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Global existence of dissipative solutions to the Camassa–Holm equation with transport noise

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Dedicated to Gui-Qiang Chen on the occasion of his sixtieth birthday

Abstract

We consider a nonlinear stochastic partial differential equation (SPDE) that takes the form of the Camassa–Holm equation perturbed by a convective, position-dependent, noise term. We establish the first global-in-time existence result for dissipative weak martingale solutions to this SPDE, with general finite-energy initial data. The solution is obtained as the limit of classical solutions to parabolic SPDEs. The proof combines model-specific statistical estimates with stochastic propagation of compactness techniques, along with the systematic use of tightness and a.s. representations of random variables on specific quasi-Polish spaces. The spatial dependence of the noise function makes more difficult the analysis of a priori estimates and various renormalisations, giving rise to nonlinear terms induced by the martingale part of the equation and the second-order Stratonovich–Itô correction term.

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Contents

1. Introduction

1.1. Background and main result

We are interested in global weak solutions of the initial-value problem for the stochastic parabolic-elliptic system

$$
\begin{aligned}
&0 = \mathrm{d}u + \left[u\,\partial_x u + \partial_x P\right]\mathrm{d}t + \sigma\,\partial_x u \circ \mathrm{d}W, \\
&-\partial_{xx}^2 P + P = u^2 + \frac{1}{2}\left|\partial_x u\right|^2, \quad \text{for } (t,x) \in (0,T)\times\mathbb{S}^1,
\end{aligned}
\tag{1.1}
$$

where $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ is the 1D torus (circle), T is a positive final time, $\sigma = \sigma(x) \in W^{2,\infty}(\mathbb{S}^1)$ is a position-dependent noise function, and W is a 1D Wiener process defined on a standard filtered probability space $\mathcal{S} = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$, henceforth called a *stochastic basis*. Formally, by the Itô–Stratonovich conversion formula, the Stratonovich differential $\sigma\,\partial_x u \circ \mathrm{d}W$ in (1.1)—known in the literature as a gradient, transport or convection noise term—can be expanded into the operational form $-\frac{1}{2}\sigma(x)\partial_x(\sigma(x)\partial_x u)\,\mathrm{d}t + \sigma(x)\partial_x u\,\mathrm{d}W$. Moreover, the elliptic equation for P can be solved to supply

$$P = P[u] := K * \left(u^2 + \frac{1}{2}|\partial_x u|^2\right), \quad K(x) = \frac{\cosh\left(x - 2\pi\,\mathrm{int}\left(\frac{x}{2\pi}\right) - \pi\right)}{2\sinh(\pi)}, \tag{1.2}$$

where K is the Green's function of $1 - \partial_{xx}^2$ on $\mathbb{S}^1$, $\mathrm{int}(x)$ is the integer part of x, and $*$ means convolution in x. Consequently, (1.5) takes the form of the nonlinear nonlocal SPDE

$$\begin{aligned} &0 = \mathrm{d}u + \left[u\,\partial_x u + \partial_x P\right]\mathrm{d}t - \frac{1}{2}\sigma\,\partial_x(\sigma\,\partial_x u)\,\mathrm{d}t + \sigma\,\partial_x u\,\mathrm{d}W, \\ &P = K * \left(u^2 + \frac{1}{2}|\partial_x u|^2\right). \end{aligned} \tag{1.3}$$

We recover the deterministic Camassa–Holm (CH) equation by setting $\sigma \equiv 0$ in (1.3). Since its introduction in the early 1980s [14,34], the CH equation has received much attention from the mathematical community. The CH equation, a nonlinear dispersive PDE modelling shallow-water waves, is nonlocal, completely integrable and may be written in (bi-)Hamiltonian form in terms of the momentum variable $m := \left(1 - \partial_{xx}^2\right)u$. Much of the excitement of the CH equation is related to its supercritical nature—coming from the competition between the dispersive and nonlinear terms—which leads to the development of singularities in finite time (blow-up via wave breaking). The question of global well-posedness of the CH equation, in different classes of appropriately defined weak solutions, is widely studied, see for example [8,9,21,41,42,63] (and the references therein). Indeed, there are two natural classes of H^1 weak solutions, *dissipative* and *conservative*, which differ in how they continue the solution past the blow-up time. Conservative solutions (see, e.g., [8]) ask that the PDE holds weakly and that the total energy is preserved. In contrast, dissipative solutions (see, e.g., [63]) are characterized by a drop in the total energy at the time of blow-up. Starting from general finite-energy data $u|_{t=0} = u_0 \in H^1$, the CH solution operator formally preserves the H^1 norm, and H^1 regularity is also needed to make distributional sense of the equation. The solution space H^1 allows for wave breaking, in the sense that the solution u remains bounded while its x-derivative $\partial_x u$ becomes (negatively) unbounded [14].

Stochastic effects, in terms of transport, forcing, or uncertain system parameters, are vital for developing models of many phenomena in fluid dynamics. The work of Holm [43] proposes a general approach to deriving SPDEs for fluid dynamics from geometric mechanics and a stochastic variational principle. In particular, he argues that "physically relevant" noise arises from a suitable perturbation of the integrated Hamiltonian of the dynamical system. The corresponding stochastic perturbation of the CH equation leads to nonlinear SPDEs like (1.3), see [23] and [4]. The works [4,23] also investigate blow-up of regular solutions. For the related stochastic Hunter–Saxton equation, see [38,39]. We refer to Appendix A for a short formal derivation of the stochastic CH equation (1.3).

Let us now turn to the mathematical analysis of the stochastic CH equation (1.3). Currently, only a few local well-posedness results are available. Most of them concern the stochastic forcing case, which corresponds to (1.1) with the transport noise $\sigma(x)\partial_x u \circ \mathrm{d}W$ replaced by a lower order Itô term $\sigma(x,u)\,\mathrm{d}W$, either in additive ($\sigma(x)\,\mathrm{d}W$) or multiplicative ($\sigma(u)\,\mathrm{d}W$) form, see the works [16,17,19,44,50,56,60,61,64,65]. See also [15] for a global existence result if $\sigma \equiv u$ and $m(0) \geq 0$.

For the CH equation perturbed by transport noise, like the term $\sigma \partial_x u \circ \mathrm{d}W$ appearing in (1.3), we refer to Albeverio, Brzeźniak, and Daletskii [1] for the first local well-posedness result (up to wave-breaking). The idea in [1] is to transform the equation into a PDE with random coefficients and apply Kato's operator theory. The work of Alonso-Orán, Rohde, and Tang [2] extends this result to a stochastic two-component CH system with transport noise (for smooth noise functions σ). Let us also draw attention to a recent study [18] that investigates the existence of weak solutions for a two-component CH equation affected by Markus pure-jump noise. A general Marcus SDE is structured as follows: $du = a\,\mathrm{d}s + b \circ \mathrm{d}W + c[u] \diamond \mathrm{d}L$, where L represents a pure jump-Lévy process, and $c[u] \diamond \mathrm{d}L$ is interpreted within the Markus framework. The study [18] zeroes in on the pure jump component $c[u] \diamond \mathrm{d}L$ in this decomposition, particularly when $c[u] = \partial_x u$. Aside from the fact that examining this case is more straightforward than dealing with the Wiener noise $\sigma(x)\partial_x u \circ \mathrm{d}W$, which is the focus of our paper, the critical difference lies in the solution class for analyzing the stochastic (two-component) CH equation. In their work, the authors of [18] devise solutions in which $\partial_x u$ is a bounded function. However, when this is confined to the context of the CH equation, such a solution class becomes overly restrictive, essentially necessitating that the initial data satisfy $u_0 - \partial_{xx}^2 u_0 \geq 0$. This limitation omits crucial solutions involving peakon-antipeakon interactions, where $\partial_x u$ could potentially blow up or become unbounded. In contrast, our result is general, applicable to any $u_0 \in H^1$ (where $\partial_x u$ may not be a bounded function). Yet, this wide applicability entails a significantly more complex analytical approach, which we will elaborate upon later in our discussion.

The global existence of properly defined weak solutions for the stochastic CH equation (1.3) is an open problem, addressed in this paper for the first time. We develop an existence theory for dissipative weak solutions for rather general "non-smooth" noise functions $\sigma \in W^{2,\infty}$. Our main result is the following theorem:

Theorem 1.1 *(Existence of dissipative solution). Let $\sigma \in W^{2,\infty}(\mathbb{S}^1)$, and fix some $p_0 > 4$. For any initial probability distribution Λ supported on $H^1(\mathbb{S}^1)$, satisfying*

$$\int_{H^1(\mathbb{S}^1)} \|v\|_{H^1(\mathbb{S}^1)}^{p_0} \Lambda(\mathrm{d}v) < \infty,$$

there exists a dissipative weak martingale solution $\left(\tilde{\mathcal{S}}, \tilde{u}, \tilde{W}\right)$ to the stochastic CH equation (1.3) with random initial data $\tilde{u}_0$ distributed according to Λ ($\tilde{u}_0 \sim \Lambda$), where $\tilde{\mathcal{S}} = \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}\right)$ is a stochastic basis. Besides, the following energy inequality holds $\tilde{\mathbb{P}}$–a.s., for a.e. $s \in [0, T)$ and every t with $s < t \leq T$,

$$\begin{aligned} &\int_{\mathbb{S}^1} \tilde{u}^2 + |\partial_x \tilde{u}|^2 \,\mathrm{d}x \Big|_s^t \\ &\quad \leq \int_s^t \int_{\mathbb{S}^1} \frac{1}{4}\partial_{xx}^2 \sigma^2 \tilde{u}^2 + \left(|\partial_x \sigma|^2 - \frac{1}{4}\partial_{xx}^2 \sigma^2\right) |\partial_x \tilde{u}|^2 \,\mathrm{d}x\,\mathrm{d}t' \\ &\qquad + \int_s^t \int_{\mathbb{S}^1} \partial_x \sigma \left(\tilde{u}^2 - |\partial_x \tilde{u}|^2\right) \mathrm{d}x\,\mathrm{d}\tilde{W}. \end{aligned} \tag{1.4}$$

Specifically, it holds for $s = 0$ and any $t \in (0, T]$, with $\int_{\mathbb{S}^1} \tilde{u}^2 + |\partial_x \tilde{u}|^2 \, dx\big|_{s=0}$ replaced by $\int_{\mathbb{S}^1} \tilde{u}_0^2 + |\partial_x \tilde{u}_0|^2 \, dx$.

Roughly speaking, by a solution to (1.3) we mean a collection $(\tilde{\mathcal{S}}, \tilde{u}, \tilde{W})$, where $\tilde{\mathcal{S}}$ is a stochastic basis, $\tilde{W}$ is a Wiener process, and $(\omega, t) \mapsto \tilde{u}(\omega, t, \cdot)$ takes values in $H^1(\mathbb{S}^1)$ and satisfies the SPDE (1.3) in the weak sense in x, see Definition 2.4 for details. Note that the solutions constructed in Theorem 1.1 are weak in the probabilistic sense, as the stochastic basis $\tilde{\mathcal{S}}$ and the Wiener process $\tilde{W}$ are parts of the unknown solution. We refer to these solutions as dissipative weak martingale solutions. The term "weak" in the quantifier "dissipative weak" indicates that the solutions are considered weak solutions in the PDE sense. Furthermore, at least in the deterministic case ($\sigma = 0$), the solutions possess the additional property that the total energy decreases over time, specifically at a wave breaking time t_0. The term "dissipative" also alludes to the methodology employed to construct these solutions, which is the vanishing viscosity method.

A manifestation of the dissipative nature of the solutions is that the total energy inequality encodes a fundamental right-continuity property; namely, we will prove that $\tilde{u}(t) \to \tilde{u}(t_0)$ in $H^1(\mathbb{S}^1)$, a.s., as $t \downarrow t_0 \in [0, T)$. In the deterministic setting $\sigma = 0$, Theorem 1.1 recovers the main result of Xin and Zhang [63].

1.2. Outline of main ideas

Let us end this introduction by briefly expounding the main ideas behind the proof of Theorem 1.1. Although the proof makes use of the vanishing viscosity method and weak convergence techniques, there are many substantial differences between the deterministic and stochastic situations. Adding the viscosity term $\varepsilon \partial_{xx}^2 u$ to (1.3), we first construct a regular solution u_ε to

$$
\begin{aligned}
&0 = du_\varepsilon + \Big[u_\varepsilon \, \partial_x u_\varepsilon + \partial_x P_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon\Big] \, dt - \frac{1}{2}\sigma_\varepsilon \partial_x \left(\sigma_\varepsilon \partial_x u_\varepsilon\right) \, dt + \sigma_\varepsilon \partial_x u_\varepsilon \, dW, \\
&P_\varepsilon = P[u_\varepsilon] := K * \left(u_\varepsilon^2 + \frac{1}{2} |\partial_x u_\varepsilon|^2\right).
\end{aligned}
\tag{1.5}
$$

This is a non-standard (nonlinear and nonlocal) parabolic SPDE. Its global-in-time well-posedness does not follow from standard parabolic SPDE theory. In [40], we prove the existence and uniqueness of pathwise H_x^m solutions for arbitrary $m \in \mathbb{N}$ (as long as the initial data are smooth). Notice that in (1.5) we have replaced σ of (1.1) with $\sigma_\varepsilon \in C^\infty(\mathbb{S}^1)$, which we require to converge to σ in $W^{2,\infty}(\mathbb{S}^1)$ as $\varepsilon \downarrow 0$. This is necessary as the well-posedness of H_x^m solutions require coefficients $\sigma_\varepsilon \in W^{m+1,\infty}(\mathbb{S}^1)$ [40, Theorem 1.2].

The relevant results from [40] are collected in Theorem 2.3 below. In particular, only a few ε-uniform statistical estimates are available (starting from smooth finite-energy initial data), including

$$
\begin{aligned}
&\mathbb{E} \, \|u_\varepsilon\|_{C_t^\theta L_x^2}^r \lesssim 1, \quad \text{for some } r > 2 \text{ and small } \theta, \\
&\|q_\varepsilon\|_{L^{2+\alpha}_{\omega,t,x}} \lesssim 1, \quad \text{for any } \alpha \in [0, 1),
\end{aligned}
\tag{1.6}
$$

see Sections 2 and 3, where the spatial gradient $q_\varepsilon := \partial_x u_\varepsilon$ satisfies the nonlinear, second-order transport-type SPDE

$$0 = \mathrm{d}q_\varepsilon + \left(\partial_x\left(u_\varepsilon\, q_\varepsilon\right) - \frac{1}{2}q_\varepsilon^2 + P_\varepsilon - u_\varepsilon^2 - \varepsilon\,\partial_{xx}^2 q_\varepsilon\right)\mathrm{d}t \\ - \frac{1}{2}\partial_x\left(\sigma_\varepsilon\,\partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\right)\mathrm{d}t + \partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\mathrm{d}W. \tag{1.7}$$

The starting point for deducing ε-uniform estimates is the SPDE satisfied by the total energy $\frac{1}{2}\left(u_\varepsilon^2 + q_\varepsilon^2\right)$, which is formally obtained by testing—via the temporal (Itô) and spatial chain rules, the SPDE (1.5) with u_ε and the SPDE (1.7) with q_ε, and then adding the resulting equations, noticing some crucial cancellations involving cubic terms of q_ε. The end result is

$$\mathrm{d}\left(\frac{u_\varepsilon^2 + q_\varepsilon^2}{2}\right) + \partial_x\left[u_\varepsilon\frac{u_\varepsilon^2 + q_\varepsilon^2}{2} + u_\varepsilon P_\varepsilon - \frac{u_\varepsilon^3}{2} - \frac{1}{4}\partial_x\sigma_\varepsilon^2\frac{q_\varepsilon^2 - u_\varepsilon^2}{2}\right]\mathrm{d}t \tag{1.8}$$
$$- \partial_{xx}^2\left[\left(\frac{1}{2}\sigma_\varepsilon^2 + \varepsilon\right)\frac{u_\varepsilon^2 + q_\varepsilon^2}{2}\right]\mathrm{d}t + \left[\partial_x\left(\sigma_\varepsilon\frac{u_\varepsilon^2 + q_\varepsilon^2}{2}\right) + \partial_x\sigma_\varepsilon\left(\frac{q_\varepsilon^2 - u_\varepsilon^2}{2}\right)\right]\mathrm{d}W$$
$$= \frac{1}{4}\partial_{xx}^2\sigma_\varepsilon^2\frac{u_\varepsilon^2}{2}\,\mathrm{d}t + \left(|\partial_x\sigma_\varepsilon|^2 - \frac{1}{4}\partial_{xx}^2\sigma_\varepsilon^2\right)\frac{q_\varepsilon^2}{2}\,\mathrm{d}t - \varepsilon\left(|\partial_x u_\varepsilon|^2 + |\partial_x q_\varepsilon|^2\right)\mathrm{d}t.$$

The second estimate in (1.6) implies, passing if necessary to a subsequence,

$$q_\varepsilon \xrightarrow{\varepsilon\downarrow 0} q \text{ in } L^p_{\omega,t,x},\ p \in [1,3), \quad q_\varepsilon^2 \xrightarrow{\varepsilon\downarrow 0} \overline{q^2} \text{ in } L^p_{\omega,t,x},\ p \in [1,3/2), \tag{1.9}$$

for some weak limits $q, \overline{q^2}$. Throughout this paper, we use overbars to denote weak limits, in spaces that often must be understood from the context. Only equipped with weak convergence of $\left\{q_\varepsilon^2 = |\partial_x u_\varepsilon|^2\right\}_{\varepsilon>0}$—because of the nonlinearity—it is not possible to pass to the limit $\varepsilon \to 0$ in (1.5), (1.7) to obtain a solution of the stochastic CH equation (1.3); strong L^2 convergence of $\{q_\varepsilon\}_{\varepsilon>0}$ is called for.

An effective (deterministic) strategy for improving the weak convergence to the required strong one is to start from a strongly convergent sequence of initial data and then attempt to propagate that strong convergence through time. This "propagation of compactness" argument is typically implemented in the context of DiPerna–Lions renormalised solutions [28]; for some applications of this strategy, see [31,49] (compressible Navier–Stokes equations) and [22,21,63] (CH equation).

The tailoring of the propagation of compactness argument to the stochastic CH equation (1.3) is rather involved. Let us explain some of the reasons for this. First, we need to use the few available estimates (1.6) to extract some strong (almost sure) compactness in the probability variable ω. Indeed, a feature of our approach is that most results are derived in a pathwise context, meaning that equations and inequalities hold almost surely (not only in the weaker statistical mean sense). The natural strategy for achieving a.s. convergence is to invoke some nontrivial results of Skorokhod, linked to the tightness (weak compactness) of probability measures and a.s. representations of random variables, see [24, Theorem 2.4] and, e.g., [5,25,32,35,36] for some applications of this approach to SPDEs. Applying this strategy to the laws $\mathcal{L}(u_\varepsilon)$ of u_ε—defined on the Polish space $C([0,T];L^2(\mathbb{S}^1))$ and whose tightness is guaranteed by the first estimate in (1.6)—we obtain new random variables $\tilde{u}_\varepsilon$—defined on a new probability space and with the same laws as the original variables u_ε—which converge almost surely to some $\tilde{u}$:

$$\tilde{u}_\varepsilon \xrightarrow{\varepsilon\downarrow 0} \tilde{u} \quad \text{in } C([0,T];L^2(\mathbb{S}^1)), \text{ almost surely.} \tag{1.10}$$

Next, we wish to apply this strategy to improve the (ω, t, x) weak convergence (1.9) to a.e. convergence in ω, weak in (t, x). The original Skorokhod construction applies to processes taking values in a Polish (complete separable metric) space. In our context the Skorokhod theorem is not directly applicable, because we have to work in spaces equipped with the weak topology, like $L^p_{t,x} - w$, which are not Polish. Therefore we use a recent version of the Skorokhod theorem—due to Jakubowski [46]—that applies to so-called quasi-Polish spaces, where *quasi-Polish* refers to a Hausdorff space that exhibits a continuous injection into a Polish space. It turns out that separable Banach spaces equipped with the weak topology as well as dual spaces of separable Banach spaces (equipped with the weak-star topology) are quasi-Polish. For relevant background material on quasi-Polish spaces, see Appendix B. We refer to Brzeźniak and Ondreját [52,12] and [7,10,11,54,59,62] for some applications of the Skorokhod–Jakubowski theorem to different SPDEs (this list is far from complete).

The second estimate in (1.6) implies that the laws $\mathcal{L}(q_\varepsilon)$ and $\mathcal{L}(q_\varepsilon^2)$ are tight as probability measures on the quasi-Polish space $L^p([0,T]\times\mathbb{S}^1) - w$, respectively for $p \in [1,3)$ (q_ε) and $p \in [1,3/2)$ (q_ε^2). An application of the Skorokhod–Jakubowski theorem supplies new random variables $\tilde{q}_\varepsilon$ and $\tilde{q}_\varepsilon^2$ defined on the same probability space as $\tilde{u}_\varepsilon$ and with the same laws as the original variables q_ε and q_ε^2, such that (extracting a subsequence if necessary and for the same values of p as before)

$$\tilde{q}_\varepsilon \xrightharpoonup{\varepsilon\downarrow 0} \tilde{q} \text{ in } L^p_{t,x}, \text{ a.s.}, \qquad \tilde{q}_\varepsilon^2 \xrightharpoonup{\varepsilon\downarrow 0} \overline{\tilde{q}^2} \text{ in } L^p_{t,x}, \text{ a.s.}, \tag{1.11}$$

for some limits $\tilde{q}$ and $\overline{\tilde{q}^2}$, see Section 4.

It is of vital importance to us that products like $S'(\tilde{q}_\varepsilon)\tilde{P}_\varepsilon$ converge weakly, for a suitable class of linearly growing nonlinearities $S(\cdot)$, where $\tilde{P}_\varepsilon$ is defined in (1.5). Since $S'(\tilde{q}_\varepsilon)$ converges weakly, $\tilde{P}_\varepsilon$ must converge strongly. This strong convergence does not follow from (1.11), as we are missing strong temporal compactness for $\tilde{q}_\varepsilon^2$. In the deterministic theory [63], one establishes directly uniform $W^{1,1}_{t,x}$ estimates for $\tilde{P}_\varepsilon$, which implies strong convergence. This strategy does not work in the stochastic setting. A natural modification of this strategy, based on the derivation of uniform Hölder continuity in t, does not seem to accomplish the task either, even if the spatial topology is weak. As a result, we cannot apply the often-used compactness approach based on tightness in the (quasi-Polish) space $C([0,T];L^p(\mathbb{S}^1) - w)$, used by many of the references above.

These obstructions have motivated us to introduce the locally convex space $L^p\big(L^p_w\big) = L^p\big([0,T];L^p(\mathbb{S}^1) - w\big)$, which is quasi-Polish (see Appendix B). The space $L^p\big(L^p_w\big)$ can account for strong temporal and weak spatial convergence of the energy variable $\tilde{q}_{\varepsilon_n}^2$. To this end, we formulate a new tightness criterion in $L^p\big(L^p_w\big)$, which we believe is of independent interest: the probability laws of a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of random variables is tight on $L^p\big(L^p_w\big)$ provided

(i) $\mathbb{E}\,\|Q_n\|_{L^p([0,T];L^p(\mathbb{S}^1))} \lesssim 1$,

(ii) $\mathbb{E}\,\|Q_n\|_{L^{\bar{p}}([0,T];L^1(\mathbb{S}^1))} \lesssim 1$, for some $\bar{p} > p$,

and, for all $\varphi \in C^\infty(\mathbb{S}^1)$ and $\vartheta > 0$,

$$\text{(iii)}\ \mathbb{E}\sup_{\tau\in(0,\vartheta)}\int_0^{T-\tau}\left|\int_{\mathbb{S}^1}\varphi(x)\big(Q_n(t+\tau,x)-Q_n(t,x)\big)\,\mathrm{d}x\right|\mathrm{d}t\lesssim_\varphi \vartheta^\alpha,$$

for some $\alpha\in(0,1)$. We verify these conditions for the energy variable $\tilde q_\varepsilon^2$, thereby supplying the following critical improvement over (1.11): $\tilde q_\varepsilon^2\to\overline{\tilde q^2}$ in $L^p\big(L^p_w\big)$ a.s.; thus, passing to a subsequence, $\tilde q_\varepsilon^2$ converges weakly in x and pointwise in $(\tilde\omega,t)$. We refer to Sections 3 and 4 for the details.

In Section 5, we prove several results that transfer the available a priori estimates and the SPDE (1.5) to the new probability space (for the new variables $\tilde u_\varepsilon$, $\tilde q_\varepsilon$, $\tilde W_\varepsilon$ and their limits). Equipped with (1.10) and (1.11), we send $\varepsilon\to0$ in the SPDE (1.5) (on the new probability space) to produce a solution $\tilde u$ of an SPDE that looks like the stochastic CH equation (1.3) but with the nonlinearity $\overline{\tilde q^2}$ instead of the required one $\tilde q^2=|\partial_x\tilde u|^2$, see Section 6.

The final Section 7 is devoted to the proof that $\overline{\tilde q^2}=\tilde q^2$ a.e. in (ω,t,x), and thereby the validity of Theorem 1.1. The proof amounts to upgrading the (t,x) weak convergence (1.11) to strong convergence via a study of the *defect measure*

$$\mathbb{D}=\mathbb{D}(\omega,t,x)=\frac12\left(\overline{\tilde q^2}-\tilde q^2\right)\ge0. \tag{1.12}$$

The idea is to derive a transport-type SPDE (up to an inequality) for the evolution of $\mathbb{D}$, so that if $\mathbb{D}$ is time-continuous at $t=0$ with $\mathbb{D}(0)=0$ (assuming strong compactness at $t=0$), then $\mathbb{D}(t)$ is zero at all later times $t>0$. Roughly speaking, an SPDE (up to an inequality) for $\frac12\big(\tilde u^2+\overline{\tilde q^2}\big)$ is obtained using (1.10), (1.11) to pass to the limit in the total energy balance (1.8) (again written on the new probability space). On the other hand, by formally repeating the derivation of the energy balance (1.8) for the limits $\tilde u$, $\tilde q$, relying on the SPDEs obtained by sending $\varepsilon\downarrow0$ in (1.5), (1.7), we arrive at an SPDE for $\frac12\big(\tilde u^2+\tilde q^2\big)$, and therefore an inequality for the defect measure $\mathbb{D}$, which takes the form

$$\begin{aligned}\partial_t\mathbb{D}+\partial_x\left(\tilde u\,\mathbb{D}-\frac14\partial_x\sigma^2\,\mathbb{D}\right)-\frac12\partial^2_{xx}\big(\sigma^2\,\mathbb{D}\big)+\big[\partial_x\left(\sigma\,\mathbb{D}\right)+\partial_x\sigma\,\mathbb{D}\big]\,\dot{\tilde W}\\ \le\left(|\partial_x\sigma|^2-\frac14\partial^2_{xx}\sigma^2-\partial_x\tilde u\right)\mathbb{D}\quad\text{in }\mathcal{D}'_{t,x},\text{ almost surely,}\end{aligned} \tag{1.13}$$

where $\mathcal{D}'_{t,x}=\mathcal{D}'([0,T)\times\mathbb{S}^1)$ denotes the space of distributions on $[0,T)\times\mathbb{S}^1$.

Unfortunately, the arguments leading up to (1.13) are only formal. Recalling (1.6), we do not have enough integrability on $\tilde q$, $\overline{\tilde q^2}$ to give sense to the terms $\tilde q^3$ and $\tilde q\,\overline{\tilde q^2}$ arising during the derivation of (1.13). The way to overcome this difficulty is to work with renormalised formulations of the SPDEs for $\tilde q_\varepsilon,\tilde q$ based on linearly growing approximations $S_\ell(v)$ of v^2 and eventually send $\ell\to\infty$. More precisely, we split v into its positive v_+ and negative parts v_- (so that $v^2=v_+^2+v_-^2$) and then work with the SPDEs satisfied by the nonlinear compositions $S_\ell\big((\tilde q_\varepsilon)_\pm\big)$, $S_\ell\big(\tilde q_\pm\big)$. In passing, let us mention that this forces us to accommodate a countable product of quasi-Polish spaces, as we need to apply the Skorokhod–Jakubowski procedure to all members of the sequence $\big\{S_\ell\big((q_\varepsilon)_\pm\big)\big\}_{\ell\in\mathbb{N}}$ simultaneously. Countable products of quasi-Polish spaces are discussed in Appendix B.

Again drawing an analogy to the deterministic theory [21,22,63], here we run into another difficulty linked to the stochastic part of the problem. Namely, the temporal irregularity of the noise induces structural changes in the equation that make it impossible to work with the familiar $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ approximations $S_\ell(v) = v^2 \mathbb{1}_{\{|v|\le \ell\}} + \ell\,(2\,|v| - \ell)\,\mathbb{1}_{\{|v|>\ell\}}$ of $v \mapsto v^2$. This adds further complications to the analysis. See Section 4 for further details.

A further intricacy arising during the derivation of (1.13) is the passage to the limit in stochastic integrals of the form $\int_0^t \int_{\mathbb{S}^1} S'(\tilde{q}_\varepsilon)\tilde{q}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tilde{W}_\varepsilon$, for some class of nonlinear functions $S(\cdot)$. Here, $\tilde{W}_\varepsilon$ is a sequence of Wiener processes converging uniformly to a limit process $\tilde{W}$, a.s., while $\int_{\mathbb{S}^1} S'(\tilde{q}_\varepsilon)\tilde{q}_\varepsilon\,\mathrm{d}x$ converges just weakly in L^p_t, a.s., towards $\int_{\mathbb{S}^1} \overline{S'(\tilde{q})\tilde{q}}\,\mathrm{d}x$. The absence of strong temporal compactness hinders the application of Lemma 2.1 of [25], which is regularly used to certify convergence of stochastic integrals. We manage this issue by once more making vital use of the quasi-Polish space $L^p\big(L^p_w\big)$ and the tightness criterion provided by the conditions (i), (ii), and (iii). The details are worked out in Section 7.

The renormalised SPDEs are derived by regularising non-smooth processes via convolution against a spatial mollifier $J_\delta(x)$. Sending $\delta \to 0$, we handle most of the error terms using standard DiPerna–Lions estimates [28], except for some unique terms coming from the interconnection between the martingale part of the equations and second-order Stratonovich–Itô correction terms. The corresponding commutator estimates—collected in Appendix C—are proved in the paper [40] by the last three authors. Similar estimates have been used recently in [54] and [38].

It remains to send $\ell \to \infty$ to recover a useful version of the SPDE inequality (1.13) for the defect measure $\mathbb{D}$. The renormalisations $S_\ell(v_\pm)$ give rise to a number of intricate error terms involving the approximation parameter ℓ, several of them linked to the stochastic nature of the problem. For the deterministic CH equation [63], this part of the analysis relies crucially on knowing that the viscous solutions q_ε obey a one-sided gradient bound of the Oleinik-type: $q_\varepsilon(t) = \partial_x u_\varepsilon(t) \lesssim 1 + \frac{1}{t}$ (that is independent of ε), a further reflection of the dissipative nature of the solutions. No such bound is currently known for the stochastic CH equation. However, let us mention that recently [39] it was discovered that dissipative solutions of the related stochastic Hunter–Saxton equation [38] satisfy a one-sided gradient bound of the form $\partial_x u(\omega,t,x) \le K(\omega,t)$, where the process $K(\omega,t) > 0$ exhibits an exponential moment bound in the sense that $\mathbb{E}\exp\big(p\int_t^T K(s)\,\mathrm{d}s\big) \lesssim t^{-2p}$ for small times t, for some $p \ge 1$. We have not been able to establish a similar bound for the stochastic CH equation. Here we will instead rely on an observation due to the third author and Coclite [22] for the deterministic CH equation, which makes it possible to rigorously derive an SPDE inequality for the “positive part” of the defect measure, $\mathbb{D}_+ = \frac{1}{2}\big(\overline{\tilde{q}_+^2} - \tilde{q}_+^2\big)$, without using an Oleinik bound. The detailed analysis of the defect measure is found in Section 7.

The remaining part of the paper is divided into six sections and three appendices, which together establishes Theorem 1.1.

2. Preliminaries and solution concepts

We refer to [20, Chapter 1] for notation and background material on stochastic analysis and SPDEs, including stochastic integrals, Itô’s chain rule, and martingale inequalities like the one of Burkholder–Davis–Gundy (BDG). For a more general context of cylindrical Wiener processes, see [24]. For some key concepts linked to probability measures (on topological spaces), weak compactness and tightness, see the book [6]. For basic properties of Bochner spaces like $L^p(\Omega;X) = L^p\big(\Omega,\mathcal{F},\mathbb{P};X\big)$, where X is a Banach space, we refer to [45, Chapters 1 & 2]. On several occasions we will use [25, Lemma 2.1] to lay the foundations for the convergence of stochastic integrals. The reader can find a primer on quasi-Polish spaces and the Skorokhod–Jakubowski theorem [46] in Appendix B. Quick background reading can be found in, e.g., [12,13,52], some results from which are quoted in the aforementioned appendix. The definition and properties of the space $C\big([0,T]; H^1(\mathbb{S}^1)-w\big)$, which is quasi-Polish and used herein, can be found in [12,13,52].

This section presents the solution concept used in Theorem 1.1 and the one used for the viscous SPDE (1.5), starting with the notion of a H^m-regular martingale solution of the viscous equation. Here $\varepsilon > 0$ is fixed, and therefore we write u instead of u_ε for the solution of (1.5).

Definition 2.1 *(H^m martingale solution of viscous SPDE).* Fix any integer $m \geq 1$ and some $p_0 > 4$. Let Λ be a probability measure on $H^m(\mathbb{S}^1)$, satisfying

$$\int_{H^m(\mathbb{S}^1)} \|v\|_{H^m(\mathbb{S}^1)}^{p_0} \, \Lambda(\mathrm{d}v) < \infty. \tag{2.1}$$

The triple $\big(\mathcal{S}, u, W\big)$ is a H^m martingale solution of (1.5) with initial law Λ if the following conditions hold:

(a) $\mathcal{S} = \big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ is a stochastic basis;
(b) W is a standard Wiener process on $\mathcal{S}$;
(c) $u : \Omega \times [0,T] \to H^1(\mathbb{S}^1)$ is adapted, with $u \in L^{p_0}\big(\Omega; C([0,T]; H^1(\mathbb{S}^1))\big)$. Moreover, $u \in L^2([0,T]; H^{m+1}(\mathbb{S}^1)) \cap L^\infty([0,T]; H^m(\mathbb{S}^1))$ a.s. and

$$u \in L^2\left(\Omega; L^2([0,T]; H^2(\mathbb{S}^1))\right).$$

(d) initial data — the law of $u_0 := u(0)$ on $H^m(\mathbb{S}^1)$ is Λ, i.e., $(u(0))_* \mathbb{P} = \Lambda$;
(e) for all $t \in [0,T]$ and all $\varphi \in C^1(\mathbb{S}^1)$, the following equation holds $\mathbb{P}$-almost surely (in the sense of Itô):

$$\begin{aligned}
&\int_{\mathbb{S}^1} u(t)\varphi \,\mathrm{d}x - \int_{\mathbb{S}^1} u_0 \varphi \,\mathrm{d}x \\
&\qquad = \int_0^t \int_{\mathbb{S}^1} -u\, \partial_x u\, \varphi + \big[P - \varepsilon \partial_x u\big] \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}s \\
&\qquad\qquad - \frac{1}{2} \int_0^t \int_{\mathbb{S}^1} \sigma \partial_x u \partial_x (\sigma\varphi) \,\mathrm{d}x \,\mathrm{d}s - \int_0^t \int_{\mathbb{S}^1} \sigma \partial_x u \varphi \,\mathrm{d}x \,\mathrm{d}W(s),
\end{aligned} \tag{2.2}$$

$$P = P[u] := K * \left(u^2 + \frac{1}{2} |\partial_x u|^2\right).$$

If $\big(\mathcal{S}, W\big)$ is not a part of the unknown solution but fixed in advance, we speak of a probabilistic strong or pathwise solution. According to the famous Yamada–Watanabe principle, a martingale solution of an SPDE is probabilistic strong if the SPDE exhibits a pathwise uniqueness result.

Definition 2.2 *(Strong H^m solution of viscous equation).* Let $u_0 \in L^{p_0}(\Omega; H^m(\mathbb{S}^1))$ for some $p_0 > 4$, and consider a fixed stochastic basis $\mathcal{S} = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$. We say that u, defined on $\mathcal{S}$, is a strong H^m solution to (1.5) with initial data $u(0) = u_0$ if, for a given Wiener process W defined on $\mathcal{S}$, the triple $\left(\mathcal{S}, u, W\right)$ constitutes a H^m martingale solution to (1.5) with initial distribution $\Lambda = (u_0)_* \mathbb{P} = P \circ u_0^{-1}$.

The viscous equation (1.5) is strongly well-posed [40]. The following theorem gathers the main results from [40].

Theorem 2.3 *(Strong well-posedness of viscous SPDE). Fix $\varepsilon>0$. Suppose $u_0 \in L^{p_0}(\Omega; H^m(\mathbb{S}^1))$ for some $p_0 > 4$. There exists a unique strong H^m solution to (1.5) with initial condition u_0. Denoting this solution by u_ε, the following properties and ε-uniform bounds hold:*

(i) *Total energy balance — for any $0 \le s \le t \le T$,*

$$\int_{\mathbb{S}^1} u_\varepsilon^2 + |\partial_x u_\varepsilon|^2 \, \mathrm{d}x \Big|_s^t + 2\varepsilon \int_s^t \int_{\mathbb{S}^1} |\partial_x u_\varepsilon|^2 + \left|\partial_{xx}^2 u_\varepsilon\right|^2 \mathrm{d}x \, \mathrm{d}t'$$
$$= \int_s^t \int_{\mathbb{S}^1} \frac{1}{4}\partial_{xx}^2 \sigma_\varepsilon^2 u_\varepsilon^2 + \left(|\partial_x \sigma_\varepsilon|^2 - \frac{1}{4}\partial_{xx}^2 \sigma_\varepsilon^2\right) |\partial_x u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t' \tag{2.3}$$
$$+ \int_s^t \int_{\mathbb{S}^1} \partial_x \sigma_\varepsilon \left(u^2 - |\partial_x u_\varepsilon|^2\right) \mathrm{d}x \, \mathrm{d}W, \quad \tilde{\mathbb{P}}\text{–almost surely.}$$

Furthermore, there exists an ε-independent positive constant

$$C = C\left(p_0, T, \|\sigma\|_{W^{2,\infty}(\mathbb{S}^1)}, \|u_0\|_{L^{p_0}(\Omega; H^1(\mathbb{S}^1))}\right)$$

such that

$$\mathbb{E}\, \|u_\varepsilon\|_{L^\infty([0,T]; H^1(\mathbb{S}^1))}^{p_0} \le C \quad \text{and}$$
$$\mathbb{E}\left|2\varepsilon \int_0^T \int_{\mathbb{S}^1} |\partial_x u_\varepsilon|^2 + \left|\partial_{xx}^2 u_\varepsilon\right|^2 \mathrm{d}x \, \mathrm{d}t\right|^{\frac{p_0}{2}} \le C. \tag{2.4}$$

(ii) *For any $\theta \in \left[0, \frac{p-2}{4p}\right)$, $p \in [2, p_0]$, there exists an ε-independent constant $C = C\big(\theta, T, \|\sigma\|_{W^{2,\infty}(\mathbb{S}^1)}, \|u_0\|_{L^2(\Omega; H^1(\mathbb{S}^1))}\big) > 0$ such that*

$$\mathbb{E}\, \|u_\varepsilon\|_{C^\theta([0,T]; L^2(\mathbb{S}^1))}^{2/(1-4\theta)} \le C. \tag{2.5}$$

(iii) *The laws of* $\{u_\varepsilon\}_{\varepsilon>0}$ *form a (uniformly in* ε*) tight sequence of probability measures on the space* $C\big([0,T]; H^1(\mathbb{S}^1)-w\big)$.

Finally, we define the solution concept used in Theorem 1.1 for the stochastic CH equation (1.3).

Definition 2.4 *(Dissipative weak martingale solution).* Let Λ be a probability measure on $H^1(\mathbb{S}^1)$ with finite p_0th moment for some $p_0>4$, i.e.,

$$\int_{H^1(\mathbb{S}^1)} \|v\|_{H^1(\mathbb{S}^1)}^{p_0}\,\Lambda(\mathrm{d}v)<\infty.$$

The triple $\big(\mathcal{S},u,W\big)$ is a dissipative weak martingale solution to the stochastic CH equation (1.3) with initial distribution Λ if:

(a) $\mathcal{S}=\big(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbb{P}\big)$ is a stochastic basis;
(b) W is a standard Wiener process on $\mathcal{S}$;
(c) $u:\Omega\times[0,T]\to L^2(\mathbb{S}^1)$ is a progressively measurable stochastic process with paths $u(\omega)\in C([0,T];L^2(\mathbb{S}^1))\cap C([0,T];H^1(\mathbb{S}^1)-w)$, for $\mathbb{P}$-a.e. $\omega\in\Omega$. Moreover, u belongs to the space $L^2\big(\Omega;L^\infty([0,T];H^1(\mathbb{S}^1))\big)$;
(d) initial data — $(u(0))_*\mathbb{P}=\Lambda$;
(e) the following equation holds in the sense of Itô, $\mathbb{P}$-almost surely, for all $t\in[0,T]$ and for all $\varphi\in C^2(\mathbb{S}^1)$,

$$\begin{aligned}\mathrm{d}\int_{\mathbb{S}^1} u\varphi\,\mathrm{d}x &= \int_{\mathbb{S}^1}\left[\frac12 u^2+P\right]\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac12\int_{\mathbb{S}^1} u\partial_x\left(\partial_x\left(\sigma\varphi\right)\sigma\right)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{S}^1} u\partial_x\left(\sigma\varphi\right)\,\mathrm{d}x\,\mathrm{d}W, \qquad (2.6)\\ P=P[u]&:=K*\left(u^2+\frac12\left|\partial_x u\right|^2\right);\end{aligned}$$

(f) temporal right-continuity in $H^1(\mathbb{S}^1)$ — for a.e. $(\omega,t_0)\in\Omega\times[0,T]$,

$$\lim_{t\downarrow t_0}\|u(t)-u(t_0)\|_{H^1(\mathbb{S}^1)}=0.$$

At a time $t=t_0$ of wave breaking, a dissipative solution u is not going to be time-continuous in H^1, but merely right-continuous. The right-continuity condition (f) in Definition 2.4 manifests the energy inequality (1.4) and the dissipative nature of the considered solution class.

Currently, no pathwise uniqueness result is known for the stochastic CH equation (1.5). As a result, we cannot rely on the Yamada–Watanabe principle to upgrade martingale solutions to strong solutions.

3. Some a-priori estimates

Recall that u_ε denotes the H^m regular solution of the viscous SPDE (1.5) with initial data $u(0) = u_0$, whose existence, uniqueness and basic properties are given by Theorem 2.3, under the assumptions that $\sigma \in W^{2,\infty}$ and $u_0 \in L^{p_0}_\omega H^m_x$ for some $p_0 > 4$. This section collects some straightforward consequences of the ε-uniform bounds listed in Theorem 2.3, which will be used in Section 7.

Lemma 3.1 *(Basic estimates). Let u_ε be the H^m regular solution of the viscous SPDE (1.5) with initial data $u(0) = u_0$ satisfying (2.1) (for an arbitrary $m > 1$). There exists a constant*

$$C = C\left(T, \|\sigma\|_{W^{2,\infty}(\mathbb{S}^1)}, \|u_0\|_{L^{p_0}(\Omega; H^1(\mathbb{S}^1))}\right),$$

independent of $\varepsilon > 0$, such that

$$\mathbb{E}\, \|u_\varepsilon\|^{p_0}_{L^\infty([0,T]\times\mathbb{S}^1)} \le C, \quad \mathbb{E}\, \|P_\varepsilon\|^{p}_{L^\infty([0,T]\times\mathbb{S}^1)} \le C,$$

for any $p \in \left[1, p_0/2\right]$, where $P_\varepsilon = P[u_\varepsilon]$ is defined in (1.5).

Proof. The first part is a direct consequence of the $L^{p_0}_\omega L^\infty_t H^1_x$ bound (2.4) and the one-dimensional embedding $H^1(\mathbb{S}^1) \hookrightarrow L^\infty(\mathbb{S}^1)$. By the definition of P_ε and because $|K(x)| \lesssim 1$ for all $x \in \mathbb{S}^1$,

$$\begin{aligned} |P_\varepsilon(\omega, t, x)|^p &\lesssim \|u_\varepsilon(\omega, t, \cdot)\|^{2p}_{L^2(\mathbb{S}^1)} + \|\partial_x u_\varepsilon(\omega, t, \cdot)\|^{2p}_{L^2(\mathbb{S}^1)} \\ &\lesssim \|u_\varepsilon(\omega, \cdot, \cdot)\|^{2p}_{L^\infty([0,T]; H^1(\mathbb{S}^1))}. \end{aligned}$$

The second estimate now follows by taking the expectation and again using (2.4), recalling the assumption $p \le p_0/2$. □

Consider any function $S \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ that satisfies

$$\begin{aligned} &|S(v)| \lesssim |v|^2, \quad \left|S'(v)\right| \lesssim |v|, \quad \left|S''\right| \lesssim 1, \\ &\text{and} \quad \left|S(v)v - \frac{1}{2}S'(v)v^2\right| \lesssim |v|^2, \quad \forall v \in \mathbb{R}. \end{aligned} \tag{3.1}$$

The goal is to compute the differential $\mathrm{d}S(q_\varepsilon)$, recalling that $q_\varepsilon = \partial_x u_\varepsilon$ is the spatial gradient of u_ε. This requires us to apply the Itô formula to (1.7). However, q_ε is known to have continuous paths only in the infinite dimensional space $L^2(\mathbb{S}^1)$, and the equation involves terms such as q_ε^2 which brings it outside the scope where standard Hilbert space-valued Itô formulas apply. Instead, we convolve (1.5) by taking φ in (2.2) to be a spatial Friedrichs mollifier J_δ. This gives us the equation for $u_{\varepsilon,\delta} = u_\varepsilon * J_\delta$, with $u_{\varepsilon,\delta}$ continuous in t for each fixed x. Taking a classical spatial derivative gives us an equation for $q_{\varepsilon,\delta} = q_\varepsilon * J_\delta$, which is (1.7) mollified against J_δ. These equations can be interpreted pointwise in x, and the real-valued Itô formula can be applied for each fixed x.

The Itô formula is classically stated for C^2 nonlinearities, but it can be extended by approximation to functions in $W^{2,\infty}$ [53, Theorem 71] (and, in fact, to even rougher functions in some cases, like in the Tanaka formula). Throughout the paper, we will be applying the Itô formula to nonlinearities S from the $W^{2,\infty}$ class satisfying (3.1).

The entire argument, including taking the mollification limit ($\delta \downarrow 0$), is executed for a similar equation across (7.47) – (7.48) in the forthcoming Lemma 7.10. That argument can be applied here to any nonlinear function S satisfying (3.1), recalling that q_ε is more regular (integrable) than the solution in Lemma 7.10. For a fixed $\varepsilon > 0$, we have $q_\varepsilon \in L^2([0,T]; H^{m-1}(\mathbb{S}^1))$, a.s., for any finite m. The only terms here not present in Lemma 7.10 are $\varepsilon S'(q_{\varepsilon,\delta})\partial^2_{xx} q_{\varepsilon,\delta}$ and $S'(q_{\varepsilon,\delta}) P_\varepsilon * J_\delta$. In view of the regularity of q_ε, we have $S'(q_{\varepsilon,\delta}) \to S'(q_\varepsilon)$, $\partial^2_{xx} q_{\varepsilon,\delta} \to \partial^2_{xx} q_\varepsilon$ a.s. in $L^2([0,T]\times\mathbb{S}^1)$. A similar reasoning applies to the other term: since P_ε belongs a.s. to $L^\infty([0,T]\times\mathbb{S}^1)$, it follows that the convolution $P_\varepsilon * J_\varepsilon$ converges to P_ε a.s. in $L^2([0,T]\times\mathbb{S}^1)$. Apart from these terms, the steps are the same, and we will not repeat them here. Similar arguments are also carried out for $\mathrm{d}\left(u_\varepsilon^2 + q_\varepsilon^2\right)$ and for the squared-difference of two solutions in [39, Theorem 7.6, Lemma 7.7] (see also Lemma D.1).

This argument, which combines mollification with the real-valued Itô formula, leads us to the SPDE

$$
\begin{aligned}
0 = \mathrm{d}S(q_\varepsilon) &+ \left(S'(q_\varepsilon)\partial_x\left(u_\varepsilon\, q_\varepsilon\right) - \frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\right)\mathrm{d}t + S'(q_\varepsilon)\left(P_\varepsilon - u_\varepsilon^2\right)\mathrm{d}t \\
&- \varepsilon S'(q_\varepsilon)\partial^2_{xx}q_\varepsilon\,\mathrm{d}t - \frac{1}{2}S'(q_\varepsilon)\partial_x\left(\sigma_\varepsilon\,\partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\right)\mathrm{d}t \\
&+ S'(q_\varepsilon)\partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\mathrm{d}W - \frac{1}{2}S''(q_\varepsilon)\left|\partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\right|^2\mathrm{d}t.
\end{aligned}
\tag{3.2}
$$

In this paper, we make repeated use of the following identities:

$$
\begin{aligned}
S'(q_\varepsilon)\partial^2_{xx}q_\varepsilon &= \partial^2_{xx}S(q_\varepsilon) - S''(q_\varepsilon)\left|\partial_x q_\varepsilon\right|^2, \\
S'(q_\varepsilon)\partial_x\left(u_\varepsilon q_\varepsilon\right) &= \partial_x\left(u_\varepsilon S(q_\varepsilon)\right) - \left(S(q_\varepsilon) - S'(q_\varepsilon)q_\varepsilon\right)\partial_x u_\varepsilon, \\
S'(q_\varepsilon)\partial_x\left(\sigma_\varepsilon q_\varepsilon\right) &= \partial_x\left(\sigma_\varepsilon S(q_\varepsilon)\right) - \left(S(q_\varepsilon) - S'(q_\varepsilon)q_\varepsilon\right)\partial_x \sigma_\varepsilon, \\
S'(q_\varepsilon)\partial_x\left(\sigma_\varepsilon\partial_x\left(\sigma_\varepsilon q_\varepsilon\right)\right) &= \partial^2_{xx}\left(\sigma_\varepsilon^2 S(q_\varepsilon)\right) - \partial_x\left(\frac{1}{2}\partial_x\sigma_\varepsilon^2\left(3S(q_\varepsilon) - 2S'(q_\varepsilon)q_\varepsilon\right)\right) \\
&\quad + \frac{1}{2}\partial^2_{xx}\sigma_\varepsilon^2\left(S(q_\varepsilon) - S'(q_\varepsilon)q_\varepsilon\right) \\
&\quad - S''(q_\varepsilon)\left(\left|\partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\right|^2 - \left|\partial_x\sigma_\varepsilon\, q_\varepsilon\right|^2\right).
\end{aligned}
\tag{3.3}
$$

Inserting (3.3) into (3.2), we obtain

$$
\begin{aligned}
0 = \mathrm{d}S(q_\varepsilon) &+ \partial_x\left[u_\varepsilon\, S(q_\varepsilon) + \frac{1}{4}\partial_x\sigma_\varepsilon^2\left(3S(q_\varepsilon) - 2S'(q_\varepsilon)q_\varepsilon\right)\right]\mathrm{d}t \\
&- \partial^2_{xx}\left[\left(\frac{1}{2}\sigma_\varepsilon^2 + \varepsilon\right)S(q_\varepsilon)\right]\mathrm{d}t + S''(q_\varepsilon)\,\varepsilon\left|\partial_x q_\varepsilon\right|^2\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[S'(q_\varepsilon)\left(P_\varepsilon-u_\varepsilon^2\right)-\left(S(q_\varepsilon)q_\varepsilon-\frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\right)\\
&\qquad-\frac{1}{4}\partial_{xx}^2\sigma_\varepsilon^2\big(S(q_\varepsilon)-S'(q_\varepsilon)q_\varepsilon\big)-\frac{1}{2}\left|\partial_x\sigma_\varepsilon\right|^2\,S''(q_\varepsilon)\,q_\varepsilon^2\Big]\,\mathrm{d}t\\
&+\Big[\partial_x\big(\sigma_\varepsilon\,S(q_\varepsilon)\big)-\partial_x\sigma_\varepsilon\,\big(S(q_\varepsilon)-S'(q_\varepsilon)q_\varepsilon\big)\Big]\,\mathrm{d}W,
\end{aligned}
\tag{3.4}
$$

noticing the cancellation of the two terms involving $\left|\partial_x\left(\sigma_\varepsilon\, q_\varepsilon\right)\right|^2$.

For use in upcoming sections, let us also state the SPDE satisfied by $S(u_e)$:

$$
\begin{aligned}
0=\mathrm{d}S(u_\varepsilon)+\partial_x&\left[u_\varepsilon\,S(u_\varepsilon)+S'(u_\varepsilon)P_\varepsilon-\int^{u_\varepsilon}S(\xi)\,\mathrm{d}\xi+\frac{1}{4}\partial_x\sigma_\varepsilon^2S(u_\varepsilon)\right]\mathrm{d}t\\
&-\partial_{xx}^2\left[\left(\frac{1}{2}\sigma_\varepsilon^2+\varepsilon\right)S(u_\varepsilon)\right]\mathrm{d}t+S''(u_\varepsilon)\,\varepsilon\left|\partial_xu_\varepsilon\right|^2\,\mathrm{d}t\\
&+\left[\frac{1}{4}\partial_{xx}^2\sigma_\varepsilon^2\big(S(u_\varepsilon)-S'(u_\varepsilon)u_\varepsilon\big)-S''(u_\varepsilon)\,q_\varepsilon\,P_\varepsilon\right]\mathrm{d}t\\
&+\Big[\partial_x\big(\sigma_\varepsilon S(u_\varepsilon)\big)-\partial_x\sigma_\varepsilon\,S(u_\varepsilon)\Big]\,\mathrm{d}W.
\end{aligned}
\tag{3.5}
$$

This equation can be derived as before, using (3.3) with q_ε replaced by u_ε, rewriting the last identity (3.3) as

$$
\begin{aligned}
S'(u_\varepsilon)\partial_x\big(\sigma_\varepsilon\,\partial_x(\sigma_\varepsilon\,u_\varepsilon)\big)&=\partial_{xx}^2\big(\sigma_\varepsilon^2S(u_\varepsilon)\big)-\partial_x\left(\frac{1}{2}\partial_x\sigma_\varepsilon^2S(q_\varepsilon)\right)\\
&\qquad-\frac{1}{2}\partial_{xx}^2\sigma_\varepsilon^2\big(S(u_\varepsilon)-S'(u_\varepsilon)u_\varepsilon\big)-S''(u_\varepsilon)\left|\sigma_\varepsilon\partial_xu_\varepsilon\right|^2.
\end{aligned}
$$

The SPDE (1.8) for the total energy balance follows from (3.4) and (3.5).

We are now in a position to derive a higher integrability property of $q_\varepsilon=\partial_xu_\varepsilon$. This property will ensure that the weak limit $\overline{q^2}$ in (1.9) does not concentrate into a measure but remains (at least) in $L^1_{\omega,t,x}$.

Proposition 3.2 *(Higher integrability). Let u_ε be the H^m regular solution of the viscous SPDE (1.5) with initial data $u(0)=u_0$ satisfying (2.1) (for an arbitrary $m>1$), and denote by $q_\varepsilon=\partial_xu_\varepsilon$ the spatial gradient of u_ε. For fixed $\alpha\in(0,1)$, there exists a constant*

$$
C=C\left(\alpha,T,\|\sigma\|_{W^{2,\infty}(\mathbb{S}^1)},\|u_0\|_{L^{p_0}(\Omega;H^1(\mathbb{S}^1))}\right),
$$

independent of $\varepsilon>0$, such that

$$
\mathbb{E}\,\|q_\varepsilon\|^{2+\alpha}_{L^{2+\alpha}([0,T]\times\mathbb{S}^1)}\le C.
\tag{3.6}
$$

Proof. Consider the function $S(v) := v\,(|v|+1)^{\alpha}$, which satisfies

$$S'(v) = (|v|+1)^{\alpha} + \alpha\,|v|\,(|v|+1)^{\alpha-1},$$

and

$$S''(v) = \alpha\,\mathrm{sgn}(v)\,(|v|+1)^{\alpha-2}\,(2+(\alpha+1)\,|v|)\,.$$

Clearly, $\left|S''(v)\right| \le C$ for all $v \in \mathbb{R}$.

Integrating the SPDE (3.4) for $S(q_\varepsilon)$ over $x \in \mathbb{R}$ gives

$$L\,\mathrm{d}t = \mathrm{d}\int_{\mathbb{S}^1} S(q_\varepsilon)\,\mathrm{d}x + (I_1+I_2)\,\mathrm{d}t + I_3\,\mathrm{d}W, \tag{3.7}$$

where

$$\begin{aligned}
L &= \int_{\mathbb{S}^1} \left(S(q_\varepsilon)q_\varepsilon - \frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\right)\mathrm{d}x,\\
I_1 &= \int_{\mathbb{S}^1} S'(q_\varepsilon)\left(P_\varepsilon - u_\varepsilon^2\right) + S''(q_\varepsilon)\,\varepsilon\,|\partial_x q_\varepsilon|^2\,\mathrm{d}x,\\
I_2 &= -\int_{\mathbb{S}^1} \big(S(q_\varepsilon) - S'(q_\varepsilon)q_\varepsilon\big)\partial_x\big(\sigma_\varepsilon\,\partial_x\sigma_\varepsilon\big) + \frac{1}{2}S''(q_\varepsilon)\,|\partial_x\sigma_\varepsilon\,q_\varepsilon|^2\,\mathrm{d}x,\\
I_3 &= -\int_{\mathbb{S}^1} \big(S(q_\varepsilon) - S'(q_\varepsilon)q_\varepsilon\big)\partial_x\sigma_\varepsilon\,\mathrm{d}x.
\end{aligned}$$

One can verify that $I_3 \in L^2(\Omega\times[0,T])$, so that $\mathbb{E}\int_0^t I_3\,\mathrm{d}W = 0$. Let us argue in some more detail for the square-integrability of I_3, observing first that

$$\left|S(v) - S'(v)v\right| = \left|\alpha v\,|v|\,(|v|+1)^{\alpha-1}\right| \lesssim_\alpha 1 + |v|^{1+\alpha} \lesssim 1 + |v|^2\,, \tag{3.8}$$

so that

$$\mathbb{E}\int_0^T \left|\int_{\mathbb{S}^1} I_3\,\mathrm{d}x\right|^2 \mathrm{d}t \lesssim_{\sigma,\alpha,T} 1 + \mathbb{E}\,\|q_\varepsilon\|^4_{L^\infty([0,T];L^2(\mathbb{S}^1))} \overset{(2.4)}{\lesssim} 1.$$

Continuing,

$$I_1 \le \int_{\mathbb{S}^1} \varepsilon S''(q_\varepsilon)\,|\partial_x q_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}^1} \left|S'(q_\varepsilon)\right|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}^1}\left(P_\varepsilon - u_\varepsilon^2\right)^2\mathrm{d}x.$$

By $\left|S''\right| \lesssim 1$, (2.4) and Lemma 3.1, we thus arrive at

$$\mathbb{E}\int_0^t |I_1|\,\mathrm{d}s \lesssim 1. \tag{3.9}$$

Next, by (3.8), $\left|S''\right| \lesssim 1$ and (2.4), we obtain

$$\mathbb{E}\int_0^t |I_2|\,\mathrm{d}s \lesssim_{\sigma,\alpha,T} 1 + \mathbb{E}\int_0^t\int_{\mathbb{S}^1} |q_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s \lesssim_{\sigma,\alpha,T} 1. \tag{3.10}$$

Finally, regarding L in (3.7), note that

$$S(v)v - \frac{1}{2}S'(v)v^2 = \frac{1}{2}v^2\left(|v|+1\right)^\alpha - \frac{\alpha}{2}|v|^3\left(|v|+1\right)^{\alpha-1} \geq \frac{1-\alpha}{2}|v|^{2+\alpha};$$

hence

$$L \geq \frac{1-\alpha}{2}\int_{\mathbb{S}^1} |q_\varepsilon|^{2+\alpha}\,\mathrm{d}x.$$

After integrating (3.7) in time, making use of the estimates (3.9), (3.10) and also $S(v) \lesssim_\alpha 1+|v|^2$, we arrive at

$$\begin{aligned}&\frac{1-\alpha}{2}\mathbb{E}\int_0^T\int_{\mathbb{S}^1} |q_\varepsilon|^{2+\alpha}\,\mathrm{d}x\\ &\qquad\lesssim_{\sigma,\alpha,T} 1 + \mathbb{E}\int_{\mathbb{S}^1} |q_\varepsilon(T,x)|^2\,\mathrm{d}x + \mathbb{E}\int_{\mathbb{S}^1} |q_\varepsilon(0,x)|^2\,\mathrm{d}x \overset{(2.4)}{\lesssim} 1.\end{aligned}$$

This concludes the proof. $\square$

The next result has no counterpart in the deterministic theory, see [21,22,63]. It is going to play an important role in the upcoming convergence analysis, as it will allow passing to the limit in products like $q_\varepsilon P_\varepsilon$ towards qP, where $P_\varepsilon = K * \left(u_\varepsilon^2 + \frac{1}{2}|q_\varepsilon|^2\right)$ and $P = K * \left(u_\varepsilon^2 + \frac{1}{2}\overline{q^2}\right)$. The deterministic approach of establishing ε-uniform $W^{1,1}_{t,x}$ estimates—to enforce strong convergence of P_ε—does not work in the stochastic setting. Besides, the nonlinear quantity q_ε^2 does not exhibit weak temporal Hölder continuity (uniformly in ε), which would be needed for a traditional stochastic compactness argument.

Proposition 3.3 *(Temporal translation estimate). Let u_ε be the H^m solution of the viscous SPDE* (1.5) *with initial data $u(0) = u_0$ satisfying* (2.1) *($m > 1$), and set $q_\varepsilon = \partial_x u_\varepsilon$. Fix a nonlinear function $S \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ that satisfies* (3.1)*. Set $Q_\varepsilon = S(q_\varepsilon)$. Then, for all $\vartheta \in (0, T \wedge 1)$ and $\varphi \in C^\infty(\mathbb{S}^1)$,*

$$\mathbb{E} \sup_{\tau\in(0,\vartheta)} \int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_\varepsilon(t+\tau)-Q_\varepsilon(t)\big)\,\mathrm{d}x \right| \mathrm{d}t \lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}\, \vartheta^{\frac{1}{2}}. \tag{3.11}$$

Remark 3.4. In view of the regularity of Q_ε (which follows from Lemma D.1), the function $\tau \mapsto \int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_\varepsilon(t+\tau)-Q_\varepsilon(t)\big)\,\mathrm{d}x \right| \mathrm{d}t$ is a.s. continuous. Therefore, it follows that the supremum can be equivalently taken over $\mathbb{Q}\cap(0,\vartheta)$, so that the resulting object is a random variable.

Proof. The nonlinear composition $Q_\varepsilon = S(q_\varepsilon)$ satisfies the SPDE (3.4). For any $t\in[0,T]$ and $\tau>0$ with $t+\tau\le T$, we obtain (easily via integration by parts)

$$\begin{aligned} &\left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_\varepsilon(t+\tau)-Q_\varepsilon(t)\big)\,\mathrm{d}x \right| \\ &\qquad \le \sum_{i=1}^{9} \int_t^{t+\tau}\int_{\mathbb{S}^1} I_\varepsilon^{(i)}\,\mathrm{d}x\,\mathrm{d}s + \sum_{i=10}^{11} \left| \int_t^{t+\tau}\int_{\mathbb{S}^1} I_\varepsilon^{(i)}\,\mathrm{d}x\,\mathrm{d}W \right|, \end{aligned}$$

where

$$\begin{aligned} &I_\varepsilon^{(1)} = |u_\varepsilon|\,|S(q_\varepsilon)|\,|\partial_x\varphi|, \quad I_\varepsilon^{(2)} = \frac{1}{4}\left|\partial_x\sigma_\varepsilon^2\right| \left|3S(q_\varepsilon)-2S'(q_\varepsilon)q_\varepsilon\right| |\partial_x\varphi|, \\ &I_\varepsilon^{(3)} = \left|\frac{1}{2}\sigma_\varepsilon^2+\varepsilon\right| |S(q_\varepsilon)| \left|\partial_{xx}^2\varphi\right|, \quad I_\varepsilon^{(4)} = \left|S''(q_\varepsilon)\right| \varepsilon\,|\partial_x q_\varepsilon|^2\,|\varphi|, \\ &I_\varepsilon^{(5)} = \left|S'(q_\varepsilon)\right| |P_\varepsilon|\,|\varphi|, \quad I_\varepsilon^{(6)} = \left|S'(q_\varepsilon)\right| \left|u_\varepsilon^2\right| |\varphi|, \\ &I_\varepsilon^{(7)} = \left|S(q_\varepsilon)q_\varepsilon - \frac{1}{2}S'(q_\varepsilon)q_\varepsilon^2\right| |\varphi|, \\ &I_\varepsilon^{(8)} = \frac{1}{4}\left|\partial_{xx}^2\sigma_\varepsilon^2\right| \left|S(q_\varepsilon)-S'(q_\varepsilon)q_\varepsilon\right| |\varphi|, \quad I_\varepsilon^{(9)} = \frac{1}{2}\,|\partial_x\sigma_\varepsilon|^2 \left|S''(q_\varepsilon)\right| \left|q_\varepsilon^2\right| |\varphi|, \\ &I_\varepsilon^{(10)} = \sigma_\varepsilon\,S(q_\varepsilon)\,\partial_x\varphi, \quad I_\varepsilon^{(11)} = \partial_x\sigma_\varepsilon\left(S(q_\varepsilon)-S'(q_\varepsilon)q_\varepsilon\right)\varphi. \end{aligned}$$

This implies that

$$\begin{aligned} &\mathbb{E} \sup_{\tau\in(0,\vartheta)} \int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_\varepsilon(t+\tau)-Q_\varepsilon(t)\big)\,\mathrm{d}x \right| \mathrm{d}t \qquad (3.12) \\ &\qquad \lesssim \vartheta \sum_{i=1}^{9} \mathbb{E}\left\| I_\varepsilon^{(i)} \right\|_{L^1([0,T]\times\mathbb{S}^1)} \end{aligned}$$

$$+\sum_{i=10}^{11}\int_0^{T-\vartheta}\mathbb{E}\sup_{\tau\in(0,\vartheta)}\left|\int_t^{t+\tau}\int_{\mathbb{S}^1} I_\varepsilon^{(i)}\,\mathrm{d}x\,\mathrm{d}W(s)\right|\mathrm{d}t.$$

In what follows, we use the trivial fact that

$$\|\varphi\|_{L^\infty(\mathbb{S}^1)},\, \|\partial_x\varphi\|_{L^\infty(\mathbb{S}^1)},\, \left\|\partial_{xx}^2\varphi\right\|_{L^\infty(\mathbb{S}^1)} \le \|\varphi\|_{C^2(\mathbb{S}^1)}.$$

For $i\in\{2,3,8,9\}$, recalling $\sigma\in W^{2,\infty}(\mathbb{S}^1)$ and (3.1),

$$I_\varepsilon^{(i)} \lesssim |q_\varepsilon|^2\,\|\varphi\|_{C^2(\mathbb{S}^1)},$$

and thus, by (2.4) (as $L_t^\infty L_x^2 \hookrightarrow L_{t,x}^2$),

$$\mathbb{E}\left\|I_\varepsilon^{(i)}\right\|_{L^1([0,T]\times\mathbb{S}^1)} \lesssim \mathbb{E}\left[\|q_\varepsilon\|_{L^2([0,T]\times\mathbb{S}^1)}^2\right]\|\varphi\|_{C^2(\mathbb{S}^1)} \lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}.$$

Similarly, for $i=7$,

$$\mathbb{E}\left\|I_\varepsilon^{(7)}\right\|_{L^1([0,T]\times\mathbb{S}^1)} \lesssim \mathbb{E}\left[\|q_\varepsilon\|_{L^2([0,T]\times\mathbb{S}^1)}^2\right]\|\varphi\|_{C^2(\mathbb{S}^1)} \lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}.$$

For $i=5$, we use (3.1), Lemma 3.1 and (2.4) (as $L_t^\infty L_x^2 \hookrightarrow L_{t,x}^1$),

$$\begin{aligned}
\mathbb{E}\left\|I_\varepsilon^{(5)}\right\|_{L^1([0,T]\times\mathbb{S}^1)} &\lesssim \mathbb{E}\left[\int_0^T\int_{\mathbb{S}^1}|q_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t\,\|P_\varepsilon\|_{L^\infty([0,T]\times\mathbb{S}^1)}\right]\|\varphi\|_{C^2(\mathbb{S}^1)} \\
&\le \left(\mathbb{E}\,\|q_\varepsilon\|_{L^1([0,T]\times\mathbb{S}^1)}^2\right)^{1/2}\left(\mathbb{E}\,\|P_\varepsilon\|_{L^\infty([0,T]\times\mathbb{S}^1)}^2\right)^{1/2}\|\varphi\|_{C^2(\mathbb{S}^1)} \\
&\lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}.
\end{aligned}$$

Similarly, for $i=6$,

$$\begin{aligned}
&\mathbb{E}\left\|I_\varepsilon^{(6)}\right\|_{L^1([0,T]\times\mathbb{S}^1)} \\
&\quad\lesssim \left(\mathbb{E}\,\|q_\varepsilon\|_{L^1([0,T]\times\mathbb{S}^1)}^2\right)^{1/2}\left(\mathbb{E}\,\|u_\varepsilon\|_{L^\infty([0,T]\times\mathbb{S}^1)}^4\right)^{1/2}\|\varphi\|_{C^2(\mathbb{S}^1)} \\
&\quad\lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}.
\end{aligned}$$

For $i=1$, by Lemma 3.1 and (2.4) (as $L_t^\infty L_x^2 \hookrightarrow L_{t,x}^2$),

$$\begin{aligned}
\mathbb{E}\left\|I_\varepsilon^{(1)}\right\|_{L^1([0,T]\times\mathbb{S}^1)} &\lesssim \mathbb{E}\left[\|u_\varepsilon\|_{L^\infty([0,T]\times\mathbb{S}^1)}\,\|q_\varepsilon\|_{L^2([0,T]\times\mathbb{S}^1)}^2\right]\|\varphi\|_{C^2(\mathbb{S}^1)} \\
&\le \left(\mathbb{E}\,\|u_\varepsilon\|_{L^\infty([0,T]\times\mathbb{S}^1)}^2\right)^{1/2}\left(\mathbb{E}\,\|q_\varepsilon\|_{L^2([0,T]\times\mathbb{S}^1)}^4\right)^{1/2}\|\varphi\|_{C^2(\mathbb{S}^1)} \\
&\lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}.
\end{aligned}$$

For $i = 4$, by the second part of (2.4),

$$\mathbb{E}\left\| I_\varepsilon^{(4)} \right\|_{L^1([0,T]\times\mathbb{S}^1)} \lesssim \mathbb{E}\int_0^T\int_{\mathbb{S}^1} \varepsilon \left|\partial_{xx}^2 u_\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}t\, \|\varphi\|_{C^2(\mathbb{S}^1)} \lesssim \|\varphi\|_{C^2(\mathbb{S}^1)}\,.$$

Finally, we turn to the stochastic integrals ($i = 11, 12$). By $\sigma \in W^{2,\infty}(\mathbb{S}^1)$, (3.1) and the BDG inequality,

$$\begin{aligned}
&\mathbb{E}\sup_{\tau\in(0,\vartheta)}\left|\int_t^{t+\tau}\int_{\mathbb{S}^1} I_\varepsilon^{(i)}\,\mathrm{d}x\,\mathrm{d}W(s)\right|\mathrm{d}t\\
&\lesssim \mathbb{E}\left[\left(\int_t^{t+\vartheta}\left(\int_{\mathbb{S}^1} q_\varepsilon^2\,\mathrm{d}x\right)^2 ds\right)^{1/2}\right]\|\varphi\|_{C^2(\mathbb{S}^1)}\\
&\le \vartheta^{\frac{1}{2}}\mathbb{E}\,\|q_\varepsilon\|^2_{L^\infty([0,T];L^2(\mathbb{S}^1))}\|\varphi\|_{C^2(\mathbb{S}^1)} \overset{(2.4)}{\lesssim} \vartheta^{\frac{1}{2}}\|\varphi\|_{C^2(\mathbb{S}^1)}\,.
\end{aligned}$$

Given (3.12), the above estimates yield (3.11). □

4. Tightness and a.s. representations

4.1. Renormalisations

As explained in the introduction, we wish to use the ε-uniform a priori estimates (2.4), (2.5), (3.6) to extract a.s. convergence properties of u_ε and of the spatial gradient $q_\varepsilon = \partial_x u_\varepsilon$, as well as of nonlinear quantities like q_ε^2. Verifying the tightness of the different probability laws, among which the one for q_ε^2 is the most challenging, we construct Skorokhod a.s. representations of u_ε, whose laws are defined on the Polish space $C_t L_x^2$, and Jakubowski a.s. representations of q_ε and several infinite sequences of nonlinear compositions of q_ε, whose laws are defined on suitable quasi-Polish spaces like $L_{t,x}^p - w$. In addition, for the energy variable q_ε^2, we construct representations in the quasi-Polish spaces $L^p(L_w^p)$, for some p, which supplies a crucial strong convergence property in t. We refer to Section B for quasi-Polish spaces and their properties.

In what follows, we fix a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Let us introduce the random mappings:

$$F_{\varepsilon_n}^q = \left(q_{\varepsilon_n}, (q_{\varepsilon_n})_+, (q_{\varepsilon_n})_-\right), \quad F_{\varepsilon_n}^{q^2} = \left(q_{\varepsilon_n}^2, (q_{\varepsilon_n})_+^2, (q_{\varepsilon_n})_-^2\right). \tag{4.1}$$

Here, we denote by f_+ and f_- the positive and negative parts of a function f, so that $f(v) = f_+(v) + f_-(v) = \max(f(v), 0) + \min(f(v), 0)$. Furthermore, the notation $(q_{\varepsilon_n})_\pm^2$ is a concise representation of $\left((q_{\varepsilon_n})_\pm\right)^2$.

To execute various renormalisation procedures, we shall need to take limits as $n \to \infty$ of infinite sequences of nonlinear compositions of q_{ε_n}, like $S_\ell(q_{\varepsilon_n})$, where $\{S_\ell(v)\}_{\ell\in\mathbb{N}}$ is a sequence that approximates $\frac{1}{2}v^2$ up to some cut-off $|v| \le \ell$. The Skorokhod–Jakubowski procedure extracts *simultaneously* the a.s. convergence of all these variables. As a preparation, we introduce several sequences of random mappings that are built from the approximation $\{S_\ell(v)\}_{\ell=1}^{\infty}$ of $\frac{1}{2}v^2$ on $\mathbb{R}$:

$$S_\ell(v) = \begin{cases} \frac{1}{2}v^2, & |v| \le \ell \\ -\frac{1}{6\ell}|v|^3 + v^2 - \frac{1}{2}\ell|v| + \frac{1}{6}\ell^2, & \ell < |v| < 2\ell \; . \\ \frac{3}{2}\ell|v| - \frac{7}{6}\ell^2, & |v| \ge 2\ell \end{cases} \tag{4.2}$$

Each function S_ℓ is convex and satisfies

$$S'_\ell(v) = \begin{cases} v, & |v| \le \ell \\ \operatorname{sgn}(v)\left(2|v| - \frac{1}{2\ell}v^2 - \frac{1}{2}\ell\right), & \ell < |v| < 2\ell \\ \frac{3}{2}\operatorname{sgn}(v)\ell, & |v| \ge 2\ell \end{cases} \tag{4.3}$$

and

$$S''_\ell(v) = \begin{cases} 1, & |v| \le \ell \\ \frac{1}{\ell}(2\ell - |v|), & \ell < |v| < 2\ell \; . \\ 0, & |v| \ge 2\ell \end{cases} \tag{4.4}$$

The random mappings that we introduce below are motivated by the need to pass to the weak limit in various nonlinear compositions of q_{ε_n}, based on

$$\begin{aligned} &S_\ell(v_\pm), \quad S_\ell(v_\pm)', \quad S_\ell(v_\pm)''v^2, \\ &S_\ell(v_\pm) - S_\ell(v_\pm)'v, \quad S_\ell(v_\pm)v - \frac{1}{2}S_\ell(v_\pm)'v^2. \end{aligned} \tag{4.5}$$

Clearly, $(v_\pm)' = \mathbb{1}_{\{|v_\pm|>0\}}$, $\left(v_\pm^2\right)' = 2v_\pm$, $\left(v_\pm^2\right)'' = 2\mathbb{1}_{\{|v_\pm|>0\}}$, and so $S(v) = v_\pm^2$ belongs to $W^{2,\infty}_{\text{loc}}(\mathbb{R})$ and satisfies (3.1). Using (4.2), (4.3), (4.4) and the chain rule, we can readily compute the following nonlinear compositions:

$$S_\ell(v_\pm)' = S'_\ell(v_\pm), \qquad S_\ell(v_\pm)'' = S''_\ell(v_\pm)\mathbb{1}_{\{|v_\pm|>0\}},$$

$$S_\ell(v_\pm) - S_\ell(v_\pm)'v = \begin{cases} -\frac{1}{2}v_\pm^2, & |v_\pm| \le \ell \\ \frac{1}{3\ell}|v_\pm|^3 - {v_\pm}^2 + \frac{1}{6}\ell^2, & \ell < |v_\pm| < 2\ell \; , \\ -\frac{7}{6}\ell^2, & |v_\pm| \ge 2\ell \end{cases}$$

$$\begin{aligned} &3S_\ell(v_\pm) - 2S_\ell(v_\pm)'v \\ &\quad = \begin{cases} -\frac{1}{2}v_\pm^2, & |v_\pm| \le \ell \\ \frac{1}{2\ell}|v_\pm|^3 - {v_\pm}^2 - \frac{1}{2}|v_\pm|\ell + \frac{1}{2}\ell^2, & \ell < |v_\pm| < 2\ell \; , \\ \frac{3}{2}\ell|v_\pm| - \frac{7}{2}\ell^2, & |v_\pm| \ge 2\ell \end{cases} \end{aligned} \tag{4.6}$$

$$
S_\ell(v_\pm)v - \frac{1}{2}S_\ell(v_\pm)'v^2
= \begin{cases} 0, & |v_\pm| \le \ell \\ \frac{1}{12\ell}|v_\pm|^3 v_\pm - \frac{1}{4}|v_\pm| v_\pm \ell + \frac{1}{6}v_\pm \ell^2, & \ell < |v_\pm| < 2\ell \\ \frac{3}{4}|v_\pm| v_\pm \ell - \frac{7}{6}v_\pm\ell^2, & |v_\pm| \ge 2\ell \end{cases}.
$$

In particular, this implies that $S_\ell \in W^{3,\infty}_{\mathrm{loc}}(\mathbb{R})$, $|S_\ell(v)| \lesssim_\ell |v|$, $\left|S'_\ell(v)\right| \lesssim_\ell 1$, $\left|S''_\ell(v)\right| \lesssim \mathbb{1}_{\{|v|\le 2\ell\}}$, and $\left|S_\ell(v)v - \frac{1}{2}S'_\ell(v)v^2\right| \lesssim_\ell |v|^2$, so that (3.1) is satisfied with $S = S_\ell$. The nonlinear compositions $v \mapsto S_\ell(v_\pm)$ belong to $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ and cater to similar bounds, see also Remark 7.11.

Notice that $v \mapsto S_\ell(v_\pm)''v^2 = S''_\ell(v_\pm)v^2$ is a continuous function (but $S_\ell(v_\pm)''$ is not). Later, we will also need to know that the function $\beta(v) = S_\ell(v_\pm)'v$ belongs to $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ (although $S_\ell(v_\pm)'$ does not) and satisfies (3.1) with $S = \beta$. Indeed,

$$
\begin{aligned}
\beta'(v) &= S''_\ell(v_\pm)\mathbb{1}_{\{|v_\pm|>0\}}v + S'_\ell(v_\pm), \\
\beta''(v) &= S'''_\ell(v_\pm)\mathbb{1}_{\{|v_\pm|>0\}}v + 2S''_\ell(v_\pm)\mathbb{1}_{\{|v_\pm|>0\}},
\end{aligned} \tag{4.7}
$$

so that $|\beta(v)| \lesssim_\ell |v|$, $\left|\beta'(v)\right| \lesssim_\ell 1$, and $\left|\beta''(v)\right| \lesssim_\ell 1$.

Remark 4.1. One may wonder about the specific choice (4.2) of renormalisations (entropies), which admittedly comes across as complicated. At this point, we run into a new difficulty compared to the deterministic CH equation [63]. The particular form of the noise in the stochastic CH equation (1.3) leads to some key structural changes in the equation satisfied by $S_\ell(q_\varepsilon)$, which prevents us from using the simple entropies of [63] (linearly growing $W^{2,\infty}$ approximations of v^2). The entropies (4.5) are carefully constructed to allow for the control of some delicate error terms involving weak limits linked to the defect measure (1.12), see Remark 7.16.

4.2. *Random mappings and path spaces*

For $\ell \in \mathbb{N}$, we introduce the random mappings

$$
F^{\xi_\ell,\pm}_{\varepsilon_n} = \xi_\ell\big|_{v=q_{\varepsilon_n}}, \quad \xi_\ell \in \mathbb{S}^\pm_\ell, \tag{4.8}
$$

where $\mathbb{S}^\pm_\ell$ denote the collections

$$
\mathbb{S}^\pm_\ell = \left\{ S_\ell(v_\pm),\ S'_\ell(v_\pm)v,\ S''_\ell(v_\pm)v^2,\ S_\ell(v_\pm)v,\ S'_\ell(v_\pm)v^2,\ S'_\ell(v_\pm) \right\} \tag{4.9}
$$

of nonlinear functions. We also make use of $F^{\mathbb{S}}_{\varepsilon_n}$ as a notation for the gathering of all these ℓ-dependent mappings:

$$
F^{\mathbb{S}}_{\varepsilon_n} = \left\{ \left\{ F^{\xi_\ell,+}_{\varepsilon_n},\ \xi_\ell \in \mathbb{S}^+_\ell \right\}_{\ell\in\mathbb{N}},\ \left\{ F^{\xi_\ell,-}_{\varepsilon_n},\ \xi_\ell \in \mathbb{S}^-_\ell \right\}_{\ell\in\mathbb{N}} \right\}. \tag{4.10}
$$

Finally, we use X_n as a collective symbol for *all* the random mappings just introduced:

$$
X_n = \left(u_{\varepsilon_n}, F^q_{\varepsilon_n}, F^{q^2}_{\varepsilon_n}, W, z_n, F^{\mathbb{S}}_{\varepsilon_n} \right), \tag{4.11}
$$

where W is the Wiener process appearing in (1.5) and $\{z_n\}_{n=1}^{\infty}$ is a sequence of C^∞ approximations of the initial data u_0, satisfying

$$z_n \in L^{p_0}\big(\Omega; C^\infty(\mathbb{S}^1)\big), \quad z_n \xrightarrow{n\uparrow\infty} u_0 \quad \text{in } L^{p_0}\big(\Omega; H^1(\mathbb{S}^1)\big), \tag{4.12}$$

recalling the assumption $p_0 > 4$ from Theorem 2.3.

The goal is to establish the tightness of the joint probability laws $\mu_n = \mathcal{L}(X_n)$ of the random mappings $X_n : \big(\Omega, \mathcal{F}, \mathbb{P}\big) \to \big(\mathcal{X}, \mathcal{B}_{\mathcal{X}}\big)$. To this end, we need to specify $\mathcal{X}$—the path space for μ_n—and the σ-algebra $\mathcal{B}_{\mathcal{X}}$. Denote the factors of the infinite vector X_n by $X_n^{(l)}$ and the corresponding factor spaces by $\mathcal{X}_l$, $l \in \mathbb{N}$. For example, $X_n^{(1)} = u_{\varepsilon_n}$ and $X_n^{(5)} = q_{\varepsilon_n}^2$, cf. (4.1). Whenever needed, we also use superscript symbols on X_n to identify the corresponding factor of X_n, for example, $X_n^u = X_n^{(1)}$ and $X_n^{q^2} = X_n^{(5)}$, while $X_n^{S_\ell(v_+)}$ would refer to $X_n^{(7+\ell)}$ with $X_n^{(7)} = \big(q_{\varepsilon_n}\big)_-^2$, see (4.8) and (4.1). Denote by $\mu^{(l)}$ the corresponding marginals of μ_n, defined on $\big(\mathcal{X}_l, \mathcal{B}_{\mathcal{X}_l}\big)$. Similarly, we will write μ_n^u instead of $\mu_n^{(1)}$ for the marginal linked to $X_n^{(1)} = u_{\varepsilon_n}$, and so forth, and the same for the factor spaces $\mathcal{X}_l$.

Remark 4.2. The notation just introduced may appear overwhelming. Fortunately, most of it will be utilised only in this section.

For a fixed number $r \in \big[1, \frac{3}{2}\big)$ (close to 3/2), we specify the following spaces for the marginals:

$$\begin{aligned}
&\mathcal{X}_u = C_t L_x^2, \quad \mathcal{X}_W = C_t, \quad \mathcal{X}_{u_0} = H_x^1,\\
&\mathcal{X}_q = L_{t,x}^{2r} - w, \quad \mathcal{X}_{q_\pm} = L_{t,x}^{2r} - w,\\
&\mathcal{X}_{q^2} = L^r\big(L_w^r\big), \quad \mathcal{X}_{q_\pm^2} = L^r\big(L_w^r\big),\\
&\mathcal{X}_\xi = L^{2r}(L_w^{2r}),\ \xi = S_\ell(v_\pm),\ S_\ell(v_\pm)'v,\ \ell \in \mathbb{N},\\
&\mathcal{X}_\xi = L_{t,x}^{2r} - w,\ \xi = S_\ell(v_\pm)',\ \ell \in \mathbb{N},\\
&\mathcal{X}_\xi = L_{t,x}^{r} - w,\ \xi = S_\ell(v_\pm)''v^2,\ S_\ell(v_\pm)v,\ S_\ell(v_\pm)'v^2,\ \ell \in \mathbb{N}.
\end{aligned} \tag{4.13}$$

Here, $C_t = C([0,T])$, $H_x^1 = H^1(\mathbb{S}^1)$, and $C_t L_x^2 = C([0,T]; L^2(\mathbb{S}^1))$ are all Polish spaces. Furthermore, $L_{t,x}^p - w = L^p([0,T]\times\mathbb{S}^1) - w$, for any $p \in [1,\infty)$, denotes the L^p space equipped with the weak topology, which is quasi-Polish. For the energy variables $q_{\varepsilon_n}^2$ and $(q_{\varepsilon_n})_\pm^2$, we use the space $L^r\big(L_w^r\big) = L^r\big([0,T]; L^r(\mathbb{S}^1) - w\big)$, which is quasi-Polish as well, see Section B and (B.1) for details. Notice that the topology of $L^r\big(L_w^r\big)$ is strong in t and weak in x. Similarly, we use $L^{2r}(L_w^{2r})$ for the variables $S_\ell\big((q_{\varepsilon_n})_\pm\big)$ and $S_\ell'\big((q_{\varepsilon_n})_\pm\big)q_{\varepsilon_n}$ (linearly growing approximations of $\frac{1}{2}(q_{\varepsilon_n})_\pm^2$).

Remark 4.3. The spaces prescribed in (4.13) reflect some minimum requirements for convergence in Section 7. The significance of the peculiar "strong-weak" spaces $L^r\big(L_w^r\big)$, $L^{2r}\big(L_w^{2r}\big)$ will become clear during the proofs of Lemmas 7.2, 7.8, and 7.9. Roughly speaking, these spaces will allow us to pass to the limit in delicate product terms like $S'(\tilde{q}_{\varepsilon_n})\,\tilde{P}_{\varepsilon_n}$ as well as in various stochastic integrals.

The path space $\mathcal{X}$ for the joint laws $\{\mu_n\}_{n\in\mathbb{N}}$ is taken as

$$\mathcal{X} = \prod_{l=1}^{\infty} \mathcal{X}_l, \qquad \mathcal{B}_{\mathcal{X}} = \mathcal{B}(\mathcal{X}), \tag{4.14}$$

which carries the product topology for its infinitely many factors.

Remark 4.4. Each factor space $\mathcal{X}_l$ in $\mathcal{X}$ is either Polish or quasi-Polish. Polish spaces are quasi-Polish and countable products of quasi-Polish spaces are quasi-Polish, see Lemma B.4. Generally, for a quasi-Polish space $\mathcal{Y}$ there are two natural candidates for the σ-algebra $\mathcal{B}_{\mathcal{Y}}$, the Borel σ-algebra $\mathcal{B}_{\mathcal{Y}} = \mathcal{B}(\mathcal{Y})$ or the σ-algebra $\mathcal{B}_{\mathcal{Y}} = \mathcal{B}_f$ generated by the separating sequence $f = \{f_l\}_{l\in\mathbb{N}}$ defining the space, see Definition B.1. In general, $\mathcal{B}_f \subset \mathcal{B}(\mathcal{Y})$, see Lemma B.9. However, for each space in (4.13) we use the Borel σ-algebra, as it happens to coincide with the one generated by the separating sequence (see Lemma B.2). The space $\mathcal{X}$ for the joint laws is equipped with the product topology. For a quasi-Polish product space like $\mathcal{X}$, the Borel σ-algebra $\mathcal{B}(\mathcal{X})$ for the product topology is likely to differ from the product of the individual Borel σ-algebras (although they do coincide if $\mathcal{X}$ is Polish), see Lemma B.9. However, as is shown in [46], this is not a problem as long as we work with random mappings with tight laws. To be specific, we take $\mathcal{B}_{\mathcal{X}} = \mathcal{B}(\mathcal{X})$.

Consider the random map $X_n = \{X_n^{(l)}\}_{l\in\mathbb{N}} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ defined by (4.11). By Theorem 2.3, one can check that each factor $X_n^{(l)}$ is a random variable (Borel measurable). We only prove this for the nonlinear parts of X_n involving q_{ε_n} (the other parts are simpler). By construction, $\Omega \overset{q_{\varepsilon_n}}{\longrightarrow} C([0,T]; H^1(\mathbb{S}^1))$ is a random variable. Since all of our nonlinear functions or entropies (here denoted by the generic placeholder β) are continuous real-valued functions and satisfy (at least) the bound $|\beta(v)| \lesssim |v|^{3-1}$, Nemytskii theory ensures then that these entropies β, when viewed as operators, are bounded and continuous from $L^3_{t,x}$ into $L^{3/2}_{t,x}$. Because $L^{3/2}_{t,x}$ embeds continuously in $L^r_{t,x} - w$ and $C_t H^1_x$ embeds continuously in $L^3_{t,x}$, the composition $\Omega \overset{\beta(q_{\varepsilon_n})}{\longrightarrow} L^r_{t,x} - w$ is a random variable.

4.3. Compactness and tightness criteria

The goal is to establish the tightness of the joint laws of X_n. The most difficult part is to verify the tightness of the laws of the energy variables $q^2_{\varepsilon_n}$ and $(q_{\varepsilon_n})^2_{\pm}$—and similarly also $S_\ell\big((q_{\varepsilon_n})_{\pm}\big)$, $S'_\ell\big((q_{\varepsilon_n})_{\pm}\big)q_{\varepsilon_n}$—which take values in a quasi-Polish space of the form $L^{p_1}\big(L^{p_2}_w\big)$, for some $p_1, p_2 \in (1,\infty)$, see (B.1). This space encodes strong temporal and weak spatial compactness. The strong t-compactness of $q^2_{\varepsilon_n}$ is essential for our analysis, noting that there is no hope of establishing uniform Hölder continuity in t, even if the spatial topology is weak. This excludes the traditional compactness approach based on tightness in the space $C([0,T]; L^p(\mathbb{S}^1) - w)$, used by many of the references listed in Section 1. Indeed, the space $L^{p_1}\big(L^{p_2}_w\big)$ was carefully singled out to resolve this particular predicament of the energy variable.

The following result, which is of independent interest, provides general criteria for compactness in $L^{p_1}\big(L^{p_2}_w\big)$. These criteria will be later used in the analysis of tightness. For the space $L^{p_1}\big(L^{p_2}_w\big)$ there exists a sequence of continuous functionals that separate points and generate the

Borel σ-algebra. This fact is discussed in Appendix B and can be found in (B.1). Based on this, Jakubowski [46, page 169] states that the notions of compactness and sequential compactness are equivalent.

The lemma stated below identifies conditions that ensure the *relative* sequential compactness of a subset in $L^{p_1}\big(L^{p_2}_w\big)$. In Appendix B.3, we show that the notions of *relative compactness* and *relative sequential compactness* are also the same in quasi-Polish spaces like $L^{p_1}\big(L^{p_2}_w\big)$. Whence, the closure $\overline{\mathcal{K}}$ of a relatively sequentially compact set $\mathcal{K}$ can be used to verify the tightness condition of Jakubowski's theorem [46] (see Theorem B.12).

Lemma 4.5 *(Compactness criterion). Fix some integrability indices $p_1, p_2 \in (1,\infty)$, and consider the space $L^{p_1}\big(L^{p_2}_w\big) = L^{p_1}\big([0,T]; L^{p_2}(\mathbb{S}^1) - w\big)$, cf. (B.1). Let $\mathcal{K}$ be a subset of $L^{p_1}\big(L^{p_2}_w\big)$ for which the following conditions hold uniformly in $Q \in \mathcal{K}$:*

(i) $\|Q\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{S}^1))} \lesssim 1$,

(ii) $\|Q\|_{L^{\bar{p}_1}([0,T];L^1(\mathbb{S}^1))} \lesssim 1$, *for some* $\bar{p}_1 > p_1$,

(iii) $\displaystyle\int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q(t+\tau,x) - Q(t,x)\big)\,\mathrm{d}x \right| \mathrm{d}t \xrightarrow{\tau\downarrow 0} 0, \quad \forall \varphi \in C^\infty(\mathbb{S}^1).$

Then $\mathcal{K}$ is relatively sequentially compact in $L^{p_1}\big(L^{p_2}_w\big)$.

Remark 4.6. Note carefully how, in (ii), some higher temporal integrability is traded for low spatial integrability. This flexibility is important for us. However, in other applications, if one is happy with the temporal integrability provided by (i), then (ii) can be dropped at the expense of getting compactness in $L^p\big(L^{p_2}_w\big)$, $\forall p < p_1$.

Proof. Consider a subset $\mathcal{K} \subset L^{p_1}\big(L^{p_2}_w\big)$ for which (i), (ii) and (iii) hold. To establish the lemma, we must demonstrate that for any sequence $\{Q_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}$, it is possible to find a subsequence that converges in $L^{p_1}\big(L^{p_2}_w\big)$.

By (i), there exists a subsequence $\big\{Q_{n_j}\big\}_{j\in\mathbb{N}}$ of $\{Q_n\}_{n\in\mathbb{N}}$ that converges weakly to some Q in $L^{p_1}([0,T]; L^{p_2}(\mathbb{S}^1))$:

$$\int_0^T \int_{\mathbb{S}^1} \psi(t)\varphi(x) Q_{n_j}(t,x)\,\mathrm{d}x\,\mathrm{d}t \xrightarrow{j\uparrow\infty} \int_0^T \int_{\mathbb{S}^1} \psi(t)\varphi(x) Q(t,x)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.15}$$

for all $\psi \in L^{p_1'}([0,T])$ and $\varphi \in L^{p_2'}(\mathbb{S}^1)$, $\frac{1}{p_1} + \frac{1}{p_1'} = \frac{1}{p_2} + \frac{1}{p_2'} = 1$.

Let J_δ be a standard (Friedrichs) mollifier in x and set

$$Q_{n_j,\delta} = Q_{n_j} * J_\delta, \quad Q_\delta = Q * J_\delta.$$

Then

$$Q_{n_j,\delta} \xrightarrow{j\uparrow\infty} Q_\delta \quad \text{in } L^{p_1}([0,T]; L^{p_2}(\mathbb{S}^1)), \text{ for each fixed } \delta,$$
$$Q_\delta \xrightarrow{\delta\downarrow 0} Q \quad \text{in } L^{p_1}([0,T]; L^{p_2}(\mathbb{S}^1)), \tag{4.16}$$

where the second convergence comes from basic properties of mollifiers (in x) and, via (i), Lebesgue's dominated convergence theorem in t. The first convergence can be proved using a basic property of the convolution product. Indeed, we have

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{S}^1} \psi(t)\varphi(x)\left(Q_\delta(t,x) - Q_{n_j,\delta}(t,x)\right) \mathrm{d}x\,\mathrm{d}t\right| \\ &\quad = \left|\int_0^T\int_{\mathbb{S}^1} \psi(t)\varphi_\delta(x)\left(Q(t,x) - Q_{n_j}(t,x)\right) \mathrm{d}x\,\mathrm{d}t\right| \xrightarrow{j\uparrow\infty} 0,\end{aligned}$$

for each fixed $\delta > 0$, recalling that the algebraic tensor product $L^{p_1'} \otimes L^{p_2'}$ is dense in $L^{p_1'}(L^{p_2'})$.

By the translation estimate (iii) with $\varphi(x) = J_\delta(y-x)$, for any $y \in \mathbb{S}^1$,

$$\int_0^{T-\tau} \left|Q_{n_j,\delta}(t+\tau, y) - Q_{n_j,\delta}(t,y)\right| \mathrm{d}t \xrightarrow{\tau\downarrow 0} 0,$$

uniformly in j. Using (i) and Vitali's convergence theorem (in y),

$$\int_0^{T-\tau}\int_{\mathbb{S}^1} \left|Q_{n_j,\delta}(t+\tau, y) - Q_{n_j,\delta}(t,y)\right| \mathrm{d}y\,\mathrm{d}t \xrightarrow{\tau\downarrow 0} 0, \tag{4.17}$$

uniformly in j.

Next, by (i), we have $\left\|\partial_x Q_{n_j,\delta}\right\|_{L^{p_1}([0,T];L^1(\mathbb{S}^1))} \lesssim_\delta 1$ and thus

$$\left\|Q_{n_j,\delta}\right\|_{L^{p_1}([0,T];W^{1,1}(\mathbb{S}^1))} \lesssim_\delta 1.$$

By (ii), we also deduce that

$$\left\|Q_{n_j,\delta}\right\|_{L^{\bar{p}_1}([0,T];L^1(\mathbb{S}^1))} \lesssim 1, \quad \text{where } \bar{p}_1 > p_1.$$

Consider the compact embedding

$$W^{1,1}(\mathbb{S}^1) \hookrightarrow L^1(\mathbb{S}^1),$$

and now note that $\left\{Q_{n_j,\delta}\right\}_{j\in\mathbb{N}}$ is bounded in

$$L^{\bar{p}_1}([0,T]; L^1(\mathbb{S}^1)) \bigcap L^1([0,T]; W^{1,1}(\mathbb{S}^1)),$$

uniformly in j, for each fixed δ. Besides, from (4.17),

$$\left\| Q_{n_j,\delta}(\cdot+\tau,\cdot) - Q_{n_j,\delta} \right\|_{L^1([t_1,t_2];L^1(\mathbb{S}^1))} \xrightarrow{\tau\downarrow 0} 0,$$

for all $0 < t_1 < t_2 < T$, uniformly in j, for each fixed δ. By [58, Theorem 4], we may therefore assume that there exists a limit $\overline{Q}_\delta \in L^{p_1}([0,T];L^1(\mathbb{S}^1))$ such that

$$Q_{n_j,\delta} \xrightarrow{j\uparrow\infty} \overline{Q}_\delta \quad \text{in } L^{p_1}([0,T];L^1(\mathbb{S}^1)), \tag{4.18}$$

for each fixed δ. However, by the uniqueness of the weak limit in (4.16),

$$\overline{Q}_\delta = Q_\delta = Q * J_\delta.$$

In fact, all subsequences extracted from $\left\{Q_{n_j,\delta}\right\}_{j\in\mathbb{N}}$ have further subsequences that converge to the same limit $Q * J_\delta$, and therefore the original sequence also converges to that limit.

Let us verify that $Q_{n_j} \xrightarrow{j\uparrow\infty} Q$ in $L^{p_1}\left(L^{p_2}_w\right)$, where Q is defined in (4.15). Fix any $\varphi \in C^\infty(\mathbb{S}^1)$. We proceed as follows:

$$\begin{aligned}
I_\varphi(j) &= \int_0^T \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q(t,x) - Q_{n_j}(t,x)\big)\,\mathrm{d}x \right|^{p_1} \mathrm{d}t \\
&\lesssim \underbrace{\int_0^T \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_\delta(t,x) - Q_{n_j,\delta}(t,x)\big)\,\mathrm{d}x \right|^{p_1} \mathrm{d}t}_{=:I_1(j,\delta)} \\
&\quad + \underbrace{\int_0^T \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q(t,x) - Q_\delta(t,x)\big)\,\mathrm{d}x \right|^{p_1} \mathrm{d}t}_{=:I_2(\delta)} \\
&\quad + \underbrace{\int_0^T \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_{n_j}(t,x) - Q_{n_j,\delta}(t,x)\big)\,\mathrm{d}x \right|^{p_1} \mathrm{d}t}_{=:I_3(\delta,j)} .
\end{aligned}$$

By (4.18),

$$I_1(j,\delta) \le \|\varphi\|_{L^\infty} \left\| Q_\delta - Q_{n_j,\delta} \right\|^{p_1}_{L^{p_1}([0,T];L^1(\mathbb{S}^1))} \xrightarrow{j\uparrow\infty} 0,$$

for each fixed $\delta > 0$. Next, we show that I_2 and I_3 tend to zero as $\delta \to 0$, uniformly in j. By Hölder's inequality, (4.16), and $C^\infty(\mathbb{S}^1) \hookrightarrow L^{p_2'}(\mathbb{S}^1)$

$$I_2(\delta) \le \|Q - Q_\delta\|^{p_1}_{L^{p_1}([0,T];L^{p_2}(\mathbb{S}^1))} \|\varphi\|^{p_1}_{L^{p_2'}(\mathbb{S}^1)} \xrightarrow{\delta\downarrow 0} 0,$$

uniformly in j. Finally, by (i), a basic property of the convolution product, Hölder's inequality, and $C^\infty(\mathbb{S}^1) \hookrightarrow L^{p_2'}(\mathbb{S}^1)$, it follows that

$$\begin{aligned} I_3(j,\delta) &= \int_0^T \left| \int_{\mathbb{S}^1} Q_{n_j}(t,x)\big(\varphi(x) - \varphi_\delta(x)\big)\,\mathrm{d}x \right|^{p_1} \mathrm{d}t \\ &\le \|Q_{n_j}\|^{p_1}_{L^{p_1}([0,T];L^{p_2}(\mathbb{S}^1))} \|\varphi_\delta - \varphi\|^{p_1}_{L^{p_2'}(\mathbb{S}^1)} \\ &\lesssim \|\varphi_\delta - \varphi\|^{p_1}_{L^{p_2'}(\mathbb{S}^1)} \xrightarrow{\delta\downarrow 0} 0, \quad \text{uniformly in } j. \end{aligned}$$

Summarising, to any given $\kappa > 0$, we can choose $\delta \le \delta_0$, for a small enough $\delta_0 = \delta_0(\kappa)$, such that $I_2(\delta) + I_3(\delta, j) \le \kappa/2$ for all j, and then choose an integer $j_0 = j_0(\delta_0)$ such that $j \ge j_0$ implies $I_1(j,\delta_0) \le \kappa/2$, and thus $I_\varphi(j) \le \kappa$ for all $j \ge j_0$. In other words, $I_\varphi(j) \xrightarrow{j\uparrow\infty} 0$, for any $\varphi \in C^\infty(\mathbb{S}^1)$. By density of $C^\infty(\mathbb{S}^1)$ in $L^{p_2'}(\mathbb{S}^1)$, this convergence holds for all $\varphi \in L^{p_2'}(\mathbb{S}^1)$, which concludes the proof. □

We use the previous lemma to formulate a tightness criterion in $L^{p_1}\big(L^{p_2}_w\big)$.

Lemma 4.7 *(Tightness criterion). Fix $p_1, p_2 \in (1,\infty)$ and consider the quasi-Polish space $L^{p_1}\big(L^{p_2}_w\big)$, cf. (B.1). Let $\{Q_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, defined on a standard probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, that take values in $L^{p_1}\big(L^{p_2}_w\big)$. Suppose the following conditions hold (uniformly in $n \in \mathbb{N}$):*

$$\text{(i)}\ \ \mathbb{E}\,\|Q_n\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{S}^1))} \lesssim 1,$$

$$\text{(ii)}\ \ \mathbb{E}\,\|Q_n\|_{L^{\bar p_1}([0,T];L^{1}(\mathbb{S}^1))} \lesssim 1, \quad \textit{for some } \bar p_1 > p_1,$$

and, for all $\varphi \in C^\infty(\mathbb{S}^1)$ and $\vartheta \in \big(0, T \wedge 1\big)$,

$$\text{(iii)}\ \ \mathbb{E} \sup_{\tau\in(0,\vartheta)} \int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi(x)\big(Q_n(t+\tau,x) - Q_n(t,x)\big)\,\mathrm{d}x \right| \mathrm{d}t \le C_\varphi \vartheta^\alpha,$$

for some $\alpha \in (0,1)$ and a constant C_φ independent of n. Then the sequence $\{\mathcal{L}(Q_n)\}_{n\in\mathbb{N}}$ of probability laws is tight on $L^{p_1}\big(L^{p_2}_w\big)$.

Proof. We will verify the tightness of the laws $\mu_n = \mathcal{L}(Q_n)$ on $L^{p_1}\big(L^{p_2}_w\big)$ by using Lemma 4.5 to produce, for each $\kappa > 0$, a compact set $\mathcal{K}_\kappa$ in $L^{p_1}\big(L^{p_2}_w\big)$ such that $\mu_n\big(\mathcal{K}_\kappa^c\big) = \mathbb{P}\big(X_n \in \mathcal{K}_\kappa^c\big) \le \kappa$, uniformly in n.

For arbitrary sequences $\{b_k\}_{k\in\mathbb{N}}$, $\{\nu_l\}_{l\in\mathbb{N}}$, $\{\vartheta_l\}_{l\in\mathbb{N}}$ of positive numbers, with $\nu_l, \vartheta_l \to 0$ as $l \to \infty$, and an arbitrary function sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ that is dense in $C^\infty(\mathbb{S}^1)$ (for the uniform topology), we introduce the set

$$K_a = \bigcap_{k=1}^{\infty} \Bigg\{ Q \in L^{p_1}\big(L^{p_2}_w\big) : \|Q\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{S}^1))} + \|Q\|_{L^{\bar{p}_1}([0,T];L^1(\mathbb{S}^1))}$$
$$+ \sup_{l\in\mathbb{N}} \frac{1}{\nu_l} \sup_{\tau\in(0,\vartheta_l)} \int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi_k(x)\big(Q(t+\tau,x) - Q(t,x)\big)\,\mathrm{d}x \right| \mathrm{d}t \le a\, b_k \Bigg\},$$

for $a > 0$. By Lemma 4.5, the set K_a is relatively compact in $L^{p_1}\big(L^{p_2}_w\big)$, for each $a > 0$. By the Chebyshev inequality and the assumptions (i), (ii) and (iii),

$$\mathbb{P}\big(Q_n \in \mathcal{K}^c_a\big) \le \sum_{k=1}^{\infty} \frac{3}{a\, b_k} \mathbb{E}\, \|Q_n\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{S}^1))} + \sum_{k=1}^{\infty} \frac{3}{a\, b_k} \mathbb{E}\, \|Q_n\|_{L^{\bar{p}_1}([0,T];L^1(\mathbb{S}^1))}$$
$$+ \sum_{k,l=1}^{\infty} \frac{3}{a\, b_k\, \nu_l} \mathbb{E} \sup_{\tau\in(0,\vartheta_l)} \int_0^{T-\tau} \left| \int_{\mathbb{S}^1} \varphi_k\big(Q_n(t+\tau,x) - Q_n(t,x)\big)\,\mathrm{d}x \right| \mathrm{d}t$$
$$\le \frac{C_1}{a} \sum_{k=1}^{\infty} \frac{1}{b_k} + \frac{1}{a} \left(\sum_{k=1}^{\infty} \frac{C_{\varphi_k}}{b_k} \right) \left(\sum_{l=1}^{\infty} \frac{\vartheta_l^\alpha}{\nu_l} \right),$$

for some n-independent constant C_1.

Particularising $b_k = 2^{k+1} \max\big(C_1, C_{\varphi_k}\big)$, and $\tau_l, \vartheta_l \to 0$ with $\nu_l = \vartheta_l^\alpha\, 2^l$, we obtain $\mathbb{P}\big(Q_n \in \mathcal{K}^c_a\big) < \frac{1}{a}$, which can be made $\le \kappa$ by taking a large. As a result, we can specify the required compact $\mathcal{K}_\kappa$ as the closure of K_a, for some $a = a(\kappa)$, such that $\mu_n\big(\mathcal{K}^c_\kappa\big) \le \kappa$. □

4.4. *Tightness and a.s. representations*

We are now in a position to verify the crucial tightness property of X_n.

Lemma 4.8 *(Tightness). Consider the random variables $X_n : \big(\Omega, \mathcal{F}, \mathbb{P}\big) \to \big(\mathcal{X}, \mathcal{B}_{\mathcal{X}}\big)$ defined by (4.11) and (4.1), (4.8), (4.10), (4.12), (4.13), (4.14). Then the sequence $\{\mu_n = \mathcal{L}(X_n)\}_{n\in\mathbb{N}}$ of joint laws is tight, as probability measures defined on $\big(\mathcal{X}, \mathcal{B}_{\mathcal{X}}\big)$.*

Proof. For each $\kappa > 0$, we must produce a compact set $\mathcal{K}_\kappa \subset \mathcal{X}$ such that

$$\mu_n\big(\mathcal{K}_\kappa\big) > 1 - \kappa \quad \Longleftrightarrow \quad \mu_n\big(\mathcal{K}^c_\kappa\big) = \mathbb{P}\big(X_n \in \mathcal{K}^c_\kappa\big) \le \kappa, \tag{4.19}$$

uniformly in n. By Tychonoff's theorem, the tightness of the joint laws on $\mathcal{X}$ follows from the tightness of the product measures $\bigotimes_{l\in\mathbb{N}} \mu_n^{(l)}$ on $\mathcal{X}$ (with the product σ-algebra). In other words, to prove (4.19) it is sufficient to find compact sets $\mathcal{K}_{l,\kappa} \subset \mathcal{X}_l$ such that $\mu_n^{(l)}\big(\mathcal{K}^c_{l,\kappa}\big) \le \kappa$, for arbitrary $\kappa > 0$, for each $l \in \mathbb{N}$.

As most of the cases can be treated similarly, we will carry out the tightness analysis of μ_n^ξ only for

$$\xi = u,\ q,\ q^2,\ u_0,\ W,\ S_\ell(q_+),\ S_\ell(q_+)'q,\ S_\ell(q_+)q,$$

thereby making up for each path space in (4.13) at least once, see also (4.1), (4.8), (4.10), and (4.12).

First, we verify the tightness on $\mathcal{X}_u = C_t L^2_x$ of the laws μ^u_n of u_{ε_n} using the $L^{p_0}_\omega L^\infty_t H^1_x$ bound (2.4) and the Hölder continuity estimate (2.5). It is enough to verify tightness via relative compactness. For $a > 0$, set

$$K(a) = \Big\{ f \in C([0,T]; L^2(\mathbb{S}^1)) \bigcap L^\infty([0,T]; H^1(\mathbb{S}^1)) : \\ \|f\|_{L^\infty([0,T];H^1(\mathbb{S}^1))} + \|f\|_{C^\theta([0,T];L^2(\mathbb{S}^1))} \le a \Big\}, \tag{4.20}$$

where $\theta \in (0, 1/4)$ is fixed (and constrained by (ii) of Theorem 2.3). By the Arzelà–Ascoli theorem [58, Lemma 1], $K(a)$ is relatively compact in $C([0,T]; L^2(\mathbb{S}^1))$. By the Chebyshev inequality,

$$\mathbb{P}\big(u_{\varepsilon_n} \in K(a)^c\big) < \frac{1}{a}\mathbb{E}\,\big\|u_{\varepsilon_n}\big\|_{L^\infty([0,T];H^1(\mathbb{S}^1))} \\ + \frac{1}{a}\mathbb{E}\,\big\|u_{\varepsilon_n}\big\|_{C^\theta([0,T];L^2(\mathbb{S}^1))} \overset{(2.4),(2.5)}{\lesssim} \frac{1}{a},$$

which can be made $\le \kappa$ by taking a large. Hence, we can specify the required compact $\mathcal{K}_{u,\kappa}$ as the closure of $K(a)$, for some $a = a(\kappa)$, such that $\mu^u_n\big(\mathcal{K}^c_{u,\kappa}\big) \le \kappa$.

Second, we consider μ^ξ_n, $\xi = S_\ell(q_+)q$, $\ell \in \mathbb{N}$. By (4.2), $|S_\ell(v_+)v| \lesssim_\ell |v|^2$. In view of Proposition 3.2,

$$\mathbb{E}\,\big\|S_\ell\big((q_{\varepsilon_n})_+\big)q_{\varepsilon_n}\big\|^r_{L^r([0,T]\times\mathbb{S}^1)} \lesssim 1, \quad \ell \in \mathbb{N}, \tag{4.21}$$

where the integrability index r appears in (4.13). Let a be a positive number and consider the set

$$K(a) = \Big\{ f \in L^r([0,T]\times\mathbb{S}^1) : \|f\|^r_{L^r([0,T]\times\mathbb{S}^1))} \le a \Big\}.$$

By the Banach–Alaoglu theorem and reflexivity of $L^r_{t,x}$, $K(a)$ is a compact subset of $\mathcal{X}_\xi = L^r_{t,x} - w$, $\xi = S_\ell(q_+)q$, cf. (4.13). By Chebyshev's inequality and (4.21),

$$\mathbb{P}\Big(S_\ell\big((q_{\varepsilon_n})_+\big)q_{\varepsilon_n} \in K(a)^c\Big) < \frac{1}{a}\mathbb{E}\,\big\|S_\ell\big((q_{\varepsilon_n})_+\big)q_{\varepsilon_n}\big)\big\|^r_{L^r([0,T]\times\mathbb{S}^1)} \lesssim \frac{1}{a},$$

which can be made $\le \kappa$ for large a. Thus, we pick $K(a)$, for some $a = a(\kappa)$, as the required compact $\mathcal{K}_{\xi,\kappa}$ for which $\mu^\xi_n\big(\mathcal{K}^c_{\xi,\kappa}\big) \le \kappa$, for $\xi = S_\ell(q_+)q$, $\ell \in \mathbb{N}$. Similarly, we can construct a compact subset $\mathcal{K}_{q,\kappa}$ of $\mathcal{X}_q = L^{2r}_{t,x} - w$ such that $\mu^q_n\big(\mathcal{K}^c_{q,\kappa}\big) \le \kappa$.

Since the law of W is tight as a Radon measure on the (Polish) space $C([0,T])$, there is a compact subset $\mathcal{K}_{W,\kappa}$ of $C([0,T])$ such that $\mu^W_n\big(\mathcal{K}^c_{W,\kappa}\big) \le \kappa$.

By the hypothesis (4.12), $\mathbb{E}\,\|z_n - u_0\|^{p_0}_{H^1(\mathbb{S}^1)} \xrightarrow{n\uparrow\infty} 0$. Therefore, by Chebyshev's inequality, we deduce the tightness of the laws of z_n, that is, there exists a compact set $\mathcal{K}_{u_0,\kappa}$ in the space $H^1(\mathbb{S}^1)$ such that $\mu^{u_0}_n\big(\mathcal{K}^c_{u_0,\kappa}\big) \le \kappa$.

Next, let us consider the tightness of $\mu_n^{q^2}$ on the "strong in t and weak in x" path space $\mathcal{X}_{q^2} = L^r\left(L^r_w\right)$. We will apply Lemma 4.7 to $Q_n = q_{\varepsilon_n}^2$ with $p_1 = p_2 = r$, recalling that $r < 3/2$ is fixed in (4.13). Note that the first condition (i) of the lemma is satisfied by the higher integrability estimate (3.6). To verify (ii), we use the estimate (2.4) as follows:

$$\left(\mathbb{E}\,\|Q_n\|_{L^p([0,T];L^1(\mathbb{S}^1))}\right)^p = \mathbb{E}\int_0^T \|Q_n(t)\|^p_{L^1(\mathbb{S}^1)}\,\mathrm{d}t$$

$$= \mathbb{E}\int_0^T \left\|q_{\varepsilon_n}(t)\right\|^{2p}_{L^2(\mathbb{S}^1)}\,\mathrm{d}t \lesssim_T \mathbb{E}\left\|q_{\varepsilon_n}\right\|^{2p}_{L^\infty([0,T];L^2(\mathbb{S}^1))} \lesssim 1,$$

for any $p \in \left[1, p_0/2\right]$, where $p_0 > 4$ is fixed in Theorem 1.1. The final condition (iii) is satisfied by Proposition 3.3 with $S(v) = v^2$.

Similarly, applying Lemma 4.7 to $Q_n = S_\ell\big((q_{\varepsilon_n})_+\big)$ with $p_1 = p_2 = 2r$, we deduce the tightness of $\mu_n^{S_\ell(q_+)}$ on $\mathcal{X}_{S_\ell(q_+)} = L^{2r}\left(L^{2r}_w\right)$. The first condition (i) is satisfied by the higher integrability estimate (3.6), recalling that $\left|S_\ell\big((q_{\varepsilon_n})_+\big)\right| \lesssim_\ell \left|q_{\varepsilon_n}\right|$, cf. (4.2). To verify (ii), note that

$$\left(\mathbb{E}\,\|Q_n\|_{L^p([0,T];L^1(\mathbb{S}^1))}\right)^p \lesssim_\ell \mathbb{E}\int_0^T \left\|q_{\varepsilon_n}(t)\right\|^p_{L^1(\mathbb{S}^1)}\,\mathrm{d}t$$

$$\lesssim_T \mathbb{E}\left\|q_{\varepsilon_n}\right\|^p_{L^\infty([0,T];L^2(\mathbb{S}^1))} \lesssim 1,$$

for any $p \in [1, p_0]$ (keep in mind that $p_0 > 4$ and $2r < 3$). The condition (iii) is satisfied by Proposition 3.3, which can be applied because $v \mapsto S_\ell(v_+) \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, see (4.6). Likewise, we can apply Lemma 4.7 to $Q_n = S_\ell\big((q_{\varepsilon_n})_+\big)' q_{\varepsilon_n}$ (still with $p_1 = p_2 = 2r$), to deduce the tightness of $\mu_n^{S_\ell(q_+)'q}$ on $\mathcal{X}_{S_\ell(q_+)'q} = L^{2r}\left(L^{2r}_w\right)$. Here, note carefully that Proposition 3.3 applies owing to the fact that the map $v \mapsto S_\ell(v_+)'v$ belongs to $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, see (4.7). □

Given Lemma 4.8 (tightness), the following theorem is an immediate consequence of the main result of Jakubowski [46], recalled as Theorem B.12 in the appendix. We refer to [10–12,52] for the first applications of the Jakubowski theorem to SPDEs. We rely on the Jakubowski version of Skorokhod's representation theorem because of the non-metrisable weak topologies in (4.14).

Proposition 4.9 *(Skorokhod–Jakubowski representations). Fix a sequence $\{\varepsilon_n\}$ of positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$, and consider the corresponding strong H^m solutions u_{ε_n} of the viscous SPDE (1.5) with initial data $u_{\varepsilon_n}(0) = z_n$, cf. (4.12). Denote the spatial gradient by $q_{\varepsilon_n} = \partial_x u_{\varepsilon_n}$. Consider the random mappings $X_n : \left(\Omega, \mathcal{F}, \mathbb{P}\right) \to \left(\mathcal{X}, \mathcal{B}_{\mathcal{X}}\right)$ defined by (4.11) and (4.1), (4.8), (4.10), (4.12), (4.13), (4.14). There exist a new probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ and $\mathcal{X}$-valued random variables*

$$\tilde{X}_n = \left(\tilde{u}_n, \tilde{F}^q_n, \tilde{F}^{q^2}_n, \tilde{W}_n, \tilde{u}_{0,n}, \tilde{F}^{\mathbb{S}}_n\right), \quad \tilde{X} = \left(\tilde{u}, \overline{\tilde{F}^q}, \overline{\tilde{F}^{q^2}}, \tilde{W}, \tilde{u}_0, \overline{\tilde{F}^{\mathbb{S}}}\right), \tag{4.22}$$

defined on $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$*, such that along a subsequence (notationally not relabelled) the joint laws of* X_n *and* $\tilde X_n$ *coincide for all n, and* $\tilde X_n \xrightarrow{n\uparrow\infty} \tilde X$ *almost surely, in the product topology on* $\mathcal{X}$*. In the first line of* (4.22)*, we have*

$$\begin{aligned} &\tilde F^{\xi_\ell,\pm}_{\varepsilon_n} = \xi_\ell\big|_{v=\tilde q_{\varepsilon_n}}, \quad \xi_\ell \in \mathbb{S}^\pm_\ell, \quad \ell\in\mathbb{N}, \\ &\tilde F^{\mathbb{S}}_{\varepsilon_n} = \left\{\left\{\tilde F^{\xi_\ell,+}_{\varepsilon_n},\ \xi_\ell\in\mathbb{S}^+_\ell\right\}_{\ell\in\mathbb{N}}, \ \left\{\tilde F^{\xi_\ell,-}_{\varepsilon_n},\ \xi_\ell\in\mathbb{S}^-_\ell\right\}_{\ell\in\mathbb{N}}\right\}, \end{aligned} \tag{4.23}$$

where $\mathbb{S}^\pm_\ell$ *denote the collections of nonlinearities given by* (4.9)*. In the second line of* (4.22)*, the "overline" should be understood as sitting over each component of the overlined quantity; for example,* $\overline{\tilde F^q} = \left(\tilde q, \overline{\tilde q_+}, \overline{\tilde q_-}\right)$ *and* $\overline{\tilde F^{q^2}} = \left(\overline{\tilde q^2}, \overline{\tilde q^2_+}, \overline{\tilde q^2_-}\right)$*, with* $\overline{\tilde q} = \tilde q$*. More explicitly, we have the following* $\tilde{\mathbb{P}}$*–a.s. convergences:*

$$\begin{aligned} &\tilde u_n \xrightarrow{n\uparrow\infty} \tilde u \text{ in } C_t L^2_x, \quad \tilde W_n \xrightarrow{n\uparrow\infty} \tilde W \text{ in } C_t, \ \tilde u_{0,n} \xrightarrow{n\uparrow\infty} \tilde u_0 \text{ in } H^1_x, \\ &\tilde q_n \xrightharpoonup{n\uparrow\infty} \tilde q, \ (\tilde q_n)_\pm \xrightharpoonup{n\uparrow\infty} \overline{\tilde q_\pm} \text{ in } L^{2r}_{t,x}, \\ &\tilde q_n^2 \xrightarrow{n\uparrow\infty} \overline{\tilde q^2} \text{ in } L^r\left(L^r_w\right), \quad (\tilde q_n)^2_\pm \xrightharpoonup{n\uparrow\infty} \overline{\tilde q^2_\pm} \text{ in } L^r\left(L^r_w\right), \\ &S_\ell\big((\tilde q_n)_\pm\big) \xrightarrow{n\uparrow\infty} \overline{S_\ell\big(\tilde q_\pm\big)} \text{ in } L^{2r}(L^{2r}_w), \quad \ell\in\mathbb{N}, \\ &S_\ell\big((\tilde q_n)_\pm\big)'\tilde q_n \xrightarrow{n\uparrow\infty} \overline{S_\ell\big(\tilde q_\pm\big)'\tilde q} \text{ in } L^{2r}(L^{2r}_w), \quad \ell\in\mathbb{N}, \\ &S_\ell\big((\tilde q_n)_\pm\big)''\tilde q_n^2 \xrightharpoonup{n\uparrow\infty} \overline{S_\ell\big(\tilde q_\pm\big)''\tilde q^2} \text{ in } L^r_{t,x}, \quad \ell\in\mathbb{N}, \\ &S_\ell\big((\tilde q_n)_\pm\big)\tilde q_n \xrightharpoonup{n\uparrow\infty} \overline{S_\ell\big(\tilde q_\pm\big)\tilde q} \text{ in } L^r_{t,x}, \quad \ell\in\mathbb{N}, \\ &S_\ell\big((\tilde q_n)_\pm\big)'\tilde q_n^2 \xrightharpoonup{n\uparrow\infty} \overline{S_\ell\big(\tilde q_\pm\big)'\tilde q^2} \text{ in } L^r_{t,x}, \quad \ell\in\mathbb{N}, \\ &S_\ell\big((\tilde q_n)_\pm\big)' \xrightharpoonup{n\uparrow\infty} \overline{S_\ell\big(\tilde q_\pm\big)'} \text{ in } L^{2r}_{t,x}, \quad \ell\in\mathbb{N}. \end{aligned} \tag{4.24}$$

Proof. An application of Theorem B.12 supplies all the claims of the proposition, except the one that the nonlinear composition variables take the explicit form (4.23). However, this follows from Lemma B.13. □

Remark 4.10. As we shall henceforth be working in the new probability space, for brevity, we drop the tilde under the overline indicating a weak limit. For example, $\overline{\tilde F^q} = (\tilde q, \overline{q_+}, \overline{q_-})$ and $\overline{\tilde F^{q^2}} = \left(\overline{q^2}, \overline{q^2_+}, \overline{q^2_-}\right)$. Similarly, instead of $\overline{S_\ell(\tilde q_\pm)}$, we write $\overline{S_\ell(q_\pm)}$, and so forth with the other nonlinear compositions.

5. Properties of a.s. representations

The strong H^m solution u_{ε_n} of the SPDE (1.5) possesses several consequential bounds, see Theorem 2.3, Lemma 3.1 and Proposition 3.2. In this section, we wish to transfer these bounds to $\tilde u_n$ (the Skorokhod–Jakubowski representation from Proposition 4.9). At the moment, we do not have the SPDE satisfied by $\tilde{u}_n$, so we cannot derive them as before. Instead, as is often done in the literature, we will appeal to the fact that u_{ε_n} and $\tilde{u}_n$ share the same probability law and invoke the Kuratowski–Lusin–Souslin (KLS) theorem. We refer to [52, Corollary A.2] and [12, Proposition C.2] for the quasi-Polish version of this theorem (cf. Lemma B.3). The KLS theorem allows one to assert that spaces of higher integrability/regularity are Borel subsets of the postulated path spaces (4.13). The law shared by u_{ε_n} and $\tilde{u}_n$ can then be integrated against over these better function spaces to derive bounds for $\tilde{u}_n$ from those of u_{ε_n}.

Lemma 5.1 *(Spatial gradient). Let $\tilde{u}_n, \tilde{q}_n, \tilde{u}, \tilde{q}$ be the Skorokhod–Jakubowski representations from Proposition 4.9. There is an event $\tilde{\Omega}_0$, with $\tilde{\mathbb{P}}(\tilde{\Omega}_0) = 1$, such that for any $\tilde{\omega} \in \tilde{\Omega}_0$ there exist sets $E_n(\tilde{\omega}), E(\tilde{\omega}) \subset [0,T] \times \mathbb{S}^1$ of full measure on which the weak spatial derivatives of $\tilde{u}_n$, $\tilde{u}$ are $\tilde{q}_n$, $\tilde{q}$, respectively, i.e., for $\tilde{\omega} \in \tilde{\Omega}_0$,*

$$\begin{aligned} \partial_x \tilde{u}_n(\tilde{\omega}, t, x) &= \tilde{q}_n(\tilde{\omega}, t, x) \text{ for a.e. } (t,x) \in E_n(\tilde{\omega}), \\ \partial_x \tilde{u}(\tilde{\omega}, t, x) &= \tilde{q}(\tilde{\omega}, t, x) \text{ for a.e. } (t,x) \in E(\tilde{\omega}). \end{aligned} \tag{5.1}$$

Proof. We first show that $\tilde{\mathbb{P}}$–a.s., for every $\psi \in C([0,T]; C^1(\mathbb{S}^1))$,

$$-\int_0^T \int_{\mathbb{S}^1} \tilde{u}_n \partial_x \psi \, dx \, dt = \int_0^T \int_{\mathbb{S}^1} \tilde{q}_n \psi \, dx \, dt, \tag{5.2}$$

which implies the first claim in (5.1). Let $\{\psi_j\}_{j=1}^\infty$ be a countable dense subset of $C([0,T]; C^1(\mathbb{S}^1))$, and consider the continuous mappings

$$\begin{aligned} &F_j : C([0,T]; L^2(\mathbb{S}^1)) \times \big(L^{2r}([0,T] \times \mathbb{S}^1) - w\big) \to \mathbb{R}, \\ &F_j(u,q) = \int_0^T \int_{\mathbb{S}^1} u \, \partial_x \psi_j \, dx \, dt + \int_0^T \int_{\mathbb{S}^1} q \, \psi_j \, dx \, dt. \end{aligned}$$

By continuity of F_j, Remark B.11, and the equality of joint laws,

$$\tilde{\mathbb{P}}\left(\left\{F_j\left(\tilde{u}_n, \tilde{q}_n\right) = 0\right\}\right) = \mathbb{P}\left(\left\{F_j\left(u_{\varepsilon_n}, q_{\varepsilon_n}\right) = 0\right\}\right) = 1.$$

Since there are countably many pairs (n, j), there is a set $\tilde{\Omega}_0$ of full $\tilde{\mathbb{P}}$–measure such that $F_j\big(\tilde{u}_n(\tilde{\omega}), \tilde{q}_n(\tilde{\omega})\big) = 0$ for all (n, j), $\tilde{\omega} \in \tilde{\Omega}_0$. This implies (5.2).

Proposition 4.9 gives the $\tilde{\mathbb{P}}$–a.s. convergences $\tilde{u}_n \xrightarrow{n\uparrow\infty} \tilde{u}$ in $C([0,T]; L^2(\mathbb{S}^1))$ and $\tilde{q}_n \xrightarrow{n\uparrow\infty} \tilde{q}$ in $L^{2r}([0,T] \times \mathbb{S}^1)$ jointly. For a fixed j, we obtain a.s. that $\int_0^T \int_{\mathbb{S}^1} \tilde{q}_n \psi_j \, dx \, dt \xrightarrow{n\uparrow\infty} \int_0^T \int_{\mathbb{S}^1} \tilde{q} \, \psi_j \, dx \, dt$. Similarly, we have the a.s. convergence $\int_0^T \int_{\mathbb{S}^1} \tilde{u}_n \partial_x \psi_j \, dx \, dt \xrightarrow{n\uparrow\infty} \int_0^T \int_{\mathbb{S}^1} \tilde{u} \, \partial_x \psi_j \, dx \, dt$. By density, we thus arrive at

$$\int_0^T \int_{\mathbb{S}^1} \psi\, \tilde{q}\, \mathrm{d}x\, \mathrm{d}t = -\int_0^T \int_{\mathbb{S}^1} \partial_x \psi\, \tilde{u}\, \mathrm{d}x\, \mathrm{d}t, \quad \tilde{\mathbb{P}}\text{–a.s.,}$$

for any $\psi \in C([0,T]; C^1(\mathbb{S}^1))$. This establishes the second claim in (5.1). □

Lemma 5.2 *(Regularity). Let $\tilde{u}_n$, $\tilde{q}_n$ be the Skorokhod–Jakubowski representations from Proposition 4.9. Then $\tilde{u}_n \in L^2([0,T]; H^m(\mathbb{S}^1))$, $\tilde{\mathbb{P}}$–a.s., for any $m \geq 1$, and thus $\tilde{q}_n(t,x) = \partial_x \tilde{u}_n(t,x)$ pointwise in (t,x), $\tilde{\mathbb{P}}$–a.s.*

Proof. Recall that u_{ε_n} is the strong solution of the viscous SPDE (1.5) with initial data $u_{\varepsilon_n}(0) = z_n$, cf. (4.12). By Definition 2.1, u_{ε_n} belongs to $L^2([0,T]; H^m(\mathbb{S}^1))$, a.s., for any m. Since the intersection $L^2([0,T]; H^m(\mathbb{S}^1)) \cap C([0,T]; L^2(\mathbb{S}^1))$ injects continuously into the path space $C([0,T]; L^2(\mathbb{S}^1))$, cf. (4.13), under the identity map, its image under the injection is Borel in $C([0,T]; L^2(\mathbb{S}^1))$, according to the KLS theorem (cf. Lemma B.3). Therefore, by the equality of laws, also the variable $\tilde{u}_n$ belongs to $L^2([0,T]; H^m(\mathbb{S}^1))$, a.s., for any m. Since $\tilde{q}_n$ is the weak x-derivative of $\tilde{u}_n$ (cf. Lemma 5.1), and we have the inclusion $H^m(\mathbb{S}^1) \hookrightarrow C^{m-1/2}(\mathbb{S}^1)$, this x-derivative is classical. □

Lemma 5.3 *(A priori estimates). Let $p_0 > 4$ be as specified in Theorem 2.3 and $r \in [1, 3/2)$ as fixed in (4.13). Let $\tilde{u}_n, \tilde{q}_n$ be the Skorokhod–Jakubowski representations from Proposition 4.9. There exists a constant C, independent of n, such that*

$$\tilde{\mathbb{E}}\, \|\tilde{u}_n\|_{L^\infty([0,T]; H^1(\mathbb{S}^1))}^{p_0} \leq C, \quad \tilde{\mathbb{E}}\, \|\tilde{q}_n\|_{L^\infty([0,T]; L^2(\mathbb{S}^1))}^{p_0} \leq C,$$
$$\textit{and} \quad \tilde{\mathbb{E}}\, \|\tilde{q}_n\|_{L^{2r}([0,T]\times\mathbb{S}^1)}^{2r} \leq C.$$

Proof. By the continuous injection of the Polish space $\mathcal{Y} = C([0,T]; H^1(\mathbb{S}^1))$ into the path space $\mathcal{X}^u = C([0,T]; L^2(\mathbb{S}^1))$, the KLS theorem ensures that $\mathcal{Y}$ is a Borel subset of $\mathcal{X}^u$, and thus the equality of laws implies the first estimate:

$$\begin{aligned}\tilde{\mathbb{E}}\, \|\tilde{u}_n\|_{L^\infty([0,T]; H^1(\mathbb{S}^1))}^{p_0} &= \int_{\mathcal{Y}} \|v\|_{L^\infty([0,T]; H^1(\mathbb{S}^1))}^{p_0}\, \mathrm{d}\mu_n^u(v) \\ &= \mathbb{E}\, \left\|u_{\varepsilon_n}\right\|_{L^\infty([0,T]; H^1(\mathbb{S}^1))}^{p_0} \overset{(2.4)}{\lesssim} 1,\end{aligned}$$

recalling that μ_n^u denotes the law of u_{ε_n}, cf. Section 4.

The second estimate is a consequence of the first and Lemmas 5.1, 5.2. Since the injection $L^{2r}([0,T] \times \mathbb{S}^1) \hookrightarrow L^{2r}([0,T] \times \mathbb{S}^1) - w$ is continuous, we can use the KLS theorem on quasi-Polish spaces (Lemma B.3) and the equality of laws to deduce the third estimate: $\tilde{\mathbb{E}}\, \|\tilde{q}_n\|_{L^{2r}([0,T]\times\mathbb{S}^1)}^{2r} = \mathbb{E}\, \left\|q_{\varepsilon_n}\right\|_{L^{2r}([0,T]\times\mathbb{S}^1)}^{2r} \overset{(3.6)}{\lesssim} 1$. □

By the a.s. convergence (4.24) and a weak compactness argument (in ω, t, x), it follows that the limit $\tilde{q} = \partial_x \tilde{u}$ continues to satisfy the third estimate of Lemma 5.3. Because of non-reflexivity, the other (L^∞) estimates are more delicate. We have the following result:

Lemma 5.4 *(A priori estimates for limits). Let $\tilde{u}$, $\tilde{q}$ be the a.s. limits from Proposition 4.9, $p_0 > 4$ be as specified in Theorem 2.3, and $r \in [1, 3/2)$ as fixed in (4.13). There exists a constant C such that*

$$\tilde{\mathbb{E}}\, \|\tilde{u}\|^{p_0}_{L^\infty([0,T];H^1(\mathbb{S}^1))} \le C. \tag{5.3}$$

Besides, $\tilde{u} \in C\big([0,T]; H^1(\mathbb{S}^1) - w\big)$, $\tilde{\mathbb{P}}$–almost surely. Finally,

$$\tilde{\mathbb{E}}\, \|\tilde{q}\|^{p_0}_{L^\infty([0,T];L^2(\mathbb{S}^1))} \le C, \quad \tilde{\mathbb{E}}\, \|\tilde{q}\|^{2r}_{L^{2r}([0,T]\times\mathbb{S}^1)} \le C. \tag{5.4}$$

Proof. The first part of (5.4) comes from (5.3) and Lemma 5.1. The second part is a corollary of the corresponding estimate in Lemma 5.3 and the considerations given before Lemma 5.4.

The rest of the proof is devoted to (5.3) and the claim about weak time-continuity. By Lemma 5.3, $\tilde{\mathbb{E}}\, \|\tilde{u}_n\|^{p_0}_{L^{\bar r}([0,T];H^1(\mathbb{S}^1))} \le C T^{p_0/\bar r}$, for any $\bar r \in [1,\infty)$. In other words, $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset_b L^{p_0}\big(\tilde\Omega; L^{\bar r}([0,T]; H^1(\mathbb{S}^1))\big)$ for any finite $\bar r$. By standard duality theory in Lebesgue–Bochner spaces (see, e.g., [27, page 98, Theorem 1]),

$$\Big(L^{p_0}\big(\tilde\Omega; L^{\bar r}([0,T]; H^1(\mathbb{S}^1))\big)\Big)^* = L^{p_0'}\big(\tilde\Omega; L^{\bar r'}([0,T]; H^{-1}(\mathbb{S}^1))\big),$$

for any $\bar r \in (1,\infty)$, $\bar r' = \frac{\bar r}{\bar r - 1}$, $p_0' = \frac{p_0}{p_0-1}$. The space $L^{p_0}\big(\tilde\Omega; L^{\bar r}([0,T]; H^1(\mathbb{S}^1))\big)$ is reflexive. Thus, by Kakutani's theorem on reflexive spaces, up to subsequences,

$$\tilde{u}_n \xrightharpoonup{n\uparrow\infty} v^{(\bar r)} \quad \text{in } L^{p_0}\big(\tilde\Omega; L^{\bar r}([0,T]; H^1(\mathbb{S}^1))\big), \tag{5.5}$$

for $\bar r \in (1,\infty)$, where the limit $v^{(\bar r)}$ depends possibly on $\bar r$. Besides,

$$\tilde{\mathbb{E}}\, \left\|v^{(\bar r)}\right\|^{p_0}_{L^{\bar r}([0,T];H^1(\mathbb{S}^1))} \le \liminf_{n\to\infty} \tilde{\mathbb{E}}\, \|\tilde{u}_n\|^{p_0}_{L^{\bar r}([0,T];H^1(\mathbb{S}^1))} \le C T^{p_0/\bar r}.$$

The continuity of the embedding

$$L^{p_0}\big(\tilde\Omega; L^{r_2}([0,T]; H^1(\mathbb{S}^1))\big) \hookrightarrow L^{p_0}\big(\tilde\Omega; L^{r_1}([0,T]; H^1(\mathbb{S}^1))\big),$$

for $1 < r_1 < r_2 < \infty$, implies that $v^{(\bar r)}$ does not, in fact, depend on $\bar r$. Therefore, we will write v instead of $v^{(\bar r)}$ in the following.

By the monotone convergence theorem,

$$\tilde{\mathbb{E}} \lim_{\bar r\to\infty} \|v\|^{p_0}_{L^{\bar r}([0,T];H^1(\mathbb{S}^1))} \le C.$$

Since the $L^{\bar r}_t$ norm depends continuously on the index $\bar r$ for any measurable function $f : [0,T] \to H^1(\mathbb{S}^1)$ for which $\lim_{\bar r\to\infty} \left(\int_0^T \|f(t)\|^{\bar r}_{H^1(\mathbb{S}^1)}\, \mathrm{d}t\right)^{1/\bar r} < \infty$, it follows that

$$\tilde{\mathbb{E}}\, \|v\|^{p_0}_{L^\infty([0,T];H^1(\mathbb{S}^1))} \le C \quad \text{and} \quad v \in L^\infty([0,T]; H^1(\mathbb{S}^1)),\ \tilde{\mathbb{P}}\text{-a.s.} \tag{5.6}$$

It remains to identify v with the $\tilde{\mathbb{P}}$–almost sure Skorokhod–Jakubowski limit $\tilde{u}$ in $C([0,T]; L^2(\mathbb{S}^1))$, see (4.24). Consider the following test functions:

$$\phi(\tilde{\omega},t,x)=\psi(\tilde{\omega})\vartheta(t,x), \quad \psi\in L^\infty(\tilde{\Omega}),\ \vartheta\in L^{\bar{r}'}([0,T];L^2(\mathbb{S}^1)). \tag{5.7}$$

From (5.5),

$$\tilde{\mathbb{E}}\left(\psi\int_0^T\int_{\mathbb{S}^1}\vartheta(t,x)\big(\tilde{u}_n(t,x)-v(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\right)\xrightarrow{n\uparrow\infty}0.$$

On the other hand, by (4.24),

$$\psi\int_0^T\int_{\mathbb{S}^1}\vartheta(t,x)\big(\tilde{u}_n(t,x)-\tilde{u}(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\xrightarrow{n\uparrow\infty}0,\quad \tilde{\mathbb{P}}\text{–a.s.}$$

By Lemma 5.3, we have the moment bound

$$\begin{aligned}&\tilde{\mathbb{E}}\left|\psi\int_0^T\int_{\mathbb{S}^1}\tilde{u}_n(\tilde{\omega},t,x)\vartheta(t,x)\,\mathrm{d}x\,\mathrm{d}t\right|^{p_0}\\&\le\|\psi\|_{L^\infty(\tilde{\Omega})}^{p_0}\|\vartheta\|_{L^1([0,T];L^2(\mathbb{S}^1))}^{p_0}\tilde{\mathbb{E}}\,\|\tilde{u}_n\|_{L^\infty([0,T];H^1(\mathbb{S}^1))}^{p_0}\le C\big(\psi,\vartheta\big),\end{aligned}$$

and so, by Vitali's convergence theorem,

$$\tilde{\mathbb{E}}\left|\psi\int_0^T\int_{\mathbb{S}^1}\vartheta(t,x)\big(\tilde{u}_n(t,x)-\tilde{u}(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\right|^{p}\xrightarrow{n\uparrow\infty}0,$$

for any $1\le p<p_0$. Consequently,

$$\tilde{\mathbb{E}}\left(\psi\int_0^T\int_{\mathbb{S}^1}\vartheta(t,x)\big(\tilde{u}(t,x)-v(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\right)=0, \tag{5.8}$$

for ψ, ϑ as in (5.7).

We use $I_z(\vartheta)$ as short-hand for $\int_0^T\int_{\mathbb{S}^1}\vartheta(t,x)z(\cdot,t,x)\,\mathrm{d}x\,\mathrm{d}t$, where $z=\tilde{u},v$. Clearly, by (5.6), $I_v(\vartheta)\in L^p(\tilde{\Omega})$, for any $1\le p<p_0$ Since (5.8) implies that $I_{\tilde{u}}(\vartheta)=I_v(\vartheta)$, almost surely, it follows that also $I_{\tilde{u}}(\vartheta)\in L^p(\tilde{\Omega})$, for each fixed ϑ. We conclude that for any $\vartheta\in L^{\bar{r}'}([0,T];L^2(\mathbb{S}^1))$, with $1<\bar{r}'<\infty$, there exists a full $\tilde{\mathbb{P}}$–measure set $\tilde{\Omega}_\vartheta$ on which $I_{\tilde{u}}(\vartheta)=I_v(\vartheta)$. By separability of $L^{\bar{r}'}([0,T];L^2(\mathbb{S}^1))$, we deduce that for any $1<\bar{r}'<\infty$ there exists a full $\tilde{\mathbb{P}}$–measure set on which the identity $I_{\tilde{u}}(\vartheta)=I_v(\vartheta)$ holds for all $\vartheta\in L^{\bar{r}'}([0,T];L^2(\mathbb{S}^1))$. We can take this set to be the countable intersection of $\tilde{\Omega}_\vartheta$ associated with a countable dense subset of ϑ in $L^{r'}([0,T];L^2(\mathbb{S}^1))$. This shows that $\tilde{u}=v$, $\tilde{\mathbb{P}}\otimes \mathrm{d}t\otimes \mathrm{d}x$–almost everywhere. We also have (5.6) for $\tilde{u}=v$, thereby concluding the proof of (5.3).

Finally, let us prove the claim that $\tilde{u}$ is weakly time-continuous. By Lemma 4.8 (tightness), see also (4.20), for any $L\in\mathbb{N}$ there exists an $a_L>0$ such that $\inf_{n\in\mathbb{N}}\mathbb{P}\big(\big\{u_{\varepsilon_n}\in K(a_L)\big\}\big)>1-1/L$. Thus, by the equality of laws,

$$\inf_{n\in\mathbb{N}}\tilde{\mathbb{P}}\big(\{\tilde{u}_n\in K(a_L)\}\big)>1-1/L.$$

Pick an arbitrary subsequence $\{n_j\}_{j\in\mathbb{N}}$ and set $A_{j,L}=\big\{\tilde{u}_{n_j}\in K(a_L)\big\}$. Then $\liminf_j \tilde{\mathbb{P}}\left(A_{j,L}\right)>1-1/L$ and so

$$\tilde{\mathbb{P}}\left(\limsup_j A_{j,L}\right)>1-1/L,$$

where $\limsup_j A_{j,L}=\bigcap_{J=1}^{\infty}\bigcup_{j>J}A_{j,L}$. Introduce the $C_tL_x^2$ convergence set

$$F=\left\{\tilde{\omega}\in\tilde{\Omega}:\tilde{u}_n(\tilde{\omega})\xrightarrow{n\uparrow\infty}\tilde{u}(\tilde{\omega})\text{ in }C_tL_x^2\right\}.$$

By the first part of (4.24), $\tilde{\mathbb{P}}(F)=1$. Therefore,

$$\tilde{\mathbb{P}}\left(F\cap\limsup_j A_{j,L}\right)>1-1/L.$$

Select an arbitrary $\tilde{\omega}_0\in F\cap\limsup_j A_{j,L}$. By construction, there is a subsequence $\{n_{j_k}\}_{k\in\mathbb{N}}$ (depending on $\tilde{\omega}_0$) such that $\tilde{u}_{n_{j_k}}(\tilde{\omega}_0)\in K(a_L)$ for all $k\in\mathbb{N}$. Besides, we have $\tilde{u}_{n_j}(\tilde{\omega}_0)\xrightarrow{j\uparrow\infty}\tilde{u}(\tilde{\omega}_0)$ in $C_tL_x^2$. This implies that $\tilde{u}_{n_{j_k}}(\tilde{\omega}_0)\xrightarrow{k\uparrow\infty}\tilde{u}(\tilde{\omega}_0)$ in $C_tL_x^2$ and whence $\tilde{u}_{n_{j_k}}(\tilde{\omega}_0)\xrightarrow{k\uparrow\infty}\tilde{u}(\tilde{\omega}_0)$ in $C_tH_x^1-w$. Since $\{n_j\}_{j\in\mathbb{N}}$ was arbitrary, and $K(a_L)$ is metrisable in $C_tH_x^1-w$, this leads us to conclude that

$$\tilde{u}_n(\tilde{\omega}_0)\xrightarrow{n\uparrow\infty}\tilde{u}(\tilde{\omega}_0)\text{ in }C_tH_x^1-w\quad\text{and}\quad\tilde{u}(\tilde{\omega}_0)\in K(a_L).$$

In other words, the convergence set

$$M_L=\left\{\tilde{\omega}\in\tilde{\Omega}:\tilde{u}_n(\tilde{\omega})\xrightarrow{n\uparrow\infty}\tilde{u}(\tilde{\omega})\text{ in }C_tH_x^1-w,\ \tilde{u}(\tilde{\omega})\in K(a_L)\right\}$$

satisfies $M_L\supseteq F\cap\limsup_j A_{j,L}$, and so $\tilde{\mathbb{P}}(M_L)>1-1/L$, for any $L\in\mathbb{N}$. Set

$$M=\left\{\tilde{\omega}\in\tilde{\Omega}:\tilde{u}_n(\tilde{\omega})\xrightarrow{n\uparrow\infty}\tilde{u}(\tilde{\omega})\text{ in }C_tH_x^1-w\right\}.$$

Then $M \supseteq M_L \supseteq F \cap \limsup_j A_{j,L} \in \tilde{\mathcal{F}}$, for any $L \in \mathbb{N}$, and therefore we obtain $M \supseteq \cup_{L=1}^{\infty}\big(F \cap \limsup_j A_{j,L}\big) \in \tilde{\mathcal{F}}$. This implies that

$$\tilde{\mathbb{P}}\left(\cup_{L=1}^{\infty}\big(F \cap \limsup_j A_{j,L}\big)\right) \geq \tilde{\mathbb{P}}\left(F \cap \limsup_j A_{j,L}\right) > 1 - 1/L,$$

for each $L \in \mathbb{N}$, i.e., $\tilde{\mathbb{P}}\left(\cup_{L=1}^{\infty}\big(F \cap \limsup_j A_{j,L}\big)\right) = 1$. By the completeness of $\big(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\big)$, it follows that $M \in \tilde{\mathcal{F}}$ and $\tilde{\mathbb{P}}(M) = 1$; thus the second claim of the lemma follows: $\tilde{u} \in C_t H_x^1 - w$, $\tilde{\mathbb{P}}$–a.s. □

Remark 5.5. The use of the path space $\mathcal{X}_{u,\text{new}} = C([0,T]; H^1(\mathbb{S}^1) - w)$ instead of $\mathcal{X}_u = C([0,T]; L^2(\mathbb{S}^1))$ could have simplified some of the work in Lemma 5.4 and other places. However, we would have needed to provide additional steps to establish the tightness on $\mathcal{X}_{u,\text{new}}$. It is also possible to operate with two different path spaces for u, each reflecting different topologies. However, this approach would necessitate additional steps to ensure that the two Skorokhod–Jakubowski representations of u_n (and their limits) in these path spaces can be properly identified. While using $\mathcal{X}_{u,\text{new}}$ instead of $\mathcal{X}_u$ might have simplified certain aspects of the analysis, there is a trade-off between simplicity and the added complexity in establishing the tightness and identifying the limits in the chosen path spaces. Therefore, we have opted to use the space $\mathcal{X}_u$.

The next result is a product of Lemmas 5.3 and 5.4, see also Lemma 3.1.

Lemma 5.6 *(Additional a priori estimates). Let $\tilde{u}_n$, $\tilde{u}$, and $\overline{q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9, see also Remark 4.10. There exists a constant C, independent of n, such that $\mathbb{E}\,\|\tilde{u}_n\|_{L^\infty([0,T]\times\mathbb{S}^1)}^{p_0} \leq C$, where $p_0 > 4$ is fixed in Theorem 2.3, and $\mathbb{E}\,\|P[\tilde{u}_n]\|_{L^\infty([0,T]\times\mathbb{S}^1)}^2 \leq C$, where $P[\cdot]$ is defined in (1.5). In particular, we have*

$$\tilde{\mathbb{E}}\left\|K * \left(\tilde{u}_n^2 + \tfrac{1}{2}\tilde{q}_n^2\right)\right\|_{L^\infty([0,T]\times\mathbb{S}^1)}^p \leq C, \quad p \in \big[1, p_0/2\big].$$

The same bounds hold with $\tilde{u}_n$ replaced by its a.s. limit $\tilde{u}$.

The final lemma of this section collects some integrability estimates (in $\tilde{\omega}, t, x$) for the a.s. weak limit $\overline{q^2}$. The second estimate will play a role in upcoming discussions about the martingale property of a stochastic integral and the weak convergence of some specific product terms.

Lemma 5.7 *(Additional a priori estimates for limits). Let $\overline{q^2} = \overline{q^2}(\tilde{\omega}, t, x)$ be the Skorokhod–Jakubowski representation from Proposition 4.9. There is a constant C such that*

$$\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1}\left|\overline{q^2}\right|^r \mathrm{d}x\,\mathrm{d}t \leq C, \quad \tilde{\mathbb{E}}\int_0^T\left\|\overline{q^2}(t)\right\|_{H^{-1}(\mathbb{S}^1)}^p \mathrm{d}t \leq C, \tag{5.9}$$

for any $p \in [1, p_0/2]$, *where* $p_0 > 4$ *and* $r \in [1, 3/2)$ *are specified in Theorem 2.3 and (4.13), respectively. Furthermore,*

$$\tilde{\mathbb{E}} \int_0^T \left\| K * \left(\tilde{u}^2 + \tfrac{1}{2}\overline{q^2} \right)(t) \right\|_{L^\infty(\mathbb{S}^1)}^p \mathrm{d}t \le C, \quad p \in [1, p_0/2]. \tag{5.10}$$

Proof. By Lemma 5.3, $\{\tilde{q}_n^2\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^r(\tilde{\Omega} \times [0,T] \times \mathbb{S}^1)$. Thus, by a weak compactness argument and (4.24), we may assume that

$$\overline{q^2} \in L^r(\tilde{\Omega} \times [0,T] \times \mathbb{S}^1) \quad \text{and} \quad \tilde{q}_n^2 \xrightharpoonup{n\uparrow\infty} \overline{q^2} \quad \text{in } L^r(\tilde{\Omega} \times [0,T] \times \mathbb{S}^1), \tag{5.11}$$

which implies that the first part of (5.9) holds.

Next, by (4.24), $\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2}$ in $L^r\left(L^r_w\right)$ a.s. Since $L^r_w(\mathbb{S}^1) \hookrightarrow H^{-1}(\mathbb{S}^1)$, this also implies the convergence

$$\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2} \quad \text{in } L^r_t\left(H^{-1}_x\right) = L^r([0,T]; H^{-1}(\mathbb{S}^1)), \text{ a.s.} \tag{5.12}$$

We can use Lemma 5.3 to deduce the n-uniform bound

$$\begin{aligned} \left\| \tilde{q}_n^2 \right\|_{L^p(\tilde{\Omega}\times[0,T];H^{-1}(\mathbb{S}^1))}^p &= \tilde{\mathbb{E}} \int_0^T \left\| \tilde{q}_n^2(t) \right\|_{H^{-1}(\mathbb{S}^1)}^p \mathrm{d}t \\ &= \tilde{\mathbb{E}} \int_0^T \sup_\varphi \left| \int_{\mathbb{S}^1} \tilde{q}_n^2 \, \varphi \, \mathrm{d}x \right|^p \mathrm{d}t \lesssim_T \tilde{\mathbb{E}} \, \|\tilde{q}_n\|_{L^\infty([0,T];L^2(\mathbb{S}^1))}^{2p} \mathrm{d}t \lesssim 1, \end{aligned} \tag{5.13}$$

for any $p \in [1, p_0/2]$. Here, the supremum runs over all $\varphi \in H^1(\mathbb{S}^1)$ for which $\|\varphi\|_{H^1(\mathbb{S}^1)} \le 1$. Since $H^1(\mathbb{S}^1) \hookrightarrow L^\infty(\mathbb{S}^1)$, we have used that $\|\varphi\|_{L^\infty(\mathbb{S}^1)} \lesssim 1$ for such functions. In other words, the sequence $\{\tilde{q}_n^2\}_{n\in\mathbb{N}}$ is bounded in the reflexive Banach space

$$L^p_{\tilde{\omega},t}\left(H^{-1}_x\right) = L^p\left(\tilde{\Omega} \times [0,T]; H^{-1}(\mathbb{S}^1)\right).$$

Hence, by a weak compactness argument and (4.24), we may assume that

$$\overline{q^2} \in L^p_{\tilde{\omega},t}\left(H^{-1}_x\right) \quad \text{and} \quad \tilde{q}_n^2 \xrightharpoonup{n\uparrow\infty} \overline{q^2} \quad \text{in } L^p_{\tilde{\omega},t}\left(H^{-1}_x\right), \quad p \in [1, p_0/2]. \tag{5.14}$$

This implies the last part of (5.9).

Finally, we prove (5.10). In view of Lemma 5.6, it is enough to consider the $\overline{q^2}$ part of (5.10). Noting that

$$\left| K * \overline{q^2}(t,x) \right| \le \|K(x - \cdot)\|_{H^1(\mathbb{S}^1)} \left\| \overline{q^2}(t) \right\|_{H^{-1}(\mathbb{S}^1)} \lesssim \left\| \overline{q^2}(t) \right\|_{H^{-1}(\mathbb{S}^1)},$$

we arrive at

$$\tilde{\mathbb{E}}\int_0^T \left\| K * \overline{q^2}(t)\right\|_{L^\infty(\mathbb{S}^1)}^p \,\mathrm{d}t \lesssim \tilde{\mathbb{E}}\int_0^T \left\|\overline{q^2}(t)\right\|_{H^{-1}(\mathbb{S}^1)}^p \,\mathrm{d}t \overset{(5.9)}{\lesssim} 1,$$

from which we infer that (5.10) holds. □

Remark 5.8. Notice how high integrability in $(\tilde{\omega}, t)$ is traded for low "integrability" in x in the second estimate in (5.9).

6. Existence of martingale solutions

Recall that $\tilde{X}_n$ and $\tilde{X}$, cf. (4.22), are collective symbols for all the Skorokhod–Jakubowski representations, which are built from a sequence $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ of strong solutions to the viscous SPDE (1.5) with initial data $u_{\varepsilon_n}(0) = z_n$, cf. (4.12), where the viscosity coefficients ε_n are positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$.

We need filtrations linked to each $\tilde{X}_n$ and the a.s. limit $\tilde{X}$. To this end, let us first introduce some notations. For $t \in [0, T]$, let $f \mapsto f|_{[0,t]}$ denote the restriction to the interval $[0, t]$ of a function f defined on $[0, T]$. Moreover, we denote by $\Sigma(E)$ the smallest σ-algebra containing a collection E of subsets of $\tilde{\Omega}$. We specify $\{\tilde{\mathcal{F}}_t^n\}_{t\in[0,T]}$ and $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$ to be the $\tilde{\mathbb{P}}$–augmented canonical filtrations of the processes $\tilde{X}_n$ and $\tilde{X}$, respectively. More precisely, for $\tilde{X}$ the filtration and corresponding stochastic basis are defined as

$$\begin{aligned} &\tilde{\mathcal{F}}_t = \Sigma\left(\Sigma\big(\tilde{X}|_{[0,t]}\big)\bigcup\big\{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\big\}\right), \quad t \in [0, T], \\ &\text{and} \quad \tilde{\mathcal{S}} = \big(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}\big). \end{aligned} \tag{6.1}$$

For $\tilde{X}_n$ the filtration $\tilde{\mathcal{F}}_t^n$ and stochastic basis $\tilde{\mathcal{S}}^n$ are defined similarly, with $\tilde{X}_n$ replacing $\tilde{X}$ and $\tilde{\mathcal{F}}_t^n$ replacing $\mathcal{F}_t$. By construction, the processes $\tilde{X}_n$ and $\tilde{X}$ are adapted to their canonical filtrations.

By the equality of laws and the Lévy martingale characterization of a Wiener process, it is clear that $\tilde{W}_n$ is a Wiener processes with respect to its own canonical filtration. Furthermore, $\tilde{W}_n$ is a Wiener process relative to the filtration $\{\tilde{\mathcal{F}}_t^n\}$ defined in (6.1). To prove this, we must verify that $\tilde{W}_n(t)$ is $\tilde{\mathcal{F}}_t^n$–measurable and $\tilde{W}_n(t) - \tilde{W}_n(s)$ is independent of $\tilde{\mathcal{F}}_s^n$, for all $s < t$. However, these properties hold because $\tilde{W}_n$ and W share the same law, and $W(t)$ is $\mathcal{F}_t$–measurable and $W(t) - W(s)$ is independent of $\mathcal{F}_s$, recalling that the unique H^m solution of the viscous SPDE (1.5), by construction, depends measurably on the initial data and the Wiener process [40].

A standard argument reveals that the a.s. limit $\tilde{W}$ of $\tilde{W}_n$, see (4.24), is a Wiener process relative to $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$ (see, e.g., [26, Lemma 4.8]).

Lemma 6.1 *($\tilde{W}$ is a Wiener process). The a.s. representation $\tilde{W}$ from Proposition 4.9 is a Wiener process defined on the stochastic basis $\tilde{\mathcal{S}}$, cf. (6.1).*

Proof. By the equality of laws and Lévy's characterisation theorem (see, e.g., [55, Theorem IV.3.6]), it remains to prove that $\tilde{W}$ is a $\tilde{\mathcal{F}}_t$ martingale.

Let $\gamma : \mathcal{X}|_{[0,s]} \to [0, 1]$ be a continuous function, where $\mathcal{X}$ is the (countable product) path space defined by (4.13), (4.14) and by $\mathcal{X}|_{[0,s]}$ we understand the same space but with $[0, T]$

replaced by $[0,s]$. Clearly, $\mathcal{X}|_{[0,s]}$ is quasi-Polish (with the product topology), and the restriction operator $\mathcal{R}_s : \mathcal{X} \to \mathcal{X}|_{[0,s]}$ is continuous (as each single component is trivially continuous). Hence, $X_n|_{[0,s]} = \mathcal{R}_s \circ X_n$ is $\mathcal{F}_s \,/\, \otimes_{l\in\mathbb{N}} \mathcal{B}_{\mathcal{X}_l}\big|_{[0,s]}$ measurable, where the countable vector X_n is defined in (4.11) and $\mathcal{B}_{\mathcal{X}_l}\big|_{[0,s]}$ denotes the Borel σ-algebra of the "restricted" space $\mathcal{X}_l|_{[0,s]}$.

Now, by the equality of laws and the $\{\mathcal{F}_t\}$-martingale property of the original Wiener process W, for any $0 \le s < t \le T$ and for any $n \in \mathbb{N}$,

$$\tilde{\mathbb{E}}\left[\gamma\big(\tilde{X}_n|_{[0,s]}\big)\big(\tilde{W}_n(t) - \tilde{W}_n(s)\big)\right] = \mathbb{E}\left[\gamma\big(X_n|_{[0,s]}\big)\big(W(t) - W(s)\big)\right] = 0,$$

where X_n is defined in (4.11). The lemma follows if we can pass to limit $n \to \infty$ in the left-hand side of the above identity. By (4.24), $\tilde{W}_n \to \tilde{W}$ in $C([0,T])$, $\tilde{\mathbb{P}}$–a.s. Moreover, by the equality of laws, $\tilde{\mathbb{E}}\left\|\tilde{W}_n\right\|^p_{C([0,T])} = \mathbb{E}\,\|W\|^p_{C([0,T])} \le C(T,p)$, for any finite p, where the last estimate comes from the BDG inequality. Hence, by Vitali's convergence theorem, $\tilde{\mathbb{E}}\left[\gamma\big(\tilde{X}|_{[0,s]}\big)\big(\tilde{W}(t) - \tilde{W}(s)\big)\right] = 0$. $\square$

The process u_{ε_n} satisfies the viscous SPDE (1.5) with initial data $u_{\varepsilon_n}(0) = z_n$, cf. (4.12). The next result shows that the Skorokhod–Jakubowski representation $\tilde{u}_n$ satisfies the same SPDE on the new probability space. There exist several approaches to proving this, see for example [5,11,52]. Here we are going to rely on a simple but general method discovered by Brzeźniak and Ondreját [11,52], and then used in several other works analysing different SPDEs, see for example [37,26,7]. To describe the idea, consider the following functional, defined for $(u,v,z) \in \mathcal{X}_u \times \mathcal{X}_{q^2} \times \mathcal{X}_{u_0}$, cf. (4.13), and $t \in [0,T]$:

$$\begin{aligned} M_n[u,v,z](t) &= \int_{\mathbb{S}^1} \varphi\, u(t)\,\mathrm{d}x - \int_{\mathbb{S}^1} \varphi\, z\,\mathrm{d}x - \varepsilon_n \int_0^t \int_{\mathbb{S}^1} \partial^2_{xx}\varphi\, u\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{S}^1} \partial_x\varphi \left(\frac{1}{2}u^2 + P[u,v]\right)\mathrm{d}x\,\mathrm{d}s - \frac{1}{2}\int_0^t \int_{\mathbb{S}^1} \partial_x\left(\partial_x\left(\sigma_{\varepsilon_n}\varphi\right)\sigma_{\varepsilon_n}\right) u\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{6.2}$$

viewing the test function $\varphi \in C^\infty(\mathbb{S}^1)$ as fixed. Here, we have augmented our usual notation, cf. (1.5), to accommodate the weak limit of $\tilde{q}_n^2$, by setting $P[u,v] = K * \big(u^2 + \frac{1}{2}v\big)$. The x-weak formulation of the SPDE for u_{ε_n}, cf. (2.2), reads

$$D_n = M_n\big[u_{\varepsilon_n}, q^2_{\varepsilon_n}, z_n\big](t) - \int_0^t \int_{\mathbb{S}^1} \partial_x\left(\varphi\sigma_{\varepsilon_n}\right) u_{\varepsilon_n}\,\mathrm{d}x\,\mathrm{d}W = 0, \qquad q_{\varepsilon_n} = \partial_x u_{\varepsilon_n}.$$

Replacing u_{ε_n}, W by $\tilde{u}_n$, $\tilde{W}_n$, respectively, we denote the corresponding quantity by $\tilde{D}_n$. The aim is to show that $D_n = 0$ implies $\tilde{D}_n = 0$. By the equality of laws, the real-valued stochastic process $\tilde{D}_n$ is a martingale (starting at 0), and if one establishes that the quadratic variation of $\tilde{D}_n$ is zero, then $\tilde{D}_n$ is zero. Since $\tilde{D}_n$ is of the form $\tilde{D}_n^{(1)} - \tilde{D}_n^{(2)}$, this boils down to computing the quadratic variation $\langle\tilde{D}_n\rangle$ as $\langle\tilde{D}_n^{(1)}\rangle - 2\langle\tilde{D}_n^{(1)}, D_n^{(2)}\rangle + \langle\tilde{D}_n^{(2)}\rangle$, where $\langle\tilde{D}_n^{(2)}\rangle(t) = \int_0^t \left|\int_{\mathbb{S}^1} \partial_x\left(\varphi\sigma_{\varepsilon_n}\right)\tilde{u}_n\,\mathrm{d}x\right|^2 \mathrm{d}s$, and the first (quadratic variation) and second (co-variation) terms can be computed via the equality of laws and properties of the corresponding terms in $\langle D_n \rangle = 0$, see (6.6) below.

Lemma 6.2 *($\tilde{u}_n$ solves SPDE). Let $\tilde{u}_n$, $\tilde{q}_n$, $\tilde{W}_n$, $\tilde{u}_{0,n}$ be the Skorokhod–Jakubowski representations from Proposition 4.9. Then, for any $\varphi \in C^\infty(\mathbb{S}^1)$ and $t \in [0, T]$,*

$$\begin{aligned}
&\int_{\mathbb{S}^1} \varphi\, \tilde{u}_n(t)\, \mathrm{d}x - \int_{\mathbb{S}^1} \varphi\, \tilde{u}_{0,n}\, \mathrm{d}x \\
&\quad = \int_0^t \int_{\mathbb{S}^1} \partial_x \varphi \left(\frac{1}{2} \tilde{u}_n^2 + \tilde{P}_n \right) \mathrm{d}x\, \mathrm{d}s + \varepsilon_n \int_0^t \int_{\mathbb{S}^1} \partial_{xx}^2 \varphi\, \tilde{u}_n\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad + \frac{1}{2} \int_0^t \int_{\mathbb{S}^1} \partial_x \left(\partial_x \left(\sigma_{\varepsilon_n} \varphi \right) \sigma_{\varepsilon_n} \right) \tilde{u}_n\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{\mathbb{S}^1} \partial_x \left(\varphi\, \sigma_{\varepsilon_n} \right) \tilde{u}_n\, \mathrm{d}x\, \mathrm{d}\tilde{W}_n,
\end{aligned} \tag{6.3}$$

$\tilde{\mathbb{P}}$–a.s., where $\tilde{P}_n = P[\tilde{u}_n] = K * \left(\tilde{u}_n^2 + \frac{1}{2} \tilde{q}_n^2 \right)$.

Proof. We follow, e.g., [7]. For notational brevity, herein we use $\langle X \rangle(t)$ to denote the quadratic variation $\langle X, X \rangle(t)$ of a process X, whilst retaining $\langle X, Y \rangle(t)$ for the co-variation between two processes X, Y.

1. Set-up and conclusion.

Given (6.2), let us also introduce the n-independent functionals

$$R[u](t) = \int_0^t \left| \int_{\mathbb{S}^1} \partial_x (\varphi \sigma)\, u\, \mathrm{d}x \right|^2 \mathrm{d}s, \quad N[u](t) = -\int_0^t \int_{\mathbb{S}^1} \partial_x (\varphi \sigma)\, u\, \mathrm{d}x\, \mathrm{d}s. \tag{6.4}$$

The proof hinges on showing that $\tilde{M}_n = M_n\big[\tilde{u}_n, \tilde{q}_n^2, \tilde{u}_{0,n}\big]$ is an $\{\tilde{\mathcal{F}}_t^n\}$-martingale with quadratic variation and covariation (with $\tilde{W}_n$) given by

$$\begin{aligned}
\langle \tilde{M}_n \rangle &= \int_0^t \left| \int_{\mathbb{S}^1} \partial_x \left(\varphi \sigma_{\varepsilon_n} \right) \tilde{u}_n\, \mathrm{d}x \right|^2 \mathrm{d}s =: \tilde{R}_n, \\
\langle \tilde{M}_n, \tilde{W}_n \rangle &= -\int_0^t \int_{\mathbb{S}^1} \partial_x \left(\varphi \sigma_{\varepsilon_n} \right) \tilde{u}_n\, \mathrm{d}x\, \mathrm{d}s =: \tilde{N}_n.
\end{aligned} \tag{6.5}$$

These identities imply that $\tilde{D}_n(t) = \tilde{M}_n(t) - \int_0^t \int_{\mathbb{S}^1} \partial_x \left(\varphi \sigma_{\varepsilon_n} \right) \tilde{u}_n\, \mathrm{d}x\, \mathrm{d}\tilde{W}_n$ has vanishing quadratic variation:

$$\left\langle \tilde{D}_n \right\rangle(t) = \left\langle \tilde{M}_n \right\rangle(t) - 2\int_0^t \int_{\mathbb{S}^1} \partial_x \left(\varphi\sigma_{\varepsilon_n}\right) \tilde{u}_n \,\mathrm{d}x \,\mathrm{d}\left\langle \tilde{M}_n, \tilde{W}_n \right\rangle$$
$$+ \left\langle \int_0^{\cdot} \int_{\mathbb{S}^1} \partial_x \left(\varphi\sigma_{\varepsilon_n}\right) \tilde{u}_n \,\mathrm{d}x \,\mathrm{d}\tilde{W}_n \right\rangle(t) = 0, \tag{6.6}$$

and $\left\langle \tilde{D}_n \right\rangle = 0$ implies $\tilde{D}_n = 0$, which is the sought-after equation (6.3).

2. Martingale properties and verification of (6.5).

We establish (6.5) by verifying the martingale property of the three processes $\tilde{M}_n$, $\tilde{M}_n^2 - \tilde{R}_n$, and $\tilde{M}_n \tilde{W}_n - \tilde{N}_n$. However, first we must check that

$$\mathcal{X}_u \times \mathcal{X}_{q^2} \times \mathcal{X}_{u_0} \ni (u, v, z) \mapsto M_n[u, v, z](t) \in \mathbb{R}, \quad t \in [0, T],$$

is measurable map. We will do this by proving continuity of M_n. Given this continuity of M_n on a finite sub-collection $\mathcal{X}_u \times \mathcal{X}_{q^2} \times \mathcal{X}_{u_0}$ of the factors of the full Cartesian product space $\mathcal{X}$, cf. (4.13) and (4.14), clearly M_n may be seen as a continuous function on the full Cartesian product $\mathcal{X}$, and then Remark B.11 supplies the desired measurability. We prove the measurability of R and N in the same way.

Continuity will follow from the estimates already established. Fix $u_i \in \mathcal{X}_u$, $v_i \in \mathcal{X}_{q^2}$ and $z_i \in \mathcal{X}_{u_0}$, see (4.13), for $i = 1, 2$. From (6.2), and by repeated applications of Hölder's inequality, we obtain

$$\begin{aligned}
&|M_n[u_1, v_1, z_1] - M_n[u_2, v_2, z_2]| \\
&\quad\lesssim \|\varphi\|_{L^2(\mathbb{S}^1)} \|u_1 - u_2\|_{C([0,T];L^2(\mathbb{S}^1))} + \|\varphi\|_{L^2(\mathbb{S}^1)} \|z_1 - z_2\|_{L^2(\mathbb{S}^1)} \\
&\qquad + \varepsilon_n \left\| \partial_{xx}^2 \varphi \right\|_{L^2(\mathbb{S}^1)} \|u_1 - u_2\|_{C([0,T];L^2(\mathbb{S}^1))} \\
&\qquad + \|\partial_x \varphi\|_{L^\infty(\mathbb{S}^1)} \|u_1 + u_2\|_{L^2([0,T]\times\mathbb{S}^1)} \|u_1 - u_2\|_{L^2([0,T]\times\mathbb{S}^1)} \\
&\qquad + \|\varphi\|_{L^\infty(\mathbb{S}^1)} \|\partial_x K\|_{L^1(\mathbb{S}^1)} \|u_1 + u_2\|_{L^2([0,T]\times\mathbb{S}^1)} \|u_1 - u_2\|_{L^2([0,T]\times\mathbb{S}^1)} \\
&\qquad + \left| \int_0^T \int_{\mathbb{S}^1} \left((\tau\varphi) * \partial_x K\right)(v_1 - v_2) \,\mathrm{d}x \,\mathrm{d}t \right| \\
&\qquad + \left\| \partial_x \left(\partial_x \left(\sigma_{\varepsilon_n}\varphi\right) \sigma_{\varepsilon_n} \right) \right\|_{L^\infty(\mathbb{S}^1)} \|u_1 - u_2\|_{L^1([0,T]\times\mathbb{S}^1)},
\end{aligned}$$

writing "$a^2 - b^2 = (a + b)(a - b)$" twice. Since $\mathcal{X}_{q^2} = L^r([0, T] \times \mathbb{S}^1) - w$ for some fixed $1 \le r < 3/2$, we have used a standard property of convolution to write

$$\int_{\mathbb{S}^1} \varphi\, \partial_x K * (v_1 - v_2) \,\mathrm{d}x = -\int_{\mathbb{S}^1} \left((\tau\varphi) * \partial_x K\right)(v_1 - v_2) \,\mathrm{d}x,$$

noting that $(\tau\varphi) * \partial_x K \in L^{r'}([0,T]\times\mathbb{S}^1)$, $1/r + 1/r' = 1$ (so $r' > 3$). Here the map τ is defined by $(\tau\varphi)(-x) = \varphi(x)$. This shows that M_n is continuous on $\mathcal{X}_u \times \mathcal{X}_{q^2} \times \mathcal{X}_{u_0}$, and thereby measurable (according to Remark B.11). Similar arguments can now be made for $N[u]$ and $R[u]$. As $u \mapsto N[u]$ is linear, continuity follows from the bound

$$|N[u]| \le \|\partial_x(\varphi\sigma)\|_{L^\infty(\mathbb{S}^1)} \|u\|_{L^1([0,T]\times\mathbb{S}^1)} .$$

For the continuity of R,

$$\begin{aligned}|R[u_1] - R[u_2]| &\le \int_0^t \left(\int_{\mathbb{S}^1} \partial_x(\varphi\sigma)(u_1 - u_2)\,\mathrm{d}x \int_{\mathbb{S}^1} \partial_x(\varphi\sigma)(u_1 + u_2)\,\mathrm{d}x \right) \mathrm{d}s \\ &\le \|\partial_x(\varphi\sigma)\|^2_{L^\infty(\mathbb{S}^1)} \|u_1 - u_2\|_{L^2([0,T];L^1(\mathbb{S}^1))} \|u_1 + u_2\|_{L^2([0,T];L^1(\mathbb{S}^1))} \\ &\lesssim_{\varphi,\sigma,T} \|u_1 + u_2\|_{L^2([0,T];L^1(\mathbb{S}^1))} \|u_1 - u_2\|_{C([0,T];L^2(\mathbb{S}^1))} .\end{aligned}$$

Finally, we verify the announced martingale properties. For any càdlàg process X on $[0,T]$ and $s,t \in [0,T]$ with $s < t$, denote by $\Delta_{s,t}X$ the difference $X(t) - X(s)$. Let $\gamma : \mathcal{X}|_{[0,s]} \to [0,1]$ be an arbitrary continuous function, where $\mathcal{X}$ is the path space defined by (4.13), (4.14). By the equality of laws in Proposition 4.9 and the martingale property of the original processes $M_n := M_n\big[u_{\varepsilon_n}, q^2_{\varepsilon_n}, z_{0,n}\big]$, $M_n^2 - R[u_n]$, and $M_n W - N[u_n]$, we obtain

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\gamma\big(\tilde{X}_n|_{[0,s]}\big)\Delta_{s,t}\tilde{M}_n\right] = 0, \quad \tilde{\mathbb{E}}\left[\gamma\big(\tilde{X}_n|_{[0,s]}\big)\left(\Delta_{s,t}\tilde{M}_n^2 - \Delta_{s,t}\tilde{R}_n\right)\right] = 0, \\ &\text{and} \quad \tilde{\mathbb{E}}\left[\gamma\big(\tilde{X}_n|_{[0,s]}\big)\left(\Delta_{s,t}\big(\tilde{M}_n\tilde{W}_n\big) - \Delta_{s,t}\tilde{N}_n\right)\right] = 0,\end{aligned} \tag{6.7}$$

which proves that $\tilde{M}_n$, $\tilde{M}_n^2 - \tilde{R}_n$, and $\tilde{M}_n\tilde{W}_n - \tilde{N}_n$ are $\{\tilde{F}_t^n\}$–martingales. □

Arguing as above, we prove next that the a.s. limit $\tilde{u}$ from Proposition 4.9 satisfies an SPDE on the new probability space that resembles the stochastic CH equation (1.3), except that the nonlinear term $\tilde{q}^2$ is replaced by the weak limit $\overline{q^2}$. Once we make the identification $\overline{q^2} = \tilde{q}^2$, which is equivalent to the strong $L^2_{\tilde{\omega},t,x}$ convergence of $\tilde{q}_n$ towards $\tilde{q}$ [51, Lemma 3.34], the proof of Theorem 1.1 is concluded. But being rather long and technical, the identification step is postponed to Section 7, which constitutes a central part of the paper.

Proposition 6.3 *(Limit $\tilde{u}$ solves SPDE). Suppose the assumptions of Theorem 1.1 hold. Let $\tilde{u}$, $\tilde{q}$, $\overline{q^2}$, $\tilde{W}$, $\tilde{u}_0$ be the Skorokhod–Jakubowski representations from Proposition 4.9, see also Remark 4.10, and let $\tilde{\mathcal{S}}$ be the stochastic basis defined in (6.1). Suppose further that the following identification holds:*

$$\overline{q^2} = \tilde{q}^2, \quad \tilde{\mathbb{P}} \otimes \mathrm{d}t \otimes \mathrm{d}x\text{–a.e. in } \tilde{\Omega} \times [0,T] \times \mathbb{S}^1. \tag{6.8}$$

Then $\big(\tilde{\mathcal{S}}, \tilde{u}, \tilde{W}\big)$ is a weak martingale solution of

$$0 = \mathrm{d}\tilde{u} + \left[\tilde{u}\,\partial_x\tilde{u} + \partial_x\tilde{P}\right]\mathrm{d}t - \frac{1}{2}\sigma\,\partial_x\left(\sigma\,\partial_x\tilde{u}\right)\mathrm{d}t + \sigma\,\partial_x\tilde{u}\,\mathrm{d}\tilde{W},$$

$$\tilde{P} = K * \left(\tilde{u}^2 + \frac{1}{2}\tilde{q}^2\right), \quad \tilde{u}(0) = \tilde{u}_0 \sim u_0,$$

in the sense that $\tilde{S}$ satisfies (a), $\tilde{W}$ satisfies (b), and $\tilde{u}$ satisfies (c), (d) of Definition 2.4. Besides, the x-weak form (2.6) holds with u, P replaced by $\tilde{u}$, $\tilde{P}$.

Proof. We continue to use the functionals M_n, N, and R defined in (6.2) and (6.4). In addition, we need the n-independent functional

$$M[u, v, z](t) = M_n[u, v, z](t) + \varepsilon_n \int_0^t \int_{\mathbb{S}^1} \partial_{xx}^2 \varphi u \,\mathrm{d}x\,\mathrm{d}s.$$

To simplify the notation, set $\tilde{M} = M\left[\tilde{u}, \tilde{q}^2, \tilde{u}_0\right]$, $\tilde{R} = R[\tilde{u}]$, and $\tilde{N} = N[\tilde{u}]$. Similarly, we continue to use $\tilde{M}_n = M_n\left[\tilde{u}_n, \tilde{q}_n^2, \tilde{u}_{0,n}\right]$, and $\tilde{R}_n$, $\tilde{N}_n$ of (6.5).

1. Set-up and conclusion.

The underlying idea of the proof is the same as before. Here we want to verify that $\tilde{M}$, $\tilde{M}^2 - \tilde{R}$, and $\tilde{M}\tilde{W} - \tilde{N}$ are all $\{\tilde{\mathcal{F}}_t\}$–martingales. The limit statements corresponding to (6.5) take the form

$$\langle\tilde{M}\rangle = \tilde{R}, \quad \langle\tilde{M}, \tilde{W}\rangle = \tilde{N}.$$

As before (6.6), these identities imply that $\tilde{D} = \tilde{M} - \int_0^\cdot \int_{\mathbb{S}^1} \partial_x(\varphi\sigma)\,\tilde{u}\,\mathrm{d}x\,\mathrm{d}\tilde{W}$ is a martingale (starting at 0) with vanishing quadratic variation, and $\tilde{D} = 0$ is the desired equation (2.6), replacing u, P by $\tilde{u}$, $\tilde{P}$. Because of Section 5, the remaining properties of $\left(\tilde{S}, \tilde{u}, \tilde{W}\right)$ are evident.

The martingale properties follow by sending $n \to \infty$ in (6.7), relying on the a.s. convergences (4.24) and the moment estimates in Lemma 5.3. Eventually, we arrive at the required martingale equalities

$$\tilde{\mathbb{E}}\left[\gamma\left(\tilde{X}|_{[0,s]}\right)\Delta_{s,t}\tilde{M}\right] = 0, \tag{6.9}$$

$$\tilde{\mathbb{E}}\left[\gamma\left(\tilde{X}|_{[0,s]}\right)\left(\Delta_{s,t}\tilde{M}^2 - \Delta_{s,t}\tilde{R}\right)\right] = 0, \tag{6.10}$$

$$\tilde{\mathbb{E}}\left[\gamma\left(\tilde{X}|_{[0,s]}\right)\left(\Delta_{s,t}\left(\tilde{M}\tilde{W}\right) - \Delta_{s,t}\tilde{N}\right)\right] = 0, \tag{6.11}$$

where $\tilde{X}$ is defined in (4.22), see also (6.1), and $\gamma : \mathcal{X}|_{[0,s]} \to [0, 1]$ is an arbitrary continuous function.

2. Passing to the limit in (6.7) to obtain (6.9).

It remains to justify the passage to the limit in each equation of (6.7). Since γ is bounded and continuous and $\tilde{X}_n \to \tilde{X}$ a.s. (Proposition 4.9), it follows that

$$\gamma\left(\tilde{X}_n|_{[0,s]}\right) \xrightarrow{n\uparrow\infty} \gamma\left(\tilde{X}|_{[0,s]}\right) \quad \text{in } L^p(\tilde{\Omega}), \text{ for any finite } p. \tag{6.12}$$

We continue with the claim that, for any $t \in [0, T]$,

$$\tilde{M}_n(t) \xrightarrow{n\uparrow\infty} M(t), \quad \tilde{\mathbb{P}}\text{–a.s.} \tag{6.13}$$

We verify (6.13) by proving the term-by-term convergence of $\tilde{M}_n$ to $\tilde{M}$. From (4.24), we have $\tilde{u}_{0,n} \to \tilde{u}_0$ in $H^1(\mathbb{S}^1)$ and $\tilde{u}_n \to \tilde{u}$ in $C([0,T]; L^2(\mathbb{S}^1))$, $\tilde{\mathbb{P}}$–almost surely. This (and $\varepsilon_n \to 0$) implies

$$\left| \int_{\mathbb{S}^1} \varphi \left(\tilde{u}_{0,n} - \tilde{u}_0\right) \mathrm{d}x \right| \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.},$$

$$\sup_{t\in[0,T]} \left| \int_{\mathbb{S}^1} \varphi \left(\tilde{u}_n - \tilde{u}\right)(t) \, \mathrm{d}x \right| \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.},$$

$$\left| \varepsilon_n \int_0^t \int_{\mathbb{S}^1} \partial_{xx}^2 \varphi \, \tilde{u}_n \, \mathrm{d}x \, \mathrm{d}s \right| \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.}$$

$$\left| \int_0^t \int_{\mathbb{S}^1} \partial_x \varphi \left(\frac{\tilde{u}_n^2}{2} - \frac{\tilde{u}^2}{2} \right) \mathrm{d}x \, \mathrm{d}s \right| \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.}$$

Using $P\big[\tilde{u}_n, \tilde{q}_n^2\big] = K * \left(\tilde{u}_n^2 + \frac{1}{2}\tilde{q}_n^2\right)$, $\tilde{u}_n^2 \to \tilde{u}^2$ in $C([0,T]; L^1(\mathbb{S}^1))$ a.s., $\tilde{q}_n^2 \rightharpoonup \overline{q^2}$ in $L^r([0,T] \times \mathbb{S}^1)$ a.s., and the weak limit identification (6.8),

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{S}^1} \partial_x \varphi \Big(P\big[\tilde{u}_n, \tilde{q}_n^2\big] - P\big[\tilde{u}, \tilde{q}^2\big] \Big) \mathrm{d}x \, \mathrm{d}s \right| \\
&\quad \le \|\partial_x \varphi\|_{L^\infty(\mathbb{S}^1)} \|K\|_{L^1(\mathbb{S}^1)} \left\| \tilde{u}_n^2 - \tilde{u}^2 \right\|_{L^1([0,T]\times\mathbb{S}^1)} \\
&\qquad + \left| \int_0^t \int_{\mathbb{S}^1} \left(\int_{\mathbb{S}^1} \partial_x \varphi(x) K(x-y) \, \mathrm{d}x \right)(y) \left(\tilde{q}_n^2(s,y) - \tilde{q}^2(s,y) \right) \mathrm{d}y \, \mathrm{d}s \right| \xrightarrow{n\uparrow\infty} 0,
\end{aligned}$$

exploiting $(s, y) \mapsto \int_{\mathbb{S}^1} \varphi(x) \partial_x K(x-y) \, \mathrm{d}x \in L^{r'}([0,T] \times \mathbb{S}^1)$ (recall $r < 3/2$ and therefore $r' > 3$).

Finally, using again $\tilde{u}_n \to \tilde{u}$ in $C([0,T]; L^2(\mathbb{S}^1))$ a.s., and the a.e. convergence $\partial_x \left(\partial_x \left(\sigma_{\varepsilon_n} \varphi\right) \sigma_{\varepsilon_n}\right) \to \partial_x \left(\partial_x (\sigma \varphi) \sigma\right)$,

$$\left| \int_0^t \int_{\mathbb{S}^1} \partial_x \left(\partial_x \left(\sigma_{\varepsilon_n} \varphi\right) \sigma_{\varepsilon_n} \right) \tilde{u}_n - \partial_x \left(\partial_x (\sigma\varphi) \sigma\right) \tilde{u} \, \mathrm{d}x \, \mathrm{d}s \right| \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.},$$

which concludes the proof of (6.13). By Lemma 5.3,

$$\mathbb{E}\left|\tilde{M}_n(t)\right|^{p_0} \lesssim_\varphi 1 \quad (\text{with } p_0 > 4). \tag{6.14}$$

Hence, by Vitali's convergence theorem,

$$\tilde{M}_n(t) \xrightarrow{n\uparrow\infty} \tilde{M}(t) \quad \text{in } L^p(\tilde{\Omega}), \text{ for any } p \in [1, p_0), t \in [0, T]. \tag{6.15}$$

In view of (6.7), (6.12), and (6.15), we see that (6.9) holds.

3. Passing to the limit in (6.7) *to obtain* (6.10).

Recalling (6.7), we consider the convergences of $\Delta_{s,t}\tilde{M}_n^2$ and $\Delta_{s,t}\tilde{R}_n$ separately. Using (6.15) again ($p_0 > 4$), we have

$$\tilde{M}_n^2(t) \xrightarrow{n\uparrow\infty} \tilde{M}^2(t) \quad \text{in } L^p(\tilde{\Omega}), \text{ for any } p \in \big[1, p_0/2\big), t \in [0, T]. \tag{6.16}$$

The convergence

$$\left| \int_0^t \left| \int_{\mathbb{S}^1} \partial_x \left(\varphi\sigma_{\varepsilon_n}\right) \tilde{u}_n \, \mathrm{d}x \right|^2 - \left| \int_{\mathbb{S}^1} \partial_x \left(\varphi\sigma\right) \tilde{u} \, \mathrm{d}x \right|^2 \mathrm{d}s \right| \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.}, \tag{6.17}$$

follows by applying the algebraic identity $a^2 - b^2 = (a+b)(a-b)$, and using the a.s. convergence $\tilde{u}_n \to \tilde{u}$ in $C([0,T]; L^2(\mathbb{S}^1))$ and the a.e. convergence $\partial_x\left(\varphi\sigma_{\varepsilon_n}\right) \to \partial_x\left(\varphi\sigma\right)$. Clearly, (6.17) implies the $\tilde{\mathbb{P}}$–a.s. convergence $\tilde{R}_n(t) \to \tilde{R}(t)$, for $t \in [0, T]$. By Lemma 5.3, $\mathbb{E}\left|\tilde{R}_n(t)\right|^{p_0/2} \lesssim_\varphi 1$, and therefore, by Vitali's convergence theorem,

$$\tilde{R}_n(t) \xrightarrow{n\uparrow\infty} \tilde{R}(t) \quad \text{in } L^p(\tilde{\Omega}), \text{ for any } p \in \big[1, p_0/2\big), t \in [0, T]. \tag{6.18}$$

Combining (6.7) with (6.12), (6.16), and (6.18), the claim (6.10) follows.

4. Passing to the limit in (6.7) *to obtain* (6.11).

From the a.s. convergence $\tilde{W}_n \to \tilde{W}$ in $C([0,T])$, cf. (4.24), along with (6.13),

$$\tilde{M}_n(t)\,\tilde{W}_n(t) \xrightarrow{n\uparrow\infty} \tilde{M}(t)\,\tilde{W}(t), \quad \tilde{\mathbb{P}}\text{–a.s.}, t \in [0, T].$$

By the Cauchy–Schwarz inequality,

$$\tilde{\mathbb{E}}\left|\tilde{M}_n\tilde{W}_n\right|^{p_0/2} \le \left(\tilde{\mathbb{E}}\left|\tilde{M}_n\right|^{p_0}\right)^{1/2}\left(\tilde{\mathbb{E}}\left|\tilde{W}_n\right|^{p_0}\right)^{1/2} \lesssim 1,$$

where we have used (6.14) and the BDG martingale inequality to bound the p_0 moment of $\tilde{W}_n$. Thus, again by Vitali's convergence theorem,

$$\tilde{M}_n(t)\,\tilde{W}_n(t) \xrightarrow{n\uparrow\infty} \tilde{M}(t)\,\tilde{M}(t) \quad \text{in } L^p(\tilde{\Omega}),\ p \in \big[1, p_0/2\big),\ t \in [0,T]. \tag{6.19}$$

The a.s. convergence of $\tilde{N}_n$ to $\tilde{N}$, cf. (6.4), follows from $\tilde{u}_n \to \tilde{u}$ in $C([0,T]; L^2(\mathbb{S}^1))$. By Lemma 5.3, $\mathbb{E}\left|\tilde{N}_n(t)\right|^{p_0} \lesssim_\varphi 1$, and thus Vitali's convergence theorem yields

$$\tilde{N}_n(t) \xrightarrow{n\uparrow\infty} \tilde{N}(t) \quad \text{in } L^p(\tilde{\Omega}), \text{ for any } p \in [1, p_0),\ t \in [0,T].$$

Combining this, (6.19) and (6.12) with (6.7), the final identity (6.11) emerges. □

7. Identification of a weak limit

In this final section we prove the crucial assumption (6.8) of Proposition 6.3, thereby concluding the proof of our main result (Theorem 1.1).

Theorem 7.1 *(Identification of weak limit). Suppose the assumptions of Theorem 1.1 hold. Let $\tilde{q}$ and $\overline{q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9, recalling that notationally we drop the tilde under the overline in $\overline{q^2}$ (see Remark 4.10). Then the weak limit identification (6.8) holds.*

For a high-level description of the proof, which is long and technical, we refer to Section 1. The proof depends on deriving Itô differential inequalities for the differences $\overline{q_\pm^2} - \tilde{q}_\pm^2 \geq 0$ (via numerous steps of truncation and regularisation). We record these inequalities over several results (see Lemmas 7.9 – 7.13). With regards to the subscripts $\pm$ on $q_\pm$, it is in fact expedient to carry out this procedure, as in the deterministic setting [21,22,63], for the positive and negative parts separately. It is a characteristic feature of dissipative solutions that q_+ does not blow up in L^∞, but q_- does. We refer to Section 1 for a discussion of the many differences between the deterministic and stochastic cases.

7.1. Energy inequalities and a right-continuity property

The differential inequalities mentioned above will serve to propagate strong compactness, assumed initially at $t = 0$, via a (yet to be established) strong temporal continuity property at $t = 0$. The existence of this strong initial trace is the content of Lemma 7.4 below, which encodes the dissipative nature of the considered solution class. However, first we need to transfer the energy balance (2.3) to the new probability space, expressed in terms of the Skorokhod–Jakubowski representations from Proposition 4.9.

Lemma 7.2 *(Energy inequality). Let $\tilde{u}_n$, $\tilde{W}_n$, $\tilde{u}_{0,n}$ be respectively the Skorokhod–Jakubowski representations of u_{ε_n}, W, z_n, where u_{ε_n} is the strong solution to the viscous SPDE (1.5) with noise W and initial data $u_{\varepsilon_n}(0) = z_n$, cf. (4.12). The energy inequality (2.3) holds with u_ε, W, ε replaced by $\tilde{u}_n$, $\tilde{W}_n$, ε_n.*

Let $\tilde{u}$, $\tilde{q}$, $\overline{q^2}$, $\tilde{W}$, $\tilde{u}_0$ be the a.s. limits from Proposition 4.9, see also Remark 4.10, and let $\tilde{S}$ be the stochastic basis defined in (6.1). Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} \tilde{u}^2 + \overline{q^2}\,\mathrm{d}x &\le \int_{\mathbb{S}^1} \frac{1}{4}\partial_{xx}^2\sigma^2\tilde{u}^2 + \left(|\partial_x\sigma|^2 - \frac{1}{4}\partial_{xx}^2\sigma^2\right)\overline{q^2}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} \partial_x\sigma\left(\tilde{u}^2 - \overline{q^2}\right)\mathrm{d}x\,\dot{\tilde{W}}, \quad \textit{in } \mathcal{D}'([0,T)),\ \tilde{\mathbb{P}}\textit{–a.s.}, \\ \int_{\mathbb{S}^1}\left(\tilde{u}^2 + \overline{q^2}\right)(0)\,\mathrm{d}x &= \int_{\mathbb{S}^1} \tilde{u}_0^2 + |\partial_x\tilde{u}_0|^2\,\mathrm{d}x.\end{aligned} \tag{7.1}$$

Remark 7.3. We emphasise that (7.1) holds in the sense of distributions on the half-open interval $[0,T)$, $\tilde{\mathbb{P}}$-a.s., whilst $\int_{\mathbb{S}^1}\tilde{u}^2 + \overline{q^2}\,\mathrm{d}x$ is understood to take the value $\int_{\mathbb{S}^1}\tilde{u}_0^2 + |\partial_x\tilde{u}_0|^2\,\mathrm{d}x$ at $t = 0$. This means that for every non-negative $\psi \in C^\infty([0,T))$,

$$\begin{aligned} &-\int_0^T \partial_t\psi \int_{\mathbb{S}^1} \tilde{u}^2 + \overline{q^2}\,\mathrm{d}x\,\mathrm{d}t - \psi(0)\int_{\mathbb{S}^1}\tilde{u}_0^2 + |\partial_x\tilde{u}_0|^2\,\mathrm{d}x \\ &\qquad \le \int_0^T \psi\int_{\mathbb{S}^1} \frac{1}{4}\partial_{xx}^2\sigma^2\tilde{u}^2 + \left(|\partial_x\sigma|^2 - \frac{1}{4}\partial_{xx}^2\sigma^2\right)\overline{q^2}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\quad + \int_0^T \psi\int_{\mathbb{S}^1}\partial_x\sigma\left(\tilde{u}^2 - \overline{q^2}\right)\mathrm{d}x\,\mathrm{d}\tilde{W}, \qquad \tilde{\mathbb{P}}\text{–a.s.}\end{aligned}$$

Proof. Recall the properties of $\tilde{u}_n$ stated in Lemmas 5.2 and 5.3. In particular, $\tilde{u}_n$ lies in the intersection $L^2([0,T]; H^m(\mathbb{S}^1)) \cap C([0,T]; H^1(\mathbb{S}^1))$ a.s., for any $m \in \mathbb{N}$. According to Lemma 6.3, $\tilde{u}_n$ satisfies the SPDE (1.5) with u_ε, W, ε replaced by $\tilde{u}_n$, $\tilde{W}_n$, ε_n, respectively. If we differentiate this equation with respect to x, cf. Lemma 5.1, then $\tilde{q}_n = \partial_x\tilde{u}_n$ satisfies the SPDE (1.7) with q_ε, W, ε replaced by $\tilde{q}_n$, $\tilde{W}_n$, ε_n. Consequently, we may apply the corresponding versions of (3.4) and (3.5) with $S(v) = v^2/2$. Adding the resulting equations yields the total energy equation (1.8) with u_ε, W, ε replaced by $\tilde{u}_n$, $\tilde{W}_n$, ε_n. Integrating this equation in x, dropping the dissipation term, and expressing the temporal differential as a time-derivative in $\mathcal{D}'([0,T))$, we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1}\tilde{u}_n^2 + \tilde{q}_n^2\,\mathrm{d}x &\le \int_{\mathbb{S}^1}\frac{1}{4}\partial_{xx}^2\sigma_{\varepsilon_n}^2\tilde{u}_n^2 + \left(\left|\partial_x\sigma_{\varepsilon_n}\right|^2 - \frac{1}{4}\partial_{xx}^2\sigma_{\varepsilon_n}^2\right)\tilde{q}_n^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1}\partial_x\sigma_{\varepsilon_n}\left(\tilde{u}_n^2 - \tilde{q}_n^2\right)\mathrm{d}x\,\dot{\tilde{W}}_n, \quad \text{in } \mathcal{D}'([0,T)),\ \tilde{\mathbb{P}}\text{–a.s.,}\end{aligned} \tag{7.2}$$

where $\int_{\mathbb{S}^1}\left(\tilde{u}_n^2 + |\tilde{q}_n|^2\right)(0)\,\mathrm{d}x = \int_{\mathbb{S}^1}\tilde{u}_{n,0}^2 + \left|\partial_x\tilde{u}_{n,0}\right|^2\mathrm{d}x$ and $\dot{\tilde{W}}_n = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{W}_n$.

Equipped with the a.s. convergences in (4.24), in particular $\tilde{u}_n \xrightarrow{n\uparrow\infty} \tilde{u}$ in $C_tL_x^2$ a.s. and $\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2}$ in $L^r\left(L_w^r\right)$ a.s., recalling that we write $\overline{q^2}$ instead of $\overline{\tilde{q}^2}$, we can send $n \to \infty$ in

(7.2) to arrive at (7.1). We refer to Lemma 7.9 for a detailed convergence proof of an inequality that is more general than (7.2).

During the derivation of (7.1), one issue was swept under the rug. Indeed, a priori, it is not clear that the process $\tilde{M}(t) = \int_0^t \int_{\mathbb{S}^1} \partial_x \sigma \overline{q^2}\, dx\, d\tilde{W}$ is a square-integrable martingale. The matter in question is that the limit $\overline{q^2}$ belongs to $L^r_{t,x}$ with merely $r < 3/2$, cf. (4.24); note carefully that we do not have this issue with the related process $\int_0^t \int_{\mathbb{S}^1} \partial_x \sigma\, \tilde{q}^2\, dx\, d\tilde{W}$, where $\tilde{q} = \partial_x \tilde{u}$ is the a.s. limit of $\tilde{q}_n = \partial_x \tilde{u}_n$, as $\tilde{u}$ satisfies (5.3). Fortunately, according to (5.9) of Lemma 5.7, we may assume that $\overline{q^2} \in L^2_{\tilde{\omega},t}\big(H^{-1}_x\big)$ and whence $\tilde{M}$ be interpreted as a square-integrable martingale, recalling that $\sigma \in W^{2,\infty}(\mathbb{S}^1)$:

$$\tilde{\mathbb{E}} \int_0^T \left| \int_{\mathbb{S}^1} \partial_x \sigma\, \overline{q^2}\, dx \right|^2 dt \le \|\partial_x \sigma\|^2_{H^1(\mathbb{S}^1)}\, \tilde{\mathbb{E}} \int_0^T \left\| \overline{q^2}(t) \right\|^2_{H^{-1}(\mathbb{S}^1)} dt < \infty. \quad \square \tag{7.3}$$

The pathwise inequality (7.1), the convergence $\tilde{u}_n \to \tilde{u}$ in $C\big([0,T]; H^1(\mathbb{S}^1) - w\big)$ a.s. (see proof of Lemma 5.4), and the strong H^1 convergence of $\tilde{u}_{0,n}$ towards $\tilde{u}_0$, imply the strong right-continuity at $t = 0$ in H^1. We have the following result:

Lemma 7.4 *(One-sided temporal continuity at $t = 0$). Let $\tilde{u}$, $\tilde{q}$, $\tilde{u}_0$, and $\overline{q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9. Then*

$$\lim_{t \downarrow 0} \|\tilde{u}(t) - \tilde{u}_0\|_{H^1(\mathbb{S}^1)} = 0, \quad \tilde{\mathbb{P}}\text{–a.s.} \tag{7.4}$$

Moreover, for the nonlinearities $S(v) = S_\ell(v_\pm)$ defined by (4.2),

$$\lim_{t \downarrow 0} \big\| S\big(\tilde{q}(t)\big) - S\big(\partial_x \tilde{u}_0\big) \big\|_{L^1(\mathbb{S}^1)} = 0, \quad \tilde{\mathbb{P}}\text{–a.s.} \tag{7.5}$$

Remark 7.5. In view of Lemma 5.4 and Vitali's convergence theorem, (7.4) and (7.5) imply

$$\lim_{t \downarrow 0} \tilde{\mathbb{E}}\, \|\tilde{u}(t) - \tilde{u}_0\|^2_{H^1(\mathbb{S}^1)} = 0, \quad \lim_{t \downarrow 0} \tilde{\mathbb{E}}\, \big\| S\big(\tilde{q}(t)\big) - S\big(\partial_x \tilde{u}_0\big) \big\|_{L^1(\mathbb{S}^1)} = 0.$$

Proof. We divide the proof into two steps.

1. One-sided temporal continuity in $H^1(\mathbb{S}^1)$, (7.4).

In the process of proving Lemma 5.4, we demonstrated that

$$\tilde{u}_n \xrightarrow{n \uparrow \infty} \tilde{u} \text{ in } C\big([0,T]; H^1(\mathbb{S}^1) - w\big), \text{ a.s.}$$

Accordingly, employing the weak lower semicontinuity of $v \mapsto \|v\|^2_{H^1(\mathbb{S}^1)}$,

$$\|\tilde{u}(t)\|^2_{H^1(\mathbb{S}^1)} \le \liminf_{n \to \infty} \|\tilde{u}_n(t)\|^2_{H^1(\mathbb{S}^1)} \le \limsup_{n \to \infty} \|\tilde{u}_n(t)\|^2_{H^1(\mathbb{S}^1)}, \quad t > 0. \tag{7.6}$$

Define

$$I_n(t) = \int_0^t \int_{\mathbb{S}^1} \frac{1}{4}\partial_x^2\sigma^2\tilde{u}_n^2 + \left(|\partial_x\sigma|^2 - \frac{1}{4}\partial_x^2\sigma^2\right)\tilde{q}_n^2\,\mathrm{d}x\,ds$$
$$+ \int_0^t \int_{\mathbb{S}^1} \partial_x\sigma\left(\tilde{u}_n^2 - \tilde{q}^2\right)\mathrm{d}x\,\mathrm{d}\tilde{W}_n,$$
$$I(t) = \int_0^t \int_{\mathbb{S}^1} \frac{1}{4}\partial_x^2\sigma^2\tilde{u}^2 + \left(|\partial_x\sigma|^2 - \frac{1}{4}\partial_x^2\sigma^2\right)\overline{q^2}\,\mathrm{d}x\,ds$$
$$+ \int_0^t \int_{\mathbb{S}^1} \partial_x\sigma\left(\tilde{u}^2 - \overline{q^2}\right)\mathrm{d}x\,\mathrm{d}\tilde{W}.$$

Arguing as in the proof of Lemma 7.2, $I_n \xrightarrow{n\uparrow\infty} I$ a.s. (with $t > 0$ fixed). By a standard deterministic argument, see for example [30, page 653], we can turn the (pathwise) distributional inequality (7.2) into the pointwise inequality

$$\|\tilde{u}_n(t)\|^2_{H^1(\mathbb{S}^1)} \le \left\|\tilde{u}_{0,n}\right\|^2_{H^1(\mathbb{S}^1)} + I_n(t), \quad t > 0. \tag{7.7}$$

Importantly, as $n \to \infty$, the right-hand side of (7.7) converges almost surely to $\|\tilde{u}_0\|^2_{H^1(\mathbb{S}^1)} + I(t)$. In view of (7.6) and (7.7), we conclude that

$$\|\tilde{u}(t)\|^2_{H^1(\mathbb{S}^1)} \le \|\tilde{u}_0\|^2_{H^1(\mathbb{S}^1)} + I(t), \quad \text{a.s., for all } t > 0. \tag{7.8}$$

Since $\tilde{u} \in C\big([0,T]; H^1(\mathbb{S}^1) - w\big)$ a.s., it is evident that

$$\tilde{u}(t) \rightharpoonup \tilde{u}_0 \quad \text{in } H^1(\mathbb{S}^1) \text{ as } t \downarrow 0. \tag{7.9}$$

Whence, again by the weak lower semicontinuity of $\|\cdot\|^2_{H^1(\mathbb{S}^1)}$,

$$\|\tilde{u}_0\|^2_{H^1(\mathbb{S}^1)} \le \liminf_{t\downarrow 0} \|\tilde{u}(t)\|^2_{H^1(\mathbb{S}^1)} \le \limsup_{t\downarrow 0} \|\tilde{u}(t)\|^2_{H^1(\mathbb{S}^1)}$$
$$\overset{(7.8)}{\le} \limsup_{t\downarrow 0}\left(\|\tilde{u}_0\|^2_{H^1(\mathbb{S}^1)} + I(t)\right) = \|\tilde{u}_0\|^2_{H^1(\mathbb{S}^1)},$$

where we have used that $I(t) \to 0$ as $t \downarrow 0$, a.s. Therefore, a.s.,

$$\lim_{t\downarrow 0} \|\tilde{u}(t)\|^2_{H^1(\mathbb{S}^1)} = \|\tilde{u}_0\|^2_{H^1(\mathbb{S}^1)}. \tag{7.10}$$

Combining (7.9) and (7.10) ("weak convergence plus convergence of norms imply strong convergence"), we attain (7.4).

2. One-sided temporal continuity for nonlinearities, (7.5).

Fix $a, b \in \mathbb{R}$ with $a < b$. Assume that $b > 0$, otherwise there would be nothing to prove. It is easy to verify that

$$\begin{aligned}|S_\ell(b_+) - S_\ell(a_+)| &= \int_{a\vee 0}^{b} S_\ell'(v)\,\mathrm{d}v \overset{(4.3)}{\le} \int_{a\vee 0}^{b} v\,\mathrm{d}v \\ &\le b^2 - (a\vee 0)^2 \le (|b| + |a|)\,|b-a|\,.\end{aligned}$$

By symmetry, the inequality holds for $b < a$, and a similar calculation establishes the inequality for $S(v_-)$.

Fix any $\tilde{\omega} \in \tilde{\Omega}$ for which (7.4) holds. For $S(v) = S_\ell(v_\pm)$ defined by (4.2), we then proceed as follows:

$$\begin{aligned}&\left\| S\big(\tilde{q}(t)\big) - S\big(\partial_x \tilde{u}_0\big)\right\|_{L^1(\mathbb{S}^1)} \\ &\quad\le \int_{\mathbb{S}^1} \big(|\tilde{q}(t,x)| + |\partial_x \tilde{u}_0(x)|\big)\, |\tilde{q}(t,x) - \partial_x \tilde{u}_0(x)|\,\mathrm{d}x \\ &\quad\le 2\,\|\tilde{q}\|_{L^\infty([0,T];L^2(\mathbb{S}^1))}\,\|\tilde{q}(t) - \partial_x\tilde{u}_0\|_{L^2(\mathbb{S}^1)} \\ &\quad\overset{(5.3)}{\lesssim_{\tilde{\omega}}} \|\tilde{q}(t) - \partial_x\tilde{u}_0\|_{L^2(\mathbb{S}^1)} \xrightarrow{(7.4)} 0 \quad \text{as } t \downarrow 0.\end{aligned}$$

This concludes the proof of (7.5). □

Once we have made the identification (6.8), the inequality (7.1) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} \tilde{u}^2 + \tilde{q}^2\,\mathrm{d}x &\le \int_{\mathbb{S}^1} \frac{1}{4}\partial_x^2\sigma^2 \tilde{u}^2 + \left(|\partial_x\sigma|^2 - \frac{1}{4}\partial_x^2\sigma^2\right)\tilde{q}^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} \partial_x\sigma\left(\tilde{u}^2 - \tilde{q}^2\right)\mathrm{d}x\,\dot{\tilde{W}}, \quad \text{in } \mathcal{D}'([0,T)),\ \tilde{\mathbb{P}}\text{–a.s.}, \\ \int_{\mathbb{S}^1}\left(\tilde{u}^2 + \tilde{q}^2\right)(0)\,\mathrm{d}x &= \int_{\mathbb{S}^1} \tilde{u}_0^2 + |\partial_x\tilde{u}_0|^2\,\mathrm{d}x,\end{aligned} \tag{7.11}$$

where $\tilde{q} = \partial_x\tilde{u}$ and so $\int_{\mathbb{S}^1} \tilde{u}^2 + \tilde{q}^2\,\mathrm{d}x = \|\tilde{u}\|^2_{H^1(\mathbb{S}^1)}$. By modifying the proof of Lemma 7.4, we can use (7.11) to establish the validity of the claim (1.4) in Theorem 1.1, and also that the limit $\tilde{u}$ satisfies part (f) of Definition 2.4. This is the content of the next lemma.

Lemma 7.6 *(Energy inequality and one-sided temporal continuity). Suppose* (7.11) *holds. Then the total energy inequality* (1.4) *holds a.s., for a.e. $s \in [0,T)$ and every t with $s < t \le T$. Specifically, it holds for $s = 0$ and any $t \in (0,T]$, with*

$$\int_{\mathbb{S}^1}\left(\tilde{u}^2 + |\partial_x\tilde{u}|^2\right)(0)\,\mathrm{d}x = \int_{\mathbb{S}^1} \tilde{u}_0^2 + |\partial_x\tilde{u}_0|^2\,\mathrm{d}x.$$

Consequently, a.s., for a.e. $t_0 \in [0, T)$,

$$\lim_{t \downarrow t_0} \|\tilde{u}(t) - \tilde{u}(t_0)\|_{H^1(\mathbb{S}^1)} = 0, \tag{7.12}$$

where the case $t_0 = 0$, for which $\tilde{u}(0) = \tilde{u}_0$, is covered by Lemma 7.4.

Remark 7.7. When we assert that a property is true "a.s. for a.e. $t \in [0, T]$", it means that for almost every $\tilde{\omega} \in \tilde{\Omega}$, under the probability measure $\tilde{\mathbb{P}}$, there exists a Lebesgue negligible subset $N = N(\tilde{\omega}) \subset [0, T]$ such that the stated property holds true for every $t \in [0, T] \setminus N(\tilde{\omega})$. Consider the inequality (1.4), which can be abstractly represented as $\mathcal{I}(\tilde{\omega}, t) \le 0$ for some function $\mathcal{I}$ on $\tilde{\Omega} \times [0, T]$ (with s fixed). For a.e. $\tilde{\omega}$, there exists a negligible set $N(\tilde{\omega}) \subset [0, T]$, such that the inequality $\mathcal{I}(\tilde{\omega}, t) \le 0$ remains valid for all $t \in [0, T] \setminus N(\tilde{\omega})$. Now note that our function $\mathcal{I}$ is integrable on the product space $\tilde{\Omega} \times [0, T]$. Considering this, we can employ the Tonelli theorem to conclude that the inequality $\mathcal{I}(\tilde{\omega}, t) \le 0$ is indeed valid for a.e. $(\tilde{\omega}, t) \in \tilde{\Omega} \times [0, T]$, i.e., $\mathcal{I}$ is nonpositive on the product space $\tilde{\Omega} \times [0, T]$.

Proof. We employ a standard deterministic argument, see for example [30, page 653]. Consider test functions $0 \le \beta_\delta \in W^{1,\infty}([0, T])$ with $\delta > 0$ taking values in a sequence converging to zero. For given s and t with $0 \le s < t \le T$, consider $\delta > 0$ such that $s + \delta < t - \delta$. For such δ, let β_δ be the continuous piecewise linear function that equals 1 on $[s + \delta, t - \delta]$, 0 on $[0, s]$ and $[t, T]$, and is linear on $[s, s + \delta]$ and $[t - \delta, t]$. Then $\beta_\delta(t') \to \mathbb{1}_{[s,t]}(t')$ for a.e. $t' \in [0, T]$. For $t' \in [0, T]$, define

$$I_\delta(t') = \int_{\mathbb{S}^1} \left(\frac{1}{4}\partial_x^2\sigma^2\tilde{u}^2 + \left(|\partial_x\sigma|^2 - \frac{1}{4}\partial_x^2\sigma^2 \right) \tilde{q}^2 \right)(t')\, \beta_\delta(t')\, \mathrm{d}x,$$

$$\Psi_\delta(t') = \int_{\mathbb{S}^1} \partial_x\sigma \left(\tilde{u}^2 - \tilde{q}^2 \right)(t')\, \beta_\delta(t')\, \mathrm{d}x.$$

By using β_δ as the test function in (7.11), we obtain the following result (a.s.):

$$\begin{aligned} &\frac{1}{\delta} \int_{t-\delta}^{t} \left\|\tilde{u}(t')\right\|_{H^1(\mathbb{S}^1)}^2 dt' - \frac{1}{\delta} \int_{s}^{s+\delta} \left\|\tilde{u}(t')\right\|_{H^1(\mathbb{S}^1)}^2 dt' \\ &\qquad \le \int_0^T I_\delta(t')\, dt' + \int_0^T \Psi_\delta(t')\, d\tilde{W}(t'). \end{aligned}$$

We apply Lebesgue's differentiation theorem to send δ to zero. As a result, we obtain the following inequality for all Lebesgue points $0 \le s < t \le T$ of the function $t' \mapsto \left\|\tilde{u}(\tilde{\omega}, t')\right\|_{H^1(\mathbb{S}^1)}^2$, which is integrable on $[0, T]$ for a.e. $\tilde{\omega}$:

$$\|\tilde{u}(\tilde{\omega}, t)\|_{H^1(\mathbb{S}^1)}^2 - \|\tilde{u}(\tilde{\omega}, s)\|_{H^1(\mathbb{S}^1)}^2 \le \int_s^t I(t')\, dt' + \int_s^t \Psi(t')\, d\tilde{W}(t'). \tag{7.13}$$

Here, I and Ψ are defined in the same manner as I_δ and Ψ_δ, respectively, but with the substitution of β_δ by 1. To be more precise, for each fixed $\tilde{\omega}$ from a set F of full $\tilde{\mathbb{P}}$–measure, there exists a subset $N(\tilde{\omega}) \subset [0, T]$ of zero Lebesgue measure such that (7.13) holds for every $t \in [0, T] \setminus N(\tilde{\omega})$.

The only distinction from the deterministic argument is the necessity to pass to the ($\delta \to 0$) limit in the stochastic integrals $\int_0^T \Psi_\delta \, d\tilde{W}$, where we clearly have $\left|\Psi_\delta - \mathbb{1}_{[s,t]}\Psi\right| \to 0$ a.e. in $\tilde{\Omega} \times [0, T]$. Furthermore, leveraging Lemma 5.4, it follows that $\left|\Psi_\delta - \mathbb{1}_{[s,t]}\Psi\right|^2 \le 4\,|\Psi|^2 \in L^1(\tilde{\Omega} \times [0, T])$. Thus, by the Lebesgue dominated convergence theorem, $\Psi_\delta \to \mathbb{1}_{[s,t]}\Psi$ in $L^2(\tilde{\Omega} \times [0, T])$. Hence, by the BDG inequality, we conclude that $\int_0^\cdot \Psi_\delta(t')\, d\tilde{W}(t') \to \int_0^\cdot \mathbb{1}_{[s,t]}(t')\Psi(t')\, d\tilde{W}(t')$ in $L^2\big(\tilde{\Omega}; C([0, T])\big)$. Passing to a subsequence, this convergence holds a.s. in $C([0, T])$.

Next, we note that (7.13) is valid for all values of t, not exclusively limited to the Lebesgue points. To see this, fix an arbitrary $t > s$, with $s \in [0, T) \setminus N(\tilde{\omega})$, $\tilde{\omega} \in F$ (s is a Lebesgue point of $\|\tilde{u}(\tilde{\omega}, \cdot)\|^2_{H^1(\mathbb{S}^1)}$). Let $t_\ell > s$, $t_\ell \in [0, T) \setminus N(\tilde{\omega})$, be a sequence of (Lebesgue) points converging to t as $\ell \to \infty$. In (7.13) we replace t by t_ℓ. Recalling that $\tilde{u} \in C\big([0, T]; H^1(\mathbb{S}^1) - w\big)$ a.s., see Lemma 5.4, which implies that $\tilde{u}$ is a.s. weakly lower semicontinuous in $H^1(\mathbb{S}^1)$, it then follows that

$$
\begin{aligned}
\|\tilde{u}(\tilde{\omega}, t)\|^2_{H^1(\mathbb{S}^1)} &\le \liminf_{\ell\to\infty} \|\tilde{u}(\tilde{\omega}, t_\ell)\|^2_{H^1(\mathbb{S}^1)} \\
&\overset{(7.13)}{\le} \|\tilde{u}(\tilde{\omega}, s)\|^2_{H^1(\mathbb{S}^1)} + \int_s^t I(t')\, dt' + \int_s^t \Psi(t')\, d\tilde{W}(t').
\end{aligned}
$$

Summarising, the inequality (7.13) holds for $\tilde{\mathbb{P}}$–a.e. $\tilde{\omega}$ (i.e., for any $\tilde{\omega} \in F$ with $\tilde{\mathbb{P}}(F) = 1$), for any time $t \in (0, T]$ and for any Lebesgue point s with $0 \le s < t \le T$ (i.e., $s \in [0, T) \setminus N(\tilde{\omega})$, $|N(\tilde{\omega})| = 0$). This proves the first part of the lemma.

The right-continuity of $\tilde{u}$ in $H^1(\mathbb{S}^1)$ at a Lebesgue point $s = t_0$ can be inferred from (7.13). More precisely, by the a.s. weak lower semicontinuity of $\tilde{u}$ and (7.13),

$$
\begin{aligned}
\|\tilde{u}(\tilde{\omega}, t_0)\|^2_{H^1(\mathbb{S}^1)} \le \liminf_{t\downarrow t_0} \|\tilde{u}(\tilde{\omega}, t)\|^2_{H^1(\mathbb{S}^1)} &\le \limsup_{t\downarrow t_0} \|\tilde{u}(\tilde{\omega}, t)\|^2_{H^1(\mathbb{S}^1)} \\
&\le \|\tilde{u}(\tilde{\omega}, t_0)\|^2_{H^1(\mathbb{S}^1)},
\end{aligned}
$$

so that $\lim_{t\downarrow t_0} \|\tilde{u}(\tilde{\omega}, t)\|^2_{H^1(\mathbb{S}^1)} = \|\tilde{u}(\tilde{\omega}, t_0)\|^2_{H^1(\mathbb{S}^1)}$, for any $\tilde{\omega} \in F$, $\tilde{\mathbb{P}}(F) = 1$, and $t_0 \in [0, T) \setminus N(\tilde{\omega})$, $|N(\tilde{\omega})| = 0$. As a result, we can employ a similar reasoning as in the proof of Lemma 7.4 to conclude that the right-continuity claim (7.12) holds.

Finally, utilizing the strong initial trace result (7.4), we can conclude that $s = 0$ is a Lebesgue point of $\|\tilde{u}(\cdot)\|^2_{H^1(\mathbb{T})}$. $\square$

7.2. *Equation for the weak limits $\overline{S(q)}$*

We will need to know that products like $S'(\tilde{q}_n)\, \tilde{P}_n$ converge weakly. Since $S'(\tilde{q}_n)$ converges weakly, it is crucial that $\tilde{P}_n$ converges strongly to $\tilde{P}$. To this end, we will make essential use of the space $L^r\big(L^r_w\big)$. First, by (5.12),

$$\left\|\tilde{q}_n^2 - \overline{q^2}\right\|_{L_t^r(H_x^{-1})}^r = \int_0^T \left\|\tilde{q}_n^2(t) - \overline{q^2}(t)\right\|_{H^{-1}(\mathbb{S}^1)}^r \, dt \xrightarrow{n\uparrow\infty} 0, \quad \text{a.s.} \tag{7.14}$$

By (5.13) and (5.11),

$$\tilde{\mathbb{E}} \int_0^T \left\|\tilde{q}_n^2(t) - \overline{q^2}(t)\right\|_{H^{-1}(\mathbb{S}^1)}^p \, dt \lesssim 1, \quad p \in \left[1, p_0/2\right]. \tag{7.15}$$

For any $\bar{p} > 1$ with $r\bar{p} \in \left[1, p_0/2\right]$ (recall that $r < 3/2$ and $p_0 > 4$), we use Hölder's inequality and (7.15) to deduce that $\tilde{\mathbb{E}} \left\|\tilde{q}_n^2 - \overline{q^2}\right\|_{L_t^r(H_x^{-1})}^{r\bar{p}} \lesssim_T 1$. Therefore, by (7.14) and Vitali's convergence theorem,

$$\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2} \quad \text{in } L_{\tilde{\omega},t}^r\left(H_x^{-1}\right). \tag{7.16}$$

Given (7.16), passing to a subsequence if necessary, we may assume that

$$\tilde{q}_n^2(\tilde{\omega}, t) \xrightarrow{n\uparrow\infty} \overline{q^2}(\tilde{\omega}, t) \quad \text{in } H^{-1}(\mathbb{S}^1), \text{ for a.e. } (\tilde{\omega}, t) \in \tilde{\Omega} \times [0, T]. \tag{7.17}$$

By Lebesgue interpolation between the convergence in $L_{\tilde{\omega},t}^r$, see (7.16), and the uniform boundedness in $L_{\tilde{\omega},t}^{p_0/2}$, see (7.15), we can improve (7.16) to

$$\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2} \quad \text{in } L_{\tilde{\omega},t}^p\left(H_x^{-1}\right), \quad p \in \left[1, p_0/2\right). \tag{7.18}$$

We can now prove the following result:

Lemma 7.8 *(Strong convergence of $\tilde{P}_n$). Let $\tilde{u}_n$, $\tilde{u}$, $\tilde{q}_n$, $\overline{q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9. Setting*

$$\begin{aligned} \tilde{P}_n &= K * \left(\tilde{u}_n^2 + \frac{1}{2}\tilde{q}_n^2\right), \quad n \in \mathbb{N}, \\ \tilde{P} &= K * \left(\tilde{u}^2 + \frac{1}{2}\overline{q^2}\right), \end{aligned} \tag{7.19}$$

the following strong convergence holds:

$$\tilde{P}_n \xrightarrow{n\uparrow\infty} \tilde{P} \quad \text{in } L^r([0, T] \times \mathbb{S}^1), \ \tilde{\mathbb{P}}\text{–a.s.}, \tag{7.20}$$

where $r \in [1, 3/2)$ is fixed in (4.13). In addition, for any $p \in \left[1, p_0/2\right)$,

$$\tilde{P}_n \xrightarrow{n\uparrow\infty} \tilde{P} \quad \text{in } L^p(\tilde{\Omega} \times [0, T] \times \mathbb{S}^1), \tag{7.21}$$

where $p_0 > 4$ is specified in Theorem 2.3.

Proof. For any (t,x),

$$\left|K*\tilde{q}_n^2 - K*\overline{q^2}\right|(t,x) = \left|\int_{\mathbb{S}^1} K(x-y)\left(\tilde{q}_n^2(t,y)-\overline{q^2}(t,y)\right)\mathrm{d}y\right| \le \|K(x-\cdot)\|_{H^1(\mathbb{S}^1)}\left\|\tilde{q}_n^2(t)-\overline{q^2}(t)\right\|_{H^{-1}(\mathbb{S}^1)},$$

where $\|K(x-\cdot)\|_{H^1(\mathbb{S}^1)} \lesssim 1$ for all x. Raising this to the rth power and then integrating in t and x, we arrive at

$$\int_0^T\int_{\mathbb{S}^1}\left|\int_{\mathbb{S}^1} K(x-y)\left(\tilde{q}_n^2(t,y)-\overline{q^2}(t,y)\right)\mathrm{d}y\right|^r \mathrm{d}x\,\mathrm{d}t \lesssim \left\|\tilde{q}_n^2-\overline{q^2}\right\|^r_{L^r([0,T];H^{-1}(\mathbb{S}^1))} \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.},$$

using (7.14). By (4.24), $\tilde{u}_n^2 \xrightarrow{n\uparrow\infty} \tilde{u}^2$ in $C_tL^1_x$ a.s. and so

$$\int_0^T\int_{\mathbb{S}^1}\left|\int_{\mathbb{S}^1} K(x-y)\left(\tilde{u}_n^2(t,y)-\tilde{u}^2(t,y)\right)\mathrm{d}y\right|^r \mathrm{d}x\,\mathrm{d}t \lesssim \left\|\tilde{u}_n^2-\tilde{u}^2\right\|^r_{L^\infty([0,T];L^1(\mathbb{S}^1))} \xrightarrow{n\uparrow\infty} 0, \quad \tilde{\mathbb{P}}\text{–a.s.}$$

Hence, (7.20) follows.

Let us now turn to the proof of (7.21). From the previous calculations,

$$\left|\tilde{P}_n(\tilde{\omega},t,x)-\tilde{P}(\tilde{\omega},t,x)\right| \lesssim \left\|\tilde{u}_n^2(\tilde{\omega},t)-\tilde{u}^2(\tilde{\omega},t)\right\|_{L^1(\mathbb{S}^1)} + \left\|\tilde{q}_n^2(\tilde{\omega},t)-\overline{q^2}(\tilde{\omega},t)\right\|_{H^{-1}(\mathbb{S}^1)} \xrightarrow{n\uparrow\infty} 0, \tag{7.22}$$

for a.e. $(\tilde{\omega},t,x)$, using (7.17) and also that $\tilde{u}_n^2(\tilde{\omega},t) \xrightarrow{n\uparrow\infty} \tilde{u}^2(\tilde{\omega},t)$ in L^1_x, uniformly in $t\in[0,T]$, $\tilde{\mathbb{P}}$–a.e. in $\tilde{\omega}$, cf. (4.24). By Lemma 5.6 and (5.10),

$$\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1}\left|\tilde{P}_n-\tilde{P}\right|^{p_0/2}\mathrm{d}x\,\mathrm{d}t \lesssim 1.$$

Combining the a.e. convergence (7.22) with this n-uniform bound in $L^{p_0/2}_{\tilde{\omega},t,x}$, the Vitali convergence theorem gives (7.21). □

The remaining part of this section is devoted to the study of the defect measure $\mathbb{D}$ defined in (1.12), which will be done by analysing the related defects $\overline{S(q)} - S(\tilde{q})$, for an appropriate class of nonlinearities S (for reasons outlined in Section 1). We compute $\overline{S(q)}$ and $S(\tilde{q})$ in this section and Section 7.3, before we put the results together in Section 7.4 to conclude that $\mathbb{D} = 0$.

Lemma 7.9 below shows that the a.s. weak limit $\overline{S(q)}$ of $S(\tilde{q}_n)$, see (4.24) and (3.4), satisfies the following pathwise inequality in $\mathcal{D}'([0,T) \times \mathbb{S}^1)$:

$$
\begin{aligned}
&\partial_t \overline{S(q)} + \partial_x \left[\tilde{u}\, \overline{S(q)} + \frac{1}{4} \partial_x \sigma^2 \left(3\overline{S(q)} - 2\overline{S'(q)q} \right) \right] \\
&\quad - \partial_{xx}^2 \left[\frac{1}{2} \sigma^2\, \overline{S(q)} \right] + \left[\overline{S'(q)} \left(P\left[\tilde{u}, \overline{q^2}\right] - \tilde{u}^2 \right) - \left(\overline{S(q)q} - \frac{1}{2} \overline{S'(q)q^2} \right) \right. \\
&\quad \left. - \frac{1}{4} \partial_{xx}^2 \sigma^2 \left(\overline{S(q)} - \overline{S'(q)q} \right) - \frac{1}{2} \left| \partial_x \sigma \right|^2 \overline{S''(q)\, q^2} \right] \\
&\quad + \left[\partial_x \left(\sigma\, \overline{S(q)} \right) - \partial_x \sigma \left(\overline{S(q)} - \overline{S'(q)q} \right) \right] \dot{\tilde{W}} \le 0,
\end{aligned}
\tag{7.23}
$$

along with the initial data $\overline{S(q)}(0) = S(\partial_x \tilde{u}_0)$. Regrettably, we cannot establish (7.23) along the lines of Proposition 6.3. The obstacle is that passing to the limit in some terms is hampered by the lack of strong temporal compactness. Instead we will furnish a "direct" weak convergence proof, relying on [25, Lemma 2.1] to establish the convergence of stochastic integrals of processes like $\int_{\mathbb{S}^1} S'(\tilde{q}_n)\tilde{q}_n \,\mathrm{d}x$. A priori, these processes only converge weakly in L_t^{2r}. However, we have devoted much effort to showing that, e.g., $S'(\tilde{q}_n)\tilde{q}_n$ converges a.s. in the strong-weak space $L^{2r}\left(L_w^{2r}\right)$, cf. (4.24). This implies that $\int_{\mathbb{S}^1} S'(\tilde{q}_n)\tilde{q}_n \,\mathrm{d}x$ converges strongly in L_t^2, which in turn allows for the application of [25, Lemma 2.1].

Lemma 7.9 *(Characterisation of weak limit). Denote by $S = S(v)$ any of the functions $S_\ell(v_\pm)$, defined by* (4.2)*, or $\frac{1}{2}v^2, \frac{1}{2}v_\pm^2$. Let $\overline{S(q)}$, $\overline{S'(q)}$, $\overline{S(q)q}$, $\overline{S'(q)q}$, $\overline{S'(q)q^2}$ and $\overline{S''(q)q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9, see also Remark 4.10, and let $\tilde{P}$ be defined by* (7.19)*. Then the inequality* (7.23) *holds weakly in (t,x), almost surely, that is, for any $0 \le \varphi \in C_c^\infty([0,T) \times \mathbb{S}^1)$,*

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{S}^1} \overline{S(q)}\, \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{S}^1} S(\partial_x \tilde{u}_0)\, \varphi(0,x) \,\mathrm{d}x \\
&\quad + \int_0^T \int_{\mathbb{S}^1} \left[\tilde{u}\, \overline{S(q)} + \frac{1}{4} \partial_x \sigma^2\, \overline{H^{(1)}(q)} \right] \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{S}^1} \frac{1}{2} \sigma^2\, \overline{S(q)}\, \partial_{xx}^2 \varphi \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T\int_{\mathbb{S}^1}\Big[\overline{S'(q)}\left(\tilde{P}-\tilde{u}^2\right)-\overline{H^{(2)}(q)} \\
&\qquad -\frac{1}{4}\partial_{xx}^2\sigma^2\,\overline{H^{(3)}(q)}-\frac{1}{2}\left|\partial_x\sigma\right|^2\overline{S''(q)\,q^2}\Big]\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_0^T\int_{\mathbb{S}^1}\sigma\,\overline{S(q)}\,\partial_x\varphi+\partial_x\sigma\,\overline{H^{(3)}(q)}\,\varphi\,\mathrm{d}x\,\mathrm{d}\tilde{W}\geq 0,\quad \tilde{\mathbb{P}}\text{–a.s.},
\end{aligned}
\tag{7.24}
$$

where we have introduced the functions

$$
\begin{aligned}
&H^{(1)}(v)=3S(v)-2S'(v)v,\quad H^{(2)}(v)=S(v)v-\tfrac{1}{2}S'(v)v^2,\\
&H^{(3)}(v)=S(v)-S'(v)v.
\end{aligned}
\tag{7.25}
$$

By the linearity of weak limits, we have $\overline{H^{(1)}(q)}=3\overline{S(q)}-2\overline{S'(q)q}$, $\overline{H^{(2)}(q)}=\overline{S(q)q}-\frac{1}{2}\overline{S'(q)q^2}$, *and* $\overline{H^{(3)}(q)}=\overline{S(q)}-\overline{S'(q)q}$.

Proof. Referring to Proposition 4.9, $\left(\tilde{u}_n,\tilde{q}_n=\partial_x\tilde{u}_n,\tilde{W}_n,\tilde{u}_{0,n}\right)$ are the Skorokhod–Jakubowski representations of $\left(u_{\varepsilon_n},q_{\varepsilon_n}=\partial_x u_{\varepsilon_n},W,z_n\right)$, respectively, where u_{ε_n} is the strong solution to the viscous SPDE (1.5) with noise W and initial function $u_{\varepsilon_n}(0)=z_n$, cf. (4.12). As in the proof of Lemma 7.2, $S(\tilde{q}_n)$ satisfies (3.4) with u_ε, q_ε, W, ε replaced by $\tilde{u}_n$, $\tilde{q}_n$, $\tilde{W}_n$, ε_n, respectively, where $\varepsilon_n\to 0$ as $n\to\infty$.

Fix a non-negative test function $\varphi\in C_c^\infty([0,T)\times\mathbb{S}^1)$. Multiply (3.4) by φ, integrate over (t,x), and then do integration-by-parts in time, keeping in mind that $S(\tilde{q}_n(0))=S(\partial_x\tilde{u}_{0,n})$. Dropping the dissipation term and employing the notation (7.25), the end result is

$$
\tilde{I}_n(\tilde{\omega})+\tilde{M}_n(\tilde{\omega})\geq 0,\quad \text{for } \tilde{\mathbb{P}}\text{–a.e. } \tilde{\omega}\in\tilde{\Omega},
\tag{7.26}
$$

where $\tilde{M}_n(\tilde{\omega})=\tilde{\mathcal{M}}_n(\tilde{\omega},T)$ with

$$
\tilde{\mathcal{M}}_n(t)=\int_0^t\int_{\mathbb{S}^1}\sigma_{\varepsilon_n}\,S(\tilde{q}_n)\,\partial_x\varphi+\partial_x\sigma_{\varepsilon_n}\,H^{(3)}(\tilde{q}_n)\,\varphi\,\mathrm{d}x\,\mathrm{d}\tilde{W}_n,\quad t\in[0,T],
\tag{7.27}
$$

and $\tilde{I}_n=\sum_{i=1}^{10}\tilde{I}_n^{(i)}$ with

$$
\begin{aligned}
&\tilde{I}_n^{(1)}=\int_0^T\int_{\mathbb{S}^1}S(\tilde{q}_n)\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t,\quad \tilde{I}_n^{(2)}=\int_{\mathbb{S}^1}S(\partial_x\tilde{u}_{0,n})\,\varphi(0,x)\,\mathrm{d}x,\\
&\tilde{I}_n^{(3)}=\int_0^T\int_{\mathbb{S}^1}\tilde{u}_n\,S(\tilde{q}_n)\,\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}_n^{(4)} &= \frac{1}{4}\int_0^T\int_{\mathbb{S}^1} \partial_x \sigma_{\varepsilon_n}^2 \, H^{(1)}(\tilde{q}_n)\, \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t, \\
\tilde{I}_n^{(5)} &= \int_0^T\int_{\mathbb{S}^1} \left(\frac{1}{2}\sigma_{\varepsilon_n}^2 + \varepsilon_n\right) S(\tilde{q}_n)\, \partial_{xx}^2 \varphi \, \mathrm{d}x \, \mathrm{d}t, \\
\tilde{I}_n^{(6)} &= -\int_0^T\int_{\mathbb{S}^1} S'(\tilde{q}_n)\, \tilde{P}_n \, \varphi \, \mathrm{d}x \, \mathrm{d}t, \quad \tilde{I}_n^{(7)} = \int_0^T\int_{\mathbb{S}^1} S'(\tilde{q}_n)\, \tilde{u}_n^2 \, \varphi \, \mathrm{d}x \, \mathrm{d}t, \\
\tilde{I}_n^{(8)} &= \int_0^T\int_{\mathbb{S}^1} H^{(2)}(\tilde{q}_n)\, \varphi \, \mathrm{d}x \, \mathrm{d}t, \quad \tilde{I}_n^{(9)} = \frac{1}{4}\int_0^T\int_{\mathbb{S}^1} \partial_{xx}^2 \sigma_{\varepsilon_n}^2 \, H^{(3)}(\tilde{q}_n)\, \varphi \, \mathrm{d}x \, \mathrm{d}t, \\
\tilde{I}_n^{(10)} &= \frac{1}{2}\int_0^T\int_{\mathbb{S}^1} \left|\partial_x \sigma_{\varepsilon_n}\right|^2 S''(\tilde{q}_n)\, \tilde{q}_n^2 \, \varphi \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{7.28}
$$

We can also write the claim (7.24) of the lemma in the form

$$
\tilde{I}(\tilde{\omega}) + \tilde{M}(\tilde{\omega}) \geq 0, \quad \text{for } \tilde{\mathbb{P}}\text{–a.e. } \tilde{\omega} \in \Omega, \tag{7.29}
$$

where $\tilde{M}(\tilde{\omega}) = \tilde{\mathcal{M}}(T)$ and $\tilde{I} = \sum_{i=1}^{10} \tilde{I}^{(i)}$ are defined as in (7.27) and (7.28) via the corresponding limit terms identified in (7.24). Below we will prove that

$$
\begin{aligned}
&\tilde{I}_n^{(i)} \xrightarrow{n\uparrow\infty} I^{(i)} \ \tilde{\mathbb{P}}\text{–a.s. and strongly in } L^2(\tilde{\Omega}), \quad \forall i \notin \{3,6,7\}, \\
&\tilde{I}_n^{(i)} \xrightarrow{n\uparrow\infty} I^{(i)} \text{ in } L^1(\tilde{\Omega}), \quad i \in \{3,6,7\}.
\end{aligned}
\tag{7.30}
$$

Moreover, we will prove that the stochastic integral term converges strongly in the sense that (for a non-relabelled subsequence)

$$
\tilde{M}_n \xrightarrow{n\uparrow\infty} \tilde{M} \text{ in } L^2(\tilde{\Omega}). \tag{7.31}
$$

Given (7.26), the convergences (7.30) and (7.31) imply that

$$
\int_{\tilde{\Omega}} \mathbb{1}_A(\tilde{\omega})\big(I(\tilde{\omega}) + M(\tilde{\omega})\big)\, \mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega}) \geq 0,
$$

for any measurable set $A \in \tilde{\mathcal{F}}$, which is enough to conclude that (7.29) holds.

It remains to verify (7.30) and (7.31). Let us start with (7.30). We do this only for the most challenging choices of S, namely $S(v) = \frac{1}{2}v^2, \frac{1}{2}v_\pm^2$. The argument is the same for $S = S_\ell$ (in fact, it is simpler because $S_\ell(v) \lesssim_\ell |v|$).

Regarding the convergences of $I_n^{(i)}$ for $i \neq \{3,6,7\}$, cf. (7.30), they are all direct consequences of the a.s. convergences in (4.24). We detail only the case $i = 1$. By (4.24) and $S(v) = \frac{1}{2}v^2, \frac{1}{2}v_\pm^2$, we have, in particular, that $S(\tilde q_n) \xrightarrow{n\uparrow\infty} \overline{S(q)}$ in $L^r([0,T]\times\mathbb{S}^1)$ a.s. and so $\tilde I_n^{(1)} \xrightarrow{n\uparrow\infty} \tilde I^{(1)}$ a.s. By Lemma 5.3, we also have the n-independent bound

$$\tilde{\mathbb{E}}\left\|\tilde I_n^{(1)}\right\|_{L^p(\tilde\Omega)}^p \lesssim_\varphi \tilde{\mathbb{E}}\left(\int_0^T\int_{\mathbb{S}^1} |\tilde q_n|^2\,\mathrm{d}x\,\mathrm{d}t\right)^p \lesssim_T \tilde{\mathbb{E}}\,\|\tilde q_n\|_{L^\infty([0,T];L^2(\mathbb{S}^1))}^{2p} \lesssim 1,$$

for $p\in\big[1,p_0/2\big)$. Thus, by Vitali's convergence theorem, $\tilde I_n^{(1)} \xrightarrow{n\uparrow\infty} \tilde I^{(1)}$ in $L^2(\tilde\Omega)$.

Let us consider the exceptional term $I_n^{(3)}$. In view of (4.24), $S(\tilde q_n) \xrightarrow{n\uparrow\infty} \overline{S(q)}$ in $L^r_{t,x}$ a.s., where $r\in[1,3/2)$ (and so $r'=\frac{r}{r-1}>3$). Given Lemma 5.3, we also have the bound $\tilde{\mathbb{E}}\,\|S(\tilde q_n)\|_{L^r_{t,x}}^r \lesssim 1$. Hence, by a weak compactness argument, we may assume that $\overline{S(q)}\in L^r(\tilde\Omega\times[0,T]\times\mathbb{S}^1)$ and

$$S(\tilde q_n) \xrightarrow{n\uparrow\infty} \overline{S(q)} \quad \text{in } L^r(\tilde\Omega\times[0,T]\times\mathbb{S}^1). \tag{7.32}$$

By (4.24), $\tilde u_n \xrightarrow{n\uparrow\infty} \tilde u$ in $C_tL^2_x$ a.s. In view of Lemma 5.6 and Vitali's convergence theorem, we thus obtain $\tilde u_n \xrightarrow{n\uparrow\infty} \tilde u$ in $L^2(\tilde\Omega\times[0,T]\times\mathbb{S}^1)$. Apart from that, Lemma 5.6 delivers the bounds $\tilde{\mathbb{E}}\,\big\|\big(\tilde u_n,\tilde u\big)\big\|_{L^{p_0}_{t,x}}^{p_0} \lesssim 1$, where $p_0>4$ and we may assume $3<r'<p_0$. Accordingly, we gather that

$$\tilde u_n \xrightarrow{n\uparrow\infty} \tilde u \quad \text{in } L^{r'}(\tilde\Omega\times[0,T]\times\mathbb{S}^1). \tag{7.33}$$

Given (7.32) and (7.33), by the weak convergence of products of strongly and weakly converging sequences,

$$\tilde u_n\, S(\tilde q_n) \xrightarrow{n\uparrow\infty} \tilde u\,\overline{S(q)} \quad \text{in } L^1(\tilde\Omega\times[0,T]\times\mathbb{S}^1),$$

which proves (7.30) for $i=3$. Next, consider the term $I_n^{(7)}$. We need to verify the weak $L^1_{\tilde\omega,t,x}$ convergence of the product $S'(\tilde q_n)\tilde u_n^2$. From (4.24), $S'(\tilde q_n) \xrightarrow{n\uparrow\infty} \overline{S'(q)}$ in $L^{2r}_{t,x}$ a.s., but Lemma 5.3 also supplies the bound $\tilde{\mathbb{E}}\,\big\|S'(\tilde q_n)\big\|_{L^{2r}_{t,x}}^{2r} \lesssim 1$. Thus, by a weak compactness argument, we may assume $\overline{S'(q)}\in L^{2r}(\tilde\Omega\times[0,T]\times\mathbb{S}^1)$ and

$$S'(\tilde q_n) \xrightarrow{n\uparrow\infty} \overline{S'(q)} \quad \text{in } L^{2r}(\tilde\Omega\times[0,T]\times\mathbb{S}^1). \tag{7.34}$$

As $2<2r<3$ (with $2r$ close to 3) and so $3/2<(2r)'<2$, arguing as for (7.33), we may assume that $\tilde u_n \xrightarrow{n\uparrow\infty} \tilde u$ in $L^{2(2r)'}(\tilde\Omega\times[0,T]\times\mathbb{S}^1)$. As a result, writing $\tilde u_n^2-\tilde u^2=(\tilde u_n-\tilde u)\,(\tilde u_n+\tilde u)$ and using the Cauchy–Schwarz inequality,

$$\tilde u_n^2 \xrightarrow{n\uparrow\infty} \tilde u^2 \quad \text{in } L^{(2r)'}(\tilde\Omega\times[0,T]\times\mathbb{S}^1).$$

Combining this with (7.34), we obtain $S'(\tilde{q}_n)\,\tilde{u}_n^2 \xrightarrow{n\uparrow\infty} \overline{S'(q)}\,\tilde{u}^2$ in $L^1_{\tilde{\omega},t,x}$, thereby establishing (7.30) for $i=7$.

Next, we turn to the weak $L^1_{\tilde{\omega},t,x}$ convergence of the delicate product term $S'(\tilde{q}_n)\,\tilde{P}_n$. Fortunately, most of the "heavy lifting" has already been done, since (7.21) implies that $\tilde{P}_n \xrightarrow{n\uparrow\infty} \tilde{P}$ in $L^2(\tilde{\Omega}\times[0,T]\times\mathbb{S}^1)$. On the other hand, given (7.34), $S'(\tilde{q}_n) \xrightarrow{n\uparrow\infty} \overline{S'(q)}$ in $L^2(\tilde{\Omega}\times[0,T]\times\mathbb{S}^1)$, and thus $S'(\tilde{q}_n)\,\tilde{P}_n \xrightarrow{n\uparrow\infty} \overline{S'(q)}\,\tilde{P}$ in $L^1(\tilde{\Omega}\times[0,T]\times\mathbb{S}^1)$. This proves (7.30) for $i=6$.

Finally, we consider the stochastic integral term (7.27), which we write as

$$\tilde{\mathcal{M}}_n(t) = \int_0^t \tilde{\mathcal{J}}_n(s)\,\mathrm{d}\tilde{W}_n, \quad \tilde{\mathcal{J}}_n = \int_{\mathbb{S}^1} \sigma_{\varepsilon_n}\,S(\tilde{q}_n)\,\partial_x\varphi + \partial_x\sigma_{\varepsilon_n}\,H^{(3)}(\tilde{q}_n)\,\varphi\,\mathrm{d}x.$$

We divide the argument into two cases, depending on the choice of S, namely $S_\ell(v_\pm)$, cf. (4.2), or $\frac{1}{2}v^2$, $\frac{1}{2}v_\pm^2$.

Let us begin with the case $S(v)=S_\ell(v_\pm)$. By (4.24), $S(\tilde{q}_n) \xrightarrow{n\uparrow\infty} \overline{S(q)}$, $S'(\tilde{q}_n)\tilde{q}_n \xrightarrow{n\uparrow\infty} \overline{S'(q)q}$, and thus $H^{(3)}(\tilde{q}_n) \xrightarrow{n\uparrow\infty} \overline{H^{(3)}(q)}$ in the strong-weak space $L^{2r}\big(L^{2r}_w\big)$ a.s. (with $2r>2$). Because of this and $\sigma\,\partial_x\varphi,\ \partial_x\sigma\varphi \in L^\infty_{t,x}$,

$$\tilde{\mathcal{J}}_n \xrightarrow{n\uparrow\infty} \int_{\mathbb{S}^1} \sigma\,\overline{S(q)}\,\partial_x\varphi + \partial_x\sigma\,\overline{H^{(3)}(q)}\,\varphi\,\mathrm{d}x =: \tilde{\mathcal{J}} \quad \text{in } L^{2r}([0,T]), \text{ a.s.}$$

This implies that $\tilde{\mathcal{J}}_n \to \tilde{\mathcal{J}}$ in $L^2([0,T])$, in probability. By (4.24), $\tilde{W}_n \to \tilde{W}$ in $C([0,T])$ a.s., and thus in probability. The assumptions of [25, Lemma 2.1] are therefore fulfilled, with the result that

$$\tilde{\mathcal{M}}_n \xrightarrow{n\uparrow\infty} \tilde{\mathcal{M}} \quad \text{in } L^2([0,T]), \text{ in probability,} \tag{7.35}$$

where $\tilde{\mathcal{M}}(t) = \int_0^t \tilde{\mathcal{J}}(s)\,\mathrm{d}\tilde{W}$. By passing to a subsequence if necessary, we may assume that this convergence holds $\tilde{\mathbb{P}}$–almost surely. Note also that the exceptional set does not depend on the particular test function $\varphi \in C_c^\infty([0,T)\times\mathbb{S}^1)$ (by the separability of C_c^∞).

Next, suppose $S(v)=\frac{1}{2}v^2$, noting that $H^{(3)}(v) = S(v) - S(v)'v$ in this case becomes $-\frac{1}{2}v^2$ and so

$$\tilde{\mathcal{M}}_n(t) = \int_0^t \tilde{\mathcal{J}}_n(s)\,\mathrm{d}\tilde{W}_n, \quad \tilde{\mathcal{J}}_n = \int_{\mathbb{S}^1} \frac{1}{2}\big(\sigma_{\varepsilon_n}\,\partial_x\varphi - \partial_x\sigma_{\varepsilon_n}\,\varphi\big)\,\tilde{q}_n^2\,\mathrm{d}x,$$

$$\tilde{\mathcal{M}}(t) = \int_0^t \tilde{\mathcal{J}}(s)\,\mathrm{d}\tilde{W}, \quad \tilde{\mathcal{J}} = \int_{\mathbb{S}^1} \frac{1}{2}\big(\sigma\,\partial_x\varphi - \partial_x\sigma\,\varphi\big)\,\overline{q^2}\,\mathrm{d}x.$$

According to our previous considerations—leading up to (7.18)—we have the crucial convergence $\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2}$ in $L^2_{\tilde{\omega},t}\big(H_x^{-1}\big)$, which amounts to strong L^2 convergence in $\tilde{\omega}, t$, see also the proof of Lemma 7.2. This implies that $\tilde{\mathcal{J}}_n \to \tilde{\mathcal{J}}$ in $L^2([0,T])$, in probability. As before, $\tilde{W}_n \to \tilde{W}$ in $C([0,T])$ a.s., and thus in probability. As a result, Lemma 2.1 of [25] supplies (7.35). By passing to a subsequence, we may assume that this convergence holds $\tilde{\mathbb{P}}$–almost surely.

The cases $S(v) = \frac{1}{2}v_\pm^2$ can be viewed in the same way, noting that $H^{(3)}(v) = S(v_\pm) - S(v_\pm)'q = -\frac{1}{2}v_\pm^2$ and that the pivotal convergence (7.18) still holds for $(\tilde{q}_n)_\pm^2$. Indeed, with the same proof, $(\tilde{q}_n)_\pm^2 \xrightarrow{n\uparrow\infty} \overline{q_\pm^2}$ in $L^2_{\tilde{\omega},t}\big(H_x^{-1}\big)$.

Finally, let us establish the sought-after convergence claim (7.31). Indeed, by the previous findings,

$$\left\|\tilde{\mathcal{M}}_n - \tilde{\mathcal{M}}\right\|^2_{L^2([0,T])} \xrightarrow{n\uparrow\infty} 0 \quad \text{a.s.}, \tag{7.36}$$

and, for any $p \in \big[2, p_0/2\big]$ (recall $p_0 > 4$),

$$\tilde{\mathbb{E}}\left\|\tilde{\mathcal{M}}_n - \tilde{\mathcal{M}}\right\|^p_{L^2([0,T])} \lesssim_T \tilde{\mathbb{E}}\sup_{t\in[0,T]}\left|\tilde{\mathcal{M}}_n(t)\right|^p + \tilde{\mathbb{E}}\sup_{t\in[0,T]}\left|\tilde{\mathcal{M}}(t)\right|^p \lesssim_T 1. \tag{7.37}$$

The last bound follows from the following calculations:

$$\begin{aligned}\tilde{\mathbb{E}}\sup_{t\in[0,T]}\left|\tilde{\mathcal{M}}_n(t)\right|^p &\lesssim \tilde{\mathbb{E}}\left[\left(\int_0^T\left|\tilde{\mathcal{J}}_n(t)\right|^2 \mathrm{d}t\right)^{p/2}\right] \\ &\lesssim_T \tilde{\mathbb{E}}\int_0^T\left|\int_{\mathbb{S}^1}\tilde{q}_n^2(t)\,\mathrm{d}x\right|^p \mathrm{d}t \lesssim_T \tilde{\mathbb{E}}\,\|\tilde{q}_n\|^{2p}_{L^\infty([0,T];L^2(\mathbb{S}^1))} \overset{\text{Lemma 5.3}}{\lesssim} 1,\end{aligned} \tag{7.38}$$

where we have used the BDG and Hölder inequalities, and similarly

$$\begin{aligned}\tilde{\mathbb{E}}\sup_{t\in[0,T]}\left|\tilde{\mathcal{M}}(t)\right|^p &\lesssim \tilde{\mathbb{E}}\left[\left(\int_0^T\left|\tilde{\mathcal{J}}(t)\right|^2 \mathrm{d}t\right)^{p/2}\right] \\ &\lesssim_T \tilde{\mathbb{E}}\int_0^T\left|\int_{\mathbb{S}^1}\overline{q^2}(t)\,\mathrm{d}x\right|^p \mathrm{d}t \le \tilde{\mathbb{E}}\int_0^T\left\|\overline{q^2}(t)\right\|^p_{H^{-1}(\mathbb{S}^1)} \mathrm{d}t \overset{(5.9)}{\lesssim} 1.\end{aligned} \tag{7.39}$$

Given (7.36) and (7.37), Vitali's convergence theorem returns

$$\tilde{\mathbb{E}}\int_0^T\left|\tilde{\mathcal{M}}_n(t) - \tilde{\mathcal{M}}(t)\right|^2 \mathrm{d}t \xrightarrow{n\uparrow\infty} 0.$$

Passing to a subsequence, we conclude that

$$D_n(t) = \tilde{\mathbb{E}} \left| \tilde{\mathcal{M}}_n(t) - \tilde{\mathcal{M}}(t) \right|^2 \xrightarrow{n\uparrow\infty} 0, \quad \text{for a.e. in } t \in [0,T]. \tag{7.40}$$

We may assume that $D_n(t) \to 0$ for *all* $t \in [0,T]$, as the function $D_n(t)$ depends continuously on $t \in [0,T]$, uniformly in n. Let us explain why. Through some straightforward manipulations and by utilising (7.38) and (7.39),

$$|D_n(t_2) - D_n(t_1)|^2 \lesssim \tilde{\mathbb{E}} \left| \tilde{\mathcal{M}}_n(t_2) - \tilde{\mathcal{M}}_n(t_1) \right|^2 + \tilde{\mathbb{E}} \left| \tilde{\mathcal{M}}(t_2) - \tilde{\mathcal{M}}(t_1) \right|^2 .$$

We will estimate the terms on the right separately. Suppose $S(v) = \frac{1}{2}v^2$. The other cases $S(v) = S_\ell(v)$, $\frac{1}{2}v_\pm^2$ can be treated similarly. For $0 \le t_1 < t_2 \le T$, the Itô isometry implies

$$\begin{aligned} \tilde{\mathbb{E}} \left| \tilde{\mathcal{M}}_n(t_2) - \tilde{\mathcal{M}}_n(t_1) \right|^2 &= \tilde{\mathbb{E}} \int_{t_1}^{t_2} \left| \tilde{\mathcal{J}}_n(t) \right|^2 \mathrm{d}t \lesssim_{\sigma,\varphi} \tilde{\mathbb{E}} \int_{t_1}^{t_2} \left| \int_{\mathbb{S}^1} \tilde{q}_n^2(t)\,\mathrm{d}x \right|^2 \mathrm{d}t \\ &\le |t_2 - t_1|\, \tilde{\mathbb{E}}\, \|\tilde{q}_n\|^4_{L^\infty([0,T];L^2(\mathbb{S}^1))} \overset{\text{Lemma 5.3}}{\lesssim} |t_2 - t_1| . \end{aligned}$$

Similarly, for any p such that $2p \in \big(1, p_0/2\big]$ ($p_0 > 4$) and $1/p + 1/p' = 1$,

$$\begin{aligned} \tilde{\mathbb{E}} \left| \tilde{\mathcal{M}}(t_2) - \tilde{\mathcal{M}}(t_1) \right|^2 &\lesssim_{\sigma,\varphi} \tilde{\mathbb{E}} \int_{t_1}^{t_2} \left\| \overline{q^2}(t) \right\|^2_{H^{-1}(\mathbb{S}^1)} \mathrm{d}t \\ &\le |t_2 - t_1|^{\frac{1}{p'}} \left(\tilde{\mathbb{E}} \int_0^T \left\| \overline{q^2}(t) \right\|^{2p}_{H^{-1}(\mathbb{S}^1)} \mathrm{d}t \right)^{1/p} \overset{(5.9)}{\lesssim} |t_2 - t_1|^{\frac{1}{p'}} . \end{aligned}$$

From the above estimations, we can infer that $|D_n(t)| \lesssim 1$ uniformly across $n \in \mathbb{N}$ and $t \in [0,T]$. Using the Arzelà–Ascoli theorem, we deduce that $D_n(t) \to D(t)$ uniformly for $t \in [0,T]$ along a subsequence, where $D \in C([0,T])$. According to (7.40), the entire sequence must converge to $D \equiv 0$. This implies that the random variable $\tilde{M}_n = \tilde{\mathcal{M}}_n(T)$ satisfies (7.31). □

Lemma 7.9 applies to the linearly growing approximations $S = S_\ell(v_\pm)$ of $\frac{1}{2}v_\pm^2$ as well as $S = \frac{1}{2}v_\pm^2$ (and $S = \frac{1}{2}v^2$). As explained in the introduction, for the deterministic CH equation [63], the analysis relies on the use of $S_\ell(v_+)$ and a one-sided gradient bound (Oleinik-type estimate) to control the error that arises when replacing v_+^2 by $S_\ell(v_+)$. As one-sided gradient bounds are not available to us, we will insist on applying Lemma 7.9 with $S = \frac{1}{2}v_+^2$ and then use some different ideas to control the defect measure. Exploiting the identities

$$\begin{aligned} &S(v) = \frac{1}{2}v_+^2, \quad S'(v) = v_+, \quad S''(v) = \mathbb{1}_{\{v>0\}}, \quad H^{(1)}(v) = -\frac{1}{2}v_+^2, \\ &H^{(2)}(v) = 0, \quad H^{(3)}(v) = -\frac{1}{2}v_+^2, \quad S''(v)\, v^2 = v_+^2, \end{aligned}$$

and the linearity of the weak limit (i.e., $\overline{a}+\overline{b}=\overline{a+b}$), the inequality (7.24) with $S(v)=\frac{1}{2}v_+^2$ simplifies into

$$\begin{aligned}
&\int_0^T\int_{\mathbb{S}^1}\frac{1}{2}\overline{q_+^2}\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{S}^1}\frac{1}{2}\left(\partial_x\tilde{u}_0\right)_+^2\,\varphi(0,x)\,\mathrm{d}x\\
&\quad+\int_0^T\int_{\mathbb{S}^1}\left(\tilde{u}-\frac{1}{4}\partial_x\sigma^2\right)\frac{1}{2}\overline{q_+^2}\,\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{S}^1}\frac{1}{4}\sigma^2\,\overline{q_+^2}\,\partial_{xx}^2\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^T\int_{\mathbb{S}^1}\left[\overline{q_+}\left(\tilde{P}-\tilde{u}^2\right)+\left(\frac{1}{4}\partial_{xx}^2\sigma^2-|\partial_x\sigma|^2\right)\frac{1}{2}\overline{q_+^2}\right]\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_{\mathbb{S}^1}\left(\sigma\,\partial_x\varphi-\partial_x\sigma\,\varphi\right)\frac{1}{2}\overline{q_+^2}\,\mathrm{d}x\,\mathrm{d}\tilde{W}\geq 0,\quad \tilde{\mathbb{P}}\text{–a.s.,}
\end{aligned}\tag{7.41}$$

for all non-negative $\varphi\in C_c^\infty([0,T)\times\mathbb{S}^1)$.

7.3. Renormalised equation for the weak limit $\tilde{q}$

According to Proposition 6.3, the a.s. limit $\tilde{u}$ from Proposition 4.9 satisfies

$$0=\mathrm{d}\tilde{u}+\left[\tilde{u}\,\partial_x\tilde{u}+\partial_x\tilde{P}\right]\mathrm{d}t-\frac{1}{2}\sigma\,\partial_x\left(\sigma\,\partial_x\tilde{u}\right)\,\mathrm{d}t+\sigma\,\partial_x\tilde{u}\,\mathrm{d}\tilde{W},\tag{7.42}$$

weakly in x, almost surely, where $-\partial_{xx}^2\tilde{P}+\tilde{P}=\tilde{u}^2+\frac{1}{2}\overline{q^2}$. By Lemma 5.1, the a.s. limit $\tilde{q}$ of Proposition 4.9 satisfies $\tilde{q}=\partial_x\tilde{u}$ weakly. Differentiating (7.42) with respect to x, we thus obtain the SPDE

$$\begin{aligned}
0=\mathrm{d}\tilde{q}&+\left(\partial_x\left(\tilde{u}\tilde{q}\right)-\frac{1}{2}\overline{q^2}+\tilde{P}-\tilde{u}^2\right)\mathrm{d}t\\
&-\frac{1}{2}\partial_x\left(\sigma\,\partial_x\left(\sigma\tilde{q}\right)\right)\,\mathrm{d}t+\partial_x\left(\sigma\tilde{q}\right)\,\mathrm{d}\tilde{W}.
\end{aligned}\tag{7.43}$$

Consider a linearly growing $S\in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ (of the type considered before). Note that, thanks to (3.3),

$$S'(\tilde{q})\left(\partial_x\left(\tilde{u}\tilde{q}\right)-\frac{1}{2}\overline{q^2}\right)=\partial_x\left(\tilde{u}S(\tilde{q})\right)-H^{(2)}(\tilde{q})-\frac{1}{2}S'(\tilde{q})\left(\overline{q^2}-\tilde{q}^2\right),$$

where $H^{(2)}$ is defined in (7.25). Formally applying Itô's formula to (7.43) as in (3.4), expressing the temporal differential as a time-derivative in $\mathcal{D}'([0,T))$, we obtain

$$
\begin{aligned}
0 = \partial_t S(\tilde q) + \partial_x \left[\tilde u\, S(\tilde q) + \frac{1}{4} \partial_x \sigma^2 H^{(1)}(\tilde q) \right] - \partial_{xx}^2 \left[\frac{1}{2} \sigma^2 S(\tilde q) \right] \\
+ \Bigg[S'(\tilde q) \left(\tilde P - \tilde u^2 \right) - H^{(2)}(\tilde q) - \frac{1}{2} S'(\tilde q) \left(\overline{q^2} - \tilde q^2 \right) \\
- \frac{1}{4} \partial_{xx}^2 \sigma^2 H^{(3)}(\tilde q) - \frac{1}{2} |\partial_x \sigma|^2\, S''(\tilde q)\, \tilde q^2 \Bigg] \\
+ \left[\partial_x \big(\sigma\, S(\tilde q) \big) - \partial_x \sigma\, H^{(3)}(\tilde q) \right] \dot{\tilde W}, \quad \text{in } \mathcal{D}'([0,T) \times \mathbb{S}^1), \text{ a.s.,}
\end{aligned}
\tag{7.44}
$$

with initial data $S(\tilde q)(0) = S(\partial_x \tilde u_0)$, here relying crucially on Lemma 7.4 (strong right-continuity at $t = 0$). Note carefully that S is assumed linearly growing in order to make sense to the product $S'(\tilde q)\big(\overline{q^2} - \tilde q^2\big)$, given the meagre integrability (4.24). This excludes the functions $S(v) = \frac{1}{2} v^2$, $\frac{1}{2} v_\pm^2$ allowed by Lemma 7.9.

The processes $\tilde u$ and $\tilde q$ appearing in (7.44) exhibit limited regularity. Specifically, $\tilde q$ does not belong to any spatial Sobolev space, as the second-order part of the SPDE (7.43) does not manifest "parabolic regularity". The rigorous derivation of (7.44) is therefore quite involved: it relies on the regularisation (by convolution) method and the real-valued Itô formula, along with non-standard DiPerna–Lions estimates to control the regularisation error linked to the second order operator and the martingale part of the equation (7.43), see Appendix C for details and Section 1 for some relevant references.

Lemma 7.10 *(Renormalisation of limit SPDE). Denote by $S(v)$ any one of the functions $S_\ell(v_\pm)$ defined by (4.2). Let $\tilde u$, $\tilde q = \partial_x \tilde u$, cf. Lemma 5.1, and $\overline{q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9, see also Remark 4.10, and $H^{(1)}, H^{(2)}, H^{(3)}$ be the functions defined in (7.25) with $S(v) = S_\ell(v_\pm)$. The SPDE (7.44) holds weakly in (t, x), almost surely, that is, $\tilde{\mathbb{P}}$–a.s.,*

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{S}^1} S(\tilde q)\, \partial_t \varphi \, \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{S}^1} S(\partial_x \tilde u_0)\, \varphi(0,x)\, \mathrm{d}x \\
&+ \int_0^T \int_{\mathbb{S}^1} \left[\tilde u\, S(\tilde q) + \frac{1}{4} \partial_x \sigma^2 H^{(1)}(\tilde q) \right] \partial_x \varphi \, \mathrm{d}x\, \mathrm{d}t \\
&+ \int_0^T \int_{\mathbb{S}^1} \frac{1}{2} \sigma^2\, S(\tilde q)\, \partial_{xx}^2 \varphi \, \mathrm{d}x\, \mathrm{d}t \\
&- \int_0^T \int_{\mathbb{S}^1} \Bigg[S'(\tilde q) \left(\tilde P - \tilde u^2 \right) - H^{(2)}(\tilde q) - \frac{1}{2} S'(\tilde q) \left(\overline{q^2} - \tilde q^2 \right) \\
&\qquad - \frac{1}{4} \partial_{xx}^2 \sigma^2 H^{(3)}(\tilde q) - \frac{1}{2} |\partial_x \sigma|^2\, S''(\tilde q)\, \tilde q^2 \Bigg] \varphi \, \mathrm{d}x\, \mathrm{d}t
\end{aligned}
\tag{7.45}
$$

$$+\int_0^T\int_{\mathbb{S}^1} \sigma\, S(\tilde{q})\, \partial_x\varphi + \partial_x\sigma\, H^{(3)}(\tilde{q})\, \varphi \, dx\, d\tilde{W} = 0,$$

for all $\varphi \in C_c^\infty([0,T)\times\mathbb{S}^1)$*, where* $\tilde{P}$ *is defined in* (7.19).

Proof. Let J_δ be a standard Friedrichs mollifier on $\mathbb{S}^1$. For $f \in L^p(\mathbb{S}^1)$, write $f_\delta = J_\delta * f$. Mollifying the limit SPDE (7.42) against J_δ, we obtain

$$\begin{aligned} 0 = d\tilde{u}_\delta + \tilde{u}_\delta\tilde{q}_\delta\, dt + E_\delta^{(1)}\, dt + \partial_x K * \left(\tilde{u}^2 + \frac{1}{2}\overline{q^2}\right) * J_\delta\, dt \\ - \frac{1}{2}\sigma\,\partial_x\left(\sigma\tilde{q}_\delta\right) dt + E_\delta^{(3)}\, dt + \left[\sigma\tilde{q}_\delta + E_\delta^{(2)}\right] d\tilde{W}, \end{aligned} \tag{7.46}$$

where $E_\delta^{(1)}$, $E_\delta^{(2)}$, $E_\delta^{(3)}$ denote the following convolution error terms:

$$\begin{aligned} &E_\delta^{(1)} = (\tilde{u}\tilde{q}) * J_\delta - \tilde{u}_\delta\tilde{q}_\delta, \quad E_\delta^{(2)} = \left(\sigma\tilde{q}\right) * J_\delta - \sigma\tilde{q}_\delta, \\ &E_\delta^{(3)} = -\frac{1}{2}\left(\sigma\,\partial_x\left(\sigma\tilde{q}\right)\right) * J_\delta + \frac{1}{2}\sigma\,\partial_x\left(\sigma\tilde{q}_\delta\right). \end{aligned}$$

Next, differentiating (7.46) with respect to x, we arrive at

$$\begin{aligned} 0 = d\tilde{q}_\delta + \left[\partial_x\left(\tilde{u}_\delta\tilde{q}_\delta\right) + K * \left(\tilde{u}^2 + \frac{1}{2}\overline{q^2}\right) * J_\delta - \left(\tilde{u}^2 + \frac{1}{2}\overline{q^2}\right) * J_\delta\right] dt \\ - \frac{1}{2}\partial_x\left(\sigma\,\partial_x\left(\sigma\tilde{q}_\delta\right)\right) dt + \partial_x\left(\sigma\tilde{q}_\delta\right) d\tilde{W} \\ + \partial_x E_\delta^{(1)}\, dt + \partial_x E_\delta^{(2)}\, d\tilde{W} + \partial_x E_\delta^{(3)}\, dt. \end{aligned} \tag{7.47}$$

Consider $S(v) = S_\ell(v_\pm)$ as in the lemma. Given (7.47), applying the standard Itô formula to $S(\tilde{q}_\delta)$, as in (7.44) (cf. also (3.3) and (3.4)), we obtain

$$0 = \int_0^T\int_{\mathbb{S}^1} S(\tilde{q}_\delta)\partial_t\varphi\, dx\, dt + \int_{\mathbb{S}^1} S(\tilde{q}_\delta(0))\varphi(0,x)\, dx + \sum_{i=1}^{6} I_\delta^{(i)}, \tag{7.48}$$

where

$$\begin{aligned} I_\delta^{(1)} = \int_0^T\int_{\mathbb{S}^1} \left(H^{(2)}(\tilde{q}_\delta) + \frac{1}{4}\,\partial_{xx}^2\sigma^2 H^{(3)}(\tilde{q}_\delta)\right)\varphi\, dx\, dt \\ + \int_0^T\int_{\mathbb{S}^1} \left(\tilde{u}_\delta S(\tilde{q}_\delta) + \frac{1}{4}\partial_x\sigma^2 H^{(1)}(\tilde{q}_\delta)\right)\partial_x\varphi\, dx\, dt \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\int_0^T \int_{\mathbb{S}^1} \sigma^2 S(\tilde{q}_\delta)\partial_{xx}^2 \varphi \,\mathrm{d}x\,\mathrm{d}t,\\
I_\delta^{(2)} &= -\int_0^T \int_{\mathbb{S}^1} S'(\tilde{q}_\delta)\overline{I}_\delta^{(2)} \varphi \,\mathrm{d}x\,\mathrm{d}t,\\
I_\delta^{(3)} &= \frac{1}{2}\int_0^T \int_{\mathbb{S}^1} |\partial_x\sigma|^2 S''(\tilde{q}_\delta)\tilde{q}_\delta^2 \varphi \,\mathrm{d}x\,\mathrm{d}t,\\
I_\delta^{(4)} &= \int_0^T \int_{\mathbb{S}^1} \sigma S(\tilde{q}_\delta)\partial_x\varphi + \partial_x\sigma H^{(3)}(\tilde{q}_\delta)\varphi \,\mathrm{d}x\,\mathrm{d}\tilde{W},\\
I_\delta^{(5)} &= \int_0^T \int_{\mathbb{S}^1} -\varphi S'(\tilde{q}_\delta)\partial_x E_\delta^{(3)} + \varphi S''(\tilde{q}_\delta)\overline{I}_\delta^{(5)} \,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^T \int_{\mathbb{S}^1} \varphi S'(\tilde{q}_\delta)\partial_x E_\delta^{(1)} \,\mathrm{d}x\,\mathrm{d}t,\\
I_\delta^{(6)} &= -\int_0^T \int_{\mathbb{S}^1} \varphi S'(\tilde{q}_\delta)\partial_x E_\delta^{(2)} \,\mathrm{d}x\,\mathrm{d}\tilde{W},
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{I}_\delta^{(2)} &= K * \left(\tilde{u}^2 + \frac{1}{2}\overline{q^2}\right) * J_\delta - \left(\tilde{u}^2 + \frac{1}{2}\overline{q^2}\right) * J_\delta + \frac{1}{2}\tilde{q}_\delta^2,\\
\overline{I}_\delta^{(5)} &= \left(\partial_x E_\delta^{(2)} \partial_x\left(\sigma\tilde{q}_\delta\right) + \frac{1}{2}\left|\partial_x E_\delta^{(2)}\right|^2\right).
\end{aligned}
$$

In deriving $\overline{I}_\delta^{(2)}$, we used the fact that K is the Green's function of $1 - \partial_{xx}^2$ on $\mathbb{S}^1$.

Denote by $I^{(i)}$ the expression corresponding to formally taking δ to zero in $I_\delta^{(i)}$, $i = 1, \dots, 6$, and the same for $\overline{I}^{(2)}$, $\overline{I}^{(5)}$.

Recall that

$$
\tilde{q} \in L_{\tilde{\omega}}^{p_0} L_t^\infty L_x^2 \cap L_{\tilde{\omega},t,x}^{2r}, \quad \overline{q^2} \in L_{\tilde{\omega},t,x}^r, \quad \tilde{u} \in L_{\tilde{\omega}}^{p_0} L_{t,x}^\infty,
$$

where $r \in [1, 3/2)$ and $p_0 > 4$, see Lemmas 5.4, 5.6, and 5.7. By a standard property of mollifiers, $\tilde{q}_\delta \xrightarrow{\delta\downarrow 0} \tilde{q}$ a.e. in $\tilde{\Omega} \times [0, T] \times \mathbb{S}^1$. Denote by $f(v)$ any one of the nonlinear functions $S(v)$, $S'(v)$, $S''(v)v^2$, $H^{(1)}(v)$, $H^{(2)}(v)$, $H^{(3)}(v)$, where we recall that $S(v) = S_\ell(v_\pm)$, cf. (4.2), and $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ are defined in (7.25) with $S(v) = S_\ell(v_\pm)$. By the continuity of $f(v)$,

$$f(\tilde{q}_\delta) \xrightarrow{\delta\downarrow 0} f(\tilde{q}) \quad \text{a.e. in } \tilde{\Omega} \times [0,T] \times \mathbb{S}^1. \tag{7.49}$$

We have the bound

$$\|f(\tilde{q}_\delta)\|_{L^\infty_{t,x}} \lesssim_\ell 1, \quad f(v) = S'(v),\ S''(v)v^2,$$

and thus, by (7.49) and Vitali's convergence theorem,

$$f(\tilde{q}_\delta) \xrightarrow{\delta\downarrow 0} f(\tilde{q}) \quad \text{in } L^p_{t,x}, \text{ a.s.,} \tag{7.50}$$

for any $1 \le p < \infty$.

Similarly, by the bounds

$$\|f(\tilde{q}_\delta)\|_{L^\infty_t L^2_x},\ \|f(\tilde{q}_\delta)\|_{L^{2r}_{t,x}} \lesssim_{\ell,\tilde{\omega}} 1, \quad f(v) = S(v),\ H^{(1)}(v),\ H^{(3)}(v),$$

we have the a.s. convergences

$$f(\tilde{q}_\delta) \xrightarrow{\delta\downarrow 0} f(\tilde{q}) \quad \text{in } L^{p_1}_t L^{p_2}_x \text{ and in } L^p_{t,x}, \tag{7.51}$$

for any $1 \le p_1 < \infty$, $1 \le p_2 < 2$, and $1 \le p < 2r$. In addition, since $f(\tilde{q}_\delta)$ is bounded in $L^{2r}_{\tilde{\omega},t,x}$ (and $2r > 2$),

$$f(\tilde{q}_\delta) \xrightarrow{\delta\downarrow 0} f(\tilde{q}) \quad \text{in } L^2_{\tilde{\omega},t,x}. \tag{7.52}$$

Next, from the bound

$$\|f(\tilde{q}_\delta)\|_{L^r_{t,x}} \lesssim_{\ell,\tilde{\omega}} 1, \quad f(v) = v^2,\ H^{(2)}(v),$$

we obtain the convergence

$$f(\tilde{q}_\delta) \xrightarrow{\delta\downarrow 0} f(\tilde{q}) \quad \text{in } L^p_{t,x}, \text{ a.s.,} \tag{7.53}$$

for any $1 \le p < r$.

Since $\overline{q^2} \in L^r_{t,x}$ a.s., we have the convergence

$$\overline{q^2} * J_\delta \xrightarrow{\delta\downarrow 0} \overline{q^2} \quad \text{in } L^p_{t,x}, \text{ a.s.} \tag{7.54}$$

Finally, since $\tilde{u} \in L^\infty_{t,x}$, a.s.,

$$\tilde{u}_\delta \xrightarrow{\delta\downarrow 0} \tilde{u}, \quad \tilde{u}^2 * J_\delta \xrightarrow{\delta\downarrow 0} \tilde{u}^2 \quad \text{in } L^p_{t,x}, \text{ a.s., for any } p \in [1,\infty). \tag{7.55}$$

1. The first two terms in (7.48).

By (7.51) and $\partial_t \varphi \in L^\infty_{t,x}$,

$$\int_0^T \int_{\mathbb{S}^1} S(\tilde{q}_\delta) \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t \xrightarrow{\delta \downarrow 0} \int_0^T \int_{\mathbb{S}^1} S(\tilde{q}) \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t, \quad \text{a.s.}$$

Regarding the initial term,

$$\int_{\mathbb{S}^1} S(\tilde{q}_\delta(0)) \varphi(0,x) \, \mathrm{d}x = \int_{\mathbb{S}^1} S(\partial_x \tilde{u}_0) \varphi(0,x) \, \mathrm{d}x + A + B_\delta,$$

where

$$A = \int_{\mathbb{S}^1} \big(S(\tilde{q}(0)) - S(\partial_x \tilde{u}_0) \varphi(0,x) \, \mathrm{d}x,$$

$$B_\delta = \int_{\mathbb{S}^1} \big(S(\tilde{q}_\delta(0)) - S(\tilde{q}(0)) \big) \varphi(0,x) \, \mathrm{d}x.$$

By Lemma 7.4 (strong initial trace of $\tilde{q} = \partial_x \tilde{u}$ in L^2), we have that $A = 0$. As in (7.51), $S(\tilde{q}_\delta(0)) \xrightarrow{\delta \downarrow 0} S(\tilde{q}(0))$ in L^1_x a.s., and hence we easily deduce that $B_\delta \xrightarrow{\delta \downarrow 0} 0$; accordingly,

$$\int_{\mathbb{S}^1} S(\tilde{q}_\delta(0)) \varphi(0,x) \, \mathrm{d}x \xrightarrow{\delta \downarrow 0} \int_{\mathbb{S}^1} S(\partial_x \tilde{u}_0) \varphi(0,x) \, \mathrm{d}x, \quad \text{a.s.}$$

2. The term $I_\delta^{(1)}$.

Since $\varphi, \partial_x \varphi, \partial_{xx}^2 \varphi, \sigma^2, \partial_x \sigma^2, \partial_{xx}^2 \sigma^2 \in L^\infty_{t,x}$, the convergences (7.51) and (7.53) ensure that $I_\delta^{(1)} \xrightarrow{\delta \downarrow 0} I^{(1)}$ a.s.

3. The term $I_\delta^{(2)}$.

In view of Young's convolution inequality and the convergences (7.53), (7.54), and (7.55), we obtain $\overline{I}_\delta^{(2)} \xrightarrow{\delta \downarrow 0} \overline{I}^{(2)}$ in $L^p_{t,x}$, a.s., for any $p \in [1, r)$, where

$$\overline{I}^{(2)} = K * \left(\tilde{u}^2 + \frac{1}{2} \overline{q^2} \right) - \tilde{u}^2 - \frac{1}{2} \left(\overline{q^2} - \tilde{q}^2 \right).$$

Moreover, by (7.50), $S'(\tilde{q}_\delta) \xrightarrow{\delta \downarrow 0} S'(\tilde{q})$ in $L^{p'}_{t,x}$, a.s., where $\frac{1}{p} + \frac{1}{p'} = 1$. Consequently, as $\varphi \in L^\infty_{t,x}$, it follows that $I_\delta^{(2)} \xrightarrow{\delta \downarrow 0} I^{(2)}$, a.s.

4. The term $I_\delta^{(3)}$.

Using (7.50), recalling that $\frac{1}{2}\left|\partial_x\sigma\right|^2\varphi \in L^\infty_{t,x}$,

$$I_\delta^{(3)} \xrightarrow{\delta\downarrow 0} \frac{1}{2}\int_0^T\int_{\mathbb{S}^1} \left|\partial_x\sigma\right|^2 S''(\tilde{q})\tilde{q}^2\varphi \,\mathrm{d}x\,\mathrm{d}s.$$

5. The term $I_\delta^{(4)}$.

By the Itô isometry and the Cauchy–Schwarz inequality,

$$\tilde{\mathbb{E}}\left|I_\delta^{(4)} - I^{(4)}\right|^2 \le \tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \left|\sigma\,\partial_x\varphi\right|^2 \left|S(\tilde{q}_\delta) - S(\tilde{q})\right|^2 \,\mathrm{d}x\,\mathrm{d}t + \tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \left|\varphi\partial_x\sigma\right|^2 \left|H^{(3)}(\tilde{q}_\delta) - H^{(3)}(\tilde{q})\right|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Both these integrals tend to nought by (7.52). Therefore, along a subsequence $\delta = \delta_j \downarrow 0$ as $j \to \infty$, $I_\delta^{(4)} \xrightarrow{j\uparrow\infty} I^{(4)}$, a.s.

6. The terms $I_\delta^{(5)}$ and $I_\delta^{(6)}$.

Since $\varphi \in L^\infty_{t,x}$, $\left|S'(\cdot)\right| \lesssim_\ell 1$, and $\left|S''(\cdot)\right| \le 1$, Lemma C.1 and Proposition C.2 allow us to directly conclude that $I_\delta^{(5)} \xrightarrow{\delta\downarrow 0} I^{(5)}$, a.s., along a subsequence $\delta = \delta_j \downarrow 0$ as $j \to \infty$. Similarly, again invoking the Itô isometry and the Cauchy–Schwarz inequality,

$$\tilde{\mathbb{E}}\left|I_\delta^{(6)}\right|^2 \le \tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \left|\varphi S'(\tilde{q}_\delta)\right|^2 \left|\partial_x E_\delta^{(2)}\right|^2 \,\mathrm{d}x\,\mathrm{d}t \xrightarrow{\delta\downarrow 0} 0,$$

by Lemma C.1. So along a subsequence $\delta = \delta_j \downarrow 0$ as $j \to \infty$, $I_\delta^{(6)} \to 0$, a.s.

This concludes the proof of the lemma. □

In the remaining part of this section, we will make hefty use of the formulas gathered in the next remark.

Remark 7.11. Recall the formulas (4.2)–(4.6) involving $S_\ell(v)$ and $S_\ell(v_\pm)$. The following identities are straightforward to verify:

$$S_\ell(v_+) = \frac{1}{2}v_+^2 - \frac{1}{6\ell}(v-\ell)^3\,\mathbb{1}_{\{\ell<v<2\ell\}} - \frac{1}{6}\left(3v^2 - 9\ell v + 7\ell^2\right)\mathbb{1}_{\{v\ge 2\ell\}},$$

$$S_\ell(v_+)' = v_+ - \frac{1}{2\ell}(v-\ell)^2\,\mathbb{1}_{\{\ell<v<2\ell\}} + \frac{1}{2}(3\ell - 2v)\,\mathbb{1}_{\{v\ge 2\ell\}},$$

$$S_\ell(v_+)'' = \mathbb{1}_{\{0<v<2\ell\}} - \frac{1}{\ell}(v-\ell)\,\mathbb{1}_{\{\ell<v<2\ell\}},$$

$$
\begin{aligned}
S_\ell(v_+) - S_\ell(v_+)'v &= -\frac{1}{2}v_+^2 + \frac{1}{6\ell}\left(2v^3 - 3\ell v^2 + \ell^3\right)\mathbb{1}_{\{\ell<v<2\ell\}}\\
&\qquad + \frac{1}{6}\left(3v^2 - 7\ell^2\right)\mathbb{1}_{\{v\geq 2\ell\}},\\
3S_\ell(v_+) - 2S_\ell(v_+)'v &= -\frac{1}{2}v_+^2 + \frac{1}{2\ell}\left(v^3 - \ell v^2 - \ell^2 v + \ell^3\right)\mathbb{1}_{\{\ell<v<2\ell\}}\\
&\qquad + \frac{1}{2}\left(v^2 + 3\ell v - 7\ell^2\right)\mathbb{1}_{\{v\geq 2\ell\}},\\
S_\ell(v_+)v - \frac{1}{2}S_\ell(v_+)'v^2 &= \frac{1}{12\ell}\left(v^4 - 3\ell^2 v^2 + 2\ell^3 v\right)\mathbb{1}_{\{\ell<v<2\ell\}}\\
&\qquad + \frac{1}{12}\left(9\ell v^2 - 14\ell^2 v\right)\mathbb{1}_{\{v\geq 2\ell\}},\\
\frac{1}{2}S_\ell(v_+)''v^2 &= \frac{1}{2}v_+^2 - \frac{1}{2\ell}v^2\left(v-\ell\right)\mathbb{1}_{\{\ell<v<2\ell\}} - \frac{1}{2}v^2\mathbb{1}_{\{v\geq 2\ell\}},
\end{aligned}
$$

and

$$
\begin{aligned}
S_\ell(v_-) &= \frac{1}{2}v_-^2 + \frac{1}{6\ell}\left(v+\ell\right)^3\mathbb{1}_{\{-2\ell<v<-\ell\}} - \frac{1}{6}\left(3v^2 + 9\ell v + 7\ell^2\right)\mathbb{1}_{\{v\leq -2\ell\}},\\
S_\ell(v_-)' &= v_- + \frac{1}{2\ell}\left(v+\ell\right)^2\mathbb{1}_{\{-2\ell<v<-\ell\}} - \frac{1}{2}\left(3\ell + 2v\right)\mathbb{1}_{\{v\leq -2\ell\}},\\
S_\ell(v_-)'' &= \mathbb{1}_{\{-2\ell<v<0\}} + \frac{1}{\ell}\left(v+\ell\right)\mathbb{1}_{\{-2\ell<v<-\ell\}},\\
S_\ell(v_-) - S_\ell(v_-)'v &= -\frac{1}{2}v_-^2 + \frac{1}{6\ell}\left(-2v^3 - 3\ell v^2 + \ell^3\right)\mathbb{1}_{\{-2\ell<v<-\ell\}}\\
&\qquad + \frac{1}{6}\left(3v^2 - 7\ell^2\right)\mathbb{1}_{\{v\leq -2\ell\}},\\
3S_\ell(v_-) - 2S_\ell(v_-)'v &= -\frac{1}{2}v_-^2 + \frac{1}{2\ell}\left(-v^3 - \ell v^2 + \ell^2 v + \ell^3\right)\mathbb{1}_{\{-2\ell<v<-\ell\}}\\
&\qquad + \frac{1}{2}\left(v^2 - 3\ell v - 7\ell^2\right)\mathbb{1}_{\{v\leq -2\ell\}},\\
S_\ell(v_-)v - \frac{1}{2}S_\ell(v_-)'v^2 &= -\frac{1}{12\ell}\left(v^4 - 3\ell^2 v^2 - 2\ell^3 v\right)\mathbb{1}_{\{-2\ell<v<-\ell\}}\\
&\qquad - \frac{1}{12}\left(9\ell v^2 + 14\ell^2 v\right)\mathbb{1}_{\{v\leq -2\ell\}},\\
\frac{1}{2}S_\ell(v_-)''v^2 &= \frac{1}{2}v_-^2 + \frac{1}{2\ell}v^2\left(v+\ell\right)\mathbb{1}_{\{-2\ell<v<-\ell\}} - \frac{1}{2}v^2\mathbb{1}_{\{v\leq -2\ell\}}.
\end{aligned}
$$

Lemma 7.10 applies to the linearly growing approximations $S_\ell(v_\pm)$ of $v_\pm^2$, but not the functions $v_\pm^2$ themselves. However, by exploiting some structural property of the SPDE (7.44), we will be able to write an SPDE—up to an inequality—for the positive part $\tilde{q}_+^2$. Together with (7.41), this observation makes it possible to control the positive part $\frac{1}{2}\left(\overline{q_+^2} - \tilde{q}_+^2\right)$ of the defect measure (1.12), without counting on a one-sided gradient estimate (available in the deterministic case [63] but not here).

Lemma 7.12 *(Characterisation of $\tilde{q}_+^2$). Let $\tilde{u}$, $\tilde{q} = \partial_x \tilde{u}$, cf. Lemma 5.1, and $\overline{q^2}$ be the Skorokhod–Jakubowski representations from Proposition 4.9, see also Remark 4.10. Then, for any nonnegative $\varphi \in C_c^\infty([0,T) \times \mathbb{S}^1)$,*

$$\begin{aligned} &\int_0^T \int_{\mathbb{S}^1} \frac{1}{2}\tilde{q}_+^2\, \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{S}^1} \frac{1}{2}\left(\partial_x \tilde{u}_0\right)_+^2 \varphi(0,x)\,\mathrm{d}x \\ &\quad + \int_0^T \int_{\mathbb{S}^1} \left(\tilde{u} - \frac{1}{4}\partial_x \sigma^2\right) \frac{1}{2}\tilde{q}_+^2\, \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{S}^1} \frac{1}{2}\sigma^2 \frac{1}{2}\tilde{q}_+^2\, \partial_{xx}^2 \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_{\mathbb{S}^1} \left[\tilde{q}_+\left(\tilde{P} - \tilde{u}^2\right) + \left(\frac{1}{4}\partial_{xx}^2 \sigma^2 - |\partial_x \sigma|^2\right)\frac{1}{2}\tilde{q}_+^2\right]\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{S}^1} \left(\sigma\, \partial_x \varphi - \partial_x \sigma\, \varphi\right)\frac{1}{2}\tilde{q}_+^2 \,\mathrm{d}x\,\mathrm{d}\tilde{W} \le 0, \quad \tilde{\mathbb{P}}\text{–a.s.}, \end{aligned} \tag{7.56}$$

where $\tilde{P}$ is defined in (7.19).

Proof. Denote the left-hand side of (7.56) by $I + M$, where M is the stochastic integral term. We will demonstrate that

$$\int_{\tilde{\Omega}} \mathbb{1}_A(\tilde{\omega})\big(I(\tilde{\omega}) + M(\tilde{\omega})\big)\,\mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega}) \le 0, \tag{7.57}$$

for any measurable set $A \in \tilde{\mathcal{F}}$.

Given (7.45) with $S(v) = S_\ell(v_+)$, observe that

$$H^{(2)}(\tilde{q}) = S(\tilde{q})\tilde{q} - \frac{1}{2}S'(\tilde{q})\tilde{q}^2 \ge 0, \qquad S'(\tilde{q})\left(\overline{q^2} - \tilde{q}^2\right) \ge 0, \tag{7.58}$$

using (7.25), Remark 7.11 and the weak convergence $\tilde{q}_n^2 \xrightarrow{n\uparrow\infty} \overline{q^2}$ in $L^r_{t,x}$ a.s., cf. (4.24) (the weak convergence implies that $\tilde{q}^2 \le \overline{q^2}$). In addition,

$$\begin{aligned} S(\tilde{q}) &= \frac{1}{2}\tilde{q}_+^2 + e_\ell^{(1)}(t,x), \\ H^{(1)}(\tilde{q}) = 3S(\tilde{q}) - 2S'(\tilde{q})\tilde{q} &= -\frac{1}{2}\tilde{q}_+^2 + e_\ell^{(2)}(t,x), \\ S'(\tilde{q}) &= \tilde{q}_+ + e_\ell^{(3)}(t,x), \\ H^{(3)}(\tilde{q}) = S(\tilde{q}) - S'(\tilde{q})\tilde{q} &= -\frac{1}{2}\tilde{q}_+^2 + e_\ell^{(4)}(t,x), \\ \frac{1}{2}S''(\tilde{q})\tilde{q}^2 &= \frac{1}{2}\tilde{q}_+^2 + e_\ell^{(5)}(t,x), \end{aligned} \tag{7.59}$$

where

$$
\begin{aligned}
e_\ell^{(1)} &= -\frac{1}{6\ell}(\tilde{q}-\ell)^3\,\mathbb{1}_{\{\ell<\tilde{q}<2\ell\}} - \frac{1}{6}\left(3\tilde{q}^2 - 9\ell\tilde{q} + 7\ell^2\right)\mathbb{1}_{\{\tilde{q}\geq 2\ell\}},\\
e_\ell^{(2)} &= \frac{1}{2\ell}\left(\tilde{q}^3 - \ell\tilde{q}^2 - \ell^2\tilde{q} + \ell^3\right)\mathbb{1}_{\{\ell<\tilde{q}<2\ell\}} + \frac{1}{2}\left(\tilde{q}^2 + 3\ell\tilde{q} - 7\ell^2\right)\mathbb{1}_{\{\tilde{q}\geq 2\ell\}},\\
e_\ell^{(3)} &= -\frac{1}{2\ell}(\tilde{q}-\ell)^2\,\mathbb{1}_{\{\ell<\tilde{q}<2\ell\}} + \frac{1}{2}(3\ell - 2\tilde{q})\,\mathbb{1}_{\{\tilde{q}\geq 2\ell\}},\\
e_\ell^{(4)} &= \frac{1}{6\ell}\left(2\tilde{q}^3 - 3\ell\tilde{q}^2 + \ell^3\right)\mathbb{1}_{\{\ell<\tilde{q}<2\ell\}} + \frac{1}{6}\left(3\tilde{q}^2 - 7\ell^2\right)\mathbb{1}_{\{\ell\geq 2\ell\}},\\
e_\ell^{(5)} &= -\frac{1}{2\ell}\tilde{q}^2(\tilde{q}-\ell)\,\mathbb{1}_{\{\ell<\tilde{q}<2\ell\}} - \frac{1}{2}\tilde{q}^2\mathbb{1}_{\{\tilde{q}\geq 2\ell\}}.
\end{aligned}
$$

By (5.3), $\tilde{q} \in L^{p_0}\big(\tilde{\Omega}; L^\infty([0,T]; L^2(\mathbb{S}^1))\big)$, where $p_0 > 4$ is fixed in Theorem 1.1. As a result, the error terms converge to zero in the sense that

$$
\begin{aligned}
&\left|e_\ell^{(i)}\right| \lesssim \mathbb{1}_{\{\tilde{q}>\ell\}}\tilde{q}^2 \xrightarrow{\ell\uparrow\infty} 0 \quad \text{a.e. in } (\tilde{\omega}, t, x), \;\; i = 1,2,4,5,\\
&\left|e_\ell^{(3)}\right| \lesssim \mathbb{1}_{\{\tilde{q}>\ell\}}\tilde{q} \xrightarrow{\ell\uparrow\infty} 0 \quad \text{a.e. in } (\tilde{\omega}, t, x).
\end{aligned}
\tag{7.60}
$$

Inserting the inequalities (7.58) and the identities (7.59) into (7.45), with $S(v) = S_\ell(v_+)$, we arrive at

$$
\begin{aligned}
&\int_{\tilde{\Omega}} \mathbb{1}_A(\tilde{\omega})\big(I(\tilde{\omega}) + M(\tilde{\omega})\big)\,\mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega})\\
&\quad \leq C_1\,\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \left|e_\ell^{(1)}\right| + \left|e_\ell^{(1)}(0)\right| + \left|e_\ell^{(2)}\right| + \left|e_\ell^{(4)}\right| + \left|e_\ell^{(5)}\right|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + C_2\,\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} |\tilde{u}|\left|e_\ell^{(1)}\right|\mathrm{d}x\,\mathrm{d}t + C_3\,\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1}\left|\tilde{P} - \tilde{u}^2\right|\left|e_\ell^{(3)}\right|\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \tilde{\mathbb{E}}\left|\int_0^T\int_{\mathbb{S}^1} \sigma\, e_\ell^{(1)}\,\partial_x\varphi + \partial_x\sigma\, e_\ell^{(4)}\varphi\,\mathrm{d}x\,\mathrm{d}\tilde{W}\right| =: \sum_{i=1}^{4} R_{i,\ell},
\end{aligned}
$$

for some ℓ-independent constants $C_1 = C_1(\sigma, \varphi, \partial_x\varphi, \partial_{xx}^2\varphi)$, $C_2 = C_2(\partial_x\varphi)$, and $C_3 = C_2(\varphi)$. Here, $e_\ell^{(1)}(0)$ refers to

$$
-\frac{1}{6\ell}(\partial_x\tilde{u}_0 - \ell)^3\,\mathbb{1}_{\{\ell<\partial_x\tilde{u}_0<2\ell\}} - \frac{1}{6}\left(3(\partial_x\tilde{u}_0)^2 - 9\ell\partial_x\tilde{u}_0 + 7\ell^2\right)\mathbb{1}_{\{\partial_x\tilde{u}_0\geq 2\ell\}}.
$$

Notice that $\left|e_\ell^{(i)}\right| \lesssim \tilde{q}^2 \in L^1_{\tilde{\omega},t,x}$, $i = 1, 2, 4, 5$, and $\left|e_\ell^{(1)}(0)\right| \lesssim (\partial_x \tilde{u}_0)^2 \in L^1_{\tilde{\omega},t,x}$, cf. (4.24). Therefore, by (7.60), and the Lebesgue dominated convergence theorem, $R_{1,\ell} \xrightarrow{\ell\uparrow\infty} 0$. The same argument applies to $R_{2,\ell}$, $R_{3,\ell}$, as

$$|\tilde{u}|\left|e_\ell^{(1)}\right| \lesssim |\tilde{u}|\,\tilde{q}^2, \qquad \left|\tilde{P}-\tilde{u}^2\right|\left|e_\ell^{(3)}\right| \lesssim \left|\tilde{P}\right|\tilde{q}^2 + |\tilde{u}|^2\,\tilde{q}^2,$$

and, by Lemma 5.6 and (5.3),

$$\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} |\tilde{u}|^p\,\tilde{q}^2\,\mathrm{d}x\,\mathrm{d}t \lesssim_T \left(\tilde{\mathbb{E}}\,\|\tilde{u}\|^{2p}_{L^\infty_{t,x}}\right)^{\frac12}\left(\tilde{\mathbb{E}}\,\|\tilde{q}\|^4_{L^\infty_t L^2_x}\right)^{\frac12} \lesssim 1, \qquad p \in \left[1, p_0/2\right],$$

$$\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \left|\tilde{P}\right|\tilde{q}^2\,\mathrm{d}x\,\mathrm{d}t \lesssim \left(\tilde{\mathbb{E}}\left\|\tilde{P}\right\|^2_{L^\infty_{t,x}}\right)^{\frac12}\left(\tilde{\mathbb{E}}\,\|\tilde{q}\|^4_{L^\infty_t L^2_x}\right)^{\frac12} \lesssim 1.$$

Finally, let us consider the stochastic integral term. By the Cauchy–Schwarz inequality and the Itô isometry,

$$\begin{aligned}\left|R_{4,\ell}\right|^2 &\le \tilde{\mathbb{E}}\int_0^T\left|\int_{\mathbb{S}^1} \sigma\, e_\ell^{(1)}\,\partial_x\varphi + \partial_x\sigma\, e_\ell^{(4)}\varphi\,\mathrm{d}x\right|^2\mathrm{d}t\\ &\lesssim_{\sigma,\varphi} \tilde{\mathbb{E}}\int_0^T\left|\int_{\mathbb{S}^1} \mathbb{1}_{\{\tilde{q}_+>\ell\}}\tilde{q}_+^2\,\mathrm{d}x\right|^2\mathrm{d}t.\end{aligned}$$

By (5.3), $\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1}\tilde{q}^2\,\mathrm{d}x\,\mathrm{d}t < \infty$ and so $\tilde{q} \in L^2(\mathbb{S}^1)$ for a.e. $(\tilde{\omega}, t)$. Thus

$$\left|\int_{\mathbb{S}^1} \mathbb{1}_{\{\tilde{q}_+>\ell\}}\tilde{q}_+^2\,\mathrm{d}x\right|^2 \xrightarrow{\ell\uparrow\infty} 0, \qquad \text{for a.e. } (\tilde{\omega}, t).$$

Besides, $\left|\int_{\mathbb{S}^1}\mathbb{1}_{\{\tilde{q}_+>\ell\}}\tilde{q}_+^2\,\mathrm{d}x\right|^2 \le \left|\int_{\mathbb{S}^1}\tilde{q}_+^2\,\mathrm{d}x\right|^2$ and

$$\tilde{\mathbb{E}}\int_0^T\left|\int_{\mathbb{S}^1}\tilde{q}_+^2\,\mathrm{d}x\right|^2\mathrm{d}t \lesssim_T \tilde{\mathbb{E}}\,\|\tilde{q}\|^4_{L^\infty([0,T];L^2(\mathbb{S}^1))} \overset{(5.3)}{\lesssim} 1. \tag{7.61}$$

Therefore, by Lebesgue's dominated convergence theorem, $\left|R_{4,\ell}\right|^2 \xrightarrow{\ell\uparrow\infty} 0$. This concludes the proof of (7.57). □

7.4. Controlling the defect measure

We define the positive part of the defect measure (1.12) by

$$\mathbb{D}^+ = \frac{1}{2}\left(\overline{q_+^2} - \tilde{q}_+^2\right) \geq 0. \tag{7.62}$$

One can construe $\mathbb{D}^+ = \mathbb{D}^+(\tilde{\omega}, t, x)$ (and similar objects introduced later on) as a random variable that assumes values in a path space of functions depending on $t \in [0, T]$ and $x \in \mathbb{S}^1$. Alternatively, $\mathbb{D}^+$ can be conceived of as a stochastic process $(\tilde{\omega}, t) \mapsto \mathbb{D}^+(\tilde{\omega}, t, \cdot)$ that takes values in some functional space (over x). If $\mathbb{D}^+$ is conceptualised as a random variable in the Lebesgue space $L^r([0, T] \times \mathbb{S}^1)$ (recall that $1 \leq r < 3/2$), then the pointwise value $\mathbb{D}^+(t)$ is only determinable modulo a set of times with zero measure in $[0, T]$. Consequently, discerning $\mathbb{D}^+$ as a traditional stochastic process proves to be a complex endeavour. Therefore, in the forthcoming discussion, we consider $\mathbb{D}^+$ (and other similar objects) as a random variable within the space $L^r([0, T] \times \mathbb{S}^1)$. In view of our previous results, $\mathbb{D}^+$ obeys an SPDE inequality. This inequality is interpreted almost surely in the distributional sense on the domain $[0, T) \times \mathbb{S}^1$, with the inclusion of the zero function as initial data in the distributional formulation.

To provide more clarity, by directly subtracting (7.41) and (7.56),

$$\begin{aligned}
&\partial_t \mathbb{D}^+ + \partial_x \left[\left(\tilde{u} - \frac{1}{4}\partial_x \sigma^2\right)\mathbb{D}^+\right] - \partial_{xx}^2 \left[\frac{1}{2}\sigma^2 \mathbb{D}^+\right] \\
&\quad + (\overline{q_+} - \tilde{q}_+)\left(\tilde{P} - \tilde{u}^2\right) + \left(\frac{1}{4}\partial_{xx}\sigma^2 - |\partial_x \sigma|^2\right)\mathbb{D}^+ \\
&\quad + \left[\partial_x\left(\sigma\, \mathbb{D}^+\right) + \partial_x \sigma\, \mathbb{D}^+\right]\dot{\tilde{W}} \leq 0 \quad \text{in } \mathcal{D}'([0, T) \times \mathbb{S}^1), \;\; \tilde{\mathbb{P}}\text{–a.s.},
\end{aligned} \tag{7.63}$$

with zero initial data (in the sense of distributions). This formulation is weak in (t, x). Employing a reasoning approach akin to the one used in the proof of Lemma 7.6, we can transform this into a formulation that is pointwise (a.e.) in $(\tilde{\omega}, t)$ and integrated in x.

Lemma 7.13 *(Positive part of defect measure). Let $\mathbb{D}^+$ be defined by (7.62) and $\tilde{P}$ by (7.19). Then, for a.e. $(\tilde{\omega}, t) \in \tilde{\Omega} \times [0, T]$,*

$$\begin{aligned}
&\int_{\mathbb{S}^1} \mathbb{D}^+(t)\, \mathrm{d}x + \int_0^t \int_{\mathbb{S}^1} (\overline{q_+} - \tilde{q}_+)\left(\tilde{P} - \tilde{u}^2\right) \mathrm{d}x\, \mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{S}^1} \left(\frac{1}{4}\partial_{xx}\sigma^2 - |\partial_x \sigma|^2\right)\mathbb{D}^+\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{\mathbb{S}^1} \partial_x \sigma\, \mathbb{D}^+\, \mathrm{d}x\, \mathrm{d}\tilde{W} \leq 0.
\end{aligned} \tag{7.64}$$

The stochastic integral is a square-integrable martingale.

Proof. Using the test function $\varphi(t, x) = \psi(t)\phi(x)$ in (7.63), with $0 \leq \psi \in C_c^\infty([0, T))$ arbitrary and $\phi \equiv 1$, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} \mathbb{D}^+ \,\mathrm{d}x + \int_{\mathbb{S}^1} (\overline{q_+} - \tilde{q}_+)\left(\tilde{P} - \tilde{u}^2\right) \mathrm{d}x \\
&\qquad + \int_{\mathbb{S}^1} \left(\frac{1}{4}\partial_{xx}\sigma^2 - |\partial_x \sigma|^2\right) \mathbb{D}^+ \,\mathrm{d}x + \int_{\mathbb{S}^1} \partial_x \sigma\, \mathbb{D}^+ \,\mathrm{d}x\, \dot{\tilde{W}} \le 0,
\end{aligned}
\tag{7.65}
$$

which holds in $\mathcal{D}'([0,T))$, a.s., with zero initial data (in the sense of distributions).

Following the proof of Lemma 7.6, for a given Lebesgue point t of the integrable function $s \mapsto \int_{\mathbb{S}^1} \mathbb{D}^+(s)\,\mathrm{d}x$ (with $\tilde{\omega}$ fixed from a set F of full $\tilde{\mathbb{P}}$–measure), consider δ such that $0 < t-\delta < T$. For such δ, let β_δ be the continuous piecewise linear function that equals 1 on $[0, t-\delta]$, 0 on $[t, T]$, and is linear on $[t-\delta, t]$. Then $\beta_\delta(s) \to \mathbb{1}_{[0,t]}(s)$ for a.e. $s \in [0,T]$. Using β_δ as test function in (7.65) gives

$$
\begin{aligned}
&\frac{1}{\delta}\int_{t-\delta}^{t} \left(\int_{\mathbb{S}^1} \mathbb{D}^+(s)\,\mathrm{d}x\right) ds + \int_0^T \int_{\mathbb{S}^1} (\overline{q_+} - \tilde{q}_+)\left(\tilde{P} - \tilde{u}^2\right)(s)\,\beta_\delta(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\mathbb{S}^1} \left(\frac{1}{4}\partial_{xx}\sigma^2 - |\partial_x\sigma|^2\right) \mathbb{D}^+(s)\,\beta_\delta(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^T \int_{\mathbb{S}^1} \partial_x\sigma\, \mathbb{D}^+(s)\,\beta_\delta(s)\,\mathrm{d}x\,\mathrm{d}\tilde{W}(s) \le 0.
\end{aligned}
$$

The stochastic integral is a square-integrable martingale on $[0,T]$, which follows from calculations like (7.3), (7.39), and (7.61).

By adhering to the proof of Lemma 7.6 and considering Remark 7.7, we can take the limit as $\delta \to 0$ in this inequality, leading us to (7.64). □

Define

$$
\mathbb{D}^-_\ell = \overline{S_\ell(q_-)} - S_\ell(\tilde{q}_-) \ge 0, \quad \ell \in \mathbb{N}, \tag{7.66}
$$

so that $\mathbb{D}^-_\ell$ approximates the negative part $\mathbb{D}^- = \frac{1}{2}\left(\overline{q_-^2} - \tilde{q}_-^2\right)$ of the defect measure (1.12). We first make explicit the approximation error by the following result:

Lemma 7.14. *Let $r' = r/(r-1)$ be the Hölder conjugate of r, recalling that $r < 3/2$. Then*

$$
\left|\tilde{\mathbb{E}} \int_0^T \int_{\mathbb{S}^1} \mathbb{D}^-_\ell - \mathbb{D}^- \,\mathrm{d}x\,\mathrm{d}t\right| \lesssim \ell^{-2(r-1)}.
$$

Proof. Using the weak convergences (5.11) and (7.32) of $\tilde{q}_n^2 \rightharpoonup \overline{q^2}$ in $L^r_{\tilde{\omega},t,x}$ and $S_\ell(\tilde{q}_n) \rightharpoonup \overline{S_\ell(q)}$ in $L^{2r}_{\tilde{\omega},t,x}$,

$$
\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \mathbb{D}_\ell^- - \mathbb{D}^- \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \lim_{n\to\infty}\tilde{\mathbb{E}}\int_0^t\int_{\mathbb{S}^1} S_\ell\big((\tilde{q}_n)_-\big) - (\tilde{q}_n)_-^2 + S_\ell(\tilde{q}_-) - \tilde{q}_-^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Remark 7.11 implies that

$$
\begin{aligned}
\Big|S_\ell\big(v_-\big) - v_-^2\Big| &\lesssim \frac{1}{\ell}\,|v+\ell|^3\,\mathbb{1}_{\{-2\ell\le v\le -\ell\}} \\
&\qquad + v^2\mathbb{1}_{\{v\le -2\ell\}} \lesssim v^2\mathbb{1}_{\{|v|\ge \ell\}} \le \ell^{-2(r-1)}v^{2r}\mathbb{1}_{\{|v|\ge\ell\}}.
\end{aligned}
$$

Therefore, by Lemma 5.3,

$$
\begin{aligned}
&\left|\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} S_\ell\big((\tilde{q}_n)_-\big) - (\tilde{q}_n)_-^2 \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad \le \ell^{-2(r-1)}\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} |\tilde{q}_n|^{2r}\,\mathbb{1}_{\{|\tilde{q}_n|\ge\ell\}}\,\mathrm{d}x\,\mathrm{d}t \lesssim \ell^{-2(r-1)}.
\end{aligned}
$$

A similar bound for $\tilde{q}$ in place of $\tilde{q}_n$ can be derived by invoking Lemma 5.4. □

Now, we introduce the functions

$$
\begin{aligned}
&H_{\ell,-}^{(1)}(v) = 3S_\ell(v_-) - 2S_\ell(v_-)'v, \quad H_{\ell,-}^{(2)}(v) = S_\ell(v_-)v - \tfrac{1}{2}S_\ell(v_-)'v^2, \\
&H_{\ell,-}^{(3)}(v) = S_\ell(v_-) - S_\ell(v_-)'v.
\end{aligned} \tag{7.67}
$$

Subtracting (7.45) from (7.24) yields

$$
\begin{aligned}
&\partial_t\mathbb{D}_\ell^- + \partial_x\left[\tilde{u}\,\mathbb{D}_\ell^- + \frac{1}{4}\partial_x\sigma^2\left(\overline{H_{\ell,-}^{(1)}(q)} - H_{\ell,-}^{(1)}(\tilde{q})\right)\right] - \partial_{xx}^2\left[\frac{1}{2}\sigma^2\,\mathbb{D}_\ell^-\right] \\
&\quad + \left(\overline{S_\ell(q_-)'} - S_\ell(\tilde{q}_-)'\right)\left(\tilde{P} - \tilde{u}^2\right) \\
&\quad - \left(\overline{H_{\ell,-}^{(2)}(q)} - H_{\ell,-}^{(2)}(\tilde{q})\right) + \frac{1}{2}S_\ell(\tilde{q}_-)'\left(\overline{q^2} - \tilde{q}^2\right) \\
&\quad - \frac{1}{4}\partial_{xx}^2\sigma^2\left(\overline{H_{\ell,-}^{(3)}(q)} - H_{\ell,-}^{(3)}(\tilde{q})\right) - \frac{1}{2}\,|\partial_x\sigma|^2\left(\overline{S_\ell(q_-)''\,q^2} - S_\ell(\tilde{q}_-)''\tilde{q}^2\right) \\
&\quad + \left[\partial_x\left(\sigma\,\mathbb{D}_\ell^-\right) - \partial_x\sigma\left(\overline{H_{\ell,-}^{(3)}(q)} - H_{\ell,-}^{(3)}(\tilde{q})\right)\right]\dot{\tilde{W}} \\
&\qquad \le 0 \quad \text{in } \mathcal{D}'([0,T)\times\mathbb{S}^1),\ \ \tilde{\mathbb{P}}\text{–a.s.,}
\end{aligned} \tag{7.68}
$$

with zero initial data: $\mathbb{D}_\ell^-(0) = 0$.

Combining (7.68) with the formulas in Remark 7.11, we obtain the following bound for the negative part $\mathbb{D}_\ell^-$ of the defect measure:

Lemma 7.15 *(Negative part of defect measure). Let $\mathbb{D}_\ell^-$ be defined by (7.66) and $\mathbb{D}^+$ by (7.62). Let $\{\tilde{u}_n\}_{n\ge1}$, $\{\tilde{P}_n\}_{n\ge1}$, and $\tilde{P}$ be as in Proposition 7.8. For any $n_0 \in \mathbb{N}$ and $L > 0$, define the measurable set*

$$A_L^{n_0} = \left\{ \tilde{\omega} \in \tilde{\Omega} : \left\| \tilde{P}_{n_0} - \tilde{u}_{n_0}^2 \right\|_{L^\infty([0,T]\times\mathbb{S}^1)} \le L \right\}, \tag{7.69}$$

which satisfies $\tilde{\mathbb{P}}\left(A_L^{n_0}\right) \to 1$ as $L \to \infty$, uniformly in n_0.

For a.e. $t \in [0,T]$ and sufficiently large ℓ (depending on L),

$$\begin{aligned}
&\int_{\mathbb{S}^1} \mathbb{D}_\ell^-(t)\,\mathrm{d}x + \int_0^t \int_{\mathbb{S}^1} \left(\overline{q_-} - \tilde{q}_-\right)\left(\tilde{P} - \tilde{u}^2\right)\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{S}^1} \left(\frac{1}{4}\partial_{xx}^2\sigma - |\partial_x\sigma|^2\right)\mathbb{D}_\ell^- - \frac{3\ell}{2}\left(\mathbb{D}_\ell^- + \mathbb{D}^+\right)\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{S}^1} \left(\overline{S_\ell(q_-)' - q_-} - \left(S_\ell(\tilde{q}_-)' - \tilde{q}_-\right)\right)\left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right)\mathrm{d}x\,\mathrm{d}s \\
&\quad + \mathcal{M}_\ell^-(t) \le 0, \qquad \text{a.s. on } A_L^{n_0},
\end{aligned} \tag{7.70}$$

where $\mathcal{M}_\ell^-(t)$ is a square-integrable martingale, with $\tilde{\mathbb{E}}\left|\mathcal{M}_\ell^-(T)\right|^2 \lesssim_\ell 1$.

Proof. Using the test function $\varphi(t,x) = \psi(t)\phi(x)$ with $0 \le \psi \in C_c^\infty([0,T))$ and $\phi \equiv 1$ in (7.68), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} \mathbb{D}_\ell^-\,\mathrm{d}x + \sum_{i=1}^4 \int_{\mathbb{S}^1} I_\ell^{(i)}\,\mathrm{d}x + \int_{\mathbb{S}^1} I_\ell^{(5)}\,\mathrm{d}x\,\dot{\tilde{W}} \le 0, \tag{7.71}$$

which holds in $\mathcal{D}'([0,T))$, $\tilde{\mathbb{P}}$–a.s., with zero initial data, where

$$\begin{aligned}
I_\ell^{(1)} &= \left(\overline{S_\ell(q_-)'} - S_\ell(\tilde{q}_-)'\right)\left(\tilde{P} - \tilde{u}^2\right), \\
I_\ell^{(2)} &= \frac{1}{2}S_\ell(\tilde{q}_-)'\left(\overline{q^2} - \tilde{q}^2\right) - \left(\overline{H_{\ell,-}^{(2)}(q)} - H_{\ell,-}^{(2)}(\tilde{q})\right), \\
I_\ell^{(3)} &= -\frac{1}{4}\partial_{xx}^2\sigma^2\left(\overline{H_{\ell,-}^{(3)}(q)} - H_{\ell,-}^{(3)}(\tilde{q})\right), \\
I_\ell^{(4)} &= -\frac{1}{2}|\partial_x\sigma|^2\left(\overline{S_\ell(q_-)''q^2} - S''(\tilde{q})\tilde{q}^2\right),
\end{aligned}$$

$$I_\ell^{(5)} = -\partial_x \sigma \left(\overline{H_{\ell,-}^{(3)}(q)} - H_{\ell,-}^{(3)}(\tilde{q}) \right),$$

and $H_{\ell,-}^{(1)}$, $H_{\ell,-}^{(2)}$, $H_{\ell,-}^{(3)}$ are defined in (7.67).

1. The term $I_\ell^{(1)}$.

In view of Remark 7.11,

$$\begin{aligned} I_\ell^{(1)} &= (\overline{q_-} - \tilde{q}_-)\left(\tilde{P} - \tilde{u}^2\right) + e_\ell^{(1)}\left(\tilde{P} - \tilde{u}^2\right) \\ &= (\overline{q_-} - \tilde{q}_-)\left(\tilde{P} - \tilde{u}^2\right) \\ &\qquad + e_\ell^{(1)}\left(\tilde{P}_{n0} - \tilde{u}_{n0}^2\right) + e_\ell^{(1)}\left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n0} + \tilde{u}_{n0}^2\right), \end{aligned}$$

where

$$\begin{aligned} e_\ell^{(1)} &= \overline{S_\ell(q_-)' - q_-} - \left(S_\ell(\tilde{q}_-)' - \tilde{q}_-\right) \\ &= \overline{f_1(q)\mathbb{1}_{\{-2\ell<q<-\ell\}} + g_1(q)\mathbb{1}_{\{q\le -2\ell\}}} \\ &\qquad - \left(f_1(\tilde{q})\mathbb{1}_{\{-2\ell<\tilde{q}<-\ell\}} + g_1(\tilde{q})\mathbb{1}_{\{\tilde{q}\le -2\ell\}}\right), \\ f_1(v) &= \frac{1}{2\ell}(v+\ell)^2, \quad g_1(v) = -\frac{1}{2}(3\ell + 2v), \end{aligned} \tag{7.72}$$

Note the real-valued mapping $r_1(v) = f_1(v)\mathbb{1}_{\{-2\ell<v<-\ell\}} + g_1(v)\mathbb{1}_{\{v\le -2\ell\}}$ is convex:

$$r_1'(v) = \frac{1}{4\ell}(v+\ell)\,\mathbb{1}_{\{-2\ell<v<-\ell\}} - \mathbb{1}_{\{v\le -2\ell\}}, \quad r_1''(v) = \frac{1}{4\ell}\mathbb{1}_{\{-2\ell<v<-\ell\}} \ge 0.$$

Because of the convexity, it follows that $e_\ell^{(1)} \ge 0$ [51, Corollary 3.33] and thus

$$\begin{aligned} I_\ell^{(1)} &\ge (\overline{q_-} - \tilde{q}_-)\left(\tilde{P} - \tilde{u}^2\right) \\ &\qquad - e_\ell^{(1)} \left\| \tilde{P}_{n0} - \tilde{u}_{n0}^2 \right\|_{L^\infty([0,T]\times\mathbb{S}^1)} + e_\ell^{(1)}\left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n0} + \tilde{u}_{n0}^2\right). \end{aligned}$$

2. The term $I_\ell^{(2)}$.

Recalling the definition (7.62) of $\mathbb{D}^+$, we next manipulate $I_\ell^{(2)}$ into the form "$C_\ell\left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right) +$ error". From Remark 7.11,

$$\frac{1}{2}\left(\overline{q_-^2} - \tilde{q}_-^2\right) = \overline{S_\ell(q_-)} - S_\ell(\tilde{q}_-) + e_\ell^{(2,1)}, \tag{7.73}$$

where

$$e_\ell^{(2,1)} = \overline{f_{2,1}(q)\mathbb{1}_{\{-2\ell<q<-\ell\}} + g_{2,1}(q)\mathbb{1}_{\{q\le -2\ell\}}} - \left(f_{2,1}(q)\mathbb{1}_{\{-2\ell<q<-\ell\}} + g_{2,1}(q)\mathbb{1}_{\{q\le -2\ell\}}\right),$$

and

$$f_{2,1}(v) = -\frac{1}{6\ell}(v+\ell)^3, \quad g_{2,1}(v) = \frac{1}{6}\left(3v^2 + 9\ell v + 7\ell^2\right),$$

recalling that we drop the tilde atop a variable sitting under an overline (see Remark 4.10). Given the identity (7.73), writing

$$\overline{q^2} - \tilde{q}^2 = \overline{q_-^2} - \tilde{q}_-^2 + \overline{q_+^2} - \tilde{q}_+^2,$$

it follows that

$$\frac{1}{2}S_\ell(\tilde{q}_-)'\left(\overline{q^2} - \tilde{q}^2\right) = \frac{1}{2}S_\ell(\tilde{q}_-)'\left(\overline{q_+^2} - \tilde{q}_+^2\right) + S_\ell(\tilde{q}_-)'\left(\overline{S_\ell(q_-)} - S_\ell(\tilde{q}_-)\right) + S_\ell(\tilde{q}_-)' e_\ell^{(2,1)}.$$

Regarding $e_\ell^{(2,1)}$, observe that $r_{2,1}(v) = f_{2,1}(v)\mathbb{1}_{\{-2\ell<v<-\ell\}} + g_{2,1}(v)\mathbb{1}_{\{v\le -2\ell\}}$ is non-negative and convex. Indeed, by construction, $r(v)$ and $r'(v)$ are continuous functions, recalling that $S_\ell(v_\pm) \in W^{3,\infty}(\mathbb{R})$, and so

$$r_{2,1}'(v) = -\frac{1}{2\ell}(v+\ell)^2\mathbb{1}_{\{-2\ell<v<-\ell\}} + \frac{1}{2}(2v+3\ell)\mathbb{1}_{\{v\le -2\ell\}},$$
$$r_{2,1}''(v) = -\frac{1}{\ell}(v+\ell)\mathbb{1}_{\{-2\ell<v<-\ell\}} + \mathbb{1}_{\{v\le -2\ell\}} \ge 0.$$

Making use of $S_\ell(\tilde{q}_-)' \ge -3\ell/2$ and the positivity (negativity) of $\overline{f(q)} - f(\tilde{q})$ for any convex (concave) f [51, Corollary 3.33]

$$I_\ell^{(2)} \ge -\frac{3\ell}{4}\left(\overline{q_+^2} - \tilde{q}_+^2\right) - \frac{3\ell}{2}\left(\overline{S_\ell(q_-)} - S_\ell(\tilde{q}_-)\right) - \frac{3\ell}{2}e_\ell^{(2,1)} - \left(\overline{H_{\ell,-}^{(2)}(q)} - H_{\ell,-}^{(2)}(\tilde{q})\right)$$
$$= -\frac{3\ell}{2}\left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right) + e_\ell^{(2)},$$

where, recalling that the expression $H_{\ell,-}^{(2)}(v) = S_\ell(v_-)v - \frac{1}{2}S_\ell(v_-)'v^2$, see (7.67), takes the explicit form calculated in Remark 7.11,

$$e_\ell^{(2)} = \overline{f_2(q)\mathbb{1}_{\{-2\ell<q<-\ell\}} + g_2(q)\mathbb{1}_{\{q\le -2\ell\}}} - \left(f_2(\tilde{q})\mathbb{1}_{\{-2\ell<\tilde{q}<-\ell\}} - g_2(\tilde{q})\mathbb{1}_{\{\tilde{q}\le -2\ell\}}\right),$$
$$f_2(v) = \frac{1}{12\ell}\left(v^4 + 3\ell v^3 + 6\ell^2 v^2 + 7\ell^3 v + 3\ell^4\right), \quad g_2(v) = -\frac{1}{12}\left(13\ell^2 v + 21\ell^3\right).$$

3. The terms $I_\ell^{(3)}$ and $I_\ell^{(4)}$.

Similarly, using (7.67) and Remark 7.11, we obtain

$$\begin{aligned} H_-^{(3)}(v) &= \Big(H_-^{(3)}(v) + S_\ell(v_-)\Big) - S_\ell(v_-) \\ &= -S_\ell(v_-) + f_3(v)\mathbb{1}_{\{-2\ell<v<-\ell\}} + g_3(v)\mathbb{1}_{\{v\le -2\ell\}}, \end{aligned}$$

where

$$f_3(v) = -\frac{1}{6\ell}v^3 + \frac{1}{2}\ell v + \frac{1}{3}\ell^2, \quad g_3(v) = -\frac{3}{2}\ell v - \frac{7}{3}\ell^2.$$

Furthermore,

$$\begin{aligned} \frac{1}{2}S_\ell(v_-)''v_-^2 &= \Big(\frac{1}{2}S_\ell(v_-)''v_-^2 - S_\ell(v_-)\Big) + S_\ell(v_-) \\ &= S_\ell(v_-) + f_4(v)\mathbb{1}_{\{-2\ell<v<-\ell\}} + g_4(v)\mathbb{1}_{\{v\le -2\ell\}}, \end{aligned}$$

where

$$f_4(v) = \frac{1}{3\ell}v^3 - \frac{1}{2}\ell v - \frac{1}{6}\ell^2, \quad g_4(v) = \frac{3}{2}\ell v + \frac{7}{6}\ell^2.$$

Therefore, if we set

$$\begin{aligned} e_\ell^{(3)} &= \overline{f_3(q)\mathbb{1}_{\{-2\ell<q<-\ell\}} + g_3(q)\mathbb{1}_{\{q\le -2\ell\}}} \\ &\qquad - f_3(\tilde q)\mathbb{1}_{\{-2\ell<\tilde q<-\ell\}} + g_3(\tilde q)\mathbb{1}_{\{\tilde q\le -2\ell\}}, \\ e_\ell^{(4)} &= \overline{f_4(q)\mathbb{1}_{\{-2\ell<q<-\ell\}} + g_4(q)\mathbb{1}_{\{q\le -2\ell\}}} \\ &\qquad - f_4(\tilde q)\mathbb{1}_{\{-2\ell<\tilde q<-\ell\}} + g_4(\tilde q)\mathbb{1}_{\{\tilde q\le -2\ell\}}, \end{aligned}$$

we get

$$\begin{aligned} I_\ell^{(3)} &= \frac{1}{4}\partial_{xx}^2\sigma^2\mathbb{D}_\ell^- - \frac{1}{4}\partial_{xx}^2\sigma^2 e_\ell^{(3)}, \\ I_\ell^{(4)} &= -\left|\partial_x\sigma\right|^2\mathbb{D}_\ell^- - \left|\partial_x\sigma\right|^2 e_\ell^{(4)}. \end{aligned}$$

4. The term $I_\ell^{(5)}$.

By Lemmas 5.3 and 5.4, recalling that $\left|S_\ell(v_-) - S_\ell(v_-)'v\right| \lesssim_\ell |v|$, cf. (4.6), we may assume that $\overline{H_{\ell,-}^{(3)}(q)}$, $H_{\ell,-}^{(3)}(\tilde q)$, cf. (7.67), and thus $I_\ell^{(5)}$ belong to $L^{2r}_{\tilde\omega,t,x}$ (with $2r>2$), for each fixed ℓ. In particular, this implies that

$$\mathcal{M}_\ell^-(t) = \int_0^t \int_{\mathbb{S}^1} I_\ell^{(5)} \,\mathrm{d}x \,\mathrm{d}\tilde{W}$$

is a square-integrable martingale on $[0, T]$.

5. The inequality (7.70).

Introduce the “total error” function

$$\begin{aligned} h_\ell(v) = \Bigg(& -f_1(v) \left\| \tilde{P}_{n_0} - \tilde{u}_{n_0}^2 \right\|_{L^\infty([0,T]\times\mathbb{S}^1)} \\ & + f_2(v) - \frac{1}{4}\partial_{xx}^2 \sigma^2 f_3(v) - |\partial_x \sigma|^2 f_4(v) \Bigg) \mathbb{1}_{\{-2\ell < v < -\ell\}} \\ & + \Bigg(-g_1(v) \left\| \tilde{P}_{n_0} - \tilde{u}_{n_0}^2 \right\|_{L^\infty([0,T]\times\mathbb{S}^1)} \\ & + g_2(v) - \frac{1}{4}\partial_{xx}^2 \sigma^2 g_3(v) - |\partial_x \sigma|^2 g_4(v) \Bigg) \mathbb{1}_{\{v \le -2\ell\}}. \end{aligned}$$

Gathering the findings of the first three steps, we deduce that

$$\begin{aligned} & I_\ell^{(1)} + I_\ell^{(2)} + I_\ell^{(3)} + I_\ell^{(4)} \\ & \ge (\overline{q_-} - \tilde{q}_-)\left(\tilde{P} - \tilde{u}^2\right) - \frac{3\ell}{2}\left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right) + \left(\frac{1}{4}\partial_{xx}^2\sigma^2 - |\partial_x\sigma|^2\right)\mathbb{D}_\ell^- \\ & \quad + \overline{h_\ell(q)} - h_\ell(\tilde{q}) + e_\ell^{(1)}\left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right), \end{aligned} \tag{7.74}$$

where the overlines denote the weak limits in n only (n_0 is kept fixed).

Recall the definition of $A_L^{n_0}$ in (7.69). We claim that $h_\ell(v)$ is convex on $A_L^{n_0}$, at least for a sufficiently large $\ell = \ell(L)$. To see this, we can compute the second derivative of h_ℓ directly on each of the two subsets $\{-2\ell < v < -\ell\}$ and $\{v \le -2\ell\}$, thanks to the continuity of h_ℓ and h_ℓ' that follows from the continuity of f_i, f_i', g_i, g_i' ($i = 1, \dots 4$), and then add the results. Indeed,

$$\begin{aligned} & f_1''(v) = \frac{1}{\ell}, \quad f_2''(v) = \frac{1}{\ell}v^2 + \frac{3}{2}v + \ell, \quad f_3''(v) = -\frac{1}{\ell}v, \\ & f_4''(v) = \frac{2}{\ell}v, \quad g_1''(v),\ g_2''(v),\ g_3''(v),\ g_4''(v) \equiv 0, \end{aligned}$$

and so on $A_L^{n_0}$, for any $v \in \mathbb{R}$ and a.e. $x \in \mathbb{S}^1$,

$$h_\ell''(v) \ge \left(\frac{1}{\ell}v^2 + \frac{3}{2}v + \ell - \frac{1}{\ell}\left\{L - \frac{1}{4}\partial_{xx}^2\sigma^2 v + 2|\partial_x\sigma|^2 v\right\}\right)\mathbb{1}_{\{-2\ell < v < -\ell\}},$$

which is non-negative for sufficiently large ℓ because the term in braces can be made small relative to the terms outside the braces.

The convexity of $h_\ell(v)$ implies that on $A_L^{n_0}$,

$$\overline{h_\ell(q)} - h_\ell(\tilde{q}) \geq 0 \quad \text{a.e. in } [0,T] \times \mathbb{S}^1. \tag{7.75}$$

Using (7.75), which holds for a sufficiently large $\ell = \ell(L)$, (7.74) becomes

$$\begin{aligned} &I_\ell^{(1)} + I_\ell^{(2)} + I_\ell^{(3)} + I_\ell^{(4)} \\ &\geq (\overline{q_-} - \tilde{q}_-)\left(\tilde{P} - \tilde{u}^2\right) - \frac{3\ell}{2}\left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right) \\ &\quad + \left(\frac{1}{4}\partial_{xx}^2 \sigma^2 - |\partial_x \sigma|^2\right)\mathbb{D}_\ell^- + e_\ell^{(1)}\left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right). \end{aligned} \tag{7.76}$$

Now we multiply (7.71) by $\mathbb{1}_{A_L^{n_0}}$ and insert (7.76), arriving at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} \mathbb{D}_\ell^- \,\mathrm{d}x + \int_{\mathbb{S}^1} (\overline{q_-} - \tilde{q}_-)\left(\tilde{P} - \tilde{u}^2\right)\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} \left(\frac{1}{4}\partial_{xx}^2\sigma - |\partial_x\sigma|^2\right)\mathbb{D}_\ell^- - \frac{3\ell}{2}\left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right)\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} e_\ell^{(1)}\left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right)\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} I_\ell^{(5)}\,\mathrm{d}x\,\dot{\tilde{W}} \leq 0, \quad \text{in } \mathcal{D}'([0,T)), \text{ a.s. on } A_L^{n_0}, \end{aligned} \tag{7.77}$$

with zero initial data (in the distributional sense). Arguing as in the proofs of Lemmas 7.4, 7.11 and 7.13, we can turn (7.77) into the inequality (7.70) that holds a.e. in $A_L^{n_0} \times [0,T]$.

Remark 7.16. Note carefully that the fifth step is rather delicate, relying on having precise control of the error terms leading up to the convexity of the total error function $h_\ell(v)$, and thus (7.75). Along the way, we exploit some crucial "coercivity" induced by the specific error term $e_\ell^{(2)}$ linked to the difference $\overline{H_{\ell,-}^{(2)}(q)} - H_{\ell,-}^{(2)}(\tilde{q})$, recalling that $H_{\ell,-}^{(2)}(v) = S_\ell(v_-)v - \frac{1}{2}S_\ell(v_-)'v^2$. It may be instructive to keep in mind that $S(v)v - \frac{1}{2}S'(v)v^2 \equiv 0$ if $S(v) = \frac{1}{2}v_\pm^2$ or v^2.

6. Properties of the set $A_L^{n_0}$.

The set $A_L^{n_0} \subset \tilde{\Omega}$ is measurable as $\|\tilde{P}_{n_0} - \tilde{u}_{n_0}^2\|_{L^\infty([0,T]\times\mathbb{S}^1)}$ is a random variable (see Lemma 5.6). Denote by $\left(A_L^{n_0}\right)^c$ the complement $\tilde{\Omega}\backslash A_L^{n_0}$. It further follows from Markov's inequality applied to the bound of Lemma 5.6 that

$$\tilde{\mathbb{P}}\left(\left(A_L^{n_0}\right)^c\right) \leq \frac{1}{L}\tilde{\mathbb{E}}\left\|\tilde{P}_{n_0} - \tilde{u}_{n_0}^2\right\|_{L^\infty([0,T]\times\mathbb{S}^1)} \lesssim \frac{1}{L}. \quad \square \tag{7.78}$$

We can now identify the weak limit $\overline{q^2}$ with $\tilde{q}^2$, thereby concluding the proof of Theorem 7.1.

Proof of Theorem 7.1. We shall be adding (7.64) and (7.70). The purpose of doing so is that using

$$\overline{q_+} + \overline{q_-} = \tilde{q} = \tilde{q}_+ + \tilde{q}_- \Longrightarrow \overline{q_+} - \tilde{q}_+ + \overline{q_-} - \tilde{q}_- = 0,$$

the term involving $\left(\tilde{P} - \tilde{u}^2\right)$ disappears, allowing us to conclude via taking an expectation and applying Gronwall's inequality, as we will demonstrate next.

We observe first that the inequality (7.64) holds a.s. on $A_L^{n_0}$, where $A_L^{n_0}$ is defined in (7.69). We now multiply each of (7.64) and (7.70) by $\mathbb{1}_{A_L^{n_0}}$, add these two equations together and then take an expectation to obtain, for all sufficiently large ℓ (with L, n_0 fixed) and a.e. $t \in [0, T]$,

$$\begin{aligned}
&\tilde{\mathbb{E}} \int_{\mathbb{S}^1} \mathbb{1}_{A_L^{n_0}} \left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right)(t)\,\mathrm{d}x \\
&\quad + \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{S}^1} f_\ell(x) \mathbb{1}_{A_L^{n_0}} \left(\mathbb{D}^+ + \mathbb{D}_\ell^-\right)(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \le -\mathbb{E} \int_0^t \int_{\mathbb{S}^1} \mathbb{1}_{A_L^{n_0}} e_\ell^{(1)} \left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right) \mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{7.79}$$

where

$$f_\ell(x) = \frac{1}{4}\partial_{xx}^2 \sigma^2 - |\partial_x \sigma|^2 - \frac{3\ell}{2}, \qquad \|f_\ell\|_{L^\infty(\mathbb{S}^1)} \le C\ell,$$

and, for brevity, we have retained the notation (7.72) for $e_\ell^{(1)}$.

Applying Gronwall's inequality to (7.79), we get for a.e. $t \in [0, T]$ that

$$\begin{aligned}
&\tilde{\mathbb{E}} \left(\mathbb{1}_{A_L^{n_0}} \left\|\mathbb{D}^+(t) + \mathbb{D}_\ell^-(t)\right\|_{L^1(\mathbb{S}^1)}\right) \\
&\qquad \le e^{Ct\ell}\, \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{S}^1} \mathbb{1}_{A_L^{n_0}}\, e_\ell^{(1)} \left(\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right) \mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Integrating the above over $[0, T]$,

$$\begin{aligned}
&\int_0^T \tilde{\mathbb{E}} \left(\mathbb{1}_{A_L^{n_0}} \left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^1(\mathbb{S}^1)}\right) \mathrm{d}t \\
&\qquad \le T e^{CT\ell}\, \tilde{\mathbb{E}} \int_0^T \int_{\mathbb{S}^1} \left|e_\ell^{(1)}\right| \left|\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right| \mathrm{d}x\,\mathrm{d}t =: T e^{CT\ell} \mathcal{J}_{n_0}.
\end{aligned}$$

Adding $\mathbb{E}\int_0^T \mathbb{1}_{\left(A_L^{n_0}\right)^c} \left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^1(\mathbb{S}^1)}\,\mathrm{d}t$ to both sides,

$$
\begin{aligned}
&\tilde{\mathbb{E}}\int_0^T \left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^1(\mathbb{S}^1)}\,\mathrm{d}t \\
&\le \mathbb{E}\int_0^T \mathbb{1}_{\left(A_L^{n_0}\right)^c} \left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^1(\mathbb{S}^1)}\,\mathrm{d}t + Te^{CT\ell}\mathcal{J}_{n_0} \\
&\lesssim_T \left(\tilde{\mathbb{E}}\int_0^T \left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^1(\mathbb{S}^1)}^r\,\mathrm{d}t\right)^{1/r} \left(\tilde{\mathbb{P}}\left(\left(A_L^{n_0}\right)^c\right)\right)^{1/r'} + e^{CT\ell}\mathcal{J}_{n_0} \\
&\lesssim_T \left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^r_{\tilde{\omega},t,x}} \left(\frac{1}{L}\right)^{1/r'} + e^{CT\ell}\mathcal{J}_{n_0},
\end{aligned}
\tag{7.80}
$$

where the final inequality follows from (7.78). The implicit constant in the final $\lesssim$ is independent of L and n_0. Also note that $\left\|\mathbb{D}^+ + \mathbb{D}_\ell^-\right\|_{L^r_{\tilde{\omega},t,x}} \lesssim 1$, uniformly in ℓ by Lemma 5.4 and the first inequality of (5.9). On the other hand, as we shall presently argue, $\mathcal{J}_{n_0} \to 0$ as $n_0 \uparrow \infty$, uniformly in ℓ. The convergence of $\mathcal{J}_{n_0}$ is a consequence of two facts. First, by the second bound of (5.4) and (7.34), we have

$$
\left\|e_\ell^{(1)}\right\|_{L^{2r}_{\tilde{\omega},t,x}} = \left\|\overline{S_\ell'(q_-) - q_-} - \left(S_\ell'(\tilde{q}) - \tilde{q}_-\right)\right\|_{L^{2r}_{\tilde{\omega},t,x}} \lesssim 1.
$$

Second, as $n_0 \uparrow \infty$, we have the strong convergences (7.21) of $\tilde{P}_{n_0} \to \tilde{P}$ in $L^p_{\tilde{\omega},t,x}$ and (7.33) of $\tilde{u}_{n_0}^2 \to \tilde{u}^2$ in $L^p_{\tilde{\omega},t,x}$, for any $p < p_0/2$. Since $p_0 > 4$, and $p := 2r/(2r-1) < 2$ for r close to $3/2$, this implies that

$$
\begin{aligned}
\mathcal{J}_{n_0} &= \tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \left|e_\ell^{(1)}\right| \left|\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \left\|e_\ell^{(1)}\right\|_{L^{2r}_{\tilde{\omega},t,x}} \left\|\tilde{P} - \tilde{u}^2 - \tilde{P}_{n_0} + \tilde{u}_{n_0}^2\right\|_{L^p_{\tilde{\omega},t,x}} \xrightarrow{n_0\uparrow\infty} 0,
\end{aligned}
$$

nullifying the harmful exponential factor in (7.80), and yielding

$$
\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{S}^1} \mathbb{D}^+ + \mathbb{D}_\ell^-\,\mathrm{d}x\,\mathrm{d}t \lesssim L^{-1/r'},
\tag{7.81}
$$

for any sufficiently large ℓ (with L fixed).

Finally, upon taking the limits $\ell \to \infty$ first—making use of Lemma 7.14—and $L \to \infty$ second in (7.81), we work out that

$$\overline{q_+^2} = \tilde{q}_+^2, \quad \overline{q_-^2} = \tilde{q}_-^2 \quad \text{a.e. in } \tilde{\Omega} \times [0, T] \times \mathbb{S}^1,$$

which concludes the proof of Theorem 7.1. □

Remark 7.17. Let us refine Remark 7.16 further by exploring the treatment of the residual "bad" error term $e^{CT\ell}\mathcal{J}_{n_0}$ in (7.80). This specific term does not lend itself to a resolution through the delicate balance of convexity and coercivity discussed in Remark 7.16. Rather, its successful management primarily depends on the strong convergence (7.21). This strong convergence, in turn, stems from the employment of the quasi-Polish strong-weak space $L^r\left(L^r_w\right)$ featured among the path spaces (4.13).

Data availability

No data was used for the research described in the article.

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Appendix A. Formal derivation of stochastic CH equation

Let us outline a formal derivation of the stochastic CH equation (1.3). Denote by M_m be the multiplication operator $M_m f = mf$, and by D the spatial derivative operator. As is well-known, the deterministic CH equation (for the momentum variable m) takes the form

$$0 = \partial_t m + M_m D \frac{\delta \tilde{h}[m]}{\delta m} + D M_m \frac{\delta \tilde{h}[m]}{\delta m}, \tag{A.1}$$

where $\tilde{h}[m] = \int_{\mathbb{S}^1} \frac{1}{2} m(t,x)\,(K * m)(t,x)\,\mathrm{d}x$ is a nonlocal Hamiltonian based on the kernel K defined in (1.2). Setting $u := K * m$, one can formally convert the bi-Hamiltonian equation (A.1) into the deterministic CH equation (for u), i.e., (1.1) with $\sigma \equiv 0$. The stochastic CH equation is obtained by considering a stochastic perturbation of the temporally-integrated Hamiltonian:

$$H[m] := \int_{\mathbb{S}^1} \left(\int_0^t \frac{1}{2} m(s,x)(K * m)(t,x)\,\mathrm{d}s + \int_0^t \big(m(s,x)\sigma(x)\big) \circ \mathrm{d}W(s) \right) \mathrm{d}x.$$

We recover the deterministic Hamiltonian $\tilde{h}$ by taking $\sigma \equiv 0$ and computing $\mathrm{d}H/\mathrm{d}t$. The first variation of $H[m]$ is $\frac{\delta H[m]}{\delta m} = u + \sigma(x)\,\dot{W}$. Note that this expression is highly irregular in time

t (of class $C^{-1/2-0}$), but at the formal level, compared with (A.1), the analogous stochastic CH equation becomes

$$0 = \mathrm{d}m + M_m D\big(u\,\mathrm{d}t + \sigma(x)\,\mathrm{d}W\big) + DM_m\big(u\,\mathrm{d}t + \sigma(x)\,\mathrm{d}W\big),$$

where the multiplication operator M here uses the Stratonovich product $\circ$; written out more explicitly, we obtain

$$0 = \mathrm{d}m + \big(m\,\partial_x u + \partial_x(mu)\big)\,\mathrm{d}t + m\,\partial_x\sigma(x) \circ \mathrm{d}W + \partial_x\big(m\sigma(x)\big) \circ \mathrm{d}W. \tag{A.2}$$

Recalling that $u = K * m$, i.e., $m = (1 - \partial^2_{xx})u$, we can formally expand (A.2) into (1.1), or (1.3) thanks to the Stratonovich–Itô conversion formula. In this paper we use (1.3) as the operational form of the stochastic CH equation.

Appendix B. Primer on quasi-Polish spaces

We detail here some definitions and results that have been applied repeatedly in our proofs. Ready references for some—but not all—of the material collected here are [46] and [12,13,52].

B.1. Examples of quasi-Polish spaces

In this subsection we give the definition and some examples of quasi-Polish spaces.

First, given two measurable spaces $(\mathfrak{X}_i, \mathcal{M}_i), i = 1, 2$, by the statement "$A$ is $\mathcal{M}_1/\mathcal{M}_2$", we mean that $A : (\mathfrak{X}_1, \mathcal{M}_1) \to (\mathfrak{X}_2, \mathcal{M}_2)$ is measurable. Let $\mathcal{A}$ be a collection of subsets, or a collection of maps. On a few occasions, see for example (6.1), by $\Sigma(\mathcal{A})$ we mean the σ-algebra generated by $\mathcal{A}$.

Definition B.1 *(Quasi-Polish space).* A topological space (Z, τ) is *quasi-Polish* if there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions $f_n : Z \to [-1, 1]$ separating points of Z.

Quasi-Polish spaces are Hausdorff. Below we exhibit some examples of commonly encountered quasi-Polish spaces, remarking specifically on the existence of a sequence of continuous functions to $[-1, 1]$ that separates points. By considering the continuous injection $(f_1, f_2, \dots) : Z \hookrightarrow [-1, 1]^{\mathbb{N}}$ of Z into the Polish space $[-1, 1]^{\mathbb{N}}$, one can recover many of the properties of Polish spaces for compact subsets of a quasi-Polish space Z, see [46, Section 2]. The topology induced by $\{f_n\}_{n\in\mathbb{N}}$, sometimes referred to as τ_f, coincides with the topology of Z on τ-compact subsets. This is the cardinal property that allows theorems on Polish spaces (such as the Skorokhod representation theorem) to carry over to quasi-Polish spaces.

Examples.

(1) If $(Z, \|\cdot\|)$ is a separable normed space (with dual Z'), then it is quasi-Polish. Indeed, let $\{\phi_n\}_{n\in\mathbb{N}} \subset Z'$ be such that

$$\|z\| = \sup_n \langle \phi_n, z\rangle, \quad z \in Z.$$

Define $f_n(z) = \frac{2}{\pi}\arctan\big(\langle \phi_n, z\rangle\big)$, $n \in \mathbb{N}$. Given $z_1 \neq z_2$, choose an integer m such that $\langle \phi_m, z_1 - z_2\rangle > \frac{1}{2}\|z_1 - z_2\| > 0$. Whereupon $\langle \phi_m, z_1\rangle > \langle \phi_m, z_2\rangle$ and hence $f_m(z_1) > f_m(z_2)$. This also shows that $Z - w$ (Z endowed with the weak topology τ_w) is quasi-Polish. Therefore, the spaces $L^p([0,T]\times\mathbb{S}^1) - w$, $1 \le p < \infty$, are quasi-Polish; they are used in (4.13) with $p = r$ and $p = 2r$, $r \in [1, 3/2)$.

(2) Let $(Z, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Equipping $C([0,T]; Z - w)$ with the locally convex topology generated by the seminorms $\|h\|_\phi := \sup_{0\le t\le T} |\langle h(t), \phi\rangle|$ for $\phi \in Z$, the space $C([0,T]; Z - w)$ becomes quasi-Polish [13, Remark 4.1]. An example is $C([0,T]; H^1(\mathbb{S}^1) - w)$, which is used on a few occasions.

(3) If $(Z, \|\cdot\|)$ is a separable Banach space, then its dual Z' endowed with the weak-star topology $\tau_\star$ is quasi-Polish. To see this, take an arbitrary countable dense subset $D \subset Z$, $D = \{d_1, d_2, \dots\}$. Given $\phi_1, \phi_2 \in Z'$, $\phi_1 \neq \phi_2$, there must exist $d_n \in D$ such that $\phi_1(d_n) \neq \phi_2(d_n)$, because, if this were not the case, then $\phi_1(d) = \phi_2(d)$, $d \in D$, and thus $\phi_1 \equiv \phi_2$. Define $f_n : Z' \to \mathbb{R}$, $\phi \mapsto \phi(d_n)$, $n \in \mathbb{N}$. Then f_n is $\tau_\star$-continuous, and we conclude that $(Z', \tau_\star)$ is quasi-Polish, with a separating sequence provided by $\{f_n\}_{n\in\mathbb{N}}$. An example is the space of Radon measure (with a separable pre-dual), equipped with the weak-star topology.

For a quasi-Polish space (Z, τ), the point-separating sequence does not always characterise the topology τ, because if Z has the topology τ_f induced by functions $\{f_n\}$ that separate points, then changing the topology of Z to the discrete topology, Z remains quasi-Polish under the maps $\{f_n\}$. In general, we have $\tau_f \subset \tau$. However, for the quasi-Polish spaces used in this paper, we will always know that $\tau_f = \tau$.

We recall the following result [45, Proposition 1.1.1], also known as "the linear characterisation of the Borel σ-algebra", which we will need below.

Lemma B.2. *Let $(Z, \|\cdot\|)$ be a separable normed space. Let $\{\phi_n\}_{n\in\mathbb{N}} \subset Z'$ be a norming sequence in the sense that*

$$\|z\| = \sup_{n\in\mathbb{N}} \big\langle \phi_n, z\big\rangle, \quad z \in Z.$$

Denote by $\mathcal{B}(Z)$ the Borel σ-algebra on Z. Then $\mathcal{B}(Z) = \Sigma\big(\phi_n, n \in \mathbb{N}\big)$.

In Section 4, we made essential use of the (topological) space

$$L^{p_1}\big(L^{p_2}_w\big) = L^{p_1}\big([0,T]; L^{p_2}(\mathbb{S}^1) - w\big), \quad p_1, p_2 \in (1,\infty). \tag{B.1}$$

To define $L^{p_1}\big(L^{p_2}_w\big)$, consider the classical Bochner space of equivalence classes of measurable functions $z : [0,T] \to L^{p_2}(\mathbb{S}^1)$ for which $\|z(\cdot)\|_{L^{p_2}(\mathbb{S}^1)} \in L^{p_1}([0,T])$, denoted by $L^{p_1}(L^{p_2}) = L^{p_1}([0,T]; L^{p_2}(\mathbb{S}^1))$. Equipping this space with the locally convex topology generated by the seminorms

$$L^{p_1}(L^{p_2}) \ni z \mapsto \left(\int_0^T \left| \int_{\mathbb{S}^1} \phi(x) z(t,x)\,\mathrm{d}x \right|^{p_1} \mathrm{d}t \right)^{\frac{1}{p_1}}, \quad \phi \in L^{p_2'}(\mathbb{S}^1), \tag{B.2}$$

where $\frac{1}{p_2} + \frac{1}{p_2'} = 1$, we denote the resulting topological space by $L^{p_1}\big(L_w^{p_2}\big)$, see (B.1).

For notational simplicity in what follows, set $Z = L^{p_1}(L^{p_2})$ and denote by τ_N the new topology (B.2). We will then write $L^{p_1}\big(L_w^{p_2}\big) = (Z, \tau_N)$. We claim that (Z, τ_N) is a quasi-Polish space. Indeed, we first notice that, given an arbitrary net $\{z_\alpha\}_\alpha \subset (Z, \tau_N)$ converging to $z \in (Z, \tau_N)$, then $z_\alpha \to z$ with respect to the standard weak topology τ_w of Z. In other words, (Z, τ_N) embeds continuously into (Z, τ_w). Trivially, τ_N is weaker than the norm topology of Z, called τ_s. Consequently, we have $\tau_w \subset \tau_N \subset \tau_s$. By the separability of Z (endowed with τ_s) and the previous discussion of this appendix, we know that (Z, τ_w) is quasi-Polish. As a result, any separating sequence of continuous functions for this space will also be a separating sequence for (Z, τ_N), thereby turning it into a quasi-Polish space. Finally, from the inclusions $\tau_w \subset \tau_N \subset \tau_s$, we also obtain that $\mathcal{B}_{\tau_N} = \mathcal{B}_{\tau_s}$, because in a separable Banach space the Borel σ-algebra $\mathcal{B}_{\tau_w}$ generated by the weak topology coincides with the strong Borel σ-algebra $\mathcal{B}_{\tau_s}$, cf. Lemma B.2, and since τ_N is an intermediate topology, this must hold for τ_N as well.

In Section 5, we used a quasi-Polish analogue of the Kuratowski–Lusin–Souslin (KLS) theorem, taken from [52, Corollary A.2] and [12, Proposition C.2]. This result is used repeatedly in Section 5.

Lemma B.3. *Let Z be a quasi-Polish space and let Y be a Polish space for which exists a continuous injection $b : Y \to Z$. For any Borel set $B \subset Y$, the set $b[B]$ is Borel in Z.*

The proof is a direct application of the KLS theorem after the injection $Z \hookrightarrow [-1, 1]^{\mathbb{N}}$, which puts us in the Polish space setting.

New quasi-Polish spaces can be constructed by forming Cartesian products of countable collections of them (see, e.g., [11] and next subsection). This fact is heavily used in Section 4. In this paper, we avoided using *intersections* of quasi-Polish spaces in our application of the Skorokhod–Jakubowski theorem [46] (see Theorem B.12). Let us consider a Skorokhod–Jakubowski representation $\{\tilde{u}_n\}$ of a sequence $\{u_n\}$, and suppose we need to know the a.s. convergence of $\{\tilde{u}_n\}$ in two different spaces Z_1 and Z_2. It is then natural to use a space Z for which

(i) Z is quasi-Polish,
(ii) compact subsets of Z can be identified, in order to verify tightness of the laws of $\{\tilde{u}_n\}$,
(iii) Z respects the topologies of Z_1 and Z_2, in the sense that a.s. convergence in Z implies a.s. convergence in Z_1 and Z_2 separately.

These three requirements are in tension. As the topology chosen with which to equip Z is strengthened, (iii) is more easily satisfied, whereas (i) and (ii) are less easily so. For the intersection $Z = Z_1 \cap Z_2$, endowed with the upper bound topology, (i) and (ii) are fulfilled as soon as Z_1 and Z_2 are quasi-Polish, since Z embeds continuously in Z_1 and Z_2 by construction. However, additional arguments are required to find compact subsets of Z to satisfy (ii) (see, e.g., [10]); the reason is that there is no general way to construct compact subsets of Z using compact subsets of Z_1 and Z_2. On the other hand, if one considers the Cartesian product with the product topology, the three requirements above are automatically satisfied. In particular, Tychonoff's theorem allows us to readily construct compact subsets of Z.

B.2. Products of quasi-Polish spaces

In Section 4, we worked systematically with random variables defined on countable products of quasi-Polish spaces.

Lemma B.4. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a countable collection of quasi-Polish spaces. Then $\mathfrak{X} = \prod_{i\in\mathbb{N}} Z_i$, endowed with the product topology, is a quasi-Polish space.*

Proof. This is immediate on invoking the definition of a quasi-Polish—that there is a countable, point-separating collection of maps $g_n : \mathfrak{X} \to [-1, 1]$. Let $\pi_i : \mathfrak{X} \to Z_i$ be the ith canonical projection. Since there is a collection $f_{i,n} : Z_i \to [-1, 1]$ for the ith factor space in $\mathfrak{X}$, the maps $\{f_{i,n} \circ \pi_i\}_{(i,n)\in\mathbb{N}^2}$ can be reordered to give $\{g_n\}_{n\in\mathbb{N}}$ on the product space $\mathfrak{X}$.[1] □

In what follows, we will continue to focus on products of quasi-Polish spaces and the measures that can be defined on them, starting with some subtle issues arising from the general non-coincidence of the Borel σ-algebra $\mathcal{B}\left(\prod_i Z_i\right)$ and the product Borel σ-algebra $\bigotimes_i \mathcal{B}(Z_i)$. To take an example, in Section 5, we implicitly identified $\tilde{u}_0 \in H^1(\mathbb{S}^1)$ with $\tilde{u}(0) \in L^2(\mathbb{S}^1)$. By the equality of laws, the probability law of $\tilde{u}(0)$ is supported on $Z = H^1(\mathbb{S}^1)$. In order to identify $\tilde{u}_0 \in Z$ with $\tilde{u}(0) \in Z$, we need to ensure that the joint law of $\left(\tilde{u}_0, \tilde{u}(0)\right)$ is supported on the diagonal $\Delta_{Z\times Z} = \{(z, z) \in Z \times Z : z \in Z\}$. For arbitrary topological spaces Z, this is not always possible, for the surprising reason that the diagonal $\Delta_{Z\times Z}$, whilst certainly in the Borel σ-algebra $\mathcal{B}(Z \times Z)$, is not necessarily in the product Borel σ-algebra $\mathcal{B}(Z) \otimes \mathcal{B}(Z)$, for large enough topologies on Z (known as *Nedoma's pathology* [57, Chapter 15.9]). However, this is no impediment in quasi-Polish spaces.

Lemma B.5. *Let Z be a quasi-Polish space. Then the diagonal $\Delta_{Z\times Z}$ belongs to $\mathcal{B}(Z) \times \mathcal{B}(Z)$, i.e., the diagonal is measurable.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be the point-separating sequence of continuous maps, $f_n : Z \to [-1, 1]$. Define the following class of subsets of Z:

$$\mathfrak{C} = \left\{ f_n^{-1}([q, r]) : q, r \in \mathbb{Q},\ 0 \le q < r \le 1,\ n \in \mathbb{N} \right\}.$$

The collection $\mathfrak{C}$ is countable and in $\mathcal{B}(Z)$, because $f_n^{-1}([q, r])$ is closed. Let $x, y \in Z, x \neq y$. Choose $m \in \mathbb{N}$ such that $f_m(y) < f_m(x)$, and two rational numbers $0 < q < r \le 1$ such that $f_m(x) \in [q, r]$ and $f_m(y) < q$, i.e., $x \in f_m^{-1}([q, r])$ and $y \notin f_m^{-1}([q, r])$. As a result, $\mathfrak{C}$ separates point of Z and, by a theorem of Dravecký [29, Theorem 1], the diagonal $\Delta_{Z\times Z}$ is measurable. □

Lemma B.6. *Consider a quasi-Polish space Z with a point-separating sequence of continuous maps $\{f_n : Z \to [-1, 1]\}_{n\in\mathbb{N}}$, and denote by $\mathcal{B}_f$ the σ-algebra generated by $\{f_n\}_{n\in\mathbb{N}}$. Let $\mu : \mathcal{B}_f \to [0, 1]$ be a tight probability measure. Define the σ-algebra*

[1] Banakh, Bogachev, and Kolesnikov [3] derives the stronger conclusion of the weak Skorokhod property instead of the weak sequential Skorokhod property derived here, under the stronger assumption of the existence of a fundamental sequence of compact sets. We do not require this assumption, and our result applies to arbitrary countable collections of quasi-Polish spaces.

$$\mathcal{B}_*(Z) := \{V \subset Z : V \cap K \in \mathcal{B}(Z),\ \forall K \subset Z\, compact\},$$

where $\mathcal{B}(Z)$ is the Borel σ-algebra of Z. Then there exists a unique Radon extension $\lambda : \mathcal{B}_*(Z) \to [0, 1]$ of μ.

Proof. Recall that the σ-algebra generated by the compact sets of Z belongs to $\mathcal{B}_f$ [46]. Then, for any $n \in \mathbb{N}$, there exists a compact $K_n \in \mathcal{B}_f$ such that $\mu(K_n^c) < n^{-1}$. Setting $K = \cup_n K_n \in \mathcal{B}_f$, it follows that

$$\mu(K^c) = \mu\left(\cap_n K_n^c\right) \leq \mu\left(K_n^c\right) < n^{-1},$$

i.e., $\mu(K^c) = 0$. Thus, the support of μ belongs to K. By [13, Section 3], there exists a unique Radon extension $\lambda : \mathcal{B}_*(Z) \to [0, 1]$ of μ. □

Remark B.7. Since for each $V \in \mathcal{B}(Z)$ and compact $K \subset Z$, $V \cap K \in \mathcal{B}(Z)$, it follows from the definition of $\mathcal{B}_*(Z)$ that $\mathcal{B}(Z) \subset \mathcal{B}_*(Z)$.

Suppose A_i, $i \in \mathbb{N}$, lives on a quasi-Polish space and is measurable. The next lemma shows that $A = \{A_i\}_{i\in\mathbb{N}}$ is measurable with respect to the product of the individual Borel σ-algebras.

Lemma B.8. *Let $\{Z_i\}_{i\in\mathbb{N}}$ be a countable collection of quasi-Polish spaces, and denote by $\mathcal{B}_i = \mathcal{B}(Z_i)$ the Borel σ-algebra on Z_i. Define $\mathfrak{X} = \prod_{i\in\mathbb{N}} Z_i$, endowed with the product topology. Let $\pi_i : \mathfrak{X} \to Z_i$ be the ith canonical projection. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $A_i : (\Omega, \mathcal{F}) \to (Z_i, \mathcal{B}_i)$, i.e., for each i, A_i is $\mathcal{F}/\mathcal{B}_i$-measurable. Finally, consider the unique map $A : \Omega \to \mathfrak{X}$ characterised by $\pi_i(A(\omega)) = A_i(\omega)\ \forall\ i$ and ω. Then A is $\mathcal{F}/\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$-measurable.*

Proof. By countability, $\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$ is generated by the family $\left\{\prod_{i\in\mathbb{N}} E_i : E_i \in \mathcal{B}_i\right\}$. It is therefore enough to check measurability for these sets only. Measurability here is evident, because

$$\left\{A \in \prod_{i\in\mathbb{N}} E_i\right\} = \bigcap_{i\in\mathbb{N}} \{A_i \in E_i\} \in \mathcal{F}. \quad \square$$

By the previous lemma, A is generally only $\mathcal{F}/\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$-measurable, and is hence precluded from being a random variable with respect to the natural σ-algebra on $\mathfrak{X}$, namely the Borel σ-algebra $\mathcal{B}(\mathfrak{X})$ (in which case the term "random mapping" is used), because generally for quasi-Polish spaces we only have $\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i \subset \mathcal{B}(\mathfrak{X})$, where $\mathcal{B}(\mathfrak{X})$ is the Borel σ-algebra on $\mathfrak{X}$ with the product topology. Fortunately, in applications with random mappings whose laws are tight, this is not a major problem, for the reason conferred about in Remark B.11 below.

We conclude this section by clarifying the relationship between the measures defined via restrictions and extensions on the hierarchy of σ-algebras introduced so far. Given a random variable A on $(\Omega, \mathbb{P})$, let us denote by $\mathbb{P}_A$ its law $\mathbb{P} \circ A^{-1}$.

Lemma B.9. *Let $\{Z_i\}_{i\in\mathbb{N}}$, $\mathfrak{X}$, $\mathcal{B}_i$, $\mathcal{B}(\mathfrak{X})$, π_i be defined as in Lemma B.8. For each $i \in \mathbb{N}$, let $\left\{f_{i,n} : Z_i \to [-1, 1]\right\}_{n\in\mathbb{N}}$ be the point-separating sequence of continuous maps linked to Z_i. For*

$i, n \in \mathbb{N}$, define $h_{i,n} = f_{i,n} \circ \pi_i$ and denote by $\mathcal{B}_f$ the σ-algebra generated by $\{h_{i,n}\}_{(i,n)\in\mathbb{N}^2}$. Finally, set

$$\mathcal{B}_*(\mathfrak{X}) = \left\{V \subset \mathfrak{X} : V \cap K \in \mathcal{B}(\mathfrak{X}),\ \forall K \subset \mathfrak{X}\, compact\right\}.$$

For each $i \in \mathbb{N}$, let $A_{i,\nu} : (\Omega, \mathcal{F}, \mathbb{P}) \to (Z_i, \mathcal{B}_i)$ be a family of random variables, indexed over $\nu \in (0, 1)$, with a corresponding tight family of laws $\{\mu_{i,\nu}\}_{0<\nu<1}$. Let $A_\nu : \Omega \to \mathfrak{X}$ be uniquely characterised by $\pi_i(A_\nu(\omega)) = A_{i,\nu}(\omega)$ for all i and ω, and denote by η_ν the law of A_ν restricted to $\mathcal{B}_f$. Finally, denote by $\mathcal{K}$ the family of compact subsets of $\mathfrak{X}$.

(i) *We have the inclusions*

$$\Sigma(\mathcal{K}) \subset \mathcal{B}_f \subset \bigotimes_{i\in\mathbb{N}} \mathcal{B}_i \subset \mathcal{B}(\mathfrak{X}) \subset \mathcal{B}_*(\mathfrak{X}).$$

(ii) *For each ν, the law η_ν of A_ν can be uniquely extended to $\mathcal{B}_*(\mathfrak{X})$ as a Radon probability measure λ_ν. The family $\{\lambda_\nu\}_{0<\nu<1}$ is tight.*
(iii) *The restriction of λ_ν to $\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$ is $\mathbb{P}_{A_\nu}$.*

Remark B.10. Part (iii) of Lemma B.9 can be summed up in the assertion that the diagram below commutes:

$$\begin{array}{ccc} \mathbb{P}_{A_\nu} & \xrightarrow{\ \text{id}\ } & \lambda_\nu\big|_{\bigotimes_i \mathcal{B}_i} \\ \Big\downarrow{\scriptstyle |\mathcal{B}_f} & & {\scriptstyle |\bigotimes_i \mathcal{B}_i}\Big\uparrow \\ \eta_\nu & \xrightarrow{\ \text{ext}\ } & \lambda_\nu \end{array}$$

Here, ext denotes the extension to $\mathcal{B}_*(\mathfrak{X})$.

Proof. We divide the proof into three natural steps.

Claim (i). Given any Borel set $B \subset [-1, 1]$,

$$\begin{aligned} h_{i,n}^{-1}(B) &= \left\{h_{i,n} \in B\right\} = \left\{\pi_i \in f_{i,n}^{-1}(B)\right\} \\ &= Z_1 \times \cdots \times Z_{i-1} \times f_{i,n}^{-1}(B) \times Z_{i+1} \times \cdots \in \bigotimes_{i\in\mathbb{N}} \mathcal{B}_i. \end{aligned}$$

By construction, we infer $\mathcal{B}_f \subset \bigotimes_{i\in\mathbb{N}} \mathcal{B}_i \subset \mathcal{B}(\mathfrak{X})$. The inclusion $\mathcal{B}(\mathfrak{X}) \subset \mathcal{B}_*(\mathfrak{X})$ is justified in the proof of Lemma B.6. The final inclusion $\Sigma(\mathcal{K}) \subset \mathcal{B}_f$ is recorded in [13, Section 3]; it follows from the fact that the topology of Z and the topology induced by the separating sequence $\{h_{i,n}\}_{(i,n)\in\mathbb{N}^2}$ coincide on compact subsets.

Claim (ii). For each fixed i, by tightness of the laws $\{\mu_{i,\nu}\}_\nu$ of $\{A_{i,\nu}\}_\nu$, for each $\varepsilon > 0$, there exists a compact set $Z_{i,\varepsilon} \subset Z_i$ such that

$$\sup_\nu \mathbb{P}_{A_{i,\nu}}\left(Z_{i,\varepsilon}^c\right) < 2^{-i}\varepsilon.$$

Set $Z_\varepsilon = \prod_{i\in\mathbb{N}} Z_{i,\varepsilon}$, which is a compact subset of $\mathfrak{X}$ by the Tychonoff theorem. Moreover, Z_ε belongs to $\mathcal{B}_f$. By the inclusion

$$\left(\prod_{i\in\mathbb{N}} Z_{i,\varepsilon}\right)^c \subset \bigcup_i \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_{i-1} \times Z_{i,\varepsilon}^c \times \mathfrak{X}_{i+1} \times \cdots$$

and the sub-additivity of measures, we deduce that, uniformly in ν,

$$\begin{aligned}\mathbb{P}_{A_\nu}\left(Z_\varepsilon^c\right) &\le \sum_{i=1}^\infty \mathbb{P}_{A_\nu}\left(\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_{i-1} \times Z_{i,\varepsilon}^c \times \mathfrak{X}_{i+1} \times \cdots\right)\\ &= \sum_{i=1}^\infty \mathbb{P}_{A_{i,\nu}}\left(Z_{i,\varepsilon}^c\right) < \varepsilon,\end{aligned}$$

where $\mathbb{P}_{A_\nu}$ is the law of A_ν on $\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$. It follows that $\left\{\mathbb{P}_{A_\nu}\right\}_{0<\nu<1}$ is tight on $\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$ and, *a fortiori*, $\left\{\mathbb{P}_{A_\nu}\big|_{\mathcal{B}_f}\right\}_{0<\nu<1}$ is tight on $\mathcal{B}_f$.

Claim (iii). Set $\bar\lambda_\nu = \lambda_\nu\big|_{\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i}$ and let $F \in \bigotimes_{i\in\mathbb{N}} \mathcal{B}_i$ be arbitrary. By definition, for any compact $K \in \mathcal{K}$,

$$\bar\lambda_\nu(K) = \lambda_\nu(K) = \eta_\nu(K) = \mathbb{P}_{A_\nu}(K).$$

In particular, if $K \subset F$, then $\mathbb{P}_{A_\nu}(F) \ge \mathbb{P}_{A_\nu}(K)$ and $\mathbb{P}_{A_\nu}(F) \ge \bar\lambda_\nu(K) = \lambda_\nu(K)$. Therefore,

$$\mathbb{P}_{A_\nu}(F) \ge \sup_{K\in\mathcal{K}, K\subset F} \bar\lambda_\nu(K) = \sup_{K\in\mathcal{K}, K\subset F} \lambda_\nu(K).$$

Since λ_ν is Radon on $\mathcal{B}_*(\mathfrak{X})$,

$$\mathbb{P}_{A_\nu}(F) \ge \lambda_\nu(F) = \bar\lambda_\nu(F), \qquad F \in \bigotimes_{i\in\mathbb{N}} \mathcal{B}_i.$$

Using the arbitrariness of F by considering $\mathfrak{X}\backslash F$ in place of F, this majorisation implies $\mathbb{P}_{A_\nu} = \bar\lambda_\nu = \lambda_\nu\big|_{\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i}$. Nevertheless, observe that this does not imply that A_ν becomes either $\mathcal{F}/\mathcal{B}_*(\mathfrak{X})$ measurable or $\mathcal{F}/\mathcal{B}(\mathfrak{X})$ measurable. □

The next remark is important and used extensively throughout the paper.

Remark B.11. Even though A_ν is not $\mathcal{F}/\mathcal{B}(\mathfrak{X})$ measurable in general, we still have the following crucial fact: because $\left\{\mathbb{P}_{A_\nu}\right\}_{0<\nu<1}$ is tight, as soon as we assume that the original probability space $\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ is complete, it follows that

$$f \circ A_\nu \quad \text{is} \quad \mathcal{F}/\mathcal{B}(\mathbb{R}^k) \text{ measurable},$$

for any continuous function f from $\mathfrak{X}$ to $\mathbb{R}^k$, see [46, page 170] for further details.

B.3. The Skorokhod–Jakubowski theorem

We recall the following result due to Jakubowski [46, Theorem 2].

Theorem B.12 *(Jakubowski). Let Z be a quasi-Polish space. Consider a sequence $Y_j : (\Omega, \mathcal{F}, \mathbb{P}) \to (Z, \mathcal{B}_Z)$ of random mappings with a tight sequence of laws μ_j, $j \in \mathbb{N}$. Then there exist a subsequence $\{Y_{j_k}\}_{k\in\mathbb{N}}$ and Z-valued random variables $V_0, V_1, V_2, \dots$, defined on $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$, where Leb is the Lebesgue measure, such that*

$$Y_{j_k} \sim V_k, \quad k \in \mathbb{N}, \qquad V_k(\xi) \xrightarrow{k\uparrow\infty} V_0(\xi) \quad \text{for a.e. } \xi \in [0,1].$$

Recently, the Jakubowski theorem was used by many authors to prove existence of solutions to various classes of SPDEs, see Section 1 for a few references. Here we only recall the first works [52,12].

The following simple but useful lemma is deployed in the proof of Theorem 4.9.

Lemma B.13 *(A.s. representations of nonlinear compositions). Let Z, W be quasi-Polish spaces, and suppose $F : Z \to W$ is a Borel function. Consider a sequence $\{Y_j\}_{j=1}^{\infty}$ of Z-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(V_k, \widetilde{F}_k)$ the a.s. representations of $(Y_j, F(Y_j))$, see Theorem B.12. Then*

$$\widetilde{F}_k = F(V_{n_k}), \quad a.s., \quad k \in \mathbb{N}.$$

Proof. We divide the proof into two steps.

Step 1. Let $(Z, \mathcal{B}_Z)$ and $(W, \mathcal{B}_W)$ be quasi-Polish spaces with σ-algebras $\mathcal{B}_Z$ and $\mathcal{B}_W$. Consider a mapping $F : Z \to W$ that is $\mathcal{B}_Z/\mathcal{B}_W$ measurable. Define the mapping $H : Z \times W \to W \times W$ by

$$(z, w) \mapsto H(z, w) = \big(H_1(z, w), H_2(z, w)\big) = (F(z), w).$$

Then H is $\mathcal{B}_Z \otimes \mathcal{B}_W/\mathcal{B}_W \otimes \mathcal{B}_W$ measurable. The validity of this claim comes from the $\mathcal{B}_Z \otimes \mathcal{B}_W/\mathcal{B}_W$ measurability of the coordinate mappings $(z, w) \overset{H_1}{\mapsto} F(z)$ and $(z, w) \overset{H_2}{\mapsto} w$, see, e.g., [47, Lemma 1.9].

Step 2. Consider three random mappings $U : (\Omega, \mathcal{F}, \mathbb{P}) \to (Z, \mathcal{B}_Z)$, $U' : (\Omega', \mathcal{F}', \mathbb{P}') \to (Z, \mathcal{B}_Z)$, and $V : (\Omega', \mathcal{F}', \mathbb{P}') \to (W, \mathcal{B}_W)$. Suppose $F : Z \to W$ is a (deterministic) mapping that is $\mathcal{B}_Z/\mathcal{B}_W$ measurable. If $\big(U, F(U)\big) \sim (U', V)$, then $V = F(U')$, a.s. It remains to prove this assertion, which implies the claim of the lemma.

By Step 1 and the measurability of compositions of measurable mappings, we conclude that $H(U, F(U))$ is $\mathcal{F}/\mathcal{B}_W \otimes \mathcal{B}_W$ and $H(U', V)$ is $\mathcal{F}'/\mathcal{B}_W \otimes \mathcal{B}_W$. Moreover, we have $H(U, F(U)) \sim H(U', V)$. Since the diagonal $\Delta_{W\times W}$ belongs to $\mathcal{B}_W \otimes \mathcal{B}_W$, cf. Lemma B.5, we obtain

$$\mathbb{P}\big(\{\omega : H(U, F(U)) \in \Delta_{W\times W}\}\big) = \mathbb{P}'\big(\{\omega' : H(U', V) \in \Delta_{W\times W}\}\big).$$

Trivially, $H(U, F(U)) \equiv \big(F(U), F(U)\big) \in \Delta_{W\times W}$, and whence

$$1 = \mathbb{P}'\big(\{\omega' : H(U', V) \in \Delta_{W\times W}\}\big) = \mathbb{P}'\big(\{\omega' : \big(F(U'), V\big) \in \Delta_{W\times W}\}\big).$$

This shows that $V = F(U')$, thereby ending the proof of the lemma. □

Remark B.14. Theorem B.12 applies to tight sequences of probability measures, where tightness implies that the global behaviour of the measures "concentrates" on a compact set. Since we are not generally working in a metric space setting, to prove that a subset K is compact, one would a priori be required to use nets rather than sequences. However, an essential property of quasi-Polish spaces is that one can restrict considerations to sequences; as a matter of fact, a subset K of a quasi-Polish space is compact if and only if it is sequentially compact [46].

The coincidence of compactness and sequential compactness is not necessarily inherited by the relativised notions of relative compactness and relative sequential compactness. Using sequences is advantageous when assessing the precompactness of subsets, as per the application of the Skorokhod–Jakubowski theorem. In these situations, we want to know that the closure of relatively sequentially compact subsets is at least sequentially compact. Let us delve into supplementary structures that quasi-Polish spaces must possess, in order to ensure the coincidence of relative compactness and relative sequential compactness.

We recall first that a subset A of a Hausdorff space $(\mathfrak{X}, \tau)$ is

a. *relatively compact* if the closure of A is compact in $\mathfrak{X}$;
b. *relatively countably compact* if each sequence in A has a cluster point in $\mathfrak{X}$;
c. *relatively sequentially compact* if each sequence in A has a convergent sub-sequence with limit in $\mathfrak{X}$.

The requisite additional structure is the following:

Definition B.15 *([33, page 30]).* A topological Hausdorff space $(\mathfrak{X}, \tau)$ is *angelic* if for every relatively countably compact set $A \subset \mathfrak{X}$ the following holds:

(i) A is relatively compact;
(ii) For each $x \in \overline{A}$ there is a sequence in A which converges to x.

Angelic spaces have several remarkable properties:

Lemma B.16 *([33, Lemma 3.1, Theorem 3.3]). In any angelic space,*

(i) *compact, countably compact and sequentially compact subsets coincide;*
(ii) *relatively compact, relatively countably compact and relatively sequentially compact subsets coincide.*

Which spaces are angelic? A theorem of Eberlein and Šmulian provides us with necessary conditions:

Lemma B.17 *([33, Theorem 3.10]). Let $(\mathfrak{X}, \tau)$ be a locally convex metrizable space. Let τ_w denote its weak topology. Then $(\mathfrak{X}, \tau_w)$ is angelic. Moreover, if τ_r is a regular topology finer than τ_w, then $(\mathfrak{X}, \tau_r)$ is angelic.*

The regularity of a topology plays a pivotal role in affirming the angelic nature of a space. For relatively sequentially compact subsets A of regular spaces, the closure $\overline{A}$ maintains sequential compactness.

As we conclude this section, we will give three pertinent examples.

- If $\mathfrak{X}$ is a normed space, then $(\mathfrak{X}, \tau_w)$ is angelic.
- $\mathfrak{X} = L^{p_1}\big([0,T]; L^{p_2}(\mathbb{S}^1) - w\big)$, with $1 < p_1, p_2 < \infty$: Recall the topologies τ_s and (B.2), denoted by τ_N, on the Bochner space $Z = L^{p_1}([0,T]; L^{p_2}(\mathbb{S}^1))$. Clearly, (Z, τ_s) is normed; therefore, when endowed with τ_w, it is angelic. Moreover, τ_N is (completely) regular, because it originates from a family of semi-norms, i.e., it is locally convex. Since $\tau_w \subset \tau_N$, (Z, τ_N) is angelic.
- $\mathfrak{X} = C([0,T]; Z - w)$ for a separable Hilbert space Z, see Example (2) of Section B.1: Let $(\varphi_n)_n \subset Z$ be such that $\|h\|_Z = \sup_n |\langle \varphi_n, h\rangle_Z|$ for all $h \in Z$. Define the following semi-norms on $C([0,T]; Z - w)$:

$$\|f\|_{\varphi_n} := \sup_{0\le t\le T} \left|\langle \varphi_n, f(t)\rangle_Z\right|, \quad f \in C([0,T]; Z-w), \quad n \in \mathbb{N}.$$

  The locally convex topology τ_0 generated by the seminorms $\|\cdot\|_{\varphi_n}$, $n \in \mathbb{N}$, is (completely) regular. The topology τ_0 is metrizable [13, Remark 4.2], and thus $\big(C([0,T]; Z-w), \tau_0\big)$ is a locally convex metrizable space. Denote by τ_w its weak topology. Then $\big(C([0,T]; Z - w, \tau_w\big)$ and $\big(C([0,T]; Z-w), \tau_0\big)$ are angelic, because trivially $\tau_0 \supset \tau_w$. Furthermore, denote by τ the regular locally convex topology generated by the semi-norms $\|\cdot\|_\phi$ in Example (2). Since $\tau \supset \tau_0$, the quasi-Polish space $\big(C([0,T]; Z-w), \tau\big)$ is angelic.

Appendix C. Regularisation errors

In Section 7, we derived the SPDE satisfied by $S(\tilde q)$, where $\tilde q$ solves the second-order transport-type SPDE (7.43) and $S : \mathbb{R} \to \mathbb{R}$ is a nonlinear function. This renormalisation step involved regularising the process $\tilde q$ by a spatial mollifier J_δ, which generates several error terms. Below we reproduce some convergence results—but not their proofs—for controlling these error terms. Analysing one of the (noise-related) terms requires a non-standard commutator estimate that goes beyond the DiPerna–Lions folklore lemma (see Proposition C.2 below). Similar estimates have been used recently in [54] and [38].

Lemma C.1 *(First order commutator errors). Consider*

$$w \in L^4\big(\Omega; L^\infty([0,T]; H^1(\mathbb{S}^1))\big),$$

and suppose $\sigma \in W^{1,\infty}(\mathbb{S}^1)$. Let J_δ be a standard Friedrichs mollifier in x, and set $w_\delta = w * J_\delta$. Define the error processes

$$\begin{aligned}
E_\delta^{(1)} &= E_\delta^{(1)}(w,v) = \big(w\,\partial_x w\big) * J_\delta - w_\delta\,\partial_x w_\delta,\\
E_\delta^{(2)} &= E_\delta^{(2)}(w) = \big(\sigma\,\partial_x w\big) * J_\delta - \sigma\,\partial_x w_\delta,\\
E_\delta^{(3)} &= E_\delta^{(3)}(w) = -\frac{1}{2}\big(\sigma\,\partial_x\left(\sigma\,\partial_x w\right)\big) * J_\delta + \frac{1}{2}\sigma\,\partial_x\left(\sigma\,\partial_x w_\delta\right).
\end{aligned}$$

The following convergences hold:

$$\begin{aligned}&\mathbb{E}\left\|\partial_x E_\delta^{(1)}\right\|_{L^1([0,T]\times\mathbb{S}^1)} \xrightarrow{\delta\downarrow 0} 0, \quad \mathbb{E}\left\|E_\delta^{(2)}\right\|^2_{L^2([0,T];H^1(\mathbb{S}^1))} \xrightarrow{\delta\downarrow 0} 0,\\ &\mathbb{E}\left\|E_\delta^{(3)}\right\|^2_{L^2([0,T]\times\mathbb{S}^1)} \xrightarrow{\delta\downarrow 0} 0.\end{aligned} \tag{C.1}$$

The first and second parts of (C.1) come from [48, Lemma 2.3] and [28, Lemma II.1], respectively. For the final part, see [40, Lemma 7.1].

To handle a regularisation error linked to the stochastic part of (7.43), we need the following proposition, which is a consequence of a second order commutator estimate found in [40, Lemma 7.3]:

Proposition C.2 *(Itô–Stratonovich related error term [40, Proposition 7.4]). Let $S \in \dot{W}^{2,\infty}(\mathbb{R})$ be such that $S'(r) = O(r)$ and $\sup_{r\in\mathbb{R}} |S''(r)| < \infty$. Let w, w_δ, $E_\delta^{(2)}$, and $E_\delta^{(3)}$ be defined as in Lemma C.1. For each $\varphi \in C^\infty([0,T]\times\mathbb{S}^1)$, the following convergence holds:*

$$\begin{aligned}\mathbb{E}\int_0^T\Bigg|\int_{\mathbb{S}^1} &-\varphi S'(\partial_x w_\delta)\partial_x E_\delta^{(3)}\\ &+\varphi S''(\partial_x w_\delta)\left(\frac{1}{2}\left|\partial_x E_\delta^{(2)}\right|^2 + \partial_x\left(\sigma\,\partial_x w_\delta\right)\partial_x E_\delta^{(2)}\right)\mathrm{d}x\Bigg|\,\mathrm{d}t \xrightarrow{\delta\downarrow 0} 0.\end{aligned}$$

Appendix D. Temporal continuity in H^1 for viscous equation

We consider the viscous equation (1.5). Since $\varepsilon > 0$ is fixed in this section, we suppress the ε-subscript. In [40, Proposition 7.8], the authors demonstrated that

$$\lim_{t\to t_0}\mathbb{E}\,\|u(t)-u(t_0)\|^2_{H^1(\mathbb{S}^1)} = 0,$$

for every t_0. Furthermore, they posited that u belongs to the space $L^{p_0}_\omega C_t H^1_x$, with $p_0 > 4$ defined in (2.1), implying that u is almost surely continuous on the interval $[0,T]$ with values in $H^1(\mathbb{S}^1)$. However, they did not provide a comprehensive proof to support this latter assertion. The aim of this appendix is to establish this temporal continuity assertion, which is utilised in this paper.

Lemma D.1 *(Temporal continuity in H^1 for viscous equation). Consider a solution u to (1.5) with initial condition u_0, as guaranteed by Theorem 2.3. Specifically, u satisfies $u \in L^{p_0}\big(\Omega; L^\infty([0,T];H^1(\mathbb{S}^1))\big)\cap L^2\big(\Omega; L^2([0,T];H^2(\mathbb{S}^1))\big)$ and, $\mathbb{P}$-a.s., $u \in C([0,T];H^1(\mathbb{S}^1)-w)$. Then we have the inclusion $u \in L^{\bar{p}}\big(\Omega; C([0,T];H^1(\mathbb{S}^1))\big)$, for any $\bar{p} < p_0$.*

Proof. Let $\{J_\delta\}_{\delta>0}$ be a spatial Friedrichs mollifier. We continue to employ the notation $f_\delta = f * J_\delta$. Using the a.s. inclusion $u \in C_t(H^1_x - w)$, we find that the quantities u_δ and $q_\delta = \partial_x u_\delta$ exhibit time-continuity, pointwise in x. It then follows quite straightforwardly that u_δ belongs to the space $C_t H^1_x$ a.s., for each fixed $\delta > 0$.

For given $\delta, \eta > 0$, applying Itô's formula to the mollified version of the SPDE (1.5) and its x-derivative, we obtain

$$\frac{1}{2}\left\|u_\delta - u_\eta\right\|_{H^1_x}^2(t) = \frac{1}{2}\left\|u_{0,\delta} - u_{0,\eta}\right\|_{H^1_x}^2 + \int_0^t \sum_{i=1}^8 I_i \, \mathrm{d}s + \int_0^t (M_1 + M_2)\, \mathrm{d}W, \tag{D.1}$$

where

$$I_1 = \varepsilon \int_{\mathbb{S}^1} \left(u_\delta - u_\eta\right) \partial_{xx}^2 \left(u_\delta - u_\eta\right) \mathrm{d}x,$$

$$I_2 = -\int_{\mathbb{S}^1} \left(u_\delta - u_\eta\right) \left((u\,\partial_x u + \partial_x P)_\delta - (u\,\partial_x u + \partial_x P)_\eta\right) \mathrm{d}x,$$

$$I_3 = \frac{1}{2}\int_{\mathbb{S}^1} \left(u_\delta - u_\eta\right) \left((\sigma\,\partial_x(\sigma\,\partial_x u))_\delta - (\sigma\,\partial_x(\sigma\,\partial_x u))_\eta\right) \mathrm{d}x,$$

$$I_4 = \frac{1}{2}\int_{\mathbb{S}^1} \left|(\sigma\partial_x u)_\delta - (\sigma\partial_x u)_\eta\right|^2 \mathrm{d}x,$$

$$I_5 = \varepsilon \int_{\mathbb{S}^1} \partial_x \left(u_\delta - u_\eta\right) \partial_{xxx}^3 \left(u_\delta - u_\eta\right) \mathrm{d}x,$$

$$I_6 = -\int_{\mathbb{S}^1} \partial_x \left(u_\delta - u_\eta\right) \partial_x \left((u\,\partial_x u + \partial_x P)_\delta - (u\,\partial_x u + \partial_x P)_\eta\right) \mathrm{d}x,$$

$$I_7 = \frac{1}{2}\int_{\mathbb{S}^1} \partial_x \left(u_\delta - u_\eta\right) \partial_x \left((\sigma\,\partial_x(\sigma\,\partial_x u))_\delta - (\sigma\,\partial_x(\sigma\,\partial_x u))_\eta\right) \mathrm{d}x,$$

$$I_8 = \frac{1}{2}\int_{\mathbb{S}^1} \left|\partial_x(\sigma\partial_x u)_\delta - \partial_x(\sigma\partial_x u)_\eta\right|^2 \mathrm{d}x,$$

$$M_1 = \int_{\mathbb{S}^1} \left(u_\delta - u_\eta\right) \left((\sigma\,\partial_x u)_\delta - (\sigma\,\partial_x u)_\eta\right) \mathrm{d}x,$$

$$M_2 = \int_{\mathbb{S}^1} \partial_x \left(u_\delta - u_\eta\right) \partial_x \left((\sigma\,\partial_x u)_\delta - (\sigma\,\partial_x u)_\eta\right) \mathrm{d}x.$$

Applying integration-by-parts, we can ascertain that $I_1, I_5 \le 0$, a.s.

We shall use repeatedly the fact that $u \in L^2_{\omega,t} H^2_x$, and thus, as $\delta \to 0$, $\left(u_\delta, \partial_x u_\delta, \partial_{xx}^2 u_\delta\right) \to \left(u, \partial_x u, \partial_{xx}^2 u\right)$ in $L^2_{\omega,t,x}$. In addition, using that $\sigma \in W^{2,\infty}$, $\left(u_\delta, (\sigma\,\partial_x u)_\delta, \partial_x(\sigma\,\partial_x u)_\delta\right) \to$

$\big(u, \sigma\, \partial_x u, \partial_x (\sigma\, \partial_x u)\big)$ in $L^2_{\omega,t,x}$. The convergences just mentioned immediately imply that $\mathbb{E}\int_0^T I_4 + I_8 \,\mathrm{d}t \xrightarrow{\delta,\eta\downarrow 0} 0$.

For I_3, we use the Cauchy–Schwarz inequality to get

$$\int_0^T |I_3| \,\mathrm{d}t \lesssim_\sigma \left\| u_\delta - u_\eta \right\|_{L^2_{t,x}} \left\| f_\delta - f_\eta \right\|_{L^2_{t,x}}, \tag{D.2}$$

where $f = \sigma\, \partial_x (\sigma\, \partial_x u) \in L^2_{\omega,t,x}$. On the right-hand side, we take an expectation. Subsequently, we utilize the Cauchy–Schwarz inequality. Each of the factors of the term on the right-hand side tend to 0 in $L^2(\Omega)$ as $\delta, \eta \downarrow 0$. After an integration by parts, the integral $\int_0^T |I_7| \,\mathrm{d}t$ can be bounded in a similar manner as stated in (D.2), but with the first factor on the right-hand side replaced by $\left\| u_\delta - u_\eta \right\|_{L^2_t H^2_x}$. Consequently, the integral can be treated using the same approach.

For the integrals I_2 and I_6, we again apply the Cauchy–Schwarz inequality and use the fact that $u \in L^2_\omega L^2_t H^2_x \cap L^2_\omega L^\infty_t H^1_x$. To be more precise, adding I_2 and I_6, and using the property that $K * f = f - \partial^2_{xx} K * f$, we find

$$\begin{aligned}
|I_2 + I_6| = \Bigg| &\int_{\mathbb{S}^1} \big(u_\delta - u_\eta\big) \big((u\,\partial_x u)_\delta - (u\,\partial_x u)_\eta\big) \,\mathrm{d}x \\
&+ \int_{\mathbb{S}^1} \partial_x \big(u_\delta - u_\eta\big)\, \partial_x \big((u\,\partial_x u)_\delta - (u\,\partial_x u)_\eta\big) \,\mathrm{d}x \\
&+ \int_{\mathbb{S}^1} \partial_x \big(u_\delta - u_\eta\big) \left(\left(u^2\right)_\delta - \left(u^2\right)_\eta + \frac{1}{2}\left(q^2\right)_\delta - \frac{1}{2}\left(q^2\right)_\eta\right) \mathrm{d}x \Bigg| \\
\le{}& \left\| u_\delta - u_\eta \right\|_{H^2_x} \left\| (u\,\partial_x u)_\delta - (u\,\partial_x u)_\eta \right\|_{L^2_x} \\
&+ \left\| \partial_x \big(u_\delta - u_\eta\big) \right\|_{L^\infty_x} \left\| \left(u^2\right)_\delta - \left(u^2\right)_\eta + \frac{1}{2}\left(q^2\right)_\delta - \frac{1}{2}\left(q^2\right)_\eta \right\|_{L^1_x},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E}\int_0^T |I_2 + I_6| \,\mathrm{d}t \\
&\le \mathbb{E}\left(\left\| u_\delta - u_\eta \right\|_{L^2_t H^2_x} \left\| (u\,\partial_x u)_\delta - (u\,\partial_x u)_\eta \right\|_{L^\infty_t L^2_x} \right) \\
&\quad + \mathbb{E}\Bigg(\left\| \partial_x \big(u_\delta - u_\eta\big) \right\|_{L^2_t L^\infty_x} \\
&\qquad\qquad \times \left\| \left(u^2\right)_\delta - \left(u^2\right)_\eta + \frac{1}{2}\left(q^2\right)_\delta - \frac{1}{2}\left(q^2\right)_\eta \right\|_{L^2_t L^1_x} \Bigg) \\
&\le \left(\mathbb{E} \left\| u_\delta - u_\eta \right\|^2_{L^2_t H^2_x} \right)^{1/2} \left(\mathbb{E} \left\| (u\,\partial_x u)_\delta - (u\,\partial_x u)_\eta \right\|^2_{L^\infty_t L^2_x} \right)^{1/2}
\end{aligned}$$

$$+\left(\mathbb{E}\left\|\partial_x\left(u_\delta-u_\eta\right)\right\|_{L_t^2L_x^\infty}^2\right)^{1/2}$$
$$\times\left(\mathbb{E}\left\|\left(u^2\right)_\delta-\left(u^2\right)_\eta+\frac{1}{2}\left(q^2\right)_\delta-\frac{1}{2}\left(q^2\right)_\eta\right\|_{L_t^2L_x^1}^2\right)^{1/2}.$$

By the Lebesgue dominated convergence theorem, the inclusions $u\in L^2_{\omega,t}H^2_x$ and $u^2,q^2\in L^2_\omega L^\infty_t L^1_x\subset L^2_{\omega,t}L^1_x$ imply that one factor in each summand above tends to zero as $\eta,\delta\downarrow 0$. At the same time, the remaining factor in each summand is bounded because of the inclusions $u\in L^{p_0}_\omega L^\infty_{t,x}$ (with $p_0>4$), $\partial_x u\in L^{p_0}_\omega L^\infty_t L^2_x$ (so that $u\,\partial_x u\in L^2_\omega L^\infty_t L^2_x$), and $u\in L^2_{\omega,t}H^2_x\subset L^2_{\omega,t}W^{1,\infty}_x$.

Next, by the BDG inequality,

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t M_1\,\mathrm{d}W\right| &\le \mathbb{E}\left(\int_0^T |M_1|^2\,\mathrm{d}s\right)^{1/2}\\ &\lesssim_\sigma \mathbb{E}\left(\left\|u_\delta-u_\eta\right\|_{L_t^\infty L_x^2}\left\|f_\delta-f_\eta\right\|_{L_t^2H_x^1)}\right)\\ &\le\left(\mathbb{E}\left\|u_\delta-u_\eta\right\|_{L_t^\infty L_x^2}^2\right)^{1/2}\left(\mathbb{E}\left\|f_\delta-f_\eta\right\|_{L_t^2L_x^2}^2\right)^{1/2},\end{aligned}$$

where $f=\sigma\,\partial_x u\in L^2_{\omega,t}H^1_x$. Similarly, with $g=\partial_x(\sigma\,\partial_x u)\in L^2_{\omega,t}L^2_x$,

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t M_2\,\mathrm{d}W\right|\lesssim_\sigma\left(\mathbb{E}\left\|u_\delta-u_\eta\right\|_{L_t^\infty H_x^1}^2\right)^{1/2}\left(\mathbb{E}\left\|g_\delta-g_\eta\right\|_{L_t^2L_x^2}^2\right)^{1/2}.$$

One factor of each term on the right in the two inequalities above tend to nought whilst the other remains bounded.

Consolidating our findings, we execute an integration of (D.1) over the interval $s\in[0,t]$. This is succeeded by taking the supremum over $t\in[0,T]$ and subsequently computing the expectation. With these steps, we arrive at

$$\begin{aligned}\mathbb{E}\left\|u_\delta-u_\eta\right\|_{C_tH_x^1}^2 &\le \mathbb{E}\left\|u_{0,\delta}-u_{0,\delta}\right\|_{H_x^1}^2+\mathbb{E}\int_0^T\left|\sum_{i\notin\{1,5\}}I_i\right|\mathrm{d}t\\ &\quad+\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\left(M_1+M_2\right)\mathrm{d}W\right|\xrightarrow{\delta,\eta\downarrow 0}0.\end{aligned}$$

This implies that $\{u_\delta\}$ is a Cauchy sequence in $L^2_\omega C_t H^1_x$. The limit U of $\{u_\delta\}$ in $L^2_\omega C_t H^1_x$ and u must coincide (ω,t,x)-a.e. Indeed, since $U\in L^2_\omega C_t H^1_x$, it is evident that $U\in L^2_\omega L^2_t H^1_x$. Furthermore, due to the fact that $p_0>2$, we can conclude that u also belongs to $L^2_\omega L^2_t H^1_x$. Therefore, we can establish that $u=U$ for a.e. (ω,t,x), indicating that they belong to the same equivalence class. □

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