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International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

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To cite this article: Kristin Krogh Arnesen & Anita Valenta (10 Dec 2023): Lesson plays as an approach to learning to teach proving, International Journal of Mathematical Education in Science and Technology, DOI: 10.1080/0020739X.2023.2288822

To link to this article: https://doi.org/10.1080/0020739X.2023.2288822

Published online: 10 Dec 2023.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
https://doi.org/10.1080/0020739X.2023.2288822

Lesson plays as an approach to learning to teach proving

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ABSTRACT

We suggest that writing lesson plays can be a valuable approach in learning to teach proving in teacher education. A lesson play is an imagined discussion between a teacher and some students about a particular mathematical content, written verbatim. In this paper, we investigate novice teachers' opportunities to work on teaching practices related to proving by generic examples through writing lesson plays. We analyse lesson plays written by 14 novice teachers, and our analysis reveals that several teaching practices are involved. Some of the teaching practices are related to teaching mathematics in general, some are more specifically related to teaching of proving, and some of the practices are particular to teaching of proving by generic examples.

ARTICLE HISTORY
Received 15 June 2023

KEYWORDS
Teacher education; approximation of practice; lesson play; proving; generic example

1. Introduction

Proving is an important part of school mathematics (eg Lampert, 2001; Stylianides, 2007), which is reflected in several curricula from different parts of the world, such as in the USA (NCTM, 2000) and Norway (Kunnskapsdepartementet, 2019). However, students face challenges when learning to prove – a fact that many studies have shown (eg Harel & Sowder, 1998, 2007). Mathematics teachers can provide crucial support for students' learning, through teaching practices such as facilitating the need to prove, helping students identify the structure of a proof, and distinguishing between correct and incorrect arguments (eg Stylianides & Ball, 2008; Zaslavsky et al., 2012). However, the teaching of proof is complicated (Stylianides et al., 2017), and studies have shown that even teachers who use resources that offer rich opportunities for proving struggle in promoting crucial aspects in the classroom (Bieda, 2010). There should therefore be an emphasis on learning how to teach proof in teacher education, thereby helping future teachers to overcome such struggles. Following Grossman, Hammerness, et al. (2009), we suggest that the work on proving in teacher education should concentrate on teaching practices.

One recommended approach to promoting teaching practices in teacher education is by using *approximations of practice* to support novices' learning to teach (Grossman, Compton, et al., 2009). Here, an approximation of practice means that novices receive the opportunity to enact some teaching practices and learn from them, but in a context of reduced complexity. Earlier approximations of teaching proving in teacher education

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include the collaborative planning of lessons, supported by a teacher educator, which is later implemented in the mentor's class for pre-service teachers (Buchbinder & McCrone, 2020; Skott et al., 2017; Stylianides et al., 2013) and with own students for in-service teachers (Herbert & Bragg, 2020; Kazemi et al., 2021). As pointed out in all these studies, this kind of approximation of practice gives the opportunity to learn about important aspects of the teaching of proving. However, regarding the approximations made with pre-service teachers, the complexity of teaching proving in the classroom is still high (Buchbinder & McCrone, 2020; Skott et al., 2017; Stylianides et al., 2013). Complexity can be reduced by the increased involvement of a teacher educator (as in Kazemi et al., 2021; see also Lampert et al., 2013), but this can be demanding and expensive (in terms of teacher educators' time spent) as the number of pre-service teachers grows. Approximations of practice, which can reduce complexity and still come at a low cost, would therefore be welcome in teacher education, particularly when it comes to complex topics such as proving.

In the present paper, we suggest that writing *lesson plays* can be an approximation of teaching proving in primary school. By lesson plays, we mean imagined discussions between a teacher and students, written verbatim. Like Grossmann, Compton et al. (2009), Herbst et al. (2011) also point out that it can be beneficial for novice teachers to engage in teaching practices in designed settings in teacher education. Lesson plays have been framed as an approximation of practice (Marmur & Zazkis, 2022; Rougée & Herbst, 2018), but these studies have not focused on the teaching of proving. Buchbinder and Cook (2018), on the other hand, use lesson plays as a tool to uncover pre-service teachers' mathematical knowledge for proving. However, no previous studies have investigated the potential of lesson plays as an approximation of teaching proving. Our study contributes to knowledge development in this area by asking the following research question:

> What teaching practices related to proving by generic examples in school surface in novice teachers' lesson plays, and how can they play out?

We use the term *novice teachers* to denote in-service and pre-service teachers who are new to teaching proving and who are engaged in learning it. Moreover, proving is a broad topic, and we have chosen to narrow the focus of our research question to proving by generic examples. Generic examples are of special relevance for primary school due to their 'reduced level of abstraction' (Dreyfus et al., 2012) and their potential to explain and convince (Rowland, 1998). To novice teachers, however, generic examples are challenging to construct (Rø & Arnesen, 2020; Yopp & Ely, 2016) and are often seen as inferior to more formal, algebraic proofs (Dogan & Williams-Pierce, 2021; Kempen & Biehler, 2019). This tension between the promotion of generic examples in primary school and the known challenges for novice teachers regarding this mode of reasoning underpins our decision to focus on proof by generic examples in this study.

To answer the research question, we analyse lesson plays written by the participants in a professional development course, where reasoning and proving was one of the study topics. Writing a lesson play leading to a generic example was part of an assignment in the course. Below, we present earlier studies on the use of lesson plays in teacher education, and we argue for why writing lesson plays can be seen as an approximation of the practice of teaching proving. This is followed by a discussion of the concept of generic examples – its structure, affordances and challenges – and what teaching practices related to generic examples can be.

2. Lesson plays in mathematics teacher education

Writing imagined dialogues as part of a proving task has been discussed in several studies in the field of mathematics education. Askevold and Lekaus (2018) asked fifth and sixth-grade students to continue an imagined discussion between some students on a given argumentation task, and they analysed the students' argumentation and understanding shown in the dialogues. Similarly, Koichu and Zazkis (2013, see also 2018) designed a task for undergraduate students to write a script for a dialogue between two people about a given historical proof. The researchers then used scripts to analyse the participants' understanding of mathematical concepts and aspects of proof. Gholamazad's (2007) pre-service teachers wrote dialogues while working on a proving task, with the goal of the design being to direct the participants' attention more to *why* than to *how* in proving. The imagined dialogues in these studies were between students or people interested in the given proof. In contrast, we are interested in imagined dialogues on particular mathematical content between a teacher and one or more students. Following Zaskis and her colleagues (2009, 2013) we call such dialogues, when written verbatim, *lesson plays*. Below, in the presentation of previous research on imagined dialogues fitting this definition, we consistently use the term lesson play, even though an alternative term (script, scriptwriting, or imagined/hypothetical/fictional classroom discussion) was used in the study being discussed.

We only found two studies that discuss lesson plays concerned with proving. Buchbinder and Cook (2018) analyse pre-service teachers' mathematical knowledge for proving, as revealed in their lesson plays concerning a proof in geometry. Additionally, Cusi and Martignone (2022) discuss the potential of lesson plays as a task in the work on proving in teacher education. Like Buchbinder and Cook (2018), Cusi and Martignone relate the writing of lesson plays to the components of mathematical knowledge for teaching. However, lesson plays have been used in several studies concerning aspects of teaching mathematics other than proving, which have shown it to be an approach that can provide an opportunity for in-depth discussions of the crucial aspects of mathematics teaching (Zazkis et al., 2009; Zazkis et al., 2013). In their study, Zazkis et al. (2013) point out that pre-service teachers' understanding of mathematical concepts and ideas had clearly appeared in their writing, together with pedagogical moves such as the types of questions, planned responses to student reasoning and inclusion of student ideas. Accordingly, lesson plays have been used to obtain insights into teachers' understanding and knowledge related to several mathematical topics (Arnesen et al., 2017; Crespo et al., 2011; Dahl et al., 2019; Kontorovich, 2018; Zazkis, 2014; Zazkis & Cook, 2018). Marmur and Zazkis (2022) use lesson plays as a tool to investigate how pre-service teachers interpret and deal with mathematical ambiguity, and Shure et al. (2022) use them to analyse teacher moves to advance students' academic literacy. On the other hand, Mamolo (2018) uses lesson plays to analyse the more general aspects of teaching mathematics, such as what pre-service teachers notice or overlook when assigning roles to the students in the lesson play, along with what kinds of expectations they have for the teacher's role.

Writing lesson plays not only provide a lens into novice teachers' knowledge, understanding, and/or interpretations, as emphasised in the studies above, but it can also be a tool in teacher education. Lim et al. (2018) point out that pre-service teachers' lesson plays improved through the repeated writing of the lesson plays and feedback from the teacher educator. In her study, Crespo (2018) shows that, through the writing and discussion of lesson plays, pre-service teachers have probed into their teaching practices, such as making sense of students' contributions and responding to students. Following Lim et al. (2018) and Crespo (2018), we consider lesson plays in the present study as an approach to work on teaching practices. As discussed above, writing lesson plays has been used in a few studies related to proving (Buchbinder & Cook, 2018; Cusi & Martignone, 2022). Instead of relating lesson plays to mathematical knowledge for teaching, as is the case for most previous research involving lesson plays, we investigate the teaching practices related to proving with which novices can engage by writing lesson plays.

As suggested by Rougée and Herbst (2018; see also Marmur & Zazkis, 2022), we argue that writing lesson plays can be considered as an approximation of practice. Hammerness et al. (2005) point out that the development of a disciplined vision of practice that can guide decision-making is crucial for novices' learning to teach. We suggest that writing lesson plays can support novices to this end. The teaching complexity is significantly reduced in writing lesson plays because the teaching itself is not enacted – no real students are making unexpected suggestions and there are no interruptions of any kind. On the other hand, by writing lesson plays, novices can still enact and rehearse many important practices, such as formulating a question to direct (imagined) students' attention or to close the discussion, and they can concentrate on learning to promote mathematical points in teaching (see, eg Crespo, 2018; Dahl et al., 2019). In the current study, we investigate the learning opportunities that are involved in using lesson plays in teacher education, particularly those opportunities to learn teaching practices related to working on generic examples. In the following, we discuss what teaching proving, particularly teaching of generic examples, can entail.

3. Theoretical background on (the teaching of) generic examples

In this paper, we study teaching practices related to the teaching of proving by generic examples. Nachlieli and Elbaum-Cohen (2021) suggest that, in the work of teaching, the teacher interprets the different situations that arise in a mathematical classroom as particular tasks, and then acts to perform these tasks. Here, the task can have both a mathematical and pedagogical nature. They argue further that we can regard a teaching practice as 'the task as seen by the performing teacher together with the procedure she executed to perform that task' (p. 7). What kinds of teaching practices can be relevant and important in teaching proving by generic examples is, even in the relatively limited framing of the current study, a highly multifaceted issue. Below, we report upon relevant knowledge from previous literature that can be related to teaching practices. We begin, however, by defining generic examples and considering some known challenges related to the learning and understanding of generic examples.

3.1. Proving by generic examples – properties and challenges

A *generic example* is an argument that is based on one or more examples, and points to some underlying mathematical structures that make the argument generalisable to all possible examples (see, eg Mason & Pimm, 1984). Many authors, including us, consider generic examples to be valid proofs, at least in an educational setting (see, eg Rowland, 1998; Rø &

Arnesen, 2020; Stylianides, 2008). An example of a proof by generic example is given in Figure 1.

We use the argument in Figure 1 to highlight some properties of generic examples. As opposed to an empirical argument (where simple checking of the claim on some examples leads to the conclusion: 'It works for the example(s), so then it must always be true'), a generic example refers to some mathematical features, such as definitions and properties that are already known, that are not particular to the example. Yopp and Ely (2016) describe this as the 'features of the example that are either defining features of the objects in the (original) claim's hypothesis or new features already established in the previous [part of the argument] (perhaps through prior results)' (p. 46). In the argument in Figure 1, mathematical features are referred to: a *definition of an even number* (a natural number is even if, and only if, it has 2 as a factor) is used twice (first to rewrite the addends, and then to identify the resulting sum as even), and the *distributive property* is used to manipulate the expression. Furthermore, in describing the process of proving, Raman (2003) emphasises the notion of key idea: 'a heuristic idea which one can map to a formal proof with appropriate sense of rigor', which 'show[s] *why* a particular claim is true' (p. 323). Identifying a key idea is crucial when constructing a generic example, even if the goal is not to write a formal proof. In the above example, the key idea is to consider the given definition of even numbers and to apply the distributive property (which can be mapped to a formal proof: the sum of any two even numbers can be written as $2n + 2m$ for some natural numbers n and m, and we get $2n + 2m = 2\,(n + m)$, which is also by definition an even number). Finally, generic examples have a 'layered' structure (Rø & Arnesen, 2020): the key idea must be generalised to all possible examples. In the argument in Figure 1, this is seen in the final sentence.

Several challenges are known when students are trying to prove by generic examples. First, there is the question of which example to choose. It would not, for instance, be particularly wise to show the claim of Figure 1 by using $2 + 2$ as an example – these numbers could be considered 'too special' (see Yopp & Ely, 2016, for an overview of this challenge). Second, it is not always easy to judge whether an example-based argument provided by a student uses the example generically or empirically. Reid and Vallejo Vargas (2018) point out that the author and reader of an argument might view the mathematical content in different ways. Being the authors of the proof in Figure 1, we considered the example in a generic way. A reader, however, might see the very same argument as 'just an example', or, in Balacheff's (1988) terms, a 'crucial experiment': it works for 14 and 8, two arbitrary

Claim: The sum of any two even numbers is an even number.

Proof: We consider the example 14 + 8 = 2 . 7 + 2 . 4 = 2 . (7 + 4) = 2 . 11

which meets the assumptions in the claim because 14 and 8 are even numbers.

14 is even because it is $2 \cdot 7$ and 8 is even because it is $2 \cdot 4$.

Thus, we have $14 + 8 = 2 \cdot 7 + 2 \cdot 4 = 2 \cdot (7 + 4) = 2 \cdot 11$, which is an even number because it has 2 as a factor. This argument would work for any sum of even numbers, because you can always factor out a 2 in the sum.

Figure 1. A proof by generic example.

even numbers, so then it probably always works. Indeed, the term 'key idea', as it appears in Raman (2003), is closely connected to the actor's thinking, which is later described as being 'based on an insight gained from an informal, private way of understanding a concept' (Hanna & Knipping, 2020, p. 9), and it is not given that the reader will perceive it in the same way.

A related challenge of teaching proving is that a student might consider an empirical argument to be a valid proof (for an overview, see Reid & Knipping, 2010). However, as pointed out by Reid and Knipping (2010, p. 62), results indicating such misconceptions do not necessarily mean that students really accept empirical arguments as mathematical proof. This is supported by studies showing that learners use empirical arguments even in cases when they know that such arguments are not mathematically valid because they struggle to find another way to verify a given statement (Stylianides & Stylianides, 2009; Weber, 2010). Although the challenge of empirical arguments is related to proving in general, it is particularly important to attend to when considering the learning of generic examples, because both generic examples and empirical arguments are example-based. As elaborated on in the previous paragraphs, it can be hard for students to see the differences between the two forms of argumentation.

3.2. Teaching practices related to the teaching of proving by generic examples

As outlined above, proofs by generic examples are arguments that make use of an example/examples to say something about a large (usually infinite) class of examples; yet many learners of proof struggle to distinguish this concept from (mathematically invalid) empirical arguments. Different design experiments have aimed to develop learners' scepticism towards empirical arguments (eg Brown, 2014; Stylianides & Stylianides, 2009). Examples can be useful in proving, but students' attention needs to be directed to searching for a key idea and understanding it as such, along with generalisation of the key idea and understanding it as such. Recently, Ellis et al. (2022) have identified several types of instructional support that can help learners to move from empirical arguments to a more structural use of examples: experiencing the need for verification, fostering contextual interpretation, fostering reflection and justification, and fostering pattern exploration. Similar teaching practices for supporting students to move from empirical arguments to proofs are also emphasised in several other studies (see, eg Brown & Coles, 2000; Valenta & Enge, 2022).

Shaughnessy et al. (2019) point out that, to give good support to students' learning, a mathematical discussion needs to be based on the students' presentation of their thinking, students need to be oriented towards each other's ideas, and, together with the teacher, the students need to contribute to a collective knowledge building towards a given learning goal.

Discussing the teaching of proving, Stylianides (2007) describes the instructional practices for supporting students in their initial arguments towards more mathematically correct arguments that evolve into proofs. He points out that the teacher's support can concern: (1) the starting point in the argument (eg helping students to identify and express definitions and properties that are taken as known from before and that can be relevant); (2) the mode of argumentation (eg supporting the move from empirical to generic example, as discussed above); and (3) supporting students' representation of the argument (eg helping students to express their thinking and generalisations). Capturing an even wider scope of mathematical reasoning and building on empirical data and literature (eg Fraivillig et al., 1999), Ellis et al. (2019) analyse teacher's moves that can support students' reasoning in the *Teacher Moves for Supporting Student Reasoning* (TMSSR) framework. They classify the identified moves into four categories: eliciting, responding to, facilitating, and extending student reasoning. Within each category, there are moves of low and high potential. For example, *encouraging evaluation* is a move of low potential, while *pressing for justification* and *pressing for generalisation* are moves that have high potential for extending student reasoning, and in particular supporting work with argumentation and proof. The TMSSR framework must be seen as a tool to 'foster an inquiry-oriented environment' (Ellis et al., 2019, p. 107) that strengthens collective reasoning, and it is not specially made for chiselling out proofs. The moves mentioned above are those that are most specifically about proof or justification in the framework.

4. Methods

In the current study, we are interested in a detailed understanding of a specific phenomenon and participant actions (see, eg Cohen et al., 2011, pp. 219–223) – that is, the phenomenon of teaching practices that can surface in novice teachers' lesson plays. In particular, we used the data – lesson plays written by novice teachers – to identify teaching practices related to the teaching of proving that can surface in such lesson plays.

4.1. Context and data

The data was collected from a professional development course for primary school. In Norway, the majority of teachers in primary schools are general teachers, and they teach several subjects. Mathematics is a compulsory subject in teacher education for primary school, and all teachers have completed at least 30 ECTS credits of mathematics/methods for teaching mathematics, and some have more than that (60 ECTS credits, or even a master's degree in mathematics education).

The course was organised for mathematics teachers in three primary schools (grades 1–7) in one Norwegian municipality, but assistants[1] (ie persons without teacher education) employed at the schools could also attend. In all, there were 14 participants on the course: 10 of them only had the compulsory course in mathematics in their education (30 ECTS credits), one had studied university mathematics for two years before becoming a teacher, and three were assistants with no teacher education or any mathematical background. All the participants worked in grades 3–7 while attending the course. As our interest in this paper is mainly on writing lesson plays as an approach to proving in teacher education – and not on aspects related to this particular group of course participants – we do not further discuss the participants' background. We call them novice teachers because they were new to the teaching of proving. The course was supposed to be organised as six one-day sessions during the school year, with assignments given between the sessions. However, most of the sessions were held online because of Covid-19 restrictions in higher education. Reasoning and proving in primary school was among the topics in the course, in addition to algebraic reasoning, teaching arithmetic and leading mathematical discussions. The session devoted to reasoning and proving was a theoretical session, where the notion of reasoning and proving was introduced with examples that can be relevant for primary school (based on the work of Stylianides, 2007), different types of arguments, both valid and not-valid, were discussed (as discussed in Stylianides, 2008). Particularly, the notion of proof by generic example (as described in Rø & Arnesen, 2020) was introduced and exemplified. Furthermore, teacher moves for supporting students' reasoning (from Ellis et al., 2019) were introduced. The literature provided for the course consisted of texts in Norwegian in which the notions were adopted from the original papers and exemplified.

After the reasoning and proving session, the participants were given an assignment to complete at home over a period of four weeks. A list of proving tasks was provided (Appendix) and the participants were asked to choose one task that was appropriate for their students and complete the following three parts of the assignment:

(1) Outline different types of arguments that can be given by students (empirical argument, generic example, demonstration and rationale[2]) in the chosen task.
(2) Let your own students[3] (a whole class or just some students) work on the task in groups of three to four, without feedback, and collect the written work produced by the groups. In the assignment, present some of the students' written work, analyse the type of arguments, and point out what is missing for the argument to be mathematically valid.
(3) Based on the written work analysed in part 2, write an imagined discussion with the students that will eventually result in an argument by generic example. The written discussion is supposed to be 'ideal' in the sense that it builds on students' work and would come up with a 'really good argument by generic example'. During the discussion, teacher moves (from Ellis et al., 2019) are to be used, and they should be marked in parentheses in the discussion. The discussion should be 1.5–2.5 pages long, and drawings/pictures of the blackboard should be included.

When designing the task, the two teacher educators giving the course – one of them being one of the authors of the present paper – made several pedagogical considerations. The overall goals of the assignment were to understand the idea of a generic example, to imagine how one can lead a class discussion towards a generic example, and to use the literature along the way. The proving tasks that the participants could choose from were designed so that intuition about the key idea should be within reach for both the participants and their students, all of whom had little experience of proving. The aim of part 1 was for the novice teachers to notice the similarities and differences between different types of arguments. The participants were teachers/assistants and thus had the opportunity to collect student work, as they were asked to do in part 2, and the teacher educators assumed that this would make the assignment and literature feel more relevant. However, the participants were *not* asked to enact the discussion in the classroom because: (1) the recording of data could be problematic because of GDPR[4], and (2) once enacted, it was the teacher educators' experience that it was more difficult to discuss the alternative progression of the discussion with novice teachers, which would be the goal of the feedback given to the novice teachers. That, as well as earlier experiences of lesson plays, was the reason why the teacher educators decided to ask for an imagined discussion instead. In this case, the teacher educators assumed that the novice teachers could focus on the mathematical content of the discussion, forget other things, and be more open to reflection after receiving feedback on the assignment.

In the present study, we were interested in teaching practices related to proving that could be found in writing lesson plays. To answer the research question, we analysed the

14 lesson plays written as part 3 of the assignment described above. However, sometimes, we also had to read parts 1 and 2 to make sense of the lesson plays.

4.2. Data analysis

The lesson plays were analysed using qualitative content analysis (Elo & Kyngäs, 2008; Mayring, 2015). Content analysis has been described as tending towards the 'quantifying' end on the 'continuum of the qualitative methodology' (Vaismoradi et al., 2013, p. 399) and as a 'hybrid qualitative-quantitative approach within the mixed methods approach' (Mayring, 2015, p. 366). Content analysis of the text therefore provided the opportunity to both describe and quantify data, here given in terms of categories that describe the phenomenon under scrutiny (Elo & Kyngäs, 2008). The analytical approach was both inductive (in the sense that we had no predefined set of teaching practices) and deductive, because we were strongly influenced by the literature described in the theoretical background.

In the process of coding, we used the operationalisation of 'teaching practices' provided by Nachlieli and Elbaum-Cohen (2021), who view teaching practices as *actions the teacher does, together with what seems to be the purpose of these actions*. Thus, we identified the action performed by the teacher in the lesson play and also considered what the teacher was trying to accomplish (Nachlieli & Elbaum-Cohen, 2021, p. 7). We read the material individually and took notes on the teaching practices we identified. Some teaching practices were informed by literature, in the sense that we looked for practices that sought to overcome known challenges of teaching of generic examples. Other teaching practices were identified because they appeared in some, but not all, lesson plays, or because their enactment varied between the lesson plays. We only recorded teaching practices that were related to the teaching of proof. After the individual analysis of the material, disagreements were resolved, which led to a minor refinement of the set of codes and their descriptions. For example, during the individual coding process, we focused on the teaching practice that involved promoting a mathematical feature in an example that could lead to a proof, trying to emphasise this feature and make it transparent to the students. However, exactly what this 'promoting' consisted of was interpreted differently. During the discussion, we agreed to call the teaching practice *Facilitating students' understanding of the key idea*, defining it as 'calling attention to a property or structure in the example that can lead to a proof by generic example, repeated in more than one turn in the discussion'. Here, students' understanding of the key idea as generic is the *purpose* of the teaching practice, and discussing the key idea in several turns is the *action*. The reason for requiring more than one turn was to identify a teaching practice with the potential to overcome the challenges described in the theoretical background, in particular keeping in mind that students may not see an example as generic even though the teacher does. For other practices, however, such as *Extending proving task*, we considered it not necessary to be discussed in several turns – the lesson play could, for example, end with the teacher suggesting a new task to investigate. After agreeing on the descriptions of the teaching practices, we updated the frequency of lesson plays that included each teaching practice. As the final step in the process of analysis, we grouped the teaching practices into three categories, reflecting whether they were intrinsic to the teaching of generic examples in particular or related to the teaching of proof or mathematics more generally. An overview is given in Table 1. The name of the teaching practice

Table 1. Teaching practices identified in the lesson plays. Numbers in parentheses indicate the number of lesson plays (out of the total of 14) in which the teaching practice was identified.

Teaching practice	Description
Teaching practices specific for teaching of generic examples	
Facilitating an example-based discussion *(9)*	Promoting an example as the starting point for discussing the key idea
Facilitating students' understanding of the key idea *(13)*	Calling attention to a property or structure in the example that can lead to a proof by generic example, repeated in more than one turn in the discussion
Facilitating students' understanding of the generalisation of the key idea to all examples *(12)*	Calling attention to the lifting of a property or structure identified in one example to all examples, repeated in more than one turn in the discussion
Teaching practices related to the teaching of proof	
Promoting scepticism towards empirical arguments *(8)*	Calling attention to the fact that empirical arguments are not valid, in more than one turn in the discussion
Supporting students' mathematical reasoning *(13)*	Using moves to elicit, respond to, facilitate and extend student reasoning (Ellis et al., 2019)
Extending proving task (*2)*	Extending the given proving task with a new general claim for the students to investigate
Promoting proof-related norms *(2)*	Introducing or strengthening proof-relevant norms
Teaching practices related to mathematics teaching	
Involving students *(10)*	Giving the students the role of main contributors to the discussion (eg key idea)
Using visual representations *(11)*	Using visual representations to emphasise some crucial mathematical aspects

indicates what we identified as the purpose of the teaching practice, while the description points to the actions.

4.3. Some methodological remarks

During the research process, we were guided by the criteria for quality in qualitative research (Tracy, 2010). In particular, related to the criterion of *rich rigour*, although the data consisted of 'only' 14 lesson plays, we were able to sufficiently answer the research question. This is because the lesson plays were comprehensive (usually two to three pages long) and because the process of analysis – as outlined above – was fine-grained, so each lesson play gave substantial contributions to the content analysis. Moreover, the present study's *credibility* and *sincerity* were ensured by reporting on the discussions between the researchers, and the researchers' roles during the process. Concerning the criterion of *resonance*, although the research was carried out in a limited context, both regarding the participants and the topic of the task, we believe that the results can be transferable to other contexts – that is, other novice teachers and other aspects of the teaching of proof. In teaching other types of proof than generic example, there will be some differences compared to generic example (eg examples need not be emphasised), but also similarities – the key idea of the proof, deduction and the nature of mathematical proof will need to be promoted, and teachers will need to support students' generalisation and development of valid arguments, no matter what kind of proof is discussed.

We remark here that in the lesson plays that we analysed, the novice teachers were explicitly asked to use teacher moves for supporting students' reasoning from the TMSSR framework (Ellis et al., 2019). It is therefore reasonable to expect that such moves would surface in our analysis (coded as the teaching practice *Supporting students' mathematical reasoning*, see Table 1). However, for a lesson play to be counted as using this teaching practice, it was necessary that it employed TMSSR moves at the right time and that the moves were specified according to the mathematical content in the discussion.

5. Findings

We identified several teaching practices for teaching of generic examples in the novice teachers' lesson plays. In Table 1, the teaching practices are listed together with a short description.

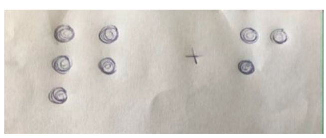
In the following, we elaborate on the teaching practices, giving examples from the lesson plays to illustrate how the teaching practices can play out. For the sake of simplicity, we typically refer to the content of the lesson plays in the manner of 'things that happened', although the lesson plays are just imagined discussions. The TMSSR moves (or 'MR moves', where MR is an abbreviation for mathematical reasoning) referred to in parentheses in the excerpts have been written by the novice teachers as part of the assignment. Apart from being translated into English, the excerpts from the lesson plays have not been edited, which means that student and teacher pseudonyms are not consistent across the excerpts. To identify the lesson plays, we have labelled the novice teachers who wrote the lesson plays with an initial, from A to N.

5.1. Facilitating an example-based discussion

This teaching practice involved the teacher promoting the development of an argument based on an example, as opposed to using examples only to exemplify an already established key idea or keeping the entire discussion at a general level. The teaching practice was included in 9 of the 14 lesson plays, and is demonstrated below by novice teacher A. So far in this lesson play, the class had collectively claimed that the conjecture – 'the sum of two odd numbers is an odd number' – was never true, and they looked at some examples that supported this claim without showing why. The discussion continued as follows:

Teacher: (…) If we think of one of the examples student A came up with. Can you repeat one of the calculations you came up with, student A?

Student A: For example, $5 + 3 = 8$.



The teacher draws the quantities on the board.

Here, we see that the teacher was asking for an example, which they then re-represented on the board. In the lesson play, this example was used to introduce and discuss a key idea: the students did not focus on one particular example by themselves, but it was the teacher's actions that made sure that one example was put on the board and subjected to further discussion.

5.2. Facilitating students' understanding of the key idea

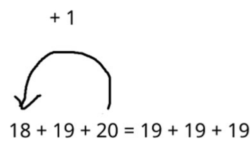
In 13 of the 14 lesson plays, the teachers spent some time discussing a key idea, often involving students in the process of both introducing the key idea and refining or repeating it. One example came from novice teacher G, who chose the task about the sum of three consecutive numbers being divisible by 3. As in the excerpt from novice teacher A above, novice teacher G directed the students' attention towards a specific example, $18 + 19 + 20$, which a student claimed to be divisible by 3 because 'there were three equal parts'. The discussion continued as follows:

Teacher: I saw something exciting in what you'd written at the end. It was that the answer was always the number in the middle. What did you mean by that? (Eliciting ideas)[5]

Student: Yes, because, in all the examples we tried, the answer was the number in the middle when we divided by 3. As here, the answer was 19 when the numbers were $18 + 19 + 20$

Teacher: If we look at the numbers $18 + 19 + 20$ one more time, are there any ways we can already make it three equal groups? (Providing guidance)

Student: If we take 1 from the largest number and move it to the smallest number, we get $19 + 19 + 19$

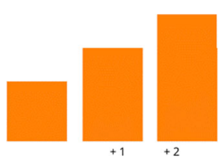


Teacher: What you are saying is this (shows on the board). (Building)

In the first line, the teacher brought up a particular key idea based on the students' written solution to the task. It appeared in the lesson play that the teacher knew that this property of the example could lead to the proof, but, as the student's next line showed, the students had only discovered the property empirically (it happened – but they did not know *why*). The teacher provided guidance for the students to discover why the example had this property, and then the student expressed the key idea. The teacher re-represented the student's answer by using the drawing. Further in the lesson play, the teacher proceeded to generalise the claim (see below). In several other lesson plays, the teacher spent even more time discussing the key idea with the students.

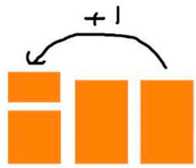
5.3. Facilitating students' understanding of the generalisation of the key idea to all examples

In 12 of the 14 lesson plays, the teachers facilitated the generalisation of the key idea. We illustrate this teaching practice by continuing the lesson play written by novice teacher G.



Teacher: Let's now say we are going to use a drawing that will be suitable for all consecutive numbers.
Does anyone know why I've written +1 under that column in the middle and +2 below the last one?

Student: Yes, because there is always 1 more in the second number and 2 more in the third number.



Teacher: What's going on here?

Student: It's exactly the same as you showed with numbers; we take 1 from the largest and move it to the least.

Teacher: I agree, it's the same, only with a drawing instead of numbers. What about this picture, what's happened now?

Student: They're all the same size.

Teacher: What would it mean if we had written it in numbers?

Student: That all the numbers are the same size, as with 19 and 19 and 19.

Novice teacher G chose to use a generalised figure to show three arbitrary consecutive numbers (there was a similar figure on the blackboard already). Here, the way in which the key idea identified in a single example can be generalised was repeated several times, involving the students, and recurringly linking to the example (18 + 19 + 20).

5.4. *Promoting scepticism towards empirical arguments*

Although this teaching practice was not required for constructing a generic example and was thus not (even implicitly) asked for in the assignment given to the novice teachers, 8 of the 14 lesson plays involved promoting scepticism towards empirical arguments. In the excerpt below, we see how novice teacher A envisioned this part of the discussion:

Teacher: Student C, can you describe in your own words what students A and B have come to? (MR move: Responding to student reasoning by Encouraging student revoicing)

Student C: Students A and B say that they have added different numbers together. They have taken two different odd numbers and added them together. They have tried this with many different odd numbers. Each time, they got an even number.

Teacher: So, what have students A and B come to?

Student C: That the statement is never true because all the answers they got are even numbers, and none of the answers is an odd number.

Teacher: But can it not be the case that, at some point, there will be a sum that is an odd number? Have you examined all the odd numbers? (MR move: Extending student reasoning by Encouraging reflection)

Student B: But we cannot check all the odd numbers in the entire world.

Teacher: No, you are absolutely right. We cannot actually check all the odd numbers in the entire world. So we need to look for another way to find this out.

We see how the teacher problematised the students' empirical approach to proving the claim by first revoicing the students' thinking and then suggesting that their argument was not sufficient and that another approach would be needed. The lesson plays usually started with the teacher's initiation of a whole-class discussion after the students had been working on the task in small groups. In 9 of the 14 lesson plays, the teachers started the discussion with empirical arguments, and in 7 of the 9 in which an empirical argument started the discussion, the teacher then pointed to the limitations of such an approach, as in the example above. In two cases, the teachers started the discussion with an empirical argument but did not promote scepticism towards this approach. Instead, they turned to students who pointed out some structure. Five of the novice teachers did not start the discussion with an empirical argument, but one of them still promoted scepticism later in the discussion.

5.5. Supporting students' mathematical reasoning

As described in the methods section, the novice teachers were asked to make use of the teacher moves from the TMSSR framework (Ellis et al., 2019), and indicate which moves they applied in parentheses along the way during the lesson play. It is not surprising to find that this teaching practice surfaced in almost all the lesson plays (13 of 14) because the novices were explicitly asked to do that. Nevertheless, our analysis shows that the moves were actively used, were closely related to the mathematical content discussed, and they were used to generate adequate questions for the students. We can also see that, in all 13 lesson plays where the TMSSR framework was used, some of the teacher's moves had a high potential for supporting students' reasoning.

The majority of novice teachers denoted the moves in parentheses along the way, as suggested in the task. In some cases, there were minor deficiencies in the novice teachers' descriptions of the teacher's moves in the lesson play, as, for example, in the following excerpt from the lesson play written by novice teacher A:

Student C: Yes, and no matter which odd numbers we use, there will always be one left over if we put them together in pairs.

Teacher: Exactly. No matter which odd numbers we use, whether it's 3, 5, 7 or 9, there will always be 1 left over. (MR moves: Responding to student reasoning, Validating a correct answer, Extending student reasoning – pressing for generalization).

The novice teacher characterised the move as extending the student's reasoning by asking for generalisation, even though the teacher only confirmed the student's generalisation.

5.6. Extending proving task

In two of the lesson plays analysed, the given proving task was extended to a new general conjecture that the students investigated. Novice teacher B extended the task through the student Magnus's voice:

Magnus: (…) When the numbers are consecutive, you can always move one from the highest number to the lowest, and then the three numbers will be equal. And three equal numbers are divisible by 3. I thus realized that the answer was *always*, so I wondered if it held for five consecutive numbers, and it did!

In the lesson play written by novice teacher E, the extension was initiated by the teacher after the construction of a generic example for the original claim. The task was then extended even more by the students.

Teacher: (…) But what would happen if three odd numbers were added?

Group 3 student: It's not going to work! You will get odd numbers in response because then, suddenly, you have 3 left and three is an odd number. That's true only if you add two odd numbers.

Student from group 2: But that's also true if you add four odd numbers.

Student from group 3: That's true only when the number of numbers you add is even, because the ones that are left over must be used to make a pair.

5.7. Promoting proof-related norms

This teaching practice was identified in two of the lesson plays. We interpret novice teacher D's summarising comment to the class as an attempt to establish a norm on proving in mathematics:

Teacher: Yes, now you have spent an hour on this assignment. Many people spent only five minutes saying that it would always be true, but they would not be able to explain why. That is often what is difficult. We are often sure that it will always be true, but there may be an example where it is not always true. Therefore, we must examine *why* things are true or not. I think you have managed to prove this nicely. You have used both drawings and things that you know from before, such as division and ten-friends.

The teacher emphasises here the need to spend time and effort when trying to conclude that a general statement is true. Furthermore, she highlights the necessity of finding out why the claim is true, and points to using different representations and what is known from before. Whilst the first point, persistence, can be seen as a norm that is important for all schoolwork, the second one is related to searching for a key idea and is particularly related to proving in mathematics.

Novice teacher E implicitly promotes another norm that we see as proof-related in the following:

Teacher: It's exciting that you lean on each other's explanations when you argue that two added odds become even. If we now merge the 'best' from all three answers, can we then try to build on the argument of group 1, or simply rewrite our arguments to make them more precise?

In the turns before this utterance, it was emphasised that one of the answers was clearer about the definition used, another was better explained by the use of drawing what happened, and the third was explained more by words. The teacher points out the need to 'fill the gaps' in the original arguments and be more precise, which are important norms in proving.

5.8. Involving students

Our operationalisation of this teaching practice required that the students be given the role of main contributors in the discussion, as opposed to the role of just answering the teacher's closed questions. To illustrate this difference, we can compare novice teacher G's lesson play to that of novice teacher L. As can be seen in the excerpts from teacher G's lesson play, presented above (under the categories related to key idea and generalisation), the teacher led the discussion and the students contributed by answering the rather closed questions, for example, as shown in the following:

Teacher: (…) Does anyone know why I've written +1 under that column in the middle and +2 below the last one?

As a contrast to the closed questions of novice teacher G, we present an excerpt from novice teacher L's lesson play, where the students have tried to prove that the sum of two odd numbers will never be an odd number:

Teacher: You investigated a certain number of odd numbers. Could it be that there is an answer that is an odd number, as you didn't check for all odd numbers?
Student 1 and Student 2: I think that it will never happen. Just give me two odd numbers.
Teacher: What makes you certain that the answer will never be an odd number? Now I want to hear what group 2 can say.
Student 3: We think that if you draw 3 apples and add 3 more apples, you get 6 apples, and 6 is an even number. We did the same with 5 and 9 and got 14, which is an even number. You can do this with any other even numbers.
Teacher: Okay, but you investigated just four odd numbers. How can you be certain that it will be true for all odd numbers?

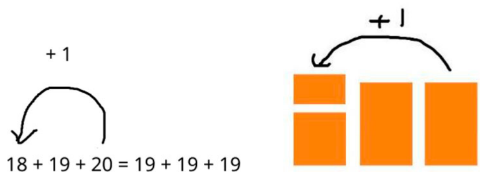
Like the teacher in novice teacher G's lesson play, this teacher intends to lead the students towards the discovery of a key idea. Unlike the first teacher, however, novice teacher L asks open questions (the key idea is not revealed or hinted at), and he asks several students in the process. In the continued lesson play, students eventually come up with a key idea. We remark that both teachers are involving the students, but it plays out differently in the two lesson plays.

5.9. Using visual representations

The lesson plays were, per se, mathematical discussions, and most of the communication was oral. However, in eleven lesson plays, the teacher used visual representations to emphasise crucial mathematical aspects. In the lesson play written by novice teacher E, the drawings from the students' written work were shown during the discussion, when one of the students said the following:

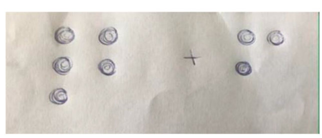
Elon: If you look at the drawing for group 3, it shows that there is an extra dot on each odd number, and when you put those dots together, they become a pair. That's how we mean it.

In ten lesson plays, this teaching practice was employed by using a blackboard to emphasise a structure or property that was crucial for the key idea. In the lesson play written by novice teacher G, several examples are given:



Three teachers used drawings, four used mainly numerical symbols, and one (novice teacher G) used both, as shown in the example above. The teachers in two of the lesson plays expressed the general statement on the blackboard by using the words 'any number': the novice teacher M used it in addition to the numerical expression for the example used, while the novice teacher I used numerical expression, drawing and the general expression.

In one of the lesson plays (written by novice teacher A), the blackboard was used, but the visual representation was not about some crucial mathematical aspects. The teacher drew the following:



The drawing was just another representation of the example of $5 + 3$, and even though the discussion went further to address pairs and putting singles in the new pair, nothing more was added to the drawing to emphasise the structure and/or the key idea visually. Therefore, in our analysis, we coded this lesson play as not showing the teaching practice of using visual representations.

6. Discussion

Our research question was as follows: What teaching practices related to proving by generic examples in school surface in novice teachers' lesson plays, and how can they play out? The analysis has revealed several teaching practices concerning proving in the lesson plays, and we have related them to three different levels: teaching of generic examples, teaching proof and teaching mathematics. In the following, we discuss the teaching practices of each level.

On the level of teaching of generic examples, we found that the novice teachers practiced facilitating an example-based discussion – that is, a discussion that started with an example. The students' arguments often contained some examples, but the teacher needed to emphasise one of them and build the argument from there. The example was further used to emphasise a key idea that could explain why the claim holds, and the argument was then generalised to construct a generic example (Rø & Arnesen, 2020). As Reid and Vallejo Vargas (2018) point out, students might not perceive an example as generic, even if the author of the argument is using it in that way. In writing lesson plays, we contend that novice teachers can have the opportunity to carefully think through how to establish both a key idea and generalisation with students. The majority of novice teachers promoted both a key idea and generalisation in several turns in the discussion with the students, also emphasising them by writing or drawing on the blackboard.

On the level of teaching proof, we noticed that 8 of the 14 novices promoted scepticism towards empirical arguments, which is a teaching practice several studies have pointed to as important in the work on proofs in school (eg Ellis et al., 2022; Valenta & Enge, 2022). Furthermore, in the lesson plays, the novices employed different moves for supporting students' mathematical reasoning – they elicited, responded, facilitated and extended student reasoning, as suggested by Ellis et al. (2019). The use of Ellis et al.'s (2019) framework for supporting students' reasoning was a requirement in the novices' assignment, so it is not surprising that they used it. However, in writing a lesson play, they had the opportunity to *practice* the moves or, more precisely, imagine how they could be used in a classroom. More surprisingly, some novice teachers included the promotion of proof-related norms and extended the proving tasks in their lesson plays. Yackel and Cobb (1996) used the notion of socio-mathematical norms to refer to classroom practices that are specific to the discipline of mathematics. In the analysed lesson plays, we have several examples of promoting norms. While promoting persistence was seen as important for schoolwork in general, norms related to the need to find out why, the need to refine arguments and the need to fill the gaps were clearly related to mathematics and proving.

Finally, some of the teaching practices we identified in the lesson plays were related to teaching mathematics (or even, just teaching) in general, yet they were still tightly connected to the particular mathematical content in the lesson plays. One of these was involving students in discussions. As Shaughnessy et al. (2019) point out, mathematical discussions need to be based on students' thinking, so students should be actively involved in knowledge-building towards a given learning goal. In the assignment, there was a requirement to build the discussion from the students' written work, but in writing the lesson plays, the novices envisioned how this could be done and how the teacher could involve students in the collective knowledge building. Furthermore, in some lesson plays (10 of 14), the students had prominent roles and contributed with important utterances, even without the teacher's explicit question; in others, it was mainly the teacher who led the discussion, and the students were answering mostly closed questions. Drageset (2014) envisions closed questions as teacher-controlled steps in the process of finding an answer to a mathematical task. Similarly, in the lesson plays, the closed questions functioned as teacher-controlled steps in the process towards establishing a proof. More open questions provide opportunities for students' active engagement, whilst closed questions can 'hinder reflection and understanding of important details' (Drageset, 2014, p. 300).

The use of several representations was another teaching practice we identified in lesson plays. In the lesson plays, this teaching practice was related to emphasising an example, the key idea, and/or the generalisation – that is, the crucial parts of a generic example.

It is important no note here that our results rely on some decisions we have taken in the analysis. For instance, we have coded a novice teacher's practice as *Facilitating students' understanding of the key idea* if the teacher in the lesson play calls attention to a property or structure in more than one turn. To be sure that it really is this practice that the novice teacher engages in, more knowledge about the teacher's intention is needed, and this kind of knowledge is not easy to gain from lesson plays only, which we use as our data here. Our analysis indicates the teaching practices novices can engage in by writing lesson plays, but it is difficult to establish the degree to which they *intentionally* engage in them, learn from them and/or use them in their teaching. Studies including interviews, reflections and/or observations in classrooms would be valuable for developing more knowledge about this.

7. Conclusions and implications

The motivation of the study was to investigate the affordances of using lesson plays as an approximation of teaching proving in primary school. As teacher educators, we were pleasantly surprised by the overall quality of the lesson plays discussed in this study. Although there were weaknesses in the lesson plays – eg some lesson plays did not really involve generic example because the discussion was not example-based, and others failed to promote the key idea or the generalisation of the key idea – all lesson plays included many relevant teaching practices, often resulting in valid arguments. Our analysis of the lesson plays indicates that writing lesson plays on proving, as exemplified by proving by generic example, gives opportunities for novice teachers to concentrate on the quality of arguments and mathematical aspects that are important in teaching proving.

Moreover, when writing lesson plays, teaching complexity is reduced, compared to approximations of practice involving enactment with real students (Buchbinder & McCrone, 2020; Skott et al., 2017; Stylianides et al., 2013). Since writing lesson plays is an approximation of practice, the novice teachers can think mainly in terms of students and discussions (and not, we believe, mainly in terms of formal mathematics, which is often the case with proving tasks in teacher education). Nevertheless, in writing lesson plays, novices must think through the crucial details related to proving in school, as Kazemi et al. (2021) describe in their study. Moreover, we hypothesise that lesson plays can hold similar potential within other aspects of reasoning and proving.

Following the above, we claim that lesson plays can support novice teachers' learning of what generic examples are and what the teaching of generic examples involves, in addition to being an affordable approximation of practice as it does not involve real students.

We close with some thoughts and remarks on the further use of lesson plays to teach the teaching of proving in teacher education. Although novices can work on important teaching practices when writing lesson plays, as our results show, we do not know to what extent they have reflected upon the teaching practices when writing – or whether they may even have 'used' a teaching practice almost accidentally, without reflecting afterwards. In Crespo's study (2018; see also González, 2018; Herbst & Milewski, 2018), pre-service teachers first wrote lesson plays, then compared and classified them, and finally revised them. This is a rather open approach, where comparing and classifying are steered by the pre-service teachers. In our study, we identified the teaching practices related to proving that can surface in the lesson plays. We can imagine an approach using the structure from Crespo's study (2018) – comparing, classifying and refining lesson plays – but focusing on (some of) the teaching practices identified in our study. For example, the teacher educator can ask novices to compare how a generalisation is promoted in some of the lesson plays, and then the novice teachers can revise their lesson plays. The variation found in how teachers involved students in their lesson plays indicates that writing, reading and revising lesson plays can give an opportunity in teacher education for in-depth discussions on student involvement in mathematical discussions. Further studies should also seek to uncover the effects on the teachers' enacted teaching in the classroom.

Notes

1. In Norwegian classrooms, an assistant's role is either to help some particular students with special needs, or to help students in general during their work on different tasks during lessons. They mostly help with practical issues, but sometimes also with the content in the tasks. Usually, assistants do not have teacher education and often do not have any academic degree. This can imply that assistants' assignments in the professional development course have some lower quality than teachers' assignments.
2. Type of arguments introduced at the lesson (notions from Stylianides, 2008), some of which are valid, and some non-valid.
3. In Norway, there were few restrictions in primary and lower secondary schools during the Covid pandemic, so it was possible for the participants to complete the assignment.
4. The General Data Protection Regulation, the EU regulation on data privacy. See https://gdpr-info.eu/.
5. We remind the reader that the references in the lesson plays to Ellis et al.'s (2019) moves to support students' mathematical reasoning were written by the novice teachers as a part of their assignment.

Acknowledgements

We wish to thank Andreas Stylianides and Farzad Radmehr for reading the manuscript and providing valuable feedback, and M. Fatih Dogan for helpful discussions along the way.

Authors' contribution statements

The two authors contributed equally to the work involved in this manuscript.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work is a part of a bigger research project, ProPrimEd – Reasoning and Proving in Primary Education, funded by the Norwegian Research Council [grant number 301 402] and Norwegian University of Science and Technology (NTNU).

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Appendix: Tasks in the assignment

Task 1. Is it always, never, or sometimes true that the sum of two odd numbers is odd?
Task 2. Is it always, never, or sometimes true that a number ending in 5 is divisible by 5?
Task 3. Is it always, never, or sometimes true that the sum of two consecutive numbers is odd?
Task 4. Is it always, never, or sometimes true that the sum of three consecutive numbers is divisible by 3?
Task 5.

$15 + 17 = 14 + 18$
$32 + 54 = 31 + 55$
$145 + 36 = 144 + 37$

Is it always true that the sum will be the same if one number goes down by 1 and the other number goes up by 1? Why/why not?