

FREE ALGEBRAS THROUGH DAY CONVOLUTION

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ABSTRACT. Building on the foundations in our previous paper, we study Segal conditions that are given by finite products, determined by structures we call cartesian patterns. We set up Day convolution on presheaves in this setting and use it to give conditions under which there is a colimit formula for free algebras and other left adjoints. This specializes to give a simple proof of Lurie’s results on operadic left Kan extensions and free algebras for symmetric ∞ -operads.

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1. INTRODUCTION

A key feature of symmetric ∞ -operads, as defined in Lurie’s book [Lur17], is that there is an explicit formula for their free algebras. More generally, for any morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ of ∞ -operads there is a formula for the corresponding left operadic Kan extension, i.e. the left adjoint to the functor $f^*: \text{Alg}_{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{C})$ between ∞ -categories of algebras given by composition with f . However, the construction of these left adjoints in [Lur17] is by a cumbersome simplex-by-simplex induction using a delicate analysis of the inert–active factorization system on finite pointed sets. Part of our goal in this paper is to give a new, simpler construction of these left adjoints in the following three steps:

- (1) We first consider algebras in the ∞ -category of spaces (with the cartesian monoidal structure). Here it is easy to see that the left adjoint is just given by an ordinary left Kan extension.
- (2) Next we consider algebras in presheaves on small symmetric monoidal ∞ -categories. Here the universal property of the Day convolution structure allows us to reduce to the previous case.
- (3) Finally, we use that any presentably symmetric monoidal ∞ -category is a symmetric monoidal localization of a presheaf ∞ -category with Day convolution. Since the localization functor is symmetric monoidal and colimit-preserving, we can use it to transport the colimit formula for the left adjoint from the previous step.

Symmetric ∞ -operads in Lurie’s sense are certain ∞ -categories over the category \mathbb{F}_* of pointed finite sets; these are, in a sense, the universal objects that have algebras in symmetric monoidal

∞ -categories. In practice, however, it can be useful to describe algebraic structures by more general ∞ -categories over \mathbb{F}_* . For example, we can sometimes find a combinatorially simpler description by using an ∞ -category that is not an ∞ -operad; as a somewhat trivial example, associative algebras can be described in terms of the simplex category Δ^{op} , which is less complicated than the symmetric associative operad. In other cases, even though it is formally known that a certain structure is described by a symmetric ∞ -operad, this object may be difficult to describe explicitly; for instance, the structure of n compatible associative algebra structures can trivially be described in terms of the product $\Delta^{n, \text{op}}$, while describing the associated ∞ -operad \mathbb{E}_n amounts to proving the Dunn–Lurie additivity theorem.

It is therefore desirable to understand when free algebras and other left adjoints can be described by an explicit formula without passing to the associated symmetric ∞ -operads. Our main goal in this paper is to obtain a simple criterion for this by following the same three steps we outlined above.

We start in §2 by making precise the class of ∞ -categories over \mathbb{F}_* we are interested in, which we call *cartesian patterns*. This builds on the foundations in our previous paper [CH21], where we studied Segal conditions in general; here we are interested in those Segal-type limit conditions that are given by finite products. We then give some examples of cartesian patterns in §3 and introduce monoidal ∞ -categories over a cartesian pattern and algebras therein in §4.

The bulk of the paper is then taken up by extending to the setting of general cartesian patterns the results we need to carry out our proof strategy:

- In §5 we show that if \mathcal{O} is a cartesian pattern and \mathcal{C} is an ∞ -category with finite products, then \mathcal{O} -monoids in \mathcal{C} are equivalent to \mathcal{O} -algebras in an \mathcal{O} -monoidal structure on \mathcal{C} given by cartesian products.
- In §6 we introduce Day convolution on presheaves for monoidal ∞ -categories over a cartesian pattern.
- In §7 we study monoidal localizations over a cartesian pattern and prove that any presentably monoidal ∞ -category is a monoidal localization of a Day convolution structure.

These foundations allow us to prove our main result in §8:

Theorem 1.1.

- (i) Suppose $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns that is extendable in the sense of Definition 8.3 and \mathcal{V} is a presentably \mathcal{P} -monoidal ∞ -category. Then the functor

$$f^*: \text{Alg}_{\mathcal{P}}(\mathcal{V}) \rightarrow \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V})$$

between ∞ -categories of algebras given by restriction along f has a left adjoint $f_!$, which for $P \in \mathcal{P}^{\text{el}}$ satisfies

$$(f_!A)(P) \simeq \text{colim}_{(O, \phi: f(O) \rightsquigarrow P) \in \mathcal{O}_{\mathcal{P}}^{\text{act}}} \phi_!A(O).$$

- (ii) Suppose \mathcal{O} is a cartesian pattern that is extendable in the sense of Definition 8.8 and \mathcal{V} is a presentably \mathcal{O} -monoidal ∞ -category. Then the functor

$$U_{\mathcal{O}}: \text{Alg}_{\mathcal{O}}(\mathcal{V}) \rightarrow \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \mathcal{V})$$

given by restriction to the subcategory \mathcal{O}^{el} of elementary objects has a left adjoint $F_{\mathcal{O}}$, which for $E \in \mathcal{O}^{\text{el}}$ satisfies

$$U_{\mathcal{O}}F_{\mathcal{O}}\Phi(E) \simeq \text{colim}_{\phi: O \rightsquigarrow E \in \text{Act}_{\mathcal{O}}(E)} \phi_!(\Phi(O_1), \dots, \Phi(O_n)).$$

Moreover, the adjunction $F_{\mathcal{O}} \dashv U_{\mathcal{O}}$ is monadic.

This theorem is a combination of Corollaries 8.13 and 8.14; we refer the reader to §§2 and 4 for the notation used. We discuss some examples of extendable cartesian patterns and morphisms to which the theorem applies in §9. The explicit formula for free algebras leads to a simple criterion for a morphism of cartesian patterns to induce equivalences on ∞ -categories of algebras, which we consider in §10 together with some easy applications.

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2. CARTESIAN PATTERNS AND MONOIDS

In this section we review some basic definitions from [CH21] and introduce the algebraic structures we will study in this paper, namely cartesian patterns and algebras and monoids over them.

Definition 2.1. An *algebraic pattern* consists of an ∞ -category \mathcal{O} equipped with a factorization system, whereby each map factors essentially uniquely as an *inert* map followed by an *active* map, together with a collection of *elementary* objects. We write \mathcal{O}^{act} and \mathcal{O}^{int} for the subcategories of \mathcal{O} containing only the active and inert morphisms, respectively, and $\mathcal{O}^{\text{el}} \subseteq \mathcal{O}^{\text{int}}$ for the full subcategory of elementary objects and inert morphisms among them. A morphism of algebraic patterns from \mathcal{O} to \mathcal{P} is a functor $f: \mathcal{O} \rightarrow \mathcal{P}$ that preserves inert and active morphisms and elementary objects.

Notation 2.2. If \mathcal{O} is an algebraic pattern, we will indicate an inert map between objects O, O' of \mathcal{O} as $O \rightsquigarrow O'$ and an active map as $O \rightsquigarrow O'$. These symbols are not meant to suggest any intuition about the nature of inert and active maps.

Example 2.3. Let \mathbb{F}_* denote a skeleton of the category of finite pointed sets with objects $\langle n \rangle := (\{0, 1, \dots, n\}, 0)$. This has an inert–active factorization system where a morphism $\phi: \langle n \rangle \rightarrow \langle m \rangle$ is

- *inert* if it is an isomorphism away from the base point, i.e. $|\phi^{-1}(i)| = 1$ if $i \neq 0$,
- *active* if it doesn't send anything except the base point to the base point, i.e. $\phi^{-1}(0) = \{0\}$.

We write \mathbb{F}_*^{b} for the algebraic pattern given by this factorization system with $\langle 1 \rangle$ as the only elementary object.

Notation 2.4. If \mathcal{O} is an algebraic pattern and O is an object of \mathcal{O} , we write

$$\mathcal{O}_{O/}^{\text{el}} := \mathcal{O}^{\text{el}} \times_{\mathcal{O}^{\text{int}}} \mathcal{O}_{O/}^{\text{int}}$$

for the ∞ -category of inert maps from O to elementary objects, and inert maps between them.

Notation 2.5. We write $\rho_i: \langle n \rangle \rightarrow \langle 1 \rangle$, $i = 1, \dots, n$, for the inert map given by

$$\rho_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i. \end{cases}$$

Then $(\mathbb{F}_*^{\text{b}})_{\langle n \rangle}^{\text{el}}$ is equivalent to the discrete set $\{\rho_1, \dots, \rho_n\}$.

Definition 2.6. A *cartesian pattern* is an algebraic pattern \mathcal{O} equipped with a morphism of algebraic patterns $|-|: \mathcal{O} \rightarrow \mathbb{F}_*^{\text{b}}$ such that for every object $O \in \mathcal{O}$ the induced map

$$\mathcal{O}_{O/}^{\text{el}} \rightarrow \mathbb{F}_{*,|O|/}^{\text{el}}$$

is an equivalence. A *morphism of cartesian patterns* is a morphism of algebraic patterns over \mathbb{F}_*^{b} .

Notation 2.7. If \mathcal{O} is a cartesian pattern and O is an object of \mathcal{O} such that $|O| \cong \langle n \rangle$, then the ∞ -category $\mathcal{O}_{O/}^{\text{el}}$ is equivalent to a discrete set consisting of n inert morphisms with source O , with an essentially unique such morphism lying over $\rho_i: \langle n \rangle \rightarrow \langle 1 \rangle$ for $i = 1, \dots, n$. We denote this inert morphism by $\rho_i^O: O \rightarrow O_i$.

Lemma 2.8. *If \mathcal{O} is a cartesian pattern, then the ∞ -category \mathcal{O}^{el} is an ∞ -groupoid.*

Proof. If $O \in \mathcal{O}$ lies over $\langle 1 \rangle$ in \mathbb{F}_* , then it follows from the definition that the ∞ -category $\mathcal{O}_{O/}^{\text{el}}$ is a contractible ∞ -groupoid. This holds in particular for $E \in \mathcal{O}^{\text{el}}$ (since E must map to the unique elementary object $\langle 1 \rangle$ in \mathbb{F}_*), so that the fibres of the cartesian fibration

$$\text{ev}_0: \text{Fun}(\Delta^1, \mathcal{O}^{\text{el}}) \rightarrow \mathcal{O}^{\text{el}}$$

are contractible ∞ -groupoids. This functor is therefore an equivalence, which means in particular that \mathcal{O}^{el} is local with respect to $\Delta^0 \rightarrow \Delta^1$; it is thus an ∞ -groupoid. \square

Definition 2.9. If \mathcal{O} is a cartesian pattern and \mathcal{C} is an ∞ -category with finite products, then a functor $F: \mathcal{O} \rightarrow \mathcal{C}$ is an \mathcal{O} -monoid if for $O \in \mathcal{O}$ lying over $\langle n \rangle$ the natural map

$$F(O) \rightarrow \prod_{i=1}^n F(O_i),$$

induced by the maps $\rho_i^O: O \rightarrow O_i$, is an equivalence. We write $\text{Mon}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathcal{O}, \mathcal{C})$ consisting of \mathcal{O} -monoids.

Remark 2.10. If \mathcal{O} is a general algebraic pattern, then in [CH21] we defined a *Segal \mathcal{O} -object* to be a functor $F: \mathcal{O} \rightarrow \mathcal{C}$ such that $F|_{\mathcal{O}^{\text{int}}}$ is a right Kan extension of $F|_{\mathcal{O}^{\text{el}}}$, or equivalently if for every object $O \in \mathcal{O}$ the canonical map

$$F(O) \rightarrow \lim_{E \in \mathcal{O}_{O/}^{\text{el}}} F(E)$$

is an equivalence. Thus an \mathcal{O} -monoid is just a special case of a Segal \mathcal{O} -object. We choose to use different terminology for this and a few other concepts to emphasize the special features of cartesian patterns and the parallels between our definitions and the special cases that are studied in [Lur17, Bar18].

Lemma 2.11. *Any morphism of cartesian patterns $f: \mathcal{O} \rightarrow \mathcal{P}$ gives by composition a functor*

$$f^*: \text{Mon}_{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Mon}_{\mathcal{O}}(\mathcal{C}).$$

Proof. Immediate, since for $O \in \mathcal{O}$ we have $|O| \cong |f(O)|$ and $f(\rho_i^O) \simeq \rho_i^{f(O)}$. \square

Remark 2.12. Let \mathcal{O} be a cartesian pattern and \mathcal{C} an ∞ -category with finite products. Then so is \mathcal{O}^{int} , and a functor $F: \mathcal{O} \rightarrow \mathcal{C}$ is an \mathcal{O} -monoid if and only if its restriction $F|_{\mathcal{O}^{\text{int}}}$ is an \mathcal{O}^{int} -monoid. Moreover, the ∞ -category $\text{Mon}_{\mathcal{O}^{\text{int}}}(\mathcal{C})$ is precisely the full subcategory of $\text{Fun}(\mathcal{O}^{\text{int}}, \mathcal{C})$ consisting of functors that are right Kan-extended from \mathcal{O}^{el} , so that right Kan extension gives an equivalence

$$\text{Mon}_{\mathcal{O}^{\text{int}}}(\mathcal{C}) \simeq \text{Fun}(\mathcal{O}^{\text{el}}, \mathcal{C}).$$

Remark 2.13. Let \mathcal{C} be an ∞ -category with sifted colimits and finite products where the cartesian product preserves sifted colimits in each variable. If \mathcal{O} is a cartesian pattern, then the full subcategory $\text{Mon}_{\mathcal{O}}(\mathcal{C})$ is closed under sifted colimits in $\text{Fun}(\mathcal{O}, \mathcal{C})$: given a sifted diagram $\phi: \mathcal{J} \rightarrow \text{Mon}_{\mathcal{O}}(\mathcal{C})$ its colimit in $\text{Fun}(\mathcal{O}, \mathcal{C})$, which is computed pointwise, satisfies

$$(\text{colim}_{\mathcal{J}} \phi)(O) \xrightarrow{\sim} \text{colim}_{\mathcal{J}} (\phi(O_1) \times \cdots \times \phi(O_n)) \simeq \left(\text{colim}_{\mathcal{J}} \phi(O_1) \right) \times \cdots \times \left(\text{colim}_{\mathcal{J}} \phi(O_n) \right).$$

It follows that for any morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ of cartesian patterns, the functor $f^*: \text{Mon}_{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Mon}_{\mathcal{O}}(\mathcal{C})$ preserves sifted colimits.

3. EXAMPLES OF CARTESIAN PATTERNS

In this section we mention some examples of cartesian patterns, and indicate where our definitions specialize to more familiar notions:

Example 3.1. For the base pattern \mathbb{F}_*^b , an \mathbb{F}_*^b -monoid is precisely a commutative monoid (in the sense considered in [Lur17], but going back to Segal’s work [Seg74] on special Γ -spaces).

Example 3.2. Let $\Delta^{\text{op},b}$ denote the simplex category with the usual inert–active factorization system (where the inert maps in Δ are the subinterval inclusions and the active maps are those that preserve the end points) and $[1]$ as the unique elementary object. This is a cartesian pattern using the map to \mathbb{F}_* given by $|[n]| = \langle n \rangle$, and with the map $|\phi|$ for $\phi: [n] \rightarrow [m]$ in Δ given by

$$|\phi|(i) = \begin{cases} j, & \phi(j-1) < i \leq \phi(j), \\ 0, & \text{otherwise.} \end{cases}$$

Then a $\Delta^{\text{op},b}$ -monoid is an associative monoid.

Example 3.3. The product $\Delta^{n,\text{op}}$ has a factorization system with inert and active maps defined componentwise. We get a cartesian pattern $\Delta^{n,\text{op},b}$ by taking $([1], \dots, [1])$ to be the unique elementary object and the map to \mathbb{F}_* to be given by the composite

$$\Delta^{n,\text{op}} \rightarrow \mathbb{F}_*^n \rightarrow \mathbb{F}_*$$

where the second map is the “smash product” of pointed finite sets. (This takes $(\langle m_1 \rangle, \dots, \langle m_n \rangle)$ to $\langle \prod_{i=1}^n m_i \rangle$; see [Lur17, Notation 2.2.5.1] for a precise description.) Then a $\Delta^{n,\text{op},b}$ -monoid can be described as an n -fold iterated associative monoid, or equivalently an \mathbb{E}_n -monoid by the Dunn–Lurie Additivity Theorem.

Example 3.4. If Φ is a perfect operator category in the sense of [Bar18] and $\Lambda(\Phi)$ is its Leinster category, then there is a cartesian pattern $\Lambda(\Phi)^b$ given by the inert–active factorization system of [Bar18], with the terminal object of Φ as the unique elementary object and the map to \mathbb{F}_* that induced by the unique operator morphism to \mathbb{F} . By choosing Φ to be the operator categories \mathbb{F} of finite sets, \mathbb{O} of finite ordered sets, and the cartesian product \mathbb{O}^n , this example specializes to the previous ones. Another example is the iterated *wreath product* $\mathbb{O}^{!n}$, whose Leinster category is Joyal’s category Θ_n^{op} of n -dimensional pasting diagrams. It is proved in [Bar18] that $\Theta_n^{\text{op},b}$ -monoids are equivalent to $\Delta^{n,\text{op},b}$ -monoids, and so are also equivalent to \mathbb{E}_n -monoids by the additivity theorem.

Example 3.5. Let \mathbb{Q} be the *dendroidal category* of [MW07], with the active–inert factorization system described in [Koc11, CHH18]. Then $\mathbb{Q}^{\text{op},b}$ denotes the cartesian pattern given by this factorization system, with the corollas as the elementary objects and the functor to \mathbb{F}_* as defined in [CH20], given by counting the number of corollas in a tree. An $\mathbb{Q}^{\text{op},b}$ -monoid in \mathcal{S} then describes a (pointed) one-object ∞ -operad.

Example 3.6. The category \mathbb{F} of acyclic connected finite graphs defined by Hackney, Robertson, and Yau in [HRY15] has an active–inert factorization system by [Koc16, 2.4.14] (where the active maps are called “refinements” and the inert maps are called “convex open inclusions”). We then write $\mathbb{F}^{\text{op},b}$ for the algebraic pattern given by this factorization system, with the graphs with exactly one vertex as the elementary objects. The functor $\mathbb{F}^{\text{op},b} \rightarrow \mathbb{F}_*^b$ given by counting the number of vertices in a graph exhibits $\mathbb{F}^{\text{op},b}$ as a cartesian pattern. According to [HR18], $\mathbb{F}^{\text{op},b}$ -monoids in \mathcal{S} model (pointed) one-object ∞ -properads.

Example 3.7. Any (symmetric) ∞ -operad $\mathcal{O} \rightarrow \mathbb{F}_*$ as in [Lur17] has an inert–active factorization system where the inert morphisms are the cocartesian morphisms lying over inert morphisms in \mathbb{F}_* and the active ones are those that lie over active morphisms in \mathbb{F}_* . If we take \mathcal{O}^b to be the algebraic pattern defined by this factorization system, with the elementary objects those that map to $\langle 1 \rangle$ (so \mathcal{O}^{el} is the underlying ∞ -groupoid $\mathcal{O}_{\langle 1 \rangle}^{\simeq}$ of the fibre at $\langle 1 \rangle$), then the given map to \mathbb{F}_* exhibits

\mathcal{O}^b as a cartesian pattern. (We will discuss a generalization of this class of cartesian patterns in Remark 4.16.)

Example 3.8. A *generalized ∞ -operad* $\mathcal{E} \rightarrow \mathbb{F}_*$ as defined in [Lur17, §2.3.2] also has an inert–active factorization system, defined in the same way as for ∞ -operads. If we again choose the elementary objects to be those that lie over $\langle 1 \rangle$, we get a cartesian pattern \mathcal{E}^b using the given functor to \mathbb{F}_* .

4. \mathcal{O} -MONOIDAL ∞ -CATEGORIES AND ALGEBRAS

In this section we define \mathcal{O} -monoidal ∞ -categories and \mathcal{O} -algebras in them.

Definition 4.1. Let \mathcal{O} be a cartesian pattern. An *\mathcal{O} -monoidal ∞ -category* is a cocartesian fibration $\mathcal{V}^\otimes \rightarrow \mathcal{O}$ whose associated functor $\mathcal{O} \rightarrow \text{Cat}_\infty$ is an \mathcal{O} -monoid. (We will often not mention the fibration explicitly and simply say that \mathcal{V}^\otimes is an \mathcal{O} -monoidal ∞ -category.)

Notation 4.2. Let $\mathcal{V}^\otimes \rightarrow \mathcal{O}$ be an \mathcal{O} -monoidal ∞ -category. If O is an object of \mathcal{O} lying over $\langle n \rangle$ in \mathbb{F}_* , we will often denote by $O(V_1, \dots, V_n)$ or just $O(V_i)$ the object of \mathcal{V}^\otimes_O that corresponds to (V_1, \dots, V_n) under the equivalence

$$\mathcal{V}^\otimes_O \simeq \prod_{i=1}^n \mathcal{V}_{O_i}$$

given by the cocartesian pushforwards over ρ_i^O .

Remark 4.3. If $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns, then it follows from Lemma 2.11 (reinterpreted through the straightening equivalence for functors to Cat_∞) that base change along f takes a \mathcal{P} -monoidal ∞ -category \mathcal{V}^\otimes to an \mathcal{O} -monoidal ∞ -category $f^*\mathcal{V}^\otimes$.

Remark 4.4. \mathcal{O} -monoidal ∞ -categories are a special case of Segal \mathcal{O} -fibrations in the terminology of [CH21].

Definition 4.5. Let \mathcal{V}^\otimes be an \mathcal{O} -monoidal ∞ -category. An *\mathcal{O} -algebra* in \mathcal{V}^\otimes is a section

$$A \begin{array}{c} \nearrow \\ \downarrow \\ \mathcal{O} \end{array} \mathcal{V}^\otimes$$

such that A takes inert morphisms in \mathcal{O} to cocartesian morphisms in \mathcal{V}^\otimes . We write $\text{Alg}_\mathcal{O}(\mathcal{V})$ for the full subcategory of $\text{Fun}_{/\mathcal{O}}(\mathcal{O}, \mathcal{V}^\otimes)$ spanned by the \mathcal{O} -algebras.

Example 4.6. Taking \mathcal{O} to be \mathbb{F}_*^b our definitions specialize to symmetric monoidal ∞ -categories and commutative algebras as defined in [Lur17], while if we take \mathcal{O} to be the cartesian pattern associated to an ∞ -operad, we get the notions of \mathcal{O} -monoidal ∞ -categories and \mathcal{O} -algebras of [Lur17]. Similarly, from $\Delta^{\text{op},b}$ we get monoidal ∞ -categories and associative algebras.

Definition 4.7. More generally, if $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns and \mathcal{V}^\otimes is a \mathcal{P} -monoidal ∞ -category, then an *\mathcal{O} -algebra* in \mathcal{V}^\otimes is a commutative triangle

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{A} & \mathcal{V}^\otimes \\ & \searrow f & \swarrow \\ & \mathcal{P} & \end{array}$$

such that A takes inert morphisms in \mathcal{O} to cocartesian morphisms in \mathcal{V}^\otimes . We write $\text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V})$ for the full subcategory of $\text{Fun}_{/\mathcal{P}}(\mathcal{O}, \mathcal{V}^\otimes)$ spanned by the \mathcal{O} -algebras; if \mathcal{P} is clear from the context, we will sometimes just write $\text{Alg}_\mathcal{O}(\mathcal{V})$. Base change along f induces a natural equivalence

$$\text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V}) \simeq \text{Alg}_\mathcal{O}(f^*\mathcal{V}).$$

We can view \mathcal{O} -algebras as a special case of morphisms of cartesian patterns, using the following canonical pattern structure on an \mathcal{O} -monoidal ∞ -category:

Definition 4.8. Let $\pi: \mathcal{V}^\otimes \rightarrow \mathcal{O}$ be an \mathcal{O} -monoidal ∞ -category. We say a morphism in \mathcal{V}^\otimes is *active* if it lies over an active morphism in \mathcal{O} , and *inert* if it is cocartesian and lies over an inert morphism in \mathcal{O} . The inert and active morphisms then form a factorization system on \mathcal{V}^\otimes by [Lur17, Proposition 2.1.2.5]; we make \mathcal{V}^\otimes an algebraic pattern using this factorization system and all the objects that lie over elementary objects in \mathcal{O} as elementary objects. The composite map $\mathcal{V}^\otimes \rightarrow \mathcal{O} \rightarrow \mathbb{F}_*$ then exhibits \mathcal{V}^\otimes as a cartesian pattern.

Remark 4.9. Let \mathcal{V}^\otimes be an \mathcal{O} -monoidal ∞ -category and $f: \mathcal{P} \rightarrow \mathcal{O}$ a morphism of cartesian patterns. Given a commutative triangle

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{F} & \mathcal{V}^\otimes \\ & \searrow f & \swarrow \\ & \mathcal{O} & \end{array}$$

the functor F is a morphism of cartesian patterns if and only if it preserves inert morphisms, i.e. if and only if it is a \mathcal{P} -algebra, since the commutativity of the diagram automatically implies that F preserves active morphisms and elementary objects.

Definition 4.10. If \mathcal{V}^\otimes and \mathcal{W}^\otimes are \mathcal{O} -monoidal ∞ -categories, then a *lax \mathcal{O} -monoidal functor* between them is a commutative triangle

$$\begin{array}{ccc} \mathcal{V}^\otimes & \xrightarrow{F} & \mathcal{W}^\otimes \\ & \searrow & \swarrow \\ & \mathcal{O} & \end{array}$$

such that F preserves inert morphisms. Equivalently, a lax \mathcal{O} -monoidal functor is a morphism of algebraic patterns over \mathcal{O} or a \mathcal{V}^\otimes -algebra in \mathcal{W}^\otimes over \mathcal{O} . If the functor from $\mathcal{V}^\otimes \rightarrow \mathcal{W}^\otimes$ preserves *all* cocartesian morphisms over \mathcal{O} , we call it an *\mathcal{O} -monoidal functor*.

Remark 4.11. Since $\text{Alg}_{\mathcal{O}}(\mathcal{V})$ is a full subcategory of $\text{Fun}_{/\mathcal{O}}(\mathcal{O}, \mathcal{V}^\otimes)$, it is not just functorial in the \mathcal{O} -monoidal ∞ -category \mathcal{V}^\otimes , but even 2-functorial: a lax \mathcal{O} -monoidal functor $F: \mathcal{V}^\otimes \rightarrow \mathcal{W}^\otimes$ gives a functor

$$F_*: \text{Alg}_{\mathcal{O}}(\mathcal{V}) \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{W})$$

given by composition with F , and a natural transformation $\eta: F \rightarrow F'$ over \mathcal{O} gives (again by composition) a natural transformation $\eta_*: F_* \rightarrow F'_*$. This means, for example, that any adjunction between \mathcal{O} -monoidal ∞ -categories induces by composition an adjunction on ∞ -categories of \mathcal{O} -algebras.

Notation 4.12. If $\mathcal{V}^\otimes \rightarrow \mathcal{O}$ is an \mathcal{O} -monoidal ∞ -category, we'll write $\mathcal{V} := \mathcal{V}^\otimes \times_{\mathcal{O}} \mathcal{O}^{\text{el}}$. In particular, we have $\mathcal{V}_E^\otimes \simeq \mathcal{V}_E$ for every $E \in \mathcal{O}^{\text{el}}$. As this notation is sometimes ambiguous, we'll also occasionally use $\mathcal{V}_{/\text{el}}^\otimes$ instead of \mathcal{V} . (Note that $\mathcal{V} \simeq \mathcal{V}_{/\text{el}}^\otimes$ must be distinguished from the ∞ -groupoid \mathcal{V}^{el} of elementary objects, which is the underlying ∞ -groupoid \mathcal{V}^\simeq of the ∞ -category \mathcal{V} .)

Notation 4.13. If \mathcal{V}^\otimes is an \mathcal{O} -monoidal ∞ -category, then we write

$$\mathcal{V}_{/\text{int}}^\otimes := \mathcal{O}^{\text{int}} \times_{\mathcal{O}} \mathcal{V}^\otimes.$$

(The ∞ -category $\mathcal{V}_{/\text{int}}^\otimes$ must be distinguished from the subcategory $(\mathcal{V}^\otimes)^{\text{int}}$ of inert morphisms in \mathcal{V}^\otimes : $\mathcal{V}_{/\text{int}}^\otimes$ contains *all* morphisms that lie over inert morphisms in \mathcal{O} , while $(\mathcal{V}^\otimes)^{\text{int}}$ contains only the *cocartesian* ones.)

Lemma 4.14. *If \mathcal{V}^\otimes is an \mathcal{O} -monoidal ∞ -category, then there is a natural equivalence*

$$\text{Alg}_{\mathcal{O}^{\text{int}}/\mathcal{O}}(\mathcal{V}^\otimes) \simeq \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \mathcal{V})$$

between \mathcal{O}^{int} -algebras and sections of $\mathcal{V} \rightarrow \mathcal{O}^{\text{el}}$.

Proof. Pulling back \mathcal{V}^\otimes to \mathcal{O}^{int} we get (since all morphisms in \mathcal{O}^{int} are inert) an equivalence

$$\text{Alg}_{\mathcal{O}^{\text{int}}/\mathcal{O}}(\mathcal{V}^\otimes) \simeq \text{Fun}_{/\mathcal{O}^{\text{int}}}^{\text{cocart}}(\mathcal{O}^{\text{int}}, \mathcal{V}_{/\text{int}}^\otimes).$$

By Remark 2.12 the cocartesian fibration $\mathcal{V}_{/\text{int}}^\otimes \rightarrow \mathcal{O}^{\text{int}}$ corresponds to a functor $\mathcal{O}^{\text{int}} \rightarrow \text{Cat}_\infty$ that is right Kan extended from \mathcal{O}^{el} . Translating the universal property of right Kan extension along the straightening equivalence, we get for every cocartesian fibration $\mathcal{E} \rightarrow \mathcal{O}^{\text{int}}$ a natural equivalence

$$\text{Map}_{/\mathcal{O}^{\text{int}}}^{\text{cocart}}(\mathcal{E}, \mathcal{V}_{/\text{int}}^\otimes) \simeq \text{Map}_{/\mathcal{O}^{\text{el}}}^{\text{cocart}}(\mathcal{E}|_{\mathcal{O}^{\text{el}}}, \mathcal{V}).$$

This upgrades to a natural equivalence of ∞ -categories

$$\text{Fun}_{/\mathcal{O}^{\text{int}}}^{\text{cocart}}(\mathcal{E}, \mathcal{V}_{/\text{int}}^\otimes) \simeq \text{Fun}_{/\mathcal{O}^{\text{el}}}^{\text{cocart}}(\mathcal{E}|_{\mathcal{O}^{\text{el}}}, \mathcal{V}),$$

since for $\mathcal{C} \in \text{Cat}_\infty$ there is a natural equivalence

$$\text{Map}_{\text{Cat}_\infty}(\mathcal{C}, \text{Fun}_{/\mathcal{O}^{\text{int}}}^{\text{cocart}}(\mathcal{E}, \mathcal{V}^\otimes)) \simeq \text{Map}_{/\mathcal{O}^{\text{int}}}^{\text{cocart}}(\mathcal{C} \times \mathcal{E}, \mathcal{V}^\otimes).$$

In our case this gives

$$\text{Fun}_{/\mathcal{O}^{\text{int}}}^{\text{cocart}}(\mathcal{O}^{\text{int}}, \mathcal{V}_{/\text{int}}^\otimes) \simeq \text{Fun}_{/\mathcal{O}^{\text{el}}}^{\text{cocart}}(\mathcal{O}^{\text{el}}, \mathcal{V}) \simeq \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \mathcal{V}),$$

where the last equivalence holds since \mathcal{O}^{el} is an ∞ -groupoid. \square

Remark 4.15. Similarly, if $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns and \mathcal{V}^\otimes is a \mathcal{P} -monoidal ∞ -category, we have a natural equivalence

$$\text{Alg}_{\mathcal{P}^{\text{int}}/\mathcal{O}}(\mathcal{V}^\otimes) \simeq \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{P}^{\text{el}}, \mathcal{V}).$$

Remark 4.16. The notion of \mathcal{O} -monoidal ∞ -category can be weakened to that of an \mathcal{O} - ∞ -operad, which is a functor $p: \mathcal{E} \rightarrow \mathcal{O}$ such that:

- (i) For every object \bar{O} in \mathcal{E} lying over $O \in \mathcal{O}$ and every inert morphism $\phi: O \rightarrow O'$ in \mathcal{O} , there exists a p -cocartesian morphism $\bar{\phi}: \bar{O} \rightarrow \bar{O}'$ lying over ϕ .
- (ii) For every object $O \in \mathcal{O}$, the functor

$$\mathcal{E}_O \rightarrow \prod_i \mathcal{E}_{O_i},$$

induced by the cocartesian morphisms over the inert maps ρ_i^O , is an equivalence.

- (iii) Given \bar{O} in \mathcal{E}_O , choose cocartesian morphisms $\rho_i^{\bar{O}}: \bar{O} \rightarrow \bar{O}_i$ over ρ_i^O . Then for any $O' \in \mathcal{O}$ and $\bar{O}' \in \mathcal{E}_{O'}$, the commutative square

$$\begin{array}{ccc} \text{Map}_{\mathcal{E}}(\bar{O}', \bar{O}) & \longrightarrow & \prod_i \text{Map}_{\mathcal{E}}(\bar{O}', \bar{O}_i) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{O}}(O', O) & \longrightarrow & \prod_i \text{Map}_{\mathcal{O}}(O', O_i) \end{array}$$

is cartesian.

This is the same as a *weak Segal fibration* over \mathcal{O} as defined in [CH21, §9]; we use the term \mathcal{O} - ∞ -operads to emphasize that this notion specializes to the (symmetric) ∞ -operads of [Lur17] over the pattern \mathbb{F}_*^b , to non-symmetric (or planar) ∞ -operads (as in [GH15] and [Lur17, §4.1.3]) over the pattern $\Delta^{\text{op}, b}$, and more generally for an operator category Φ to Φ - ∞ -operads in the sense of [Bar18] for the pattern $\Lambda(\Phi)^b$. If \mathcal{E} is an \mathcal{O} - ∞ -operad, then it has a canonical pattern structure by [CH21, Lemma 9.4], where the inert morphisms are those that are cocartesian and lie over inert morphisms in \mathcal{O} , and the active morphisms are those that lie over active morphisms in \mathcal{O} ; this is a cartesian pattern via the composite $\mathcal{E} \rightarrow \mathcal{O} \xrightarrow{\text{h1}} \mathbb{F}_*$.

5. MONOIDS AS ALGEBRAS

Let \mathcal{O} be a cartesian pattern. Our goal in this section is to prove that \mathcal{O} -monoids in an ∞ -category \mathcal{C} with finite products are equivalent to \mathcal{O} -algebras in an \mathcal{O} -monoidal ∞ -category determined by the cartesian product in \mathcal{C} . More precisely, we will prove the following generalization of [Lur17, Proposition 2.4.1.7]:

Proposition 5.1. *Suppose \mathcal{C} is an ∞ -category with finite products. Let $\mathcal{C}^\times \rightarrow \mathbb{F}_*$ be the corresponding cartesian symmetric monoidal ∞ -category. If \mathcal{O} is a cartesian pattern, then there is a natural equivalence*

$$\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}/\mathbb{F}_*}(\mathcal{C}^\times).$$

Here the cartesian symmetric monoidal ∞ -category \mathcal{C}^\times is defined as in [Lur17], and before we prove the proposition we need to recall this definition, which we phrase using the terminology of algebraic patterns:

Definition 5.2. Let Γ^\times (cf. [Lur17, Notation 2.4.1.2]) denote the full subcategory of $\mathbb{F}_*^{\Delta^1}$ spanned by the inert morphisms in \mathbb{F}_* . Let $\mathrm{ev}_0, \mathrm{ev}_1: \Gamma^\times \rightarrow \mathbb{F}_*$ denote the functors given by evaluation at 0 and 1, respectively.

Lemma 5.3. *$\mathrm{ev}_0: \Gamma^\times \rightarrow \mathbb{F}_*$ is a cartesian fibration.*

Proof. This follows from the uniqueness of inert–active factorizations, by the dual of the proof of [CH21, Proposition 7.2]. \square

Definition 5.4. We can apply the construction of [Lur09, Corollary 3.2.2.12] to the cartesian fibration ev_0 and the cocartesian fibration $\mathcal{C} \times \mathbb{F}_* \rightarrow \mathbb{F}_*$ to obtain a cocartesian fibration $\bar{\mathcal{C}}^\times \rightarrow \mathbb{F}_*$ with the universal property that there is a natural equivalence

$$\mathrm{Map}_{/\mathbb{F}_*}(K, \bar{\mathcal{C}}^\times) \simeq \mathrm{Map}(K \times_{\mathbb{F}_*} \Gamma^\times, \mathcal{C})$$

for any ∞ -category K over \mathbb{F}_* . In particular the ∞ -category $\bar{\mathcal{C}}^\times$ is given fibrewise over $\langle n \rangle$ by $\mathrm{Fun}(\Gamma_{\langle n \rangle}^\times, \mathcal{C})$. (And by the dual of [GHN17, Proposition 7.3] this is indeed the corresponding functor.)

If \mathcal{C} is an ∞ -category with finite products, we define \mathcal{C}^\times to be the full subcategory of $\bar{\mathcal{C}}^\times$ whose objects over $\langle n \rangle$ are the functors $F: \Gamma_{\langle n \rangle}^\times \rightarrow \mathcal{C}$ such that for every object $\phi: \langle n \rangle \rightarrow \langle m \rangle$, the map $F(\phi) \rightarrow \prod_{i=1}^m F(\rho_i \phi)$ is an equivalence.

Proposition 5.5 (Lurie, [Lur17, Proposition 2.4.1.5]). *If \mathcal{C} is an ∞ -category with products, then the restricted functor $\mathcal{C}^\times \rightarrow \mathbb{F}_*$ is a symmetric monoidal ∞ -category.* \square

Remark 5.6. From the definition it follows that any functor $g: \mathcal{C} \rightarrow \mathcal{D}$ induces a natural morphism of cocartesian fibrations $\bar{g}^\times: \bar{\mathcal{C}}^\times \rightarrow \bar{\mathcal{D}}^\times$; if g preserves products, then this restricts to a natural symmetric monoidal functor $g^\times: \mathcal{C}^\times \rightarrow \mathcal{D}^\times$.

Definition 5.7. We say a morphism

$$\begin{array}{ccc} \langle m \rangle & \longrightarrow & \langle m' \rangle \\ \downarrow & & \downarrow \\ \langle n \rangle & \longrightarrow & \langle n' \rangle \end{array}$$

in Γ^\times is inert or active if the horizontal maps in the square are inert or active, respectively.

Lemma 5.8. *The inert and active morphisms determine a factorization system on Γ^\times .*

Proof. Given a morphism as above, then by the factorization system on \mathbb{F}_*^b we get horizontal inert-active factorizations

$$\begin{array}{ccccc} \langle m \rangle & \xrightarrow{\quad} & \langle m'' \rangle & \rightsquigarrow & \langle m' \rangle \\ \downarrow & & \vdots & & \downarrow \\ \langle n \rangle & \xrightarrow{\quad} & \langle n'' \rangle & \rightsquigarrow & \langle n' \rangle. \end{array}$$

The existence of a factorization system on Γ^\times is equivalent to the existence of an inert morphism indicated by the dotted arrow. The map $\langle m'' \rangle \rightsquigarrow \langle m' \rangle \rightarrow \langle n' \rangle$ factors into $\langle m'' \rangle \rightarrow \langle q \rangle \rightsquigarrow \langle n' \rangle$. The essential uniqueness of factorizations implies that $\langle m \rangle \rightarrow \langle n \rangle \rightarrow \langle n'' \rangle \rightsquigarrow \langle n' \rangle$ coincides with $\langle m \rangle \rightarrow \langle m'' \rangle \rightarrow \langle q \rangle \rightsquigarrow \langle n' \rangle$. Hence, there is an equivalence $\langle n'' \rangle \simeq \langle q \rangle$ which proves the existence of the dotted morphism in the diagram above. \square

Definition 5.9. We give Γ^\times an algebraic pattern structure using this factorization system, with $\text{id}_{\langle 1 \rangle}$ as the only elementary object.

Remark 5.10. It is clear from the definition of Γ^\times that the evaluations at 0 and 1 give morphisms of algebraic patterns $\text{ev}_0, \text{ev}_1: \Gamma^\times \rightarrow \mathbb{F}_*^b$. Moreover, the evaluation at 1 exhibits Γ^\times as a cartesian pattern.

Remark 5.11. If \mathcal{O} is a cartesian pattern, then we can equip the pullback $\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times$ over evaluation at $\{0\}$ with a canonical pattern structure (which according to [CH21, Corollary 5.5] gives the fibre product in the ∞ -category of algebraic patterns). Here a morphism in $\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times$ that is given by $O \rightarrow O'$ and a commutative square

$$\begin{array}{ccc} |O| & \longrightarrow & |O'| \\ \downarrow & & \downarrow \\ \langle n \rangle & \longrightarrow & \langle n' \rangle \end{array}$$

is inert or active if $O \rightarrow O'$ and the horizontal maps in the square are inert or active, respectively, and the elementary objects are the pairs $(E, \text{id}_{\langle 1 \rangle})$ with $E \in \mathcal{O}^{\text{el}}$. The composite

$$\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times \rightarrow \Gamma^\times \xrightarrow{\text{ev}_1} \mathbb{F}_*$$

exhibits $\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times$ as a cartesian pattern.

Definition 5.12. Let $i: \mathbb{F}_* \rightarrow \Gamma^\times$ be the functor that takes $\langle n \rangle$ to $\text{id}_{\langle n \rangle}$ (given by composition with $\Delta^1 \rightarrow \Delta^0$); this is fully faithful and a morphism of cartesian patterns. If \mathcal{O} is a cartesian pattern, we define $i_{\mathcal{O}}: \mathcal{O} \rightarrow \mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times$ by pulling back i , so that $i_{\mathcal{O}}$ takes $O \in \mathcal{O}$ to $(O, \text{id}_{|O|})$. This is again fully faithful, and since the target is a fibre product of patterns it is also a morphism of cartesian patterns.

Remark 5.13. An active morphism $(O, |O| \rightarrow \langle n \rangle) \rightsquigarrow i_{\mathcal{O}}(O') \simeq (O', |O'| = |O'|)$ is given by an active map $O \rightsquigarrow O'$ together with a commutative square

$$\begin{array}{ccc} |O| & \rightsquigarrow & |O'| \\ \downarrow & & \parallel \\ \langle n \rangle & \rightsquigarrow & |O'|. \end{array}$$

The uniqueness of the inert-active factorization then implies that $|O| \cong \langle n \rangle$, and hence $i_{\mathcal{O}}$ has unique lifting of active morphisms in the sense of [CH21, Definition 6.1].

Lemma 5.14. Let \mathcal{O} be a cartesian pattern and \mathcal{C} an ∞ -category with finite products. Composition with $i_{\mathcal{O}}$ and right Kan extension along it restrict to an adjoint equivalence

$$i_{\mathcal{O}}^*: \text{Mon}_{\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times}(\mathcal{C}) \xrightleftharpoons[\sim]{\sim} \text{Mon}_{\mathcal{O}}(\mathcal{C}) : i_{\mathcal{O},*}.$$

Proof. Since $i_{\mathcal{O}}$ is a morphism of cartesian patterns, the functor $i_{\mathcal{O}}^*$ restricts to the full subcategories of monoids. Moreover, since $i_{\mathcal{O}}$ has unique lifting of active morphisms, its right adjoint $i_{\mathcal{O},*}$ also restricts to monoids by [CH21, Proposition 6.3], and it is fully faithful since $i_{\mathcal{O}}$ is fully faithful. It remains to show that $i_{\mathcal{O},*}$ is essentially surjective on monoids. Let $M: \mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times \rightarrow \mathcal{C}$ be a monoid, and let $(O, j: |O| \rightarrow \langle n \rangle)$ be an object of $\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times$. We must show that the canonical map $M(O, j) \rightarrow (i_{\mathcal{O},*} i_{\mathcal{O}}^* M)(O, j)$ is an equivalence. But we have a commutative square

$$\begin{array}{ccc} M(O, j) & \longrightarrow & (i_{\mathcal{O},*} i_{\mathcal{O}}^* M)(O, j) \\ \downarrow & & \downarrow \\ \prod_{i=1}^n M(O_i, \text{id}_{\langle 1 \rangle}) & \longrightarrow & \prod_{i=1}^n (i_{\mathcal{O},*} i_{\mathcal{O}}^* M)(O_i, \text{id}_{\langle 1 \rangle}), \end{array}$$

where the vertical maps are equivalences since M and $i_{\mathcal{O},*} i_{\mathcal{O}}^* M$ are monoids. Moreover, the bottom horizontal map is an equivalence since $i_{\mathcal{O},*}$ is fully faithful and the objects $(O_i, \text{id}_{\langle 1 \rangle}) \simeq i_{\mathcal{O}}(O_i)$ are in the image of $i_{\mathcal{O}}$. The top horizontal map is therefore also an equivalence. \square

Lemma 5.15. *Under the natural equivalence*

$$\text{Fun}_{/\mathbb{F}_*}(\mathcal{O}, \bar{\mathcal{C}}^\times) \simeq \text{Fun}(\mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times, \mathcal{C}),$$

the full subcategory $\text{Mon}_{\mathcal{O} \times_{\mathbb{F}_} \Gamma^\times}(\mathcal{C})$ is identified with $\text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\mathcal{C}^\times)$.*

Proof. By definition, a functor $F: \mathcal{O} \rightarrow \bar{\mathcal{C}}^\times$ over \mathbb{F}_* lies in $\text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\mathcal{C}^\times)$ if and only if F factors through the full subcategory \mathcal{C}^\times , and F takes inert morphisms to cocartesian morphisms. We can reformulate these requirements in terms of the corresponding functor $F': \mathcal{O} \times_{\mathbb{F}_*} \Gamma^\times \rightarrow \mathcal{C}$ as the following pair of conditions:

(1) The map

$$F'(O, \phi) \rightarrow \prod_{i=1}^n F'(O, \rho_i \phi)$$

is an equivalence for every object $(O, \phi: |O| \rightarrow \langle n \rangle)$.

(2) For every inert map $\psi: O' \rightarrow O$ in \mathcal{O} the morphism

$$F'(O', \phi|\psi) \rightarrow F'(O, \phi)$$

is an equivalence.

By the definition of \mathcal{C}^\times , condition (1) holds if and only if F factors through \mathcal{C}^\times , while the description of cocartesian morphisms in \mathcal{C}^\times in [Lur17, Proposition 2.4.1.5.(2)] shows that F takes inert morphisms to cocartesian morphisms if and only if condition (2) holds.

On the other hand, F' is a monoid if for every object (O, ϕ) the map

$$F'(O, \phi) \rightarrow \prod_{i=1}^n F'(O_{\phi^{-1}i}, \text{id}_{\langle 1 \rangle})$$

is an equivalence. To see that this is equivalent to the first pair of conditions, observe that we have commutative triangles

$$\begin{array}{ccc} F'(O, \phi) & \xrightarrow{\quad\quad\quad} & \prod_{i=1}^n F'(O, \rho_i \phi) \\ & \searrow & \swarrow \\ & \prod_{i=1}^n F'(O_{\phi^{-1}i}, \text{id}_{\langle 1 \rangle}), & \\ \\ F'(O', \phi|\psi) & \xrightarrow{\quad\quad\quad} & F'(O, \phi) \\ & \searrow & \swarrow \\ & \prod_{i=1}^n F'(O_{\phi^{-1}i}, \text{id}_{\langle 1 \rangle}). & \end{array}$$

First suppose conditions (1) and (2) hold. In the first triangle, the horizontal map is an equivalence by (1), while the right diagonal map is an equivalence by (2) using the identity $\rho_i \phi = \text{id}_{\langle 1 \rangle} |\rho_i^{\mathcal{O}}|$. Hence the left diagonal map is also an equivalence, which shows that F' is a monoid.

Conversely, if F' is a monoid, then both diagonal maps in the second triangle are equivalences (using the equivalences $O'_{(\phi|\psi|)^{-1}i} \simeq O_{\phi^{-1}i}$). Thus the horizontal map is also an equivalence, which gives condition (2). In the first triangle we then have that the left diagonal map is an equivalence since F' is a monoid, and the right diagonal map is an equivalence by condition (2) applied to the inert maps $\rho_i^{\mathcal{O}}$. Thus the horizontal map in the first triangle is an equivalence, which gives condition (1). \square

Proof of Proposition 5.1. Combine Lemmas 5.14 and 5.15. \square

Remark 5.16. If $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns, then it is clear from the proof that the equivalence of Proposition 5.1 is natural. Thus if \mathcal{C} is an ∞ -category with finite products, composition with f gives a commutative square

$$\begin{array}{ccc} \text{Mon}_{\mathcal{P}}(\mathcal{C}) & \xrightarrow{\sim} & \text{Alg}_{\mathcal{P}/\mathbb{F}_*}(\mathcal{C}^\times) \\ \downarrow f^* & & \downarrow f^* \\ \text{Mon}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\sim} & \text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\mathcal{C}^\times) \end{array}$$

Moreover, if \mathcal{D} is another ∞ -category with finite products and $g: \mathcal{C} \rightarrow \mathcal{D}$ is a product-preserving functor, then g and the symmetric monoidal functor $g^\times: \mathcal{C}^\times \rightarrow \mathcal{D}^\times$ from Remark 5.6 fit in a commutative square

$$\begin{array}{ccc} \text{Mon}_{\mathcal{O}}(\mathcal{C}) & \xrightarrow{\sim} & \text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\mathcal{C}^\times) \\ \downarrow g_* & & \downarrow g_*^\times \\ \text{Mon}_{\mathcal{O}}(\mathcal{D}) & \xrightarrow{\sim} & \text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\mathcal{D}^\times), \end{array}$$

which combines with composition with f to give a natural commutative cube.

6. DAY CONVOLUTION OVER CARTESIAN PATTERNS

In this section we introduce Day convolution for presheaves of spaces on \mathcal{O} -monoidal ∞ -categories, where \mathcal{O} is any cartesian pattern. We generalize a simple and elegant approach to Day convolution for presheaves due to Heine [Hei18, §6.1]. Other constructions of Day convolution (which work for functors to more general targets than just spaces) are due to Glasman [Gla16] and Lurie [Lur17]. We begin by describing a useful model of the universal cocartesian fibration, using right fibrations:

Notation 6.1. Let RFib denote the full subcategory of $\text{Cat}_{\infty}^{\Delta^1}$ spanned by the right fibrations.

Proposition 6.2. *The functor $\text{ev}_1: \text{RFib} \rightarrow \text{Cat}_{\infty}$, given by evaluation at $1 \in \Delta^1$, is a cartesian and cocartesian fibration, and the corresponding contravariant functor is equivalent to*

$$\text{P}(-) := \text{Fun}((-)^{\text{op}}, \mathcal{S}): \text{Cat}_{\infty}^{\text{op}} \rightarrow \widehat{\text{Cat}}_{\infty}.$$

Proof. The ∞ -category Cat_{∞} has pullbacks, so the functor $\text{ev}_1: \text{Fun}(\Delta^1, \text{Cat}_{\infty}) \rightarrow \text{Cat}_{\infty}$ is a cartesian fibration, with cartesian morphisms given by pullback squares. Since the pullback of a right fibration is again a right fibration, the full subcategory RFib inherits cartesian morphisms from $\text{Fun}(\Delta^1, \text{Cat}_{\infty})$, which implies that $\text{ev}_1: \text{RFib} \rightarrow \text{Cat}_{\infty}$ is a cartesian fibration. To show that it is also a cocartesian fibration it then suffices by [Lur09, Corollary 5.2.2.5] to check that for any morphism $f: \mathcal{C}' \rightarrow \mathcal{C}$ in Cat_{∞} , the cartesian pullback functor

$$f^*: \text{RFib}_{\mathcal{C}} \rightarrow \text{RFib}_{\mathcal{C}'}$$

has a left adjoint. Under the straightening equivalence, this functor corresponds to the functor $f^*: \text{P}(\mathcal{C}) \rightarrow \text{P}(\mathcal{C}')$ given by composition with f^{op} , which indeed has a left adjoint given by left Kan extension along f^{op} .

To identify the corresponding functor we use the naturality of the straightening equivalence, as discussed in [GHN17, Appendix A]. By a variant of [GHN17, Corollary A.32], the pseudo-naturality of straightening on the model category level induces a natural equivalence

$$\mathrm{RFib}_{(-)} \xrightarrow{\simeq} \mathrm{Fun}((-)^{\mathrm{op}}, \mathcal{S}).$$

Here the functoriality in the domain is induced by strict pullbacks on the model category level using the covariant model structures on slices of simplicial sets [Lur09, §2.1].

It thus suffices to identify this functor with that corresponding to our cartesian fibration. We can model RFib as a relative category by the full subcategory $\mathrm{Fun}(\Delta^1, \mathrm{Set}_\Delta)_{\mathrm{RFib}}$ of $\mathrm{Fun}(\Delta^1, \mathrm{Set}_\Delta)$ spanned by the right fibrations between quasicategories in the sense of [Lur09], with the weak equivalences inherited from the Joyal model structure on Set_Δ . Evaluation at 1 gives a cartesian fibration of relative categories $\mathrm{Fun}(\Delta^1, \mathrm{Set}_\Delta)_{\mathrm{RFib}} \rightarrow \mathrm{Set}_\Delta$, whose associated functor from Set_Δ to relative categories induces on the ∞ -level the functor $\mathrm{RFib}_{(-)}$ we considered above. In this situation Hinich [Hin16] has shown that the functor associated to the induced cartesian fibration of ∞ -categories is that obtained from the associated functor to relative categories by inverting weak equivalences. \square

Definition 6.3. Let RSI denote the full subcategory of RFib spanned by the right fibrations given by slice categories, i.e. right fibrations of the form $\mathcal{C}_{/c} \rightarrow \mathcal{C}$.

Corollary 6.4. $\mathrm{ev}_1: \mathrm{RSI} \rightarrow \mathrm{Cat}_\infty$ is a cocartesian fibration, and the inclusion $\mathrm{RSI} \rightarrow \mathrm{RFib}$ preserves cocartesian morphisms. The corresponding functor $\mathrm{Cat}_\infty \rightarrow \mathrm{Cat}_\infty$ is the identity, and the inclusion corresponds to the Yoneda embeddings $\mathcal{C} \hookrightarrow \mathrm{P}(\mathcal{C})$.

Proof. Under the straightening equivalence $\mathrm{RFib}_{\mathcal{C}} \simeq \mathrm{P}(\mathcal{C})$, the slice fibration $\mathcal{C}_{/c} \rightarrow \mathcal{C}$ corresponds to the representable presheaf $\mathrm{Map}_{\mathcal{C}}(-, c)$. Moreover, for a functor $f: \mathcal{C} \rightarrow \mathcal{C}'$ the cocartesian pushforward $f_!: \mathrm{RFib}_{\mathcal{C}} \rightarrow \mathrm{RFib}_{\mathcal{C}'}$ corresponds to the functor $f_!: \mathrm{P}(\mathcal{C}) \rightarrow \mathrm{P}(\mathcal{C}')$ given by left Kan extension along f^{op} . This lives in a commutative square

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathrm{P}(\mathcal{C}) \\ f \downarrow & & \downarrow f_! \\ \mathcal{C}' & \hookrightarrow & \mathrm{P}(\mathcal{C}') \end{array}$$

where the horizontal maps are the Yoneda embeddings. In particular, $f_!$ takes the representable presheaf $\mathrm{Map}_{\mathcal{C}}(-, c)$ to $\mathrm{Map}_{\mathcal{C}'}(-, f(c))$. In terms of right fibrations, this means

$$f_!(\mathcal{C}_{/c} \rightarrow \mathcal{C}) \simeq (\mathcal{C}'_{/f(c)} \rightarrow \mathcal{C}');$$

thus RSI inherits cocartesian morphisms from RFib , and this makes $\mathrm{ev}_1: \mathrm{RSI} \rightarrow \mathrm{Cat}_\infty$ a cocartesian fibration. The corresponding functor $\mathrm{Cat}_\infty \rightarrow \widehat{\mathrm{Cat}}_\infty$ is equivalent to that taking \mathcal{C} to the full subcategory of $\mathrm{P}(\mathcal{C})$ spanned by the representable presheaves; the naturality of the Yoneda embedding shows that this is equivalent to the identity of Cat_∞ , as required, and that the inclusion $\mathrm{RSI} \hookrightarrow \mathrm{RFib}$ corresponds to the Yoneda embedding. \square

Corollary 6.5. Suppose $\pi: \mathcal{F} \rightarrow \mathcal{B}$ is the cocartesian fibration corresponding to a functor $F: \mathcal{B} \rightarrow \mathrm{Cat}_\infty$. Then there is a pullback square of ∞ -categories

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathrm{RSI} \\ \pi \downarrow & & \downarrow \mathrm{ev}_1 \\ \mathcal{B} & \xrightarrow{F} & \mathrm{Cat}_\infty. \end{array}$$

In other words, $\mathrm{ev}_1: \mathrm{RSI} \rightarrow \mathrm{Cat}_\infty$ is the universal cocartesian fibration.

Proof. Pullback of cocartesian fibrations corresponds to composition of functors, so $F^*\mathrm{RSI} \rightarrow \mathcal{B}$ is the cocartesian fibration for $\mathrm{id} \circ F \simeq F$. \square

Our next goal is to prove an \mathcal{O} -monoidal version of Corollary 6.5, which needs some preliminaries.

Remark 6.6. The ∞ -categories \mathbf{RFib} and \mathbf{RSI} are both closed under cartesian products as full subcategories of $\mathbf{Fun}(\Delta^1, \mathbf{Cat}_\infty)$. These ∞ -categories therefore have cartesian products, and the functor to \mathbf{Cat}_∞ given by evaluation at 1 in Δ^1 preserves products. We hence have induced symmetric monoidal functors

$$\mathrm{ev}_1^\times: \mathbf{RFib}^\times, \mathbf{RSI}^\times \rightarrow \mathbf{Cat}_\infty^\times.$$

Our next goal is to show that these functors are both cocartesian fibrations, and indeed exhibit \mathbf{RFib}^\times and \mathbf{RSI}^\times as $\mathbf{Cat}_\infty^\times$ -monoidal ∞ -categories. To see this we apply the following general criterion:

Proposition 6.7. *Let \mathcal{O} be a cartesian pattern. Suppose $F: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ is an \mathcal{O} -monoidal functor between \mathcal{O} -monoidal ∞ -categories such that*

- (1) *for every $E \in \mathcal{O}^{\mathrm{el}}$, the functor on fibres $F_E: \mathcal{C}_E \rightarrow \mathcal{D}_E$ is a cocartesian fibration,*
- (2) *for every active morphism $\phi: O \rightsquigarrow E$ in \mathcal{O} with $E \in \mathcal{O}^{\mathrm{el}}$, in the commutative square*

$$\begin{array}{ccc} \prod_i \mathcal{C}_{O_i} & \xrightarrow{\phi_!^{\mathcal{C}}} & \mathcal{C}_E \\ \prod_i F_{O_i} \downarrow & & \downarrow F_E \\ \prod_i \mathcal{D}_{O_i} & \xrightarrow{\phi_!^{\mathcal{D}}} & \mathcal{D}_E, \end{array}$$

the functor $\phi_!^{\mathcal{C}}$ takes $\prod_i F_{O_i}$ -cocartesian morphisms (i.e. tuples of F_{O_i} -cocartesian morphisms) to F_E -cocartesian morphisms.

Then:

- (i) *F is a cocartesian fibration that exhibits \mathcal{C}^\otimes as a \mathcal{D}^\otimes -monoidal ∞ -category.*
- (ii) *Given a morphism $\psi: O(D_i) \rightarrow O'(D'_j)$ in \mathcal{D}^\otimes lying over $\phi: O \rightarrow O'$ in \mathcal{O} and an object $O(C_i)$ in \mathcal{C}^\otimes over $O(D_i)$, the cocartesian morphism over ψ with this domain is the composite*

$$O(C_i) \rightarrow \phi_! O(C_i) \rightarrow \psi'_! \phi_! O(C_i)$$

where $O(C_i) \rightarrow \phi_! O(C_i)$ is cocartesian over ϕ , the map ψ factors uniquely (since F preserves cocartesian morphisms) as

$$O(D_i) \rightarrow F(\phi_! O(C_i)) \xrightarrow{\psi'} O'(D'_j),$$

where ψ' lies in the fibre $\mathcal{D}_{O'}^\otimes$, and $\phi_! O(C_i) \rightarrow \psi'_! \phi_! O(C_i)$ is $F_{O'}$ -cocartesian over ψ' .

- (iii) *If $f: \mathcal{P} \rightarrow \mathcal{D}^\otimes$ is a morphism of cartesian patterns, we have a natural cartesian square*

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{P}/\mathcal{D}^\otimes}(\mathcal{C}^\otimes) & \longrightarrow & \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C}^\otimes) \\ \downarrow & & \downarrow \\ \{f\} & \longrightarrow & \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{D}^\otimes) \end{array}$$

Proof. To prove that F is a cocartesian fibration we use the criterion of (the dual of) [HMS19, Lemma A.1.8], which requires us to check that

- (a) *for every $O \in \mathcal{O}$, the functor $F_O: \mathcal{C}_O^\otimes \rightarrow \mathcal{D}_O^\otimes$ is a cocartesian fibration,*
- (b) *for every morphism $\phi: O \rightarrow O'$ in \mathcal{O} , in the commutative square*

$$\begin{array}{ccc} \mathcal{C}_O^\otimes & \xrightarrow{\phi_!^{\mathcal{C}}} & \mathcal{C}_{O'}^\otimes \\ \downarrow F_O & & \downarrow F_{O'} \\ \mathcal{D}_O^\otimes & \xrightarrow{\phi_!^{\mathcal{D}}} & \mathcal{D}_{O'}^\otimes, \end{array}$$

the functor $\phi_!^{\mathcal{C}}$ takes F_O -cocartesian morphisms to $F_{O'}$ -cocartesian morphisms.

Condition (a) is clear, since F_O is equivalent to the product

$$\prod_i F_{O_i} : \prod_i \mathcal{C}_{O_i} \rightarrow \prod_i \mathcal{D}_{O_i},$$

where each F_{O_i} is by assumption a cocartesian fibration. Moreover, a morphism in \mathcal{C}_O^\otimes is F_O -cocartesian if and only if under the equivalence with $\prod_i \mathcal{C}_{O_i}$ it corresponds to a tuple of F_{O_i} -cocartesian morphisms. If ϕ is an inert morphism in \mathcal{O} , then $\phi_!^{\mathcal{C}}$ corresponds to a projection to some factors in this product, hence condition (b) is immediate for inert morphisms. Using the factorization system we can then reduce condition (b) to the case of an active morphism $O \rightsquigarrow E$ with $E \in \mathcal{O}^{\text{el}}$, where it holds by assumption. The description of the cocartesian morphisms in the proof of [HMS19, Lemma A.1.8] also gives (ii).

To see that F exhibits \mathcal{C}^\otimes as \mathcal{D}^\otimes -monoidal, observe that for $O \in \mathcal{O}$ the commutative square

$$\begin{array}{ccc} \mathcal{C}_O^\otimes & \xrightarrow{\sim} & \prod_i \mathcal{C}_{O_i} \\ \downarrow F_O & & \downarrow F_{O_i} \\ \mathcal{D}_O^\otimes & \xrightarrow{\sim} & \prod_i \mathcal{D}_{O_i} \end{array}$$

is cartesian, since the horizontal maps are equivalences. For any $O(D_i) \in \mathcal{D}_O^\otimes$ we therefore have an equivalence on fibres

$$\mathcal{C}_{O(D_i)}^\otimes \xrightarrow{\sim} \prod_i \mathcal{C}_{D_i},$$

as required.

To prove (iii), observe that we have a pullback square

$$\begin{array}{ccc} \text{Fun}_{/\mathcal{D}^\otimes}(\mathcal{P}, \mathcal{C}^\otimes) & \longrightarrow & \text{Fun}_{/\mathcal{O}}(\mathcal{P}, \mathcal{C}^\otimes) \\ \downarrow & & \downarrow \\ \{f\} & \longrightarrow & \text{Fun}_{/\mathcal{O}}(\mathcal{P}, \mathcal{D}^\otimes). \end{array}$$

This restricts to the full subcategories of algebras because (ii) implies that a functor from $\mathcal{P} \rightarrow \mathcal{C}^\otimes$ over \mathcal{D}^\otimes is an algebra if and only if the underlying functor over \mathcal{O} is an algebra (since every inert morphism in \mathcal{C}^\otimes is F -cocartesian over an inert morphisms in \mathcal{D}^\otimes). \square

Corollary 6.8. *Suppose $\pi: \mathcal{E} \rightarrow \mathcal{B}$ is a cocartesian fibration such that*

- (1) \mathcal{E} and \mathcal{B} have finite products,
- (2) π preserves these,
- (3) if $e_i \rightarrow e'_i$ ($i = 1, 2$) are π -cocartesian morphisms in \mathcal{E} then $e_1 \times e_2 \rightarrow e'_1 \times e'_2$ is again π -cocartesian.

Then the induced symmetric monoidal functor $\pi^\times: \mathcal{E}^\times \rightarrow \mathcal{B}^\times$ is a cocartesian fibration that exhibits \mathcal{E}^\times as \mathcal{B}^\times -monoidal. \square

Lemma 6.9. *Suppose $\mathcal{E} \rightarrow \mathcal{A}$ and $\mathcal{F} \rightarrow \mathcal{B}$ are right fibrations corresponding to functors $E: \mathcal{A}^{\text{op}} \rightarrow \mathcal{S}$, $F: \mathcal{B}^{\text{op}} \rightarrow \mathcal{S}$. Then the right fibration $\mathcal{E} \times \mathcal{F} \rightarrow \mathcal{A} \times \mathcal{B}$ corresponds to the functor $E \times F: \mathcal{A}^{\text{op}} \times \mathcal{B}^{\text{op}} \rightarrow \mathcal{S}$ (taking (a, b) to $E(a) \times F(b)$).*

Proof. Consider the right fibrations

$$\mathcal{A} \times \mathcal{E}, \mathcal{F} \times \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B}.$$

These are the pullbacks of \mathcal{E} and \mathcal{F} along the projections from $\mathcal{A} \times \mathcal{B}$ to \mathcal{A} and \mathcal{B} , respectively. Since pullbacks of right fibrations correspond to compositions of functors, the associated functors are therefore

$$\begin{array}{ccc} \mathcal{A}^{\text{op}} \times \mathcal{B}^{\text{op}} & \rightarrow & \mathcal{A}^{\text{op}} \xrightarrow{E} \mathcal{S}, \\ \mathcal{A}^{\text{op}} \times \mathcal{B}^{\text{op}} & \rightarrow & \mathcal{B}^{\text{op}} \xrightarrow{F} \mathcal{S}, \end{array}$$

respectively. Now we can identify $\mathcal{E} \times \mathcal{F}$ with $(\mathcal{E} \times \mathcal{B}) \times_{\mathcal{A} \times \mathcal{B}} (\mathcal{A} \times \mathcal{F})$, which is a product of right fibrations over $\mathcal{A} \times \mathcal{B}$. Since straightening preserves limits, the corresponding functor is the product $E \times F$ as required. \square

Corollary 6.10. *The functors*

$$\mathrm{ev}_1^\times: \mathrm{RSI}^\times, \mathrm{RFib}^\times \rightarrow \mathrm{Cat}_\infty^\times$$

are cocartesian fibrations that exhibit their domains as $\mathrm{Cat}_\infty^\times$ -monoidal ∞ -categories. The inclusion $\mathrm{RSI}^\times \hookrightarrow \mathrm{RFib}^\times$ is $\mathrm{Cat}_\infty^\times$ -monoidal (i.e. preserves cocartesian morphisms over $\mathrm{Cat}_\infty^\times$).

Proof. The only non-obvious condition in Corollary 6.8 is that products of cocartesian morphisms in RFib are again cocartesian (since the cocartesian morphisms in RSI are inherited from RFib , it is enough to consider the case of RFib).

Suppose $\mathcal{E} \rightarrow \mathcal{A}$ and $\mathcal{F} \rightarrow \mathcal{B}$ are right fibrations corresponding to functors $E: \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{S}$, $F: \mathcal{B}^{\mathrm{op}} \rightarrow \mathcal{S}$. Then the product $\mathcal{E} \times \mathcal{F} \rightarrow \mathcal{A} \times \mathcal{B}$ corresponds to $E \times F$ by Lemma 6.9. Given $\alpha: \mathcal{A} \rightarrow \mathcal{A}'$ and $\beta: \mathcal{B} \rightarrow \mathcal{B}'$, the cocartesian pushforward $(\alpha \times \beta)_! (\mathcal{E} \times \mathcal{F})$ corresponds to the left Kan extension of $E \times F$ along $(\alpha \times \beta)^{\mathrm{op}}$, given by

$$(x, y) \in (\mathcal{A}' \times \mathcal{B}')^{\mathrm{op}} \mapsto \mathrm{colim}_{(a,b) \in (\mathcal{A}_x \times \mathcal{B}_y)^{\mathrm{op}}} E(a) \times F(b).$$

Since the product in \mathcal{S} commutes with colimits in each variable, this is equivalent to

$$(x, y) \in (\mathcal{A}' \times \mathcal{B}')^{\mathrm{op}} \mapsto \left(\mathrm{colim}_{a \in (\mathcal{A}_x)^{\mathrm{op}}} E(a) \right) \times \left(\mathrm{colim}_{b \in (\mathcal{B}_y)^{\mathrm{op}}} F(b) \right)$$

which by Lemma 6.9 is the functor corresponding to $\alpha_! \mathcal{E} \times \beta_! \mathcal{F}$, as required.

For the last claim, note that cocartesian morphisms in RSI are inherited from RFib . The description of cocartesian morphisms in Proposition 6.7(ii) therefore shows that the inclusion $\mathrm{RSI}^\times \hookrightarrow \mathrm{RFib}^\times$ preserves cocartesian morphisms over $\mathrm{Cat}_\infty^\times$. \square

Remark 6.11. Using Proposition 6.7(ii) we can describe the cocartesian morphisms in RFib^\times as follows: Given a morphism $(\mathcal{C}_1, \dots, \mathcal{C}_n) \rightarrow \mathcal{D}$ in $\mathrm{Cat}_\infty^\times$, which corresponds to a functor $\Phi: \mathcal{C}_1 \times \dots \times \mathcal{C}_n \rightarrow \mathcal{D}$, and an object of RFib^\times over $(\mathcal{C}_1, \dots, \mathcal{C}_n)$, which we can identify with a family of presheaves $F_i: \mathcal{C}_i^{\mathrm{op}} \rightarrow \mathcal{S}$, ($i = 1, \dots, n$), the cocartesian morphism over Φ takes this to the left Kan extension $\Phi_!(\prod_i F_i): \mathcal{D}^{\mathrm{op}} \rightarrow \mathcal{S}$ along Φ^{op} of the product

$$\prod_i F_i: \prod_i \mathcal{C}_i^{\mathrm{op}} \xrightarrow{(F_i)} \mathcal{S}^{\times n} \xrightarrow{\times} \mathcal{S}.$$

Proposition 6.12. *Let \mathcal{O} be a cartesian pattern, and suppose $\pi: \mathcal{C}^\otimes \rightarrow \mathcal{O}$ is an \mathcal{O} -monoidal ∞ -category, corresponding to an \mathcal{O} -monoid M in Cat_∞ . Let $A: \mathcal{O} \rightarrow \mathrm{Cat}_\infty^\times$ be the corresponding \mathcal{O} -algebra under the equivalence of Proposition 5.1. Then there is a pullback square*

$$\begin{array}{ccc} \mathcal{C}^\otimes & \longrightarrow & \mathrm{RSI}^\times \\ \downarrow & & \downarrow \mathrm{ev}_1^\times \\ \mathcal{O} & \xrightarrow{A} & \mathrm{Cat}_\infty^\times. \end{array}$$

Remark 6.13. We can interpret this as exhibiting the pullback $\mathcal{O} \times_{\mathbb{F}_*} \mathrm{RSI}^\times \rightarrow \mathcal{O} \times_{\mathbb{F}_*} \mathrm{Cat}_\infty^\times$ as the universal cocartesian fibration of \mathcal{O} -monoidal ∞ -categories.

Proof of Proposition 6.12. By Corollary 6.5 we have a pullback square

$$\begin{array}{ccc} \mathcal{C}^\otimes & \xrightarrow{\overline{M}} & \mathrm{RSI} \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{M} & \mathrm{Cat}_\infty, \end{array}$$

where M is by assumption an \mathcal{O} -monoid. The functor \overline{M} takes an object $X \simeq O(X_1, \dots, X_n) \in \mathcal{C}^\otimes$ to the slice $(\mathcal{C}^\otimes)_{/X}$. But the equivalence $\mathcal{C}^\otimes \simeq \prod_{i=1}^n \mathcal{C}_{O_i}$ induces an equivalence

$$(\mathcal{C}^\otimes)_{/X} \simeq \prod_{i=1}^n \mathcal{C}_{O_i/X_i}.$$

This shows that the functor \overline{M} is a \mathcal{C}^\otimes -monoid. By the naturality of the equivalence between monoids and algebras, as discussed in Remark 5.16, our pullback square therefore corresponds to a commutative square

$$\begin{array}{ccc} \mathcal{C}^\otimes & \xrightarrow{\overline{A}} & \mathbf{RSI}^\times \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{A} & \mathbf{Cat}_\infty^\times, \end{array}$$

where \overline{A} is the \mathcal{C}^\otimes -algebra corresponding to \overline{M} . It then remains to verify that this square is cartesian.

Here the vertical maps are cocartesian fibrations, and we first observe that \overline{A} preserves cocartesian morphisms. Since \overline{A} is an algebra it suffices to check this in the case of a cocartesian morphism $\overline{\phi}: O(C_1, \dots, C_n) \rightarrow C'$ over an active morphism $\phi: O \rightsquigarrow E$ with $E \in \mathcal{O}^{\text{el}}$. Here $C' \simeq M(\phi)(O(C_1, \dots, C_n))$, and $\overline{A}(\overline{\phi})$ is by construction the morphism

$$(\mathcal{C}_{O_1/C_1}, \dots, \mathcal{C}_{O_n/C_n}) \rightarrow \mathcal{C}_{E/C'}$$

corresponding to the functor

$$\overline{M}(\overline{\phi}): \prod_i \mathcal{C}_{O_i/C_i} \simeq \mathcal{C}_{O/O(C_1, \dots, C_n)}^\otimes \rightarrow \mathcal{C}_{E/C'}.$$

By definition of \overline{M} as a pullback it preserves cocartesian morphisms, so this is a cocartesian morphism in \mathbf{RSI} . The description of cocartesian morphisms in \mathbf{RSI}^\times in Remark 6.11 now implies that $\overline{A}(\overline{\phi})$ is therefore cocartesian in \mathbf{RSI}^\times , as required.

To prove that the square is cartesian it now suffices see that it induces equivalences on fibres over all $O \in \mathcal{O}$. But since \overline{A} preserves cocartesian morphisms, we have for $O \in \mathcal{O}$ a commutative square

$$\begin{array}{ccc} \mathcal{C}_O^\otimes & \longrightarrow & \mathbf{RSI}_{A(O)}^\times \\ \downarrow \sim & & \downarrow \sim \\ \prod_i \mathcal{C}_{O_i} & \longrightarrow & \prod_i \mathbf{RSI}_{M(O_i)} \end{array}$$

where the vertical maps are equivalences since \mathcal{C}^\otimes is \mathcal{O} -monoidal and \mathbf{RSI}^\times is $\mathbf{Cat}_\infty^\times$ -monoidal, while the bottom horizontal map is an equivalence since \mathcal{C}^\otimes is pulled back from \mathbf{RSI} along M . \square

Definition 6.14. Suppose \mathcal{C}^\otimes is an \mathcal{O} -monoidal ∞ -category. By Proposition 5.1 this corresponds to an \mathcal{O} -algebra $C: \mathcal{O} \rightarrow \mathbf{Cat}_\infty^\times$. The (contravariant) Day convolution of \mathcal{C}^\otimes is the \mathcal{O} -monoidal ∞ -category given by the pullback

$$\begin{array}{ccc} \mathbf{P}_\mathcal{O}(\mathcal{C})^\otimes & \longrightarrow & \mathbf{RFib}^\times \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{C} & \mathbf{Cat}_\infty^\times. \end{array}$$

Remark 6.15. Using Remark 6.11 we can describe the cocartesian morphisms in $\mathbf{P}_\mathcal{O}(\mathcal{C})^\otimes$: given $\phi: O \rightarrow E$ in \mathcal{O} active with $E \in \mathcal{O}^{\text{el}}$ and $F_i \in \mathbf{P}(\mathcal{C}_{O_i})$, we take the left Kan extension along

$$\phi_!^{\text{op}}: \prod_i \mathcal{C}_{O_i}^{\text{op}} \simeq (\mathcal{C}^\otimes)^{\text{op}} \rightarrow \mathcal{C}_E^{\text{op}}$$

of the product

$$\prod_i F_i: \prod_i \mathcal{C}_{O_i}^{\text{op}} \xrightarrow{(F_i)} \prod_i \mathcal{S} \xrightarrow{\times} \mathcal{S}.$$

Proposition 6.16. *The Yoneda embedding gives a natural \mathcal{O} -monoidal functor $\mathcal{C}^\otimes \rightarrow P_{\mathcal{O}}(\mathcal{C})^\otimes$.*

Proof. By Corollary 6.10 the inclusion $\text{RSI} \hookrightarrow \text{RFib}$ induces a Cat_∞^\times -monoidal functor $\text{RSI}^\times \hookrightarrow \text{RFib}^\times$. Pulling this back along the \mathcal{O} -algebra $C: \mathcal{O} \rightarrow \text{Cat}_\infty^\times$ corresponding to \mathcal{C}^\otimes we get using Proposition 6.12 an \mathcal{O} -monoidal functor $\mathcal{C}^\otimes \rightarrow P_{\mathcal{O}}(\mathcal{C})^\otimes$. Over $E \in \mathcal{O}^{\text{el}}$ it follows from Corollary 6.4 that this functor is given by the Yoneda embedding $\mathcal{C}_E \hookrightarrow P(\mathcal{C}_E)$. \square

Notation 6.17. Suppose $\mathcal{C}^\otimes \rightarrow \mathcal{O}$ is an \mathcal{O} -monoidal ∞ -category, corresponding to an \mathcal{O} -monoid $M: \mathcal{O} \rightarrow \text{Cat}_\infty$. Since $(-)^{\text{op}}$ is an automorphism of Cat_∞ , the composite

$$\mathcal{O} \xrightarrow{M} \text{Cat}_\infty \xrightarrow{(-)^{\text{op}}} \text{Cat}_\infty$$

is also an \mathcal{O} -monoid. We write $\mathcal{C}^{\text{op},\otimes} \rightarrow \mathcal{O}$ for the corresponding cocartesian fibration. (Note that if we write $\mathcal{C}_\otimes \rightarrow \mathcal{O}^{\text{op}}$ for the *cartesian* fibration for M , then $\mathcal{C}^{\text{op},\otimes} \simeq (\mathcal{C}_\otimes)^{\text{op}}$.) To avoid confusion we will avoid the notation \mathcal{C}^{op} for the pullback of $\mathcal{C}^{\text{op},\otimes}$ to \mathcal{O}^{el} (since this is not the opposite ∞ -category of \mathcal{C} , but the fibrewise opposite) and instead write $\mathcal{C}_{/\text{el}}^{\text{op},\otimes}$.

We can now prove the universal property for mapping into the Day convolution:

Proposition 6.18. *Suppose $\mathcal{C}^\otimes \rightarrow \mathcal{O}$ is an \mathcal{O} -monoidal ∞ -category, then there is an equivalence*

$$\text{Alg}_{\mathcal{O}}(P_{\mathcal{O}}(\mathcal{C})^\otimes) \simeq \text{Mon}_{\mathcal{C}^{\text{op},\otimes}}(\mathcal{S}).$$

Proof. Let $M: \mathcal{O} \rightarrow \text{Cat}_\infty$ be the monoid corresponding to \mathcal{C}^\otimes , and let $A: \mathcal{O} \rightarrow \text{Cat}_\infty^\times$ be the corresponding algebra. Using the definition of $P_{\mathcal{O}}(\mathcal{C})^\otimes$ as a pullback we then have natural equivalences

$$\begin{aligned} \text{Alg}_{\mathcal{O}}(P_{\mathcal{O}}(\mathcal{C})^\otimes) &\simeq \text{Alg}_{\mathcal{O}/\text{Cat}_\infty^\times}(\text{RFib}^\times) && \text{(by pullback)} \\ &\simeq \{A\} \times_{\text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\text{Cat}_\infty^\times)} \text{Alg}_{\mathcal{O}/\mathbb{F}_*}(\text{RFib}^\times) && \text{(by Proposition 6.7(iii))} \\ &\simeq \{M\} \times_{\text{Mon}_{\mathcal{O}}(\text{Cat}_\infty)} \text{Mon}_{\mathcal{O}}(\text{RFib}), && \text{(by Proposition 5.1)} \end{aligned}$$

where the right-hand side is the full subcategory of $\text{Fun}_{/\text{Cat}_\infty}(\mathcal{O}, \text{RFib})$ spanned by the monoids. Since the cartesian fibration $\text{ev}_1: \text{RFib} \rightarrow \text{Cat}_\infty$ corresponds to the functor $\text{Fun}((-)^{\text{op}}, \mathcal{S})$ by [GHN17, Proposition 7.3], we have an equivalence

$$\text{Fun}_{/\text{Cat}_\infty}(\mathcal{O}, \text{RFib}) \simeq \text{Fun}(\mathcal{O} \times_{\text{Cat}_\infty} \mathcal{E}, \mathcal{S}),$$

where $\mathcal{E} \rightarrow \text{Cat}_\infty$ is the cocartesian fibration corresponding to the functor $\text{op}: \text{Cat}_\infty \rightarrow \text{Cat}_\infty$. By definition, the pullback $\mathcal{O} \times_{\text{Cat}_\infty} \mathcal{E}$ is precisely $\mathcal{C}^{\text{op},\otimes} \rightarrow \mathcal{O}$. We have therefore identified $\text{Alg}_{\mathcal{O}}(P_{\mathcal{O}}(\mathcal{C})^\otimes)$ with a full subcategory of $\text{Fun}(\mathcal{C}^{\text{op},\otimes}, \mathcal{S})$, and we need to check that this is precisely the full subcategory of $\mathcal{C}^{\text{op},\otimes}$ -monoids.

Under the equivalence of [GHN17, Proposition 7.3], a functor $\phi: \mathcal{C}^{\text{op},\otimes} \rightarrow \mathcal{S}$ corresponds to the functor $\Phi: \mathcal{O} \rightarrow \text{RFib}$ that takes $O \in \mathcal{O}$ to the right fibration for the presheaf $\phi|_{\mathcal{C}_O^\otimes}: (\mathcal{C}_O^\otimes)^{\text{op}} \rightarrow \mathcal{S}$. We then observe that this gives an \mathcal{O} -monoid in RFib precisely when ϕ is a $\mathcal{C}^{\text{op},\otimes}$ -monoid, using the commutative squares

$$\begin{array}{ccc} \Phi(O) & \longrightarrow & \prod_i \Phi(O_i) \\ \downarrow & & \downarrow \\ \mathcal{C}_O^\otimes & \xrightarrow{\sim} & \prod_i \mathcal{C}_{O_i}^\otimes. \end{array}$$

Here the vertical maps are right fibrations, so that the top horizontal map is an equivalence if and only if the square is cartesian, which is equivalent to the map on fibres being an equivalence for every $O(C_i) \in \mathcal{C}_O^\otimes$. The map on fibres we can identify with

$$\phi(O(C_i)) \rightarrow \prod_i \phi(C_i),$$

so all of these are equivalences precisely when ϕ is a $\mathcal{C}^{\text{op},\otimes}$ -monoid. \square

Corollary 6.19. *Let \mathcal{C}^\otimes be a small \mathcal{O} -monoidal ∞ -category. We say a $\mathcal{C}^{\text{op},\otimes}$ -monoid*

$$M: \mathcal{C}^{\text{op},\otimes} \rightarrow \mathcal{S}$$

is representable if for every $E \in \mathcal{O}^{\text{el}}$, the restriction

$$M|_{\mathcal{C}_E^{\text{op}}} : \mathcal{C}_E^{\text{op}} \simeq (\mathcal{C}^{\text{op},\otimes})_E \rightarrow \mathcal{S}$$

is a representable presheaf. There is a natural equivalence

$$\text{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \text{Mon}_{\mathcal{C}^{\text{op},\otimes}}^{\text{rep}}(\mathcal{S}).$$

Proof. The \mathcal{O} -monoidal inclusion $\mathcal{C}^\otimes \hookrightarrow \text{P}_{\mathcal{O}}(\mathcal{C})^\otimes$ identifies $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ with the full subcategory of $\text{Alg}_{\mathcal{O}}(\text{P}_{\mathcal{O}}(\mathcal{C}))$ spanned by the \mathcal{O} -algebras A such that $A(E) \in \text{P}(\mathcal{C}_E)$ is representable for every $E \in \mathcal{O}^{\text{el}}$. This full subcategory is identified with $\text{Mon}_{\mathcal{C}^{\text{op},\otimes}}^{\text{rep}}(\mathcal{S})$ under the equivalence of Proposition 6.18. \square

Lemma 6.20. *Suppose $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns and $\mathcal{C}^\otimes \rightarrow \mathcal{P}$ is a \mathcal{P} -monoidal ∞ -category. Then there is a natural equivalence*

$$\text{P}_{\mathcal{O}}(f^*\mathcal{C})^\otimes \simeq f^*\text{P}_{\mathcal{P}}(\mathcal{C})^\otimes$$

of \mathcal{O} -monoidal ∞ -categories.

Proof. By definition we have a commutative diagram

$$\begin{array}{ccccc} f^*\text{P}_{\mathcal{P}}(\mathcal{C})^\otimes & \longrightarrow & \text{P}_{\mathcal{P}}(\mathcal{C})^\otimes & \longrightarrow & \text{RFib}^\times \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P} & \xrightarrow{C} & \text{Cat}_\infty^\times, \end{array}$$

where C is the algebra corresponding to the \mathcal{P} -monoidal ∞ -category \mathcal{C}^\otimes , and both squares are cartesian. Then the composite square is also cartesian. On the other hand, by Remark 5.16 the composite $C \circ f$ is the \mathcal{O} -algebra corresponding to $f^*\mathcal{C}^\otimes$, and so the pullback of RFib^\times along $C \circ f$ is by definition $\text{P}_{\mathcal{O}}(f^*\mathcal{C})^\otimes$. \square

Corollary 6.21. *Let $f: \mathcal{O} \rightarrow \mathcal{P}$ be a morphism of cartesian pattern. We have natural equivalences of ∞ -categories*

$$\text{Alg}_{\mathcal{O}/\mathcal{P}}(\text{P}_{\mathcal{P}}(\mathcal{C})^\otimes) \simeq \text{Mon}_{f^*\mathcal{C}^{\text{op},\otimes}}(\mathcal{S}).$$

Proof. By Lemma 6.20 pulling back along f gives natural equivalences

$$\text{Alg}_{\mathcal{O}/\mathcal{P}}(\text{P}_{\mathcal{P}}(\mathcal{C})^\otimes) \simeq \text{Alg}_{\mathcal{O}}(f^*\text{P}_{\mathcal{P}}(\mathcal{C})^\otimes) \simeq \text{Alg}_{\mathcal{O}}(\text{P}_{\mathcal{O}}(f^*\mathcal{C})^\otimes).$$

Since we also have $(f^*\mathcal{C})^{\text{op},\otimes} \simeq f^*(\mathcal{C}^{\text{op},\otimes})$, the result now follows from Proposition 6.18. \square

Remark 6.22. As a special case, for the inclusion $\mathcal{O}^{\text{int}} \rightarrow \mathcal{O}$ we get a commutative square of equivalences

$$\begin{array}{ccc} \text{Alg}_{\mathcal{O}^{\text{int}}/\mathcal{O}}(\text{P}_{\mathcal{O}}(\mathcal{C})^\otimes) & \xrightarrow{\sim} & \text{Mon}_{\mathcal{C}^{\text{op},\otimes}/\mathcal{O}^{\text{int}}}(\mathcal{S}) \\ \downarrow \sim & & \downarrow \sim \\ \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \text{P}_{\mathcal{O}}(\mathcal{C})) & \xrightarrow{\sim} & \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{C}^{\text{op},\otimes}, \mathcal{S}) \end{array}$$

where the top horizontal map is an equivalence by Corollary 6.21, the right vertical map by Remark 2.12, the left vertical map by Lemma 4.14, and the bottom horizontal map by a trivial version of Corollary 6.21 or by [GHN17, Proposition 7.3].

We will next prove that every \mathcal{O} -monoidal functor from a small \mathcal{O} -monoidal ∞ -category extends to the Day convolution, provided the target is compatible with small colimits in the following sense:

Definition 6.23. If \mathcal{K} is some class of ∞ -categories, we say that an \mathcal{O} -monoidal ∞ -category $\mathcal{C}^\otimes \rightarrow \mathcal{O}$ is *compatible with \mathcal{K} -colimits* if the ∞ -categories \mathcal{C}_E for $E \in \mathcal{O}^{\text{el}}$ have \mathcal{K} -shaped colimits, and for every active map $\phi: O \rightarrow E$ in \mathcal{O} with $E \in \mathcal{O}^{\text{el}}$, the functor

$$\phi_!: \prod_i \mathcal{C}_{O_i} \simeq \mathcal{C}_O^\otimes \rightarrow \mathcal{C}_E$$

preserves \mathcal{K} -shaped colimits in each variable. If \mathcal{K} is the class of all small ∞ -categories, we say that \mathcal{C}^\otimes is *compatible with (small) colimits* or *cocontinuously \mathcal{O} -monoidal*.

Remark 6.24. For any small \mathcal{O} -monoidal ∞ -category \mathcal{C}^\otimes , the Day convolution $P_{\mathcal{O}}(\mathcal{C}^\otimes)$ is compatible with small colimits. This is easy to see using the description of cocartesian morphisms in terms of products and left Kan extensions, since products in \mathcal{S} preserve colimits in each variable.

Definition 6.25. Suppose \mathcal{C}^\otimes and \mathcal{D}^\otimes are \mathcal{O} -monoidal ∞ -categories that are compatible with small colimits. We say that an \mathcal{O} -monoidal functor $F: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ is *cocontinuous* if the underlying functors $F_E: \mathcal{C}_E \rightarrow \mathcal{D}_E$ preserve small colimits for $E \in \mathcal{O}^{\text{el}}$.

Proposition 6.26. *Let \mathcal{C}^\otimes be a small \mathcal{O} -monoidal ∞ -category and \mathcal{V}^\otimes be an \mathcal{O} -monoidal ∞ -category compatible with small colimits. Then every \mathcal{O} -monoidal functor $F: \mathcal{C}^\otimes \rightarrow \mathcal{V}^\otimes$ induces a cocontinuous \mathcal{O} -monoidal functor*

$$F_!: P_{\mathcal{O}}(\mathcal{C}^\otimes) \rightarrow \mathcal{V}^\otimes$$

such that the composite

$$\mathcal{C}^\otimes \xrightarrow{y^\otimes} P_{\mathcal{O}}(\mathcal{C}^\otimes) \xrightarrow{F_!} \mathcal{V}^\otimes$$

is equivalent to F , and $F_{!,E}: P(\mathcal{C}_E) \rightarrow \mathcal{V}_E$ for $E \in \mathcal{O}^{\text{el}}$ is the unique cocontinuous functor extending F_E along the Yoneda embedding of \mathcal{C}_E . If in addition \mathcal{V}^\otimes is locally small, then $F_!$ has a lax \mathcal{O} -monoidal right adjoint F^* , given over $E \in \mathcal{O}^{\text{el}}$ by the restricted Yoneda embedding

$$F_E^*: \mathcal{V}_E \rightarrow P(\mathcal{V}_E) \rightarrow P(\mathcal{C}_E).$$

Proof. The \mathcal{O} -monoidal functor $F: \mathcal{C}^\otimes \rightarrow \mathcal{V}^\otimes$ corresponds under the equivalence of Proposition 5.1 to a morphism of \mathcal{O} -algebras in $\widehat{\text{Cat}}_\infty^\times$, i.e. a natural transformation $\phi: \mathcal{O} \times \Delta^1 \rightarrow \widehat{\text{Cat}}_\infty^\times$ over \mathbb{F}_* . We can pull back the cocartesian fibration $\widehat{\text{RFib}}^\times \rightarrow \widehat{\text{Cat}}_\infty^\times$ along this to obtain a cocartesian fibration $\mathcal{E}_+ \rightarrow \mathcal{O} \times \Delta^1$. Let \mathcal{E} be the full subcategory of \mathcal{E}_+ containing those objects over 0 that lie in $P_{\mathcal{O}}(\mathcal{C}^\otimes)$ and those objects over 1 that lie in \mathcal{V}^\otimes . We claim that the restricted projection $\mathcal{E} \rightarrow \mathcal{O} \times \Delta^1$ is again cocartesian (but the inclusion into \mathcal{E}_+ does not preserve all cocartesian morphisms). Given an object $\Phi \in P(\mathcal{C}_E)$, which we can write as a small colimit $\text{colim}_{x \in \mathcal{J}} y(\phi(x))$ of representable presheaves, the cocartesian morphism in \mathcal{E}_+ over $(\text{id}_E, 0 \rightarrow 1)$ takes $\Phi \in P(\mathcal{C}_E)$ to the colimit $\text{colim}_{x \in \mathcal{J}} y(F\phi(x))$ computed in (large) presheaves on \mathcal{V}_E . Considering maps in $\widehat{P}(\mathcal{V}_E)$ from this to representable presheaves, we see there is an initial one, given by the same colimit computed in the ∞ -category \mathcal{V}_E . This gives a cocartesian morphism in \mathcal{E} over $(\text{id}_E, 0 \rightarrow 1)$, and combining this observation with the compatibility of \mathcal{V}^\otimes with small colimits we see easily that \mathcal{E} is a cocartesian fibration. Unstraightening this over Δ^1 we get a commutative diagram

$$\begin{array}{ccc} P_{\mathcal{O}}(\mathcal{C}^\otimes) & \xrightarrow{F_!} & \mathcal{V}^\otimes \\ & \searrow & \swarrow \\ & \mathcal{O} & \end{array}$$

where $F_!$ preserves cocartesian morphisms, as required. To get the right adjoint we apply the criterion of [HMS19, Lemma A.1.10] to see that the composite $\mathcal{E} \rightarrow \mathcal{O} \times \Delta^1 \rightarrow \Delta^1$ is also a cartesian fibration, and the cartesian morphisms lie over equivalences on \mathcal{O} . Since we already know that the composition $\mathcal{E} \rightarrow \mathcal{O}$ is a cocartesian fibration, the only thing to check is that for every $O \in \mathcal{O}$ the cocartesian fibration $\mathcal{E}_O \rightarrow \Delta^1$ is a cartesian fibration. By identifying this cocartesian fibration with the functor $P_{\mathcal{O}}(\mathcal{C}^\otimes)_O \rightarrow \mathcal{V}_O^\otimes$, it suffices to show that it has a right adjoint. Since this functor is equivalent to $\prod_i P(\mathcal{C}_{O_i}) \rightarrow \prod_i \mathcal{V}_{O_i}$ and products of adjoints are adjoints, we only need to see that

each component $F_{O_i, !}: \mathcal{P}(\mathcal{C}_{O_i}) \rightarrow \mathcal{V}_{O_i}$ has a right adjoint. If \mathcal{V}_{O_i} is locally small, then $F_{O_i, !}$ has a right adjoint $F_{O_i}^*: \mathcal{V}_{O_i} \rightarrow \mathcal{P}(\mathcal{V}_{O_i}) \rightarrow \mathcal{P}(\mathcal{C}_{O_i})$ given by the composite of the Yoneda embedding and the precomposition with F_{O_i} . This gives a commutative diagram

$$\begin{array}{ccc} \mathcal{V}^\otimes & \xrightarrow{F^*} & \mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes \\ & \searrow & \swarrow \\ & \mathcal{O}, & \end{array}$$

and it only remains to show that F^* preserves inert morphisms. Given an inert morphism $\phi: O \rightarrow O'$ we want to show that the canonical natural transformation

$$F_{O'}^* \phi! \rightarrow \phi! F_O^*,$$

which arises as the mate transformation of the square

$$\begin{array}{ccc} \mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes & \xrightarrow{F_{O, !}} & \mathcal{V}_\mathcal{O}^\otimes \\ \phi! \downarrow & & \downarrow \phi! \\ \mathcal{P}_{\mathcal{O}'}(\mathcal{C})^\otimes & \xrightarrow{F_{O', !}} & \mathcal{V}_{\mathcal{O}'}^\otimes, \end{array}$$

is an equivalence. We can identify this with the square

$$\begin{array}{ccc} \prod_i \mathcal{P}(\mathcal{C}_{O_i}) & \xrightarrow{\prod_i F_{O_i, !}} & \prod_i \mathcal{V}_{O_i} \\ \downarrow & & \downarrow \\ \prod_j \mathcal{P}(\mathcal{C}_{O'_j}) & \xrightarrow{\prod_j F_{O'_j, !}} & \prod_j \mathcal{V}_{O'_j}, \end{array}$$

where the vertical maps are given by projections to the same subset of factors in the product. It is then clear that the mate square also commutes, since the right adjoint of $\prod_i F_{i, O_i}$ is the product $\prod_i F_{O_i}^*$. \square

Remark 6.27. The extension of $F: \mathcal{C}^\otimes \rightarrow \mathcal{V}^\otimes$ to $\mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes$ is in fact unique. Since we do not need this universal property, we will only give a sketch of the argument: Given a cocontinuous \mathcal{O} -monoidal functor $\Phi: \mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes \rightarrow \mathcal{V}^\otimes$, we can compose this with the Yoneda embedding $\mathcal{C}^\otimes \rightarrow \mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes$; this data we can interpret as a 2-simplex in the ∞ -category of large \mathcal{O} -monoidal ∞ -categories, which corresponds to a 2-simplex of algebras

$$\mathcal{O} \times \Delta^2 \rightarrow \widehat{\text{Cat}}_\infty^\times.$$

We can pull back $\widehat{\text{RFib}}^\times$ along this to obtain a cocartesian fibration $\mathcal{E}_+ \rightarrow \mathcal{O} \times \Delta^2$. Then we consider the full subcategory \mathcal{E} whose objects over 0 are those in $\mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes$, whose objects over 1 are those in $\mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes$ (but now viewed inside $\widehat{\mathcal{P}}_\mathcal{O}(\mathcal{P}_\mathcal{O}(\mathcal{C}))^\otimes$), and whose objects over 2 are those in \mathcal{V}^\otimes . As before, we can check that $\mathcal{E} \rightarrow \mathcal{O} \times \Delta^2$ is a cocartesian fibration; its fibre over $\Delta^{\{0,2\}}$ corresponds to the extension $(\Phi|_{\mathcal{C}^\otimes})!$, its fibre over $\Delta^{\{1,2\}}$ to Φ , and using the universal property for mapping out of presheaves its fibre over $\Delta^{\{0,1\}}$ is an \mathcal{O} -monoidal equivalence between two versions of $\mathcal{P}_\mathcal{O}(\mathcal{C})^\otimes$.

7. \mathcal{O} -MONOIDAL LOCALIZATIONS AND PRESENTABILITY OF ALGEBRAS

In this section we first discuss \mathcal{O} -monoidal localizations and then consider *presentably* \mathcal{O} -monoidal ∞ -categories, in the following sense:

Definition 7.1. We say an \mathcal{O} -monoidal ∞ -category \mathcal{V}^\otimes is *presentably \mathcal{O} -monoidal* if it is compatible with small colimits and the ∞ -categories \mathcal{V}_E for $E \in \mathcal{O}^{\text{el}}$ are all presentable.

Our main result is that every presentably \mathcal{O} -monoidal ∞ -category is an \mathcal{O} -monoidal localization of a Day convolution \mathcal{O} -monoidal structure on a small full subcategory. We apply this to show that if \mathcal{V} is presentably \mathcal{O} -monoidal, then the ∞ -category $\text{Alg}_{\mathcal{O}}(\mathcal{V})$ is presentable.

We begin by proving a general existence result for \mathcal{O} -monoidal adjunctions, in the following sense:

Definition 7.2. Consider a commutative triangle

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ & \searrow p & \swarrow q \\ & & \mathcal{B}. \end{array}$$

We say that G has a left adjoint *relative to* \mathcal{B} if G has a left adjoint $F: \mathcal{D} \rightarrow \mathcal{C}$ and the unit map $d \rightarrow GFd$ maps to an equivalence in \mathcal{B} for all $d \in \mathcal{D}$.

Remark 7.3. If F is a left adjoint relative to \mathcal{B} as above, then it follows that $pF(d) \simeq qGF(d) \simeq q(d)$, so that F is a functor over \mathcal{B} . Moreover, the counit map $FGc \rightarrow c$ also lies over an equivalence in \mathcal{B} : applying p is the same as applying qG , and the map $qGFGc \rightarrow qGc$ is an equivalence since the composite

$$qGc \rightarrow qGFGc \rightarrow qGc$$

is the identity by one of the adjunction equivalences, and the first unit map is an equivalence by assumption. Thus G has a left adjoint relative to \mathcal{B} if and only if it has a left adjoint in the $(\infty, 2)$ -category of ∞ -categories over \mathcal{B} .

Remark 7.4. If F is a left adjoint relative to \mathcal{B} as above, then F takes any q -cocartesian morphism in \mathcal{D} to a p -cocartesian morphism in \mathcal{C} : If $\delta: d \rightarrow d'$ is a morphism in \mathcal{D} , then the relative adjunction data lets us identify the following pair of commutative squares:

$$\begin{array}{ccc} \text{Map}_{\mathcal{C}}(Fd', c) & \xrightarrow{(F\delta)^*} & \text{Map}_{\mathcal{C}}(Fd, c) & & \text{Map}_{\mathcal{D}}(d', Gc) & \xrightarrow{\delta^*} & \text{Map}_{\mathcal{D}}(d, Gc) \\ \downarrow p & & \downarrow p & & \downarrow q & & \downarrow q \\ \text{Map}_{\mathcal{B}}(pFd', pc) & \xrightarrow{(pF\delta)^*} & \text{Map}_{\mathcal{B}}(pFd, pc) & & \text{Map}_{\mathcal{B}}(qd', qGc) & \xrightarrow{(q\delta)^*} & \text{Map}_{\mathcal{B}}(qd, qGc). \end{array}$$

If δ is q -cocartesian then the second square is cartesian for any $c \in \mathcal{C}$, hence so is the first square, which says precisely that $F\delta$ is p -cocartesian.

Definition 7.5. Suppose $G: \mathcal{C}^{\otimes} \rightarrow \mathcal{D}^{\otimes}$ is a lax \mathcal{O} -monoidal functor. We say that G has an \mathcal{O} -monoidal left adjoint if it has a left adjoint relative to \mathcal{O} ; this is then automatically \mathcal{O} -monoidal by Remark 7.4.

Proposition 7.6. Consider a commutative triangle

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{G} & \mathcal{D}^{\otimes} \\ & \searrow p & \swarrow q \\ & & \mathcal{O}, \end{array}$$

where p and q are \mathcal{O} -monoidal ∞ -categories and G is lax \mathcal{O} -monoidal. Suppose

- (1) $G_E: \mathcal{C}_E \rightarrow \mathcal{D}_E$ admits a left adjoint F_E for all $E \in \mathcal{O}^{\text{el}}$,
- (2) for every active map $\phi: O \rightarrow E$ in \mathcal{O} with $E \in \mathcal{O}^{\text{el}}$, the natural transformation

$$F_E \circ \phi_!^{\mathcal{C}} \rightarrow \phi_!^{\mathcal{D}} \circ \prod_i F_{O_i}$$

of functors $\prod_i \mathcal{C}_{O_i} \rightarrow \mathcal{D}_E$, is an equivalence.

Then the functor G admits an \mathcal{O} -monoidal left adjoint F .

Proof. This is the \mathcal{O} -monoidal analogue of [Lur17, Corollary 7.3.2.12] and follows from the criterion for existence of relative left adjoints in [Lur17, Proposition 7.3.2.11]. To apply this we must show the following conditions hold:

- (a) For every $O \in \mathcal{O}$, the functor $G_O: \mathcal{C}_O^\otimes \rightarrow \mathcal{D}_O^\otimes$ admits a left adjoint F_O .
 (b) for every morphism $\phi: O \rightarrow O'$ in \mathcal{O} the natural transformation

$$F_{O'} \circ \phi_!^{\mathcal{C}} \rightarrow \phi_!^{\mathcal{D}} \circ F_O$$

is an equivalence.

Condition (a) follows from (1) since the functor G_O is equivalent to the product $\prod_i G_{O_i}: \prod_i \mathcal{C}_{O_i} \rightarrow \prod_i \mathcal{D}_{O_i}$, which has left adjoint $\prod_i F_{O_i}$. Condition (b) is obvious for inert maps ϕ (since then $\phi_!^{\mathcal{C}}$ and $\phi_!^{\mathcal{D}}$ are both projections unto the same collection of factors in a product), so using the factorization system it is enough to consider ϕ active. But then we can write $\phi_!^{\mathcal{C}}$ as the product $\prod_i \phi_{i,!}^{\mathcal{C}}$ where $\phi_i: O_i \rightsquigarrow O'_i$ comes from the factorization

$$\begin{array}{ccc} O & \xrightarrow{\phi} & O' \\ \downarrow & & \downarrow \rho_i^{O'} \\ O_i & \xrightarrow{\phi_i} & O'_i, \end{array}$$

where we know the natural transformation for ϕ_i is an equivalence by assumption (2). The map in question is therefore a product of maps we know are equivalences. \square

Remark 7.7. For any morphism of cartesian patterns $f: \mathcal{P} \rightarrow \mathcal{O}$, a relative adjunction between \mathcal{O} -monoidal ∞ -categories as in Proposition 7.6 induces via composition an adjunction on ∞ -categories of algebras,

$$F_*: \text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{D}) \rightleftarrows \text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C}) : G_*.$$

This follows from the 2-functoriality of $\text{Alg}_{\mathcal{P}/\mathcal{O}}(-)$ as discussed in Remark 4.11.

Definition 7.8. An \mathcal{O} -monoidal localization is an \mathcal{O} -monoidal functor $\mathcal{V}^\otimes \rightarrow \mathcal{U}^\otimes$ that has a right adjoint relative to \mathcal{O} which is fully faithful.

Remark 7.9. Suppose $L: \mathcal{V}^\otimes \rightarrow \mathcal{U}^\otimes$ is an \mathcal{O} -monoidal localization with right adjoint i relative to \mathcal{O} . Then i is automatically a lax \mathcal{O} -monoidal functor. Indeed, more generally if $F: \mathcal{V}^\otimes \rightarrow \mathcal{U}^\otimes$ is any \mathcal{O} -monoidal functor that has a right adjoint G relative to \mathcal{O} , then G is lax \mathcal{O} -monoidal. This is because when $\phi: O \rightarrow O'$ is inert, the cocartesian pushforward functor $\phi_!: \mathcal{V}_O^\otimes \rightarrow \mathcal{V}_{O'}^\otimes$ is given by projecting to some subset of the factors in $\mathcal{V}_O^\otimes \simeq \prod_{i \in |O|} \mathcal{V}_{O_i}$. The right adjoint G_O of $F_O \simeq \prod_i F_{O_i}$ is given by the product $\prod_i G_{O_i}$, and so the canonical map

$$\phi_! G_O \rightarrow G_{O'} \phi_!$$

is an equivalence in this case, since both sides are identified with $\prod_{j \in |O'|} G_{O'_j}$.

Remark 7.10. Let $L: \mathcal{V}^\otimes \rightarrow \mathcal{U}^\otimes$ be an \mathcal{O} -monoidal localization with right adjoint $i: \mathcal{U}^\otimes \hookrightarrow \mathcal{V}^\otimes$. For any morphism of cartesian patterns $f: \mathcal{P} \rightarrow \mathcal{O}$, we get an induced localization of ∞ -categories of algebras: as in Remark 7.7 we have an induced adjunction

$$L_*: \text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{V}) \rightleftarrows \text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{U}) : i_*$$

given by composition with L and i , and the equivalence $L_* i_* \simeq (Li)_* \simeq \text{id}$ implies that i_* is fully faithful. Since a \mathcal{P} -algebra $A: \mathcal{O} \rightarrow \mathcal{V}^\otimes$ factors through the full subcategory \mathcal{U}^\otimes if and only if $A(E) \in \mathcal{U}_E$ for every $E \in \mathcal{O}^{\text{el}}$, we see that $\text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{U})$ is the full subcategory of $\text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{V})$ spanned by the algebras with this property. We can interpret this as the commutative square

$$\begin{array}{ccc} \text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{U}) & \hookrightarrow & \text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \text{Fun}_{\mathcal{P}^{\text{el}}/\mathcal{O}^{\text{el}}}(\mathcal{U}) & \hookrightarrow & \text{Fun}_{\mathcal{P}^{\text{el}}/\mathcal{O}^{\text{el}}}(\mathcal{V}) \end{array}$$

being cartesian.

Notation 7.11. Suppose \mathcal{V}^\otimes is an \mathcal{O} -monoidal ∞ -category. Given a collection of full subcategories $\mathcal{U}_E \subseteq \mathcal{V}_E$ for $E \in \mathcal{O}^{\text{el}}$, the full subcategory $\mathcal{U}^\otimes \subseteq \mathcal{V}^\otimes$ generated by $(\mathcal{U}_E)_{E \in \mathcal{O}^{\text{el}}}$ is that spanned by the objects over $O \in \mathcal{O}$ that lie in the full subcategory of \mathcal{V}_O^\otimes that is identified with $\prod_i \mathcal{U}_{O_i}$ under the equivalence $\mathcal{V}_O^\otimes \simeq \prod_i \mathcal{V}_{O_i}$. (Note that in general \mathcal{U}^\otimes is not an \mathcal{O} -monoidal ∞ -category, though it is an \mathcal{O} - ∞ -operad.)

Corollary 7.12. Let \mathcal{V}^\otimes be an \mathcal{O} -monoidal ∞ -category, and suppose given full subcategories $\mathcal{U}_E \subseteq \mathcal{V}_E$ for all $E \in \mathcal{O}^{\text{el}}$ such that

- (1) each inclusion $i_E: \mathcal{U}_E \hookrightarrow \mathcal{V}_E$ has a left adjoint L_E ,
- (2) for every active morphism $\phi: O \rightsquigarrow E$ in \mathcal{O} with $E \in \mathcal{O}^{\text{el}}$, the natural map

$$\phi_!^\vee O(X_1, \dots, X_{O_n}) \rightarrow \phi_!^\vee O(L_{O_1} X_1, \dots, L_{O_n} X_n)$$

is taken to an equivalence by L_E .

If $\mathcal{U}^\otimes \subseteq \mathcal{V}^\otimes$ is the full subcategory generated by $(\mathcal{U}_E)_{E \in \mathcal{O}^{\text{el}}}$, then:

- (i) The restricted functor $\mathcal{U}^\otimes \rightarrow \mathcal{O}$ is an \mathcal{O} -monoidal ∞ -category.
- (ii) The inclusion $i: \mathcal{U}^\otimes \hookrightarrow \mathcal{V}^\otimes$ is lax \mathcal{O} -monoidal.
- (iii) The lax monoidal functor i exhibits \mathcal{U}^\otimes as an \mathcal{O} -monoidal localization. In other words, the functor i has a left adjoint $L: \mathcal{V}^\otimes \rightarrow \mathcal{U}^\otimes$ relative to \mathcal{O} (which is then automatically an \mathcal{O} -monoidal functor).

Proof. For $O \in \mathcal{O}$, let $L_O: \mathcal{V}_O^\otimes \rightarrow \mathcal{U}_O^\otimes$ denote the functor corresponding to the product $\prod_i L_{O_i}$, which is left adjoint to the inclusion i_O . We claim that for $\phi: O \rightarrow O'$ in \mathcal{O} and $O(U_i) \in \mathcal{U}_{O_i}^\otimes$, the morphism $O(U_i) \rightarrow L_{O'} \phi_!^\vee O(U_i)$ is cocartesian. It is easy to see that it is locally cocartesian, and then condition (2) implies that these locally cocartesian morphisms compose. This proves (i). If ϕ is inert, then we do not have to apply $L_{O'}$ (since $\phi_!^\vee$ just projects to some factors in a product), so the inclusion i preserves inert morphisms, which gives (ii). This means all the conditions of Proposition 7.6 hold for i , which gives (iii). \square

Our next goal is to show that every presentably \mathcal{O} -monoidal ∞ -category can be described as an \mathcal{O} -monoidal localization of a Day convolution. We start by briefly discussing the more general case of *accessibly* \mathcal{O} -monoidal ∞ -categories:

Definition 7.13. We say an \mathcal{O} -monoidal ∞ -category \mathcal{V}^\otimes is *accessibly* \mathcal{O} -monoidal if the ∞ -categories \mathcal{V}_E for $E \in \mathcal{O}^{\text{el}}$ are all accessible, and for every active map $\phi: O \rightarrow E$ with $E \in \mathcal{O}^{\text{el}}$, the functor $\phi_!^\vee: \prod_i \mathcal{V}_{O_i} \rightarrow \mathcal{V}_E$ is accessible. We say that \mathcal{O} is κ -*accessibly* \mathcal{O} -monoidal for some regular cardinal κ if the ∞ -categories \mathcal{V}_E are all κ -accessible and for every active map $\phi: O \rightarrow E$ with $E \in \mathcal{O}^{\text{el}}$, the functor $\phi_!^\vee: \prod_i \mathcal{V}_{O_i} \rightarrow \mathcal{V}_E$ preserves κ -filtered colimits. We say that \mathcal{V}^\otimes is κ -*presentably* \mathcal{O} -monoidal for some regular cardinal κ if \mathcal{V}^\otimes is both presentably and κ -accessibly \mathcal{O} -monoidal.

Remark 7.14. If \mathcal{V}^\otimes is accessibly (presentably) \mathcal{O} -monoidal, then we can always choose a regular cardinal κ such that \mathcal{V}^\otimes is κ -accessibly (κ -presentably) \mathcal{O} -monoidal.

Proposition 7.15. Suppose \mathcal{V}^\otimes is an accessibly \mathcal{O} -monoidal ∞ -category. Then there exists a regular cardinal κ such that \mathcal{V}^\otimes is κ -accessibly \mathcal{O} -monoidal, the full subcategory $\mathcal{V}^{\kappa, \otimes}$ generated by the collection $(\mathcal{V}_E^\kappa)_{E \in \mathcal{O}^{\text{el}}}$ of κ -compact objects is an \mathcal{O} -monoidal ∞ -category, and the inclusion $\mathcal{V}^{\kappa, \otimes} \hookrightarrow \mathcal{V}^\otimes$ is \mathcal{O} -monoidal.

Proof. We can choose a regular cardinal λ such that \mathcal{V}_E is λ -accessible for each $E \in \mathcal{O}^{\text{el}}$ and the functor $\phi_!^\vee: \prod_i \mathcal{V}_{O_i} \rightarrow \mathcal{V}_E$ preserves λ -filtered colimits in each variable. We can then choose a regular cardinal $\kappa \gg \lambda$ such that for every such active map ϕ , we have

$$\phi_!^\vee \left(\prod_i \mathcal{V}_{O_i}^\lambda \right) \subseteq \mathcal{V}_E^\kappa.$$

By [CH20, Lemma 2.6.11], any object of \mathcal{V}_E^κ is the colimit of a κ -small λ -filtered diagram in \mathcal{V}_E^λ . Since $\phi_!^\vee$ preserves λ -filtered colimits in each variable, for $O(v_i) \in \mathcal{V}_O^{\kappa, \otimes}$ we can write $\phi_!^\vee(O(v_i))$ as a

κ -small colimit of κ -compact objects, and hence this object is also κ -compact by [Lur09, Corollary 5.3.4.15]. The full subcategory $\mathcal{V}^{\kappa, \otimes}$ therefore inherits cocartesian morphisms from \mathcal{V}^{\otimes} , which means that it is an \mathcal{O} -monoidal ∞ -category and the inclusion into \mathcal{V}^{\otimes} is an \mathcal{O} -monoidal functor. \square

Corollary 7.16. *Suppose \mathcal{V}^{\otimes} is a presentably \mathcal{O} -monoidal ∞ -category. Then there exists a regular cardinal κ such that \mathcal{V}^{\otimes} is an \mathcal{O} -monoidal localization of $\mathcal{P}_{\mathcal{O}}(\mathcal{V}^{\kappa})^{\otimes}$.*

Proof. By Proposition 7.15 we can choose a regular cardinal κ such that the full subcategory $\mathcal{V}^{\kappa, \otimes}$ is an \mathcal{O} -monoidal ∞ -category and the inclusion $i: \mathcal{V}^{\kappa, \otimes} \hookrightarrow \mathcal{V}^{\otimes}$ is \mathcal{O} -monoidal. Since \mathcal{V}^{\otimes} is compatible with small colimits, by Proposition 6.26 there then exists a cocontinuous \mathcal{O} -monoidal functor

$$L: \mathcal{P}_{\mathcal{O}}(\mathcal{V}^{\kappa})^{\otimes} \rightarrow \mathcal{V}^{\otimes}$$

extending i along the Yoneda embedding $\mathcal{V}^{\kappa, \otimes} \hookrightarrow \mathcal{P}_{\mathcal{O}}(\mathcal{V}^{\kappa})^{\otimes}$, and this has a lax \mathcal{O} -monoidal right adjoint $R: \mathcal{V}^{\otimes} \rightarrow \mathcal{P}_{\mathcal{O}}(\mathcal{V}^{\kappa})^{\otimes}$. It remains to show that R is fully faithful. This amounts to showing that for $\phi: O \rightarrow E$ in \mathcal{O} active with $E \in \mathcal{O}^{\text{el}}$ and $O(V_i) \in \mathcal{V}_{\mathcal{O}}^{\otimes}$, the natural map

$$L_E \phi_1^{\mathcal{P}_{\mathcal{O}}(\mathcal{V}^{\kappa})} R_O O(V_i) \rightarrow \phi_1^{\mathcal{V}} O(V_i)$$

is an equivalence. This follows since it is true when $O(V_i)$ lies in $\mathcal{V}_{\mathcal{O}}^{\kappa, \otimes}$ and all the functors preserve κ -filtered colimits in each variable. (For R_O this is true since it is equivalent to the product of the restricted Yoneda embeddings $\mathcal{V}_{\mathcal{O}_i} \hookrightarrow \mathcal{P}(\mathcal{V}_{\mathcal{O}_i}^{\kappa})$, which tautologically preserve κ -filtered colimits.) \square

Remark 7.17. Corollary 7.16 says in particular that for any presentably \mathcal{O} -monoidal ∞ -category \mathcal{V}^{\otimes} there exists a small \mathcal{O} -monoidal ∞ -category \mathcal{C}^{\otimes} and an \mathcal{O} -monoidal localization

$$\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \xrightarrow{L} \mathcal{V}^{\otimes}.$$

Our next goal is to obtain another description of localizations of Day convolutions, in terms of localizing at classes of maps compatible with the \mathcal{O} -monoidal structure:

Notation 7.18. Let \mathcal{C} be a small ∞ -category and \mathbb{S} a set of maps in $\mathcal{P}(\mathcal{C})$. We write $\mathcal{P}_{\mathbb{S}}(\mathcal{C})$ for the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by the \mathbb{S} -local objects, i.e. the objects Φ such that $\text{Map}_{\mathcal{P}(\mathcal{C})}(-, \Phi)$ takes the morphisms in \mathbb{S} to equivalences.

Definition 7.19. Let \mathcal{C}^{\otimes} be a small \mathcal{O} -monoidal ∞ -category. A collection $\mathbb{S} = (\mathbb{S}_E)_{E \in \mathcal{O}^{\text{el}}}$ of sets of morphisms \mathbb{S}_E in \mathcal{C}_E is *compatible with the \mathcal{O} -monoidal structure* if for every active morphism $\phi: O \rightsquigarrow E$ in \mathcal{O} with $E \in \mathcal{O}^{\text{el}}$, the functor

$$\phi_1^{\mathcal{P}_{\mathcal{O}}(\mathcal{C})}: \prod_i \mathcal{P}(\mathcal{C}_{O_i}) \simeq \mathcal{P}_{\mathcal{O}}(\mathcal{C})_{\mathcal{O}}^{\otimes} \rightarrow \mathcal{P}(\mathcal{C}_E)$$

takes a morphism $(\text{id}, \dots, \text{id}, s, \text{id}, \dots, \text{id})$ with $s \in \mathbb{S}_{O_i}$ into the strongly saturated class $\overline{\mathbb{S}}_E$ generated by \mathbb{S}_E . We then write $\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C})^{\otimes}$ for the full subcategory generated by the collection of full subcategories $\mathcal{P}_{\mathbb{S}_E}(\mathcal{C}_E)$ of \mathbb{S}_E -local objects.

By Corollary 7.12 we get an \mathcal{O} -monoidal localization of $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes}$:

Corollary 7.20. *Let \mathcal{C}^{\otimes} be a small \mathcal{O} -monoidal ∞ -category and \mathbb{S} a collection of sets of morphisms compatible with the \mathcal{O} -monoidal structure. Then there is an \mathcal{O} -monoidal localization*

$$\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes} \xrightarrow{L_{\mathbb{S}}} \mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C})^{\otimes}$$

left adjoint to the inclusion $\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C})^{\otimes} \hookrightarrow \mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes}$.

Proof. It suffices to verify the two conditions in Corollary 7.12. Since \mathbb{S}_E is a set for every $E \in \mathcal{O}^{\text{el}}$, the inclusion $\mathcal{P}_{\mathbb{S}_E}(\mathcal{C}_E) \hookrightarrow \mathcal{P}(\mathcal{C}_E)$ has a left adjoint $L_{\mathbb{S}_E}$, which exhibits $\mathcal{P}_{\mathbb{S}_E}(\mathcal{C}_E)$ as the localization at the strongly saturated class of maps generated by \mathbb{S}_E . Hence, the first condition is satisfied, and for the second condition we observe that the compatibility of the maps in \mathbb{S} with the \mathcal{O} -monoidal structure implies that for every active map $\phi: O \rightsquigarrow E$, the map $\phi_1^{\mathcal{P}_{\mathcal{O}}(\mathcal{C})} \mathcal{O}(X_1, \dots, X_i, \dots, X_n) \rightarrow \phi_1^{\mathcal{P}_{\mathcal{O}}(\mathcal{C})} \mathcal{O}(X_1, \dots, L_{\mathbb{S}, O_i} X_i, \dots, X_n)$ lies in $\overline{\mathbb{S}}_E$. In particular, the map $\phi_1^{\mathcal{P}_{\mathcal{O}}(\mathcal{C})} \mathcal{O}(X_1, \dots, X_n) \rightarrow$

$\phi_!^{\mathcal{P}_\circ(\mathcal{C})} \mathcal{O}(L_{\mathbb{S}, O_1} X_1, \dots, L_{\mathbb{S}, O_n} X_n)$ is taken to an equivalence by $L_{\mathbb{S}, E}$, which is the second condition of Corollary 7.12. \square

Remark 7.21. Let us say an \mathcal{O} -monoidal localization is *accessible* if the component of the fully faithful right adjoint at each object of \mathcal{O}^{el} is an accessible functor. Then it follows from the classification of accessible localizations of presheaf ∞ -categories in [Lur09, §5.5.4] that every accessible \mathcal{O} -monoidal localization of $\mathcal{P}_\circ(\mathcal{C})^\otimes$ is of the form $\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C})^\otimes$ for some collection of sets of morphisms \mathbb{S} compatible with the \mathcal{O} -monoidal structure.

Remark 7.22. Suppose \mathcal{V}^\otimes is a presentably \mathcal{O} -monoidal ∞ -category and κ is a regular cardinal such that \mathcal{V}^\otimes is κ -presentably \mathcal{O} -monoidal and the full subcategory $\mathcal{V}^{\kappa, \otimes}$ is \mathcal{O} -monoidal. Then \mathcal{V}^\otimes is equivalent to $\mathcal{P}_{\mathcal{O}, \mathbb{S}^\kappa}(\mathcal{V}^\kappa)^\otimes$ where \mathbb{S}_E^κ consists of the maps

$$\text{colim}_J y(\phi) \rightarrow y(\text{colim}_J \phi)$$

in $\mathcal{P}(\mathcal{V}_E^\kappa)$ where $\phi: J \rightarrow \mathcal{V}_E^\kappa$ ranges over a set of representatives of κ -small colimit diagrams.

Definition 7.23. Let \mathcal{C}^\otimes be a small \mathcal{O} -monoidal ∞ -category and \mathbb{S} a collection of sets of morphisms compatible with the \mathcal{O} -monoidal structure on $\mathcal{P}_\circ(\mathcal{C})^\otimes$. We say that a $\mathcal{C}^{\text{op}, \otimes}$ -monoid $M: \mathcal{C}^{\text{op}, \otimes} \rightarrow \mathcal{S}$ is \mathbb{S} -local if for every $E \in \mathcal{O}^{\text{el}}$ the restriction

$$M_E: \mathcal{C}_E^{\text{op}} \simeq (\mathcal{C}^{\text{op}, \otimes})_E \rightarrow \mathcal{S}$$

is \mathbb{S}_E -local. We write $\text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}^{\mathbb{S}}(\mathcal{S})$ for the full subcategory of $\text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}(\mathcal{S})$ spanned by the \mathbb{S} -local monoids.

Proposition 7.24. *Let \mathcal{C}^\otimes and \mathbb{S} be as in Definition 7.23. Then there is a natural equivalence*

$$\text{Alg}_\circ(\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C})) \simeq \text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}^{\mathbb{S}}(\mathcal{S}).$$

Proof. By Remark 7.10 the inclusion $\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C})^\otimes \hookrightarrow \mathcal{P}_\circ(\mathcal{C})^\otimes$ identifies $\text{Alg}_\circ(\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C}))$ with the full subcategory of $\text{Alg}_\circ(\mathcal{P}_\circ(\mathcal{C}))$ spanned by the algebras A such that $A(E)$ lies in $\mathbb{P}_{\mathbb{S}_E}(\mathcal{C}_E)$ for every $E \in \mathcal{O}^{\text{el}}$. Under the equivalence

$$\text{Alg}_\circ(\mathcal{P}_\circ(\mathcal{C})) \simeq \text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}(\mathcal{S})$$

of Proposition 6.18 this full subcategory is identified with $\text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}^{\mathbb{S}}(\mathcal{S})$. \square

Remark 7.25. Proposition 7.24 implies that the ∞ -category $\text{Alg}_\circ(\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C}))$ is equivalent to the full subcategory of $\text{Fun}(\mathcal{C}^{\text{op}, \otimes}, \mathcal{S})$ spanned by objects that are local with respect to a set of maps. Thus $\text{Alg}_\circ(\mathcal{P}_{\mathcal{O}, \mathbb{S}}(\mathcal{C}))$ is an accessible localization of a presheaf ∞ -category and so is in particular a presentable ∞ -category. Since every presentably \mathcal{O} -monoidal ∞ -category is equivalent to one of this form by Corollary 7.16, we have proved the following:

Corollary 7.26. *Suppose \mathcal{V}^\otimes is a presentably \mathcal{O} -monoidal ∞ -category. Then the ∞ -category $\text{Alg}_\circ(\mathcal{V})$ is presentable.* \square

8. EXTENDABILITY AND FREE ALGEBRAS

In this section we recall the notion of *extendability* for a morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ of cartesian patterns, and show that if \mathcal{V} is a presentably \mathcal{P} -monoidal ∞ -category, then the left adjoint

$$f_!: \text{Alg}_\circ(\mathcal{V}) \rightarrow \text{Alg}_\mathcal{P}(\mathcal{V})$$

to the functor given by composition with f can be described by an explicit colimit formula. In particular, if $\mathcal{O}^{\text{int}} \rightarrow \mathcal{O}$ is extendable (in which case we just say that \mathcal{O} is extendable), then we get an explicit formula for free \mathcal{O} -algebras.

Definition 8.1. A morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ of cartesian patterns has *unique lifting of inert morphisms* if for every inert morphism $\phi: f(O) \rightsquigarrow P$ in \mathcal{P} there is a unique lift to an inert morphism $\psi: O \rightsquigarrow O'$ in \mathcal{O} such that $f(\psi) \simeq \phi$. In other words, the induced map of ∞ -groupoids

$$(\mathcal{O}_{O'}^{\text{int}})^{\simeq} \rightarrow (\mathcal{P}_{f(O)'}^{\text{int}})^{\simeq}$$

is an equivalence.

Remark 8.2. If $f: \mathcal{O} \rightarrow \mathcal{P}$ has unique lifting of inert morphisms, then by [CH21, Corollary 7.4] we can define a functor $\mathcal{P} \rightarrow \text{Cat}_{\infty}$ that takes $P \in \mathcal{P}$ to $\mathcal{O}_{/P}^{\text{act}}$. A morphism $\alpha: P \rightarrow P'$ in \mathcal{P} is sent to the functor $\alpha_!: \mathcal{O}_{/P}^{\text{act}} \rightarrow \mathcal{O}_{/P'}^{\text{act}}$, that takes a pair $(O, f(O) \rightsquigarrow P)$ to the pair $(O', f(O') \rightsquigarrow P')$ given by first forming the commutative square

$$\begin{array}{ccc} f(O) & \rightsquigarrow & Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{\alpha} & P' \end{array}$$

by taking the inert–active factorization of $f(O) \rightsquigarrow P \xrightarrow{\alpha} P'$ and then lifting the inert map $f(O) \rightsquigarrow Q$ to a unique inert map $O \rightsquigarrow O'$ in \mathcal{O} .

Definition 8.3. A morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ of cartesian patterns is *extendable* if

- (1) f has unique lifting of inert morphisms,
- (2) for $P \in \mathcal{P}$ over $\langle n \rangle$, the functor

$$\mathcal{O}_{/P}^{\text{act}} \rightarrow \prod_{i=1}^n \mathcal{O}_{/P_i}^{\text{act}}$$

taking $(O, \phi: f(O) \rightarrow P)$ to $(\rho_{i,1}^P(O, \phi))_i$ is cofinal.

Remark 8.4. The general definition of an extendable morphism in [CH21, Definition 7.7] has a third condition, but this is automatic in the case of cartesian patterns: Namely, given an active morphism $\phi: f(O) \rightsquigarrow P$, we can use the unique lifting of inert morphisms to define a functor $\mathcal{P}_{/P}^{\text{el,op}} \rightarrow \text{Cat}_{\infty}$ taking $\alpha: P \rightarrow E$ to $\mathcal{O}_{\alpha_! O'}^{\text{el}}$. If $\mathcal{O}^{\text{el}}(\phi) \rightarrow \mathcal{P}_{/P}^{\text{el}}$ denotes the corresponding cartesian fibration, then there is a functor $\mathcal{O}^{\text{el}}(\phi) \rightarrow \mathcal{O}_{/O'}^{\text{el}}$ that takes $(\alpha, \alpha_! O \rightarrow E')$ to $O \rightarrow \alpha_! O \rightarrow E'$. The condition is that this functor should induce an equivalence

$$\lim_{\mathcal{O}_{/O'}^{\text{el}}} F \rightarrow \lim_{\mathcal{O}^{\text{el}}(\phi)} F$$

for every functor $F: \mathcal{O}^{\text{el}} \rightarrow \mathcal{S}$. However, if f is a morphism of cartesian patterns, then this functor is necessarily an equivalence: if $\langle n \rangle = |O|$ and $\langle m \rangle = |P|$, then $\mathcal{O}_{/O'}^{\text{el}}$ and $\mathcal{P}_{/P}^{\text{el}}$ are isomorphic to the discrete sets $\{\rho_i^O: i = 1, \dots, n\}$ and $\{\rho_i^P: i = 1, \dots, m\}$, while $\mathcal{O}_{\rho_{i,1}^P, O'}^{\text{el}}$ is isomorphic to the set $|\phi|^{-1}(i)$. Thus $\mathcal{O}^{\text{el}}(\phi)$ is the set of pairs $\{(i, j): 1 \leq i \leq m, j \in |\phi|^{-1}(i)\}$ and the map to $\mathcal{O}_{/O'}^{\text{el}}$ is the obvious isomorphism of this with $\{1, \dots, n\}$ (implied by ϕ being active).

Example 8.5. If $f: \mathcal{O} \rightarrow \mathcal{P}$ is an extendable morphism of cartesian patterns, then for any commutative square

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F} & \mathcal{D}^{\otimes} \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P}, \end{array}$$

where \mathcal{C}^{\otimes} is an \mathcal{O} -monoidal ∞ -category, \mathcal{D}^{\otimes} is a \mathcal{P} -monoidal ∞ -category, and f preserves cocartesian morphisms, then the morphism F is extendable. This is a special case of [CH21, Proposition 9.5].

In particular, for any \mathcal{P} -monoidal ∞ -category \mathcal{D}^\otimes , in the pullback square

$$\begin{array}{ccc} f^*\mathcal{D}^\otimes & \xrightarrow{\bar{f}} & \mathcal{D}^\otimes \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P}, \end{array}$$

the morphism $\bar{f}: f^*\mathcal{D}^\otimes \rightarrow \mathcal{D}^\otimes$ is extendable.

Proposition 8.6. *Suppose $f: \mathcal{O} \rightarrow \mathcal{P}$ is an extendable morphism of cartesian patterns. Then the functor $f^*: \text{Mon}_{\mathcal{P}}(\mathcal{S}) \rightarrow \text{Mon}_{\mathcal{O}}(\mathcal{S})$ has a left adjoint $f_!$, given by left Kan extension along f , which satisfies*

$$f_!M(P) \simeq \text{colim}_{O \in \mathcal{O}_{/P}^{\text{act}}} M(O).$$

Proof. This is a special case of [CH21, Proposition 7.13]. We give a brief sketch of the proof, as it is particularly simple in the case of cartesian patterns. Since $f^*: \text{Fun}(\mathcal{P}, \mathcal{S}) \rightarrow \text{Fun}(\mathcal{O}, \mathcal{S})$ has a left adjoint $f_!$ given by left Kan extension, it suffices to show that if M is an \mathcal{O} -monoid, then the left Kan extension $f_!M$ is an \mathcal{O} -monoid. We have natural equivalences

$$\begin{aligned} f_!M(P) &\simeq \text{colim}_{O \in \mathcal{O}_{/P}} M(O) \\ &\simeq \text{colim}_{O \in \mathcal{O}_{/P}^{\text{act}}} M(O) && \text{(Definition 8.3(1) and [CH21, 7.2])} \\ &\simeq \text{colim}_{(O_i) \in \prod_i \mathcal{O}_{/P_i}^{\text{act}}} \prod_i M(O_i) && \text{(Definition 8.3(2))} \\ &\simeq \prod_i \text{colim}_{O_i \in \mathcal{O}_{/P_i}^{\text{act}}} M(O_i) && (\mathcal{S} \text{ cartesian closed}) \\ &\simeq \prod_i f_!M(P_i), \end{aligned}$$

as required. □

Remark 8.7. The same result is true more generally for monoids in any ∞ -category where the cartesian product commutes with colimits indexed by the ∞ -categories $\mathcal{O}_{/P}^{\text{act}}$.

Definition 8.8. Let \mathcal{O} be a cartesian pattern. We write $\text{Act}_{\mathcal{O}}(O)$ for the ∞ -groupoid of active morphisms to O in \mathcal{O} . We say \mathcal{O} is *extendable* if the functor

$$\text{Act}_{\mathcal{O}}(O) \rightarrow \prod_i \text{Act}_{\mathcal{O}}(O_i),$$

taking $\phi: O' \rightsquigarrow O$ to the morphism $\rho_{i,!}^O \phi$ given by the inert-active factorization

$$\begin{array}{ccc} O' & \xrightarrow{\phi} & O \\ \downarrow & & \downarrow \rho_i^O \\ \rho_{i,!}^O O' & \xrightarrow{\rho_{i,!}^O \phi} & O_i, \end{array}$$

is an equivalence.

Remark 8.9. Since the equivalences are precisely the morphisms that are both active and inert, we can identify $\text{Act}_{\mathcal{O}}(O)$ with $(\mathcal{O}^{\text{int}})^{\text{act}}_{/O}$. Thus \mathcal{O} is extendable if and only if the inclusion $\mathcal{O}^{\text{int}} \rightarrow \mathcal{O}$ is an extendable morphism of cartesian patterns (since unique lifting of inert morphisms is tautological in this case).

Corollary 8.10. *Suppose \mathcal{O} is an extendable cartesian pattern. Then the functor*

$$U_{\mathcal{O}}: \text{Mon}_{\mathcal{O}}(\mathcal{S}) \rightarrow \text{Fun}(\mathcal{O}^{\text{el}}, \mathcal{S})$$

given by restriction to \mathcal{O}^{el} has a left adjoint $F_{\mathcal{O}}$ given by right Kan extension along $\mathcal{O}^{\text{el}} \hookrightarrow \mathcal{O}^{\text{int}}$ followed by left Kan extension along $\mathcal{O}^{\text{int}} \rightarrow \mathcal{O}$. This satisfies

$$F_{\mathcal{O}}(\Phi)(O) \simeq \text{colim}_{O' \rightsquigarrow O \in \text{Act}_{\mathcal{O}}(O)} \prod_i \Phi(O'_i).$$

Proof. Combine Proposition 8.6 with Remark 2.12. \square

Example 8.11. If \mathcal{O} is an extendable cartesian pattern, then for any morphism of \mathcal{O} - ∞ -operads (i.e. weak Segal \mathcal{O} -fibrations in the terminology of [CH21])

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ & \searrow & \swarrow \\ & \mathcal{O}, & \end{array}$$

the morphism f is extendable by [CH21, Corollary 9.16]. In particular, any \mathcal{O} - ∞ -operad is an extendable cartesian pattern.

We now want to extend these descriptions of left adjoints from monoids to algebras, starting with the special case of Day convolution:

Proposition 8.12. *Suppose \mathcal{C} is a small \mathcal{P} -monoidal ∞ -category, and $f: \mathcal{O} \rightarrow \mathcal{P}$ is an extendable morphism of cartesian patterns. Then the functor*

$$f^*: \text{Alg}_{\mathcal{P}}(\text{P}_{\mathcal{P}}(\mathcal{C})^{\otimes}) \rightarrow \text{Alg}_{\mathcal{O}/\mathcal{P}}(\text{P}_{\mathcal{P}}(\mathcal{C})^{\otimes})$$

has a left adjoint $f_!$, which for $P \in \mathcal{P}^{\text{el}}$ satisfies

$$(f_!A)(P) \simeq \text{colim}_{(O, \phi: f(O) \rightsquigarrow P) \in \mathcal{O}_{/P}^{\text{act}}} \phi_!A(O).$$

Proof. Consider the pullback square

$$\begin{array}{ccc} f^* \mathcal{C}^{\text{op}, \otimes} & \xrightarrow{\bar{f}} & \mathcal{C}^{\text{op}, \otimes} \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P}, \end{array}$$

where the morphism $\bar{f}: f^* \mathcal{C}^{\text{op}, \otimes} \rightarrow \mathcal{C}^{\text{op}, \otimes}$ is an extendable morphism of cartesian patterns by Example 8.5. From Proposition 8.6 we therefore get a left adjoint

$$\bar{f}_!: \text{Mon}_{f^* \mathcal{C}^{\text{op}, \otimes}}(\mathcal{S}) \rightarrow \text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}(\mathcal{S}),$$

given by left Kan extension; for $X \in \mathcal{C}^{\text{op}, \otimes}$ and $M \in \text{Mon}_{f^* \mathcal{C}^{\text{op}, \otimes}}(\mathcal{S})$ this satisfies

$$\bar{f}_!M(X) \simeq \text{colim}_{(Y, \bar{f}(Y) \rightarrow X) \in (f^* \mathcal{C}^{\text{op}, \otimes})_{/X}^{\text{act}}} M(Y).$$

If X lies over $P \in \mathcal{P}$, we see from the proof of [CH21, Proposition 9.5] that the canonical projection $(f^* \mathcal{C}^{\text{op}, \otimes})_{/X}^{\text{act}} \rightarrow \mathcal{O}_{/P}^{\text{act}}$ is a cocartesian fibration, whose fibre at $(O, f(O) \rightsquigarrow P)$ is

$$(f^* \mathcal{C}^{\text{op}, \otimes})_O \times_{\mathcal{C}^{\text{op}, \otimes}} \mathcal{C}_{P/X}^{\text{op}, \otimes} \simeq \mathcal{C}_{f(O)}^{\text{op}, \otimes} \times_{\mathcal{C}^{\text{op}, \otimes}} \mathcal{C}_{P/X}^{\text{op}, \otimes} \simeq \mathcal{C}_{f(O)/X}^{\text{op}, \otimes}.$$

Thus we can rewrite the formula for $\bar{f}_!M(X)$ as

$$\bar{f}_!M(X) \simeq \text{colim}_{(O, f(O) \rightsquigarrow P) \in \mathcal{O}_{/P}^{\text{act}}} \text{colim}_{(Y, \phi_!Y \rightarrow X) \in \mathcal{C}_{f(O)/X}^{\text{op}, \otimes}} M(Y),$$

where we are omitting notation for the equivalence $(f^* \mathcal{C}^{\otimes})_O \simeq \mathcal{C}_{f(O)}^{\otimes}$. Here we can identify $\text{colim}_{(Y, \phi_!Y \rightarrow X) \in \mathcal{C}_{f(O)/X}^{\text{op}, \otimes}} M(Y)$ with the value at X of the left Kan extension of $M|_{\mathcal{C}_{f(O)}^{\text{op}, \otimes}}$ along

$\phi_! : \mathcal{C}_{f(O)}^{\text{op}, \otimes} \rightarrow \mathcal{C}_P^{\text{op}, \otimes}$, which is the cocartesian pushforward in $\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})$. Under the natural equivalences

$$\text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}(\mathcal{S}) \simeq \text{Alg}_{\mathcal{P}}(\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})), \quad \text{Mon}_{f^* \mathcal{C}^{\text{op}, \otimes}}(\mathcal{S}) \simeq \text{Alg}_{\mathcal{O}}(\mathcal{P}_{\mathcal{O}}(f^* \mathcal{C}^{\otimes})) \simeq \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes}))$$

this formula therefore corresponds to the one above. \square

Corollary 8.13. *Suppose \mathcal{V} is a presentably \mathcal{P} -monoidal ∞ -category, and $f : \mathcal{O} \rightarrow \mathcal{P}$ is an extendable morphism of cartesian patterns. Then the functor*

$$f^* : \text{Alg}_{\mathcal{P}}(\mathcal{V}) \rightarrow \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V})$$

has a left adjoint $f_!$, which for $P \in \mathcal{P}^{\text{el}}$ satisfies

$$(f_! A)(P) \simeq \underset{(O, \phi : f(O) \rightsquigarrow P) \in \mathcal{O}_{/P}^{\text{act}}}{\text{colim}} \phi_! A(O).$$

Proof. Since \mathcal{V} is presentably \mathcal{P} -monoidal, by Corollary 7.16 there exists a small \mathcal{P} -monoidal ∞ -category \mathcal{C} and a \mathcal{P} -monoidal localization

$$L : \mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes}) \rightarrow \mathcal{V}^{\otimes},$$

with a fully faithful lax \mathcal{P} -monoidal right adjoint $i : \mathcal{V}^{\otimes} \rightarrow \mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})$. From Remark 7.10 we then have a commutative square

$$\begin{array}{ccc} \text{Alg}_{\mathcal{P}}(\mathcal{V}) & \xrightarrow{f_{\mathcal{V}}^*} & \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V}) \\ \downarrow i_* & & \downarrow i_* \\ \text{Alg}_{\mathcal{P}}(\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})) & \xrightarrow{f^*} & \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})), \end{array}$$

where the vertical functors are both fully faithful, with left adjoints given by L_* . It follows that we have a commutative square of left adjoints

$$\begin{array}{ccc} \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})) & \xrightarrow{f_!} & \text{Alg}_{\mathcal{P}}(\mathcal{P}_{\mathcal{P}}(\mathcal{C}^{\otimes})) \\ \downarrow L_* & & \downarrow L_* \\ \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V}) & \xrightarrow{f_{\mathcal{V},!}} & \text{Alg}_{\mathcal{P}}(\mathcal{V}). \end{array}$$

Since the right adjoint i_* is fully faithful, the counit $L_* i_* \xrightarrow{\sim} \text{id}$ is invertible, and combining this with the equivalence from the square we get

$$f_{\mathcal{V},!} \simeq f_{\mathcal{V},!} L_* i_* \simeq L_* f_! i_*.$$

Since L preserves colimits and cocartesian morphisms, this implies that $f_{\mathcal{V},!}$ satisfies

$$f_{\mathcal{V},!} A(P) \simeq L \left(\underset{(O, \phi : f(O) \rightsquigarrow P) \in \mathcal{O}_{/P}^{\text{act}}}{\text{colim}} \phi_! i(A(O)) \right) \simeq \underset{(O, \phi : f(O) \rightsquigarrow P) \in \mathcal{O}_{/P}^{\text{act}}}{\text{colim}} \phi_! A(O),$$

as required. \square

Corollary 8.14. *Suppose \mathcal{O} is an extendable cartesian pattern, and \mathcal{V} is a presentably \mathcal{O} -monoidal ∞ -category. Then the restriction*

$$U_{\mathcal{O}} : \text{Alg}_{\mathcal{O}}(\mathcal{V}) \rightarrow \text{Alg}_{\mathcal{O}^{\text{int}}/\mathcal{O}}(\mathcal{V}) \simeq \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \mathcal{V})$$

has a left adjoint $F_{\mathcal{O}} : \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \mathcal{V}) \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{V})$, which for $\Phi : \mathcal{O}^{\text{el}} \rightarrow \mathcal{V}$ and $E \in \mathcal{O}^{\text{el}}$ is given by

$$F_{\mathcal{O}} \Phi(E) \simeq \underset{(\phi : O \rightarrow E) \in \text{Act}_{\mathcal{O}}(E)}{\text{colim}} \phi_! (\Phi(O_1), \dots, \Phi(O_n)).$$

Moreover, the adjunction $F_{\mathcal{O}} \dashv U_{\mathcal{O}}$ is monadic.

Proof. The existence of the left adjoint follows from Corollary 8.13 applied to the map $\mathcal{O}^{\text{int}} \rightarrow \mathcal{O}$ (together with the equivalence of Lemma 4.14). To see the adjunction is monadic we apply the monadicity theorem for ∞ -categories, [Lur17, Theorem 4.7.3.5]. We then need to show that $U_{\mathcal{O}}$ detects equivalences, which is clear, and that $U_{\mathcal{O}}$ -split simplicial objects have colimits and these are preserved by $U_{\mathcal{O}}$. Suppose therefore that we have a $U_{\mathcal{O}}$ -split simplicial diagram $\phi: \Delta^{\text{op}} \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{V})$. Since \mathcal{V}^{\otimes} is presentably \mathcal{O} -monoidal, it is an \mathcal{O} -monoidal localization of a Day convolution $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes}$, which gives a commutative diagram

$$\begin{array}{ccc} \text{Alg}_{\mathcal{O}}(\mathcal{V}) & \longleftarrow & \text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}(\mathcal{S}) \\ \downarrow U_{\mathcal{O}} & & \downarrow U'_{\mathcal{O}} \\ \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, \mathcal{V}) & \longleftarrow & \text{Fun}(\mathcal{C}_{/\text{el}}^{\text{op}, \otimes}, \mathcal{S}). \end{array}$$

Here the functor $U'_{\mathcal{O}}$ is a monadic right adjoint by [CH21, Corollary 8.2] and so the colimit of the $U'_{\mathcal{O}}$ -split simplicial diagram that is the image of ϕ exists and is preserved by $U'_{\mathcal{O}}$. (Alternatively, this follows from the description of sifted colimits of monoids in Remark 2.13.) But by Remark 7.10 the commutative square above is cartesian, hence the colimit in $\text{Mon}_{\mathcal{C}^{\text{op}, \otimes}}(\mathcal{S})$ actually lies in the full subcategory $\text{Alg}_{\mathcal{O}}(\mathcal{V})$. Thus the $U_{\mathcal{O}}$ -split simplicial diagram ϕ has a colimit and this is preserved by $U_{\mathcal{O}}$, as required. \square

Remark 8.15. We can remove the presentability condition in Corollary 8.14: since a small \mathcal{O} -monoidal ∞ -category \mathcal{C}^{\otimes} is always a full \mathcal{O} -monoidal subcategory of the presentably \mathcal{O} -monoidal ∞ -category $\mathcal{P}_{\mathcal{O}}(\mathcal{C})^{\otimes}$, we can embed any large \mathcal{O} -monoidal ∞ -category \mathcal{V}^{\otimes} in a presentably \mathcal{O} -monoidal ∞ -category in a larger universe. Moreover, we can do so in a way that preserves small colimits, in which case we see that the left adjoint from Corollary 8.14 restricts to the full subcategory of \mathcal{O} -algebras in \mathcal{V}^{\otimes} provided the \mathcal{O} -monoidal structure is compatible with colimits of shape $\text{Act}_{\mathcal{O}}(E)$ for $E \in \mathcal{O}^{\text{el}}$.

Remark 8.16. If \mathcal{O} is an extendable cartesian pattern, then the formula for the free \mathcal{O} -monoid monad on $\text{Fun}(\mathcal{O}^{\text{el}}, \mathcal{S})$ shows that this is an *analytic* monad in the sense of [GHK17], and hence corresponds by the results of that paper to an ∞ -operad (in the sense of a not necessarily complete dendroidal Segal space). We expect that this observation can be strengthened: there should be a canonical morphism $\mathcal{O} \rightarrow \mathcal{O}_{\text{opd}}$ of cartesian patterns where \mathcal{O}_{opd} is a symmetric ∞ -operad, such that $\text{Mon}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{\sim} \text{Mon}_{\mathcal{O}_{\text{opd}}}(\mathcal{S})$ and $\mathcal{O}^{\text{el}} \rightarrow \mathcal{O}_{\text{opd}}^{\text{el}}$ is an epimorphism (i.e. is surjective on π_0). We hope to address this question elsewhere.

9. EXAMPLES OF EXTENDABILITY

In this section we will give some examples of extendable patterns and morphisms, and spell out what our results from the previous section amount to in these examples.

Example 9.1. The pattern \mathbb{F}_*^{\flat} is extendable: The category $\mathbb{F}_*^{\text{act}}$ can be identified with the category \mathbb{F} of (unpointed) finite sets, so the desired equivalence

$$\text{Act}(\langle n \rangle) \xrightarrow{\sim} \prod_{i=1}^n \text{Act}(\langle 1 \rangle)$$

corresponds to underlying equivalence of groupoids from the (“straightening”) equivalence

$$\mathbb{F}/\mathbf{n} \rightarrow \prod_{i=1}^n \mathbb{F}$$

between sets over $\mathbf{n} := \{1, \dots, n\}$ and families of sets indexed by \mathbf{n} , given by taking fibres at $i \in \mathbf{n}$. The groupoid $\text{Act}_{\mathbb{F}_*}(\langle 1 \rangle)$ is equivalent to the groupoid \mathbb{F}^{\simeq} of finite sets and bijections, i.e.

$\coprod_{n=0}^{\infty} B\Sigma_n$, and we recover the expected formula for free commutative algebras in a presentably symmetric monoidal ∞ -category \mathcal{V}^{\otimes} :

$$U_{\mathbb{F}_*} F_{\mathbb{F}_*}(V) \simeq \operatorname{colim}_{\phi: \langle n \rangle \rightarrow \langle 1 \rangle \in \operatorname{Act}_{\mathbb{F}_*}(\langle 1 \rangle)} \phi!(V, \dots, V) \simeq \prod_{n=0}^{\infty} V_{h\Sigma_n}^{\otimes n}.$$

Example 9.2. By Example 8.11, every morphism $f: \mathcal{O} \rightarrow \mathcal{O}'$ of symmetric ∞ -operads is extendable. If \mathcal{V}^{\otimes} is a presentably \mathcal{O}' -monoidal ∞ -category, we recover the formula for the operadic left Kan extension $f_!: \operatorname{Alg}_{\mathcal{O}/\mathcal{O}'}(\mathcal{V}) \rightarrow \operatorname{Alg}_{\mathcal{O}'}(\mathcal{V})$ from [Lur17]: for $X \in \mathcal{O}'_{\langle 1 \rangle}$ and $A \in \operatorname{Alg}_{\mathcal{O}/\mathcal{O}'}(\mathcal{V})$, we have

$$f_!A(X) \simeq \operatorname{colim}_{(O, \phi) \in \mathcal{O}_{/X}^{\operatorname{act}}} f(\phi)_!A(O).$$

In particular, if \mathcal{V}^{\otimes} is a symmetric monoidal ∞ -category, then we have

$$f_!A(X) \simeq \operatorname{colim}_{(O, \phi) \in \mathcal{O}_{/X}^{\operatorname{act}}} |\phi|_!A(O) \simeq \operatorname{colim}_{(O, \phi) \in \mathcal{O}_{/X}^{\operatorname{act}}} \bigotimes_i A(O_i).$$

Example 9.3. As a special case of the previous example, every symmetric ∞ -operad \mathcal{O} is an extendable cartesian pattern, and our results recover the expected formula for free \mathcal{O} -algebras in a presentably symmetric monoidal ∞ -category \mathcal{V}^{\otimes} : the forgetful functor $U_{\mathcal{O}}: \operatorname{Alg}_{\mathcal{O}}(\mathcal{V}) \rightarrow \operatorname{Fun}(\mathcal{O}^{\operatorname{el}}, \mathcal{V})$ has a left adjoint $F_{\mathcal{O}}$, which for $E \in \mathcal{O}^{\operatorname{el}}$ satisfies

$$F_{\mathcal{O}}\Phi(E) \simeq \operatorname{colim}_{\mathcal{O} \xrightarrow{\phi} E \in \operatorname{Act}_{\mathcal{O}}(E)} \bigotimes_i \Phi(O_i).$$

If $\mathcal{O}^{\operatorname{el}} := \mathcal{O}_{\langle 1 \rangle}^{\simeq} \simeq *$ we can define $\mathcal{O}(n)$ to be the fibre of $\operatorname{Act}_{\mathcal{O}}(*) \rightarrow \mathbb{F}^{\simeq}$ at the point of $B\Sigma_n$ (with its canonical Σ_n -action), and then rewrite the formula in the more familiar form

$$F_{\mathcal{O}}V \simeq \prod_{n=0}^{\infty} \mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

Remark 9.4. The formula in the previous example does not agree with that given in [Lur17, §3.1.3]. This is because [Lur17] uses the term “free algebras” in a non-standard way: instead of considering the operadic left Kan extension to \mathcal{O} from the ∞ -operad $\mathcal{O}^{\operatorname{int}}$ containing only the inert morphisms in \mathcal{O} , Lurie considers the extension from the ∞ -operad $\mathcal{O} \times_{\mathbb{F}_*} \mathbb{F}_*^{\operatorname{int}}$ containing those morphisms that map to inert morphisms in \mathbb{F}_* (but are not necessarily cocartesian). The difference is that $\mathcal{O} \times_{\mathbb{F}_*} \mathbb{F}_*^{\operatorname{int}}$ remembers *all* the unary operations in \mathcal{O} , while $\mathcal{O}^{\operatorname{int}}$ remembers only the *invertible* ones.

Example 9.5. Suppose $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of *generalized* symmetric ∞ -operads (in the sense of [Lur17, §2.3.2], or equivalently weak Segal fibrations over \mathbb{F}_*^{\natural} in the terminology of [CH21]). This certainly has unique lifting of inert morphisms, and so is extendable as a morphism of cartesian patterns if and only if for every $P \in \mathcal{P}$ over $\langle n \rangle$, the functor

$$\mathcal{O}_{/P}^{\operatorname{act}} \rightarrow \prod_{i=1}^n \mathcal{O}_{/P_i}^{\operatorname{act}}$$

is cofinal. In particular, a generalized ∞ -operad \mathcal{O} is extendable if and only if

$$\operatorname{Act}_{\mathcal{O}}(O) \rightarrow \prod_{i=1}^n \operatorname{Act}_{\mathcal{O}}(O_i)$$

is an equivalence for $O \in \mathcal{O}$ over $\langle n \rangle$. By [CH21, Proposition 9.15], we do have an equivalence between $\operatorname{Act}_{\mathcal{O}}(O)$ and the iterated fibre product

$$\operatorname{Act}_{\mathcal{O}}(O) \rightarrow \operatorname{Act}_{\mathcal{O}}(O_1) \times_{\operatorname{Act}_{\mathcal{O}}(\sigma_1 O)} \cdots \times_{\operatorname{Act}_{\mathcal{O}}(\sigma_1 O)} \operatorname{Act}_{\mathcal{O}}(O_n),$$

where σ denotes the unique map $\langle n \rangle \rightarrow \langle 0 \rangle$. Since the only active map to $\langle 0 \rangle$ in \mathbb{F}_* is the identity, for $X \in \mathcal{O}_{\langle 0 \rangle}$ the ∞ -groupoid $\operatorname{Act}_{\mathcal{O}}(X)$ is equivalent to $\mathcal{O}_{\langle 0 \rangle / X}^{\simeq}$. If $\mathcal{O}_{\langle 0 \rangle}$ is an ∞ -groupoid, then $\operatorname{Act}_{\mathcal{O}}(X)$ is therefore contractible for all $X \in \mathcal{O}_{\langle 0 \rangle}$. This shows that a generalized symmetric ∞ -operad \mathcal{O}

such that $\mathcal{O}_{\langle 0 \rangle}$ is an ∞ -groupoid is always extendable. More generally, if $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of generalized ∞ -operads such that $\mathcal{O}_{\langle 0 \rangle}$ and $\mathcal{P}_{\langle 0 \rangle}$ are ∞ -groupoids and $f_{\langle 0 \rangle}: \mathcal{O}_{\langle 0 \rangle} \rightarrow \mathcal{P}_{\langle 0 \rangle}$ is an equivalence, then f is extendable, since we have

$$\mathcal{O}_0^{\text{act}} \times_{\mathcal{P}_0^{\text{act}}} \mathcal{P}_{0/P}^{\text{act}} \simeq *$$

for every $P \in \mathcal{P}_0$. (Note, however, that a more general morphism between generalized ∞ -operads whose fibres at $\langle 0 \rangle$ are ∞ -groupoids may still fail to be extendable.)

Example 9.6. The pattern $\Delta^{\text{op},b}$ is extendable: We can identify the category $\Delta^{\text{op},\text{act}}$ with the category \mathbb{O} of finite ordered sets; then the desired equivalence

$$\text{Act}([n]) \xrightarrow{\sim} \prod_{i=1}^n \text{Act}([1])$$

becomes underlying equivalence of groupoids arising from the obvious equivalence

$$\mathbb{O}_{/n} \rightarrow \prod_{i=1}^n \mathbb{O}_{/1}$$

that takes an ordered set over n to its fibres at the points of n . Since every object of Δ^{op} has a unique active map to $[1]$, the groupoid $\text{Act}_{\Delta^{\text{op}}}([1])$ is isomorphic to the set $\{0, 1, \dots\}$. If \mathcal{V}^{\otimes} is a presentably Δ^{op} -monoidal ∞ -category we get the expected formula for free associative algebras:

$$T_{\Delta^{\text{op}}}(V) \simeq \underset{\phi: [n] \rightarrow [1] \in \text{Act}_{\Delta^{\text{op}}}([1])}{\text{colim}} \phi!(V, \dots, V) \simeq \prod_{n=0}^{\infty} V^{\otimes n}.$$

We also get analogues of Examples 9.2, 9.3 and 9.5:

- every morphism of non-symmetric ∞ -operads is extendable,
- every non-symmetric ∞ -operad is extendable,
- every generalized non-symmetric ∞ -operad whose fibre at $[0]$ is an ∞ -groupoid is extendable,
- every morphism of generalized non-symmetric ∞ -operads whose fibres at $[0]$ are ∞ -groupoids and whose restriction to these is an equivalence, is extendable.¹

10. MORITA EQUIVALENCES

In this section we use our results on extendable cartesian patterns to give a condition for a morphism of cartesian patterns to induce equivalences on ∞ -categories of algebras, i.e. to be a *Morita equivalence* in the following sense:

Definition 10.1. We say that a morphism of cartesian patterns $f: \mathcal{O} \rightarrow \mathcal{P}$ is a *Morita equivalence* if for every \mathcal{P} -monoidal ∞ -category \mathcal{V}^{\otimes} the functor

$$f^*: \text{Alg}_{\mathcal{P}}(\mathcal{V}) \rightarrow \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V}),$$

given by composition with f , is an equivalence.

Remark 10.2. If f is a Morita equivalence, then as a special case (taking \mathcal{V}^{\otimes} to be $\text{Cat}_{\infty}^{\times}$) we have that pullback along f gives an equivalence between \mathcal{P} -monoidal and \mathcal{O} -monoidal ∞ -categories.

Our discussion of free algebras leads to a checkable criterion for a morphism of extendable cartesian patterns to be a Morita equivalence:

Proposition 10.3. *Suppose \mathcal{O} and \mathcal{P} are extendable cartesian patterns and $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of cartesian patterns such that*

- (1) $f^{\text{el}}: \mathcal{O}^{\text{el}} \rightarrow \mathcal{P}^{\text{el}}$ is an equivalence of ∞ -groupoids,

¹The existence of operadic left Kan extensions in this case was used in [GH15].

(2) for every $E \in \mathcal{O}^{\text{el}}$ the functor

$$\text{Act}_{\mathcal{O}}(E) \rightarrow \text{Act}_{\mathcal{P}}(f(E)),$$

induced by f , is an equivalence of ∞ -groupoids.

Then f is a Morita equivalence.

Proof. We first consider the case where \mathcal{V}^{\otimes} is presentably \mathcal{P} -monoidal. Then we have a commutative diagram

$$\begin{array}{ccc} \text{Alg}_{\mathcal{P}}(\mathcal{V}) & \xrightarrow{f^*} & \text{Alg}_{\mathcal{O}/\mathcal{P}}(\mathcal{V}) \\ \downarrow U_{\mathcal{P}} & & \downarrow U_{\mathcal{O}} \\ \text{Fun}_{/\mathcal{P}^{\text{el}}}(\mathcal{P}^{\text{el}}, \mathcal{V}) & \xrightarrow{f^{\text{el},*}} & \text{Fun}_{/\mathcal{O}^{\text{el}}}(\mathcal{O}^{\text{el}}, f^*\mathcal{V}) \end{array}$$

where the vertical maps are monadic right adjoints by Corollary 8.14 and the bottom horizontal map is an equivalence by assumption (1). Using [Lur17, Corollary 4.7.3.16] we see that f^* is an equivalence if and only if for every $\xi \in \text{Fun}_{/\mathcal{P}^{\text{el}}}(\mathcal{P}^{\text{el}}, \mathcal{V})$ the natural map

$$F_{\mathcal{O}} f^{\text{el},*} \xi \rightarrow f^* F_{\mathcal{P}} \xi$$

is an equivalence. Since $U_{\mathcal{P}}$ detects equivalences, it suffices to show that the induced map

$$(F_{\mathcal{O}} f^{\text{el},*} \xi)(E) \rightarrow (F_{\mathcal{P}} \xi)(f(E))$$

is an equivalence for all $E \in \mathcal{P}^{\text{el}}$. Since \mathcal{O} and \mathcal{P} are extendable we have colimit formulas for $F_{\mathcal{O}}$ and $F_{\mathcal{P}}$, which identify this map with the map

$$\alpha: \text{colim}_{\mathcal{O} \rightarrow E \in \text{Act}_{\mathcal{O}}(E)} \alpha_1 \xi(f(O)) \rightarrow \beta: \text{colim}_{\mathcal{P} \rightarrow f(E) \in \text{Act}_{\mathcal{P}}(f(E))} \beta_1 \xi(P),$$

which is the natural map of colimits arising from the morphism $\text{Act}_{\mathcal{O}}(E) \rightarrow \text{Act}_{\mathcal{P}}(f(E))$ induced by f . This is an equivalence by assumption (2). Thus f^* is an equivalence for every presentably \mathcal{P} -monoidal ∞ -category. Since we can embed any \mathcal{P} -monoidal ∞ -category fully faithfully in a presentably \mathcal{P} -monoidal one (possibly after passing to a larger universe) this completes the proof. \square

Remark 10.4. If $f: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism between extendable cartesian patterns such that $f^{\text{el}}: \mathcal{O}^{\text{el}} \rightarrow \mathcal{P}^{\text{el}}$ is an equivalence, then this proof in the case of \mathcal{S} shows that condition (2) is also *necessary* for f to be a Morita equivalence.

In some cases this criterion can be used to identify the ∞ -operad corresponding to a cartesian pattern without using the highly technical machinery of *approximations* from [Lur17, §2.3.3], as we will now demonstrate in some simple examples:

Example 10.5 (Associative algebras). Let $\text{Ass} \rightarrow \mathbb{F}_*$ denote the (symmetric) associative ∞ -operad. As in [Lur17, Remark 4.1.1.4] we can think of this as a category whose objects are the pointed finite sets $\langle n \rangle \in \mathbb{F}_*$, with a morphism $\langle n \rangle \rightarrow \langle m \rangle$ given by a morphism $\phi: \langle n \rangle \rightarrow \langle m \rangle$ in \mathbb{F}_* together with linear orderings \leq_i of the preimages $\phi^{-1}(i)$, $1 \leq i \leq m$. The composite of $(\phi, \leq_i): \langle n \rangle \rightarrow \langle m \rangle$ and $(\psi, \leq'_j): \langle m \rangle \rightarrow \langle k \rangle$ is given by the composite $\psi\phi$ in \mathbb{F}_* with the ordering \leq''_t of $(\psi\phi)^{-1}(t)$ given by

$$i \leq''_t i' \iff \phi(i) \leq'_t \phi(i') \text{ and } i \leq_s i' \text{ if } s = \phi(i) = \phi(i').$$

There is a functor $\text{cut}: \Delta^{\text{op}} \rightarrow \text{Ass}$ that takes $[n] \in \Delta^{\text{op}}$ to $\langle n \rangle$ and a morphism $\phi: [m] \rightarrow [n]$ in Δ to the morphism $\langle n \rangle \rightarrow \langle m \rangle$ given by

$$i \mapsto \begin{cases} j, & \phi(j-1) < i \leq \phi(j), \\ 0, & \text{if no such } j \text{ exists} \end{cases}$$

with the linear ordering of $\text{cut}(\phi)^{-1}(j)$ that given by identifying this with $\{i: \phi(j-1) < i \leq \phi(j)\}$. It is easy to see that $\text{cut}: \Delta^{\text{op}} \rightarrow \text{Ass}$ is a morphism of cartesian patterns, and we claim that it is a Morita equivalence: Both patterns are extendable by Examples 9.3 and 9.6, with $\Delta^{\text{op,el}} \simeq \text{Ass}^{\text{el}} \simeq *$. Moreover, $\text{Act}_{\Delta^{\text{op}}}([1])$ is the discrete set $\{[n] \rightarrow [1]: n = 0, 1, \dots\}$ while $\text{Act}_{\text{Ass}}(\langle 1 \rangle)$ can be identified

with the disjoint union over n of the contractible groupoid of linear orderings of $\{1, \dots, n\}$. The conditions of Proposition 10.3 therefore hold, and so we get for any (Ass-)monoidal ∞ -category \mathcal{V}^\otimes an equivalence

$$\mathrm{Alg}_{\Delta^{\mathrm{op}}/\mathrm{Ass}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{Ass}}(\mathcal{V}).$$

Example 10.6 (Bimodules). Let $\Delta_{/[1]}^{\mathrm{op},b}$ denote the category $\Delta_{/[1]}^{\mathrm{op}} := (\Delta_{/[1]})^{\mathrm{op}}$ with the inert/active factorization system lifted from Δ^{op} (along the left fibration $\Delta_{/[1]}^{\mathrm{op}} \rightarrow \Delta^{\mathrm{op}}$) and the three maps $[1] \rightarrow [1]$ as elementary objects. We can think of a morphism $[n] \rightarrow [1]$ as a sequence (i_0, \dots, i_n) with $0 \leq i_0 \leq \dots \leq i_n \leq 1$; then the elementary objects are $(0, 0)$, $(0, 1)$, and $(1, 1)$. We also let $\mathrm{Bimod} \rightarrow \mathbb{F}_*$ denote the (symmetric) bimodule operad (whose algebras are given by a pair of associative algebras and a bimodule between them). This can be described (cf. [Lur17, Notation 4.3.1.5]) as a category where

- objects are lists $(\langle n \rangle, (a_1, b_1), \dots, (a_n, b_n))$ where $0 \leq a_i \leq b_i \leq 1$,
- a morphism $(\langle n \rangle, (a_1, b_1), \dots, (a_n, b_n)) \rightarrow (\langle m \rangle, (a'_1, b'_1), \dots, (a'_m, b'_m))$ is given by a morphism $(\phi, \leq_i): \langle n \rangle \rightarrow \langle m \rangle$ in Ass such that for $j = 1, \dots, m$, if $\phi^{-1}(j) = \{i_1 <_j i_2 <_j \dots <_j i_k\}$, then

$$a'_j = a_{i_1} \leq b_{i_1} = a_{i_2} \leq \dots \leq a_{i_k} \leq b_{i_k} = b'_j.$$

We then define a functor $\Delta_{/[1]}^{\mathrm{op}} \xrightarrow{\mathrm{cut}'} \mathrm{Bimod}$ by

- $\mathrm{cut}'(i_0, \dots, i_n) = (\langle n \rangle, (i_0, i_1), \dots, (i_{n-1}, i_n))$,
- for a morphism $\phi: (i_0, \dots, i_n) \rightarrow (j_0, \dots, j_m)$, which is given by $\phi: [m] \rightarrow [n]$ in Δ such that $j_s = i_{\phi(s)}$, we set $\mathrm{cut}'(\phi) = \mathrm{cut}(\phi)$, which satisfies the required condition.

The functor cut' then fits in a commutative square

$$\begin{array}{ccc} \Delta_{/[1]}^{\mathrm{op}} & \xrightarrow{\mathrm{cut}'} & \mathrm{Bimod} \\ \downarrow & & \downarrow \\ \Delta^{\mathrm{op}} & \xrightarrow{\mathrm{cut}} & \mathrm{Ass}, \end{array}$$

and is a morphism of cartesian patterns (since the factorization systems are lifted from those on Δ^{op} and Ass). The pattern $\Delta_{/[1]}^{\mathrm{op},b}$ is extendable, e.g. by the non-symmetric analogue of Example 9.5, while Bimod is extendable by Example 9.3. We have $\Delta_{/[1]}^{\mathrm{op},\mathrm{el}} = \{(0, 0), (0, 1), (1, 1)\}$ which is isomorphic to $\mathrm{Bimod}^{\mathrm{el}} = \{(\langle 1 \rangle, (0, 0)), (\langle 1 \rangle, (0, 1)), (\langle 1 \rangle, (1, 1))\}$. The ∞ -groupoids $\mathrm{Act}_{\Delta_{/[1]}^{\mathrm{op}}}((0, 0))$ and $\mathrm{Act}_{\Delta_{/[1]}^{\mathrm{op}}}((1, 1))$ are discrete sets isomorphic to \mathbb{N} (with n corresponding to the unique active morphisms $(0, \dots, 0) \rightsquigarrow (0, 0)$ and $(1, \dots, 1) \rightsquigarrow (1, 1)$, respectively, with the source lying over $[n]$), while $\mathrm{Act}_{\Delta_{/[1]}^{\mathrm{op}}}((0, 1))$ is isomorphic to $\mathbb{N} \times \mathbb{N}$ (with (n, m) corresponding to the unique active map $(0, \dots, 0, 1, \dots, 1) \rightarrow (0, 1)$ with $(n+1)$ 0's and $(m+1)$ 1's). On the other hand, the ∞ -groupoid $\mathrm{Act}_{\mathrm{Bimod}}((\langle 1 \rangle, (i, j)))$ we can describe as a coproduct of contractible groupoids indexed by the set $\mathrm{Act}_{\Delta_{/[1]}^{\mathrm{op}}}(i, j)$. Hence Proposition 10.3 applies, and so we get for any Bimod -monoidal ∞ -category \mathcal{V}^\otimes an equivalence

$$\mathrm{Alg}_{\Delta_{/[1]}^{\mathrm{op}}/\mathrm{Bimod}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{Bimod}}(\mathcal{V}).$$

Example 10.7 (Modules over commutative algebras). Let $\mathbb{F}_{*,\langle 1 \rangle}^b$ denote the slice category $\mathbb{F}_{*,\langle 1 \rangle}/$ with the inert/active factorization system lifted from \mathbb{F}_* (along the left fibration $\mathbb{F}_{*,\langle 1 \rangle}/ \rightarrow \mathbb{F}_*$) with the two maps $\langle 1 \rangle \rightarrow \langle 1 \rangle$ as elementary objects; then $\mathbb{F}_{*,\langle 1 \rangle}^b$ is a cartesian pattern. We can also think of the objects as pairs $(\langle n \rangle, i)$ with $i \in \langle n \rangle$, with a morphism $(\langle n \rangle, i) \rightarrow (\langle m \rangle, j)$ given by a morphism $\phi: \langle n \rangle \rightarrow \langle m \rangle$ in \mathbb{F}_* such that $\phi(i) = j$. The cartesian pattern $\mathbb{F}_{*,\langle 1 \rangle}^b$ is extendable: The ∞ -groupoid $\mathrm{Act}_{\mathbb{F}_{*,\langle 1 \rangle}^b}((\langle n \rangle, i))$ we can describe as the groupoid of pairs $(\phi: \langle m \rangle \rightarrow \langle n \rangle, j \in \phi^{-1}(i))$

with ϕ active, and so in the commutative square

$$\begin{array}{ccc} \mathrm{Act}_{\mathbb{F}_{*,\langle 1 \rangle /}}(\langle n \rangle, i) & \longrightarrow & \prod_{j=1}^n \mathrm{Act}_{\mathbb{F}_{*,\langle 1 \rangle /}}(\langle 1 \rangle, \rho_j(i)) \\ \downarrow & & \downarrow \\ \mathrm{Act}_{\mathbb{F}_*}(\langle n \rangle) & \xrightarrow{\sim} & \prod_{j=1}^n \mathrm{Act}_{\mathbb{F}_*}(\langle 1 \rangle), \end{array}$$

the map on fibres over each active map $\phi \in \mathrm{Act}_{\mathbb{F}_*}(\langle n \rangle)$ is an isomorphism; this is therefore a pullback square, so the top horizontal morphism is an equivalence, as required.

We let $\mathrm{CMod} \rightarrow \mathbb{F}_*$ denote the (symmetric) ∞ -operad whose algebras are a commutative algebra together with a module over it. This can be described as a category with

- objects lists $(\langle n \rangle, i_1, \dots, i_n)$ with $i_s \in \{0, 1\}$ (with $(\langle 1 \rangle, 0)$ representing the algebra and $(\langle 1 \rangle, 1)$ the module,
- a morphism $(\langle n \rangle, i_1, \dots, i_n) \rightarrow (\langle m \rangle, j_1, \dots, j_m)$ is given by a morphism $\phi: \langle n \rangle \rightarrow \langle m \rangle$ in \mathbb{F}_* such that for all $s = 1, \dots, m$, we have

$$\sum_{t \in \phi^{-1}(s)} i_t = j_s.$$

We can define a functor $\mu: \mathbb{F}_{*,\langle 1 \rangle} \rightarrow \mathrm{CMod}$ given on objects by

$$\mu(\langle n \rangle, i) = (\langle n \rangle, \delta_{1i}, \dots, \delta_{ni}),$$

where $\delta_{ji} = 1$ if $j = i$, and 0 otherwise. Given a morphism $(\langle n \rangle, i) \rightarrow (\langle m \rangle, j)$ over $\phi: \langle n \rangle \rightarrow \langle m \rangle$, we assign to it the morphism $\mu(\langle n \rangle, i) \rightarrow \mu(\langle m \rangle, j)$ over ϕ , which indeed exists. It is clear that μ is a morphism of cartesian patterns, and identifies $\mathbb{F}_{*,\langle 1 \rangle /}$ with the full subcategory of CMod spanned by the objects with a most one copy of 1. We claim that μ is a Morita equivalence: $\mathbb{F}_{*,\langle 1 \rangle}^{\mathrm{el}}$ and $\mathrm{CMod}^{\mathrm{el}}$ are both the 2-element set containing $(\langle 1 \rangle, i)$ ($i = 0, 1$), and all active morphisms to $(\langle 1 \rangle, i)$ in CMod are in the image of μ , so that

$$\mathrm{Act}_{\mathbb{F}_{*,\langle 1 \rangle /}}(\langle 1 \rangle, i) \simeq \mathrm{Act}_{\mathrm{CMod}}(\langle 1 \rangle, i).$$

Since CMod is extendable by Example 9.3, we can apply Proposition 10.3 to get for any CMod -monoidal ∞ -category \mathcal{V}^{\otimes} an equivalence

$$\mathrm{Alg}_{\mathbb{F}_{*,\langle 1 \rangle /} / \mathrm{CMod}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{CMod}}(\mathcal{V}).$$

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