; R^*(x)) - g_0'(x; R^*(x))x \equiv h(x)$ has a unique solution $x^* > \hat{x}$. To do this, we notice that

$$g_0'(x; R^*(x)) = \frac{(1+\epsilon)u\lambda_B}{(r-\mu)^2} - \frac{(1+\epsilon)u\lambda_A(R^*(x))}{(r-\mu)^2}, \quad (19)$$

and, consequently, $\lim_{x\to+\infty} g_0'(x; R^*(x)) = \frac{(1+\epsilon)u}{(r-\mu)^2}(\lambda_B - \lambda_\infty)$, where $\lambda_\infty = \lim_{x\to+\infty} \lambda_A(x)$. This implies that

$$\lim_{x\to+\infty}\left(\frac{\beta_1 g_0(x; R^*(x)) - xg_0'(x; R^*(x))}{x}\right) = (\beta_1 - 1)\frac{(1+\epsilon)u}{(r-\mu)^2}(\lambda_B - \lambda_\infty) > 0.$$

Therefore, $\lim_{x\to+\infty} h(x) = +\infty$. Additionally, one can notice that the function h can be written as

$$h(x) = (\beta_1 - 1)\frac{(1+\epsilon)ux}{(r-\mu)^2}\left(\lambda_B - \lambda_A(R^*(x))\right) - \beta_1(R^*)'(x),$$

which implies that $h(\hat{x}) = 0$. Additionally,

$$h'(x) = (\beta_1 - 1)\frac{(1+\epsilon)u}{(r-\mu)^2}\left(\lambda_B - \lambda_A(R^*(x))\right) - (R^*)'(x),$$

and $h'(\hat{x}) = -(R^*)'(\hat{x}) < 0$. Therefore, the result follows if $h''(x) > 0$, because in this case $h(x) < 0$ for $x \in]\hat{x}, x^*[$ and $h(x) > 0$ for $x \in]x^*, +\infty[$.

In fact,

$$h''(x) = -(\beta_1 - 1)\frac{(1+\epsilon)u}{(r-\mu)^2}\lambda_A'(R^*(x))(R^*)'(x) - (R^*)''(x) > 0.$$

To prove that V_0 satisfies the HJB equation, we need to check that

$$rg_0(x; R^*(x)) - \mu x g_0'(x; R^*(x)) - \frac{\sigma^2}{2}x^2 g_0''(x; R^*(x)) \geq 0, \text{ for } x > x^* \quad (20)$$

$$Ax^{\beta_1} - g_0(x; R^*(x)) \geq 0, \text{ for } x < x^*. \quad (21)$$

To verify (20), we notice that

$$\begin{aligned} rg_0(x; R^*(x)) &- \mu x g_0'(x; R^*(x)) - \frac{\sigma^2}{2}x^2 g_0''(x; R^*(x)) \\ &= (r-\mu)\frac{(1+\epsilon)xu}{(r-\mu)^2}(\lambda_B - \lambda_A(R^*(x))) \\ &- rR^*(x) - \frac{\sigma^2}{2}x(R^*)'(x). \end{aligned}$$

Taking into account that

$$r = -\frac{\sigma^2}{2}\beta_1\beta_2 \quad \text{and} \quad \mu = \frac{\sigma^2}{2}(1 - \beta_1 - \beta_2), \quad (22)$$

where $\beta_2 = \frac{\left(\frac{\sigma^2}{2}-\mu\right) - \sqrt{\left(\frac{\sigma^2}{2}-\mu\right)^2 + 2\sigma^2 r}}{\sigma^2}$, one can check that (20) is equivalent to

$$-(\beta_1 - 1)(\beta_2 - 1)\frac{x(1+\epsilon)u}{(r-\mu)^2}(\lambda_B - \lambda_A(R^*(x))) + \beta_1\beta_2 R^*(x) - x(R^*)'(x) \geq 0$$

$$-\beta_2 h(x) + xh'(x) \geq 0,$$

where, in light of the arguments used to prove the uniqueness of solution to the equation $h(x) = 0$, it follows that the previous inequality is true for values $x > x^*$.

To prove the inequality (21) one may notice that, for $0 < x \leq \hat{x}$, we have $g_0(x; R^*(x)) = 0$, and, consequently,

$$Ax^{\beta_1} - g_0(x; R^*(x)) = (x^*)^{-\beta_1} g_0(x^*; R^*(x^*))x^{\beta_1} > 0,$$

because g_0' is increasing, as one may check in Eq. (19), and $g_0(\hat{x}; R^*(\hat{x})) = 0$. For $\hat{x} < x \leq x^*$, we have that

$$Ax^{\beta_1} - g_0(x; R^*(x)) = x^{\beta_1}\left((x^*)^{-\beta_1} g_0(x^*; R^*(x^*)) - x^{-\beta_1} g_0(x; R^*(x))\right).$$

Taking into account that

$$A\hat{x}^{\beta_1} - g_0(\hat{x}; R^*(\hat{x})) > 0, \quad A(x^*)^{\beta_1} - g_0(x^*; R^*(x^*)) = 0 \quad \text{and}$$

$$\left((x^*)^{-\beta_1} g_0(x^*; R^*(x^*)) - x^{-\beta_1} g_0(x; R^*(x))\right)' = x^{-\beta_1 - 1}h(x) < 0$$

inequality (21) is proved.

To finalize the proof, we highlight that $V_0(\cdot, n) \in C^2(]0, +\infty[\backslash\{x^*\})$ and $V_0(\cdot, n)$ is C^1 at the point x^*, because the parameter A and the threshold x^* satisfy the system of equations.

References

Akyildirim, E., Corbet, S., O'Connell, J.F., Sensoy, A., 2021. The influence of aviation disasters on engine manufacturers: An analysis of financial and reputational contagion risks. Int. Rev. Financ. Anal. 74, 101630.

Araman, V.F., Caldentey, R., 2009. Dynamic pricing for nonperishable products with demand learning. Oper. Res. 57 (5), 1169–1188.

Babich, V., Burnetas, A.N., Ritchken, P.H., 2007. Competition and diversification effects in supply chains with supplier default risk. Manuf. Serv. Oper. Manag. 9 (2), 123–146.

Bakshi, N., Kleindorfer, P., 2009. Co-opetition and investment for supply-chain resilience. Prod. Oper. Manage. 18 (6), 583–603.

Čokorilo, O., Gvozdenović, S., Vasov, L., Mirosavljević, P., 2010. Costs of unsafety in aviation. Technol. Econ. Dev. Econ. 16 (2), 188–201.

Dangl, T., 1999. Investment and capacity choice under uncertain demand. European J. Oper. Res. 117, 415–428.

Dawid, H., Muehlheusser, G., 2022. Smart products: Liability, investments in product safety, and the timing of market introduction. J. Econom. Dynam. Control 134, 104288.

Defourny, B., 2018. Optimality of an affine intensity policy for maximizing the probability of an arrival count in point-process intensity control. Oper. Res. Lett. 46 (1), 51–55.

Dixit, A., Pindyck, R., 1994. Investment under Uncertainty. Princeton University Press.

Garg, A., Deshmukh, S., 2006. Maintenance management: Literature review and directions. J. Qual. Maint. Eng. 12 (3), 205–238.

Hay, B., Spier, K.E., 2005. Manufacturer liability for harms caused by consumers to others. Amer. Econ. Rev. 95 (5), 1700–1711.

Huberts, N.F.D., Huisman, K.J.M., Kort, P.M., Lavrutich, M.N., 2015. Capacity choice in (strategic) real options models: A survey. Dynam. Games Appl. 5 (4), 424–439.

Huisman, K.J., Kort, P.M., 2015. Strategic capacity investment under uncertainty. Rand J. Econ. 46 (2), 376–408.

Jin, X., Li, L., Ni, J., 2009. Option model for joint production and preventive maintenance system. Int. J. Prod. Econ. 119 (2), 347–353.

Kim, S.-H., Tomlin, B., 2013. Guilt by association: Strategic failure prevention and recovery capacity investments. Manage. Sci. 59 (7), 1631–1649.

Knemeyer, A.M., Zinn, W., Eroglu, C., 2009. Proactive planning for catastrophic events in supply chains. J. Oper. Manage. 27 (2), 141–153.

Kort, P., Lavrutich, M., Nunes, C., Oliveira, C., 2021. Preventing environmental disasters in investment under uncertainty. Environ. Res. Econ. 1–22.

Marucheck, A., Greis, N., Mena, C., Cai, L., 2011. Product safety and security in the global supply chain: Issues, challenges and research opportunities. J. Oper. Manage. 29 (7), 707–720, Special Issue: Product Safety and Security on the Global Supply Chain.

Rivera-Gómez, H., Gharbi, A., Kenné, J.P., 2013. Joint production and major maintenance planning policy of a manufacturing system with deteriorating quality. Int. J. Prod. Econ. 146 (2), 575–587.

Sassone, A., Grosso, M., Poncino, M., Macii, E., 2016. Smart electronic systems: An overview. Smart Syst. Integr. Simul. 5–21.

Shavell, S., 2020. On the redesign of accident liability for the world of autonomous vehicles. J. Legal Stud. 49 (2), 243–285.

Squalli, J., Saad, M., 2006. Accidents airline safety perceptions and consumer demand. J. Econ. Finance 30 (3), 297–305.

Tomlin, B., 2006. On the value of mitigation and contingency strategies for managing supply chain disruption risks. Manage. Sci. 52 (5), 639–657.

Tomlin, B., 2009. Disruption-management strategies for short life-cycle products. Nav. Res. Logist. 56 (4), 318–347.

Wang, Y., Gilland, W., Tomlin, B., 2010. Mitigating supply risk: Dual sourcing or process improvement? Manuf. Serv. Oper. Manag. 12 (3), 489–510.

Wang, H., Pham, H., 2006. Reliability and Optimal Maintenance. Springer-Verlag, London, UK.

Xu, Y., Zhu, L., Pinedo, M., 2020. Operational risk management: A stochastic control framework with preventive and corrective controls. Oper. Res. 68 (6), 1804–1825.