

NOTES ON THE SHORT \mathbb{C}^k 'S

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ABSTRACT. Domains that are increasing union of balls (up to biholomorphism) and on which the Kobayashi metric vanishes identically arise inexorably in complex analysis. In this article we show that in higher dimensions these domains have infinite volume and the Bergman spaces of these domains are trivial. As a consequence they fail to be strictly pseudo-convex at each of their boundary points although these domains are pseudo-convex by definition. These domains can be of different types and one of them is Short \mathbb{C}^k 's. In pursuit of identifying the Runge Short \mathbb{C}^k 's (up to biholomorphism), we introduce a special class of Short \mathbb{C}^k 's, called Loewner Short \mathbb{C}^k 's. These are those Short \mathbb{C}^k 's which can be exhausted in a continuous manner by a strictly increasing parametrized family of open sets, each of which is biholomorphically equivalent to the unit ball and therefore, they are Runge up to biholomorphism. Although, the question of whether all Short \mathbb{C}^k 's are Runge (up to biholomorphism), or whether all Short \mathbb{C}^k 's are Loewner remains unsettled, we show that the typical Short \mathbb{C}^k 's are Loewner. In the final section, we construct a bunch of non-autonomous basins of attraction, which serve as interesting examples of Short \mathbb{C}^2 's.

1. INTRODUCTION

For $k \geq 2$, let $\Omega \subseteq \mathbb{C}^k$ be an increasing union of unit balls, i.e.,

$$\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_k \subseteq \Omega_{k+1} \subseteq \cdots \subseteq \Omega = \bigcup_{i=1}^{\infty} \Omega_i,$$

where each Ω_i is biholomorphic to the unit ball in \mathbb{C}^k .

Definition 1.1. An increasing union of unit balls $\Omega \subseteq \mathbb{C}^k$, for $k \geq 2$, is called Short \mathbb{C}^k if the Kobayashi metric vanishes identically on Ω but there exists a non-constant bounded above plurisubharmonic function (psh) defined on Ω .

Genesis of these Short \mathbb{C}^k 's lies in the *union problem*: Is an increasing union of Stein manifolds always Stein? In [6], the first author settled this question by showing that in dimension three onwards, there exist increasing sequences of balls whose final unions are not Stein (also see [17]). Another related theme is to look for the model domains for increasing unions of balls. The possible model domains for increasing unions of balls when the Kobayashi metric does not vanish identically therein, was discussed in [8] (also see [9]). In particular, it follows (from [8]) that in dimension two, up to biholomorphism any such domain Ω is either the unit ball or the product domain $\Delta \times \mathbb{C}$, where Δ is the unit disc. On the other hand, when the Kobayashi metric on an increasing union of balls Ω vanishes identically, one can immediately see that Ω could be all of \mathbb{C}^k or a biholomorphic copy of \mathbb{C}^k , the so called Fatou-Bieberbach (FB) domain. Further, in [7], another possible model domain for Ω was found when the Kobayashi metric vanishes identically on Ω : It was shown that Ω could be a Short \mathbb{C}^k which is evidently never biholomorphic to any FB domain (or \mathbb{C}^k). These domains appear naturally in complex dynamics as non-autonomous basins of attraction of automorphisms of \mathbb{C}^k .

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Although, recently many interesting examples of Short \mathbb{C}^k 's are found ([2], [4], [5]), mainly on par with its sibling FB domains, the general theory of Short \mathbb{C}^k 's is not yet well-developed. The present article started off with an aim to recognize the fundamental properties of Short \mathbb{C}^k 's. Of our particular interest is to compare the properties of Short \mathbb{C}^k 's and \mathbb{C}^k (or FB domains). Being a biholomorphic copy of \mathbb{C}^k , any FB domain has infinite volume. So a natural question is whether all Short \mathbb{C}^k 's have infinite volume or not. During the course of seeking the answer, we ended up proving a more general result which answers the before-mentioned question in the affirmative.

Theorem 1.2. *For $k \geq 2$, let $\{\Omega_i\}_{i \geq 1}$ be a sequence of domains in \mathbb{C}^k , each of which is biholomorphic to the unit ball, such that $\Omega_i \subseteq \Omega_{i+1}$ for $i \geq 1$ and the Kobayashi metric vanishes identically on Ω . Then the volume of Ω is infinite and the Bergman space*

$$A^2(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{C} : \int_{\Omega} |f|^2 < \infty \right\}$$

consists only of the function $f \equiv 0$.

As an immediate corollary we get the following.

Corollary 1.3. *Short \mathbb{C}^k 's have infinite volume. Further the Bergman space of any Short \mathbb{C}^k is trivial.*

Next we discuss some other intriguing by-products of Theorem 1.2.

A remark on Bedford conjecture: We address Problem 24 in [1]. Let $\{F_j\}_{j \geq 1}$ be a sequence of automorphisms of \mathbb{C}^k such that $F_j(0) = 0$, for all $j \geq 1$. Further, assume that F_j 's satisfy *uniform bound condition*, i.e., there exist $0 < c < d < 1$ such that

$$c\|z\| \leq \|F_j(z)\| \leq d\|z\|,$$

for all $z \in B(0,1)$, the unit ball in \mathbb{C}^k . Then the *Bedford conjecture* (see [1]) states that the non-autonomous basin of attraction

$$\Omega_{\{F_j\}} = \{z \in \mathbb{C}^k : F_n \circ \dots \circ F_1(z) \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

is biholomorphic to \mathbb{C}^k . While the conjecture still remains open, it is well-known that $\Omega_{\{F_j\}}$ is an increasing union of balls and the Kobayashi metric vanishes identically on $\Omega_{\{F_j\}}$. Therefore by Theorem 1.2, $\Omega_{\{F_j\}}$ has infinite volume and has trivial Bergman space.

In [15], it was shown that the dimension of the Bergman space of any domain in \mathbb{C} is either 0 or ∞ . He also showed that the result is not true in higher dimensions by finding explicit examples of domains with finite dimensional Bergman space. However, the domains he constructed are (Reinhardt but) not pseudoconvex. This triggered the question whether there exists a pseudoconvex domain in higher dimension with finite dimensional Bergman space. This question is still unsettled. However, it was addressed in some recent works (see [11], [12], [14]). In particular, in [12] Gallagher–Harz–Herbort gave a sufficient condition for pseudo-convex domains to have infinite dimensional Bergman spaces in terms of their cores. Further they found three different classes of pseudo-convex domains which satisfy the “core” condition and thus have infinite dimensional Bergman spaces. We discuss these domains briefly here:

- Let Ω be a domain and let $p \in \partial\Omega$ be such that there exists a continuous psh function φ_p defined on a one-sided open neighbourhood \mathcal{N}_p such that $\varphi_p^*(z) < 0$ for all $z \in \overline{\mathcal{N}_p} \setminus \{p\}$ and $\varphi_p^*(p) = 0$ where $\varphi_p^*(z) = \limsup_{z' \rightarrow z} \varphi_p(z')$. Thus p is a local peak point for the class of continuous psh functions. In this case, $\dim A^2(\Omega) = \infty$;
- If there exists a point on the boundary of Ω near which $\partial\Omega$ is C^∞ smooth and of finite type in the sense of D’Angelo, then $\dim A^2(\Omega) = \infty$;

- If there exists a point on the boundary of Ω near which the domain is strictly pseudo-convex, then $\dim A^2(\Omega) = \infty$.

In light of the above discussion, we get:

Corollary 1.4. *For $k \geq 2$, let $\Omega \subseteq \mathbb{C}^k$ be an increasing union of balls on which the Kobayashi metric vanishes identically (in particular, if Ω is a Short \mathbb{C}^k), then it has neither a local peak point nor any point of finite type on its boundary. Further, $\partial\Omega$ fails to be strictly pseudo-convex at each of its point and Ω cannot have any strictly pseudo-convex neighbourhood.*

For $k \geq 2$, the primary source of examples of Short \mathbb{C}^k 's are the non-autonomous basins of attraction of sequence of automorphisms F_j 's of \mathbb{C}^k which are of the form:

$$(1.1) \quad F_j(z_1, \dots, z_k) := \left(z_1^d + P_{1,j}(z_1, \dots, z_k), P_{2,j}(z_1, \dots, z_k), \dots, P_{k,j}(z_1, \dots, z_k) \right),$$

where $d \geq 2$ and the degree of $P_{i,j}$'s are at most $(d-1)$. Further, the moduli of the coefficients of $P_{i,j}$'s are at most $\eta_j = a_j^{dj}$ with $0 \leq a_{j+1} \leq a_j < 1$, for $j \geq 1$. These Short \mathbb{C}^k 's turn out to be the 0-sublevel sets of global psh functions, i.e.,

$$\Omega_{\{F_j\}} = \{z \in \mathbb{C}^k : \varphi(z) < 0\},$$

for some psh function φ on \mathbb{C}^k which can be obtained by modifying the Green's functions of the sequences of automorphisms $\{F_j\}$'s. In dimension 2, another source of these domains are the sublevel sets of Green functions of Hénon maps which are the most important class of polynomial automorphisms of \mathbb{C}^2 . They are of the form $(z, w) \mapsto (p(z) + \delta w, z)$ where p is a polynomial in z of degree $d \geq 2$ and $\delta \neq 0$ (see [7] for details). Further the Green's function for a Hénon map H is

$$G_H^+(z, w) = \frac{1}{d^n} \log^+ \|H^n(z, w)\|,$$

for $(z, w) \in \mathbb{C}^2$, where $\log^+(x) = \max\{\log x, 0\}$. Now both of these classes of examples are Runge, i.e., holomorphic functions on these domains can be approximated uniformly on compact sets by polynomials, simply because they are sublevel sets of global psh functions. However, there exist examples of Short \mathbb{C}^k 's which are not Runge. In [16], Wold gave an example of a Fatou Bieberbach domain $U \subseteq \mathbb{C}^2$ which is not Runge. He showed that there exists a compact set $K \subseteq U$ whose polynomial convex hull contains points outside the domain U . Now let F be a biholomorphism between U and \mathbb{C}^2 . There are plenty of Short \mathbb{C}^2 's, belonging to the large class of before-mentioned examples, each of which contains $F(K)$. Now carrying back these Short \mathbb{C}^2 's via F^{-1} we get Short \mathbb{C}^2 's in U containing K . Clearly these Short \mathbb{C}^2 's are non-Runge but biholomorphic to Runge domains. In fact, to best of our knowledge, all known examples of Short \mathbb{C}^k 's are either Runge or biholomorphic to Runge domains. Rather a bit more can be said: If $\Omega \subseteq \mathbb{C}^k$ is an increasing union of Ω_i 's, for $i \geq 1$ and if each Ω_i is biholomorphic to the unit ball via a global automorphism of \mathbb{C}^k , then by Prop. 1.2 in [10], Ω is Runge (this criterion holds for all known examples of Short \mathbb{C}^k 's). These observations prompts the question: If not Runge, whether a Short \mathbb{C}^k always biholomorphic to a Runge one? In an attempt to answer this question, we introduce a sub-class of Short \mathbb{C}^k 's called *Loewner Short \mathbb{C}^k 's*. They are those Short \mathbb{C}^k 's which can be parametrized on the positive real axis by strictly monotonically increasing domains each of which is biholomorphically equivalent either to the unit ball or to the unit polydisc. Being a biholomorphic copy of \mathbb{C}^k , any FB domain enjoys this kind of holomorphic exhaustion by balls. So while comparing FB domains and Short \mathbb{C}^k 's it is natural to be inquisitive about whether Short \mathbb{C}^k 's can also be exhausted by a holomorphically varying parametrized family of strictly increasing balls. This serves as another motivation to define Loewner Short \mathbb{C}^k 's.

Definition 1.5. A Short \mathbb{C}^k Ω is *Loewner* if the following holds:

- (1) $\Omega = \cup_{t \geq 0} \Omega_t$, where either the domain Ω_t is biholomorphic to the unit ball in \mathbb{C}^k , for each $t \geq 0$ or for each $t \geq 0$, the domain Ω_t is biholomorphic to the unit polydisc in \mathbb{C}^k .
- (2) for any $0 \leq t < s$, $\Omega_t \subset \subset \Omega_s$.
- (3) For all $t > 0$, $\Omega_t = \cup_{t' < t} \Omega_{t'}$.
- (4) For all $t' < t$, $\overline{\Omega}_{t'} = \cap_{t > t'} \Omega_t$.

Proposition 1.6. *Any Loewner Short \mathbb{C}^k is biholomorphic to a Runge domain.*

Using Thm. 4.2 in [3] we get that for each $n \geq 0$, the pair (Ω_n, Ω_{n+1}) is a Runge pair and then Prop. 1.6 follows using Thm. 3.4 in [3], Thus our previously posed question on Runge embedding of Short \mathbb{C}^k 's can be modified as follows:

Question: Is any Short \mathbb{C}^k Loewner?

Although we could not settle the question, we identify a large class of Loewner Short \mathbb{C}^k 's.

Theorem 1.7. *Let $0 < a_{n+1} \leq a_n < 1$ for all $n \geq 1$. Let $F_n(z, w) = (z^d + q_n(z) + \delta_n w, \delta_n z)$ with $1 \leq \deg(q_n) \leq (d-1)$ such that the moduli of the coefficients of q_n and δ_n are bounded by $a_n^{d^n}$. Then the basin of attraction*

$$\Omega = \{(z, w) : F_n \circ \cdots \circ F_1(z, w) \rightarrow 0\}$$

is a Loewner Short \mathbb{C}^2 .

From [7, Thm. 1.4] it follows that Ω is a Short \mathbb{C}^k . Further it follows (from the proof of [7, Thm. 1.4]) that for $r > 0$ sufficiently small, $\Omega = \bigcup_{n=1}^{\infty} F(n)^{-1}(\Delta(0; r, r))$, where $F(n) = F_n \circ \cdots \circ F_1$ and $\Delta(0; r, r) = \{(z, w) : |z|, |w| < r\}$. Moreover, $\Omega_n = F(n)^{-1}(\Delta(0; r, r)) \subseteq F(n+1)^{-1}(\Delta(0; r, r)) = \Omega_{n+1}$, for all n sufficiently large. To prove Ω to be Loewner it is sufficient to show that Ω_n can be stretched to Ω_{n+1} via a holomorphic family of strictly increasing open sets, each of which biholomorphic to polydisc. In fact, for all $n \geq 1$, that $F_n(\Delta(0; r, r))$ can be expanded to $\Delta(0; r, r)$ in the above-mentioned way suffices the purpose. It is straightforward to see that for all $n \geq 1$, $\Delta(0; r, r)$ can be shrunk holomorphically to $F_n(\Delta(0; r, r))$, but a priori it is not clear whether this can be performed in a monotonic manner too. The key step in the proof of Thm. 1.7 is to construct a parametrized family of automorphisms $F_{n,t}$ for each $n \geq 1$ and for all $0 \leq t \leq 1$ such that $F_{n,0} \equiv F_n$, $F_{n,1} \equiv \text{Id}$ and $F_{n,t}(\Delta(0; r, r)) \subset \subset F_{n,s}(\Delta(0; r, r))$, whenever $s > t$. If one looks carefully at the proof of Thm. 1.7, then it becomes apparent that the particular form of F_n 's plays the most crucial role in constructing $F_{n,t}$'s with desired properties. Nevertheless, we can stretch the family of F_n 's a bit to higher dimensions.

Let a_n 's be as before and let for $k \geq 3$, $F_n(z_1, z_2, \dots, z_k) = (z_1^d + q_n(z_1) + \eta_n z_k, \eta_n z_1, \dots, \eta_n z_{k-1})$ with $1 \leq \deg(q_n) \leq (d-1)$ such that the moduli of the coefficients of q_n and η_n are bounded by $a_n^{d^n}$. Then the basin of attraction

$$\Omega = \{(z, w) : F_n \circ \cdots \circ F_1(z, w) \rightarrow 0\}$$

is a Loewner Short \mathbb{C}^k .

If we consider any non-autonomous basin of attraction Ω of polynomial automorphisms of the form (1.1) (which is known to be Short \mathbb{C}^k from [7, Thm. 1.4]), it is not clear how to construct the intermediate automorphisms $F_{n,t}$'s. Nevertheless, we believe that these Ω 's are also Loewner.

Next we prove that the sublevel sets of Green's functions of Hénon maps are Loewner. As before the form of Hénon maps plays the pivotal role. That any sublevel set of the Green's function of a Hénon map can be realized as the basin of attraction of a sequence of Hénon maps with rapidly decaying coefficients (in other words, hypotheses of Thm. 1.7 are satisfied) leads us to the following theorem.

Theorem 1.8. *Let H be a Hénon map and let G_H^+ be the Green's function of H . Then for any $r > 0$, the set*

$$\Omega_r = \{z \in \mathbb{C}^2 : G_H^+(z) < r\},$$

is a Loewner Short \mathbb{C}^2 .

We conclude the article by constructing new examples of Short \mathbb{C}^2 's.

Example 1: Instead of a single Hénon map, we consider a random sequence of Hénon maps $\{F_n\}_{n \geq 1}$ such that the moduli of the coefficients of F_n 's are uniformly bounded above. Then it follows from [13, Prop. 1.1] that the corresponding random Green's function

$$G_{\{F_n\}}^+(z, w) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|F_n \circ \cdots \circ F_1(z, w)\|,$$

for $(z, w) \in \mathbb{C}^2$, exists where $d = \deg(F_n)$ for all $n \geq 1$ and $\log^+(x) = \max\{\log x, 0\}$. In spirit of [7, Thm. 1.12] (which shows that the sublevel sets of the Green's function of a single Hénon map are Short \mathbb{C}^2 's), we prove the following:

Theorem 1.9. *Let $\{F_n\}$ be a sequence of Hénon maps of the form $F_n(z, w) = (z^d + q_n(z) + \delta_n w, z)$ where q_n 's are polynomials of degree at most $(d-1) \geq 1$ and $\delta_n \neq 0$, for all $n \geq 1$. Further, assume that the moduli of the coefficients of q_n 's and δ_n 's are uniformly bounded above. Then for any $r > 0$, the sublevel set*

$$\Omega_{\{F_n\}, r} = \left\{ (z, w) \in \mathbb{C}^2 : G_{\{F_n\}}^+(z, w) < r \right\}$$

is a Short \mathbb{C}^2 .

Example 2: For $0 < c < 1$, let $F_n(z, w) = (z^2 + c^{2n} w, c^{2n} z)$ for all $n \geq 1$. Then by [7, Thm. 1.4], it follows that the non-autonomous basin of attraction of F_n 's at the origin is always a Short \mathbb{C}^2 . The following theorem records that if we replace the coefficients of the linear terms of F_n 's by c^{t_n} with t_n converging to 2, the corresponding basin of attraction is not necessarily always a Short \mathbb{C}^2 . The following theorem can also be compared with [7, Thm. 1.10].

Theorem 1.10. *For $n \geq 1$, let $F_n(z, w) = (z^2 + c^{t_n} w, c^{t_n} z)$ where $t_n \rightarrow 2$ and $0 < c < 1$. Let*

$$\Omega = \{(z, w) : F_n \circ \cdots \circ F_1(z) \rightarrow 0\}$$

be the non-autonomous basin of attraction of the sequence of automorphisms $\{F_n\}$ at the origin. Then,

- A. *There exists a sequence $t_n \rightarrow 2$ such that Ω is a Short \mathbb{C}^2 .*
- B. *There exists a sequence $t_n \rightarrow 2$ such that Ω is a Fatou-Bieberbach domain.*

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2. VOLUME OF SHORT \mathbb{C}^k 'S: PROOF OF THM. 1.2

Proof. Let $\Omega = \cup_{i=1}^{\infty} \Omega_i$. Let $B(0, 1)$ be the unit ball in \mathbb{C}^2 . We choose biholomorphisms $\phi_i : B(0, 1) \rightarrow \Omega_i$. We can assume that $0 \in \Omega_1$ and hence in all Ω_i 's. After pre-composing with automorphisms of the unit ball, we can assume that $\phi_i(0) = 0$ for all i .

Pick an $\epsilon > 0$. Let ξ denote a tangent vector at 0 of length 1. Since the Kobayashi metric of Ω vanishes at 0 identically, there exists a holomorphic map $\psi : \Delta \rightarrow \Omega$ so that $|\psi'(0)| = \lambda \xi$ for some constant $\lambda > \frac{4}{\epsilon}$, where Δ is the unit disc in \mathbb{C} . Hence there is a holomorphic map $\psi_1 : \bar{\Delta} \rightarrow \Omega$

such that $\psi'_1(0) = \lambda\xi/2$. Then the image is a compact subset of some Ω_i . After small rotation we then know that there is a relatively open neighbourhood of ξ in the unit sphere so that for each ξ' in this neighbourhood there is a map $f : \Delta \rightarrow \Omega_i$ so that $f(0) = 0, f'(0) = \mu\xi'$ with $\mu > 2/\epsilon$. This then is true for all larger i . Hence by compactness of the unit sphere, there is an integer j so that for any tangent vector ξ at zero of length one, there is a holomorphic map $f : \Delta \rightarrow \Omega_j$ so that $f(0) = 0$ and $f'(0) = \lambda\xi$ for some $\lambda > 2/\epsilon$. This is true for any large j .

Next consider the composite maps $g = \phi_j^{-1} \circ f$. These are holomorphic maps from the unit disc to the unit ball. Moreover, $g(0) = 0$ and

$$g'(0) = (\phi_j^{-1})'(f(0))f'(0) = (\phi_j^{-1})'(f(0))(\lambda\xi).$$

Then by Schwarz lemma it follows that

$$\|(\phi_j^{-1})'(f(0))(\lambda\xi)\| \leq 1, \text{ i.e., } \|(\phi_j^{-1})'(f(0))(\xi)\| \leq 1/\lambda < \epsilon/2.$$

Let $\tilde{\xi}$ be the unit vector in the direction of $(\phi_j^{-1})'(f(0))(\xi)$. Then $(\phi_j^{-1})'(f(0))(\xi) = \sigma\tilde{\xi}$ and $\sigma < \epsilon/2$. Therefore $\phi'_j(0)(\tilde{\xi}) = \frac{\xi}{\sigma}$. Thus it follows that $\|\phi'_j(0)(\tilde{\xi})\| \geq \frac{2}{\epsilon}$. But as ξ runs over all unit tangent vectors, the same is true for the vectors $\tilde{\xi}$. Therefore, we have shown: For any large enough j , we have that if $\tilde{\xi}$ is any unit tangent vector at the origin, then $\|\phi'_j(0)(\tilde{\xi})\| \geq \frac{2}{\epsilon}$.

Let A be a 2 by 2 matrix so that $\|A(x)\| \geq 2\|x\|/\epsilon$ for every vector x . Then the inverse matrix B satisfies $\|B(y)\| \leq \epsilon\|y\|/2$. Hence each entry in the matrix of y has size at most $\epsilon/2$. Then the Jacobian of the matrix can be at most $\epsilon^2/2$. But this implies that the Jacobian of A is at least $2/\epsilon^2$.

To finish the proof we only need to show that there is no non-trivial L^2 holomorphic function on Ω . In particular this shows that the function $f \equiv 1$ is not in $L^2(\Omega)$ which implies that the volume of Ω is infinite. So assume to the contrary that there is such an f on Ω . We could have chosen the point 0 differently, so we can assume that $f(0) = c \neq 0$. Then for every i , we have that $\int_{\Omega_i} |f|^2 < 1$. By the change of variable formula this shows that

$$\int_{\mathbb{B}(0,1)} |(f \circ \phi_i) \times \text{Jac}(\phi_i)|^2 dV < 1.$$

Let $g_i = (f \circ \phi_i) \times \text{Jac}(\phi_i)$. Then g_i is a holomorphic function on the unit ball with $|g_i(0)| \geq 2|c|/\epsilon^2$ and $\int_{\mathbb{B}(0,1)} |g_i|^2 < 1$ for all large enough i .

After writing $g_i(z) = g_i(0) + \text{h.o.t.}$, if we estimate the integral of $|g_i|^2$ we get an estimate from below by integrating only the constant term. This gives a contradiction if we choose ϵ small enough. \square

3. LOEWNER SHORT \mathbb{C}^k 'S

Proof of Theorem 1.7. By Thm. 1.4 in [7], it follows that Ω is a Short \mathbb{C}^2 . Now we prove that Ω is Loewner.

Test Case: To make the idea of the proof transparent, we first deal with the simplest case: Namely, Ω is non-autonomous basin of attraction of quadratic Hénon maps F_n 's where $F_n(z, w) = (z^2 + a_n^{2n}w, a_n^{2n}z)$ with $0 < a_{n+1} \leq a_n < 1$, for all $n \geq 1$.

For $r > 0$ sufficiently small, let $\Delta^2(0; r, r)$ be the bidisc of radius r and

$$\Omega_n = F_1^{-1} \circ \dots \circ F_n^{-1}(\Delta^2(0; r, r)).$$

It follows from Thm 1.4 in [7] that $\Omega_n \subseteq \Omega_{n+1}$ and $\Omega = \cup_{n=1}^{\infty} \Omega_n$. We show that for any $n \geq 1$, $F_n(\Delta^2(0; r, r))$ can be deformed holomorphically and monotonically to $\Delta^2(0; r, r)$.

For $0 \leq t \leq 1$, define $\Delta_t^2 = \Delta^2(0, (1+t)r, (1+2t)r)$ and $F_{n,t}$ on Δ_t^2 as follows:

$$F_{n,t}(z, w) = ((1-t)z^2 + [a_n^{2n} + rt]w, a_n^{2n}z).$$

Note that when $t = 0$, then $F_{n,0} \equiv F_n$ and $\Delta_0^2 = \Delta^2(0, r, r)$. For $t = 1$, $F_{n,1}(z, w) = ((a_n^{2n} + r)w, a_n^{2n}z)$ and the image of Δ_1^2 under the map $F_{n,1}$ becomes $\Delta^2(0, (a_n^{2n} + r)3r, a_n^{2n}2r)$.

First we show that as t increases, the images of Δ_t^2 's under the map $F_{n,t}$'s strictly increase, i.e., if $0 \leq t < s \leq 1$, then

$$(3.1) \quad F_{n,t}(\Delta_t^2) \subset\subset F_{n,s}(\Delta_s^2).$$

Fix a z_0 with $|z_0| < (1+t)r$. Now we consider the vertical slice $L_{z_0} = \{w \in \mathbb{C} : |w| < (1+2t)r\}$. The image

$$F_{n,t}(L_{z_0}) = \{((1-t)z_0^2 + [a_n^{2n} + rt]w, a_n^{2n}z_0) : |w| < (1+2t)r\},$$

is the horizontal disc at height $a_n^{2n}z_0$ with center $(1-t)z_0^2$ and radius $(a_n^{2n} + rt)(1+2t)r$. The image of L_{z_0} under the map $F_{n,s}$ gives the horizontal disc with center $(1-s)z_0^2$ and radius $(a_n^{2n} + rs)(1+2s)r$ at the same height. The condition of strict monotonicity is the following:

$$(1-s)|z_0|^2 + (a_n^{2n} + rs)(1+2s)r > (1-t)|z_0|^2 + (a_n^{2n} + rt)(1+2t)r,$$

which is equivalent to

$$(a_n^{2n} + rs)(1+2s)r - (a_n^{2n} + rt)(1+2t)r > (1-t)|z_0|^2 - (1-s)|z_0|^2,$$

i.e. to

$$[a_n^{2n}(1+2s)r - a_n^{2n}(1+2t)r] + [r^2s(1+2s) - r^2t(1+2t)] > (s-t)|z_0|^2.$$

Thus it suffices to prove that

$$r^2s(1+2s) - r^2t(1+2t) > (s-t)|z_0|^2,$$

which is equivalent to

$$r^2(s-t) + 2r^2(s^2 - t^2) > (s-t)|z_0|^2,$$

in other words, to

$$r^2(1+2s+2t) > |z_0|^2.$$

Now since $|z_0| < (1+t)r$ and $s > t$, it suffices to prove that $(1+4t) > (1+t)^2$ which is clearly true since $t < 1$.

For $0 \leq t \leq 1$, let $\Omega_t = F_{n,t}(\Delta_t^2)$. We now show that $\Omega_t = \bigcup_{t' < t} \Omega_{t'}$. Since (3.1) holds, clearly $\bigcup_{t' < t} \Omega_{t'} \subseteq \Omega_t$. To prove the reverse containment, let $(z_0, w_0) \in \Omega_t$. Then

$$(z_0, w_0) = F_{n,t}(z, w) = ((1-t)z^2 + [a_n^{2n} + rt]w, a_n^{2n}z)$$

for some $(z, w) \in \Delta_t^2$. Thus $z = w_0/a_n^{2n}$ and $(1-t)z^2 + [a_n^{2n} + rt]w = z_0$. If there exists $t' < t$ such that $F_{n,t'}(z', w') = (z_0, w_0)$ for some $(z', w') \in \Delta_{t'}^2$, then

$$(1-t)z^2 + [a_n^{2n} + rt]w = (1-t')z'^2 + [a_n^{2n} + rt']w',$$

i.e.,

$$(t' - t)z^2 + (a_n^{2n} + rt)w = (a_n^{2n} + rt')w'$$

which is equivalent to

$$w' = w + \frac{(t-t')}{(a_n^{2n} + rt')} [rw - z^2].$$

To check the existence of some w' with $|w'| < (1+2t')r$, it is sufficient to check if $t' < t$ can be chosen so that

$$|w| + \frac{(t-t')}{(a_n^{2n} + rt')} [r|w| + |z|^2] < (1+2t')r,$$

i.e.,

$$(3.2) \quad |w| \left(1 + \frac{(t-t')r}{(a_n^{2n} + rt')} \right) + \frac{(t-t')}{(a_n^{2n} + rt')} |z|^2 < (1+2t')r.$$

To check whether $t' < t$ exists so that (3.2) holds, it is enough to see whether the following holds for t' sufficiently close to t :

$$(3.3) \quad |w| \left(1 + \frac{t-t'}{a_n^{2^n}} \right) + \frac{(t-t')}{a_n^{2^n}} |z|^2 < (1+2t')r.$$

Now there exists $\tilde{t} < t$ such that $|w| < (1+2\tilde{t})r$. Thus it is possible to choose $t' < t$ such that (3.3) holds. This proves

$$\Omega_t = \bigcup_{t' < t} \Omega_{t'}.$$

Next we prove that $\bar{\Omega}_t = \bigcap_{t' > t} \Omega_{t'}$. That $\bar{\Omega}_t \subseteq \bigcap_{t' > t} \Omega_{t'}$ follows from (3.1). Now let $(z_0, w_0) \in \Omega_{t'}$ for all $t' > t$. Therefore,

$$(z_0, w_0) = ((1-t)z_{t'}^2 + [a_n^{2^n} + rt]w_{t'}, a_n^{2^n} z_{t'})$$

for some $(z_{t'}, w_{t'}) \in \Delta_{t'}^2$. This implies $z = z_{t'} = w_0/a_n^{2^n}$ for all $t' > t$ and

$$(1-t') \frac{w_0^2}{a_n^{2^{n+1}}} + (a_n^{2^n} + rt') w_{t'} = z_0,$$

i.e.,

$$w_{t'} = \left[z_0 - (1-t') \frac{w_0^2}{a_n^{2^{n+1}}} \right] \frac{1}{(a_n^{2^n} + rt')}$$

for all $t' > t$. Now clearly $(z, w_t) = \lim_{n \rightarrow \infty} (z, w_{t'}) \in \bar{\Delta}_t^2$ and $F_{n,t}(z, w_t) = (z_0, w_0)$. Therefore $\bigcap_{t' > t} \Omega_{t'} \subseteq \bar{\Omega}_t$.

Till now we proved that for all $n \geq 1$, $\Omega_0 = F_n(\Delta^2(0; r, r))$ can be expanded monotonically to $\Omega_1 = \Delta^2(0, (a_n^{2^n} + r)3r, a_n^{2^n} 2r)$ via a holomorphically varying family of domains Ω_t for $0 \leq t \leq 1$ that satisfies the properties enlisted in Definition 1.5. Now it is easy to see that the bidisc $\Omega'_0 = \Delta^2(0, (a_n^{2^n} + r)3r, a_n^{2^n} 2r)$ can be expanded to the bidisc $\Omega'_1 = \Delta^2(0; r, r)$ monotonically and holomorphically via bidisc $\Omega'_t = (0; tr + (1-t)(a_n^{2^n} + r)3r, tr + (1-t)a_n^{2^n} 2r)$ for $0 \leq t' \leq 1$. Further, this family of bidiscs satisfies the properties indicated in Definition 1.5. Therefore, for each $n \geq 1$, we showed existence of domains $\{\Omega_{n,t}\}_{0 \leq t' \leq 1}$ such that $\Omega_{n,0} = F_n(\Delta^2(0; r, r))$ and $\Omega_{n,1} = \Delta^2(0; r, r)$. Further $\Omega_{n,t} \subset \subset \Omega_{n,s}$ if $0 \leq t < s \leq 1$, $\Omega_{n,t} = \bigcup_{t' < t} \Omega_{n,t'}$ and $\bar{\Omega}_{n,t} = \bigcap_{t' > t} \Omega_{n,t'}$. In turn for each n , we have

$$(3.4) \quad \begin{aligned} \Omega_{n-1} &= F_1^{-1} \circ \dots \circ F_n^{-1}(F_n(\Delta^2(0; r, r))) \subseteq \dots \subseteq F_1^{-1} \circ \dots \circ F_n^{-1}(\Omega_{n,t}) \\ &\subseteq \dots \subseteq F_1^{-1} \circ \dots \circ F_n^{-1}(\Omega_{n,s}) \subseteq \dots \subseteq F_1^{-1} \circ \dots \circ F_n^{-1}(\Delta^2(0; r, r)) = \Omega_n. \end{aligned}$$

This proves Ω is Loewner when for each $n \geq 1$, $\deg(F_n) = 2$ and a_n is real.

General Case. For each $n \geq 1$, $F_n(z, w) = (p_n(z) + \delta_n w, \delta_n z)$, where $p_n(z) = z^d + q_n(z)$ with q_n a polynomial of degree $d-1$. For each $n \geq 1$, we modify F_n as follows. Let $0 \leq t_n < 2\pi$ be such that $\delta'_n = e^{it_n} \delta_n > 0$. Define

$$\tilde{F}_n(z, w) = (\tilde{p}_n(z) + \delta'_n w, \delta'_n z)$$

where $\tilde{p}_n(z) = p_n((\delta'_n/\delta_n)z)$. As before we can show that for any $n \geq 1$, the bidisc $\Delta^2(0; r, r)$ can be deformed to $\tilde{F}_n(\Delta^2(0; r, r))$. Choose $M \gg 2^d + d/r^d$. For each $n \geq 1$, define $F_{n,t}$ on $\Delta_t^2 = \Delta^2(0, (1+t)r, (1+2t)r)$ as follows:

$$\tilde{F}_{n,t}(z, w) = \left((1-t)\tilde{p}_n(z) + [\delta'_n + Mr^{d-1}t]w, \delta'_n z \right).$$

As before, for fix a z_0 with $|z_0| < (1+t)r$, the condition of strict monotonicity, i.e., for $0 \leq t < s \leq 1$, $\tilde{F}_{n,t}(\Delta_t^2) \subset \subset \tilde{F}_{n,s}(\Delta_s^2)$ is

$$\left(\delta'_n + Mr^{d-1}t \right) (1+2t)r + (s-t)|\tilde{p}_n(z_0)| < \left(\delta'_n + Mr^{d-1}s \right) (1+2s)r,$$

which is equivalent to showing

$$\delta'_n [(1+2s) - (1+2t)]r + Mr^{d-1}s(1+2s)r - Mr^{d-1}t(1+2t)r > (s-t)|\tilde{p}_n(z_0)|.$$

Thus it is sufficient to show

$$Mr^d(s-t) + 2Mr^d(s^2 - t^2) > (s-t)|\tilde{p}_n(z_0)|,$$

or equivalently,

$$(3.5) \quad Mr^d(1+2s+2t) > |\tilde{p}_n(z_0)|.$$

Now since $|z_0| < (1+t)r$, $|\tilde{p}_n(z_0)| < (1+t)^d r^d + a_n^{2n} \left((1+t)^{d-1} r^{d-1} + \dots + (1+t)r + 1 \right) < 2d r^d + d$. Thus clearly (3.5) holds.

Thus $\tilde{F}_{n,1}(\Delta^2(0; r, r))$ and thus $\Delta^2(0; r, r)$ can be deformed holomorphically and monotonically to $\tilde{F}_{n,0}(\Delta^2(0; r, r))$. We claim that $\tilde{F}_{n,0}(\Delta^2(0; r, r)) = F_{n,0}(\Delta^2(0; r, r))$. Fix a z_0 with $|z_0| < r$, then $F_n(z_0, w) = \tilde{F}_n((\delta_n/\delta'_n)z_0, (\delta_n/\delta'_n)w)$. Thus $F_{n,0}(\Delta^2(0; r, r)) \subseteq \tilde{F}_{n,0}(\Delta^2(0; r, r))$. Similarly it follows that $\tilde{F}_{n,0}(\Delta^2(0; r, r)) \subseteq F_{n,0}(\Delta^2(0; r, r))$. The rest of the proof follows as before.

Proof of Thm. 1.8. We complete the proof in three steps.

Step 1: Let $\deg(H) = d$ and let $r_n \uparrow r$ as $n \rightarrow \infty$. Let $\delta < 1$ and for each $n \geq 1$, let us choose $m(n) \geq n$ such that

$$(3.6) \quad e^{(r_{n+1}-r_{n+2})d^{m(n)}} < \frac{\delta}{n}.$$

Let $\|(z, w)\|_1 = \max\{|z|, |w|\}$. For each $n \geq 1$, set

$$\Omega_n = \left\{ (z, w) \in \mathbb{C}^2 : \|H^{m(n)}(z, w)\|_1 < e^{r_{n+1}d^{m(n)}} \right\}.$$

Now since (3.6) holds,

$$\begin{aligned} (r_{n+1} - r_{n+2})d^{m(n)} &< \log \left(\frac{\delta}{n} \right) < -\log n \\ \Rightarrow (r_{n+2} - r_{n+1}) &> \frac{\log n}{d^{m(n)}} \\ (3.7) \quad \Rightarrow r_{n+1} + \frac{\log n}{d^{m(n)}} &< r_{n+2}. \end{aligned}$$

Claim: Ω_r is increasing union of Ω_n 's.

There exists $L > 1$ such that

$$(3.8) \quad G_H^+(z, w) \leq \log^+ \|(z, w)\|_1 + L$$

for all $(z, w) \in \mathbb{C}^2$. Now

$$\frac{1}{d^{m(n)}} \log^+ \|H^{m(n)}(z, w)\|_1 < r_{n+1}$$

for all $(z, w) \in \Omega_n$. Thus by (3.8),

$$\begin{aligned} G_H^+ \circ H^{m(n)}(z, w) &\leq \log^+ \|H^{m(n)}(z, w)\|_1 + L \\ (3.9) \quad \Rightarrow G_H^+(z, w) &\leq \frac{1}{d^{m(n)}} \log^+ \|H^{m(n)}(z, w)\|_1 + \frac{L}{d^{m(n)}} < r_{n+2} < r. \end{aligned}$$

The second last inequality follows from (3.7). We can choose $m(n+1)$ large enough such that

$$\frac{1}{d^{m(n+1)}} \log^+ \|H^{m(n+1)}(z, w)\|_1 < \frac{1}{d^{m(n)}} \log^+ \|H^{m(n)}(z, w)\|_1 + \frac{L'}{d^{m(n)}} < r_{n+2} < r$$

for some $L' > 0$ and for all $(z, w) \in \Omega_n$. Therefore,

$$\Omega_1 \subseteq \Omega_2 \subseteq \dots = \bigcup_{n=1}^{\infty} \Omega_n \subseteq \Omega_r.$$

To prove the converse, let $G_H^+(z, w) < r$. This implies $G_H^+(z, w) < r_k$ for some $k \geq 1$. Therefore,

$$\frac{1}{d^n} \log^+ \|H^n(z, w)\|_1 < r_k$$

for all $n \geq n_0$. Therefore, $(z, w) \in \Omega_n$ for some large n . Thus the claim follows.

Step 2: Note that for each $n \geq 1$, the global map $\Phi_n = \varphi_{m(n)} \circ H^{m(n)}$ maps Ω_n to $\Delta^2(0; 1, 1)$ where $\varphi_{m(n)}(z, w) = \left(e^{-r_{n+1}d^{m(n)}} z, e^{-r_{n+1}d^{m(n)}} w \right)$. Now we claim that

$$\Omega_r = \{(z, w) \in \mathbb{C}^2 : \Phi_n(z, w) \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

For any $(z, w) \in \Omega_r$, there exists n_0 large enough such that $(z, w) \in \Omega_n$ for all $n \geq n_0$. Now if (z, w) is in the non-escaping set $K_H^+ = \{(z, w) \in \mathbb{C}^2 : \|H^n(z, w)\| < M \text{ for all } n \geq 1\}$, then clearly, $\Phi_n(z, w) \rightarrow 0$ as $n \rightarrow \infty$. Let $(z, w) \notin K_H^+$ and let $(z, w) \in \Omega_n$ for all $n \geq n_0$, then there exists a large natural number, without loss of generality, $m(n_0)$ say, such that $H^{m(n_0)}(z, w) \in V_R^+$ where $V_R^+ = \{(z, w) \in \mathbb{C}^2 : |z| \geq \max\{|w|, R\}\}$. Now

$$\begin{aligned} \left\| H^{m(n)}(z, w) \right\|_1 &= \left\| H^{m(n)-m(n_0)} \left(H^{m(n_0)}(z, w) \right) \right\|_1 \sim \left| \left(H^{m(n_0)}(z, w) \right) \right|_1^{d^{m(n)-m(n_0)}} \\ &< \left(e^{r_{n_0+1}d^{m(n_0)}} \right)^{d^{m(n)-m(n_0)}} = e^{r_{n_0+1}d^{m(n)}}. \end{aligned}$$

Thus

$$\|\Phi_n(z, w)\| = \left\| \varphi_{m(n)} \circ H^{m(n)}(z, w) \right\| < e^{(-r_{n+1}+r_{n_0+1})d^{m(n)}} \rightarrow 0$$

as $n \rightarrow \infty$. Conversely, it is straightforward to see that if $\Phi_n(z, w) \rightarrow 0$ as $n \rightarrow \infty$, then $(z, w) \in \Omega_r$.

Now without loss of generality, we assume that $\Delta^2(0; c, c) \subseteq \Omega_r$, for $c > 0$ sufficiently small. Let $\Phi_n(z, w) \in \Delta^2(0; c, c)$. This implies

$$\left\| H^{m(n)}(z, w) \right\|_1 < ce^{r_{n+1}d^{m(n)}}.$$

Therefore combining (3.7) and (3.9), we get

$$\begin{aligned} \frac{1}{d^{m(n)}} \log^+ \left\| H^{m(n)}(z, w) \right\|_1 &< \frac{\log c}{d^{m(n)}} + r_{n+1} \\ \Rightarrow \frac{1}{d^{m(n+1)}} \log^+ \left\| H^{m(n+1)}(z, w) \right\|_1 &< \frac{\log c}{d^{m(n)}} + r_{n+1} + \frac{L'}{d^{m(n)}} < \frac{\log c}{d^{m(n+1)}} + r_{n+2}. \end{aligned}$$

Thus $\Phi_{n+1}(z, w) \in \Delta^2(0; c, c)$. This shows $\Phi_n^{-1}(\Delta^2(0; c, c)) \subseteq \Phi_{n+1}^{-1}(\Delta^2(0; c, c))$ for all n . Also we have $\Omega_r = \bigcup_{n=0}^{\infty} \Phi_n^{-1}(\Delta^2(0; c, c))$

Step 3: Without loss of generality we further assume that $\Delta^2(0; 1, 1) \subseteq \Omega_r$. Let $\Phi_0 \equiv \text{Id}$. Then

$$\Phi_n = (\Phi_n \circ \Phi_{n-1}^{-1}) \circ (\Phi_{n-1} \circ \Phi_{n-2}^{-1}) \circ \cdots \circ (\Phi_2 \circ \Phi_1^{-1}) \circ (\Phi_1 \circ \Phi_0^{-1}) \circ \Phi_0 = F_n \circ \cdots \circ F_1 \circ F_0$$

where $F_n = \Phi_n \circ \Phi_{n-1}^{-1} = \varphi_{m(n)} \circ H^{m(n)-m(n-1)} \circ \varphi_{m(n-1)}^{-1}$ for $n \geq 1$ and $F_0 \equiv \text{Id}$. Note that $F(n)^{-1}(\Delta^2(0; c, c)) \subseteq F(n+1)^{-1}(\Delta^2(0; c, c))$ where $F(n) = F_n \circ \cdots \circ F_1 \circ F_0$ for all n . Further,

$$\Omega_r = \bigcup_{n=0}^{\infty} F(n)^{-1}(\Delta^2(0; c, c)).$$

Now we show that for each n , $\Delta^2(0; c, c)$ can be distorted monotonically to $F_n(\Delta^2(0; c, c))$ which in turn proves that Ω_r is Loewner as in Thm. 1.7. For each n , consider arbitrary sequence of real numbers satisfying

$$r_n = r_{m(n-1)} < r_{m(n-1)+1} < r_{m(n-1)+2} < \cdots < r_{m(n)-1} < r_{m(n)} = r_{n+1}.$$

Further let

$$\varphi_{m(n-1)+k}(z, w) = \left(e^{-r_{[m(n-1)+k]}d^{[m(n-1)+k]}} z, e^{-r_{[m(n-1)+k]}d^{[m(n-1)+k]}} w \right)$$

Now note that

$$\begin{aligned} & \varphi_{m(n)} \circ H^{m(n)-m(n-1)} \circ \varphi_{m(n-1)}^{-1} \\ &= \left[\varphi_{m(n)} \circ H \circ \varphi_{m(n)-1}^{-1} \right] \circ \left[\varphi_{m(n)-1} \circ H \circ \varphi_{m(n)-2}^{-1} \right] \circ \cdots \circ \left[\varphi_{m(n-1)+1} \circ H \circ \varphi_{m(n-1)}^{-1} \right] \\ &= L_{m(n)} \circ L_{m(n)-1} \circ \cdots \circ L_{m(n-1)+1}. \end{aligned}$$

Now as in Thm. 1.7, one can show that $\Delta^2(0; c, c)$ can be distorted monotonically to $L_j(\Delta^2(0; c, c))$'s and therefore, $\Delta^2(0; c, c)$ can be distorted monotonically to $\varphi_{m(n)} \circ H^{m(n)-m(n-1)} \circ \varphi_{m(n-1)}^{-1}(\Delta^2(0; c, c)) = F_n(\Delta^2(0; c, c))$. This completes the proof.

4. EXAMPLES OF SHORT \mathbb{C}^2 'S

Proof of Theorem 1.9. The proof follows a similar line of arguments as in Thm. 1.12 in [7]. We start the proof with a claim.

Claim: For any compact set $K \subseteq \Omega_{\{F_n\}, r}$ and for any $\epsilon > 0$, there exist an open set $U \subseteq \Omega_{\{F_n\}, r}$ and an automorphism Φ of \mathbb{C}^2 such that $\Phi(U) = B(0, 1)$ and $\Phi(K) \subseteq B(0; \epsilon)$ where $B(0; 1)$ and $B(0; \epsilon)$ are balls with center at the origin and of radius 1 and ϵ , respectively.

We denote the random Green's function corresponding to the sequence of Hénon maps $\{F_n\}_{n \geq m}$ by $G_{\{F_n\}, m}^+$ and following the notation in the introduction of this article we have $G_{\{F_n\}, 1}^+ \equiv G_{\{F_n\}}^+$. By Prop. 1.1 in [13], we have

$$(4.1) \quad G_{\{F_n\}, m+1}^+ \circ F_m \circ \cdots \circ F_1(z, w) = d^m G_{\{F_n\}}^+(z, w),$$

for all $(z, w) \in \mathbb{C}^2$ and for all $m \geq 0$. Further, there exists $L > 1$ such that

$$(4.2) \quad G_{\{F_n\}, m}^+(z, w) \leq \log^+ \|(z, w)\| + L,$$

for all $(z, w) \in \mathbb{C}^2$ and for all $m \geq 1$. Now choose $0 < r_1 < r_2 < r$ such that $K \subseteq \Omega_{\{F_n\}, r_1}$. Further, choose n sufficiently large such that $e^{(r_1-r_2)d^n} < \epsilon$ and set

$$U = \left\{ (z, w) \in \mathbb{C}^2 : (z, w) \in F_1^{-1} \circ \cdots \circ F_n^{-1} \left(B \left(0, e^{r_2 d^n} \right) \right) \right\} \subseteq \Omega_{\{F_n\}, r}.$$

This choice is possible since (4.1) and (4.2) hold.

Let $\varphi : (z, w) \mapsto (z/e^{r_2 d^n}, w/e^{r_2 d^n})$ for $(z, w) \in \mathbb{C}^2$ and let $\Phi = \varphi \circ F_n \circ \cdots \circ F_1$. Then $\Phi(U) = B(0, 1)$ and $\Phi(K) \subseteq B(0, \epsilon)$.

Let $r_n \uparrow r$ and $R_n \rightarrow \infty$, as $n \rightarrow \infty$. Now $\Omega_{\{F_n\}, r} = \bigcup_{n \geq 1} K_n$ where $K_n = \overline{\Omega}_{r_n} \cap V_{R_n}$ and $V_{R_n} = \{(z, w) \in \mathbb{C}^2 : |z|, |w| \leq R_n\}$. For each $n \geq 1$, there exists $m(n) > 1$ such that $e^{(r_{n+1}-r_{n+2})d^{m(n)}} < 1/n$ and

$$U_n = \left\{ (z, w) \in \mathbb{C}^2 : (z, w) \in F_1^{-1} \circ \cdots \circ F_{m(n)}^{-1} \left(B \left(0, e^{r_{n+2} d^{m(n)}} \right) \right) \right\} \subseteq \Omega_{\{F_n\}, r}.$$

Thus there exists a sequence of automorphisms $\Phi_n = \varphi_{m(n)} \circ F_{m(n)} \circ \cdots \circ F_1$ such that

$$\Phi_n(K_n) \subseteq B \left(0, \frac{1}{n} \right) \text{ and } \Phi_n(U_n) = B(0, 1).$$

where $\varphi_{m(n)}(z, w) = \left(z/e^{r_{n+2} d^{m(n)}}, w/e^{r_{n+2} d^{m(n)}} \right)$.

Let $p \in \Omega_{\{F_n\}, r}$ and ζ be a tangent vector at p . Without loss of generality, assume $p = 0$. Let $p \in U_n$ for all $n \geq n_0$. Now $\Phi_n(p) = 0$ for all n . Let $(F_1^{-1} \circ \cdots \circ F_{m(n_0)}^{-1})'(0)(\eta_0) = \zeta$. Now $(F_1^{-1} \circ \cdots \circ F_{m(n)}^{-1})'(0)(\eta_n) = \zeta$ where $\eta_n = (F_{m(n)} \circ \cdots \circ F_{m(n_0+1)})'(0)(\eta_0)$. Clearly, $|\eta_n| \leq |\eta_0|$

for all $n \geq n_0$. So for any $R > 0$, we can choose n sufficiently large such that the map $\rho(\Delta) \subseteq B(0, e^{r_{n+2}d^n})$ where $\rho(z) = R\eta_n z$ and Δ is the unit disc. Now consider the map $T_n = F_1^{-1} \circ \dots \circ F_{m(n)}^{-1} \circ \rho : \Delta \rightarrow \Omega_r$. Note that $T_n(0) = 0$ and $T'_n(0) = R\zeta$. Thus the Kobayashi metric vanishes identically on $\Omega_{\{F_n\}, r}$. This completes the proof.

4.1. Proof of Theorem 1.10: We first show the existence of a sequence $t_n \rightarrow 2$ such that the corresponding basin of attraction is a Short \mathbb{C}^2 .

Proof of A. Let $0 < r' < r < 1$ and $r_l = r(r')^l$. Then

$$(4.3) \quad F_{n+l}(\Delta^2(0; r_l)) \subseteq \Delta^2(0; r_l^2 + c^{t_{n+l}}).$$

Claim: For large n , $t_n^l > \frac{l+1}{2}$, for all $l \geq 1$.

Let n be so large such that $t_n > 1.9$, then for $l = 1$, the claim follows. Assuming that it holds for l , we shall prove that it holds for $l + 1$. Note that

$$t_n^{l+1} = t_n^l t_n > \frac{l+1}{2} 1.9 = 1.9 \frac{l}{2} + \frac{1.9}{2} > \frac{l}{2} + \frac{.9l}{2} + \frac{1.9}{2} \geq \frac{l}{2} + \frac{2.8}{2} > \frac{l+2}{2}.$$

Therefore,

$$\begin{aligned} \log c^{t_{n+l}^{n+l}} &= t_{n+l}^{n+l} \log c \\ &= (t_{n+l}^n)(t_{n+l}^l) \log c \leq (t_{n+l}^n) \frac{(l+1)}{2} \log c \\ &\leq (t_{n+l}^n) \frac{(l+1)}{4} \log c + (t_{n+l}^n) \frac{(l+1)}{4} \log c \\ &\leq \log r(1-r) + (l+1) \log r' \end{aligned}$$

for n sufficiently large and for all $l \geq 1$. Thus

$$c^{t_{n+l}^{n+l}} \leq r(1-r)r'^{(l+1)} \leq r(r')^{l+1} - r^2(r')^{l+1} \leq r(r')^{l+1} - r^2(r')^{2l}$$

which gives

$$r_l^2 + c^{t_{n+l}^{n+l}} \leq r_{l+1}.$$

This proves that for large n ,

$$(4.4) \quad F_{n+l}(\Delta^2(0; r_l)) \subseteq \Delta^2(0; r_{l+1})$$

for all $l \geq 1$. Once we prove (4.4), that Ω is an increasing union of balls and the Kobayashi metric on Ω vanishes identically, follow exactly in the same way as in Thm. 1.4 in [7].

Choose a sequence $\{t_n\}_{n \geq 1}$ such that

$$(4.5) \quad t_n^n \geq 2t_{n-1}^{n-1}.$$

One such example is given by $t_n = 2^{(1-\frac{\epsilon_n}{n})}$ with $0 < \epsilon_{n+1} \leq \epsilon_n < 1$, for all $n \geq 1$. Now let $F(n) = F_n \circ \dots \circ F_1$ for all $n \geq 1$ and

$$\varphi_n(z, w) = \max\{|(F(n))_1(z, w)|, |(F(n))_2(z, w)|, c^{t_{n-1}^{n-1}}\}.$$

If $\varphi_n(z) \leq 1$, then

$$\begin{aligned} \varphi_{n+1}(z, w) &= \max\left\{((F(n))_1(z, w))^2 + c^{t_n}((F(n))_2(z, w)), c^{t_n}((F(n))_1(z, w)), c^{t_n}\right\} \\ &\leq \max\left\{\varphi_n^2(z, w) + c^{t_n}\varphi_n(z, w), c^{t_n}\varphi_n(z, w), c^{t_n}\right\}. \end{aligned}$$

Therefore it follows from (4.5) that if $\varphi_n(z, w) \leq 1$, then

$$\varphi_{n+1}(z, w) \leq 2\varphi_n^2(z, w).$$

If $\varphi_n(z, w) \geq 1$, then it is easy to see that

$$\varphi_{n+1}(z, w) \leq 2\varphi_n^2(z, w).$$

Therefore we have

$$\varphi_{n+1}(z, w) \leq 2\varphi_n^2(z, w).$$

for all $(z, w) \in \mathbb{C}^2$ and

$$\frac{1}{2^{n+1}} \log \varphi_{n+1}(z, w) \leq \frac{\log 2}{2^{n+1}} + \frac{1}{2^n} \log \varphi_n(z, w).$$

Thus the sequence of psh function

$$\frac{1}{2^n} \log \varphi_n(z, w) + \sum_{j>n} \frac{\log 2}{2^j}$$

monotonically decreases to a psh function φ in \mathbb{C}^2 .

It can be easily checked that if $\varphi(z, w) < 0$ for any $(z, w) \in \mathbb{C}^2$, then $(z, w) \in \Omega$. Now if $(z, w) \in \Omega$, then $(F_n \circ \dots \circ F_1)(z, w) \rightarrow 0$ as $n \rightarrow \infty$ and thus

$$\frac{1}{2^n} \log \varphi_n(z, w) = \max \left\{ \frac{1}{2^n} \log |F_1^n(z, w)|, \frac{1}{2^n} \log |F_2^n(z, w)|, \frac{t^{n-1}}{2^n} \log c \right\} \leq \frac{1}{2^n} \log c$$

for sufficiently large n . This implies

$$\frac{1}{2^n} \log \varphi_n(z, w) + \sum_{j>n} \frac{\log 2}{2^j} \leq \frac{1}{2^n} \log c + \frac{1}{2^n} \log 2 = \frac{1}{2^n} \log(2c) < 0$$

for large n . Therefore, $\varphi(z, w) < 0$ for all $(z, w) \in \Omega$. It remains to show that φ is non-constant on Ω which follows applying the same argument as in Thm. 1.4 in [7]. Thus we omit the proof here.

Proof of B. We prove a series of lemmas to show the existence of a sequence $\{t_n\} \rightarrow 2$ such that the corresponding non-autonomous basin of attraction turns out to be a Fatou-Bieberbach domain.

We use the following notations: For each $n \geq 1$, let $F(n) = F_n \circ \dots \circ F_1$. For $(z, w) \in \Omega$, let $(z_n, w_n) = F(n)(z, w)$. For a compact set K in the basin of the attraction Ω , set $\delta_n = \max\{|z_n|, |w_n|\} : (z, w) \in K\}$.

Lemma 4.1. *Assume that $a_n = c^{t^n}$ for $0 < c < 1$ and $1 < t < 2$. Fix $0 < b < 1$. Set $F_n(z, w) = (z^2 + a_n w, a_n z)$. Then there exists an n_0 such that $\delta_{n+k} \leq a_{n+k+1} b^k$ for all $n \geq n_0$ and for all $k \geq 0$.*

Proof. There are two steps. First we prove that there exists an arbitrarily large n , so that $\delta_n \leq a_{n+1}$. Suppose on the contrary there is some n_0 so that $\delta_n > a_{n+1}$ for all $n > n_0$. Then for all such n 's, $\delta_{n+1} \leq \delta_n^2 + a_{n+1} \delta_n \leq 2\delta_n^2$. Hence $\delta_{n+k} \leq 2^{1+2+\dots+2^{k-1}} (\delta_n)^{2^k}$. So $a_{n+k+1} \leq \delta_{n+k} \leq (2\delta_n)^{2^k}$. This implies that $(c^{t^{n+1}})^{t^k} \leq (2\delta_n)^{2^k}$. But we can assume that $2\delta_n < 1$. Then this will fail for all large k 's. The second step is to fix a large such n and then prove that $\delta_{n+k} \leq a_{n+k+1} b^k$ for all positive k . This is true for $k = 0$. We will show that this inequality holds inductively. So suppose that $\delta_{n+k} \leq a_{n+k+1} b^k$. Then we have that $\delta_{n+k+1} \leq (a_{n+k+1})^2 (b^{2k} + b^k)$. Hence we would like to say that $(a_{n+k+1})^2 (b^{2k} + b^k) \leq a_{n+k+2} b^{k+1}$. This means that

$$\left(c^{t^{n+k+1}} \right)^{2-t} \leq \frac{b}{b^k + 1}.$$

This works as long as we have fixed a large n . □

In the next lemma, we assume $1 < t_n < 2$ for all n .

Lemma 4.2. *We set $a_n = c^{t_n}$. If there exists an arbitrarily large $n \geq 1$ and a $k(n) \geq 1$ (depending on n) so that $t_{n+k(n)+1} \leq 2^{\frac{k(n)}{n+k(n)+1}}$, then there exists an arbitrarily large n so that $\delta_n \leq a_{n+1}$.*

Proof. If the claim of the lemma does not hold, then $\delta_n > a_{n+1}$ for all large n . Therefore, $\delta_{n+1} \leq \delta_n^2 + a_{n+1}\delta_n \leq 2\delta_n^2$ and inductively we get $\delta_{n+k} \leq 2^{1+2+\dots+2^{k-1}}\delta_n^{2^k}$ for all $k \geq 1$. Hence $a_{n+k+1} < (2\delta_n)^{2^k}$ and consequently, $c^{t_{n+k+1}} < (2\delta_n)^{2^k}$. If n is large enough, then $2\delta_n < c$. Thus $t_{n+k+1} > 2^k$ for all large n and for all $k \geq 1$. Hence we have $2^{k(n)} < t_{n+k+1} < 2^{k(n)}$ which is contradiction. This completes the proof. \square

Lemma 4.3. *Let $0 < b, c < 1$. Set $t_n = 2^{1-\frac{\epsilon_n}{n}}$ where $\epsilon_{n+1} \geq \epsilon_n + \frac{1}{2^{n/2} \ln 2}$ and $\epsilon_n/n \rightarrow 0$. Let $a_n = c^{t_n}$ and let $F_n = (z^2 + a_n w, a_n z)$. Suppose there exists an arbitrarily large n and $k(n) \geq 1$ so that $t_{n+k(n)+1} \leq 2^{\frac{k(n)}{n+k(n)+1}}$ (This means that $\epsilon_{n+k(n)+1} \geq n+1$). Then there exists an arbitrarily large n so that for all $k \geq 0$, $\delta_{n+k} \leq a_{n+k+1}b^k$.*

Proof. By Lemma 4.2, there exists an arbitrarily large n so that $\delta_n \leq a_{n+1}$. Pick such a large n . As in the proof of Lemma 4.1, we would like to have the estimate $(a_{n+k+1})^2(b^{2^k} + b^k) \leq a_{n+k+2}b^{k+1}$ to prove the lemma for all k . This means that we want

$$c^{2[2^{(1-\frac{\epsilon_{n+k+1}}{n+k+1})}]^{n+k+1}}(b^{2^k} + b^k) \leq c^{[2^{(1-\frac{\epsilon_{n+k+2}}{n+k+2})}]^{n+k+2}}b^{k+1},$$

i.e.,

$$c^{2^{1+n+k+1-\epsilon_{n+k+1}}} \leq c^{2^{n+k+2-\epsilon_{n+k+2}}} \frac{b}{1+b^k},$$

i.e.,

$$c^{(2^{n+k+2}[2^{-\epsilon_{n+k+1}} - 2^{-\epsilon_{n+k+2}}])} \leq \frac{b}{1+b^k},$$

i.e.,

$$c^{2^{n+k+2} \cdot 2^{-\epsilon_{n+k+2}} [2^{\epsilon_{n+k+2}-\epsilon_{n+k+1}} - 1]} \leq \frac{b}{1+b^k}.$$

The above follows if

$$c^{2^{n+k+2} \cdot 2^{-\epsilon_{n+k+2}} [\ln 2 (\epsilon_{n+k+2} - \epsilon_{n+k+1})]} \leq \frac{b}{1+b^k},$$

which again follows if

$$c^{2^{n+k+2} \cdot 2^{-\epsilon_{n+k+2}} \frac{1}{2^{(n+k+1)/2}}} \leq \frac{b}{1+b^k}.$$

The above follows if n is large enough. \square

We continue to use previous notations and suppose that $\epsilon_n \rightarrow \infty$. Then we prove that Ω is a Fatou-Bieberbach domain.

Since $\epsilon_n \rightarrow \infty$, for every $n \geq 1$ there exists an integer $k(n) \geq 1$ so that $\epsilon_{n+k(n)+1} \geq n+1$. Then applying Lemma 4.3, we have an arbitrarily large n so that for all $k \geq 0$, $\delta_{n+k} \leq a_{n+k+1}b^k$. We then need to show that $a_1 \cdots a_n \leq c^{n/2}a_{n+1}$ for all large n . This is equivalent to the inequality

$$t_1^1 + t_2^2 + \cdots + t_n^n \geq n/2 + t_{n+1}^{n+1}.$$

We write this as $ls_n \geq rs_n$. Here the n th left side is $ls_n = t_1^1 + t_2^2 + \cdots + t_n^n$ and the n th right side is $rs_n = \frac{n}{2} + t_{n+1}^{n+1}$. We compare the growth of the left side and the right side:

$$G_n = (ls_{n+1} - ls_n) - (rs_{n+1} - rs_n) = (t_{n+1}^{n+1}) - (1/2 + t_{n+2}^{n+2} - t_{n+1}^{n+1}) = 2t_{n+1}^{n+1} - t_{n+2}^{n+2} - 1/2.$$

Thus

$$\begin{aligned} G_n &= 2(2^{1-\frac{\epsilon_{n+1}}{n+1}})^{n+1} - (2^{1-\frac{\epsilon_{n+2}}{n+2}})^{n+2} - 1/2 = 2^{n+2}(2^{-\epsilon_{n+1}} - 2^{-\epsilon_{n+2}}) - 1/2 \\ &= 2^{n+2}2^{-\epsilon_{n+2}}(2^{\epsilon_{n+2}-\epsilon_{n+1}} - 1) - 1/2. \end{aligned}$$

Hence

$$G_n \geq 2^{n+2}2^{-\epsilon_{n+2}} \ln 2(\epsilon_{n+2} - \epsilon_{n+1}) - 1/2$$

which implies

$$G_n \geq 2^{3(n+2)/4} \frac{1}{2^{(n+1)/2}} - 1/2 \rightarrow \infty.$$

Now this is the first step in the proof of Lemma 3.13 in [7]. Now the rest of the proof of Lemma 3.13 goes through and then to show that Ω is a Fatou-Bieberbach domain, one can follow the same steps as in the the last paragraph of [7].

REFERENCES

- [1] Abbondandolo, A. et al.: *A survey on non-autonomous basins in several complex variables*, <https://arxiv.org/pdf/1311.3835.pdf>
- [2] Arosio, L.; Boc-Thaler, L.; Peters, H.: *A transcendental Hénon map with an oscillating wandering Short \mathbb{C}^2* , *Mathematische Zeitschrift* (2021), <https://doi.org/10.1007/s00209-020-02677-4>.
- [3] Arosio, L.; Bracci, F.; Wold, E. F.: *Solving the Loewner PDE in complete hyperbolic starlike domains of \mathbb{C}^n* , *Adv. Math.* **242** (2013), 209–216.
- [4] Bera, S.: *Examples of non-autonomous basins of attraction-II*, *J. Ramanujan Math. Soc.* **34** (2019), no. 3, 343–363.
- [5] Bera, S.; Pal, R.; Verma, K.: *Examples of non-autonomous basins of attraction*, *Illinois J. Math.* **61** (2017), 531–567.
- [6] Fornæss, J. E.: *An increasing sequence of Stein manifolds whose limit is not Stein*, *Math. Ann.* **223** (1976), 275–277.
- [7] Fornæss, J. E.: *Short \mathbb{C}^k* , *Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday*, 95–108, *Adv. Stud. Pure Math.*, **42** (2004), Math. Soc. Japan, Tokyo
- [8] Fornæss, J. E.; Sibony, N.: *Increasing sequences of complex manifolds*, *Math. Ann.* **255** (1981), 351–360.
- [9] Fornæss, J.E., Stout, E.L.: *Polydiscs in complex manifolds*, *Math. Ann.* **227** (1977), 145–153
- [10] Forstneric, F.; Rosay, J-P.: *Approximation of biholomorphic mappings by automorphisms of \mathbb{C}^n* , *Invent Math* **112** (1993), 323–349
- [11] Jucha, P.: *A remark on the dimension of the Bergman space of some Hartogs domains*, *J. Geom. Anal.* **22**, 23–37
- [12] Gallagher, A.-K.; Harz, T.; Herbot, G.: *On the dimension of the Bergman space for some unbounded domains*, *The Journal of Geometric Analysis* **27**, (2017), 1435–1444.
- [13] Pal, R.; Verma, K.: *Ergodic properties of families of Hénon maps*, *Ann. Polon. Math.* **121** (2018), no. 1, 45–71.
- [14] Peter, P. ; Włodzimierz, Z.: *L_h^2 -Functions in Unbounded Balanced Domains*. *The Journal of Geometric Analysis* volume **27** (2017), 2118–2130.
- [15] Wiegerinck, J. J. O. O.: *Domains with finite dimensional Bergman space*, *Mathematische Zeitschrift* **187** (1984), 559–562
- [16] Wold, E. F.: *A Fatou-Bieberbach domain in \mathbb{C}^2 which is not Runge*, *Math. Ann.* **340** (2008), no. 4, 775–780.
- [17] Wold, E. F.: *A long \mathbb{C}^2 which is not Stein*, *Ark. Mat.* **48** (2010), no. 1, 207–210.

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