CHAPTER 2

Mathematical Modelling Using Study and Research Paths

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Introduction

Mathematical Modelling Using Study and Research Paths (MA3001, n.d.) is a 7.5 ECTS master's course in mathematics education at the Norwegian University of Science and Technology in Trondheim (NTNU). It is designed for students enrolled in the master's programme "Natural Science with Teacher Education", geared towards Grades 8–13. Inquiries involving mathematical modelling are the focus of the course, aiming to answer generating questions. The inquiries are shaped in a new didactic paradigm—that of *questioning the world* (Chevallard, 2015)—, which is rooted in the *anthropological theory of the didactic*, the ATD (Chevallard, 2019). This framework introduces a unique methodology for questioning the world, referred to as *Study and Research Paths* (SRPs).

The course involves two inquiries structured as SRPs, both focusing on *climate change*. In addition, the course encompasses several other inquiries that explore various systems, including the thermal insulation capabilities of two brands of thermoses, the likelihood of having a disease in the event of a positive test result, and the braking distance of a vehicle suddenly braking on a horizontal surface, among others. In teacher education, an SRP is a tool with a dual purpose. Firstly, student teachers shall become aware of the raisons d'être of the knowledge they must become sufficiently acquainted with in order to study questions. Secondly, it endeavours to equip them with key capacities to employ SRPs as a didactic tool, particularly when guiding pupils in the study of open modelling questions.

The author of this chapter served as the lecturer of the course in 2022 when it was first launched. Nine students completed the course, all of whom had mathematics as their main subject and either physics (5), computer science (2), biology (1), or chemistry (1) as their second subject in the programme (with the number of students who had each subject shown in parentheses). This chapter provides a broad account of the inaugural run of the course.

Infrastructure of the Course

Knowledge Goals

These are the interrelated knowledge goals of the course:

- Mathematical modelling of systems: the notions of system and model; design of modelling tasks.
- Elementary algebra as a modelling tool: formulas as algebraic models; formulas as equations with parameters.

- The anthropological theory of the didactic: the questioning of the world paradigm; study and research paths; the Herbartian schema.

Upon successful completion of the course, the student will be capable of modelling systems with parameters and have skills in addressing questions about these systems using both elementary algebra and additional modelling tools. Furthermore, the student will be capable of crafting modelling tasks for secondary school, concentrating on the interplay between systems and models. Next, the student will have the competence to explore generating questions through inquiries formatted as SRPs, underpinned by principles and tools derived from the ATD.

Teaching Methods and Study Activities

The course has different teaching and study formats: lectures, exercise classes, inquiries in terms of SRPs, and seminars. The lectures are given to initiate study of theoretical knowledge in mathematics as well as mathematics education, including methods of inquiry and modelling. The exercise classes are structured in a way to facilitate the discussion of tasks or problems, as provided by the lecturer, that the students have already worked on and provided answers to. The SRPs are organised as follows: The lecturer, serving as the supervisor of the class's inquiries, introduces a generating question, *Q*. Students collaborate in teams to study and answer *Q*. Regular meetings take place between the class and the lecturer to review the class's progress on *Q*, and to verify or recalibrate the direction of the inquiries. Seminars are also convened, wherein the teams present interim reports and gain feedback on their developing answers from both other teams and the lecturer. These seminars also serve as a platform to potentially receive suggestions on references they may find useful in their ongoing investigation.

The following activities are mandatory: participation in lectures and exercise classes; implementation and oral presentation of an $\text{SRP}_{\text{pilot}}$; execution of an SRP; composition and oral presentation of a halfway SRP report. Additionally, students are required to provide feedback on another team's halfway SRP report, write a final SRP report, and design a poster based on the conducted SRP.

Requirements, Syllabus, and Assessment

The course requires admission to the study programme "Natural Science With Teacher Education" at NTNU and a minimum of 60 ECTS in mathematics. Mandatory reading applicable to all students is specified at the onset of the semester. In addition to this shared literature, each team will need to utilise resources specifically selected for their teams' inquiries; these are considered obligatory for each respective team. For further details about the common required reading, refer to Appendix A.

The course utilises two types of assessment. Formative assessment is carried out at two stages: first, by fellow students' and the lecturer's feedback on a preliminary (halfway) SRP report; second, by the lecturer's written feedback on a draft of the final SRP report. Summative assessment is carried out in terms of an individual oral examination, where the student brings a poster designed by the team, based on the conducted SRP. The grade scale for the oral examination consists of letters (A–F).

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Course Roadmap: Structure and Content with Materials

Unit 1

Title	**A new didactic paradigm and modelling of systems using algebra as a tool**	
Duration[1]	6 hours.	
Organisation	5 × 45 min [Lectures] + 3 × 45 min [Exercise classes].	
Literature	- Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Springer. - Niss, M. (2015). Prescriptive modeling: Challenges and opportunities. In G. Stillman et al. (Eds.), *Mathematical modeling in education research and practice: Cultural, social and cognitive influences* (pp. 67–79). Springer. - Strømskag, H., & Chevallard, Y. (2022). Elementary algebra as a modelling tool: A plea for a new curriculum. *Recherches en Didactique des Mathématiques*, *42*(3), 371–409.	
Topics	- The paradigm of questioning the world - Modelling of systems - Elementary algebra as a modelling tool	
Questions	- What is the thermal insulating capacity of two brands of thermoses? - How can we determine the probability that a person, who has tested positive for a disease, indeed has the disease?	
Materials	PowerPoints	- Introduction (Appendix C1) - Elementary algebra as a modelling tool (Appendix C2) - The ATD and the notions of model and system (Appendix C3) - The ATD and a new didactic paradigm (Appendix C4)
	Tasks	- Modelling of insulating capacity of thermoses (Appendix D1) - Proposed solution to task on insulating capacity of thermoses (Appendix D2) - Modelling of probability (Appendix D3)

Unit 2

Title	**Some tools from the anthropological theory of the didactic**
Duration	3 hours.
Organisation	2 × 45 min [Lecture] + 2 × 45 min [Exercise classes].
Literature	- Bosch, M., & Gascón, J. (2014). Introduction to the anthropological theory of the didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 67–83). Springer. - Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114.

[1] For each of the five units, the time spent on homework is not counted in the "Duration" column. However, time allocated for team work on the SRPs is included in "Duration", and further details are provided in the "Organisation" column.

	- Markulin, K. et al. (2021). Project-based learning in statistics: A critical analysis. *Caminhos da Educação Matemática em Revista*, *11*(1), 200–220.	
Topics	- Didactic system - Study and research path - Herbartian schema	
Question	How do study and research paths differ from problem- / project-based learning?	
Materials	PowerPoint	- SRPs and the Herbartian schema (Appendix C5)
	Task	- Comparison of SRPs and problem- / project-based learning approaches (Appendix D4)

Unit 3

Title	**Climate change**	
Duration	5.25 hours.	
Organisation	1 × 45 min [Lecture] + 4 × 45 min [SRP_{pilot}] + 2 × 45 min [Presentation and discussion].	
Literature	Literature selected by each team during their SRP	
Topic	Basic knowledge on climate change	
Generating question	What is climate change, and why is it happening?	
Materials	PowerPoint	- Pilot SRP on climate change by a student group (Appendix C6)
	Task	- A small-scale inquiry into climate change (Appendix D5)

Unit 4

Title	**Carbon capture and storage**	
Duration	30 hours.	
Organisation	6 × 45 min [Lectures and question time[2]] + 30 × 45 min [SRP] + 4 × 45 min [Presentation and discussion].	
Literature	- Strømskag, H. (2022, 21 January). *A note on the Herbartian schema: A dynamic model for a study of a generating question*. Dep. of Mathematical Sciences, NTNU. - Literature selected by each team during their SRP	
Topics	- Models constructed by scientists on CCS - Parameters and their interrelationships	
Generating question	How is carbon capture and storage modelled in the literature? What mathematics is involved in these models? Which parameters are included, and what are the relationships between them?	
Materials	PowerPoints	- Some background to carbon capture and storage (Appendix C7) - Some aspects related to SRPs in teacher education (Appendix C8)
	Tasks	- Inquiry into carbon capture and storage (Appendix D6) - Hydrogeological modelling (Appendix D7)

[2] "Question time" refers to a teaching format wherein the lecturer provides prepared answers to questions that have been submitted in advance.

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Unit 5

Title	**The role of modelling in secondary school and design of modelling tasks**	
Duration	3.75 hours.	
Organisation	3 × 45 min [Lectures and question time] + 2 × 45 min [Exercise classes].	
Literature	Strømskag, H., & Chevallard, Y. (2022). Elementary algebra as a modelling tool: A plea for a new curriculum. *Recherches en Didactique des Mathématiques*, *42*(3), 371–409.	
Topics	- Systems in the natural and the social world - Instruction of mathematical modelling in secondary school	
Question	In which ways can modelling tasks be structured to help students gain knowledge about the systems involved?	
Materials	PowerPoint	- Modelling of systems and design of modelling tasks (Appendix C9)
	Manuscript	- Note on the dialectic of systems and models (Appendix B)
	Tasks	- Modelling of the Celsius-Fahrenheit relationship (Appendix D8) - Materials for an answer to the modelling of temperature scales (Appendix D9) - Modelling of braking distance of a vehicle (Appendix D10) - Design of modelling tasks for Grades 8–13 (Appendix D11)

Theoretical Framework

Key Principles of the Anthropological Theory of the Didactic

The course addressed in this chapter is fundamentally rooted in the Anthropological Theory of the Didactic (the ATD), a research framework developed since the 1980s, mainly by Yves Chevallard. This section provides a concise summary of the core concepts of the ATD. The presentation primarily draws upon Chevallard's 2019 article, "Introducing the Anthropological Theory of the Didactic: An attempt at a principled approach". For a more condensed overview of the ATD, one can also refer to Chevallard and Bosch (2019).

The ATD concerns the *didactic*, meaning that the spotlight is on specific actions that individuals or institutions perform or decide, with the potential effect that someone learns something. To study such actions, the concept of *praxeology* has been developed as an analytical tool. It begins with the idea of a *type of tasks*[3] that can be solved using certain *techniques*—task type and technique make up the *praxis* block (knowhow) of a praxeology. Further, every technique requires an explanation or discourse, called *technology*, which in turn has a justifying discourse at a higher level, called *theory*—technology and theory make up the *logos* block (knowledge) of a praxeology. A praxeology is thus a model both of the way we perform, and of the way we explicate and justify, our actions.

The ATD includes a theory of cognition, which proposes that a person has learned something about an object if the person's relation to the object has changed. (Objects are material and non-material components of the world we describe.) Within the framework of the ATD, Chevallard (2015) puts

[3] Solving a quadratic equation, creating a semester plan for a course, or shovelling snow are examples of different types of tasks.

forward a new didactic paradigm, called the "paradigm of questioning the world". This paradigm presumes a break with the prevailing didactic paradigm, known as the "paradigm of visiting works", where (in the latter) mathematics teaching is based on the "transmission" of already existing knowledge, followed by students using it to solve some relatively standardised tasks. The knowledge being taught has in this way lost its raison d'être, namely the questions it contributes to answer. The mathematics being studied is then often perceived by the students as decontextualised and purposeless. In the new didactic paradigm, students are given the opportunity to develop genuine knowledge—genuine in the sense that they should also know the purpose and utility of the knowledge. This is the ethos for teaching and learning in the course in question.

Study and Research Paths and the Herbartian Schema

The methodology developed in the ATD to operate in the new paradigm concerns a type of study and research process created by a generating question, where there is a dialectic between research and study (Bosch, 2018). "Research" refers to investigations or problem-solving, while "study" denotes consultation of existing (and available) knowledge that is initiated not only by the teacher but also by the students. The path created in the execution of a study and research process is referred to as a "path"—hence the term *study and research path* (SRP).

A praxeological function that inquiring into a question Q assumes is to lead to studying all sorts of works (including derived questions Q_i). How can we describe what happens in a didactic system $\mathcal{S}$ when a class X studies a question Q under the supervision of teacher(s) Y? The model provided by the ATD is the *reduced Herbartian schema*: $S(X, Y, Q)$ å A (Chevallard, 2019).[4] Here, A is the answer to the question Q that the didactic system is expected to produce. The answer A is usually written with a heart ♥ in superscript: $S(X, Y, Q)$ å $A^{♥}$—the heart hints at the fact that this answer will be "at the heart" of the didactic system, the "authorised" answer to question Q (at least for some time).

In the following, I explain how the reduced Herbartian schema is developed into a (full-fledged) Herbartian schema. The first step involves introducing the didactic milieu, M, which consists of the material and non-material tools that the class collects to conduct its study of the question Q. The reduced Herbartian schema then becomes the semi-reduced Herbartian schema:

$$[S(X, Y, Q) \text{ ➡ } M] \text{ ➥ } A^{♥}.$$[5]

This is understood in such a way that the didactic system creates the milieu M and generates the answer $A^{♥}$ with the help of the milieu denoted by

$$M = \{A_1^{\Diamond}, A_2^{\Diamond}, \ldots, A_m^{\Diamond}, W_1, W_2, \ldots, W_n, Q_1, Q_2, \ldots, Q_p, D_1, D_2, \ldots, D_q\},$$

comprising the following elements:

- $A_i^{\Diamond}$ are existing answers—found in the literature and multimedia resources—that are provided by other persons or institutions. $A_i^{\Diamond}$ is read as "A_i diamond", where the diamond signifies the "brand" of an institution or person. Therefore, a teacher (in direct instruction), a textbook, or a webpage are institutions that, in reality, "brand" their answers to the questions they tackle.

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[4] The adjective *Herbartian* refers to the German philosopher and pedagogue Johann Friedrich Herbart (1776–1841).

[5] The arrow ➡ is read "creates", and the arrow ➥ is read "generates".

- W_j are all types of work (theories, experimental plans, historical studies, reports, etc.) that must be used in order to study and understand all the other components of M. To be able to use these works, they must be studied. Studying a work W_j (which is not a question itself) consists of formulating and studying a series of derived questions Q_k to investigate W_j.
- Q_k are questions generated by the study of Q and all the other components in M. Thus, the study of any component in M boils down to studying questions. These derived questions depend on the generating question Q and on the direction the investigation of Q evolves in.
- D_l are datasets that are collected through various types of research during the study of Q.

An SRP will produce partial answers, A_r. These will be part of a "stock taking" that results in the final (though provisional) answer $A^{\heartsuit}$. Therefore, the complete Herbartian scheme, which is a dynamic model of the SRP, is symbolised as follows:

$$[S(X, Y, Q) \Rightarrow \{A_1^{\diamond}, A_2^{\diamond}, \ldots, A_m^{\diamond}, W_1, W_2, \ldots, W_n, Q_1, Q_2, \ldots, Q_p, D_1, D_2, \ldots, D_q\}] \Rightarrow A^{\heartsuit}.$$

Numerous inquiries in the format of SRPs have been carried out and published across various educational levels and countries. Examples include upper secondary schools in Denmark (Jessen, 2017, 2019) and France (Bourgade et al., 2020), teacher education programmes in Mexico and Spain (Barquero et al., 2018; Barquero et al., 2019), as well as Norway (Strømskag, forthcoming). Further examples include engineering education in Spain (Bartolomé et al., 2018, 2021; Amer et al., 2022).

Mathematical Modelling in the Course

The term *model*, as used in the course, is based on the notion of *system*, where a system is anything that has a reality subject to its own laws. A (geometric) sphere is an example of a system; the spread of a virus in a population is another example. Let S be a system. We say that S' is a model of S if, by studying the model S', we can answer certain questions about the system S. One tries to build a model S' of S, which makes it easier, safer, and faster to answer the questions about the system S, by studying the model S' rather than by studying S "directly". Strømskag and Chevallard (2022) have outlined four principles for using elementary algebra when modelling phenomena and objects in the natural and social world:

- Students start from a system S and a question Q raised about it, which appears to require mathematical elements for an adequate treatment.
- They develop a model S' of S in relation to question Q, which is built with elementary algebra and incorporates as many parameters as necessary.
- The students engage with S' to derive an answer considered adequate for Q.
- Concurrently, inspired by this process of inquiring about S, the students uncover the resources of algebra, and study or restudy them to effectively utilise the tools thus acquired.

In the course under consideration, there are two distinct operations that pertain to mathematical modelling: one involves students' own construction of models of systems, while the other involves students' examination of models of systems, created by scientists in various fields. In both instances, the objective is to comprehend systems; however, only in the former case are students engaged in the modelling process themselves. When students model systems, they generate mathematical models that facilitate their ability to respond to inquiries regarding the systems in question. An example of a task of this type is "Modelling of the probability of having a disease given a positive test result". A solution demonstration is presented in the subsequent section.

On the other hand, when students investigate models crafted by scientists, they study relationships between parameters in those models, employing knowledge in mathematics and other fields, either already mastered or essential to acquire during the investigation. Prior to the main SRPs on scientists' models in the field of climate change, pilot SRPs were conducted to achieve two objectives: generating broad knowledge about climate change and gaining experience with a small-scale SRP.[6] The generating question for the main SRPs was about carbon capture and storage (CCS), a technology involving capture of carbon dioxide (CO_2) from industrial sources and power plants, and storage of it in geological formations under the seabed.

The subsequent two sections present digests of inquiries conducted by students in the course, showcasing the two modelling operations.

Inquiry 1: Probability of Having a Disease Given a Positive Test Result

The inquiry presented in this section concerns students' own mathematical modelling.

The System to be Modelled

The starting point is a system, *S*, consisting of a population with an infectious disease and a screening test for the disease with reliability 95%. It is supposed that the test does not produce false negatives (i.e., everyone who has the disease will test positive for it). In the media, it is indicated that screening tests may be completely *illusory* in the sense that a person declared to be infected by the disease may have extremely low risk of actually having the disease.[7] Since the reliability of the test is quite high, this sounds paradoxical, and the question is whether the hint of illusion can really be true. More precisely, the question *Q* to be answered is: How can we ascertain the probability that a person, who has tested positive for a disease, indeed has the disease? (The specific task is detailed in Appendix D3.)

Constructing a model of this system was primarily focused on illustrating the importance of elementary algebra as a modelling tool. Although the students answered *Q* by modelling *S*, the account made here is based on the lecturer's proposed solution. Its purpose in class was to serve as a benchmark, underlining vital attributes of a modelling process and the crucial role played by the parameters of the system under investigation.

Creating a Model and Producing Knowledge about S

These are the parameters of *S* we choose to implement:

- a is the incidence rate (i.e., the relative frequency) of the disease in the population, where $0 \le a \le 1$;
- N is the population size;
- p is the probability of having the disease when having tested positive, where $0 \le p \le 1$;
- r is the reliability of the screening test, where $0 \le r \le 1$.

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[6] The assignment for the pilot inquiry is detailed in Appendix D5, while one possible result, as determined by one of the student groups, is displayed in Appendix B6.
[7] This has been done, for example, by the French physicist and populariser, Étienne Klein (Klein, 2020).

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We have that aN is the number of people infected by the disease, and $(1-a)N$ is the number of non-infected people. Of the non-infected, $(1-r)(1-a)N$ are found positive for the disease. This gives that the total number of positives is equal to $aN + (1-r)(1-a)N$. Therefore, the probability of having the disease when having tested positive is given by:

$$p = \frac{aN}{aN + (1-r)(1-a)N} = \frac{a}{a + (1-r)(1-a)}.$$

Simplification of this expression yields $p = \frac{a}{a + (1-r)(1-a)} = \frac{a}{1-r+ra} = \frac{a}{a\left(\frac{1}{a} - \frac{r}{a} + r\right)} = \frac{1}{\frac{1}{a} - \frac{r}{a} + r} = \frac{1}{r + \frac{1-r}{a}}$. That is, the following equality is a model of the probability sought:

$$p = \frac{1}{r + \frac{1-r}{a}} \qquad (S').$$

S' displays the relationships between the parameters of the system S and provides an answer to the question Q. First, we can observe on S' that p is independent of the population size, N. Second, the following observation explains the apparent paradox in S:

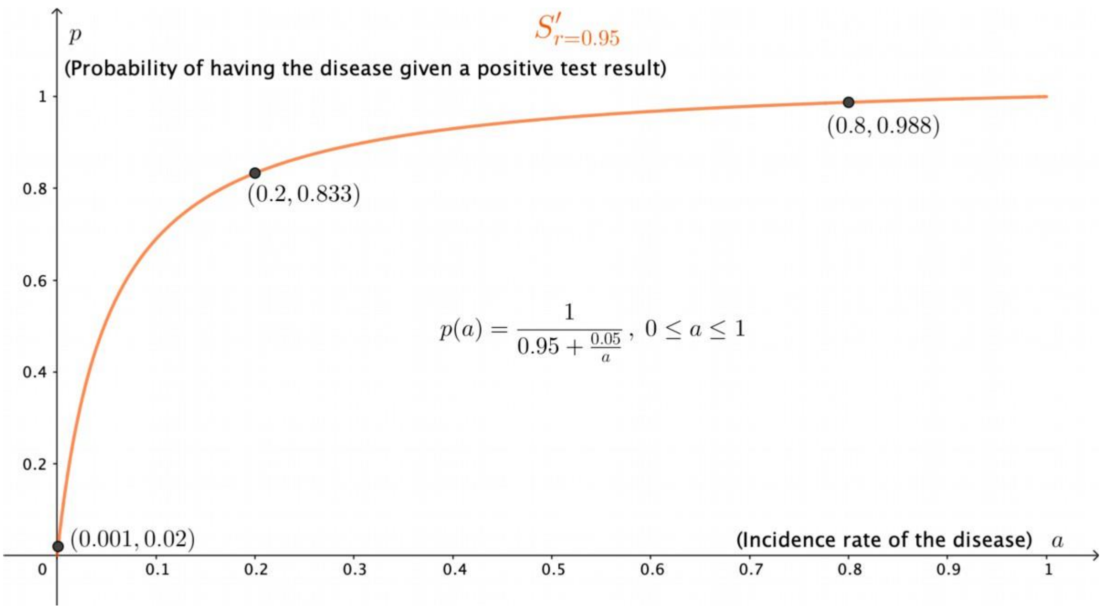
> For a fixed r, we see that when a increases, the fraction $\frac{1-r}{a}$ becomes smaller, and hence the denominator $r + \frac{1-r}{a}$ gets smaller too, so that the probability p increases.

Moreover, this relationship shows that p depends not only on r but also on the relative frequency of the disease, a. Specifically, small values of a yield small values of p. Let us fix $r = 95\% = 0.95$ and do calculations on $S'_{r=0.95}$ to display the relationship between p and a. Our model $S'_{r=0.95}$ is given by the function below, the graph of which is depicted in Figure 1.

$$p(a) = \frac{1}{0{,}95 + \frac{0{,}05}{a}}, \text{ for } 0 \le a \le 1 \qquad (S'_{r=0.95}).$$

Figure 1

A Model of S With r = 0.95



The points plotted on the graph display the following relationships between *a* and *p*:

- $a = 0.1\%$ corresponds to $p \approx 2\%$;
- $a = 20\%$ corresponds to $p \approx 83\%$;
- $a = 80\%$ corresponds to $p \approx 99\%$.

For further reading, the article "Why Every Clinician Should Know Bayes' Rule" by Tiemens et al. (2020) is recommended.

To conclude, probabilistic reasoning and elementary algebra was used to create S', a model that relates the probability of being sick, given a positive test result, to the incidence rate of the disease and the reliability of the test. Algebra was used to explore how changes in one parameter at a time affected the system, S. Further, algebraic transformations were used to derive new formulas based on the original model, S'. ($S'_{r=0.95}$ is one of several formulas developed during the inquiry.) Generally, by using algebra as a tool to model real-world phenomena, we can explore the behaviour of systems in a quantitative and precise way. This can help us to develop new insights into complex systems and to make accurate predictions and sensible decisions.

Inquiry 2: SRPs on Carbon Capture and Storage

The inquiry outlined in this section deals with students' examination of models created by scientists. The generating question for the class's inquiry was this:

> Q. "*How is carbon capture and storage modelled in the literature? What mathematics is involved in these models? Which parameters are included, and what are the relationships between them?*"

Guidelines for the SRP report is found in Appendix D5.

Some basic knowledge for investigating models of carbon capture and storage (CCS) was taught in the course as a blend of lectures and seminars. Contextual knowledge for CCS was introduced as a starting point, and additional knowledge was later taught based on needs emerging during the study and research process. While some publications were proposed by the lecturer, most of the resources the students used originated from the need to study derived questions during their inquiry. A summary of the knowledge taught for the CCS inquiries is presented in the following two subsections.

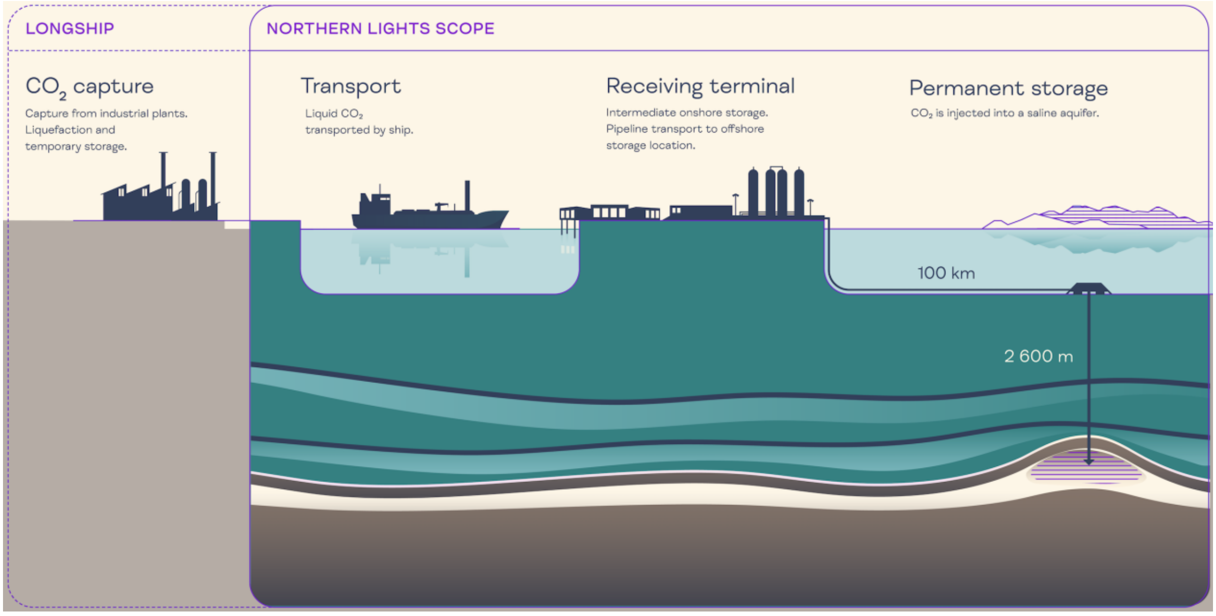
What is Carbon Capture and Storage?

CCS involves two processes: first, the capture of carbon dioxide (CO_2) emissions from industrial processes, such as steel and cement production or from power plants; second, the transport via ship or pipeline and storage of the captured CO_2 in deep underground geological formations. The goal of CCS is to reduce the amount of CO_2 that is released into the atmosphere, to help mitigate climate change.

The Norwegian government decided in 2020 to invest in full-scale CO_2 capture at Norcem's cement factory in Brevik (Norway) and in the transport and storage project Northern Lights (Report to the Norwegian Parliament No. 33, 2019–2020). Norcem's capture facility is the first of its kind in the world, and Northern Lights JV, a collaboration between Equinor, Shell, and TotalEnergies, is the first CO_2 storage facility in Europe that is open to European capture operators. The CO_2 will be transported by ship to a land terminal in Western Norway before it is sent on through a pipeline to an offshore CO_2 storage facility 2,600 meters below the seabed in the North Sea for permanent storage (see Figure 2).

Figure 2

The CCS Process



Note. The figure is taken from "About the Longship Project" (n.d.). Reprinted with permission.

This venture, called the Longship Project, is planned to be fully operative in 2024 (see "Carbon Capture, Utilisation and Storage," 2023).

At an early stage of the class's inquiry into CCS, the lecturer and students decided to delimit the models studied to either carbon capture or carbon storage. Below, I provide a brief background on carbon storage, which will lay the groundwork for a later summary of one team's inquiry into this component of the CCS technology.

Carbon Storage: A Quick Overview

There are four processes related to carbon storage (e.g., Niemi et al., 2017): thermal, hydrological, chemical, and mechanical processes. Team A chose to focus on *hydrological* processes, which involve the movement and behaviour of fluids such as water and CO_2. These processes may include fluid flow through porous rock formations, the displacement of brine and other fluids by injected CO_2, and the effects of hydrostatic pressure on rock properties and geomechanical stability. The next paragraphs briefly describe what dissolution of CO_2 in a background hydrological flow entails.

Normally, when gas is injected into a rock layer, it tends to accumulate in structural traps, such as anticlines. An anticline is a type of geological formation where a rock layer is folded upward into a hill-like shape. Gas can get trapped in these structures, and it doesn't necessarily spread out or dissolve throughout the surrounding rock formation. However, in the case of CO_2 injection into porous rocks, the water present within those rocks is constantly flowing due to background groundwater flow. As the water flows, it continuously dissolves the CO_2 that has been injected into the rock formation, rather than just allowing it to accumulate in a trap. Central in this dissolution process is Darcy's Law, used in geology, hydrology, and civil engineering to model the flow of groundwater, oil, and other fluids through porous media.[8] Darcy's Law states that the rate of flow of a fluid through a porous medium is proportional to the hydraulic gradient and the effective permeability of the medium (Fleurant & Bodin-Fleurant, 2019). Mathematically, this scientific law can be expressed as

$$\dot{V} = \frac{dV}{dt} = -kA\frac{dh}{dl} \quad (1).$$

The parameters of Equation (1) are: $\dot{V}$ is the volumetric flow rate of the fluid (i.e., the volume of fluid that passes through a given cross-sectional area per unit of time); k is the effective permeability of the porous medium (i.e., its hydraulic conductivity); A is the cross-sectional area of the medium through which the fluid is flowing, $\frac{dh}{dl}$ is the hydraulic gradient, representing the change in hydraulic head (pressure difference) per unit of distance along the direction of flow; and the negative sign indicates that the fluid flows from higher hydraulic head to lower hydraulic head. Appendix D6 contains a task entitled "Modelling in Hydrogeology," which utilises Darcy's Law.

An Inquiry Into Carbon Storage

This section presents a condensed version of Team A's SRP. The system S examined is carbon storage in underground saline formations that extend over a relatively large area that can potentially store significant amounts of CO_2. The concepts utilised in their report are the following:[9]

- *Aquifer* denotes a geological formation in the ground where the rocks or sediments have a large content of groundwater.
- *CO_2 plume* in aquifers refers to a volume of CO_2 that has been injected into a deep saline aquifer for the purpose of carbon storage. The CO_2 plume spreads out through the pore spaces in the rock, displacing brine and filling the available pore space.
- *Injection well* is a type of well used to inject fluids or other substances into the ground.
- *Injection array* is a cluster of injection wells used together.
- *Mobility ratio* in geology is a measure of how easily a fluid flows through a porous medium.
- *Permeability* is a measure of how easily a gas or liquid can penetrate a porous medium.
- *Porosity* is the ratio between the volume of voids in a material and the total volume.
- *Saturation* refers to the fraction of the pore volume that is occupied by a specific fluid.
- *Supercritical CO_2* refers to a state of CO_2 above its critical temperature (31.1°C) and above its critical pressure (72.9 atm). Here, CO_2 exhibits both gas-like and liquid-like properties.
- *Viscosity* is a measure of a fluid's resistance to flow.

Team A presented an answer to Q_0 that was made up of a model to calculate the CO_2 storage capacity in deep, saline aquifers at regional level, proposed by MIT environmental engineering experts, Szulczewski and Juanes (2009):

$$C = \frac{2M\Gamma^2(1-S_{WC})}{\Gamma^2+(2-\Gamma)(1-M+M\Gamma)}\rho_{CO_2}\phi HWL_{\text{total}} \quad (2).$$

[8] "Darcy's Law" originates from the work of Henry Darcy (1803–1858), a French engineer renowned for his significant contributions to hydraulics.

[9] Excerpts from Team A's report, originally written in Norwegian, are translated into English by the author.

Equation (2) is an analytic model represented by an explicit, closed-form expression, which is an existing answer to Q_0. The model is a rational equation with 9 parameters: C is the mass of trapped CO_2; M is the mobility ratio, measuring fluidity / viscosity of a substance through a porous medium; Γ is the trapping coefficient of CO_2; s_{WC} is the saturation of the connate (i.e., saline, naturally occurring) water in the reservoir; ρ_{CO_2} is the density of CO_2; ϕ is the porosity of the reservoir; H is the net sandstone thickness of the reservoir; W is the length of the injection array in the reservoir; and L_{total} is the total extent of the CO_2 plume after it is trapped (Szulczewski & Juanes, 2009, p. 3309). As for the parameters included, Team A displayed formulas for Γ and M, taken from the same publication. The trapping coefficient, Γ, was defined as the ratio of the residual saturation of CO_2 (s_{rg}), referring to the fraction of pore space that is occupied by trapped CO_2, to the fraction of pore space that is not occupied by trapped connate water ($1 - s_{WC}$):

$$\Gamma = \frac{s_{rg}}{1-s_{WC}} \quad (3).$$

The mobility ratio, M, was formulated in terms of the viscosity of brine (μ_W), the viscosity of CO_2 (μ_g), and the endpoint relative permeability of CO_2 (k^*_{rg}):[10]

$$M = \frac{\frac{1}{\mu_W}}{\frac{k^*_{rg}}{\mu_g}} \quad (4).$$

Interdependence Between Parameters

Identifying parameters involved in S and their interconnectedness was part of Q_0. However, Q_0 did not induce the students to examine the *mathematical aspects* of these relationships, which was neither done in the article they drew on. Therefore, building on insights from Bachu (2015) and Ketzer et al. (2009), the author will apply quantitative reasoning to enrich the understanding of Equations (3) and (4) in the following paragraphs.

The Trapping Coefficient. In Equation (3), Γ represents the fraction of injected CO_2 that is effectively trapped within the subsurface reservoir. The denominator, $1–s_{WC}$, represents the "free" space within the reservoir that is available for CO_2 to be trapped, and the numerator, s_{rg}, represents the fraction of that space that is actually occupied by the trapped CO_2. So, Equation (3) is essentially comparing the amount of trapped CO_2 (s_{rg}) to the amount of available space for trapping ($1–s_{WC}$) to determine the fraction of injected CO_2 that is effectively trapped (Γ). In other words, if the saturation of fossil groundwater s_{WC} is relatively high, then there is less "free" space available for CO_2 trapping, so the trapping coefficient Γ would also be high. A high s_{WC} means less pore space for CO_2 trapping, but also more water displacement by CO_2, which can increase trapping efficiency. Furthermore, connate water may react with CO_2 to form carbonic acid, which can dissolve the minerals in the reservoir rock and create new pore space for CO_2. This is known as mineral trapping, an important mechanism for long-term storage of CO_2 in the formation.

If s_{rg} is low, it means that a smaller fraction of the available space within the underground reservoir is occupied by trapped CO_2. This could be due to factors such as the geological characteristics

[10] The *endpoint relative permeability* of CO_2 is determined experimentally by measuring the flow of CO_2 and groundwater through a sample of the porous medium under fixed pressure and temperature.

of the reservoir, the injection strategy, or the properties of the injected CO_2. If s_{rg} is low, the trapping coefficient Γ would also be low, indicating that a smaller fraction of the injected CO_2 is effectively trapped. This could have implications for the overall effectiveness of CCS as a mitigation strategy for climate change, as a lower trapping coefficient would mean that a larger proportion of the injected CO_2 could potentially leak into the atmosphere over time. In general, a high trapping coefficient is desirable for effective and long-term storage of injected CO_2, as it indicates a higher degree of CO_2 retention within the bedrock formation. Overall, Equation (3) provides a simple way to estimate the trapping coefficient of injected CO_2 in a subsurface reservoir, based on the properties of the reservoir itself.

The Mobility Ratio. In Equation (4), *M* is a dimensionless quantity that represents the ratio of fluid mobilities between the displaced fluid (brine) and the injected fluid (CO_2), in a subsurface reservoir. It provides insight into how the two fluids interact and move through the reservoir during processes such as carbon storage. The numerator, $\frac{1}{\mu_W}$, represents the inverse of the viscosity of brine (i.e., its fluidity). As the viscosity of brine increases, the mobility ratio decreases, indicating that it is harder for CO_2 to flow through the reservoir. Conversely, as the viscosity of brine decreases, the mobility ratio increases, indicating that it is easier for CO_2 to flow through the reservoir. The denominator of Equation (4), $\frac{k^*_{rg}}{\mu_g}$, represents the ratio of the endpoint relative permeability of CO_2 (k^*_{rg})—which is a measure of how easily CO_2 can flow through the reservoir relative to brine—, to the viscosity of CO_2. As k^*_{rg} increases, the mobility ratio, *M*, decreases, indicating that it is easier for CO_2 to flow through the reservoir relative to brine. Conversely, as k^*_{rg} decreases, the mobility ratio *M* increases, indicating that it is harder for CO_2 to flow through the reservoir relative to brine.

A higher mobility ratio suggests that the injected fluid has greater mobility compared to the displaced fluid. This means that the injected CO_2 can flow more easily through the reservoir, potentially resulting in faster movement and potentially less efficient displacement of brine. Conversely, a lower mobility ratio indicates that the injected fluid has lower mobility relative to the displaced fluid. This suggests that CO_2 faces more resistance or has a harder time flowing through the reservoir, which can result in slower movement and potentially more efficient displacement of the brine.

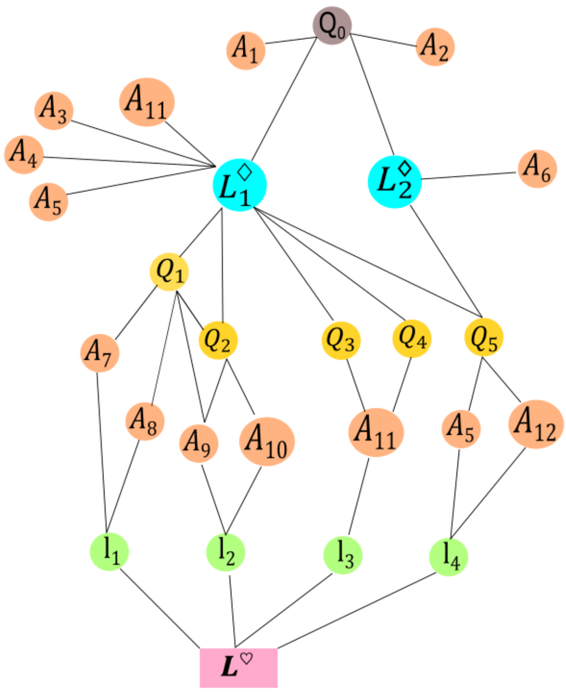
To conclude about *M*, the relationship between the parameters in Equation (4) is complex and depends on the specific conditions of the reservoir. However, in general, a lower mobility ratio is desirable for effective CO_2 storage, as it indicates that CO_2 is less likely to migrate and potentially leak out of the subsurface storage.

A Pathway Diagram of Team A's Inquiry

Figure 3 is a directed graph illustrating the course of Team A's SRP, accompanied by Table 1, describing the elements of the milieu created and utilised during the inquiry. Note that publications cited in Table 1 are not included in the reference list, since the table is just meant to illustrate the correspondence between the directed graph and the table that describes its nodes.

Figure 3

The SRP on Carbon Storage Carried out by Team A



Note. Q_0 = generating question; Q_j = derived questions; $L_i^{\diamondsuit}$ = existing answers (*L* for "løsning" in Norwegian); A_k = works (*A* for "arbeid" in Norwegian); l_m = partial answers; $L^{\heartsuit}$ = final answer to Q_0. Reproduced with permission.

Table 1

The Milieu Created During the Inquiry Conducted by Team A

Generating question
Q_0: How is carbon capture and storage modelled in the literature? What mathematics is involved in these models? Which parameters are included, and what are the relationships between them?
Existing answers $L_i^\diamond$
$L_1^\diamond$: Calculation of CO_2 storage capacity in deep, saline aquifers (Szulczewski & Juanes, 2009).
$L_2^\diamond$: Estimation of CO_2 storage capacity (Bachu et al., 2007).
Derived questions Q_j
Q_1: What is the relationship between viscosity and permeability in CO_2 storage?
Q_2: What is the mobility ratio about, and why is this an important component in CO_2 storage models?
Q_3: What simplifications of the models have been made, and what are the consequences of them?
Q_4: Are the parameters typically calculated by theory or are they due to empirical work?
Q_5: How important is pressure in calculating storage capacity?
Works A_k
A_1: "This is what you need to know about capture and storage of CO_2" (Sintef, 2019)
A_2: Storage and transport of CO_2 on the continental shelf (Norwegian Ministry of Petroleum and Energy, 2014)
A_3: CO_2 trapping mechanisms (CCP, n.d.)
A_4: Article on capillary capture for geological CO_2 storage (Krevor et al., 2015).
A_5: Video about CO_2 storage by an expert on the topic (Benson, 2021).
A_6: Norwegian website on terminology for the petroleum industry (Petroleumstilsynet, 2022)
A_7: Darcy's law for fluid flow in a porous medium ("Darcy's law", 2022).
A_8: Master's thesis on CO_2 storage in sandstone and limestone (Kvinge, 2012)
A_9: Reservoir engineering (Satter & Iqbal, 2016)
A_{10}: Article on mobility ratios (Bamidele et al., 2009)
A_{11}: Mathematical model of the CO_2 plume footprint in deep, saline aquifers (MacMinn & Juanes, 2009)
A_{12}: How can CO_2 be stored under the Earth? (National Energy Technology Laboratory, 2022)

Note. Partial answers (l_m) are synthesised into $L^\heartsuit$ elsewhere in Team A's report. Reproduced with permission.

Navigating the Interplay of Systems and Models in Task Design

In the preceding section, two distinct cases of mathematical modelling within teacher education were explored. This section transitions into the subsequent phase, which involves instructing student teachers on how to craft modelling activities tailored for secondary education. The capability to develop effective modelling tasks, which enhance understanding of the interplay between systems and models, is a critical skill for student teachers preparing to teach mathematics in Grades 8–13. The task design approach employed in this course is elucidated through a commentary on one of the assignments given to the student teachers. A comprehensive examination of this assignment, including principled explanations and a suggested solution, is given by Strømskag (2023), a manuscript found in Appendix B.[11] What is offered in the following paragraphs is merely a brief summary of the assignment and its solution.

The exercise began with a typical task extracted from a Norwegian textbook (depicted in Figure 4).

Figure 4

Task for Modelling of Stopping Distance

1.130

The stopping distance for a car in motion hinges on both the driver's response time and the braking distance.[12] The table below outlines the stopping distance, denoted by $S(x)$, in metres corresponding to certain speeds in kilometres per hour for a specified car and a specified driver.

x (km/h)	40	60	80	100
$S(x)$ (m)	24	45	73	108

a) Plot the data points from the table in a coordinate system and elucidate why a quadratic function seems to be a suitable fit.
b) Determine the quadratic function, S, that most accurately represents the given data. Sketch the graph incorporating the data points. Ensure that the expression of the function is accurate to three decimal places.
c) Find graphically the speed that would result in a stopping distance of 150 metres.
d) Find graphically the stopping distance corresponding to a speed of 90 km/h.

Note. The task is taken from a Grade 11 textbook, *Sinus 1T*, written by Oldervoll et al. (2020, p. 375). It has been translated into English by the author.

In the textbook task, students were instructed to plot stopping distances against varying vehicle speeds, with the intention of creating a second-degree polynomial model using regression analysis. The specific method for model creation was predetermined by the task's placement under the heading "Polynomial Regression". While the developed model, $S(x) = 0.009x^2 + 0.175x + 3$, did accurately reflect the given data, it fell short in fostering a thorough understanding of the system's underlying mechanics. A clear disconnect existed between the real-world dynamics of the system being studied and the parameters of the model, which subsequently constrained the educational value of the exercise.

In an effort to amplify the educational value of investigating the system in question, the lecturer redirected the task towards principles of inquiry and exploration, with the goal of illuminating the system's defining properties. This involved gaining a nuanced understanding of how system parameters, like the friction coefficient (a critical determinant of braking distance), and speed interact. As part of the refined task, student teachers were asked to model a system where a vehicle, travelling at a specific speed (72 km/h) on a dry, horizontal asphalt surface, needs to make a sudden stop. The question to be answered was about the vehicle's braking distance, which hinged on their ability to understand the

[11] Strømskag (2023) is an extension of a proposed solution originally presented during the course examined here. Its intent is to assist fellow educators who wish to incorporate this assignment into their own course curricula.
[12] Stopping distance = Reaction distance + Braking distance.

implications of friction when brakes are applied. Furthermore, they were prompted to discuss how varying speeds and different road conditions might impact the braking distance.

To calculate the braking distance of a vehicle, the first step involved understanding the dynamics of the forces involved and the vehicle's acceleration. Drawing on Grimenes et al. (2011) and grounded in Newton's laws of motion, the procedure can be outlined as follows: In the vertical direction, the force of gravity (G) and the normal force (N) balance each other out, as per Newton's first law ($N = G$). Meanwhile, the friction force, acting opposite to the vehicle's motion, is proportional to the normal force ($F_{friction} = \mu N$), where μ is the friction coefficient. Given $N = G = mg$, we derive $F_{friction} = \mu mg$. In the horizontal direction, the friction force is the sole force working against the movement. Using Newton's second law ($\Sigma F = ma$), where ΣF is equal to $-F_{friction}$, we find the acceleration a as follows: $-\mu mg = ma$, simplifying to $a = -\mu g$.

With the acceleration known, then the braking distance could be computed, using the equation of motion for uniform acceleration: $v^2 - v_0^2 = 2ad$ (Grimenes et al., 2011, p. 32). The involved parameters were defined as follows:

- a denoted the uniform acceleration of the vehicle (which in the case of braking is negative);
- g denoted the acceleration resulting from gravity (commonly approximated as 9.81 m/s^2 at sea level on Earth);
- m denoted the mass of the vehicle;
- d denoted the distance travelled (which in this case was equal to the braking distance);
- v denoted the velocity of the vehicle in the final state (which in this case was equal to 0);
- v_0 denoted the speed in the initial state (which was the speed at the moment the braking starts);
- μ denoted the coefficient of friction (which depends on the properties of the materials in contact, in this case, the rubber of the tyres and the road surface).

The resultant model was the second-degree polynomial, $d = \frac{{v_0}^2}{2\mu g}$. (For a breakdown of how this formula was derived and further insights into its components, refer to Appendix B.) Besides answering the specific question raised about the system, this model facilitated the generation of general knowledge about the system, such as: the braking distance is independent of the vehicle's mass, and the braking distance is proportional to the square of the initial speed. It was highlighted that this model, as it assumes uniform acceleration throughout braking and ideal conditions, might not yield accurate braking distances in more complex scenarios. However, these very limitations can urge students to reflect on other parameters that could potentially influence braking distance, such as road and tyre conditions, vehicle design, and driver reaction time.

Designed with students having a basic understanding of mechanics in mind, the updated task's final phase focused on tailoring it to students at varying levels. The lecturer proposed a reconfiguration of the task to bypass the need for extensive knowledge of the inherent mechanics of the system. This strategy involved supplying two distinct datasets—one for dry and one for wet road conditions—that correlated the vehicle's speed with its resultant braking distance. Concurrently, students would be introduced to the principle that braking distance is proportional to the square of the speed at the point of braking.

Concluding Comments

The modelling activities implemented in the course provided several advantages for the students. These included enhanced commitment and research capabilities, utilising algebra as a modelling instrument, and cultivating didactic knowledge of task design, particularly focusing on the relationship between systems and models. This chapter demonstrates the integral role that elementary algebra plays in the modelling of systems. Elementary algebra provides the means to express interdependence between parameters, analyse change, optimise performance, solve problems, and make predictions. By using elementary algebraic methods and quantitative reasoning, we can gain a deeper understanding of the world around us and make informed decisions based on quantitative data.

To operate in the paradigm of questioning the world—particularly with open-ended inquiries like the ones on CCS—is challenging for both teachers and students. A new *didactic contract* (Brousseau, 1997) needs to be implemented, because the teacher is generally not an "expert" on the systems studied. The shift away from seeing the teacher as the ultimate authority on knowledge may be difficult for both parties. It calls for students to assume a greater degree of responsibility for their own learning, and, not least, that the teacher allows them to do so. In addition, adoption of the new didactic paradigm entails the introduction of novel tasks for both teachers and students to tackle: in the parlance of the ATD, this translates to a shift in the *topos* of each party. From the teacher's perspective, the ability to pose relevant and thought-provoking generating questions becomes a crucial, albeit demanding, new role. As for the students, they are now expected to focus intensively on one question over a significant duration. This is not just an unfamiliar territory for them, but it can also become a source of frustration, especially if their perseverance wavers.

In conclusion, the course presented in this chapter underlines the critical role of mathematical modelling in questioning and understanding real-world phenomena. This directly aligns with the paradigm of questioning the world (Chevallard, 2015), which encourages not only students, but citizens at large, to interrogate to better understand their surrounding world. The ambition has been to offer useful insights into the effective incorporation of mathematical modelling, with a focus on the dialectic of systems and models, into teacher education. It is hoped that these insights, together with the appendices referenced in the course roadmap, extend their applicability beyond the confines of the course explored in this examination.

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2.1. Didactic Commentary

Didactic Analysis of MA3001: Insights from the Post-Course Survey

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Introduction

In Chapter 2 (this volume), I examined the inaugural run of the course MA3001 – "Mathematical Modelling Using Study and Research Paths" (MA3001, n.d.), which took place in 2022 at the Norwegian University of Science and Technology. In this chapter, I conduct a didactic analysis of the course based on student feedback from a post-course survey, using the Anthropological Theory of the Didactic (ATD) (Chevallard, 2015, 2019). Nine students completed the course, and eight of them conducted the survey.

The objectives of the course were diverse. Firstly, it aimed to teach mathematical modelling of systems using elementary algebra as a modelling tool, where parameters played a crucial role. Secondly, it sought to conduct inquiries in the format of study and research paths (SRPs), emphasising the tenets of the paradigm of questioning the world. Finally, it examined models of climate change found in the literature, exploring how system parameters and their interrelationships can elucidate the properties of the modelled systems. The main SRPs in class were about carbon capture and storage (CCS). In each of these areas, meta-level instruction was incorporated, given that the course was integral to a teacher education programme. The bedrock of the course lay in the ATD, whose principles were crucial to the treatment of the course contents. For an account of the scientific elements of the course commented on in this chapter (algebra, ATD, CCS, modelling, SRP), refer to Chapter 2 (this volume).

The subsequent analysis is segmented into two sections: "Mathematical Modelling of Systems Using Elementary Algebra" and "The ATD, Study and Research Paths, and Carbon Capture and Storage". Within each section, the questions and corresponding student responses are displayed. Comments not directly relevant to the topic have been excluded for brevity and clarity. Student feedback is marked with an "S" followed by a numeric identifier (e.g., S1, S2, …, S8). The survey was conducted non-anonymously to evaluate the potential impact of students' prior knowledge. It is noteworthy that S3, S6, S7, and S8 were in their 3rd year of the teacher education programme, while S1, S2, S4, and S5 were in their 4th year. Furthermore, it is relevant that S1, S4, and S5 had previously enrolled in the course MA3061 (n.d.), which introduced the ATD and provided guidance on conducting SRPs focused on the differential calculus present in the Norwegian upper secondary curriculum. In contrast, S2, S3, S6, S7, and S8 had not taken MA3061.

After detailing the students' survey responses, I present my analyses for each of the two sections.

Mathematical Modelling of Systems Using Elementary Algebra

1 How relevant have you found the topics in MA3001 to be for your professional development as a future) mathematics teacher?

	Irrelevant	Not very relevant	Relevant	Very relevant
Subject matter related to modelling	0	0	1	7
Subject matter related to algebra	0	0	2	6

1.1 Comments regarding your answer to the question about the course's relevance to the mathematics teaching profession.

S1. The course has broadened my perspective on what it means to model in mathematics. This has so far been missing (as far as I remember) in other subjects in the programme.

S2. What has been taught about algebra and modelling is particularly relevant because it provides concrete focus areas when teaching these topics. You get a good insight into what is important.

S3. The materials on algebra, modelling and ATD are what I think is most relevant for developing as a mathematics teacher. You learn methods to use within these topics, in addition to seeing the new paradigm from a maths perspective.

2 How suitable do you believe the assignments and ways of working in MA3001 have been in terms of fostering professionalism as a mathematics teacher?

	Unsuitable	Not very suitable	Suitable	Very suitable
Modelling assignments	0	0	2	6
Assignments on algebra	0	0	2	6

2.1 Comments on my answer about assignments and ways of working in the course.

S1. Owing to the fact that "modelling" is one of the core elements of LK20 [National Curriculum for Grades 1–13], the tasks in this course have been very appropriate for developing our expertise in this area. Similarly, algebra is a topic that falls under the core element "Mathematical Knowledge Area". We have also become aware of the importance of algebra as a tool for the core element "Abstraction and Generalisation", emphasising the significance of parameters.

S3. Recognising the value of using parameters within algebra has perhaps been the most pertinent aspect for me. The new curriculum offers less focus on algebra, so it has been beneficial to see how algebra can be used in an exploratory manner where students need to construct, manipulate, and evaluate formulas.

S4. The short tasks we got were very good and useful.

S5. Nice to work on concrete tasks that can be used in teaching in school.

S7. The contents on algebra felt a bit on the side of the main topic [SRP on CCS?], but was very informative.

4 What was particularly interesting or relevant to you in this course (within mathematics, mathematics didactics, physics, geology, economics, chemistry, etc.)?

S2. I particularly found the instruction in algebra and modelling engaging. Several facets of mathematics instruction were revealed that I hadn't previously reflected upon. These insights I will carry forward in my career as a mathematics teacher.

S4. I found it extremely engaging to learn about modelling. It's something I have never been fond of and never truly grasped. This course gave me a clearer understanding of what a model can be, and I was quite taken with the system-model theory we employed. It was succinct and straight to the point.

S5. It was exciting to learn how algebra teaching has evolved over time and how this influences student learning.

S8. It has also been relevant to understand more about how algebra can act as a modelling tool and how one can set the stage for students to attempt construction, manipulation, and evaluation of formulas.

10 Through MA3001, have you developed a new understanding of any bodies of knowledge (different from your previous understanding)? If so, what does this change entail?

S2. Yes, I have developed a new comprehension of algebra, modelling, and parameters. This change can be described as acquiring tools for how students can purposefully work with algebra to find solutions to the questions they wish to answer.

S3. I have gained a greater insight into how to craft algebraic problems where students have to focus on more than just using a formula to find an answer to a problem. I feel that I have a better grasp of how vital it is to use parameters so that students develop skills around using formulas through manipulation and construction in addition to the evaluation aspect.

S4. Yes, refer to my answer to Question 4.

S5. Parameters.

S8. I have learnt a bit more about how algebra can be used as a modelling tool and about different models.

Analysis of Student Survey Responses on Modelling and Algebra

Perspective on Mathematical Modelling and Algebra

The students' responses exhibit a holistic understanding and appreciation of the mathematical modelling and algebra conveyed in the course. As S1 noted, the course filled a significant gap in their prior knowledge, broadening their understanding of mathematical modelling. This hints at the epistemological dimension of the ATD (Florensa et al., 2015), where students recognise a richer set of practices and knowledge related to mathematical modelling.

Student Engagement and New Praxeologies

Responses to Question 4, particularly by S4, shed light on their journey from disinterest or unfamiliarity to a point of clarity and interest. In ATD terms, this transformative experience can be viewed as the result of building new relations to the knowledge at stake (Chevallard & Bosch, 2019). This progression equips students to perform new types of tasks, using different techniques, justified by new technologies, in a new theoretical framework (the ATD). The systematic approach to modelling, as mentioned by S4, might imply the appreciation of a more refined and task-oriented practice.

S5's reflection on the evolution of algebra teaching connects to the historical-epistemological dimension of the ATD. It is imperative to understand not just the "how" (i.e., the functioning) but also the "why" (i.e., the utility) of mathematical practices. By knowing the historical context of algebra, the students can better understand its current pedagogical practices.

Utilising Algebra as a Tool

The responses, especially those of S8, S3, and S2 in various questions, bring forth a nuance in understanding algebra not just as an abstract subject, that is, as a praxeological complex considered from the point of view of its *structure* and (partially) its *functioning*, but as a tool, that is, from a praxeological complex considered from the point of view of its possible *uses*—specifically, as a modelling tool. This resonates with the anthropological notion in the ATD, where algebra becomes a human practice, deeply intertwined with real-world problem solving—and let me stress, this includes mathematical problem solving (Strømskag & Chevallard, 2022).

Modelling Assignments and Assignments on Algebra

The quantitative data gives that 6 of the 8 students found the modelling assignments and assignments on algebra to be "very suitable" and 2 found them suitable for fostering professionalism. This suggests that the students were able to engage in praxeologies of mathematical modelling and algebra. S1's response emphasises the importance of modelling and algebra as core elements in the curriculum. The mention of the core element "Abstraction and Generalisation" and the significance of parameters suggests that the course provided them with the techniques and tools necessary for mathematics teaching in school. S3's comment points to the technique of using parameters in algebra and their usefulness in constructing, manipulating and evaluating formulas, thus emphasising its praxeological value.

Emerging Understandings

The responses to Question 10, especially from S2 and S3, highlight a deepened appreciation of the processes behind the crafting of algebraic problems for Grades 8–13. Their newfound understanding of the role of parameters and the manipulation, construction, and evaluation of formulas underscores the course's effectiveness in altering preconceived notions, which in ATD terminology means that they have developed new relations to objects. This underlines the transformative dimension of the ATD, where prior relations to objects are reshaped through didactic interventions.

In conclusion about modelling and algebra, through the ATD lens, the feedback reveals a deeper institutional and personal transformation in students' engagement with mathematical modelling and algebra. Parameters emerge as a significant component, bridging the gap between algebraic technique and technological depth in modelling. The course seems to have effectively embedded these nuanced elements, fostering a comprehensive understanding of their role in the broader mathematical landscape.

Institutional Dimension: Relevance to the Mathematics Teaching Profession

The institutional dimension refers here to how knowledge is structured, organized, and delivered within an educational institution. The mention of the LK20 National Curriculum for Grades 1–13 in S1's response indicates the alignment of the course content with the institutional demands. By preparing students for the curricular expectations, the course is preparing them for their teaching roles. S3's remark about the "new curriculum" indicates that while institutional priorities might shift (like less focus on algebra), the course was beneficial in showcasing how algebra remains a vital tool. S4's and S5's feedback might suggest that the course's practical and applicable nature, referring to short tasks and concrete tasks, is institutionally relevant, as it prepares them for their roles in classrooms.

S7's remark suggests that algebra seemed somewhat tangential to the primary focus, which I infer to be the SRP on CCS. This implies that the relationship between algebra and the intricate CCS models they delved into might not have been evident to S7. Consequently, there appears to be a need to enhance the teaching of algebra with respect to parameters and their interconnectedness in mathematical models presented in scholarly works (e.g., models developed by scientists). The weakness experienced by S7 is likely related to the limitation discussed in Chapter 2 (this volume). The generating question of the SRPs on CCS was the following:

Q. How is carbon capture and storage modelled in the literature? What mathematics is involved in these models? Which parameters are included, and what are the relationships between them?

However, the students' answers to the last part of the question, about relationships between parameters in the models they studied, were less developed. This indicates that it is necessary to explain what we mean by describing interrelationships between parameters. To exemplify how this may be done, in Chapter 2 (this volume, pp. 29–30), I used quantitative reasoning to explain system properties based on two equations presented in Team A's SRP report—one equation with three and the other with four parameters. Such analyses of models, whether they are constructed by the students themselves or found in the literature, is at the heart of the new didactic paradigm, where the point is to produce knowledge about the systems under investigation, and not just stop when a model is constructed or found. This can be challenging since it requires some knowledge about the system parameters, which in the case of CCS mainly belonged to geology and physics. In the context of teaching modelling, it is important to explain that identifying relationships between parameters means using mathematics (e.g., quantitative or probabilistic reasoning) to explain some of the properties of the system at stake.

The ATD, Study and Research Paths, and Carbon Capture and Storage

1 How relevant have you found the topics in MA3001 to be for your professional development as a (future) mathematics teacher?

	Irrelevant	Not very relevant	Relevant	Very relevant
The subject matter related to the ATD	0	0	1	7
The subject matter related to CCS	0	0	3	5

1.1 Comments regarding your answer to the question about the course's relevance to the mathematics teaching profession.

S1. Would like to have more lectures on the ATD like we had in MA3061 [a course S1 had taken the previous semester]. Comprehensive theory.

S2. ATD and carbon capture and storage are also relevant, but since ATD is not the current didactic paradigm, there are many things that can be difficult to implement in schools.

S3. The materials on algebra, modelling and ATD are what I think is most relevant for developing as a mathematics teacher. You learn methods to use within these topics, in addition to seeing the new paradigm from a maths perspective. The subject matter related to carbon capture and storage is also relevant in that students will develop an ethical view of the environmental problems facing the world. The reason why I see this as slightly less relevant is that the subject matter is very complex, and it can be difficult to understand for upper secondary school students.

S4. In general, I found the course very relevant to my future teaching profession.

S8. Some of the subject matter has been a bit heavy to read since there were long articles in English, but I think the subject matter in the course has generally been relevant for further development of mathematics teachers.

2 How suitable do you believe the assignments and working methods in MA3001 have been in terms of fostering professionalism as a mathematics teacher?

	Unsuitable	Not very suitable	Suitable	Very suitable
Study and research path on CCS	0	0	3	5

2.1 Comments on my answer about assignments and working methods in the course.

S1. I'm keen to try out the SRP working method with my own students, and CCS is particularly relevant due to sustainable development.

S2. The reason I deemed the SRP only "suitable" is that many other subjects are incorporated within CCS than just mathematics, making SRP demanding to conduct as a teacher. Apart from that, I find SRP to be an intriguing form of project-based learning and believe it has been taught very well.

S6. While I appreciate the SRP, it's not exactly my preferred way of working, but it was fun. The tasks were relevant and enjoyable.

S8. I found it very enlightening to learn a new way of working with a problem. Both the work with the SRP and tasks carried out in class are something I will incorporate into my future role as a mathematics teacher.

3 Was there anything you found surprising in MA3001? If yes, please explain.

S2. I was surprised by how involved the students were in the course and how much influence we had. This is something I am not used to from previous courses. The lecturer seemed very genuinely interested in our input, and organised the teaching accordingly.

S4. At the start of the SRP, I thought the topic Heidi [the lecturer] had chosen was surprisingly advanced. Most of us got a little sweaty when we started studying carbon capture and storage. Fortunately, it became more manageable as we realised that we were not supposed to go in depth on everything or "becoming geologists".

4 What was particularly interesting/relevant to you in this course (within mathematics, mathematics didactics, physics, geology, economics, chemistry, etc.)?

S1. ATD as a theory was particularly relevant (especially since I consider writing a master's thesis on it). At the same time, I found it intriguing to explore the phenomenon of CCS and see the significant roles different disciplines play in the entire process.

S2. It was also insightful to partake in group work that demanded such intense collaboration. Throughout our working period, we met various requirements and gave three joint presentations. From prior experiences with group assignments, I'm not accustomed to such high collaborative demands. This was quite positive from my viewpoint, especially since I got on well with my teammate.

S3. Personally, the most captivating part for me was learning about the SRP research method, and how I can utilise it as a teacher. It offered fresh perspectives on, for example, working with interdisciplinarity, which is a primary element in the curriculum. From my practical experiences, many teachers find it challenging to devise schemes of work around this.

S4. I also appreciated the interdisciplinary nature of the SRP, making it highly pertinent for us student teachers.

S7. It was exciting to study both ATD and SRP. It presented a different approach to work that was quite engaging, something I would be keen to delve more into.

S8. I have found it highly engaging to learn and work in a novel way with SRPs.

5 What has been of great importance to your learning in MA3001?

S1. Collaboration has undoubtedly been of great significance for learning. We weren't quite sure where to begin with SRP, but after discussing the task, it all fell into place. Regular seminars have contributed to maintaining the progression of the project.

S2. I would argue that the lecturer and her adaptability to the students have been crucial. I would also highlight my SRP partner. We have had different approaches to the subject and have learned a great deal from one another because of this.

S3. It has been beneficial having seminar-based instruction. Moreover, one doesn't feel overwhelmed with lectures; instead, we have had productive teaching sessions packed with quality content. I think we have been shown numerous exemplary instances of the various topics we have covered, providing us with a clear vision of how we can use modelling, algebra, and SRPs in our future teaching roles. In our work on SRP regarding carbon capture and storage, I feel the lecturer provided excellent feedback and consistent guidance. This has played a role in the motivation for the task. It has also been essential to understand how the task on carbon capture and storage relates to our teaching profession.

S5. Having a passionate lecturer who displays a keen interest in the subject and provides relevant teaching for us as future teachers is invaluable. Fellow students who are also eager to learn and engage enhance the experience.

S6. The opportunity to discuss with others.

S7. The teacher has been incredibly skilled and understanding, recognising that this is new territory for many and genuinely wanting us, the students, to achieve meaningful learning outcomes.

S8. There has been ample opportunity to ask questions about any uncertainties along the way. Collaborative group work has also been advantageous. It's been beneficial to share experiences and pose questions to fellow students.

6 What challenges have you encountered in MA3001?

S1. One of the challenges has been the geological terminology in CCS. Additionally, understanding models with limited information has been tough. Another challenge was adhering to the maximum word count, which, while crucial for maintaining precision in phrasing, proved difficult. The same applied to the exam; 15 minutes pass quickly when there is much to tell.

S2. The most significant challenges stemmed from this being a hectic term. An 8-week placement [in school] concurrent with the subject resulted in limited time for project work at certain junctures. Delving into CCS was also challenging due to its intricate nature and my minimal prior knowledge.

S3. Working with the SRP was notably challenging, as this method was unfamiliar territory for someone in their third year [of the teacher education programme]. It was a different approach compared to what one might be accustomed to, but it became more effective as familiarity grew. The complex nature of the content on carbon capture and storage was also particularly daunting at the onset of the project. There was a lot to grasp, but collaboration with peers and the lecturer made the process more manageable over time. Working on the topic continuously was undoubtedly beneficial, culminating in a feeling of accomplishment when the final report was submitted.

S5. While the topic was unfamiliar, it also meant I learned a great deal about geology and carbon storage.

S7. It was a novel approach, which became a little bit demotivating when progress wasn't swift.

S8. Initially, it was somewhat challenging to genuinely delve into answering the generating question and to pinpoint where to begin. Furthermore, commitments in other modules meant there wasn't ample time to immerse oneself in the SRP, making it challenging to get fully acquainted with the task as getting into a good flow took time. Maintaining motivation was also a bit of a hurdle, especially during periods of stagnation.

7 If challenges in the course have been weakened or eliminated during the semester, what has contributed to this?

S1. In terms of geological terminology, it mostly involved reading up on the subject. Regarding the maximum word count, we have been critical of what should be included, simplified our sentences, and received assistance (from you) in eliminating non-essential elements.

S2. My partner's interest in CCS, or a strong desire to understand the field, has motivated me to spend more time reading. Once I finished my placement, I also had more time for the subject.

S4. It was nice to meet the rest of the class and present to one another.

S5. The lecturer provided valuable feedback on our work, guiding us in a way that made us feel we were on the right track.

S7. The further we delved into the SRP, the easier it became to work with. Receiving feedback from the teacher also helped, reassuring us that our efforts were valuable.

S8. As one got into a good workflow, it became easier. It was also more straightforward to maintain motivation once we had found some answers and began working with the SRP. Being able to ask questions along the way when unsure has also been immensely helpful.

8 Please indicate your level of agreement with the following statements about the didactic paradigm of questioning the world.

	Strongly disagree	Slightly disagree	Somewhat agree	Totally agree
Not knowing the answer to a generating question in advance would be a serious obstacle for me as a future teacher.	0	8	0	0
I will probably try to collaborate with teachers in other subjects to implement SRPs in school.	0	0	4	4
I lack the tools to operate as a teacher in the new didactic paradigm.	2	4	2	0
I liked the fact that in an SRP you come across unexpected facts and ideas that you probably wouldn't have come across otherwise.	0	0	3	5

9 Comments on my answers to the statements in the previous point.

S1. Regarding the first question, about not knowing the answer, it would likely feel different from "usual" teaching. One just has to be confident that students may end up with more knowledge this way.

S2. I wouldn't say that not knowing the answer to a generating question in advance would be a major obstacle, but rather a disadvantage because I want to be in control of what my students get in terms of learning outcomes.

S3. Collaborating with other teachers on SRPs in the school, I believe, is a good way to execute interdisciplinary projects. It doesn't require any extra equipment, but it demands that we, as teachers, embrace a far more open-ended task than we're used to, where together with students we can find answers to, for instance, generating questions about climate challenges.

S6. I plan to get teachers on board who are keen to test out SRP as a method; if not, this can also be applied in other contexts.

S7. I'm not entirely sure how I would specifically set something up in school based on the ATD. It would have been nice to get some concrete examples while we were learning about it.

S8. I would like to use SRPs in teaching, but will probably have some other teachers with me in the beginning. It can be slightly uncomfortable not knowing all the answers in advance, but it will indeed be a good learning experience for me.

10 Through MA3001, have you developed a new understanding of any bodies of knowledge (different from your previous understanding)? If so, what does this change entail?

S1. I have gained a deeper understanding of ATD. I actually thought ATD was a theory resulting from the paradigm of questioning the world, but I realise it's the other way around.

S4. Yes, refer my answer to Question 4.

S5. New understanding within carbon capture and storage, and more knowledge in executing an SRP where we don't know what the final results will be.

S8. Working with SRPs also offers more opportunities for students to think critically and engage in problem-solving.

11 Is there anything you wish were done differently in MA3001? If yes, please explain.

S1. It would have been interesting to know how the ATD differs from the didactics we learn at ILU [Department of teacher education, NTNU]. I can provide the reading list from the didactic course if necessary.

S2. I wish the course had started earlier in the semester, reducing the workload during my placement period.

S4. Regarding the content related to ATD, it might have been insightful to discuss how to introduce it to a secondary school class. The didactic contract for an SRP was touched upon, but a practical discussion on its implementation would have been helpful.

S7. I would have appreciated a concrete example of how one could use ATD and SRPs in schools. Just briefly, as carrying out an SRP [in the course] was very educational, but I'm unsure how one would approach it in a school setting.

12 Free comments.

S2. I am very pleased with this course and found the tasks and themes intriguing. The lecturer has been academically proficient, skilled at adapting, and motivating.

S3. I would say that this has been the didactic course from which I have learned the most. It's the course that has provided the most academic relevance for the job I'll have after completing my studies. It has been educational both in learning how to use modelling and algebra in the exploratory aspect of mathematics, and the SRP has brought new ideas to interdisciplinary work in schools (it can obviously be used within the subject too).

Analysis of Student Survey Responses on the ATD, SRPs, and CCS

The ATD and Working Methods: Relevance to the Mathematics Teaching Profession

The majority (7 of 8 students) found the subject matter related to the ATD to be very relevant. This shows a high appreciation and understanding of the anthropological perspective within mathematics education. The comments emphasize the importance and relevance of the ATD for their development as prospective mathematics teachers. Of the 8 students, 5 found the SRP on CCS to be very suitable as working method in the course. This showcases a general appreciation of the SRP method when it comes to fostering professionalism. The comments on assignment and working methods reflect a range of perspectives on the SRP method: S1 finds SRP relevant especially in the context of sustainable development, which can be seen as an application of the ATD where societal challenges are integrated into mathematical practices. S2 acknowledges the complexity of integrating subjects other than mathematics within CCS, hinting at the intricate balance needed in praxeologies. S6 provides mixed feedback, suggesting that while SRPs might be fun and relevant, it might not align with everyone's "teaching preference."

Surprising Elements in MA3001

S2 and S4 highlighted unexpected aspects of the course that turned out to be positive: S2 was pleasantly surprised by the level of involvement and influence the students had in shaping the course. S4 was initially taken aback by the complexity of the SRP topic, but the apprehension faded as the objectives became clearer.

With respect to interests in and relevance of MA3001, the majority of responses point towards the appreciation of the interdisciplinary approach of the course and the various teaching methods. S1 found the ATD and the multidisciplinary nature of CCS interesting. S2 and S8 appreciated the group work and collaboration in the course. S3 and S4 emphasized the value of the SRP research method and its applicability in teaching. S7 highlighted both the ATD and SRP as engaging different approaches to learning.

Foundational Studying Factors in MA3001

The team-centred approach and the seminar discussions across teams stand out as a recurring theme among the students. S1, S2, S3, and S8 spoke about the significance of collaboration, group work, and the interactions with their peers in enhancing their learning experience. The role of the lecturer was highlighted by several students (S2, S3, S5, and S7), with emphasis on her adaptability, passion, and proficiency in guiding students. S3 and S5 mentioned the seminar-based instruction and quality content as critical contributors to their understanding and engagement. S6 valued discussions with peers, suggesting the course fostered an environment of active engagement and peer learning. S7 praised the instructor's empathetic and skilled teaching approach, acknowledging the novelty of the subject for the students. S8 stressed the importance of open communication and the opportunity to ask questions, reflecting a supportive studying environment.

Overcoming the Hurdles in MA3001

Praxeological Context: The praxeology of the course, involving methodologies like SRP and intricate content like CCS, was a significant source of challenge. However, this challenge was alleviated through various tools, including self-study, feedback from the instructor, and peer collaboration.

Ecosystem of Support: It is evident that the course ecosystem, which encouraged open communication, feedback, and peer support, played a vital role in aiding students to navigate challenges. The lecturer's proactive role and the collaborative spirit of the class were pivotal.

Motivational Aspects: The motivational challenges were countered through peer influence, achieving clarity in the subject, and receiving guidance and reassurance from the lecturer. This indicates that motivation, while internal, can be nurtured by external influences and a supportive learning environment.

In the context of the ATD, understanding these challenges and resolutions helps identify the praxeological obstacles students might face in such courses and offers insights into crafting more effective didactic strategies for similar educational settings in the future.

Beyond Traditional Teaching: Challenges Navigating a New Didactic Paradigm

Praxeological Context: The responses and comments reflect a student body that is adapting to a new teaching paradigm, one that emphasises exploration, team-work, and interdisciplinarity. While there is acknowledgment of its value, there are also concerns about maintaining desired learning outcomes and a clear transition from theoretical understanding to practical application.

Personal and Institutional Adaptation: The personal comfort and confidence of students play a crucial role in determining their readiness to embrace the new didactic paradigm. There is a sense of cautious optimism, coupled with a desire for more tangible examples and support in implementing SRPs as a new method of teaching (supervision) and learning (study) in their future profession as teachers.

Interdisciplinary Collaboration: There is a clear inclination towards interdisciplinary collaboration. This signals a move away from isolated subject teaching towards a more integrated approach that mirrors real-world problem solving.

The students' responses and comments about challenges encountered in the course suggest that while there is a recognition of the value of the paradigm of questioning the world, there is also a need for more robust support systems, practical examples, and collaboration to ease the transition for future teachers.

Strengthened Understanding of Bodies of Knowledge

In their reflections on MA3001, students have demonstrated both a deepening and reshaping of their understanding of certain topics. S1 underwent a pivotal shift in understanding the relationship between the ATD and the paradigm of questioning the world. S4's response, while referring to a previous comment, suggests an evolved grasp of the course's content. S5's insights span a broad spectrum, encompassing both the intricate subject of carbon capture and storage and the intricate pedagogical process of SRPs where outcomes are unpredictable. Lastly, S8 perceives SRPs not just as educational strategies but as instrumental tools to cultivate critical thinking and problem-solving skills in students. This reflection embodies the praxeological dimension of the ATD, highlighting the importance of understanding the types of tasks—mathematical as well as didactic tasks—and the techniques and technologies to solve them.

Students' Desires and Appreciations for MA3001

Through the ATD lens, feedback from MA3001 students highlights key insights into knowledge learnability (partially pertaining to epistemology) and a keenness to translate into the school context acquired ways of studying (pertaining to praxeological analysis and engineering). These insights can be condensed into four main themes, which are briefly outlined below.

Shifting Epistemological Foundations: Student responses underscore their new relations to teaching and learning, marked by a willingness to embrace open-endedness and interdisciplinary teamwork. Several comments indicate students' recognition that engaging in interdisciplinary approaches, like SRPs, can lead to a richer, more integrated perspective on knowledge and problem-solving.

Praxeological Considerations: Feedback points to a desire for practical applications and real-world teaching scenarios, emphasising the praxeological dimension of the ATD. Students are eager to understand how to apply in secondary education the techniques and technologies they have acquired. They seek clarity on transitioning from theory to practice, underlining the importance of tangible examples and techniques for introducing SRPs in schools.

Links With Other Theoretical Frameworks: The comparison between the ATD and the didactics studied at ILU (Department of Teacher Education, NTNU) emerges as a significant theme. One of the students exhibits curiosity about how the ATD complements or contrasts with other theoretical frameworks they have been exposed to. This cross-referencing is open to progress and yet to come.

Affirmation and Satisfaction: Notably, while there are suggestions for improvement, several students communicated profound satisfaction with the course. Students appreciated its academic depth and the versatility it brought to their future teaching careers. The blend of modelling, algebra, and the introduction of SRPs has significantly impacted their perspective on interdisciplinary work in schools. Furthermore, the course's ability to stimulate interest and maintain engagement, backed by supposedly relevant and adaptable lecturing, reinforces its efficacy and resonance with students.

Concluding Comments

Students manifested a comprehensive appreciation of mathematical modelling, with several highlighting how the course has enhanced their understanding in this domain. A transformative learning experience was evident as students navigated from initial unfamiliarity to deeper comprehension, attributed to the ATD framework. Algebra was perceived not merely as an abstract domain but as a vital modelling tool, echoing the anthropological dimension of the ATD. Feedback on modelling assignments suggested the course's efficacy in fostering praxeological practices, with a notable emphasis on parameters. These parameters, vital in bridging algebraic techniques and technological depth in modelling, were seen as crucial in the wider mathematical context. The chapter also explored the course's alignment with institutional expectations, particularly the LK20 National Curriculum. While the course was largely deemed beneficial, some feedback highlighted areas for enhancement, especially regarding the clear integration of algebra and its applicability in broader subjects, like Carbon Capture and Storage (CCS).

As mentioned in the introductory section, S1, S4, and S5 had taken the MA3061 course (n.d.), which introduced the ATD and SRPs, while S2, S3, S6, S7, and S8 had not. Most of the challenges and desires for changes were voiced by S6, S7, and S8. This aligns with the expectation that they might find the course content and methodologies more demanding compared to the 4th year students who had previously taken MA3061. S2, though a 4th year student, had not taken MA3061. However, she faced no significant challenges, possibly due to collaborating with a teammate who had completed MA3061. Similarly, S3, a 3rd year student, experienced few challenges, which might be attributed to his 4th year teammates who had completed MA3061. These observations underscore the potential complexities of teaching a course to a mixed group of students, where some have prior knowledge of the theoretical framework and methodologies, while others do not.

The MA3001 course offered a deep engagement with content and didactic methodologies. Student feedback reveals an overall positive reception but indicates areas for enhancement, particularly concerning translating theoretical concepts into practical applications—which constitutes a research programme to be considered. The blending of modelling, algebra, and SRPs significantly impacted students' views on interdisciplinary teaching, making the course a notable experience in their academic journey.

Finally, I would like to briefly comment on the prerequisites for operating within the paradigm of questioning the world. Regarding supervision and execution of an SRP, neither the teacher nor the students are expected to have studied in advance the fields of knowledge pertinent to the generating question. For the SRPs on CCS, this encompassed geology, physics, and chemistry. The teacher and students study and learn side by side, each embodying what the ATD describes as a Herbartian and procognitive attitude (Chevallard, 2015). Such joint exploration can be challenging, as it redefines traditional roles and expectations. It requires both parties to venture into unfamiliar territories, fostering collaboration and ongoing discovery. This shift necessitates a new didactic contract and a novel *topos* for both teacher and students, topics I have discussed in the concluding section of Chapter 2 (this volume).

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Appendix

Appendix A – Reading List

Literature for Mathematical Modelling Using Study and Research Paths

Core texts (for all students)

Bosch, M., & Gascón, J. (2014). Introduction to the anthropological theory of the didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 67–83). Springer. https://doi.org/10.1007/978-3-319-05389-9_5 [Pages to read: 67–73.]

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Springer. https://doi.org/10.1007/978-3-319-12688-3_13

Niss, M. (2015). Prescriptive modeling: Challenges and opportunities. In G. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modeling in education research and practice: Cultural, social and cognitive influences* (pp. 67–79). Springer. https://doi.org/10.1007/978-3-319-18272-8_5

Strømskag, H. (2022, 21 January). *Et notat om det herbartiske skjemaet – En dynamisk modell for studiet av et spørsmål* [A note on the Herbartian schema—A dynamic model for the study of a question]. Department of Mathematical Sciences, NTNU.

Strømskag, H. (2022, 23 March). *Et notat om begrepet «didaktisk kontrakt»* [A note on the concept of "didactic contract"]. Department of Mathematical Sciences, NTNU.

Strømskag, H. (2022, 8 April). *Om dialektikken mellom systemer og modeller: Et eksempel med bremselengde* [On the dialectic of systems and models: The case of braking distance]. Department of Mathematical Sciences, NTNU.

Strømskag, H., & Chevallard, Y. (2022a). Conditions for revitalizing the elementary algebra curriculum. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education* (pp. 620–627). Free University of Bozen-Bolzano and ERME. https://hal.science/hal-03745443v1/document

Supplementary texts (for all students)

Barquero, B., Bosch, M., & Wozniak, F. (2019). Modelling praxeologies in teacher education: the cake box. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education* (pp. 1144–1151). Utrecht University and ERME. https://hal.archives-ouvertes.fr/CERME11/hal-02408705v1

Bauman, Z. (2007). Forbrukersamfunn [Consumer society] (A. Peretz, Trans.). In G. E. Schjelderup & M. W. Knudsen (Eds.), *Forbrukersosiologi: Makt, tegn og mening i forbrukersamfunnet* (pp. 229–250). Cappelen.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114. https://www.jasme.jp/hjme/download/05_Yves Chevallard.pdf

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education* (6th ed., pp. 201–204). Routledge.

Doerr, H. (2007). What knowledge do teachers need for teaching mathematics through applications and modelling? In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 69–78). Springer. https://doi.org/10.1007/978-0-387-29822-1_5

Education is key to addressing climate change. (n.d.). United Nations. Climate Action. https://www.un.org/en/climatechange/climate-solutions/education-key-addressing-climate-change

Ely, R., & Adams, A. E. (2012). Unknown, placeholder, or variable: What is *x*? *Mathematics Education Research Journal*, *24*, 19–38. https://doi.org/10.1007/s13394-011-0029-9

Markulin, K., Bosch, M., & Florensa, I. (2021). Project-based learning in statistics: A critical analysis. *Caminhos da Educação Matemática em Revista*, *11*(1), 200–220.

Oversby, J. (2015). Teachers' learning about climate change education. *Procedia – Social and Behavioral Sciences*, *167*(8), 23–27. https://doi.org/10.1016/j.sbspro.2014.12.637

Strømskag, H., & Chevallard, Y. (2022b). Elementary algebra as a modelling tool: A plea for a new curriculum. *Recherches en Didactique des Mathématiques*, *42*(3), 371–409. https://revue-rdm.com/2022/elementary-algebra-as-a-modelling-tool-a-plea-for-a-new-curriculum/

Thagaard, T. (2009). *Systematikk og innlevelse: En innføring i kvalitative metoder* [Systematics and empathy: An introduction to qualitative methods] (3rd ed.). Fagbokforlaget. [Pages to read: 62–63.]

Texts for the individual team

In addition to the texts listed above, the required reading for each team includes the literature and other resources utilized in their SRP on carbon capture and storage.

Appendix B

On the Dialectic of Systems and Models: The Case of Braking Distance

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Introduction

The rapid, abrupt halt of a vehicle in motion: a phenomenon that has saved countless lives and yet, remains shrouded in mystery to many. What really goes into this process? How does the intricate interplay between speed, road surface, and friction determine the braking distance of a vehicle? In this manuscript,[13] I try to unravel this seemingly complex system and delve into its basic mechanics. The ultimate goal is to demonstrate how the design and execution of modelling tasks can enhance students' understanding of the dialectic of systems and models.[14] This interplay between the real-world dynamics of systems being studied on the one hand, and properties of models being constructed, on the other, plays a pivotal role in the didactic paradigm of *questioning the world*, proposed by Chevallard (2015). The principles of this paradigm, foundational to the mathematics education course, "Mathematical Modelling Using Study and Research Paths," taught at the Norwegian University of Science and Technology, are the focus of this discussion.

This manuscript navigates mathematical modelling in teacher education, specifically focusing on the interaction between the system being investigated and the model under construction. The discussion primarily takes its basis from the study titled "Elementary algebra as a modelling tool: A plea for a new curriculum" by Strømskag and Chevallard (2022). Starting with an exploration of a distinctive modelling task on braking distance, sourced from a textbook for Grade 11 (students aged 16–17), it dissects the solutions and critically assesses the interplay between the system and its model within the task. The manuscript further ventures into enhancing the conditions that promote knowledge creation about the investigated system, integrating a concise overview of the phenomenon of *friction* to illuminate the dynamics between the system under investigation and the model under construction. This leads to the unveiling of a revised task about a vehicle's braking distance, meticulously adapted for student teachers, along with a proposed solution. Continuing its trajectory, the manuscript assesses potential adaptations of the revised modelling task to align with varying educational levels, signifying a move towards more sophisticated task design. This advancement is further highlighted in the closing section, proposing a novel task design suitable for Grades 8–13.

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[13] This manuscript builds upon a note I developed in 2022, as part of our exploration of braking distance modelling in the course.

[14] In this manuscript, the term "students" is used to encompass both university students and school pupils.

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A Modelling Task From a School Textbook

In the Norwegian textbook for Grade 11, *Sinus 1T* (Oldervoll et al., 2020, p. 375)[15], under the headline "Polynomial Regression", the following task is found:

1.130

The stopping distance for a car in motion hinges on both the driver's response time and the braking distance.[16] The table below outlines the stopping distance, denoted by $S(x)$, in metres corresponding to certain speeds in kilometres per hour for a specified car and a specified driver.

x (km/h)	40	60	80	100
$S(x)$ (m)	24	45	73	108

a) Plot the data points from the table in a coordinate system and elucidate why a quadratic function seems to be a suitable fit.

b) Determine the quadratic function, S, that most accurately represents the given data. Sketch the graph incorporating the data points. Ensure that the expression of the function is accurate to three decimal places.

c) Find graphically the speed that would result in a stopping distance of 150 metres.

d) Find graphically the stopping distance corresponding to a speed of 90 km/h.

A Condensed Solution to the Task

Given that the task is located in the textbook under the headline "Polynomial regression", it is clear that regression analysis is the expected technique for identifying a second-degree polynomial that best fits the given data set. We use GeoGebra and arrive at the following function: $S(x) = 0.009x^2 + 0.175x + 3$. The graph is depicted in Figure 1, where solutions to the final two subtasks are illustrated by the points (90, 89.6) and (120, 150).

[15] The task has been translated into English by the author.

[16] Stopping distance = Reaction distance + Braking distance.

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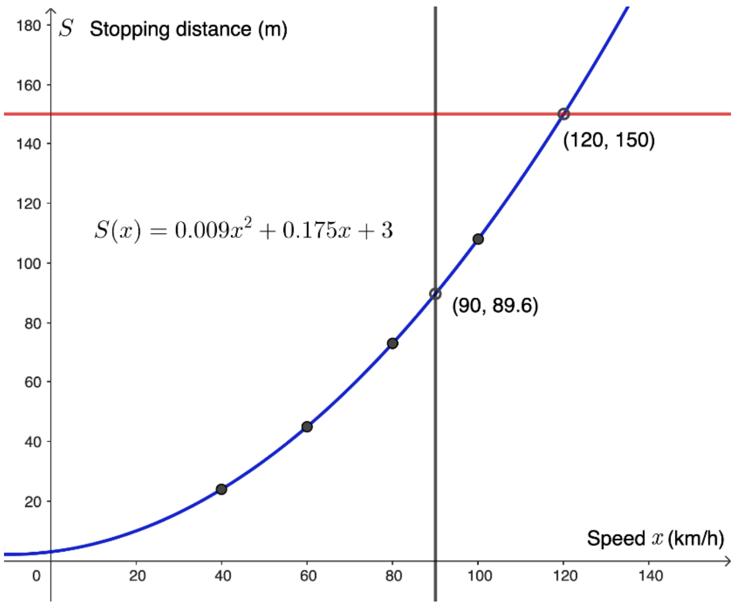
Figure 1

A Graphical Solution to the Textbook Task



Examining Relationships Between the System and the Model in the Task

The model constructed is a second-degree polynomial, $S(x) = ax^2 + bx + c$, with parameters $a = 0.009$, $b = 0.175$, and $c = 3$. While this model fits the given data points, its parameters present limitations as they lack clear physical interpretation. When the speed is zero ($x = 0$), the model counterintuitively suggests a stopping distance of 3 meters ($c = 3$), which opposes real-world scenarios; a stationary vehicle should have no stopping distance.

For positive values of a and b, the quadratic term ax^2 and the linear term bx both suggest that the stopping distance increases with speed, which is generally true. However, the specific coefficients ($a = 0.009$ and $b = 0.175$) have no clear interpretation in the system being studied. They fail to directly correlate with physical factors influencing stopping distance like brake efficiency, tyre grip, and variations in driving conditions, such as road surface. Additionally, these coefficients do not explicitly relate to the complex interplay of psychological perception and physiological response involved in the driver's reaction time.

While the model, shaped by regression analysis, may conform well to the given data, it provides limited insight into the underlying system being modelled. There is a discernible disconnect between the parameters of the model and the real-world dynamics of the system. As a result, the second-degree polynomial model is inapplicable for producing knowledge about the system being investigated, thereby constraining the educational potential of this modelling activity. Might we not then ponder the true essence of this task's significance?

Refined Conditions for Generating Knowledge About the System Studied

In the context of mathematical modelling in education, a fundamental consideration is how we—as mathematics teachers and educators—can amplify the educational value of a modelling activity. One key strategy in this endeavour involves *questioning*. Indeed, asking insightful questions about the system being investigated creates opportunities for exploration and learning by illuminating the defining properties of the system. These properties, or system parameters, and their interrelationships, form the crux of our understanding. Thus, encouraging a study of these parameters aids in unearthing the intrinsic structure and behaviour of the system. The aim is therefore to design modelling tasks—that is, to create conditions—so that students may construct models that *produce knowledge* about the systems under consideration (Chevallard, 1989). This is related to the first stage of the modelling process: the delineation of the system we intend to study, specifying the attributes that are relevant to the study we want to make of this system.

Let us go back to the system where a vehicle in motion brakes abruptly on a horizontal surface. Now, our aim is to create opportunities for students to construct a model that allows them to generate knowledge about the system under investigation. In this scenario, the exclusive focus is placed on the braking distance, as it allows us to isolate and better understand the effects of physical forces at play when the brakes are applied. The reaction distance, although important in real-world situations, introduces additional parameters like human perception and response time, which are outside the scope of our current physics-based exploration.

A key factor in the physics of braking distances is the *friction coefficient*. The friction force between the vehicle's tyres and the road surface is what allows a vehicle to stop. When you apply the brakes, you are essentially using the vehicle's brake system to convert kinetic energy into heat via friction. Changes in this coefficient will greatly affect the system being examined. For instance, on wet or icy roads (where the friction coefficient would be lower), the braking distance will be significantly longer. The next section provides some background knowledge about this phenomenon.

Friction: A Quick Overview

This section provides a concise synopsis of key concepts about friction, largely based on the work of Grimenes et al. (2011). Friction is a force that opposes the movement of one solid object sliding or rolling over another. There are several types of friction, including static friction (friction between objects that are not moving relative to each other), kinetic or sliding friction (friction between objects that are moving relative to each other), and rolling friction (friction that acts on rolling objects).

It is friction that naturally halts various forms of everyday motion unless other forces intervene. We should appreciate the existence of friction, as it is indispensable for many of our daily activities. For instance, without friction, we would be unable to walk, cycle, or drive a vehicle. When we walk, friction provides the necessary grip that allows our feet to push against the ground without slipping. This interaction propels us forward, a consequence of Newton's third law of motion. It is the friction between our feet and the ground that enables us to maintain our balance and manoeuvre in desired directions. In cycling or driving, friction plays a comparable role. The tyres of a bicycle, or a vehicle, grip the surface due to friction, propelling the vehicle forward when power is applied. Additionally, the friction between the bicycle's brake pads and the wheels, or a vehicle's brake system and its wheels, provides the stopping power when required. In the absence of friction, the tyres would slide uncontrollably along the surface, rendering steering and control of the vehicle virtually impossible.

Mathematical Representation of Friction

The force of friction can be represented by the following formula: $F = \mu N$, where F is the force of friction, μ (mu) is the coefficient of friction, which is a dimensionless scalar value that describes the ratio of the force of friction between two bodies to the force pressing them together. N is the normal force, or the perpendicular force with which the surfaces push against each other. In most introductory physics problems, the scenario is something like a box (object) on a flat surface (another object, like a table or the Earth). In these cases, the normal force is equal to the weight of the box, because the surface needs to exert an upward force equal to the weight of the box to prevent it from going through the surface, in accordance with Newton's third law of motion.

Coefficient of Friction

The coefficient of friction (μ) is dependent on the materials involved in the frictional interaction. Typically, the value of μ is established empirically. Table 1 presents a selection of friction coefficient values across different material pairs.

Table 1

Typical Friction Coefficients Between Various Materials

Material 1	Material 2	μ (Friction coefficient)
Steel	Steel	0.6
Steel	Ice	0.05
Steel	Teflon	0.04
Ice	Ice	0.03
Rubber	Dry asphalt	0.7
Rubber	Wet asphalt	0.2
Rubber	Ice	0.02
Wood	Wood	0.3
Hip joint	Hip joint	0.003

Note. The table is adapted from Grimenes et al. (2011, p. 68).

Spotlight on the Interaction Between System and Model

Building upon the analysis of the textbook task, the vision of modelling, and synopsis of friction from the three preceding sections, the following task—inspired by one found in Grimenes et al. (2011, p. 69)—serves as a reformulation of the textbook task. It delves deeper into the theoretical underpinnings of a scenario involving a vehicle braking and sliding on a horizontal surface. This endeavour is geared towards exemplifying a modelling task and its potential solutions, with an emphasis on the intricate interplay between the system under study and the model developed.

Braking Distance of a Vehicle: A Revised Task for Student Teachers

A vehicle is travelling at the speed of 72 km/h on a horizontal road. The vehicle needs to brake suddenly, and the driver presses the brake pedal so hard that the brakes lock. The vehicle then slides along the road until it comes to a stop. We assume that the coefficient of friction between the tyres and the road during sliding is 0.7—that is, the system to be studied involves frictional interaction between rubber and dry asphalt (according to Table 1).

a) Develop a model that allows you to provide answers in response to the following prompts: How long is the braking distance in the described scenario? Provide examples of how different road conditions and varying speeds can impact the braking distance of a vehicle sliding after brake lock.

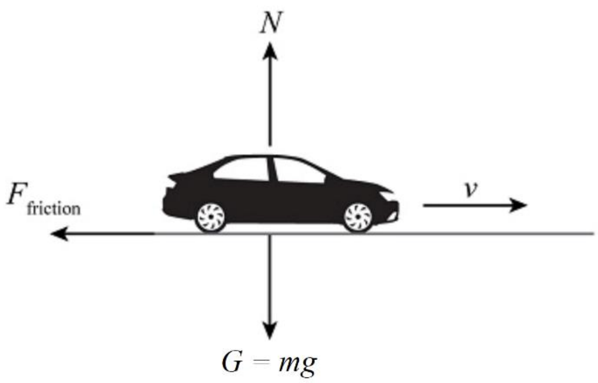
b) Examine the interplay between the system and the model: What insights does the constructed model yield about the system under investigation? Can you describe any specific constraints or limitations associated with this model?

Proposed Solution to the Task

a) In order to determine the braking distance, it is imperative to understand the vehicle's acceleration. This understanding can be gained through Newton's laws of motion. As a first step, we study the forces acting on the vehicle, illustrated in Figure 2. For additional insight, you might consider this tutorial from the website "The Physics Classroom":[17] https://www.physicsclassroom.com/Physics-Video-Tutorial/Newtons-Laws/Force-Of-Friction/Video.

Figure 2

Forces Acting on the Vehicle



In the vertical direction, the force of gravity (G) acts downwards, and the normal force (N) acts upwards. Since there is no acceleration in the vertical direction, Newton's first law states that these forces are equal, $N = G$. As for the direction of the friction force for bodies sliding, it is opposite to the direction of motion. Experiments show that the frictional force acting on a body sliding on a surface has little dependence on speed and on the contact surface area. Experiments have shown that the friction force, $F_{friction}$, is approximately proportional to the normal force on the body from the surface, that is, $F_{friction} = \mu N$, where μ is the friction coefficient for sliding friction. Furthermore, because $N = G = mg$, we have that $F_{friction} = \mu mg$.

[17] "The Physics Classroom" is an online, free to use physics website developed by Tom Henderson, primarily for beginning physics students and their teachers. For more details, see at https://www.physicsclassroom.com/.

In the horizontal direction, only the friction force acts against the movement. We then determine the acceleration, a, using Newton's second law: $\Sigma F = ma$. In this case, $\Sigma F = -F_{friction}$. Therefore, $-F_{friction} = ma$, and considering that $F_{friction} = \mu N$ and $N = mg$, it follows that $-\mu mg = ma$. This can be simplified to: $a = -\mu g$.

We can now find the braking distance using the equation of motion for the constant acceleration case: $v^2 - v_0^2 = 2ad$ (Grimenes et al., 2011, p. 32). The parameters are defined as follows:

- a denotes the constant acceleration of the object (which in the case of braking is negative);
- g denotes the acceleration resulting from gravity (commonly approximated as 9.81 m/s^2 at sea level on Earth);
- m denotes the mass of the object (which in this case is the mass of the vehicle);
- d denotes the distance travelled (which in this case is equal to the braking distance);
- v denotes the velocity of the object in the final state (which in this case is equal to 0);
- v_0 denotes the speed in the initial state (which is the speed at the moment the braking starts);
- μ denotes the coefficient of friction (which depends on the properties of the materials in contact, in this case, the rubber of the tyres and the road surface).

We transform the equation $v^2 - v_0^2 = 2ad$, and get d expressed by the other parameters:

$2ad = -v_0^2$ which gives that $d = \frac{-v_0^2}{2a} = \frac{-v_0^2}{2\cdot(-\mu g)} = \frac{v_0^2}{2\mu g}$. This equation is our model for the braking distance of a vehicle travelling on a horizontal road and braking suddenly at speed v_0:

$$d = \frac{v_0^2}{2\mu g}.$$

Using the given values, we can find the braking distance, d. We want to calculate the braking distance in metres and therefore need to convert the speed to m/s. By converting 72 km/h to metres per second, we get 72,000 metres/3,600 seconds, which corresponds to a speed of 20 m/s. We insert this into the equation and get the following result: $d = \frac{(20\text{ m/s})^2}{2\cdot 0.70 \cdot 9.81\text{ m/s}^2} \approx 29$ m. This means that at a speed of 72 km/h, the braking distance on dry asphalt ($\mu = 0.7$) will be approximately 29 m.

Different Situations With Varying Parameter Values

We can use the model to calculate the braking distance on wet asphalt at the same speed, 20 m/s. According to Table 1, $\mu = 0.2$ in this case, and we get $d = \frac{(20\text{ m/s})^2}{2\cdot 0.2 \cdot 9.81\text{ m/s}^2} \approx 102$ m.

Let us calculate the braking distance on dry asphalt at the speed of 90 km/h. This speed is equivalent to 25 m/s (as 90,000 metres/3,600 seconds = 25 m/s). Substituting into the model, we find that $d = \frac{(25\text{ m/s})^2}{2\cdot 0.7 \cdot 9.81\text{ m/s}^2} \approx 45.5$ m. This indicates a braking distance of approximately 45.5 meters.

What if we brake suddenly on wet asphalt at a speed of 90 km/h? This will give a braking distance of $d = \frac{(25\text{ m/s})^2}{2\cdot 0.2 \cdot 9.81\text{ m/s}^2} \approx 159$ m. The resulting braking distance of 159 metres is alarmingly long, emphasising the importance of reduced speeds and increased caution when driving on wet or slippery road conditions.

b) The model $d = \frac{{v_0}^2}{2\mu g}$ permits generation of the following knowledge:

From our model, we can deduce that the braking distance is independent of the vehicle's mass. This suggests that for any vehicle, given the same speed and friction coefficient, the braking distance will remain constant. Furthermore, the model shows that the braking distance increases with the square of the initial speed (i.e., the speed at which the brakes are applied). To put it another way, if the initial speed changes by a factor of k (from v_0 to kv_0), the braking distance will change by the square of this factor (from d to k^2d).

Let us calculate the braking distance during sudden braking on wet asphalt, when the speed is reduced from 20 m/s to 10 m/s. With a speed of 10 m/s (36 km/h), we get $d = \frac{(10\text{ m/s})^2}{2\cdot 0.2\cdot 9.81\text{ m/s}^2} \approx 25.5$ m. We see here that when the speed is reduced from 20 m/s to $\frac{1}{2}\cdot 20$ m/s, the braking distance is reduced from 102 m to 25.5 m = $\frac{1}{4}\cdot 102$ m. This provides a concrete example of the quadratic proportionality property of the model.

Some Constraints and Limitations

The model $d = \frac{{v_0}^2}{2\mu g}$ is based on the assumption that there is no ABS (Anti-lock Braking System) operating in the system being studied. ABS is a technology that prevents the wheels from locking during braking. Such a mechanism helps to maintain road grip and vehicle stability during braking and can therefore reduce braking distance.

A limitation of the model is that it relies on simplifications and assumptions to describe the braking distance. It assumes uniform acceleration throughout braking and ideal conditions such as a constant coefficient of friction and an even surface. In reality, braking distance can be affected by several factors, including road conditions, tyre condition, vehicle design and driver reaction time. The model therefore provides a simplified description of the system and may have limitations when used to predict accurate braking distances in complex situations.

Lastly, it should be noted that the values of the friction coefficients used in the model are empirically determined and may carry some uncertainty. These coefficients are based on previous observations and experiments under various driving conditions.

Tailoring the Revised Modelling Task to Various Educational Levels

This section delves into the adaptation of the task for different educational levels. The task and its solution, outlined in the previous section, are optimally designed for students already equipped with fundamental knowledge in mechanics. However, its inherent flexibility enables its modification to serve as a learning tool for students just beginning their journey in basic mechanics. This adaptability largely depends on the specific objectives of the modelling task and the time allocation for its execution.

Moreover, the task can be altered in a manner that sidelines the need for an understanding of the mechanics intrinsic to the system being studied. One potential strategy could involve offering two distinct sets of data for system analysis, customised for dry and wet asphalt conditions. These datasets would correlate the vehicle's speed with its respective braking distance under different speed scenarios. For instance, the observations displayed in Table 2 could serve this purpose. (Keep in mind that while students will need to convert speed values from km/h to m/s, it would be beneficial for them to identify this requirement on their own.) Concurrently, students are taught about the phenomenon where a vehicle's braking distance is directly proportional to the square of its speed upon braking.

Table 2

Speed vs Braking Distance on Varying Road Conditions

Speed (km/h)	Braking distance on dry asphalt (m)	Braking distance on wet asphalt (m)
20	2,25	7,87
40	8,99	31,46
60	20,23	70,79
80	35,96	125,85
100	56,18	196,64

This joint presentation of empirical data and underlying principles offers a suitable framework for students to develop the model $d = kv^2$ and calculate the proportionality constant k for both conditions. Such a process not only enhances students' practical application skills in mechanics, but also deepens their understanding of the nature of scientific inquiry and the process of transforming real-world phenomena into mathematical models.

Advancing Task Design

This section introduces a task which centres around the crafting of modelling assignments for secondary school. Mirroring the structure of the braking distance assignment, the student teachers were encouraged to initiate from a textbook task. However, in this instance, the selection of the task was left to their discretion, with the objective to revise and enhance the connection between the system and the model outlined in the original task scenario. The exact wording of the task, including detailed subtasks, is provided in the following section.

Design of Modelling Tasks for Grades 8–13

Identify a modelling task in a secondary school textbook that, in your opinion, inadequately supports the generation of knowledge about the system it aims to model. Your assignment is to design a new task, pertaining to the same system, in a way that strengthens the connection between the system under study and the model to be constructed. Subtasks a–c formulated below are meant to help you in studying the system to be examined, in order to prepare your own task design. Moreover, the note with a proposed solution on braking distance is available for your perusal, and could serve as a useful reference or spark of inspiration in your own work.[18]

a) Delineate the system *S* that will be the focus of your study. State the initial question *Q* that you aim to answer regarding *S*. Define the relevant parameters that are crucial for studying *S*, along with their respective relationships.

[18] The note referenced herein is elucidated in Footnote 1, found on page 1 of this manuscript.

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b) Build up a suitable model S' that can effectively address the initial question Q about the system S. Explain how this model S' can not only answer Q but also produce additional knowledge about the system S. Provide a brief overview of the background knowledge required for constructing the model S'.
c) Discuss the relationships between the system studied (S) and the model constructed (S') including the affordances (strengths and advantages) and constraints (limitations and assumptions) of the chosen model in representing the real-world dynamics of S.
d) Utilising the knowledge generated through the previous subtasks, devise a modelling task suitable for students in Grades 8–13 (select one of the grades). The task should explore the same system (S) and answer a question (Q), concerning S. Q can either be the same or different from the one you addressed previously. Strive to design the task in a way that enables students to develop knowledge about the real-world dynamics of S, while ensuring it is both appropriately challenging and manageable for the chosen grade level.

In closing, it should be recognised that the eventual impact of the proposals and discussions within this manuscript, as they pertain to secondary education, represents an open question, one that awaits further scholarly exploration.

References

Chevallard, Y. (1989). Le passage de l'arithmétique à l'algébrique dans l'enseignement des mathématiques au collège—Deuxième partie. Perspectives curriculaires : la notion de modelisation. *Petit x*, *19*, 45–75. https://publimath.univ-irem.fr/numerisation/PX/IGR89002/IGR89002.pdf

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Springer. https://doi.org/10.1007/978-3-319-12688-3_13

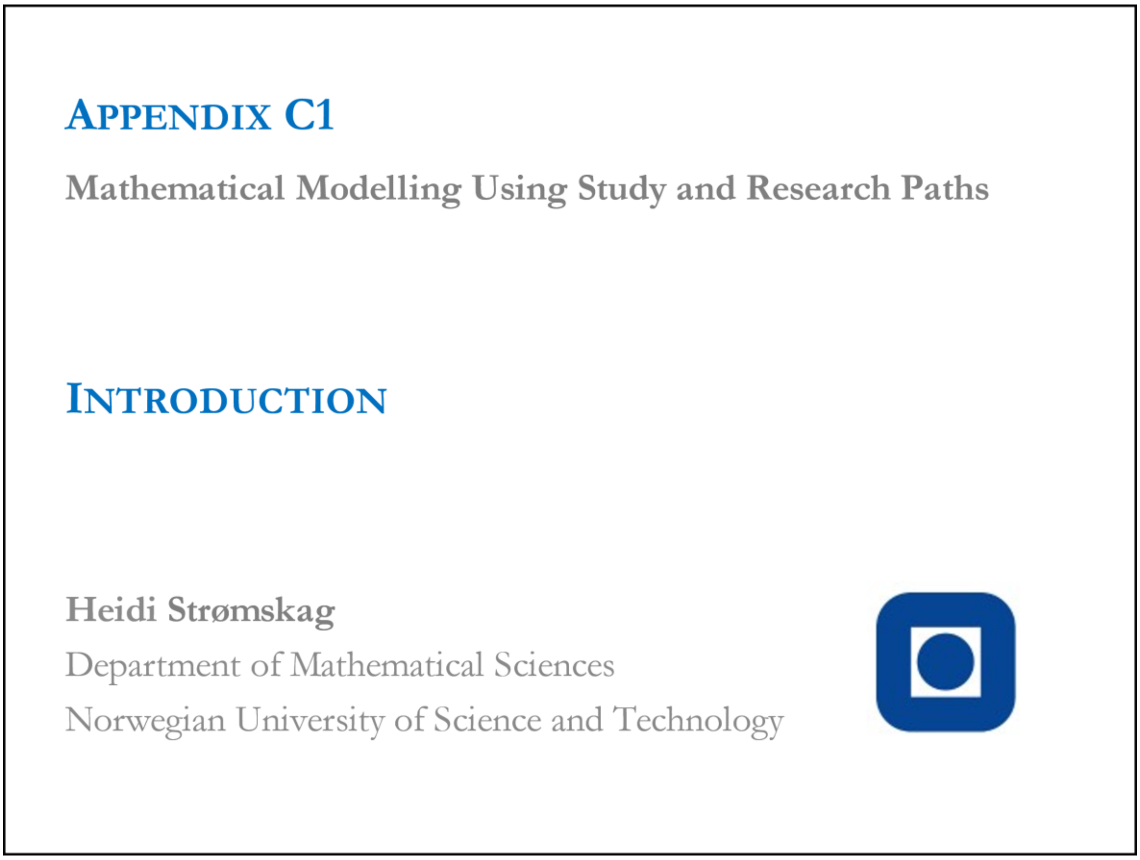
Grimenes, A. A., Jerstad, P., & Sletbak, B. (2011). *Grunnleggende fysikk for universitet og høgskole* [Basic physics for universities and university colleges]. Oslo: Cappelen Damm.

Oldervoll, T., Svorstøl, O., Vestergaard, B., Gustafsson, E., Osnes, E. R., Jacobsen, R. B., & Pedersen, T. (2020). *Sinus 1T: Matematikk* (4th ed.). Oslo: Cappelen Damm.

Strømskag, H., & Chevallard, Y. (2022). Elementary algebra as a modelling tool: A plea for a new curriculum. *Recherches en Didactique des Mathématiques*, *42*(3), 371–409. https://revue-rdm.com/2022/elementary-algebra-as-a-modelling-tool-a-plea-for-a-new-curriculum/

Appendix C – PowerPoint-Slides

C1 – Introduction



Constituents of the course

Unit 1	**Modelling of systems using algebra as a modelling tool A new didactic paradigm**
Duration	8 × 45 min.
Organization	5 × 45 min. (lectures) + 3 × 45 min. (exercise classes).
Topics	The paradigm of questioning the world. Modelling of systems. Algebra as a modelling tool.

Unit 2	**Some tools from the anthropological theory of the didactic**
Duration	2 × 45 min.
Organization	2 × 45 min. (lecture).
Topics	Didactic system; Study and research paths; Herbartian schema.

Unit 3	**Climate change**
Duration	7 × 45 min.
Organization	1 × 45 min. (lecture) + 4 × 45 min. (SRP_{pilot}) + 2 × 45 min. (presentation and discussion).
Topic	Basic knowledge on climate change.

Unit 4	**Carbon capture and storage**
Duration	40 × 45 min.
Organization	6 × 45 min. (lectures / question time) + 30 × 45 min. (SRP) + 4 × 45 min. (presentation and discussion).
Topics	Models constructed by scientists on CCS. Parameters and their interrelationships.

Unit 5	**The role of modelling in school mathematics Design of modelling tasks**
Duration	5 × 45 min.
Organization	3 × 45 min. (lectures) + 2 × 45 min. (exercise classes).
Topics	Systems in the natural and the social world.

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Study and research paths (SRPs)

- A type of problem-based learning
- Rooted in the *Anthropological Theory of the Didactic* (ATD)
  - Initiated and mainly developed by Yves Chevallard (Bosch & Gascón, 2014)
- Towards a new didactic paradigm (Chevallard, 2015)
  - Involves asking questions about the world around us (the *Paradigm of Questioning the World*)
  - The current didactic paradigm (which is in decline)?
  - Why do we need a new paradigm?
  - What are the challenges of the new paradigm?
  - What does the national curriculum (LK20) say about mathematics teaching and learning?

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Generating question *Q*

Carbon capture and storage is crucial to achieving the Paris Agreement's target of a maximum global temperature increase of 1.5 °C this century.

Q. How is carbon capture and storage modelled in the literature?

What mathematics are included in these models? What parameters are involved, and what is the relationship between them?

Heidi Strømskag Appendix C1 4

An SRP to answer a generating question *Q*

Find existing answers to *Q* (in the literature and multimedia)
To understand these existing answers, one must:
Study works (study part)
To understand these works, one must:
Ask and answer derived questions (research part)
To answer the derived questions, one must:
Find existing answers to the derived questions (in the literature and multimedia)
To understand these existing answers, one must:
Study works (study part)
To understand these works, one must:
Ask and answer new derived questions (research part)
To answer the new derived questions, one must:
… etc.

Heidi Strømskag Appendix C1 5

Theoretical tools to tackle mathematical teaching and learning processes

- Didactic system
- Model – system
- Didactic milieu
- Herbartian schema
- Directed graph

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Pilot-SRP

Generating question:

Q. What is climate change, and why is it happening?

- Preparation: Read about the Herbartian schema (Strømskag, 2022)
- Work in self-selected groups of two or three

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Pilot-SRP (cont.)

Generating question:

Q. What is climate change, and why is it happening?

Procedure for the pilot study

1. Search for sources that can provide *existing answers* to *Q*.
2. As you study these existing answers, formulate and answer *derived questions*, Q_i, to understand the existing answers. Search for *works* (articles, books, reports, or multimedia resources) that help you understand existing answers and help you answer the Q_i.
3. Write a mini-report from the inquiry, which includes:
    - An answer to *Q* in terms of a synthesis of the answers to the Q_i.
    - A representation of the pathway of the inquiry, including an overview of the Q_i and works you have studied to arrive at the final answer to *Q*. This can be done through a directed graph accompanied by a table that explains its nodes.
4. Based on the mini-report, create a PowerPoint that presents the SRP you have conducted. All teams will present their inquiry on Friday 11 February. Time frame for presentation: 5 min. + discussion.

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A modelling task

- *Comparison of thermoses*
- Assignment to be found in Blackboard
- To be discussed on 9th February
- Relevant article: Niss (2015)

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Appendix C – PowerPoint-Slides

C2 – Elementary Algebra as a Modelling Tool

APPENDIX C2

Mathematical Modelling Using Study and Research Paths

ELEMENTARY ALGEBRA AS A MODELLING TOOL

Heidi Strømskag
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Elementary algebra (Grade 8–13)

- The role of algebra in school mathematics
- Some history
- Didactic transposition
- What does it take to revitalize school algebra?

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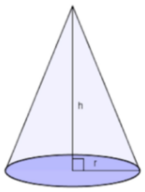
Some history

From the end of the 16th century, it became apparent how algebra was a modelling tool to generate ***formulas***

- via mathematicians like e.g., François Viète, Thomas Harriot, René Descartes

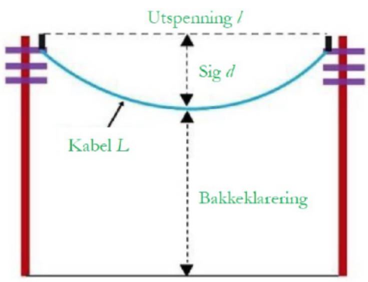
1) First in the capacity of ***formulas as algebraic models***:



Ex 1

$$V = \frac{1}{3}\pi r^2 h$$

(formula for the volume *V* of a coin with radius *r* and height *h*)



Ex 2

$$L = l + \frac{8d^2}{3l}$$

(formula for the length *L* of a cable with span *l* and sag *d*)

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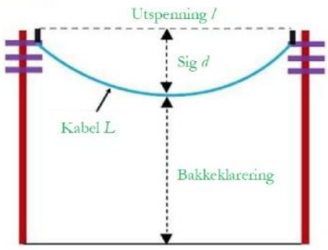
Some history (cont.)

2) Then in the capacity of **formulas as equations with parameters:**

- Solving the equation for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, <u>with respect to</u> $\boldsymbol{r}$ or $\boldsymbol{h}$.
  - $r = \sqrt{\frac{3V}{\pi h}}$
  - $h = \frac{3V}{\pi r^2}$
- Solving for the cable length, $L = l + \frac{8d^2}{3l}$, <u>with respect to</u> $\boldsymbol{d}$ *or* $\boldsymbol{l}$.
  - $d = \sqrt{\frac{3l(L-l)}{8}}$
  - $l^2 - Ll + \frac{8d^2}{3} = 0 \Rightarrow l = \frac{L \pm \sqrt{L^2 - 4\frac{8d^2}{3}}}{2} = \frac{L}{2} \pm \sqrt{\frac{L^2}{4} - \frac{8d^2}{3}}$



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The significance of algebra

"One of the most important applications of elementary Algebra is to the use of formulae. In every form of applied science and mathematics … formulae are constantly employed, and their interpretation and manipulation are essential." (Abbott, 1942/1971, s. 69)

- *Teach yourself algebra* (by Percival Abbott, 1869–1954) in the series "Teach Yourself" (publisher Hodder & Stoughton) was a *bestseller*.
- Percival Abbott was an influential British mathematics teacher (he also wrote *Teach Yourself Calculus*)

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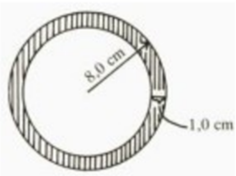
Formulas in algebra

- Abbott explains that work on formulas involves 3 operations (Abbott, 1942/1971, p. 69):
  - **Construction**
  - **Manipulation**
  - **Evaluation**

Example (adapted from an exercise inErstad et al., 1984, p. 127)



In this task you will need that a circle with radius r has area πr^2. The figure shows a cross section of a water pipe. The outer radius of the pipe is equal to 8.0 cm and the thickness of the pipe material is equal to 1.0 cm.

a) Calculate the area of the pipe material (shaded in the figure). — Evaluation

Let the outer radius of the pipe be equal to R and the thickness of the pipe be equal to x.

b) Show that the area A of the pipe material is given by $A=2\pi Rx-\pi x^2$. — Construction

c) Find the outer radius R and the thickness x expressed by the other parameters. — Manipulation

d) Calculate the thickness x when $A=88$ cm^2 and $R=8.0$ cm. — Evaluation

Heidi Strømskag Appendix C2 6

A key concept: Didactic transposition

- The theory of ***didactic transposition*** was introduced in 1985 by YvesChevallard (Chevallard, 1985).
- Refers to the ***transformations*** a body of knowledge undergoes from the moment it is produced by scientists, to the moment it is chosen and designed by people from the "noosphere" (curriculum designers, textbook authors, and others) to be taught, until it is actually taught and studied in a given educational institution (Chevallard & Bosch, 2014).

Figure 1. Didactic transposition processes (adapted from Chevallard og Bosch (2014, p. 171)

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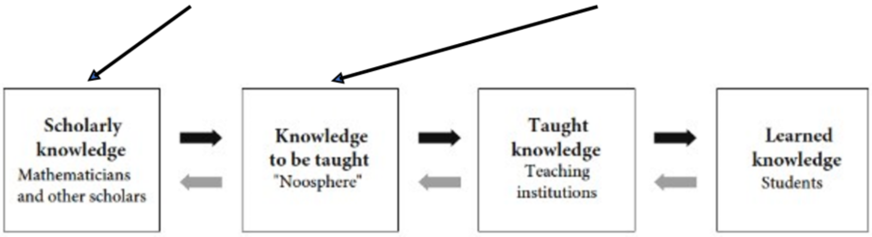
Analysis of didactic transposition processes

- To study teaching and learning in the classroom, it is not enough to only study what happens in the classroom (e.g., how teachers and students *think* and *do*)
- *The knowledge taught must itself be the object of analysis*!
- The researcher studies transformation processes that knowledge undergoes *between the instances* in the figure below.
  - Anna Karina Kristianslund's master's thesis (NTNU, 2022): Analysis of the didactic transposition processes that differential calculus has undergone from the textbook *Kalkulus* (university) to the textbook *Mønster R1* (Grade 12).



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Examples of didactic transformations

- Simplifications
  - Some types of tasks disappear
- Distortions
  - For example, theorems are "reshaped" into definitions
- ...

Purpose?

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Result of a *simplification* of algebraic knowledge (Example 1)

Physical law: When resistors R_1, R_2, and R_3 are er connected in parallel, the total resistance, R, is calculated using the equation:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

- To solve this equation with respect to R_i is no longer part of the algebra curriculum for Grade 8–13 → This type of task has disappeared.

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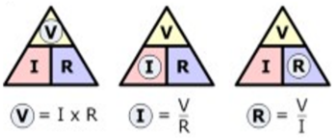
Result of a *distortion* of algebraic knowledge (Example 2a)

Background knowledge:

- Ohm's law says that the electric **voltage** (U) is equal to the product of **resistance** (R) and **amperage** (I). The voltage is the force needed to move the current through a circuit. The resistance tells us how much the material the electricity moves through slows the current.
- $\boldsymbol{U = R \cdot I}$, where U is measured in volts, R is measured in Ohm, and I is measured in amperes.
    - Which of U, R, and I are parameters?

Distortion:

Instead of learning algebraic calculations to solve equations with respect to one of the other parameters in an equation (here the parameters R and I), students in school learn a mnemonic technique (a «triangle trick») – which is a distortion of the knowledge.



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Result of a *distortion* of algebraic knowledge (Example 2b)

From the website ***Matematikkens verden*** **[The World of Mathematics]**

https://www.matematikkensverden.no/2015/07/brker-med-symboler.html

Distance, speed, and time

"Sometimes you will come across fractions with letters. The letters are symbols for different values."

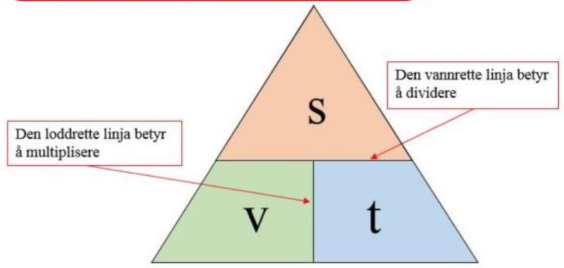
***Comment*:**
The triangle trick below involves rote learning of what the horizontal and vertical lines shall mean. But what if we put *v* or *t* on top of the triangle instead of *s*?

$v = \frac{s}{t}$ Denne brøken kan vi snu på, alt ettersom hva vi skal finne ut.

s er strekning og måles i m eller km

t er tid og måles i timer, minutter og sekunder

v er fart og måles i km/t (km/h) eller m/s



Skal vi finne farten blir regnestykket slik:

$v = \frac{s}{t}$

Skal vi finne strekningen blir regnestykket slik:

$s = v \cdot t$

Skal vi finne tiden blir regnestykket slik:

$t = \frac{s}{v}$

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Result of a *distortion* of algebraic knowledge (Example 3)

From the textbook *Sinus 1T*:

A formula gives us the value of a variable by help of the value of one or several other variables. The volume V of a sphere is given by

$$V = \frac{4}{3}\pi r^3.$$

In this formula, we find the value of V when we know the value of the variable r, which is the radius. The variable r we call the independent variable and V we call the dependent variable. We choose values for the independent variable and calculate the value of the dependent variable. The formula above also contains the constant $\pi \approx 3.14$.

Sometimes we need values of two variables to calculate the third. The volume V of a cylinder is given by V=πr2h. … In this case we have two independent variables and one dependent variable. In most of the formulas we will work on in this book, we will have one independent and one dependent variable.

$$V = \pi r^2 h.$$

In this case we have two independent variables and one dependent variable. In most of the formulas we will work on in this book, we will have one independent and one dependent variable. (Oldervoll et al., 2020, p. 24)

Parameters have been displaced here: Formulas are fixed, rigid, immobile expressions– they have ceased to be fuel of algebraic work!

Heidi Strømskag Appendix C2 13

Algebra at work

- **Formulas** (in terms of **equations with parameters**) are fundamental in modelling of systems in the world around us.

Construction

Manipulation →

Evaluation

Manipulations are lacking because equations with parameters have disappeared through the didactic transposition of elementary algebra.
This is a serious shortcoming when it comes to modelling of mathematical and extramathematical systems (i.e., systems in nature and in society).

Measure:
To be able to answer questions through modelling of systems in the world around us, we need to *revitalize the algebra curriculum* in school.

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Modelling of systems in the world around us

Algebra as an effective *modelling tool* in school:

1. Students start from a system S and a question Q about S
2. Students build a model S' of S, relative to Q, based on elementary algebra—and which will include as many parameters as necessary.
3. They work on S' and arrive at an answer to Q.
4. Through the process of inquiry, they discover the resources of algebra, studying them and then making effective use of them.

(Strømskag & Chevallard, 2022)

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Task (temperature scales)

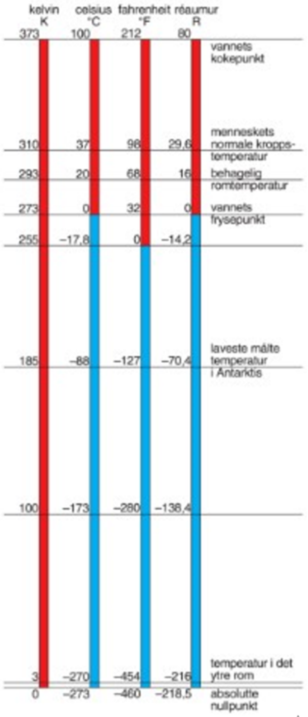

Figure taken from *Store norske leksikon*: https://snl.no/temperaturskala

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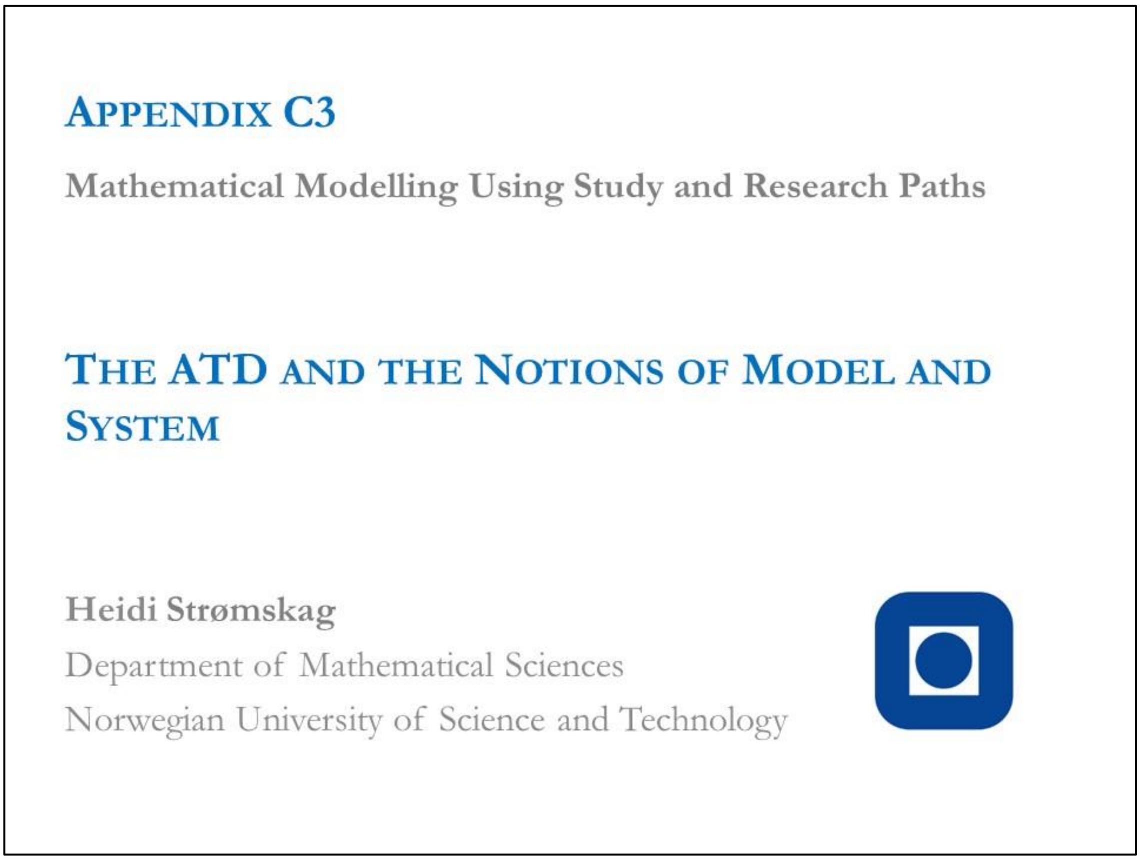
Appendix C – PowerPoint-Slides

C3 – The ATD and the Notions of Model and System



Research framework

- The Anthropological Theory of the Didactic, **ATD** (Bosch & Gascón, 2014; Chevallard, 2019)
- **In the ATD, there are four decades of research on the teaching of algebra**

  Selected publications: Chevallard (1984, 1985, 1989, 1990, 1994); Gascón (1993, 1999, 2011); Bosch & Chevallard (1999); Bolea, Bosch & Gascón (1999, 2001, 2004); Ruiz-Munzón, Bosch & Gascón (2007, 2011, 2015, 2020); Ruiz-Munzón (2010); Chevallard & Bosch (2012); Ruiz-Munzón, Matheron, Bosch & Gascón (2012); Bosch (2015); Strømskag & Chevallard (2022a, 2022b, 2023).

Heidi Stromskag Appendix C3 2

Bringing elementary algebra back to full life

Redefining the secondary algebra curriculum in terms of what is *needed* to questioning, understanding and changing the world.

Example 1: Algebra of percentages

For $r > 0$, we have $q\left(1+\frac{r}{100}\right)\left(1-\frac{r}{100}\right) = q\left(1-\frac{r^2}{10\,000}\right) < q$

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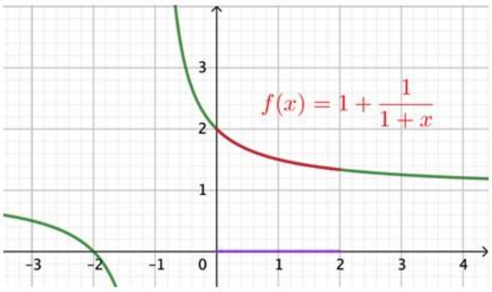
Bringing elementary algebra back to full life (cont.)

Example 2: An algebraic model of $\sqrt{2}$

$x = \sqrt{2} \Rightarrow x^2 - 1 = 1 \Rightarrow x = 1 + \frac{1}{x+1} = f(x)$, with f a contraction mapping. We have that f is a continuous and decreasing function on the interval [0, 2], and $\sqrt{2}$ is a fixed point of f since $f(\sqrt{2}) = \sqrt{2}$.

Therefore, **if** $a < \sqrt{2} <$ b, **then** $f(b) < \sqrt{2} < f(a)$.

$$1 < \sqrt{2} < 2 \Rightarrow 1 + \tfrac{1}{3} < \sqrt{2} < 1 + \tfrac{1}{2} \Rightarrow 1 + \tfrac{2}{5} < \sqrt{2} < 1 + \tfrac{3}{7} \Rightarrow \cdots$$
$$\Rightarrow 1.4142132 < \sqrt{2} < 1.41421393 \Rightarrow \cdots$$



Heidi Stromskag Appendix C3 4

The notions of *system* and *model* in the ATD

A ***system*** is a fragment of reality that has its own laws. Let us consider a system $\mathcal{S}$:

A system $\mathcal{S}'$ is said to be a ***model*** of $\mathcal{S}$ if, by studying $\mathcal{S}'$, one can produce knowledge about $\mathcal{S}$

- studying *questions* Q about $\mathcal{S}$ by asking $\mathcal{S}'$ about these questions
- choosing models $\mathcal{S}'$ of $\mathcal{S}$ whose study of question Q is easier, safer, quicker than by a "direct" study of $\mathcal{S}$

$\mathcal{S}'$

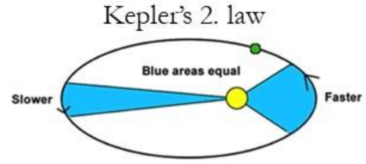
$\mathcal{S}$

For example, if the radius r of a sphere increases by 20%, the new surface area A' will become $4\pi(1.2r)^2 = 1.44A$, so that the surface area will increase by 44% – a tricky result to obtain experimentally.

Heidi Stromskag Appendix C3 5

Descriptive versus normative/prescriptive models

- **Descriptive models: models *of* something**
  - Kepler's laws of planetary motion
  - Galileo's laws for bodies in free fall
  - Newton's law of cooling
  - Geometric models of objects



- **Normative models: models *for* something**
  - Templates in clothing production
  - The pension formula in public pension schemes
  - The formulae for elections and mandates in the distribution of power after elections
  - Gini coefficient for income inequality
  - The BMI formula

$$\text{BMI} = \frac{\text{weight (kg)}}{[\text{height (m)}]^2}$$

(Hilgers, n.d.; Niss, 2015)

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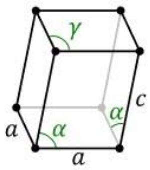
Descriptive versus normative (uses of) models

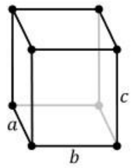
Given the ordered pair $(\mathcal{S}, \mathcal{S}')$

$\mathcal{S}'$ $\mathcal{S}$



$\mathcal{S}'$ is a *descriptive* model of $\mathcal{S}$ if $\mathcal{S}'$ is modelled on (or after) $\mathcal{S}$ so that $\mathcal{S}'$ is regarded as a **model *of* $\mathcal{S}$**

$\mathcal{S}'$ $\mathcal{S}$



$\mathcal{S}'$ is a *normative* model of $\mathcal{S}$ if $\mathcal{S}$ is modelled on (or after) $\mathcal{S}'$, that is, $\mathcal{S}'$ is a **model *for* $\mathcal{S}$**

Depending on the *situations studied*, models have descriptive and normative ***uses***

Heidi Stromskag Appendix C3 7

The Herbartian schema – A model of an inquiry

- A **didactic system** is a triplet $S(X, Y, k)$
- The **Herbartian schema** is a dynamic model of a didactic system's inquiry into a generating question Q

$$[S(X, Y, Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}$$

↷ creates

↪ generates

(Chevallard, 2019)

Heidi Strømskag Appendix C3 8

The didactic system related to *Q*

$S(X, Y, Q)$

- $X = \{x_1, \ldots, x_9\}$
- $Y = \{\text{lecturer}\}$
- Q: ***How is carbon capture and storage modelled in the literature?***

  What mathematics is involved in these models?

  What are the parameters involved, and what are the relationships between them?

- Students x_i working in 4 teams to answer Q

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The milieu *M* for the study of a generating question *Q*

$M = \{A_1^{\Diamond}, A_2^{\Diamond}, \ldots, A_m^{\Diamond}, W_1, W_2, \ldots, W_n, Q_1, Q_2, \ldots, Q_p, D_1, D_2, \ldots, D_q\}$

Existing answers found in the literature / on the Internet	Works to study and understand $A_i^{\Diamond}$	Questions derived from the study of $Q, A_i^{\Diamond}$, and W_j	Dataset collected through various types of research during the study of Q

Heidi Stromskag Appendix C3 10

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Heidi Stromskag Appendix C3 11

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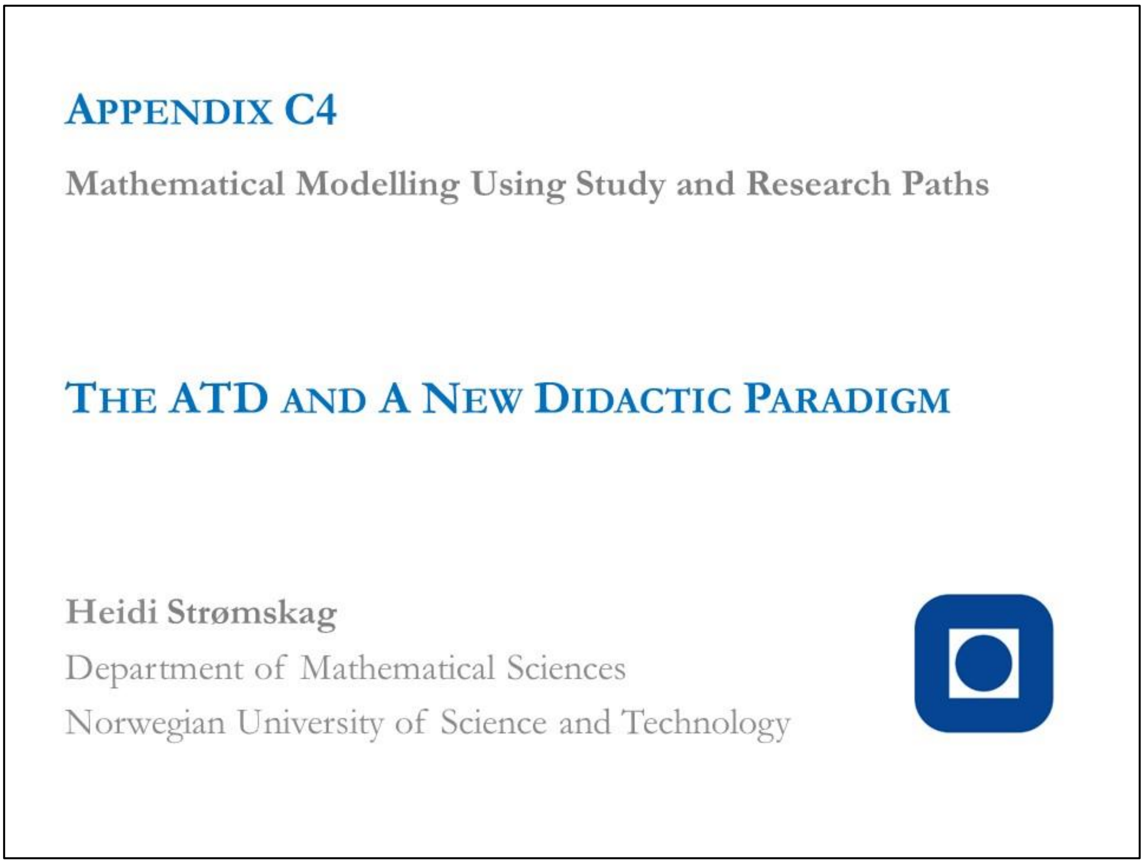
Appendix C – PowerPoint-Slides

C4 – The ATD and a New Didactic Paradigm



The anthropological theory of the didactic

Founder and main developer of the ATD since the 1980s is Yves Chevallard, professor emeritus at Aix-Marseille Université, France.

Another important contributor is Marianna Bosch, professor at Universitat de Barcelona, Spain.

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Briefly on the ATD

ATD is about the ***didactic***

- The focus is on concrete actions that people or institutions perform or decide, which are intended to *teach someone something*.
- ***Praxeology*** is an analytical tool for studying such actions – it is a two-part model of the way we perform and explain human actions:

1) A ***type of tasks*** that can be solved by certain ***techniques*** ➤ task types and techniques make up the *praxis block* of a praxeology (how actions are performed).

2) Any technique needs an explanation called ***technology*** which in turn has a justifying explanation at a higher level, called ***theory*** ➤ technology and theory make up the *logos block* of a praxeology (the knowledge behind actions).

- Examples of different types of tasks: solving a differential equation, designing a task for students, writing a text, making pancake batter, skiing down a hill.

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Paradigm

- A paradigm is a kind of *contract* that governs a specific type of human activity
- The "clauses" in such a contract are usually not explicitly formulated
  - Jean-Jacques Rousseau (1762/1988) wrote about the *social contract*:

    "The clauses of this contract is so determined by the nature of the act that the slightest modification would render them null or void, so that, **although they have never perhaps been formally enunciated, they are everywhere the same, everywhere tacitly accepted and recognized**." (p. 92)

- Thomas Kuhn: *The Structure of Scientific Revolutions* (1962)
  - ➢ on scientific paradigms and paradigm shifts

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The current didactic paradigm

The paradigm of "*visiting works*" ("knowledge monuments"):

- The reason to study a work *w* is for *w*'s own sake:
  - *w* is considered valuable in its own right—*w* is recognised in the culture and in the curriculum
  - *w* is not studied because of what it allows us to think or do...
- Studying *w* is like visiting a museum
- Any examples you've experienced like this?

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Why do we need a new didactic paradigm?

Heidi Stromskag Appendix C4 6

The new didactic paradigm

The paradigm of “questioning the social and natural world around us”, rooted in ATD (Chevallard, 2015)

Three principles:

1. Every community has obligations to its members. One of them is to define a curriculum that ensures that community members are able **to think and act appropriately**, in a way that serves themselves and others in the various social settings in which they participate (especially in terms of family, profession and citizenship)
2. The curriculum should enable members to (individually or collectively) **identify, formulate and respond to issues** they face.
3. To achieve Point 2, the community should define (and regularly review) a **core curriculum made up of questions** to which community members have the right and duty to respond.

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Philosophical inspiration

- German philosopher and founder of pedagogy:
  Johan Friedrich Herbart (1776–1841): a scientist's attitude to the world
  - seeking answers to unanswered questions (for those concerned)

- Investigative approach (enquiry):
  - "Herbartian"
  - "precognitive"
  - "exoteric"

Getty Images/Hulton Archive

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The 2020 Norwegian curriculum in mathematics (LK20)?

Homework:

When reading Chevallard (2015), study the curriculums for Mathematics 1P and Mathematics R1 and note down attributes from the two paradigms ("visiting works" and "questioning the world").

Google queries:

- Matematikk P kompetansemål
- Matematikk R kompetansemål

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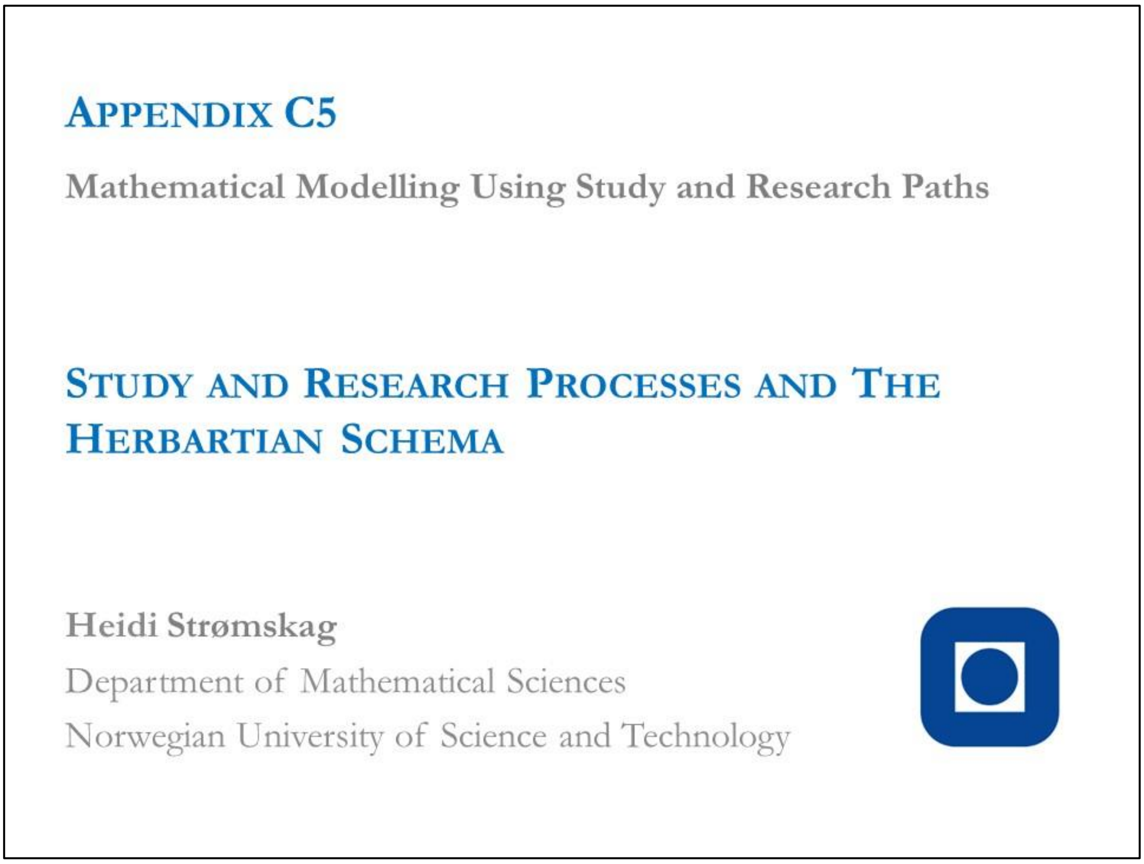
Appendix C – PowerPoint-Slides

C5 – Study and Research Processes and The Herbartian Schema



An SRP to answer a generating question Q

Find existing answers to Q (in the literature, on the Internet)
To understand these existing answers, one must:
  Study works (study part)
  To understand these works, one must:
    Ask and answer derived questions (research part)
    To answer the derived questions, one must:
      Find existing answers to the derived questions (in the literature, on the Internet)
      To understand these existing answers, one must:
        Study works (study part)
        To understand these works, one must:
          Ask and answer new derived questions (research part)
          To answer the new derived questions, one must:
            … etc.

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The Herbartian schema – A model of an inquiry

- A **didactic system** is a triplet $\boldsymbol{S(X, Y, \mathcal{k})}$
- The **Herbartian schema** is a dynamic model of a didactic system's inquiry into a generating question Q

$$[S(X, Y, Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}$$

$\curvearrowright$ creates
$\hookrightarrow$ generates

(Chevallard, 2019)

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The didactic system related to Q

$S(X, Y, Q)$

- $X = \{x_1, \ldots, x_9\}$
- $Y = \{\text{lecturer}\}$
- Q: ***How is carbon capture and storage modelled in the literature?***

  What mathematics is involved in these models?

  What are the parameters involved, and what are the relationships between them?
- Students x_i working in 4 teams to answer Q

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The milieu M for the study of a generating question Q

$$M = \{A_1^{\diamond}, A_2^{\diamond}, \ldots, A_m^{\diamond}, W_1, W_2, \ldots, W_n, Q_1, Q_2, \ldots, Q_p, D_1, D_2, \ldots, D_q\}$$

Existing answers found in the literature and multi-media resources	Works to study and understand $A_i^{\diamond}$	Questions derived from the study of $Q, A_i^{\diamond}$, and W_j	Dataset collected through various types of research during the study of Q

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Displaying the path of an inquiry

(example from an SRP on differential calculus, MA3061)

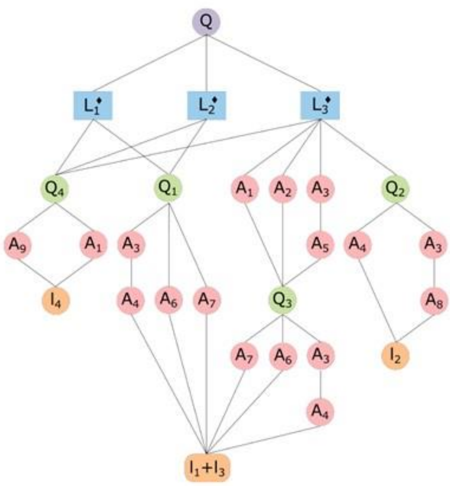

Figure 1. Directed graph displaying the **path of an inquiry** (Team 4, MA3061)

Existing answers, $L_i^\diamond$
$L_1^\diamond$: Sinus 1T and Sinus R1 (publisher Cappelen Damm)
$L_2^\diamond$: Mathematics 1T and Mathematics R1 (publisher Aschehoug)
$L_3^\diamond$: Articles on derivation at National Digital Learning Arena's website for Mathematics 1T
Works, A_i
A_1: Calculus 1 by Adams & Essex (2013)
A_2: Video on limits from NTNU Undervisning (YouTube Channel)
A_3: "Pupils' understanding of the derivative: A case study of R2 pupils' understanding of graphical representations of the derivative." Master's thesis by Fandrem (2016)
A_4: "Students' understanding of differentiation" by Orton (1983)
A_5: "Concept image and concept definition in mathematics with particular reference to limits and continuity" by Tall & Vinner (1981)
A_6: "The historical development of the calculus" by Edwards (1979)
A_7: "The changing concept of change: The derivative from Fermat to Weierstrass" by Grabiner (1983)
A_8: "Introduction to diagnostic teaching in mathematics" by Brekke (1995)
A_9: "Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm" by Chevallard (2015)
Derived questions, Q_i
Q_1. What strength does a graphical approach have and what strength does an algebraic approach have, and how do they supplement each other in an introduction to derivation?
Q_2. Do simplifications appear about derivation in the textbooks $L_1^\diamond$, $L_2^\diamond$, and at $L_3^\diamond$, which can create confusion, and how could they have been avoided?
Q_3. Should pupils first get a proper understanding of limits before they are introduced to derivation?
Q_4. Is "growth speed" an appropriate concept for instantaneous rate of change in a point?

Table 1. **Elements in the milieu** – nodes of the graph in Figure 1 (Team 4, MA3061)

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Reference

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://www.jasme.jp/hjme/download/05_Yves Chevallard.pdf

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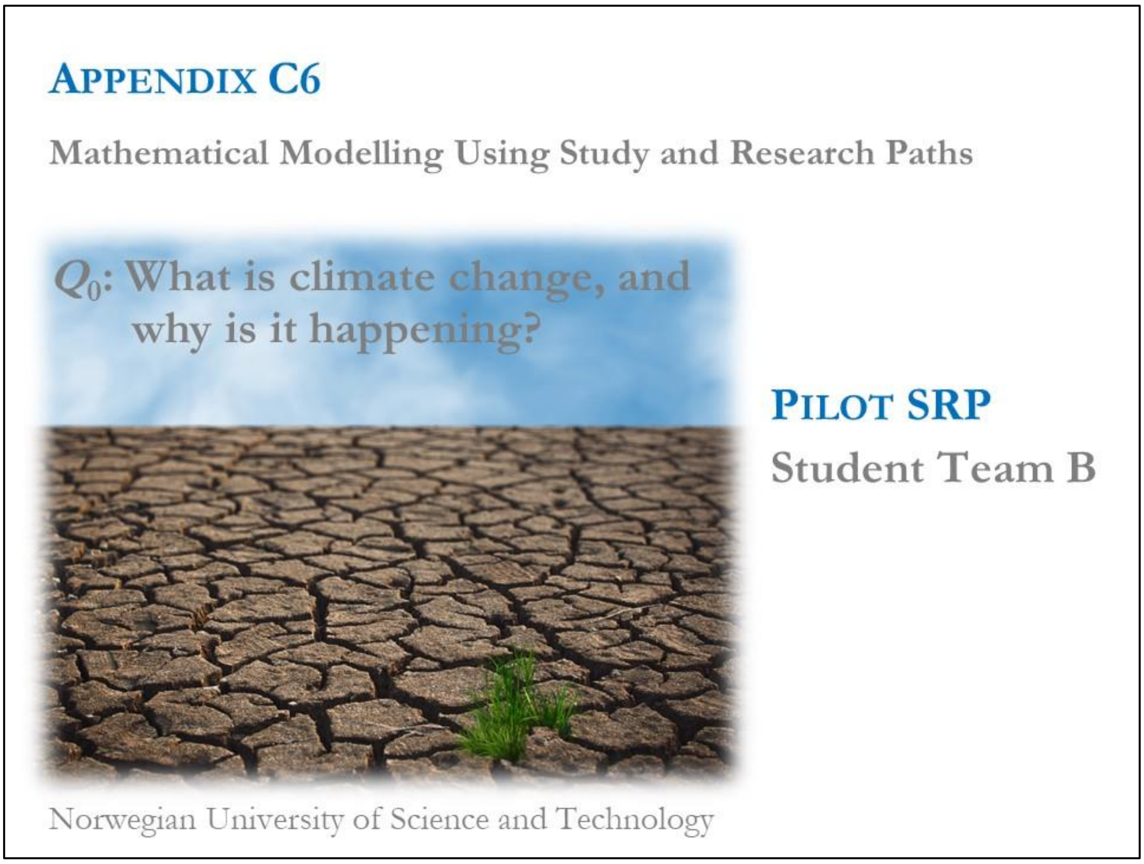
Appendix C – PowerPoint-Slides

C6 – What is climate change, and why is it happening?



Existing answers

- FN-sambandet. (2021). *Klimaendringer* [Climate change]. https://www.fn.no/tema/klima-og-miljoe/klimaendringer
- Miljødirektoratet. (2021). *Klimaendringene skjer her og nå* [Climate change is happening here and now] https://miljostatus.miljodirektoratet.no/tema/klima/

- Earth's temperature is rising
- Ice at the poles is melting
- Sea level rises and becomes more acidic
- More extreme weather

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Derived questions and answers

Q_1: What is climate, and why does it change?

- **W_1:** FN-sambandet. (2021). *Klimaendringer* [Climate change]. https://www.fn.no/tema/klima-og-miljoe/klimaendringer
- **a_1:** Climate is an average of the weather measured over a long period of time. It changes as a result of the Earth emitting more greenhouse gases than it naturally should do. This contributes to an increased greenhouse effect and means that less heat escapes through the atmosphere. Result: increasing temperatures.

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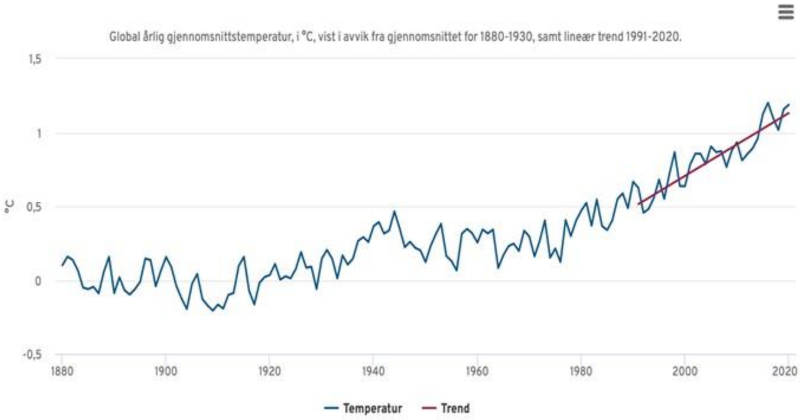
Derived questions and answers (cont.)

Q_2: How has the global temperature changed over the last 200 years?

- W_2: Energi og klima. (2021). *En varmere klode* [A warmer planet]. https://energiogklima.no/klimavakten/global-temperatur
- a_2:



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Derived questions and answers (cont.)

Q_3: Why is there more extreme rainfall than before?

- W_3: Sorteberg et al. (2019). https://www.uib.no/klimaenergi/123978/derfor-f%C3%A5r-vi-ekstremnedb%C3%B8r-knyttet-til-klimaendringene
- a_3: Simplified: A warmer atmosphere can retain more water vapour than a cold atmosphere. The amount of water vapour increases exponentially as a function of temperature, resulting in increased condensation at higher temperatures.

Q_4: How has the coronavirus pandemic affected climate change?

- W_4: Le Quéré et al. (2020). Temporary reduction in daily global CO_2 emissions during the COVID-19 forced confinement. https://www.nature.com/articles/s41558-020-0797-x.pdf
- a_4: Despite reduced CO_2 emissions (up to 17% in April 2020), this has had an equivalent zero effect on climate change.

Q_5: How are changes in sea level measured?

- W_5: Lindsey, R. (2020). *Climate change: Global sea level.* https://www.climate.gov/news-features/understanding-climate/climate-change-global-sea-level
- a_5: Sea level is measured using two main methods: tide gauges and satellite altimeters.

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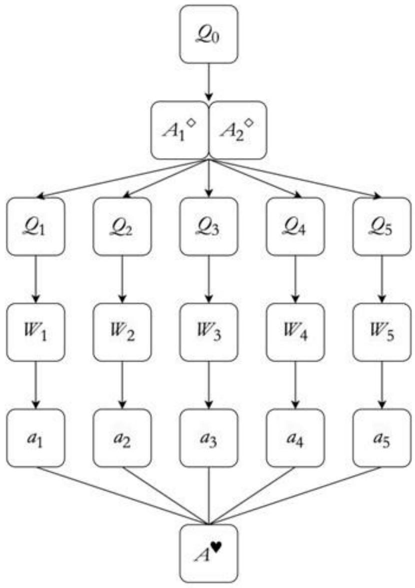
Herbartian schema



Q_0 generating question

$A_i^\diamond$ existing answers

W_j works to be used

a_k partial answers

$A^\heartsuit$ final answer

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Final (though provisional) answer to Q_0

Our final answer $A^\heartsuit$ will be a combination of the existing answers and a synthesis of derived questions:

- **Changes in the average weather over a longer period of time.**
  Unfortunately, a reduction in a short period (like COVID-19) has little effect overall
- **Main cause: Changes in the amount of greenhouse gases in the atmosphere**
  As a result of increased CO_2 in the atmosphere, the pH value of the ocean becomes lower → more acidic (consequences for the ecosystem)
- **Temperature rise causes the ice to melt and the ocean to warm up.**
  The sea is rising: both as a result of "warm" water taking up more space and ice melting
- **More rainfall due to more water vapour in the atmosphere (especially in Norway)**
  This can trigger both floods and landslides

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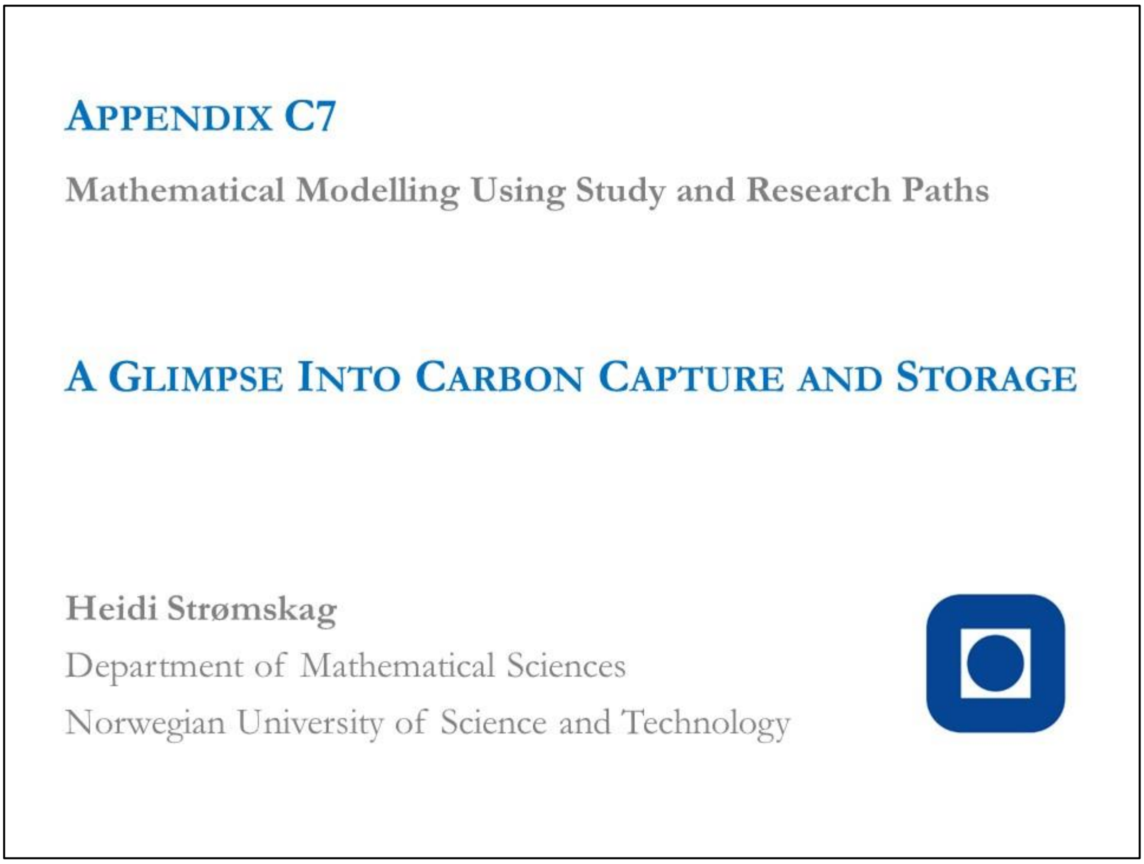
Appendix C – PowerPoint-Slides

C7 – A Glimpse Into Carbon Capture and Storage



Some elements on the topic

- What does Carbon Capture and Storage (CCS) involve?
- History of CCS in Norway
- Legislation
- Models

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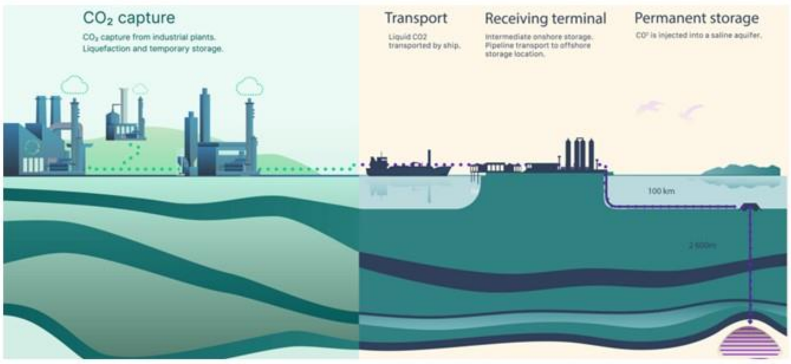
The process towards carbon storage

Phase 1: **Carbon capture**. Here, CO_2 is separated from the rest of the emission gases at the combustion plant.

Phase 2: **Compression** of CO_2 into liquid form.

Phase 3: **Transport** of the compressed CO_2 gas to where it will be stored.

Phase 4: **Carbon storage**. Here, CO_2 is pumped underground in so-called reservoirs. An example could be empty oil reservoirs.



https://northernlightsccs.com/

Karbonfangst og -lagring. (2021, 4 November). Ung Energi. https://ungenergi.no/miljoteknologi/ovrig-miljoteknologi/karbonfangst/

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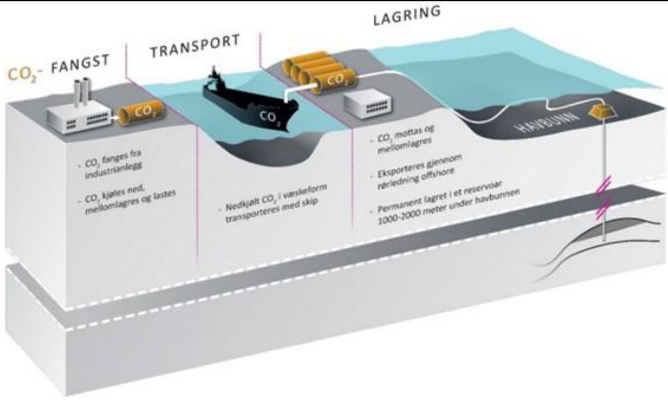

Illustration: Equinor

- Norcem's capture plant is the first of its kind in the world (ca. 900,000 tonnes of CO_2 per year)
- Northern Lights: collaboration between Equinor, Shell and Total
  - The first CO_2 storage facility in Europe that will be opened to European capture operators
  - The CO_2 will be transported by ship to a land terminal in Øygarden in Western Norway, before being piped to an offshore CO_2 storage facility under the seabed in the North Sea for permanent storage.
  - The CO_2 storage facility is located ca. 2500 m below the seabed, south of the Troll field.

https://www.nho.no/tema/energi-miljo-og-klima/artikler/co2-fangst-og-lagring-ccs/

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Carbon capture and storage (CCS) – Norway's history

- Podcast from NRK Radio, 9 November 2021 (18 min.):
- https://radio.nrk.no/podkast/oppdatert/l_b463b28d-e9f9-4d15-a3b2-8de9f91d15f2

2007 Mongstad – "Moon landing" project
2012 Realisation of test facility
2013 Testing terminated
2020 New trial - Norcem Brevik

Norcem Brevik cement factory

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Regulations on storage and transport of CO2 on the shelf (Legislative Data)

Department	Norwegian Ministry of Petroleum and Energy
Published	In 2014, Booklet 15
Entry into force	5 December 2014
Applies to	Norway
Authorisation	LOV-1963-06-21-12-§3
Announced	8 December 2014
Revised	20 January 2015 (§ 6-2)

§ 1-1. ***Purpose***

The purpose of these regulations is to contribute to sustainable energy and industrial production, by facilitating the utilisation of subsea reservoirs on the continental shelf for environmentally safe storage of CO_2 as a measure to counteract climate change.

§ 1-2. ***The right to subsea reservoirs for storage of CO_2***

The Norwegian state has ownership rights to subsea reservoirs on the continental shelf for utilisation of these for storage of CO_2 and the exclusive right to manage them.

https://lovdata.no/dokument/LTI/forskrift/2014-12-05-1517

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Regulations on storage and transport of CO_2 on the continental shelf

Chapter overview:

Chapter 1. Introductory provisions (§§ 1-1–1-11).
Chapter 2. Exploration licence (§§ 2-1–2-6).
Chapter 3. Exploration licence (§§ 3-1–3-5).
Chapter 4. Licence to exploit a subsea reservoir for injection and storage of CO2 (exploitation licence (§§ 4-1–4-15).
Chapter 5. Injection and storage of CO_2 (§§ 5-1–5-13).
Chapter 6. Transport etc. of CO_2 (§§ 6-1–6-3).
Chapter 7. Termination of injection and storage of CO2 (§§ 7-1–7-6).
Chapter 8. Liability for compensation for pollution damage (§§ 8-1–8-8).
Chapter 9. Special rules on compensation to Norwegian fishermen (§§ 9-1–9-6).
Chapter 10. Special requirements for safety (§§ 10-1–10-6).
Chapter 11. General provisions (§§ 11-1–11-24).
Chapter 12. Infringement (§ 12-1).

Appendix I. Criteria for description and assessment of the possible storage site and surrounding area mentioned in § 1-10 of these regulations.

Appendix II. Criteria for establishing and updating the monitoring plan in § 5-4 and the follow-up plan in § 5-7.

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What is a model of a system? (Recap from lecture on 11 Feb.)

- A ***system*** is a fragment of reality with its own laws.
- Let S be a given system. We search for an answer to ***questions*** Q about S.
- A system S' is a ***model*** of S if we can produce knowledge of S by studying S'.

We construct/select a model S' of S such that studying Q is easier, safer and faster than studying S "directly".

Remark: There is no such thing as a universal model of a system: a model is *relative* to the question(s) we seek answers to!

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Opportunities for studies

- What are the **parameters** included in the categories below, describing the geology of the storage location?
- Search the literature for **mathematical models** that include these parameters.

The static geological model or models shall characterise the complex with respect to:

- Geological structure of the physical trap
- Geomechanical, geochemical and flow properties of the reservoir overburden (cap rocks, seals, porous and permeable layers) and surrounding formations
- Characterisation of fractures and faults and presence of natural and man-made flow paths
- Area and vertical extent of the storage complex
- Pore volume (including distribution of porosity)
- Initial fluid distribution
- Other relevant characteristics

(quoted from Legislative Data)

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Some involved concepts

- **Flow** in porous rocks
- Transport
- Dissolution of gas
- Partial differential equation:
  - contains an unknown function of several independent variables and the partial derivatives of the function with respect to these variables
  - used to solve problems involving the spreading of sound or heat, electrodynamics, **fluid mechanics**, elasticity, etc.
- Finite element method:
  - Numerical method for solving partial differential equations in several variables with initial conditions

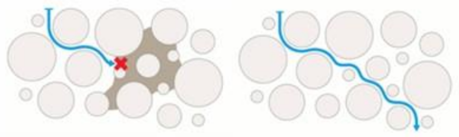

Figure adapted from "Understanding Porosity and Permeability" (2021)

Understanding porosity and permeability. (2021, 2 June). Earth Resources. Victoria State Government. https://earthresources.vic.gov.au/projects/victorian-gas-program/onshore-conventional-gas/porosity-permeability

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Some background knowledge

- When CO_2 is injected into layers of porous rock, it dissolves into the groundwater.
- The background flow of groundwater causes the gas to continuously dissolve into the water rather than just dissolving vertically into an aquifer (water-bearing rock) or rock layer.
- This is because the gas usually collects in structural traps in rock layers, for example in anticlines (geological term for an uplifted part of a folded rock).
- Using mathematics, partial differential equations describing this movement can be solved so that the concentration and velocity profiles of the aquifer can be predicted.
- This provides information on how much CO_2 can ultimately dissolve in the aquifer, the timescales over which this occurs and the mechanisms that control transport.
- Techniques such as finite element methods allow the analysis to be completed using computers.

Assertion: In a typical aquifer used for CO2 storage, it would take around one million years to dissolve the gas injected into it (Unwin et al., 2016).

Question: What model and mathematical calculations underpin this claim? Which parameters are included?

Unwin, H. J. T., Wells, G. N., & Woods, A. W. (2016). CO2 dissolution in a background hydrological flow. *Journal of Fluid Mechanics*, *789*, 768–784. https://doi.org/10.1017/jfm.2015.752

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Some background knowledge (cont.)

THKM (thermo-hydro-chemical-mechanical) processes relevant for CCS:

- T – Thermo: thermal simulations (**energy balance equation**) to estimate the change in temperature in the storage and surrounding formations. Temperature change affects the mechanical stresses and how the fluid behaves.
- H – Hydro: flow simulations (**mass balance equation**) to estimate the dispersion of the injected CO_2 and the pore pressure from the injection.
- K – Chemical: chemical effects (**mass balances, chemical equilibria/reactions, kinetics, equations of state**, etc.), there may be geochemical reactions between CO_2 and the formations, dissolution of CO_2 in water, pressure/temperature dependence of CO_2 and formation water, swelling of the formation, etc.
- M – Mechanical: mechanical simulations (**momentum balance equation**) to estimate stress changes due to the injection (which can be compared with fracture limits to check that the bearing is not compromised).

(T. I. Bjornarå, personal communication, 28 February 2022)

Bjornarå, T. I. (2018). *Model development for efficient simulation of CO2 storage* [Doctoral dissertation, University of Bergen, Norway]. https://bora.uib.no/bora-xmlui/handle/1956/17695

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Some background knowledge (cont.)

Interaction between the processes THKM on the previous slide means that the mathematical models quickly become complicated:

- For example, if you model hydro flow (H), you calculate the pore pressure, and the pore pressure affects the stress field (M), but the compressibility of the formation (mechanical deformation) also affects the flow so that the two processes are coupled (HM).
- Temperature (T) affects the mechanical stresses (TM) which in turn affect the flow (THM), and they also affect the fluid properties (TH).

Assumptions are often made to eliminate interaction processes to simplify the modelling. For instance, temperature effects are often ignored, and if stresses are not of interest, you "only" need to solve for the flow of CO_2 in the reservoir.

(T. I. Bjørnarå, personal communication, 28 February 2022)

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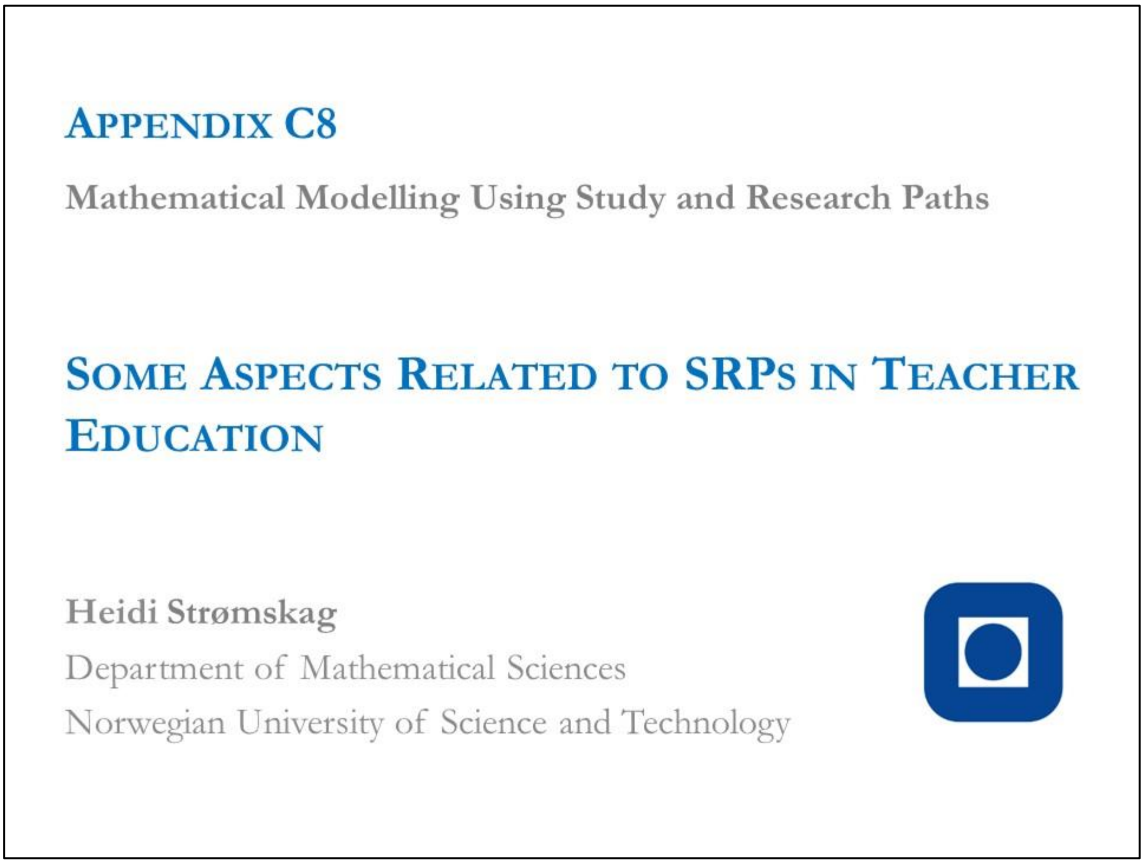
Appendix C – PowerPoint-Slides

C8 – Some Aspects Related to SRPs in Teacher Education



Final report on the SRPs on CCS – some aspects

Information and ideas complementing the guidelines presented earlier

Introduction

- Carbon Capture and Storage: What is it, and why is it important?
- Responsibility of the educational sector: The United Nations Framework Convention on Climate Change (UNFCCC)
- Briefly on SRP as a method to study a question – problem-based teaching/learning format

Theory

- Concepts from the ATD: New didactic paradigm, SRP, Herbartian schema, etc.: Chevallard (2015), Strømskag (2022a)
- System – model: Strømskag and Chevallard (2022)
- Concepts related to the generating question Q (from geology, physics, mathematics, etc.)
- Various kinds of mathematical models

Method

- SRP: Strømskag (2022a)
- Document study (~ content analysis): Cohen et al. (2007), Thagaard (2009)

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Final report on the SRP on CCS – some aspects (cont.)

Results

- Mathematical models in the literature on CCS (parameters and relations between them)
- $A_i^\lozenge$, W_j, Q_k, a_m, $A^\heartsuit$ presented with the help of a directed graph and an associated table

Discussion

- On the method used: SRP involves a new *didactic contract* (Brousseau, 1997; Strømskag, 2022b)
- On the topic studied: Challenges related to teaching about climate change (Oversby, 2015)
- What have you learned from conducting this inquiry?

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Responsibility of the educational sector

The United Nations Framework Convention on Climate Change, UNFCCC:

- Education is a key stakeholder in addressing the issue of climate change
- The UNFCCC assigns responsibility to parties of the convention to conduct climate change awareness campaigns in education and other public activities to ensure public participation in programmes and access to information on climate change

Education can encourage people to change their attitudes and behavior; it also helps them to make informed decisions. In the classroom, young people can be taught the impact of global warming and learn how to adapt to climate change. Education empowers all people, but especially motivates the young to take action. (Education is Key to Addressing Climate Change, n.d.).

Ethical considerations on the climate issue: human consumption, growth (Bauman, 2007)

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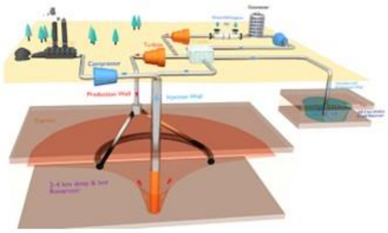
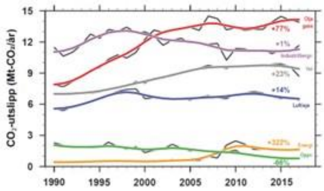
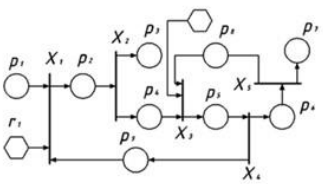
Some types of mathematical models

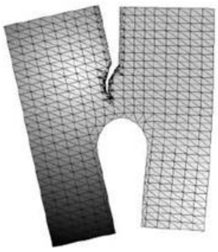
Graphical model (diagram, graph, chart, etc.)







Geometric model (describing shapes)



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Some types of mathematical models (cont.)

Analytic model

- Formula
- Equation
- Inequality
- Function
- Differential equation
- Integral
- Linear system
- …

Model with which we can do *calculations*

Analytic model on *closed form*:

A mathematical model that uses a limited number of standard operations. It can contain constants, variables, certain well-known operations (e.g., $+ - \times \div$) and functions (e.g., *n*th root, power, logarithm, trigonometric functions and inverse hyperbolic functions), but no limits, differentials (derivatives) or integrals.

[ELEMENTARY ALGEBRA]

Analytic model on open form (open expression)

A mathematical model involving boundary values, derivation or integration. These types of expressions involve variables or functions that cannot be expressed using a limited number of algebraic operations. Instead, they are defined by a process, such as calculating a boundary value or performing a derivation or integral, that results in a function or expression that depends on one or more variables.

[CALCULUS / ANALYSIS]

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"Closed-Form Expression" (2021)

v·t·e	Arithmetic expressions	Polynomial expressions	Algebraic expressions	Closed-form expressions	Analytic expressions	Mathematical expressions
Constant	Yes	Yes	Yes	Yes	Yes	Yes
Elementary arithmetic operation	Yes	Addition, subtraction, and multiplication only	Yes	Yes	Yes	Yes
Finite sum	Yes	Yes	Yes	Yes	Yes	Yes
Finite product	Yes	Yes	Yes	Yes	Yes	Yes
Finite continued fraction	Yes	No	Yes	Yes	Yes	Yes
Variable	No	Yes	Yes	Yes	Yes	Yes
Integer exponent	No	Yes	Yes	Yes	Yes	Yes
Integer nth root	No	No	Yes	Yes	Yes	Yes
Rational exponent	No	No	Yes	Yes	Yes	Yes
Integer factorial	No	No	Yes	Yes	Yes	Yes
Irrational exponent	No	No	No	Yes	Yes	Yes
Logarithm	No	No	No	Yes	Yes	Yes
Trigonometric function	No	No	No	Yes	Yes	Yes
Inverse trigonometric function	No	No	No	Yes	Yes	Yes
Hyperbolic function	No	No	No	Yes	Yes	Yes
Inverse hyperbolic function	No	No	No	Yes	Yes	Yes
Root of a polynomial that is not an algebraic solution	No	No	No	No	Yes	Yes
Gamma function and factorial of a non-integer	No	No	No	No	Yes	Yes
Bessel function	No	No	No	No	Yes	Yes
Special function	No	No	No	No	Yes	Yes
Infinite sum (series) (including power series)	No	No	No	No	Convergent only	Yes
Infinite product	No	No	No	No	Convergent only	Yes
Infinite continued fraction	No	No	No	No	Convergent only	Yes
Limit	No	No	No	No	No	Yes
Derivative	No	No	No	No	No	Yes
Integral	No	No	No	No	No	Yes

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Example: Szulczewski & Juanes (2009) – model for storage capacity of CO_2 in saline aquifers at large scale

$$C = \left[\frac{2M\Gamma^2(1-S_{wc})}{\Gamma^2+(2-\Gamma)(1-M+M\Gamma)}\right]\rho_{CO_2}\phi HWL_{total}, \quad (1)$$

$$M = \frac{1/\mu_w}{k^*_{rg}/\mu_g} \quad (2)$$

$$\Gamma = \frac{S_{rg}}{1-S_{wc}} \quad (3)$$

$$L_{max} = \left[\frac{(2-\Gamma)(1-M(1-\Gamma))}{(2-\Gamma)(1-M(1-\Gamma))+\Gamma^2}\right]L_{total} \quad (4)$$

$$L_{inj} = L_{total} - L_{max} \quad (5)$$

Heidi Stromskag   Appendix C8   8

Didactic contract (Brousseau, 1997)

- The mutual expectations and obligations that teachers and students have towards each other
- Some rules in the contract are implicit, others are explicit

(see note, Strømskag, 2022b)

Heidi Stromskag   Appendix C8   9

Teaching climate change

Complexity related to:

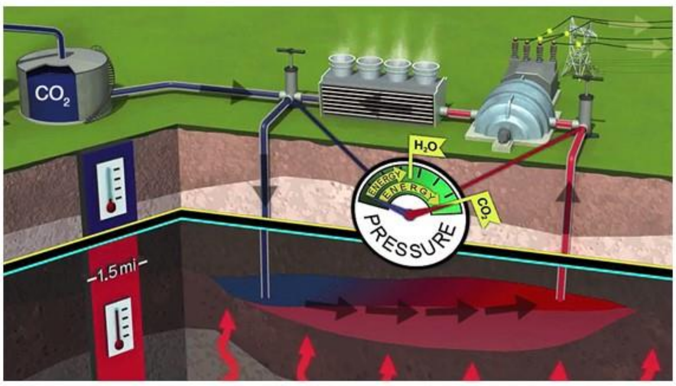
- Claims/knowledge in the field are based on modelling using uncertain and partial data → challenges traditional views of what a science is
- Encompasses expertise in many disciplines (interdisciplinarity)
- Attitudes to environmental issues and consumption are central (see Bauman, 2007)
- Strong links to personal and collective action, often political
- Characterisation of learning within the topic is multifaceted and often beyond the expertise of many teachers

(Oversby, 2015)

Heidi Stromskag Appendix C8 10

Future prospect: Storage of CO_2 for geothermal energy production?

- CO_2 Plume Geothermal (CPG)
- https://www.youtube.com/watch?v=x7fPNLY6h0U&ab_channel=GeothermalEnergyandGeofluidsgroup%2CETHZurich



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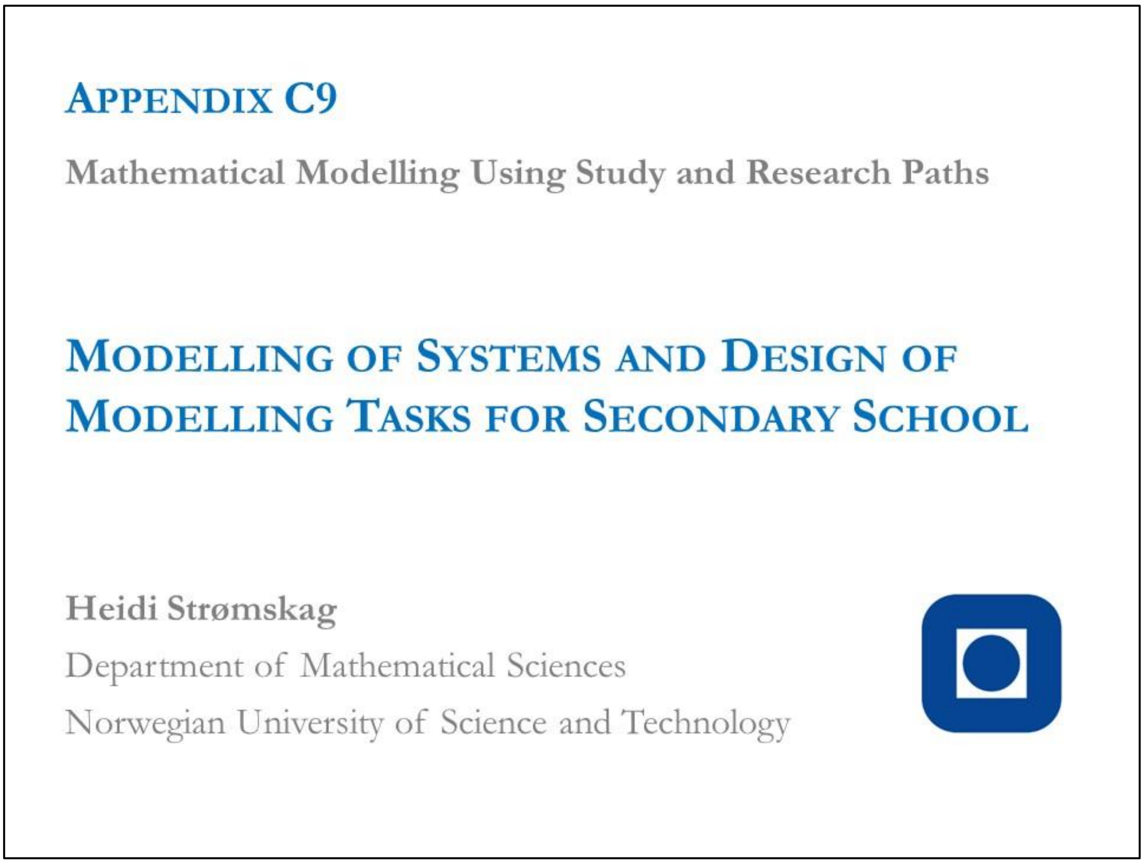
Appendix C – PowerPoint-Slides

C9 – Modelling of Systems and Design of Modelling Tasks for Secondary School



Inquiry as a pathway: Aligning goals for civilization, society, and education

Goals for the civilization

- Climate conservation: Actively preserving and restoring Earth's climate for future generations.
- Resource redistribution: Facilitating equitable and sustainable sharing of global resources.
- Conflict mitigation: Promoting diplomacy and understanding to prevent wars and other conflicts.

Goals for society

- Citizens being capable of answering questions about the world around them in order to address societal as well as personal needs: Needs relating to technological innovation, social progression, and a comprehensive understanding of natural and social phenomena.

Heidi Stromskag Appendix C9 2

Inquiry as a pathway: Aligning goals for civilization, society, and education (cont.)

Goals for mathematics education

- Educating pupils to tackle questions Q about the intra- and extramathematical world by studying existing answers to Q and posing new, derived questions, Q_i.
- This involves providing opportunities for pupils to become:
  - ***Herbartian*** (i.e., having intellectual curiosity);
  - ***procognitive*** (i.e., being ready to study and learn fields of knowledge new to them);
  - ***exoteric*** (i.e., embracing continuous studying and learning).

(Chevallard, 2015)

Heidi Stromskag Appendix C9 3

Inquiry as a pathway: Aligning goals for civilization, society, and education (cont.)

Goals for mathematics teacher education

- Providing prospective teachers with didactic tools that help them achieve the above goals for compulsory mathematics education. Examples of tools from the ATD (introduced in this course):
  - mathematical modelling based on the notions of a model and system in reference to questions about the system
  - the methodology of inquiries in the format of SRPs
  - the Herbartian schema – a dynamic model for the study of a question
  - directed graph + associated table – a pathway diagram of an inquiry governed by questions
  - formulas and parameters
  - three operations on formulas (construction, transformation, and evaluation)
- How are these tools interrelated? Discuss in groups.

Heidi Stromskag Appendix C9 4

The notions of ***system*** and ***model*** in the ATD (recap)

A ***system*** is a fragment of reality that has its own laws. Let us consider a system $\mathcal{S}$.

A system $\mathcal{S}'$ is said to be a ***model*** of $\mathcal{S}$ if, by studying $\mathcal{S}'$, one can produce knowledge about $\mathcal{S}$

- studying *questions* Q about $\mathcal{S}$ by asking $\mathcal{S}'$ about these questions
- choosing models $\mathcal{S}'$ of $\mathcal{S}$ whose study of question Q is easier, safer, quicker than by a "direct" study of $\mathcal{S}$

$\mathcal{S}'$

$\mathcal{S}$

(Strømskag & Chevallard, 2022)

Heidi Stromskag Appendix C9 5

Modelling driven by questions

We use a model to produce knowledge about a system

Main stages in the modelling process:

- Delineation of the system we intend to study, specifying the attributes that are relevant to the study we want to make of this system
- Constructing the model, building on old and new knowledge
- Using the model to produce knowledge about the system under study

(Chevallard, 1989)

Heidi Stromskag Appendix C9 6

Modelling driven by questions

Initial question about a system in the world

AIM: Constructing a model to produce knowledge the system under study

Main stages in the modelling process

- **Delineation of the system** we intend to study, specifying the attributes that are relevant to the study we want to make of this system — Questions about the system
- **Constructing the model**, building on both prior knowledge and knowledge developed as part of the modelling process — Questions about the model
- Using the model to **produce knowledge about the system** under study — Questions about the system-model interplay

Final answer and new questions

Heidi Stromskag Appendix C9 7

A modelling task from a mathematics textbook

A typical task from a Norwegian textbook for Grade 11 (Oldervoll et al., 2020):

1.130
The stopping distance for a car in motion hinges on both the driver's response time and the braking distance. The table below outlines the stopping distance, denoted by $S(x)$, in metres corresponding to certain speeds in kilometres per hour for a specified car and a specified driver.

x (km/h)	40	60	80	100
$S(x)$ (m)	24	45	73	108

a) Plot the data points from the table in a coordinate system and elucidate why a quadratic function seems to be a suitable fit.
b) Determine the quadratic function, S, that most accurately represents the given data. Sketch the graph incorporating the data points. Ensure that the expression of the function is accurate to three decimal places.
c) Find graphically the speed that would result in a stopping distance of 150 metres.
d) Find graphically the stopping distance corresponding to a speed of 90 km/h.

Heidi Stromskag Appendix C9 8

A modelling task from a mathematics textbook (cont.)

Discuss in groups the following questions about the modelling task from the textbook:

1. What is the *question* asked about the *system* S being studied?
2. Delineate S. Which parameters are relevant to consider?
3. Describe the model S' aimed at in the task.
4. What knowledge can be generated about S with the help of S'?
5. Give an overall comment about the modelling task you have examined.

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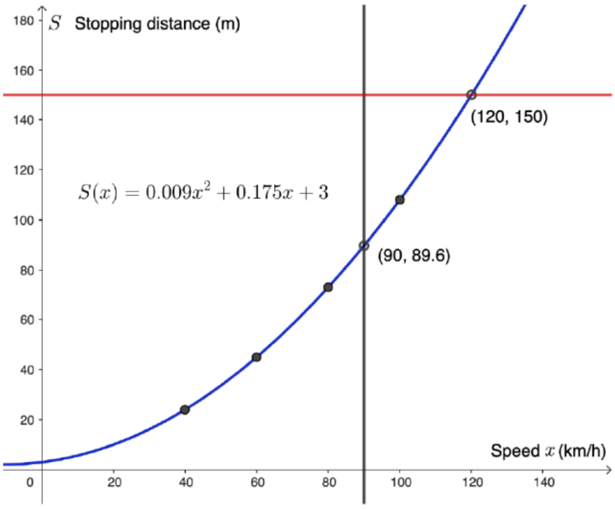
A modelling task from a mathematics textbook (cont.)

The expected technique: Using regression analysis in GeoGebra to determine a second-degree polynomial that fits the given set of data points: $f(x) = 0.009x^2 + 0.175x + 3$.



Examination of the resulting model

- The model provides a good fit for the given data.
- The model offers minimal understanding of the underlying system being modelled.
- There is no direct interpretation of the parameters in the model related to the real-world system.

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Spotlight on the interaction between system and model: A new task pertaining to the same system

Braking distance of a car: A modelling task

A car is travelling at a speed of 72 km/h on a horizontal road. The car needs to brake suddenly, and the driver presses the brake pedal so hard that the brakes lock. The car then slides along the road until it comes to a stop. We assume that the coefficient of friction between the tyres and the road during sliding is 0.7—that is, the system to be studied involves frictional interaction between rubber and dry asphalt (according to Table 1).

a) Develop a model that allows you to provide answers in response to the following prompts: How long is the braking distance in the described scenario? Provide examples of how different road conditions and varying speeds can impact the braking distance of a vehicle sliding after brake lock.
b) What knowledge does the constructed model allow us to produce about the system under study?
c) Discuss limitations of the model in relation to the system under study.

Material 1	Material 2	μ (Friction coefficient)
Steel	Steel	0.6
Steel	Ice	0.05
Steel	Teflon	0.04
Ice	Ice	0.03
Rubber	Dry asphalt	0.7
Rubber	Wet asphalt	0.2
Rubber	Ice	0.02
Wood	Wood	0.3
Hip joint	Hip joint	0.003

Friction coefficients between various materials. Adapted from Grimenes et al. (2011, p. 68)

Heidi Stromskag Appendix C9 11

Spotlight on the interaction between system and model

Design of modelling tasks for Grades 8–13

Identify a modelling task in a secondary school textbook that, in your opinion, inadequately supports the generation of knowledge about the system it aims to model. Your assignment is to design a new task, pertaining to the same system, in a way that strengthens the connection between the system under study and the model to be constructed. Subtasks a–c formulated below are meant to help you in studying the system to be examined, in order to prepare your own task design. Moreover, the note with a proposed solution on braking distance is available for your perusal, and could serve as a useful reference or spark of inspiration in your own work.

a) Delineate the system S that will be the focus of your study. State the initial question Q that you aim to answer regarding S. Define the relevant parameters that are crucial for studying S, along with their respective relationships.

b) Build up a suitable model S' that can effectively address the initial question Q about the system S. Explain how this model S' can not only answer Q but also produce additional knowledge about the system S. Provide a brief overview of the background knowledge required for constructing the model S'.

c) Discuss the relationships between the system studied (S) and the model constructed (S') including the affordances (strengths and advantages) and constraints (limitations and assumptions) of the chosen model in representing the real-world dynamics of S.

d) Utilising the knowledge generated through the previous subtasks, devise a modelling task suitable for students in Grades 8–13 (select one of the grades). The task should explore the same system (S) and address a question (Q), concerning S. Q can either be the same or different from the one you addressed previously. Strive to design the task in a way that enables students to generate knowledge about the real-world dynamics of S, while ensuring it is both appropriately challenging and manageable for the chosen grade level.

Heidi Strømskag Appendix C9 12

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Appendix D – Tasks

D1 – Modelling of *thermal insulating capacity* of thermoses

The picture below shows two 1-liter thermoses, with Sarek on the left and Eva Solo on the right.

The question to be answered is:
What are the thermal insulating capacities of the Sarek and Eva Solo thermoses?

Newton's law of cooling will play a role in the modelling of the temperature in the two containers. It states that:

> *"The rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings".*

An experiment has been done where boiling water has been poured into the two containers, and the temperature has been measured at certain points in time after filling. The temperature in the room where the containers were placed during the experiment was 24.2 °C. The temperature of the water just before the lids were screwed on the thermoses was 94.3 °C.

a) For each thermos, build a model that gives the temperature of its water as a function of time after filling of hot water. Use this information: after 3 hours, the water in Sarek measured 88.9 °C, and the water in Eva Solo measured 83.7 °C.

b) Explain whether the models you have built are descriptive or normative.

c) Use the models to designate a ***descriptor*** of the thermal insulating capacity of each thermos. Give this descriptor an explanatory name that may be used by vendors of thermoses.

d) Several observations were made, as shown in the table below. Discuss the validity of your models in light of these measurements.

Time passed after filling (hours)	Sarek	Eva Solo
0	94,3 °C	94,3 °C
1	92,6 °C	89,0 °C
2	90,3 °C	85,8 °C
3	88,9 °C	83,7 °C
5	85,9 °C	78,7 °C
7	82,9 °C	74,6 °C
11	77,8 °C	67,8 °C
12	76,5 °C	66,5 °C
53	47,5 °C	37,0 °C
75	39,9 °C	32,1 °C
96	34,8 °C	28,9 °C

Table 1. Temperatures of the water at points in time after filling of water

Appendix D – Tasks

D2 – Proposed solution: Modelling of thermal insulating capacity of thermoses

a) Let $S(t)$ and $E(t)$ denote the temperature in the Sarek and Eva Solo thermos respectively at time t (in hours). We use Newton's law of cooling to set up a differential equation in each case.

Sarek (k_s is the proportionality factor):

$\frac{dS}{dt} = k_s(S - 24.2)$. Initial condition: $S(0) = 94.3$ og $S(3) = 88.9$.

Eva Solo (k_e is the proportionality factor):

$\frac{dE}{dt} = k_e(E - 24.2)$. Initial condition: $E(0) = 94.3$ og $E(3) = 83.7$.

Solving the differential equation for Sarek:

$$\frac{dS}{dt} = k_s(S - 24.2) \Leftrightarrow \int \frac{dS}{S-24.2} = \int k_s\, dt \Leftrightarrow \ln|S - 24.2| = kt + C$$

$$\Leftrightarrow S - 24.2 = e^{k_s t + C} \Leftrightarrow S(t) = C_1 e^{k_s t} + 24.2.$$

Using the initial conditions to decide C_1 og k_s:
$S(0) = 94.3 \Leftrightarrow C_1 = 70.1$.

That is to say, we have $S(t) = 70.1e^{k_s t} + 24.2$.

Solving the differential equation for Eva Solo:

Because the temperature at $t = 0$ is the same for Eva Solo, we have:
$E(t) = 70.1e^{k_e t} + 24.2$.

We find the proportionality factor in the two cases:

Sarek:
$$S(3) = 88{,}9 \Leftrightarrow 70.1e^{3k_s} + 24.2 = 88{,}9 \Leftrightarrow e^{3k_s} = \frac{64.7}{70.1} \Leftrightarrow 3k_s = \ln\left(\frac{564.7}{70.1}\right)$$
$$\Leftrightarrow k_s \approx -0.0267.$$

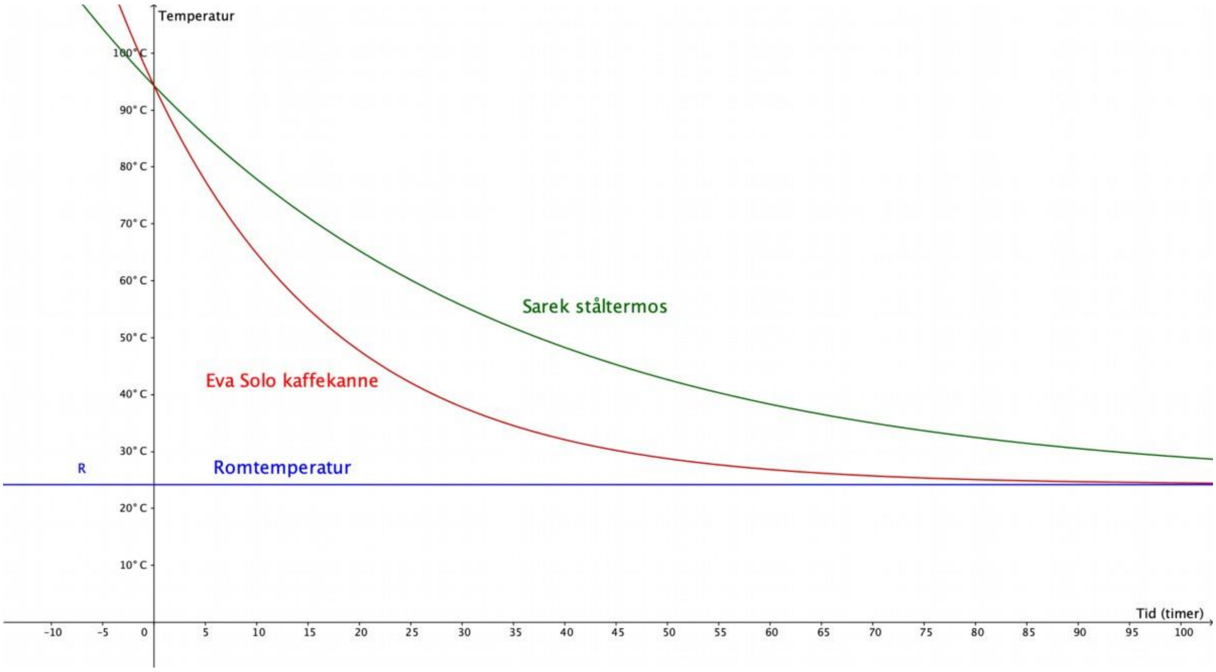
That is to say, we have: $\underline{S(t) = 70.1e^{-0.0267t} + 24.2}$

Eva Solo:
$$E(3) = 83.7 \Leftrightarrow 70.1e^{3k_e} + 24.2 = 83.7 \Leftrightarrow e^{3k_e} = \frac{59.5}{70.1}$$

$\Leftrightarrow\ 3k_e = \ln\left(\frac{59.5}{70.1}\right) \Leftrightarrow\ k_e \approx -0.0546.$

That is to say, we have: $\underline{E(t) = 70.1e^{-0.0546t} + 24.2}$

The functions *S*(*t*) and *E*(*t*) are *analytic models*, depending on the choice of initial conditions. Below are *graphical models* for the temperature development in the two thermoses.



The proportionality factor is negative in both cases, and the one with the largest absolute value will give the greatest reduction in temperature per unit of time. We see that the temperature is best maintained in Sarek. This is consistent with the fact that the proportionality factor in the model for Sarek has a smaller absolute value than the model for Eva Solo.

b) These are *descriptive* models. They also have the further property of being *deterministic*, in that they are constructed on the basis of a physical law for temperature development for bodies placed in an environment with a constant temperature (Newton's law of cooling).

c) *Suggestion*. A measure of insulating properties can be called the insulation coefficient, *I*. This will be a ***descriptor*** (see Niss, 2015) and can be calculated according to the equation $I = 100(1 - |k|)$. Note that this equation is an *invention*.

This gives the following insulation coefficients:
Sarek: *I* = 97,3
Eva Solo: *I* = 94,5
Furthermore, the insulation coefficient can be linked to intervals that qualitatively describe the insulating properties of thermoses, for example:

$I \in [100, 95)$	very good
$I \in [95, 90)$	good
$I \in [90, 85)$	acceptable
$I \leq 85$	poor

Note: The systems we model differ in terms of how far into the future it is appropriate to consider. For example, a thermos for use in an indoor party does not need to retain heat for as many hours as a thermos for bringing on a longer journey in sub-zero temperatures.

d) Factors contributing to inaccuracy in the data: measurement inaccuracy (the tool used), heat loss when opening-closing, not equal time for measurement in the two thermoses, etc. A *data logger* could have been used to get more accurate data.

Finally, it is possible to use *regression analysis* (e.g., in GeoGebra) to find a model for each thermos. In general, regression models are derived from actual data. They are used to describe the relationship between variables and predict future outcomes. Regression models do not necessarily represent the underlying physics or theoretical principles of a system, but instead provide a statistical relationship between variables.

Reference

Niss, M. (2015). Prescriptive modeling: Challenges and opportunities. In G. Stillman, W. Blum, & M. S. Biembengut (Eds.), Mathematical modeling in education research and practice: Cultural, social and cognitive influences (pp. 67–79). Springer. https://doi.org/10.1007/978-3-319-18272-8_5

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Appendix D – Tasks

D3 – Modelling of probability of having a disease given a positive test result

The system *S* is a population with an infectious disease, where the relative frequency of the disease is equal to *a*, and the reliability of a screening test for the disease is equal to *r*. Moreover, the screening test is designed in a way that ensures that everyone who has the disease tests positive (i.e., no false negatives).

a) Create a model of the given system that gives you the answer to this question, *Q*: What is the probability, *p*, that a person with a positive test result really has the disease?

b) Explore the model you have created for different values of *p*. Find out, for example, what relative frequency of the disease corresponds to 98% probability of having the disease after receiving a positive test result.

Appendix D – Tasks

D4 – Comparison of study and research paths (SRPs) and problem-/project-based learning approaches

For this investigation, your primary resource will be the article by Markulin et al. (2021). Here are the questions you will study and answer:

a) How does the methodology of SRPs differ from problem-/project-based learning approaches? What are the principles upon which each of these approaches are founded?

b) Imagine a hypothetical question (*Q*) about a system, relevant to study by secondary school students. Explain how *Q* could potentially be tackled differently by the two approaches: SRPs and problem-/project-based learning.

Reference

Markulin, K., Bosch, M., & Florensa, I. (2021). Project-based learning in statistics: A critical analysis. *Caminhos da Educação Matemática em Revista*, *11*(1), 200–220.

Appendix D – Tasks

D5 - Study and research path on climate change: A pilot study

Generating question

Q_{pilot}. *What is climate change, and why is it happening?*

- Preparation: Read about the Herbartian schema (Strømskag, 2022). The note explains the process of an SRP and some tools for presenting such inquiries. Even if you do not need to use all the terminology introduced in this note when presenting the results, the content of the note will still be helpful in the study and research process you will be conducting.
- Work in self-selected groups of two or three.

Procedure for the pilot study

1. Search for sources that can provide *existing answers* to Q.
2. As you study these existing answers, formulate and answer *derived questions*, Q_i, to understand the existing answers. Search for *works* (articles, books, reports, or multimedia resources) that help you understand existing answers and help you answer the Q_i.
3. Write a mini-report from the inquiry, which includes:
   - An answer to Q in terms of a synthesis of the answers to the Q_i.
   - A representation of the pathway of the inquiry, including an overview of the Q_i and works you have studied to arrive at the final answer to Q. This can be done through a directed graph accompanied by a table that explains its nodes.
4. Based on the mini-report, create a PowerPoint that presents the SRP you have conducted. All teams will present their inquiry on Friday 11 February. Time frame for presentation: 5 min. + discussion.

“Hand keep clode”—image downloaded from Pixabay

Appendix D – Tasks

D6 – A study and research process into Carbon Capture and Storage

Generating question

Q_0. *How is carbon capture and storage modelled in the literature?*
What mathematics is involved in these models? Which parameters are included, and what are the relationships between them?

Guidelines for the final SRP report

Length: Max. **6 500 words** (excluding reference list, table of contents, and any appendices).

Note 1: The chapter headings below may well be replaced by more informative headings. Chapters that span several pages should be divided into sub-chapters with suitable headings.

Note 2: APA 7 style should be used for both references in the text itself and in the formatting of the reference list.

The final SRP report shall be given an appropriate **title** and shall contain the following parts:

Table of contents (titles of chapters and sub-chapters)

1. Introduction

- What is this study about? Why is the topic important? Why should it be a topic in education?

2. Didactic framework and theory

- Brief account of the didactic paradigm in which the study is anchored
- Presentation of theoretical tools (e.g., model – system; Herbartian schema)
- Possible glossary of technical terms (briefly explained)

3. Methodology

- Presentation of the research strategy used: SFL and documentary study.
- Presentation of didactic tools in the form of a directed graph with an accompanying table showing the course of the study with the following elements: the generating question, existing solutions, derived questions, works, partial answers.

4. Results

- Presentation of existing answers $A_i^\diamond$ and works W_j that are central to understanding $A_i^\diamond$.
- Presentation of derived questions Q_k and partial answers a_m that are results of studying Q_k, $A_i^\diamond$, and W_j.
- Discussion of the partial answers a_m and presentation of the final (though provisional) answer $A^\heartsuit$.

5. Discussion and concluding remarks

- Comments on the investigation process. Include an overview of derived questions and work that has been omitted (for various reasons that do not need to be explained). What have you learnt from conducting this inquiry? What has been challenging? What are open questions that need further study?

6. References

Milestones/deadlines:

18 March at 23.55:	Submission of a preliminary report to response team and Heidi.
22–23 March:	Presentation of a preliminary report in class. System for feedback.
11 April at 23.55:	Submission of a draft of final report to Heidi for feedback.
13 May at 23.55:	Submission of a final report.

Appendix D – Tasks

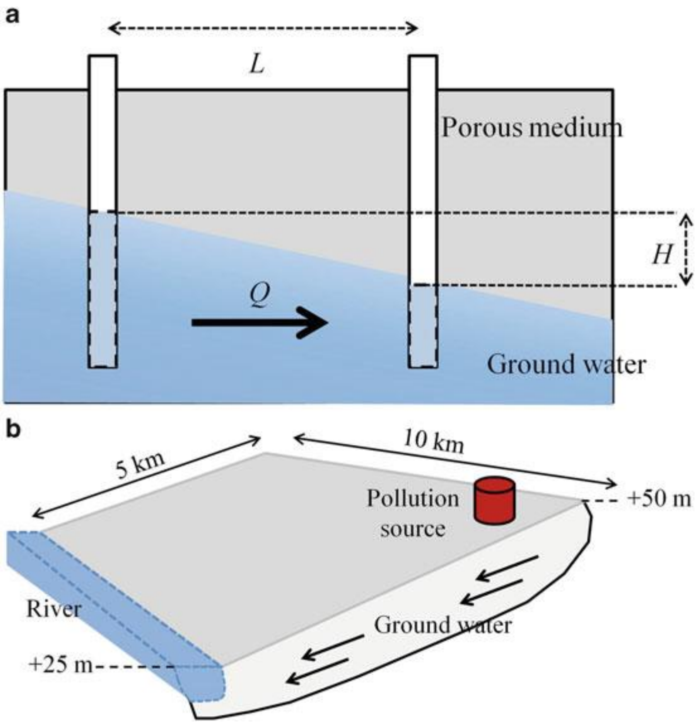
D7 – Hydrogeological modelling: Evaluating aquifer flow and contaminant transport[19]

To determine the flow rate of water in an aquifer, Darcy's law can be used. It gives the relation between the flow through this medium (Q), the hydraulic gradient i, the surface perpendicular to the flow A, and a parameter that characterizes the aquifer k: permeability (expressed in m/s).

Darcy's law is given by the relation $Q = A \times k \times i$. The hydraulic gradient is the ratio between the difference in water depth in two boreholes and the distance between these two boreholes. The hydraulic gradient is $i = \frac{H}{L}$ (see Figure 1).

Figure 1

Principle of flow in an aquifer: The flow rate is proportional to the hydraulic gradient



Note. The figure (1) is taken from Fleurant and Bodin-Fleurant (2019, p. 53) and is reproduced with permission. This figure is not covered by the chapter's CC BY-NC 4.0 license and should not be reproduced without obtaining separate permission from the copyright holder, Springer.

1) What is the unit of measure of the gradient? Of the flow?

2) Justify mathematically and physically that $Q = 0\ m^3/s$ when $i = 0$.

[19] The task is taken from Fleurant and Bodin-Fleurant (2019, p. 52).

3) It is assumed that this aquifer consists of limestone whose estimated permeability is 8.64 m/day, and that the river is 15 m deep. From the graph in Figure 1, determine the flow (in L/s) that goes from the aquifer to the river.
4) By imagining that a soluble contaminant escapes from the tank shown in the figure by a cylinder, calculate the time it will take the contaminant to reach the river.

Reference

Fleurant, C., & Bodin-Fleurant, S. (2019). *Mathematics for Earth science and geography*. Cham: Springer. https://doi.org/10.1007/978-3-319-69242-5

Appendix D – Tasks

D8 – Modelling of temperature scales and designing a Fahrenheit-Celsius conversion task

The modelling task below is motivated by the following exercise, found in the three-volume textbook by Søgaard and Tambs Lyche (1939/1969, p. 104):

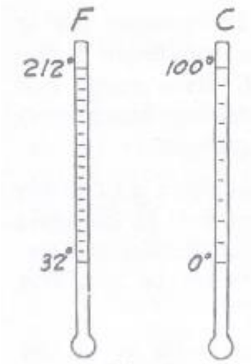


Fig. 17.

Exercise 231. Look at the figure and explain that we will get the Fahrenheit temperature F expressed in terms of the Celsius temperature C by the formula $F = \frac{180}{100}C + 32$.

Use this formula and find C in terms of F.

Take $F = 100$, 50, 41, 32, 14, 0, and calculate C in each case.

Here is your assignment, the answers to which will be discussed in class in the upcoming exercise class.

1) Consider Figure 1 on the next page. There is a linear relationship between the temperature scales in Fahrenheit and Celsius. Why is that?

2) What does it mean to have a measurement *scale* or *level*? What types of measurement scales and levels exist, and what are they used for?

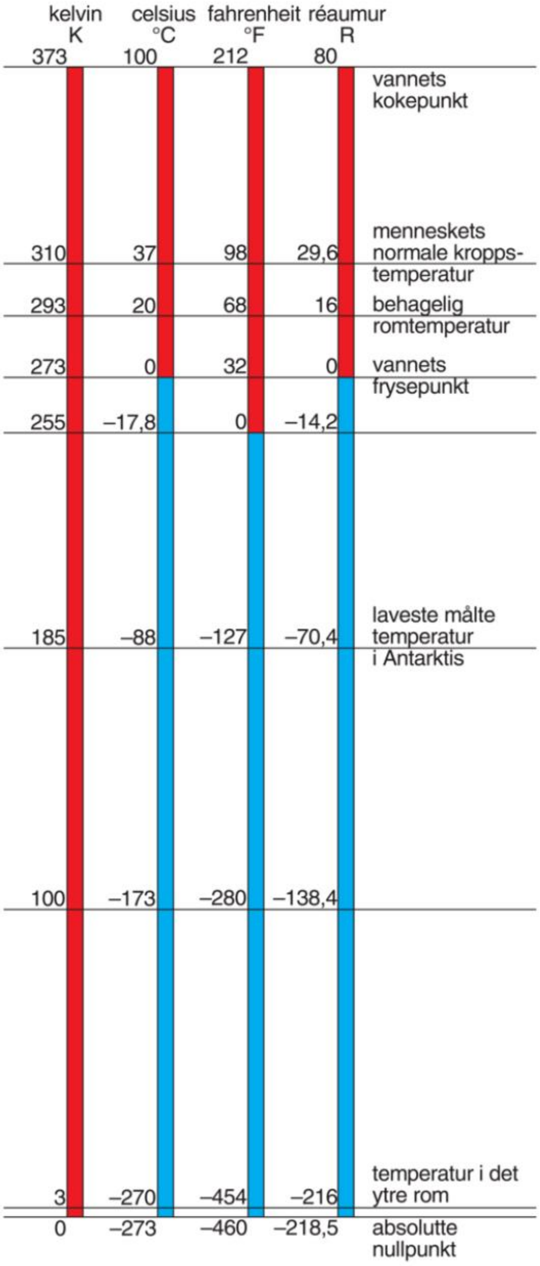
3) Design a task intended for upper secondary school, which involves modelling the system with the two temperature scales. You may take the exercise from Søgaard and Tambs Lyche (1939/1969) as a starting point, but ensure that your task explicitly involves incorporating parameters. Moreover, the task should contain elements that makes it necessary for students to carry out construction, transformation, and evaluation of formulas (in the sense of Abbott, 1942/1971, as cited in Strømskag & Chevallard, 2022). Create a proposed solution to the task you have designed.

Reference

Strømskag, H., & Chevallard, Y. (2022). Elementary algebra as a modelling tool: A plea for a new curriculum. *Recherches en Didactique des Mathématiques*, *42*(3), 371–409. https://revue-rdm.com/2022/elementary-algebra-as-a-modelling-tool-a-plea-for-a-new-curriculum/

Figure 1

Temperature scales



Note. The figure is taken from *Store Norske Leksikon*. https://snl.no/temperaturskala. Reproduced with permission.

Appendix D – Tasks

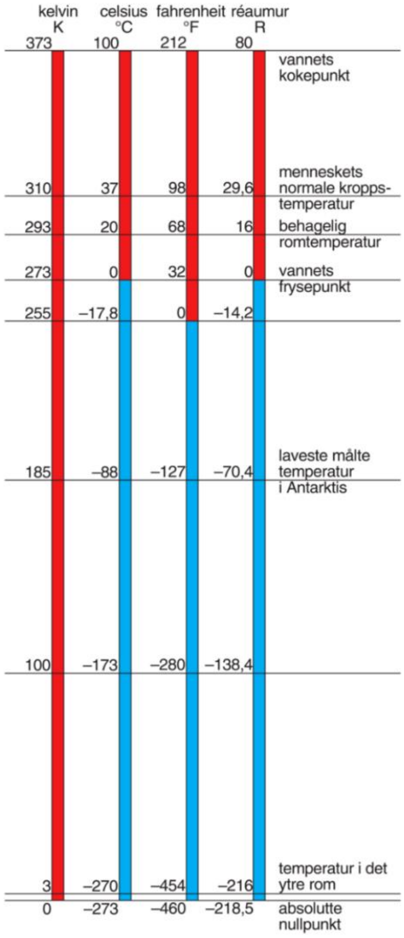
D9 – Materials for an answer to modelling of a relationship between Celsius and Fahrenheit grades

About temperature scales: All temperature scales are calibrated according to the thermal properties of a particular substance or device. Usually this is established by fixing two well-defined temperature points and defining temperature increments via a linear function of the response of the thermometric device. For example, both the Celsius scale and the Fahrenheit scale were originally based on the *linear evolution of a narrow mercury column within a limited temperature range*, each using different reference points and scale increments.

See https://en.wikipedia.org/wiki/Scale_of_temperature

Figure 1

Temperature scales



Note. The figure is taken from *Store Norske Leksikon*. https://snl.no/temperaturskala. Reproduced with permission.

See also https://en.wikipedia.org/wiki/Level_of_measurement. Particularly important here is the following:

Level of measurement, or *scale of measure*, is a classification that describes the *nature of information* within the values assigned to variables. The theory was first developed in 1946 by psychologist Stanley Stevens (1906–1973):

- Nominal level
- Ordinal scale
- Interval scale
- Ratio scale

1) So, we are searching for a linear equation: $\boldsymbol{F = aC + b}$, where F represents the degree number in Fahrenheit and C represents the degree number in Celsius. We have that F, C, a, and b are all parameters. We shall determine a and b from the system given in the exercise.

   Possible contextual reformulation of the textbook exercise:
   Anne is in the US and feels sick. She finds a forehead thermometer that measures temperature in degrees Fahrenheit. She measures her body temperature to be equal to 102 degrees Fahrenheit. Anne doesn't remember exactly how to convert from Fahrenheit to Celsius, but she knows the following: There is a linear relationship between the Celsius and Fahrenheit scales. Furthermore, she knows that 0 degrees Celsius corresponds to 32 degrees Fahrenheit and that 100 degrees Celsius corresponds to 212 degrees Fahrenheit. How can she find out what 102 degrees Fahrenheit is in degrees Celsius?

2) *Construction of a formula.*

   $F = aC + b$

   $32 = a \cdot 0 + b$
   $212 = 100a + b$

   $b = 32$
   $212 = 100a + 32$

   $b = 32$
   $a = \frac{9}{5}$

   That is to say, we have the following formula that is an *equation*: $F = \frac{9}{5}C + 32$ or $\underline{F = 1{,}8C + 32}$.

3) *Transformation.*

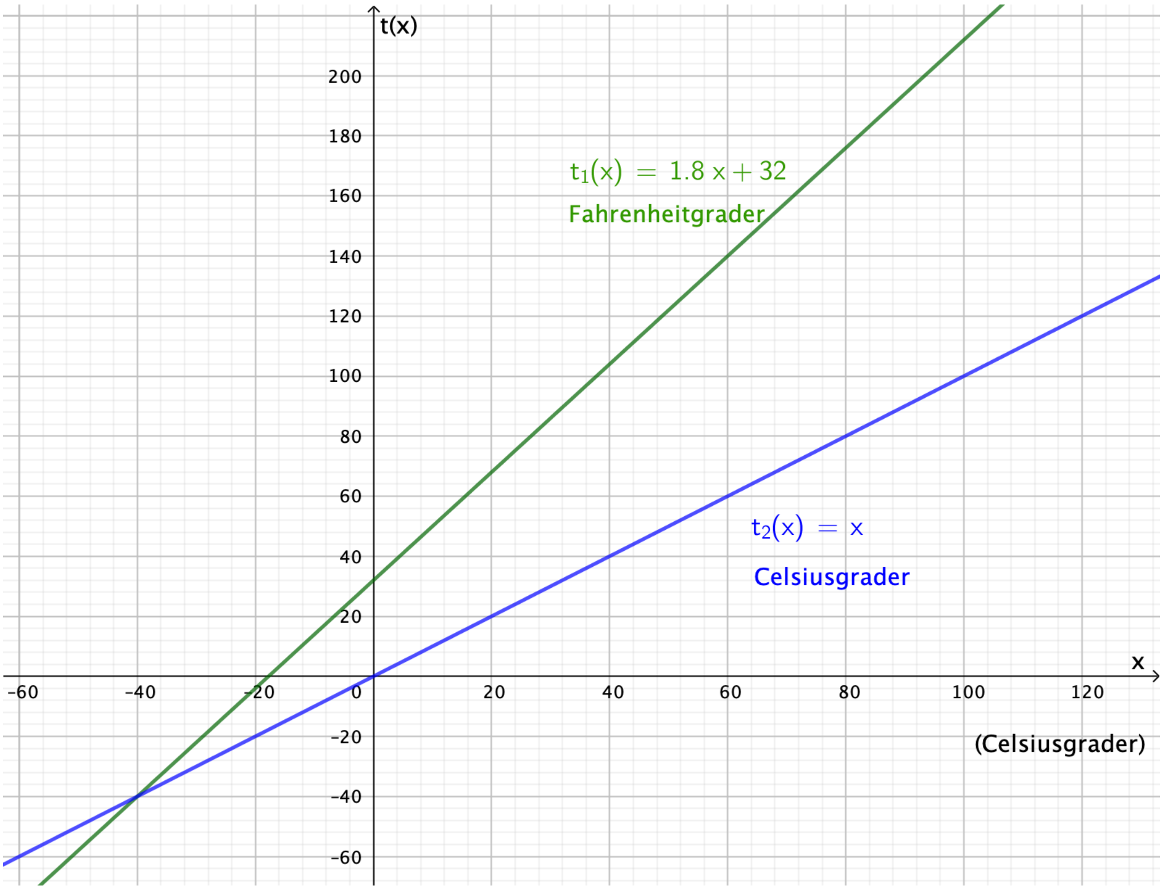
Finding C expressed in terms of F:

$$\frac{9}{5}C = F - 32 \quad \Leftrightarrow \quad \underline{C = \frac{5}{9}(F - 32)}.$$

4) *Evaluation*.

How many degrees Celsius is 100 degrees Fahrenheit?
How many degrees Fahrenheit is 20 degrees Celsius?
Are there any degree numbers where the two temperature scales have the same measure?

Algebraic solution: $F(C) = C \Leftrightarrow 1{,}8\,C + 32 = C \Leftrightarrow \underline{C = -40}$
Graphical solution:



Note. Percival Abbott's (1942/1971) three categories—construction, manipulation, and evaluation—are a *framework of principles* that describe operations on formulae and equations in algebra. It is important to re-establish manipulation of formulae/equations (a neglected area). This is important from a modelling point of view, when working with modelling in school and using algebra as a tool.

Appendix D – Tasks

D10 – Task: Modelling of braking distance

The following task is inspired by Grimenes et al. (2011, p. 69).[20]
A car is travelling at a speed of 72 km/h on a horizontal road. The car needs to brake suddenly, and the driver presses the brake pedal so hard that the brakes lock. The car then slides along the road until it comes to a stop. We assume that the coefficient of friction between the tyres and the road during sliding is 0.7—that is, the system to be studied involves frictional interaction between rubber and dry asphalt, according to Table 1.

Table 1

Typical friction coefficients between various materials

Material 1	Material 2	μ (Friction coefficient)
Steel	Steel	0.6
Steel	Ice	0.05
Steel	Teflon	0.04
Ice	Ice	0.03
Rubber	Dry asphalt	0.7
Rubber	Wet asphalt	0.2
Rubber	Ice	0.02
Wood	Wood	0.3
Hip joint	Hip joint	0.003

Note. The table is adapted from Grimenes et al. (2011, p. 68).

a) What is the system S under investigation, and what is the initial question Q to be answered about S? Delineate S by specifying the parameters that are relevant to the study you will make of it. What knowledge do you need to explain the relationships between the parameters of S?

b) Develop a model S' that can effectively answer the question Q and, further, generate additional knowledge about S. Provide examples of how different road conditions and varying speeds can impact the braking distance of a vehicle sliding after brake lock.

c) What knowledge does the constructed model allow you to produce about the system under investigation? Discuss relationships between the system S and the model S'.

[20] A proposed solution to the task is included in Appendix B. This information was not provided in the original task given to the student teachers. They got a note including a proposed solution after they had worked on the task.

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Appendix D – Tasks

D11 – Design of modelling tasks for Grades 8–13

Identify a modelling task in a secondary school textbook that, in your opinion, inadequately supports the generation of knowledge about the system it aims to model. Your assignment is to design a new task, pertaining to the same system, in a way that strengthens the connection between the system under study and the model to be constructed. Subtasks a–c formulated below are meant to help you in studying the system to be examined, in order to prepare your own task design. Moreover, the note with a proposed solution on braking distance is available for your perusal, and could serve as a useful reference or spark of inspiration in your own work.[21]

e) Delineate the system S that will be the focus of your study. State the initial question Q that you aim to answer regarding S. Define the relevant parameters that are crucial for studying S, along with their respective relationships.

f) Build up a suitable model S' that can effectively address the initial question Q about the system S. Explain how this model S' can not only answer Q but also produce additional knowledge about the system S. Provide a brief overview of the background knowledge required for constructing the model S'.

g) Discuss the relationships between the system studied (S) and the model constructed (S') including the affordances (strengths and advantages) and constraints (limitations and assumptions) of the chosen model in representing the real-world dynamics of S.

h) Utilising the knowledge generated through the previous subtasks, devise a modelling task suitable for students in Grades 8–13 (select one of the grades). The task should explore the same system (S) and address a question (Q), concerning S. Q can either be the same or different from the one you addressed previously. Strive to design the task in a way that enables students to develop knowledge about the real-world dynamics of S, while ensuring it is both appropriately challenging and manageable for the chosen grade level.

[21] The note referenced herein is a theoretically substantiated solution proposal that I developed in 2022, as part of our exploration related to modelling of a vehicle's braking distance.

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