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Lax monoidal adjunctions, two-variable fibrations and the calculus of mates

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Abstract

We provide a calculus of mates for functors to the ∞-category of ∞-categories and extend Lurie's unstraightening equivalences to show that (op)lax natural transformations correspond to maps of (co)cartesian fibrations that do not necessarily preserve (co)cartesian edges. As a sample application, we obtain an equivalence between lax symmetric monoidal structures on right adjoint functors and oplax symmetric monoidal structures on the left adjoint functors between symmetric monoidal ∞-categories that is compatible with both horizontal and vertical composition of such structures. As the technical heart of the paper, we study various new types of fibrations over a product of two ∞-categories. In particular, we show how they can be dualised over one of the two factors and how they encode functors out of the Gray tensor product of $(\infty, 2)$-categories.

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1 | INTRODUCTION

The goal of the present paper is to establish a version of the *calculus of mates* for diagrams of ∞-categories, encoded in terms of (co)cartesian fibrations. Recall that if we have an *oplax square* in a 2-category, that is a diagram of the form

$$\begin{array}{ccc} \bullet & \xrightarrow{L} & \bullet \\ {\scriptstyle f}\big\downarrow & \overset{\alpha}{\nearrow} & \big\downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[L']{} & \bullet \end{array}$$

given by a 2-morphism $\alpha : L'f \to gL$, where L and L' have right adjoints R and R', then we can form the *mate square*, which is the lax square

$$\begin{array}{ccc} \bullet & \xrightarrow{R} & \bullet \\ {\scriptstyle g}\big\downarrow & \overset{\beta}{\swarrow} & \big\downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[R']{} & \bullet, \end{array}$$

where β is the *Beck–Chevalley transformation*

$$fR \longrightarrow R'L'fR \xrightarrow{\alpha} R'gLR \longrightarrow R'g$$

defined using the unit of the adjunction $L' \dashv R'$ and the counit of $L \dashv R$. It is not hard to show that this procedure and its dual give an isomorphism between oplax squares with horizontal left adjoints and lax squares with horizontal right adjoints. Moreover, this *mate correspondence* is compatible with compositions of squares both vertically and horizontally. A useful way to encode some of this functoriality is in terms of *(op)lax natural transformations*. If C, D are 2-categories, a lax transformation ϕ between functors $F, G : C \to D$ has lax naturality squares

$$\begin{array}{ccc} F(x) & \xrightarrow{\phi_x} & G(x) \\ {\scriptstyle F(f)}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow[\phi_y]{} & G(y) \end{array}$$

for each morphism $f : x \to y$ in C, and similarly for oplax transformations. By taking mates, we obtain an equivalence between oplax transformations given pointwise by left adjoints and lax transformations given pointwise by right adjoints.

In this paper, we will prove an ∞-categorical version of this equivalence in the special case where C is an ∞-category and D is the $(\infty, 2)$-category of ∞-categories. This arises as a combination of our two main results: on the one hand, we will prove a version of the mate correspondence for (co)cartesian fibrations (Theorem B), and on the other hand, we will provide a fibrational description of (op)lax transformations (Theorem E).

1.1 | Lax monoidal adjunctions

Since many ∞-categorical structures are conveniently encoded using fibrations, we believe that this fibrational approach to mates is very useful in its own right. Before we discuss our general results in more detail, we will look at a concrete special case that illustrates this point: the left adjoint of a lax symmetric monoidal functor between ∞-categories admits a canonical oplax symmetric monoidal structure, and vice versa. To state this more precisely, let us briefly recall the definitions of symmetric monoidal ∞-categories and (op)lax monoidal functors among them:

If X is an ∞-category with finite products and Fin_* denotes the category of finite pointed sets, a *commutative monoid* in X can be defined as a functor $M : \mathrm{Fin}_* \to X$ such that the Segal maps $M_{\langle n\rangle} \to \prod_{i=1}^{n} M_{\langle 1\rangle}$ are equivalences; here $\langle n\rangle = \{0, \dots, n\}$ is pointed by 0 and the map is induced by the projections $\langle n\rangle \to \langle 1\rangle$ that send all but one element to the base point. (The underlying object of the monoid is $M_{\langle 1\rangle}$.) Such a commutative monoid in the ∞-category Cat of small ∞-categories is then a *symmetric monoidal ∞-category*. Moreover, a homomorphism of commutative monoids is simply a natural transformation of functors $\mathrm{Fin}_* \to X$; in the case of symmetric monoidal ∞-categories, these correspond to *strong* symmetric monoidal functors.

A functor $B \to \mathrm{Cat}$ can be encoded as either a cocartesian fibration over B or a cartesian fibration over B^{op}. A symmetric monoidal structure on an ∞-category C can therefore be described either by a cocartesian fibration $C^{\otimes} \to \mathrm{Fin}_*$ or a cartesian fibration $C_{\otimes} \to \mathrm{Fin}_*^{\mathrm{op}}$. In terms of these fibrations, a strong monoidal functor from C to D corresponds to a commutative triangle of either of the forms

$$\begin{array}{ccc} C^{\otimes} & \xrightarrow{f^{\otimes}} & D^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{Fin}_*, & \end{array} \qquad \begin{array}{ccc} C_{\otimes} & \xrightarrow{f_{\otimes}} & D_{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{Fin}_*^{\mathrm{op}}, & \end{array}$$

where $f^{\otimes}$ preserves cocartesian morphisms and $f_{\otimes}$ preserves cartesian morphisms.

By weakening the conditions on $f^{\otimes}$ and $f_{\otimes}$, we obtain good definitions of lax and oplax symmetric monoidal functors. To this end, recall that a morphism $\phi : \langle n\rangle \to \langle m\rangle$ in Fin_* is called *inert* if it restricts to an isomorphism $\langle n\rangle \setminus \{\phi^{-1}(0)\} \xrightarrow{\sim} \langle m\rangle \setminus \{0\}$. In [23], Lurie defines a lax symmetric monoidal functor to be a commutative triangle as on the left where $f^{\otimes}$ only preserves the cocartesian morphisms that cover inert morphisms in Fin_*. Informally, we can think of an object of $C^{\otimes}$ over $\langle n\rangle$ as a list $(c_1, \dots, c_n)$ of objects in C, and this condition should be thought of as requiring the image of this object in $D^{\otimes}$ to be the list $(f(c_1), \dots, f(c_n))$. For objects $x, y \in C$, we have a cocartesian morphism starting at $(x, y) \in C^{\otimes}_{\langle 2\rangle}$ covering the map $\langle 2\rangle \to \langle 1\rangle$ that takes both 1 and 2 to 1. Its target encodes (essentially by definition) the tensor product $x \otimes y$. This morphism gets sent to a morphism $(fx, fy) \to f(x \otimes y)$ in $D^{\otimes}$, which may no longer be cocartesian; taking a cocartesian factorisation, we then obtain the lax monoidal structure map $fx \otimes fy \to f(x \otimes y)$ in D.

If we instead decide to similarly relax the conditions on $f_{\otimes}$ and require that it only preserves the cartesian morphisms covering inerts, we obtain Lurie's definition of an oplax symmetric monoidal functor. Namely, now the tensor product of x and y in C is encoded by a cartesian morphism $x \otimes y \to (x, y)$ in $C_{\otimes}$ covering the same map as before. Its image in $D_{\otimes}$ factors into a cartesian edge preceded by a map $f(x \otimes y) \to f(x) \otimes f(y)$, which by definition is the oplax structure map of f. We remark that the opposite functor $(C_{\otimes})^{\mathrm{op}} \to \mathrm{Fin}_*$ is actually the cocartesian fibration which encodes the natural symmetric monoidal structure on C^{op}, and so an oplax monoidal functor can also be described as a lax monoidal functor $C^{\mathrm{op}} \to D^{\mathrm{op}}$, as the name suggests.

In this fibrational context, we can easily deduce the following from our general results.

Proposition A. *Given two symmetric monoidal ∞-categories C and D, the extraction of adjoints gives inverse equivalences between the ∞-category of lax symmetric monoidal right adjoints $C^{\otimes} \to D^{\otimes}$ and the opposite of the ∞-category of oplax monoidal left adjoints $D_{\otimes} \to C_{\otimes}$.*

In fact, we will produce an equivalence of $(\infty, 2)$-categories that also encodes the compatibility of taking mates with compositions of (op)lax monoidal functors. We note that since the first version of this paper appeared, another proof of Proposition A has been given by Torii in [30, 31], based on the two universal properties of Day convolution on presheaves. Let us also say immediately that in [23] Lurie already proved that the right adjoint of a strong monoidal functor admits a lax monoidal structure, which suffices for a great many applications. Moreover, Torii has more generally produced a lax monoidal structure (but none of the accompanying coherences) on the right adjoint of an oplax monoidal functor in [29], by means of a span category construction. (We compare his construction to ours in [14].)

1.2 Parametrised adjunctions

With the example of lax monoidal adjunctions in mind, it hopefully seems reasonable that the fibrational calculus of mates should relate fibrewise right adjoint functors between cocartesian fibrations and fibrewise left adjoint functors between the corresponding cartesian fibrations. Our first main result shows that this holds generally:

Theorem B. *Let B be an ∞-category. Then there is an equivalence of $(\infty, 2)$-categories*

$$\mathbf{Cocart}^{\mathrm{lax,R}}(B) \simeq \left(\mathbf{Cart}^{\mathrm{opl,L}}(B^{\mathrm{op}})\right)^{(1,2)-\mathrm{op}}$$

extracting adjoints fibrewise; here the left-hand side denotes the $(\infty, 2)$-category with cocartesian fibrations over B as objects, fibrewise right adjoint functors (that need not preserve cocartesian lifts) as 1-morphisms, and natural transformations between these as 2-morphisms. The right-hand side is defined dually, using cartesian fibrations and fibrewise right adjoints, with the directions of 1- and 2-morphisms reversed by the superscript. Furthermore, these equivalences are natural in pulling back along the base.

Taking B to be an ∞-operad and taking appropriate subcategories cut out by the Segal conditions, this specialises to give the following.

Corollary C. *For any ∞-operad O, the extraction of adjoints gives a canonical equivalence of $(\infty, 2)$-categories*

$$\mathbf{MonCat}_{O}^{\mathrm{lax,R}} \simeq \left(\mathbf{MonCat}_{O}^{\mathrm{opl,L}}\right)^{(1,2)-\mathrm{op}},$$

where the left-hand side denotes the $(\infty, 2)$-category of O-monoidal ∞-categories, lax O-monoidal functors that admit (objectwise) left adjoints and O-monoidal transformations; the right-hand side is defined dually using oplax O-monoidal functors that admit right adjoints. Furthermore, these equivalences are natural in pulling back along operad maps in the base.

Proposition A is contained in this statement by taking $O = \mathbb{E}_\infty$ and examining the morphism ∞-categories between two symmetric monoidal ∞-categories C and D. We also use Corollary C to extend a result of Hinich: We show that the internal mapping functor

$$[-,-]\colon C^{\mathrm{op}} \times C \longrightarrow C$$

in a closed $\mathbb{E}_{n+1}$-monoidal ∞-category C admits a canonical lax $\mathbb{E}_n$-monoidal structure, where $1 \leqslant n \leqslant \infty$; in [18], Hinich established the case $n = \infty$ by different means.

In order to prove Theorem B, we need to describe functors to the full subcategories $\mathrm{Cocart}^{\mathrm{lax}}(B)$ and $\mathrm{Cart}^{\mathrm{opl}}(B)$ of cocartesian and cartesian fibration in Cat/B in terms of fibrations.

We show that functors $A \to \mathrm{Cocart}^{\mathrm{lax}}(B)$ correspond under covariant unstraightening to functors $p = (p_1, p_2)\colon E \to A \times B$ such that

(1) p_1 is a cocartesian fibration,
(2) p_1-cocartesian morphisms map to equivalences under p_2,
(3) for every $a \in A$, the functor $(p_2)_a\colon E_a \to B$ on fibres over a is a cocartesian fibration.

We call such a functor a *Gray fibration*, for reasons that will become clear in a moment. Dually, functors $A^{\mathrm{op}} \to \mathrm{Cocart}^{\mathrm{lax}}(B)$ correspond under contravariant unstraightening to functors $p = (p_1, p_2)\colon E \to A \times B$ such that

(1) p_1 is a cartesian fibration,
(2) p_1-cartesian morphisms map to equivalences under p_2,
(3) for every $a \in A$, the functor $(p_2)_a\colon E_a \to B$ on fibres over a is a cocartesian fibration.

We call these functors *curved orthofibrations*. While the notion of Gray fibrations admits a cartesian dual, *op-Gray fibrations*, which encode functors $A^{\mathrm{op}} \to \mathrm{Cart}^{\mathrm{opl}}(B)$, the key point for our proofs is that curved orthofibrations are self-dual in the sense that they can also be characterised by p_2 being a cocartesian fibration, p_2-cocartesian morphisms mapping to equivalences under p_1 and the functors $(p_1)_b\colon E_b \to A$ being cartesian fibrations. They can therefore also be straightened covariantly in the second variable, and hence correspond to functors $B \to \mathrm{Cart}^{\mathrm{opl}}(A)$.

Combining these one-variable straightenings, we obtain the following 'dualisation' equivalences.

Theorem D. *There is a natural equivalence of ∞-categories*

$$\mathrm{Gray}(A,B) \simeq \mathrm{CrvOrtho}(A^{\mathrm{op}},B) \quad \textit{and} \quad \mathrm{OpGray}(A,B) \simeq \mathrm{CrvOrtho}(A,B^{\mathrm{op}});$$

here $\mathrm{Gray}(A,B)$ *and* $\mathrm{OpGray}(A,B)$ *are the ∞-category of Gray fibrations and op-Gray fibrations over $A \times B$, respectively, whereas* $\mathrm{CrvOrtho}(A^{\mathrm{op}},B)$ *is the ∞-category of curved orthofibrations over $A^{\mathrm{op}} \times B$ and in both cases, the morphisms are required to preserve the defining (co)cartesian morphisms.*

Special cases of this duality were already known. For example, bifibrations are precisely those curved orthofibration whose fibres are ∞-groupoids, and under the equivalences above, they correspond precisely to the left and right fibrations, respectively. An equivalence of this kind was first established by Stevenson in [27] by different means.

To see the relation of these results on two-variable fibrations to Theorem B, let us explain how the equivalence we build acts on a morphism $f: D \to E$ of cartesian fibrations over B which is given fibrewise by left adjoints:

(1) Firstly, f can be covariantly unstraightened to a curved orthofibration over $B \times [1]$.
(2) Using Theorem D, this corresponds to a Gray fibration over $B^{\mathrm{op}} \times [1]$. Furthermore, because f is given fibrewise by left adjoints, this Gray fibration is also a curved orthofibration over $[1] \times B^{\mathrm{op}}$.
(3) Therefore, this new curved orthofibration can be covariantly straightened to a functor $[1]^{\mathrm{op}} \to \mathrm{Cocart}^{\mathrm{lax}}(B^{\mathrm{op}})$, corresponding to a morphism $E^{\vee} \to D^{\vee}$ over B^{op} between the cocartesian fibrations dual to the cartesian fibrations we started with.

Here the second half of the second step is the parametrised analogue of the statement that adjunctions among ∞-categories can be encoded by functors to [1] that are both cocartesian and cartesian fibrations, with the left and right adjoints obtained by cocartesian and cartesian straightening, respectively. At the level of spaces of objects, a very similar argument appears in work of Ayala, Mazel-Gee and Rozenblyum, where the base B is allowed to be an $(\infty, 2)$-category [2, Lemma B.5.7]. Their work also contains Corollary C at the level of objects, as [2, Remark 4.1.7].

1.3 | Unstraightening lax natural transformations

So far we have explained the fibrational perspective on lax natural transformation and how in this context, we produce an analogue of the calculus of mates. However, there is a second perspective on lax natural transformations which is more intrinsic to $(\infty, 2)$-category theory. To explain it, recall that in terms of diagrams of ∞-categories, a lax natural transformation f between two functors $F, G: B \to \mathrm{Cat}$ should be encoded by maps $F(b) \to G(b)$ for each b, together with lax commuting squares

$$\begin{array}{ccc} F(b) & \xrightarrow{f_b} & F(b) \\ {\scriptstyle \beta_!}\downarrow & \swarrow_{f_\beta} & \downarrow{\scriptstyle \beta_!} \\ F(b') & \xrightarrow[f_{b'}]{} & F(b') \end{array}$$

for each map $\beta: b \to b'$ in B (together with coherence data for compositions). For oplax transformations, the direction of the natural transformation f_β is reversed. The $(\infty, 2)$-categorical approach to defining such (op)lax natural transformations is to use the Gray tensor product $\boxtimes$, for which a good model has been introduced by Gagna, Harpaz and Lanari in [7]. Indeed, one can define lax and oplax natural transformations between functors of $(\infty, 2)$-categories $F, G: \mathbf{X} \to \mathbf{Y}$ in general as functors $[1] \boxtimes \mathbf{X} \to \mathbf{Y}$ and $\mathbf{X} \boxtimes [1] \to \mathbf{Y}$ restricting to F and G under the two embedding $[0] \to [1]$, respectively. To see why this is reasonable, we observe that the Gray tensor product $[1] \boxtimes [1]$ of the 1-simplex with itself is exactly the lax square

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \swarrow & \downarrow \\ \bullet & \longrightarrow & \bullet. \end{array}$$

Therefore, a functor $f : [1] \boxtimes \mathbf{X} \to \mathbf{Y}$ in particular encodes the data of a lax square

$$\begin{array}{ccc} F(b) & \xrightarrow{f_b} & F(b) \\ {\scriptstyle \beta_!}\big\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle f_\beta} & \big\downarrow{\scriptstyle \beta_!} \\ F(b') & \xrightarrow[f_{b'}]{} & F(b') \end{array}$$

for every 1-morphism β in $\mathbf{X}$. The Gray tensor product $[1] \boxtimes \mathbf{X}$ further encodes all of the higher coherences that these lax squares should satisfy.

Our second main result relates these definitions to the fibrational approach discussed above. To explain the statement, let us introduce two $(\infty, 2)$-categories $\mathbf{Fun}^{\mathrm{lax}}(\mathbf{X}, \mathbf{Y})$ and $\mathbf{Fun}^{\mathrm{opl}}(\mathbf{X}, \mathbf{Y})$ universally defined via

$$\mathrm{Fun}(-, \mathbf{Fun}^{\mathrm{lax}}(\mathbf{X}, \mathbf{Y})) \simeq \mathrm{Fun}(- \boxtimes \mathbf{X}, \mathbf{Y}) \quad \text{and} \quad \mathrm{Fun}(-, \mathbf{Fun}^{\mathrm{opl}}(\mathbf{X}, \mathbf{Y})) \simeq \mathrm{Fun}(\mathrm{X} \boxtimes -, \mathbf{Y})$$

as functors $\mathrm{Cat}_2 \to \mathrm{Cat}$. By definition, the 1-morphisms in the $(\infty, 2)$-category $\mathbf{Fun}^{\mathrm{lax}}(\mathbf{X}, \mathbf{Y})$ correspond to lax natural transformations and analogously for oplax transformations. By means of Lurie's locally cocartesian unstraightening equivalence [22], we show the following.

Theorem E. *There are natural equivalences of ∞-categories*

$$\mathrm{Gray}(A, B) \simeq \mathrm{Fun}(A \boxtimes B, \mathbf{Cat}),$$

and consequently natural equivalences of $(\infty, 2)$-categories

$$\mathbf{Cocart}^{\mathrm{lax}}(B) \simeq \mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat}), \quad \mathbf{Cart}^{\mathrm{opl}}(B) \simeq \mathbf{Fun}^{\mathrm{opl}}(B^{\mathrm{op}}, \mathbf{Cat}),$$

given on objects by straightening of (co)cartesian fibrations; here the targets are defined as above, and so have functors as objects, (op)lax natural transformations as morphisms, and modifications between these as 2-morphisms.

In particular, this implies that the cocartesian unstraightening of a functor $B \longrightarrow \mathrm{Cat}$ has the universal property of the *lax colimit*: it is given by the left adjoint of the constant diagram functor $\mathbf{Cat} \longrightarrow \mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})$ (see Observation 5.3.3). Furthermore, using Theorem E, we obtain the following reformulation of Theorem B.

Corollary F. *Extracting adjoints gives an equivalence of $(\infty, 2)$-categories*

$$\mathbf{Fun}^{\mathrm{lax,R}}(B, \mathbf{Cat}) \simeq \left(\mathbf{Fun}^{\mathrm{opl,L}}(B, \mathbf{Cat})\right)^{(1,2)-\mathrm{op}}$$

for every ∞-category B, where the superscript R *denotes the locally full (or 1-full) sub-2-category of* $\mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})$ *spanned by those lax natural transformations that admit pointwise left adjoints, and dually for the right-hand side. Furthermore, these equivalences are natural for restriction in B.*

We will show that the equivalence of $(\infty, 2)$-categories in Corollary F is given by a higher categorical form of the calculus of mates, in the following sense: on 1-morphisms, it takes a lax natural transformation $\theta^R : [1] \boxtimes B \to \mathbf{Cat}$ such that θ^R_b is a right adjoint for every $b \in B$ to an oplax transformation $\theta^L : B \boxtimes [1] \to \mathbf{Cat}$ such that the lax naturality squares for θ^R are the mates of the oplax naturality squares for θ^L. This procedure of taking mates should make sense in any $(\infty, 2)$-category, which suggests that a version of Corollary F should hold for any two $(\infty, 2)$-categories in lieu of B and $\mathbf{Cat}$. This generality does not seem to be directly within reach of our methods.

Furthermore, we provide two elaborations on the equivalences of Theorem B and Corollary F. Firstly, we describe the (co)unit of a fibrewise adjunction fibrationally, and consequently also the passage to adjoint morphisms in families. Secondly, we provide a characterisation of fibrewise adjoints in terms of mapping functors, to identify these in practice.

Let us finally point out that the book of Gaitsgory and Rozenblyum contains Theorem E as a consequence of their general 2-categorical straightening procedure [9, Corollary 11.1.2.6]. It also outlines a version of Corollary F with B and $\mathbf{Cat}$ replaced by arbitrary $(\infty, 2)$-categories [9, Section 12.3], using constructions somewhat similar to the ones presented here; in particular, our notions of curved orthofibrations and op-Gray fibrations are also introduced in Sections 12.2.1 and 12.2.3 of [9]. However, as far as we can tell they deduce both results using arguments different from ours, which at some points seem to rely on some of the unproven statements from [9, Chapter 10], notably about models for $(\infty, 2)$-categories in terms of bisimplicial spaces of lax squares.

Remark. This article is one part of a recombination of our earlier preprints [12] and [17], which contain many of the results we present here, most of them twice; the other part is [14]. For the reader interested in archaeology, we mention that Theorems 1.1, 1.2 and 1.3 from [12] are now contained in Corollary F, Proposition A and Proposition 3.4.9, whereas Theorems A, B and C from [17] are now part of Proposition A, Theorem D and Corollary C, respectively.

1.4 | Organisation

Section 2 introduces curved orthofibrations and Gray fibrations in more detail and establishes their basic properties. In particular, we deduce Theorem D as Theorem 2.5.1. In Section 3, we then introduce and study parametrised adjunctions in fibrational form. We prove Theorem B as Theorem 3.1.11, and deduce Proposition A and Corollary C as Corollary 3.4.8 and Theorem 3.4.7, respectively. In between, this section also contains the identification of the functor in Theorem B on morphisms with the Beck–Chevalley construction and the characterisation of parametrised adjoints in terms of mapping ∞-groupoids. Section 4 then discusses units and counits for parametrised adjunctions and derives the functoriality of the passage to adjoint morphisms in the parametrised context. Finally, in Section 5, we establish the connection to lax natural transformations, prove Theorem E as a combination of Corollary 5.2.10 and Theorem 5.3.1 and lastly deduce Corollary F as Theorem 5.3.5.

1.5 | Conventions

As mentioned above, in order to declutter notation, we will write Gpd, Cat and Cat_2 for the ∞-categories of ∞-groupoids (or spaces), ∞-categories and $(\infty, 2)$-categories, respectively. By default, we use complete 2-fold Segal spaces as the definition of the latter, but we will also need to discuss other models in Section 5.

The letter ι will denote the core of an ∞-category, that is, the ∞-groupoid spanned by its equivalences, and $|\cdot|$ its realisation. By a subcategory of an ∞-category, we mean a functor such that the induced morphisms on mapping ∞-groupoids and cores are inclusions of path components. A subcategory is *full* if the functor furthermore induces equivalences on mapping ∞-groupoids, while it is *wide* if the functor induces an equivalence on cores. Similarly, a sub-2-category of an $(\infty, 2)$-category is a functor inducing subcategory inclusions on mapping ∞-categories and a inclusion of path components on underlying ∞-groupoids; we say that such a sub-2-category is *1-full* if it is locally full, that is, given by full subcategory inclusions on mapping ∞-categories.

Throughout, we shall use small caps such as CAT to indicate the large variants of ∞-categories and boldface such as **Cat** to indicate the $(\infty, 2)$-categorical variants. We have also reserved sub- and superscripts on category names to refer to changes on morphisms, for example, $\mathrm{Cart}(A) \subseteq \mathrm{Cart}^{\mathrm{opl}}(A)$.

We will write $\mathrm{Ar}(C)$ for the arrow ∞-category $\mathrm{Fun}([1], C)$ of an ∞-category C, and $\mathrm{Tw}^{\ell}(C)$ and $\mathrm{Tw}^{r}(C)$ for the two versions of the twisted arrow category, geared so that the combined source-target map defines a left fibraton in the former, and a right fibration in the latter case, see 2.5.8.

2 | TWO-VARIABLE FIBRATIONS

Our main goal in the present section is to introduce two new classes of fibrations over a product of two ∞-categories, namely curved orthofibrations and Gray fibrations, and describe how they can be unstraightened over one of the two factors, and consequently dualised. We first recall some basic material about (co)cartesian fibrations in §2.1. Then in §2.2, we discuss functors to a product of two ∞-categories that behave like a (co)cartesian fibration in one of the two variables; both curved orthofibrations and Gray fibrations are special cases of such functors. In §2.3, we then introduce (curved) orthofibrations and study their partial unstraightenings. We consider Gray fibrations in §2.4 and characterise cocartesian and left fibrations among these. Finally, in §2.5, we record the various ways in which these fibrations can be dualised. In particular, we prove Theorem D there.

2.1 | Background

For the reader's convenience, we begin by briefly reviewing some basic material on (co)cartesian morphisms and fibrations.

Definition 2.1.1. Let $p : X \to S$ be a functor of ∞-categories. Then a morphism $\alpha : y \to z$ of X is *p-cartesian* if the square

$$\begin{array}{ccc} \mathrm{Map}_X(x, y) & \xrightarrow{\alpha_*} & \mathrm{Map}_X(x, z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_S(p(x), p(y)) & \xrightarrow{p(\alpha)_*} & \mathrm{Map}_S(p(x), p(z)) \end{array}$$

is a pullback square in Gpd for every $x \in X$. Dually, the morphism α is *p-cocartesian* if it is p^{op}-cartesian when regarded as a morphism in X^{op}, or in other words if for every $x \in X$, the square

$$\begin{array}{ccc} \mathrm{Map}_X(z,x) & \xrightarrow{\alpha^*} & \mathrm{Map}_X(y,x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_S(p(z),p(x)) & \xrightarrow{p(\alpha)^*} & \mathrm{Map}_S(p(y),p(x)) \end{array}$$

is cartesian.

Notation 2.1.2. To make diagrams more readable, for $p : X \to B$, we will sometimes indicate a p-cocartesian morphism of X by $x \rightarrowtail y$ and a p-cartesian morphism of X by $x \twoheadrightarrow y$.

Definition 2.1.3. Let $p : X \to S$ be a functor of ∞-categories. If T is a subcategory of S, we say that X *has all p-cartesian lifts* over T if for every morphism $f : a \to b$ in T and every object x such that $p(x) \simeq b$, there exists a filler in the commutative square

$$\begin{array}{ccc} [0] & \xrightarrow{x} & X \\ {\scriptstyle d_0}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ [1] & \xrightarrow{f} & S \end{array}$$

which is a p-cartesian morphism. Dually, X *has all p-cocartesian lifts over* T if X^{op} has all p^{op}-cartesian lifts over T^{op}. The functor $p : X \to S$ is a *cartesian fibration* if X has all p-cartesian lifts over S, and a *cocartesian fibration* if X has all p-cocartesian lifts over S.

Notation 2.1.4. We will write $\mathrm{Cocart}^{\mathrm{lax}}(S)$ and $\mathrm{Cart}^{\mathrm{opl}}(S)$ for the full subcategories of $\mathrm{Cat}_{/S}$ spanned by the cocartesian and cartesian fibrations, respectively, and $\mathrm{Cocart}(S)$ and $\mathrm{Cart}(S)$ for the wide subcategories thereof in which morphisms are required to preserve (co)cartesian edges.

Remark 2.1.5. The definition above is an invariant version of the definition for quasi-categories given by Lurie in [21, Definition 2.4.2.1]. More precisely, a map p between quasi-categories corresponds to a (co)cartesian fibration in our sense if and only if for some (and then any) factorisation of p into a categorical equivalence followed by a categorical fibration the latter is a (co)cartesian fibration in Lurie's sense.

Definition 2.1.6. Let $p : X \to S$ be a functor of ∞-categories. A morphism $\alpha : y \to z$ in X is *locally p-(co)cartesian* if it is a (co)cartesian morphism for the pullback $X \times_S [1] \to [1]$ of p along $p(\alpha) : [1] \to S$. The functor p is a *locally (co)cartesian fibration* if the pullback $X \times_S [1] \to [1]$ is a (co)cartesian fibration for every map $[1] \to S$.

Notation 2.1.7. We write $\mathrm{LocCocart}^{\mathrm{lax}}(S)$ and $\mathrm{LocCart}^{\mathrm{opl}}(S)$ for the full subcategories of $\mathrm{Cat}_{/S}$ spanned by the locally cocartesian and locally cartesian fibrations, respectively. We also denote by $\mathrm{LocCocart}(S)$ and $\mathrm{LocCart}(S)$ the wide subcategories of these where morphisms are required to preserve locally (co)cartesian morphisms.

Definition 2.1.8. We call a functor $p : X \to S$ that is both a cartesian and a cocartesian fibration a *bicartesian fibration*. We write $\mathrm{Bicart}^{\mathrm{(op)lax}}(S)$ for the full subcategory of $\mathrm{Cat}_{/S}$ spanned by the bicartesian fibrations.

Remark 2.1.9. In the category theory literature, our ‘bicartesian fibrations’ are often called ‘bifibrations’; we will instead use the latter term as in [21], see Definition 2.3.14.

We recall the following characterisation from [21, Lemma 2.4.2.7] of cartesian morphisms in a locally cartesian fibration, which will be used repeatedly below.

Proposition 2.1.10. *Suppose $p\colon E \to B$ is a locally cartesian fibration. Then the following are equivalent for a locally p-cartesian morphism $f\colon x \to y$ in E:*

(1) *f is a p-cartesian morphism.*
(2) *For every locally p-cartesian morphism $g\colon z \to x$, the composite $fg\colon z \to y$ is also locally p-cartesian.*

Corollary 2.1.11. *Suppose $p\colon E \to B$ is a locally cartesian fibration. Then p is a cartesian fibration if and only if any composite of locally p-cartesian morphisms is locally p-cartesian.*

The following is [21, Proposition 2.4.2.4].

Lemma 2.1.12. *The following conditions on a cartesian fibration $p\colon X \to S$ are equivalent:*

(1) *the fibres X_s are ∞-groupoids for all s in S,*
(2) *all morphisms in X are p-cartesian,*
(3) *p is conservative.*

Definition 2.1.13. A *right fibration* is a cartesian fibration satisfying the equivalent conditions of the previous lemma. A *left fibration* is a functor whose opposite is a right fibration.

Finally, let us briefly discuss functoriality. Consider the functor $t\colon \mathrm{Ar}(\mathrm{Cat}) \to \mathrm{Cat}$ extracting the target of a morphism. Its cartesian edges are precisely the pullback squares, so since Cat is complete we obtain a functor

$$\mathrm{Cat}^{\mathrm{op}} \longrightarrow \mathrm{C{\small AT}}, \quad S \longmapsto \mathrm{Cat}/S$$

by cartesian unstraightening, where CAT denotes the ∞-category of *large* ∞-categories. By [21, Proposition 2.4.1.3], the pullback of a (co)cartesian fibration is a (co)cartesian fibration and the structure map in a pullback preserves cocartesian edges. Therefore, one obtains subfunctors

$$\mathrm{LFib}, \mathrm{RFib}, \mathrm{Cocart}, \mathrm{Cart}\colon \mathrm{Cat}^{\mathrm{op}} \longrightarrow \mathrm{C{\small AT}}$$

via the construction above. Combining Lurie’s unstraightening equivalence with [8, Appendix A], one finds inverse equivalences

$$\mathrm{Str}^{\mathrm{ct}}\colon \mathrm{Cart} \rightleftarrows \mathrm{Fun}(-^{\mathrm{op}}, \mathrm{Cat}) : \mathrm{Un}^{\mathrm{ct}} \quad \text{and} \quad \mathrm{Str}^{\mathrm{cc}}\colon \mathrm{Cocart} \rightleftarrows \mathrm{Fun}(-, \mathrm{Cat}) : \mathrm{Un}^{\mathrm{cc}}$$

which restrict to equivalences

$$\mathrm{RFib} \rightleftarrows \mathrm{Fun}(-^{\mathrm{op}}, \mathrm{Gpd}) \quad \text{and} \quad \mathrm{LFib} \rightleftarrows \mathrm{Fun}(-, \mathrm{Gpd}).$$

The resulting equivalence between cartesian and cocartesian fibrations we shall denote

$$\mathrm{D}^{\mathrm{cc}}: \ \mathrm{Cart}(-^{\mathrm{op}}) \rightleftarrows \mathrm{Cocart} \ : \mathrm{D}^{\mathrm{ct}}.$$

Its restriction to left and right fibrations is simply given by taking opposites, but this is not true in general, since D^{ct} and D^{cc} are given by the identity on $\mathrm{Cart}(*) \simeq \mathrm{Cat} \simeq \mathrm{Cocart}(*)$; an explicit description of the equivalence in the general case is the main result of [3].

2.2 | Straightening in one variable

Before we introduce the main classes of fibrations we are interested in, here we will consider the most general kinds of functors to a product of ∞-categories that can be straightened over one of the two factors. The basic observation we need for this is the following.

Proposition 2.2.1. *Given a functor of ∞-categories $p = (p_1, p_2) : X \to A \times B$, a morphism $\alpha : x \to y$ in X such that $p_1(\alpha)$ is an equivalence is p-cocartesian if and only if it is p_2-cocartesian.*

Proof. Since equivalences are always cocartesian, this is an immediate consequence of [21, Proposition 2.4.1.3(3)], which says that given $X \xrightarrow{q} Y \xrightarrow{r} Z$, a morphism in X whose image in Y is r-cocartesian is q-cocartesian if and only if it is rq-cocartesian. □

Corollary 2.2.2. *The following are equivalent for a functor $p = (p_1, p_2) : X \to A \times B$:*

(1) *X has all p-cocartesian lifts over $A \times \iota B$.*
(2) *p_1 is a cocartesian fibration and all p_1-cocartesian morphisms lie over equivalences in B.*
(3) *In the commutative triangle,*

$$\begin{array}{ccc} X & \xrightarrow{p} & A \times B \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & A & \end{array}$$

the map p_1 is a cocartesian fibration, and p takes p_1-cocartesian morphisms to pr_1-cocartesian morphisms.

Proof. The equivalence of (1) and (2) is immediate from Proposition 2.2.1, while that of (2) and (3) amounts to the observation that the cocartesian morphisms for $\mathrm{pr}_1 : A \times B \to A$ are precisely those morphisms that project to equivalences in B. □

Definition 2.2.3. We say that a functor $p : X \to A \times B$ is *cocartesian over the left factor*, or simply *cocartesian over A* when no confusion can arise, if it satisfies the equivalent conditions of Corollary 2.2.2. Dually, we say that p is *cartesian over A* if p^{op} is cocartesian over A^{op}. We write $\mathrm{LCocart}(A, B)$ and $\mathrm{LCart}(A, B)$ for the subcategories of $\mathrm{Cat}/(A \times B)$ whose objects are (co)cartesian over A, with the morphisms required to preserve the (co)cartesian morphisms over $A \times \iota B$. Similarly, we write $\mathrm{RCocart}(A, B)$ and $\mathrm{RCart}(A, B)$ for the subcategories of $\mathrm{Cat}/(A \times B)$ whose objects are (co)cartesian over the right factor B, with the morphisms required to preserve the (co)cartesian morphisms over $\iota A \times B$.

Of course, we obtain equivalences

$$\mathrm{RCocart}(A,B) \simeq \mathrm{LCocart}(B,A), \quad \mathrm{RCart}(A,B) \simeq \mathrm{LCart}(B,A)$$

by restricting the obvious equivalence $\mathrm{Cat}/(A \times B) \simeq \mathrm{Cat}/(B \times A)$.

From the third condition in Corollary 2.2.2, we immediately see:

Corollary 2.2.4. *We write* $\mathrm{pr}_1 : A \times B \to A$ *for the projection to* A*. The equivalence* $\mathrm{Cat}/(A \times B) \simeq (\mathrm{Cat}/A)/\mathrm{pr}_1$ *restricts to equivalences of subcategories*

$$\mathrm{LCocart}(A,B) \simeq \mathrm{Cocart}(A)/\mathrm{pr}_1, \quad \mathrm{LCart}(A,B) \simeq \mathrm{Cart}(A)/\mathrm{pr}_1. \tag{2.2.5}$$

Combining these equivalences with straightening over A*, we get natural equivalences*

$$\mathrm{LCocart}(A,B) \simeq \mathrm{Fun}(A, \mathrm{Cat}/B), \quad \mathrm{LCart}(A,B) \simeq \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Cat}/B), \tag{2.2.6}$$

since $\mathrm{pr}_1 : A \times B \to A$ *straightens over* A *to the constant functor with value* B*.*

For later use, we also note the following consequence of Proposition 2.2.1 here.

Corollary 2.2.7. *Let* I *be an* ∞*-groupoid and* C *an* ∞*-category. The following are equivalent for a functor* $p: X \to I \times C$*:*

(1) *p is a cocartesian fibration.*
(2) *For every $i \in I$, the morphism on fibres $p_i : X_i \to C$ is a cocartesian fibration.*
(3) *The composite $X \xrightarrow{p} I \times C \to C$ is a cocartesian fibration.*

Proof. The equivalence of (1) and (3) follows from Proposition 2.2.1, while (1) implies (2) since cocartesian fibrations are closed under base change. Finally, applying the criterion of [15, Lemma A.1.8] to the commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{p} & I \times C \\ & \searrow \quad \swarrow_{\mathrm{pr}_1} & \\ & I, & \end{array}$$

shows that (2) implies (1). □

Notation 2.2.8. Given a functor $p: X \to A \times B$, we define p_ℓ and p_r by the cartesian squares

$$\begin{array}{ccccc} X_\ell & \longrightarrow & X & \longleftarrow & X_r \\ {\scriptstyle p_\ell}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_r} \\ A \times \iota B & \longrightarrow & A \times B & \longleftarrow & \iota A \times B. \end{array}$$

To make diagrams more readable, we will sometimes indicate a p_ℓ-cartesian edge of X by $x \twoheadrightarrow y$ and a p_r-cocartesian edge of X by $x \rightarrowtail y$.

Remark 2.2.9. From Corollary 2.2.7, it follows immediately that for a functor $p: X \to A \times B$, the pullback p_ℓ is a (co)cartesian fibration if and only if for every $b \in B$, the map on fibres $p_b: X_b \to A$ is a (co)cartesian fibration, and similarly for p_r.

2.3 | Curved orthofibrations

If we combine our conditions from the previous subsection for a functor to $A \times B$ to straighten contravariantly over A and covariantly over B, we obtain the following definition.

Definition 2.3.1. A *curved orthofibration* is a functor of ∞-categories $p: X \to A \times B$ such that p is cartesian over A and cocartesian over B, that is, X has all p-cartesian lifts over $A \times \iota B$ and all p-cocartesian lifts over $\iota A \times B$. We write $\mathrm{CrvOrtho}(A, B)$ for the subcategory of $\mathrm{Cat}/(A \times B)$ whose objects are the curved orthofibrations, with the morphisms required to preserve both cartesian morphisms over A and cocartesian morphisms over B.

Let us record two alternative characterisations:

Observation 2.3.2. Using Corollary 2.2.2, we can reformulate the definition of a curved orthofibration as a functor $p = (p_1, p_2): X \to A \times B$ such that

(1) p_1 is a cartesian fibration,
(2) p_2 is a cocartesian fibration,
(3) every p_1-cartesian morphism in X lies over an equivalence in B,
(4) every p_2-cocartesian morphism in X lies over an equivalence in A.

Proposition 2.3.3. *The following are equivalent for a functor $p = (p_1, p_2): X \to A \times B$:*

(1) *p is a curved orthofibration.*
(2) *In the commutative triangle*

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & A \times B \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & A, & \end{array}$$

p_1 is a cartesian fibration, p takes p_1-cartesian morphisms to pr_1*-cartesian morphisms, and for every $a \in A$, the map on fibres $X_a \to B$ is a cocartesian fibration.*
(3) *p is cartesian over A and $p_r: X_r \to \iota A \times B$ is a cocartesian fibration.*
(4) *In the commutative triangle*

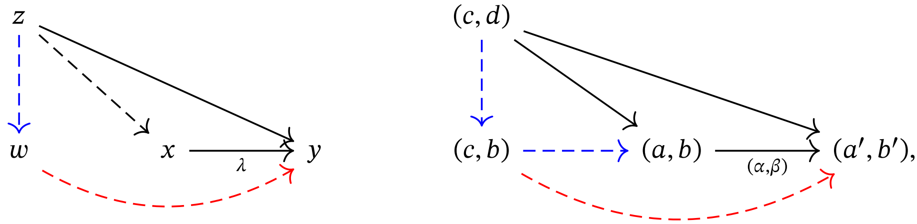
$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & A \times B \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & B, & \end{array}$$

p_2 is a cocartesian fibration, p takes p_2-cocartesian morphisms to pr_2*-cocartesian morphisms, and for every $b \in B$, the map on fibres $X_b \to A$ is a cartesian fibration.*
(5) *p is cocartesian over B and $p_\ell: X_\ell \to A \times \iota B$ is a cartesian fibration.*

Proof. The equivalence of (2) and (3), as well as of (4) and (5), follows from Remark 2.2.9 and Corollary 2.2.2. It thus remains to show that (1) is equivalent to one of these pairs, since they correspond to each other under taking opposites.

If p is a curved orthofibration, then it is immediate from the definition that p_ℓ is a cartesian and p_r a cocartesian fibration, that is, (1) implies (3) and (5). Conversely, the implication (2) ⇒ (1) is [15, Lemma A.1.10]. For completeness, we also include a brief argument that (5) implies (1): We need to show that a p_ℓ-cartesian lift $\lambda : x \to y$ in X of an arrow $(\alpha, \beta) : (a, b) \to (a', b')$, for which $\beta : b \to b'$ is an equivalence, is automatically p-cartesian.

Consider thus the black part of the diagram



which is a lifting problem in which one has to find a black dashed arrow in an essentially unique manner. First take an (essentially unique) p-cocartesian lift $z \longrightarrow w$ of $(c, d) \to (c, b)$. Since this arrow is cocartesian in all of X, there is an essentially unique dotted red arrow lifting the outer triangle on the right. Since the lower horizontal part of the diagram lives over $A \times \iota B$, there now exists an essentially unique map $w \longrightarrow x$ (not drawn) lifting the lower triangle. The composition with $z \longrightarrow x$ is the desired black dotted map, and using that $z \longrightarrow w$ is p-cocartesian, one can then complete the diagram in an essentially unique way. The essential uniqueness of the map $z \longrightarrow x$ is seen by reading the argument in reverse. □

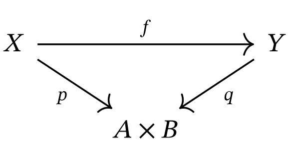
By definition, a functor to $A \times B$ is a curved orthofibration if and only if it lies in both $\mathrm{LCart}(A, B)$ and $\mathrm{RCocart}(A, B)$, and thus, by Corollary 2.2.4, a curved orthofibration can be straightened in either of the two variables. The previous lemma allows us to see precisely *what* a curved orthofibration straightens to.

Corollary 2.3.4. *Straightening over A and B give natural equivalences*

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(B))^{\mathrm{cc}} \simeq \mathrm{CrvOrtho}(A, B) \simeq \mathrm{Fun}(B, \mathrm{Cart}^{\mathrm{opl}}(A))^{\mathrm{ct}},$$

respectively, where $\mathrm{Fun}(B, \mathrm{Cart}^{\mathrm{opl}}(A))^{\mathrm{ct}}$ *denotes the wide subcategory of* $\mathrm{Fun}(B, \mathrm{Cart}^{\mathrm{opl}}(A))$ *in which the morphisms are natural transformations whose components all preserve cartesian morphisms over A, and similarly, for* $\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(B))^{\mathrm{cc}}$.

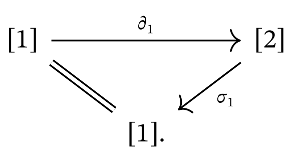
Proof. (2) and (4) of Proposition 2.3.3 immediately imply the result at the level of objects. We will exhibit the claim on morphisms for the left equivalence, the right being dual. Suppose that



is the unstraightening of a map $\eta \in \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(B))$. Because it is given by cartesian unstraightening, it will necessarily preserve cartesian edges over A. Naturality of unstraightening implies that the components of η preserving cocartesian edges is equivalent to the maps f_a on fibres preserving cocartesian edges. Proposition 2.2.1 then implies that f_r is a map of cocartesian fibrations. However, we have seen in Proposition 2.3.3 that for a curved orthofibration, the p_r-cocartesian edges agree with the p-cocartesian edges over B. Therefore, η preserves cocartesian edges pointwise if and only if f preserves cocartesian edges over B, and is thus a map of curved orthofibrations. □

We saw above that curved orthofibrations over $A \times B$ can be straightened to functors $A^{\mathrm{op}} \to \mathrm{Cocart}^{\mathrm{lax}}(B)$ or $B \to \mathrm{Cart}^{\mathrm{opl}}(A)$. Our next goal is to introduce a further condition that will ensure that these functors actually land in the subcategories $\mathrm{Cocart}(B)$ and $\mathrm{Cart}(A)$ (and thus encode functors $A^{\mathrm{op}} \times B \to \mathrm{Cat}$), giving the notion of *orthofibrations*. We also specialise further to *bifibrations* as considered by Lurie in [21, Section 2.4.7], and studied in detail, for example, in Stevenson [27] or [15, Appendix A]; these straighten to functors $A^{\mathrm{op}} \to \mathrm{LFib}(B)$ and $B \to \mathrm{RFib}(A)$.

Construction 2.3.5. Suppose that $A = [1]^{\mathrm{op}}$ and $B = [1]$, and let us write α, β for the unique non-degenerate simplices in A and B. Consider the diagram $\rho\colon [1] \longrightarrow \mathrm{Cocart}^{\mathrm{lax}}([1])$ corresponding to the map of cocartesian fibrations (between ordinary categories)



(2.3.6)

Note that this diagram is characterised by a universal property in $\mathrm{Ar}(\mathrm{Cocart}^{\mathrm{lax}}([1]))^{\mathrm{cc}}$, which is our slightly shorter notation for $\mathrm{Fun}([1], \mathrm{Cocart}^{\mathrm{lax}}([1]))^{\mathrm{cc}}$: for each object $g\colon X \longrightarrow Y$ in $\mathrm{Ar}(\mathrm{Cocart}^{\mathrm{lax}}([1]))^{\mathrm{cc}}$, evaluation at $\{0\} \in [1] = \rho(0)$ yields a natural equivalence to the fibre of X over 0

$$\mathrm{Map}_{\mathrm{Ar}(\mathrm{Cocart}^{\mathrm{lax}}([1]))^{\mathrm{cc}}}(\rho, g) \simeq \iota X_0.$$

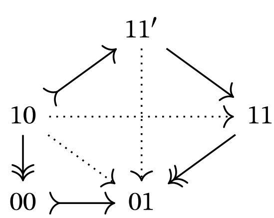
Indeed, unravelling the definitions shows that a natural transformation $\rho \Rightarrow g$ whose components preserve cocartesian arrows is given by a cocartesian arrow $\tilde{\beta}\colon x \longrightarrow \beta_! x$ in X over β, together with a factorisation of $g(\tilde{\beta})$ into a cocartesian morphism followed by a fibrewise one,

$$g(\tilde{\beta})\colon g(x) \rightarrowtail \beta_! g(x) \xrightarrow{\rho_\beta(x)} g(\beta_! x). \tag{2.3.7}$$

Now the cartesian unstraightening of ρ over $A = [1]$ is the curved orthofibration

$$q\colon Q \longrightarrow [1]^{\mathrm{op}} \times [1],$$

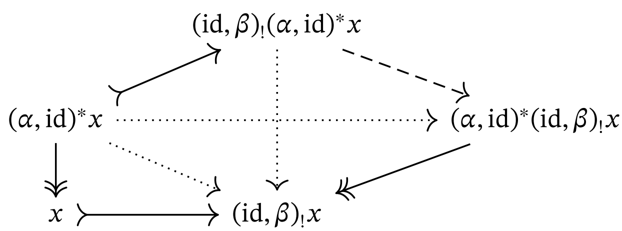
where Q is the poset given by



and the projection is the evident one, sending $11' \longrightarrow 11$ to the identity. Then $Q \longrightarrow [1]^{\mathrm{op}} \times [1]$ has the following universal property: for every curved orthofibration $p : X \longrightarrow A \times B$ and every $\alpha : a' \longrightarrow a$ and $\beta : b \longrightarrow b'$, there is a natural equivalence between the ∞-groupoid $\iota(X_{(a,b)})$ of objects in the fibre over (a, b) and the ∞-groupoid of maps of curved orthofibrations

$$\begin{array}{ccc} Q & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ [1]^{\mathrm{op}} \times [1] & \xrightarrow{\alpha\times\beta} & A \times B. \end{array}$$

Let us refer to such diagrams as *p-interpolating diagrams*. Explicitly, the p-interpolating diagram associated to $x \in X_{(a,b)}$ is given by



where we choose p-(co)cartesian morphism and (dotted) compositions as indicated, and finally, the dashed arrow is given by either factoring the horizontal dotted morphisms through the cocartesian morphism over (id, β), or equivalently by factoring the vertical dotted morphism through the cartesian morphism over (α, id).

Definition 2.3.8. Let $p : X \longrightarrow A \times B$ be a curved orthofibration. We will refer to a morphism in X as *p-interpolating* if it arises as the evaluation at $11' \to 11$ of a p-interpolating diagram $Q \to X$.

Remark 2.3.9. The p-interpolating edges in X are precisely the edges that arise under unstraightening from the morphisms $\rho_\beta(x)$ described in (2.3.7).

Definition 2.3.10. A functor $p = (p_1, p_2) : X \to A \times B$ is an *orthofibration* if it is a curved orthofibration and all p-interpolating morphisms in X are invertible, that is, for every pair of morphisms $\alpha : a' \to a$ in A and $\beta : b \to b'$ in B and every object x in X over (a, b), the interpolating morphism

$$(\mathrm{id}, \beta)_!(\alpha, \mathrm{id})^*x \longrightarrow (\alpha, \mathrm{id})^*(\mathrm{id}, \beta)_!x$$

is an equivalence. We write $\mathrm{Ortho}(A, B)$ for the full subcategory of $\mathrm{CrvOrtho}(A, B)$ spanned by the orthofibrations. (Note that our orthofibrations are the same as the *two-sided fibrations* defined in [26, Section 7.1].)

Proposition 2.3.11. *The following are equivalent for a curved orthofibration $p = (p_1, p_2) : X \to A \times B$:*

(1) *p is an orthofibration.*

(2) *For every morphism* $\alpha : a' \to a$ *in* A, *the cartesian transport functor* $\alpha^* : X_a \to X_{a'}$ *preserves* p_2*-cocartesian morphisms.*
(3) *For every morphism* $\beta : b' \to b$ *in* B, *the cocartesian transport functor* $\beta_! : X_{b'} \to X_b$ *preserves* p_1*-cartesian morphisms.*

Proof. Let us consider a curved orthofibration p, morphisms $\alpha : a' \longrightarrow a$ and $\beta : b \longrightarrow b'$, and $x \in X_{(a,b)}$. By Construction 2.3.5, the associated interpolating morphism fits into commuting triangles

$$\begin{array}{ccc} (\alpha, \mathrm{id})^* x & \rightarrowtail & (\mathrm{id}, \beta)_! (\alpha, \mathrm{id})^* x \\ & \searrow^{\alpha^*(\tilde{\beta})} & \downarrow \\ & & (\alpha, \mathrm{id})^* (\mathrm{id}, \beta)_! x \end{array} \qquad \begin{array}{ccc} (\mathrm{id}, \beta)_! (\alpha, \mathrm{id})^* x & \longrightarrow & (\alpha, \mathrm{id})^* (\mathrm{id}, \beta)_! x \\ & \searrow^{\beta_!(\tilde{\alpha})} & \downarrow \\ & & (\mathrm{id}, \beta)_! x. \end{array}$$

Here, the diagonal morphisms are the image of the cocartesian morphism $\tilde{\beta} : x \longrightarrow (\mathrm{id}, \beta)_! x$ under $\alpha^* : X_a \longrightarrow X_{a'}$ and the image of the cartesian morphism $(\alpha, \mathrm{id})^* : (\mathrm{id}, \alpha)^* x \longrightarrow x$ under $\beta_! : X_b \longrightarrow X_{b'}$. It follows that the interpolating morphism is an equivalence if and only if these images remain cocartesian and cartesian, respectively. This shows that condition (1) is equivalent to both (2) and (3). □

From this, we see that restricting the equivalence of Corollary 2.3.4 to orthofibrations gives the following.

Corollary 2.3.12. *Straightening over* A *and* B *gives natural equivalences*

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Cocart}(B)) \simeq \mathrm{Ortho}(A, B) \simeq \mathrm{Fun}(B, \mathrm{Cart}(A)).$$

Of course, one can now apply another instance of the straightening functor on both outer terms. We will discuss the result in §2.5 below. For now, let us instead specialise the discussion further. Since interpolating edges always lie over equivalences in $A \times B$, we find the following.

Proposition 2.3.13. *For a functor* $p : X \to A \times B$, *the following are equivalent:*

(1) *p is a conservative curved orthofibration.*
(2) *p is a curved orthofibration whose fibres are ∞-groupoids.*
(3) *p is a curved orthofibration and p_ℓ is a left fibration.*
(4) *p is a curved orthofibration and p_r is a right fibration.*
(5) *p_1 is a cartesian fibration and a morphism in X is p_1-cartesian if and only if it is sent to an equivalence by p_2, and p_2 is a cocartesian fibration and a morphism in X is p_2-cocartesian if and only if it is sent to an equivalence by p_1.*

If these conditions are satisfied, then p is in particular an orthofibration.

Definition 2.3.14. A *bifibration* is a functor $p = (p_1, p_2) : X \to A \times B$ satisfying the equivalent conditions of the previous proposition.

Restricting the equivalence of Corollary 2.3.12 to bifibrations gives:

Corollary 2.3.15. *Straightening over A and B give natural equivalences*

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathrm{LFib}(B)) \simeq \mathrm{Bifib}(A, B) \simeq \mathrm{Fun}(B, \mathrm{RFib}(A)).$$

2.4 | Gray fibrations

We saw above that curved orthofibrations over $A^{\mathrm{op}} \times B$ could be unstraightened to functors $A \to \mathrm{Cocart}^{\mathrm{lax}}(B)$. We can also consider the functors to $A \times B$ that correspond to such functors under *cocartesian* unstraightening over A, which leads to the following definition.

Definition 2.4.1. A *Gray fibration* over (A, B) is a functor $p : X \to A \times B$ such that p is cocartesian over A and $p_r : X_r \to \iota A \times B$ is a cocartesian fibration. We write $\mathrm{Gray}(A, B)$ for the subcategory of $\mathrm{Cat}/(A \times B)$ whose objects are the Gray fibrations, with morphisms required to preserve both types of cocartesian morphisms.

Dually, we say $p : X \to A \times B$ is an *op-Gray fibration* if p^{op} is a Gray fibration over $(A^{\mathrm{op}}, B^{\mathrm{op}})$, and denote the ∞-category they span by $\mathrm{OpGray}(A, B)$.

We will see in Corollary 5.2.10 below that Gray fibrations over (A, B) encode functors of $(\infty, 2)$-categories $A \boxtimes B \to \mathbf{Cat}$, where $\boxtimes$ denotes the Gray tensor product, which is the reason for the name. In particular, just as the Gray tensor product is not symmetric, let us point out that a Gray fibration $(p_1, p_2) : X \longrightarrow A \times B$ typically does *not* determine a Gray fibration $(p_2, p_1) : X \longrightarrow B \times A$.

Observation 2.4.2. From Corollary 2.2.2 and Remark 2.2.9, we see that a functor $p = (p_1, p_2) : X \to A \times B$ is a Gray fibration if and only if in the commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\;p\;} & A \times B \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & A, & \end{array}$$

p_1 is a cocartesian fibration, p takes p_1-cocartesian morphisms to pr_1-cocartesian morphisms, and for every $a \in A$, the map on fibres $X_a \to B$ is a cocartesian fibration.

Combining this observation with Corollary 2.2.4, and the same analysis as in Corollary 2.3.4, we see the following:

Corollary 2.4.3. *Straightening over A gives a natural equivalence*

$$\mathrm{Gray}(A, B) \simeq \mathrm{Fun}(A, \mathrm{Cocart}^{\mathrm{lax}}(B))^{\mathrm{cc}}.$$

Our next goal is to give an alternative characterisation of Gray fibrations, namely as those locally cocartesian fibrations that are cocartesian over certain triangles in the base. This characterisation will be the key to relating them to Gray tensor products below in §5.2. We first observe that Gray fibrations are in particular locally cocartesian fibrations:

Lemma 2.4.4. *Let $p : X \to A \times B$ be a Gray fibration. Then every morphism in X over (α,β) of the form $x \to (\mathrm{id},\beta)_! x \to (\alpha,\mathrm{id})_!(\mathrm{id},\beta)_! x$, where the first morphism is p-cocartesian over (id,β) and the second is p_r-cocartesian over (α,id), is locally p-cocartesian. In particular, p is a locally cocartesian fibration where all locally p-cocartesian morphisms are of this form, and we have a fully faithful inclusion*

$$\mathrm{Gray}(A,B) \subseteq \mathrm{LocCocart}(A \times B).$$

Proof. It follows from Proposition 2.1.10 that the morphisms of the given form are locally p-cocartesian, since any p_r-cocartesian morphism in X_r is in particular locally p-cocartesian. Thus, X has all locally p-cocartesian lifts, that is, p is a locally cocartesian fibration. Moreover, all locally p-cocartesian morphisms are of the given form by uniqueness.

It remains to show that a morphism $f : X \to Y$ between Gray fibrations over $A \times B$ preserves locally cocartesian morphisms if and only if it lies in $\mathrm{Gray}(A,B)$, which is immediate from the description of locally p-cocartesian morphisms in terms of the two types of cocartesian morphisms for a Gray fibration. □

Remark 2.4.5. It is immediate from the definition that any cocartesian fibration over $A \times B$ is a Gray fibration. Since $\mathrm{Cocart}(A \times B)$ is also a full subcategory of $\mathrm{LocCocart}(A \times B)$, it follows that we have a fully faithful inclusion

$$\mathrm{Cocart}(A \times B) \subseteq \mathrm{Gray}(A,B).$$

The following characterisation pins down exactly how Gray fibrations fit in between cocartesian and locally cocartesian fibrations.

Lemma 2.4.6. *A locally cocartesian fibration $p : X \longrightarrow A \times B$ is a Gray fibration if and only if it restricts to a cocartesian fibration over each triangle $\sigma : [2] \longrightarrow A \times B$ of one of the following forms:*

$$\begin{array}{ccc} (a,b) & \xrightarrow{(\mathrm{id},\beta)} & (a,b') \\ & {\scriptstyle (\mathrm{id},\beta'\beta)}\searrow & \downarrow{\scriptstyle (\mathrm{id},\beta')} \\ & & (a,b'') \end{array} \qquad \begin{array}{ccc} (a,b) & \xrightarrow{(\alpha,\mathrm{id})} & (a',b) \\ & {\scriptstyle (\alpha'\alpha,\mathrm{id})}\searrow & \downarrow{\scriptstyle (\alpha',\mathrm{id})} \\ & & (a'',b) \end{array} \qquad \begin{array}{ccc} (a,b) & \xrightarrow{(\alpha,\mathrm{id})} & (a',b) \\ & {\scriptstyle (\alpha,\beta)}\searrow & \downarrow{\scriptstyle (\mathrm{id},\beta)} \\ & & (a',b') \end{array}$$

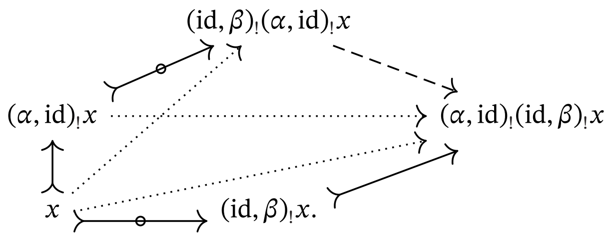
Proof. For a locally cocartesian fibration p, being cocartesian over the first type of triangle is equivalent to p_r being a cocartesian fibration by Corollary 2.1.11. Using Proposition 2.1.10, being cocartesian over the second and third types of triangles is equivalent to p being a locally cocartesian fibration such that for any two locally cocartesian arrows $x \longrightarrow x'$ and $x' \longrightarrow x''$ covering (α,id) and (α',β), respectively, their composition is locally cocartesian as well. By Proposition 2.1.10, this means precisely that p admits cocartesian lifts over $A \times \iota B$. □

We see that the difference between Gray and cocartesian fibrations lies in the fact that in a Gray fibration, the locally cocartesian lifts of the three edges in a diagram of the form

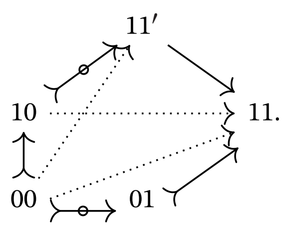
$$\begin{array}{ccc} (a,b) & \xrightarrow{(\mathrm{id},\beta)} & (a,b') \\ & {\scriptstyle (\alpha,\beta)}\searrow & \downarrow{\scriptstyle (\alpha,\mathrm{id})} \\ & & (a',b') \end{array}$$

need not form a commutative diagram. We now analyse the relationship between Gray fibrations and cocartesian fibrations over a product more closely.

Construction 2.4.7. Let $p : X \to A \times B$ be a Gray fibration. Consider an edge $(\alpha, \beta) : (a, b) \to (a', b')$ in $A \times B$ as above. Given a lift x of the source of this edge, we can choose p_r- and p-cocartesian lifts as in the solid part of



(Recall that tailed arrows denote p-cocartesian edges and tailed arrows marked by a circle denote p_r-cocartesian edges.) Now by Proposition 2.1.10, the composition $x \to (\alpha, \mathrm{id})_! x \to (\mathrm{id}, \beta)_!(\alpha, \mathrm{id})_! x$ along the top is still locally p-cocartesian, whence there exists an essentially unique dashed arrow as indicated making the diagram commute. More formally, consider the functor $\rho : [1] \longrightarrow \mathrm{Cocart}^{\mathrm{lax}}([1])$ from Construction 2.3.5. Then the cocartesian unstraightening of ρ over $[1]$ can be identified with the Gray fibration $Q' \longrightarrow [1] \times [1]$, where Q' is the poset



and the projection is the evident one, sending $11' \to 11$ to the identity. Then just as in Construction 2.3.5, evaluation at 00 induces an equivalence between the ∞-groupoid of maps of Gray fibrations

$$\begin{array}{ccc} Q' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ [1] \times [1] & \xrightarrow{\alpha \times \beta} & A \times B \end{array}$$

and $\iota X_{(a,b)}$.

Definition 2.4.8. If $p : X \longrightarrow A \times B$ is a Gray fibration, then a morphism $Q' \longrightarrow X$ of Gray fibrations is said to be a *p-interpolating diagram*. A morphism in X is said to be *p-interpolating* if it arises as the restriction of a p-interpolating diagram to $11' \longrightarrow 11$.

Note that we do not include in the notation whether an edge in X is regarded as an interpolating edge for a Gray or curved orthofibration, assuming that p is both (a situation we will have to explicitly consider later). We will be more explicit when the need arises.

Proposition 2.4.9. *Let $p: X \to A \times B$ be a Gray fibration. Then the following are equivalent:*

(1) *p is a cocartesian fibration,*
(2) *p restricts to a cocartesian fibration over each triangle $\sigma: [2] \to A \times B$ of the form*

$$\begin{array}{ccc} (a,b) & \xrightarrow{(\mathrm{id},\beta)} & (a,b') \\ & \underset{(\alpha,\beta)}{\searrow} & \downarrow{\scriptstyle (\alpha,\mathrm{id})} \\ & & (a',b'), \end{array}$$

(3) *Every p-interpolating edge in X is an equivalence.*
(4) *For every morphism $\alpha: a \to a'$ in A, the cocartesian transport functor $\alpha_!: X_a \to X_{a'}$ preserves cocartesian morphisms over B.*
(5) *The functor $A \to \mathrm{Cocart}^{\mathrm{lax}}(B)$ obtained by straightening p over A factors through the wide subcategory $\mathrm{Cocart}(B)$.*

Proof. Condition (1) immediately implies (2). Conversely, since p is a Gray fibration, it follows from (2) that locally p-cocartesian morphisms in X are closed under composition: Given morphisms $(\alpha, \beta): (a, b) \to (a', b')$ and $(\alpha', \beta'): (a', b') \to (a'', b'')$ in $A \times B$, we must show that for $x_{00} \in E_{(a,b)}$ the composite

$$x_{00} \to x_{11} \to x_{22}$$

of locally p-cocartesian morphisms over (α, β) and (α', β') is again locally p-cocartesian over $(\alpha'\alpha, \beta'\beta)$. We can expand this to a composite

$$x_{00} \to x_{10} \to x_{11} \to x_{12} \to x_{22}$$

of locally p-cocartesian morphisms over (α, id), (id, β), (id, β') and (α', id). Then the composite $x_{10} \to x_{12}$ is locally p-cocartesian over $(\mathrm{id}, \beta'\beta)$, and so the composite morphism $x_{00} \to x_{12}$ can alternatively be factored as $x_{00} \to x_{02} \to x_{12}$ where these morphisms are locally p-cocartesian over $(\mathrm{id}, \beta'\beta)$ and (α, id). Then $x_{02} \to x_{12} \to x_{22}$ is locally p-cocartesian over $(\alpha'\alpha, \mathrm{id})$, and so finally $x_{00} \to x_{02} \to x_{22}$ is locally cocartesian over $(\alpha'\alpha, \beta'\beta)$ as required. Hence p is a cocartesian fibration by Corollary 2.1.11.

From Construction 2.4.7, we see that (2) implies (3), since the p-interpolating morphisms are now obtained by factoring a morphism that is already locally p-cocartesian. Moreover, if all p-interpolating morphisms are invertible, we can also conclude that the composite of locally p-cocartesian morphisms over a triangle as in (2) factors as a locally p-cocartesian morphism followed by an equivalence, and hence is again locally p-cocartesian.

Since the cocartesian edges in X_a over B are precisely the locally p-cocartesian edges in X that lie over id_a, the equivalence of (3) and (4) is immediate from the definition of p-interpolating edges, while (5) is just a rephrasing of (4). □

The interpolating edges of a Gray fibration $p: X \to A \times B$ map to $\iota(A \times B)$ by construction. From the analogous assertion for cocartesian fibrations, we therefore immediately obtain the following.

Corollary 2.4.10. *A Gray fibration $p : X \to A \times B$ is a left fibration if and only if it is conservative, or equivalently if its fibres are ∞-groupoids.*

2.5 | Dualisation of curved orthofibrations and Gray fibrations

In the present section, we put together the pieces and analyse the dualisation equivalence promised in Theorem D.

Theorem 2.5.1. *Cocartesian straightening followed by cartesian unstraightening over A provides a natural equivalence*

$$\mathrm{D}^{\mathrm{ct}} : \mathrm{Gray}(A, B) \rightleftarrows \mathrm{CrvOrtho}(A^{\mathrm{op}}, B) : \mathrm{D}^{\mathrm{cc}} \tag{2.5.2}$$

between Gray fibrations over $A \times B$ and curved orthofibrations over $A^{\mathrm{op}} \times B$. It restricts to the identity if $A = *$ and is the usual dualisation equivalence between cartesian and cocartesian fibrations if $B = *$. In particular, for all $(a, b) \in A \times B$, there are canonical equivalences

$$\mathrm{D}^{\mathrm{ct}}(p)_{(a,b)} \simeq X_{(a,b)} \quad \text{and} \quad \mathrm{D}^{\mathrm{cc}}(q)_{(a,b)} \simeq Y_{(a,b)} \tag{2.5.3}$$

for every Gray fibration $p : X \to A \times B$ and curved orthofibration $q : Y \to A^{\mathrm{op}} \times B$.

Dually, there is an equivalence

$$\mathrm{D}^{\mathrm{ct}} : \mathrm{CrvOrtho}(A, B) \rightleftarrows \mathrm{OpGray}(A, B^{\mathrm{op}}) : \mathrm{D}^{\mathrm{cc}} \tag{2.5.4}$$

with the analogous properties.

Proof. Combine the straightening equivalence of Corollary 2.4.3 with the first equivalence of Corollary 2.3.4 to obtain

$$\mathrm{Gray}(A, B) \simeq \mathrm{Fun}(A, \mathrm{Cocart}^{\mathrm{lax}}(B))^{\mathrm{cc}} \simeq \mathrm{CrvOrtho}(A^{\mathrm{op}}, B).$$

The addenda are all immediate from the construction, and the dual case is obtained by using the equivalence from Corollary 2.3.4 combined with the dual of Corollary 2.4.3. □

Proposition 2.5.5. *Let $p : X \longrightarrow A \times B$ be a Gray fibration and let $q : Y \longrightarrow A^{\mathrm{op}} \times B$ be the dual curved orthofibration. For each $\alpha : a \longrightarrow a'$, $\beta : b \longrightarrow b'$ and $x \in X_{a',b} \simeq Y_{a',b}$, the canonical equivalence $X_{a,b'} \simeq Y_{a,b'}$ identifies the associated p-interpolating morphism from Definition 2.4.8 with the associated q-interpolating morphism from Definition 2.3.8.*

Proof. The statement immediately reduces to the case where $A = B = [1]$. Now by construction, the Gray fibration $Q' \longrightarrow [1] \times [1]$ from Definition 2.4.8 is dual to the curved orthofibration $Q \longrightarrow [1]^{\mathrm{op}} \times [1]$ from Construction 2.3.5, so it follows that the ∞-groupoid of interpolating diagrams $Q' \longrightarrow X$ and $Q \longrightarrow Y$ are equivalent. Since dualisation identifies the morphism $11' \longrightarrow 11$ in Q with $11' \longrightarrow 11$ in Q', dualisation preserves interpolating morphisms as well. □

Corollary 2.5.6. *The equivalences from Theorem 2.5.1 restrict to equivalences*

$$\mathrm{Cocart}(A\times B) \rightleftarrows \mathrm{Ortho}(A^{\mathrm{op}}, B) \qquad \textit{and} \qquad \mathrm{LFib}(A\times B) \rightleftarrows \mathrm{Bifib}(A^{\mathrm{op}}, B)$$

and dually

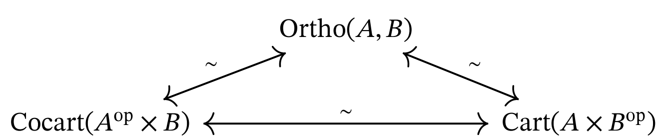
$$\mathrm{Ortho}(A, B) \rightleftarrows \mathrm{Cart}(A\times B^{\mathrm{op}}) \qquad \textit{and} \qquad \mathrm{Bifib}(A, B) \rightleftarrows \mathrm{RFib}(A\times B^{\mathrm{op}}).$$

Proof. The left-hand equivalences follow by replacing the use of Corollary 2.3.4 and Corollary 2.4.3 in Theorem 2.5.1 with Corollary 2.3.12 and straightening for (co)cartesian fibrations. Alternatively, using the previous proposition, they follow from characterisation (3) of Proposition 2.4.9. The statement about left and bifibrations follows by inspecting fibres. □

Remark 2.5.7.

(1) Equivalences as on the right were first constructed by Stevenson, by comparing both $\mathrm{Bifib}(A, B)$ and $\mathrm{LFib}(A^{\mathrm{op}}\times B)$ to an ∞-category of correspondences [27, Theorems C & D]. In the companion paper [14], we will prove a uniqueness result for the equivalences above that in particular shows that our equivalences restrict to those of Stevenson.

(2) From Corollary 2.5.6, we obtain a diagram of equivalences



where the lower maps are given by dualisation in a single variable (i.e. over $A^{\mathrm{op}}\times B$). It is not a priori clear that this diagram commutes, but this will also be a consequence of the results in [14]. Combined with the usual straightening equivalences for (co)cartesian fibrations, we similarly obtain two a priori different equivalences

$$\mathrm{Ortho}(A, B) \simeq \mathrm{Fun}(A^{\mathrm{op}}\times B, \mathrm{Cat}),$$

given by straightening first over A and then over B, or vice versa. Both restrict to equivalences

$$\mathrm{Bifib}(A, B) \simeq \mathrm{Fun}(A^{\mathrm{op}}\times B, \mathrm{Gpd})$$

and their agreement seems to be new even in this latter case.

(3) By restricting to one of the two legs in the previous point, the dualisation of bifibrations is also discussed in detail in [16, Section 5], [15, Section A.1] and [5, Section 7.1].

(4) In [14], we also supply a more explicit description of the equivalences in Theorem 2.5.1 based on span ∞-categories, generalising the work of Barwick, Glasman and Nardin [3] in the single-variable case.

As a typical example of the dualisation procedure above, consider the bifibration $(s,t)\colon \mathrm{Ar}(C) \to C\times C$. Its duals are the twisted arrow categories of C; let us briefly recall these to fix conventions.

Notation 2.5.8. For an ∞-category C, we write $\mathrm{Tw}^{\ell}(C)$ and $\mathrm{Tw}^{r}(C)$ for the left and right *twisted arrow ∞-category* of C. These are characterised by the natural equivalences

$$\mathrm{Map}([n], \mathrm{Tw}^{r}(C)) \simeq \mathrm{Map}([n] \star [n]^{\mathrm{op}}, C), \quad \mathrm{Map}\big([n], \mathrm{Tw}^{\ell}(C)\big) \simeq \mathrm{Map}([n]^{\mathrm{op}} \star [n], C),$$

so that $\mathrm{Tw}^{r}(C) = \mathrm{Tw}^{\ell}(C)^{\mathrm{op}}$. The natural inclusions of $[n]$ and $[n]^{\mathrm{op}}$ correspond to functors

$$(s,t) : \ \mathrm{Tw}^{\ell}(C) \longrightarrow C^{\mathrm{op}} \times C, \qquad (s,t) : \ \mathrm{Tw}^{r}(C) \longrightarrow C \times C^{\mathrm{op}},$$

which are a left fibration and a right fibration, respectively, both straightening to the mapping functor

$$\mathrm{Map}_C : \ C^{\mathrm{op}} \times C \to \mathrm{Gpd}.$$

Remark 2.5.9. Informally, the objects of $\mathrm{Tw}^{r}(C)$ are the morphisms in C. For morphisms $f : x \to y$, $f' : x' \to y'$ in C, a morphism from f to f' in $\mathrm{Tw}^{r}(C)$ is a commutative diagram

$$\begin{array}{ccc} x & \longrightarrow & x' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ x' & \longleftarrow & y'. \end{array}$$

Example 2.5.10. There are canonical equivalences

$$\mathrm{D}^{\mathrm{cc}}(\mathrm{Ar}(C) \to C \times C) \simeq \mathrm{Tw}^{\ell}(C) \to C^{\mathrm{op}} \times C$$

and

$$\mathrm{D}^{\mathrm{ct}}(\mathrm{Ar}(C) \to C \times C) \simeq \mathrm{Tw}^{r}(C) \to C \times C^{\mathrm{op}}.$$

This is proved, for example, in [15, Corollary A.2.5], based on Lurie's recognition criterion for twisted arrow categories [21, Corollary 5.2.1.2]. We supply another proof in [14] and also extend the statement to the (op)lax arrow and twisted arrow categories of an $(\infty, 2)$-category.

3 | PARAMETRISED AND MONOIDAL ADJUNCTIONS

In the present section, we will use the results of Section 2 to study the operation of taking adjoint functors in families. The statements we prove in §3.1 boil down to the fact that for any lax natural transformation $g : F \Rightarrow G$ between two diagrams of ∞-categories, such that each component of g is a right adjoint, the pointwise left adjoints assemble into an oplax natural transformation $f : G \Rightarrow F$. For the moment, we shall, however, stay in the fibrational picture and instead consider maps between the associated (co)cartesian fibrations. In particular, we will prove Theorem B from the introduction. The translation of this statement into the mate correspondence for (op)lax transformations will be delayed till §5.3. In §3.2 and §3.3, we carry out two consistency checks: In the former, we show that for each morphism, our functorial passage to fibrewise adjoints is given by the Beck–Chevalley construction on morphisms, and in the latter, we prove that the fibrewise adjoints we produce in the fibrational picture are characterised by the expected relation on morphism ∞-groupoids from the left to the right adjoint.

In §3.4, we then finally specialise the discussion to maps of ∞-operads and produce the correspondence between lax O-monoidal structures on a right adjoint functor and oplax O-monoidal structures on its left adjoint. In particular, we will prove Proposition A and Corollary C here.

3.1 | Parametrised adjunctions

We start by considering adjunctions in families over a base ∞-category B:

Definition 3.1.1. A map $g: C \longrightarrow D$ in $\mathrm{Cocart}^{\mathrm{lax}}(B)$ is said to be a *B-parametrised right adjoint* if it induces right adjoint functors between the fibres over each $b \in B$. Dually, a map $f: D \longrightarrow C$ in $\mathrm{Cart}^{\mathrm{opl}}(B)$ is said to be a *B-parametrised left adjoint* if it induces left adjoint functors between the fibres.

Let us write $\mathrm{Cocart}^{\mathrm{lax,R}}(B)$ and $\mathrm{Cart}^{\mathrm{opl,L}}(B)$ for the wide subcategories of $\mathrm{Cocart}^{\mathrm{lax}}(B)$ and $\mathrm{Cart}^{\mathrm{opl}}(B)$ whose maps are B-parametrised right and left adjoints, respectively.

As defined the categories $\mathrm{Cocart}^{\mathrm{lax,R}}(B)$ and $\mathrm{Cart}^{\mathrm{opl,L}}(B)$ are oblivious to the fact that there are non-invertible transformations between B-parametrised adjoints. As it is often important not to forget these when passing to adjoints (in the specialisation to symmetric monoidal categories in §3.4, they correspond to symmetric monoidal natural transformations, for example), we first enhance the ∞-categories from Definition 3.1.1 to $(\infty, 2)$-categories that encode natural transformations as their 2-morphisms.

For this, we will use the description of $(\infty, 2)$-categories as complete 2-fold Segal ∞-groupoids.

Definition 3.1.2. A *complete 2-fold Segal ∞-groupoid* is a functor $X: \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Gpd}$ such that

(1) the simplicial ∞-groupoids $X_{n,\bullet}$ and $X_{\bullet,m}$ satisfy the Segal condition for all n, m,
(2) the simplicial ∞-groupoid $X_{0,\bullet}$ is constant,
(3) the Segal ∞-groupoids $X_{\bullet,0}$ and $X_{1,\bullet}$ (and hence $X_{n,\bullet}$ for all n) are complete.

Note that by [20, Lemma 2.8], these conditions imply that $X_{\bullet,m}$ is also complete for all m.

We use the following general construction to enhance our ∞-categories to $(\infty, 2)$-categories.

Proposition 3.1.3. *Suppose $F: \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a limit-preserving functor such that for every ∞-category B, the functor $F(|B|) \to F(B)$ arising from the canonical map $B \to |B|$ induces a monomorphism $\iota F(|B|) \to \iota F(B)$ on underlying ∞-groupoids. If we define $F_{\mathrm{lc}}(B) \subseteq F(B)$ to be the full subcategory spanned by the image of $F(|B|)$ under this functor, then F_{lc} is also a limit-preserving functor $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, and the bisimplicial ∞-groupoid*

$$([n], [m]) \mapsto \mathrm{Map}_{\mathrm{CAT}}([n], F_{\mathrm{lc}}([m]))$$

is a complete 2-fold Segal space.

Proof. Since the map $B \to |B|$ is a natural transformation in B, we see that F_{lc} is a subfunctor of F. Note that the condition that $\iota F(|B|) \to \iota F(B)$ is a monomorphism implies that $\iota F(|B|) \to \iota F_{\mathrm{lc}}(B)$ is an equivalence. For any colimit of ∞-categories $B \simeq \mathrm{colim}_i B_i$, we then have a commutative diagram

$$\begin{array}{ccc} F(|B|) & \xrightarrow{\sim} & \lim_i F(|B_i|) \\ \downarrow & & \downarrow \\ F_{\mathrm{lc}}(B) & \longrightarrow & \lim_i F_{\mathrm{lc}}(B_i) \\ \downarrow & & \downarrow \\ F(B) & \xrightarrow{\sim} & \lim_i F(B_i), \end{array}$$

where the top and bottom horizontal maps are equivalences since F preserves limits (and $|-|: \mathrm{Cat} \to \mathrm{Gpd} \hookrightarrow \mathrm{Cat}$ preserves colimits), and the bottom right vertical map is fully faithful since fully faithful maps are closed under limits. Hence, the middle horizontal functor is also fully faithful, by the 2-of-3 property for equivalences applied to mapping ∞-groupoids. In the top square, the vertical morphisms are both given by equivalences on underlying ∞-groupoids, since this condition is also closed under limits. By the 2-of-3 property, it follows that the middle horizontal functor is also an equivalence on underlying ∞-groupoids, and hence, it is an equivalence. Thus, F_{lc} preserves limits.

It follows that the functor

$$([n],[m]) \mapsto \mathrm{Map}_{\mathrm{CAT}}([n], F_{\mathrm{lc}}([m]))$$

satisfies the Segal and completeness conditions levelwise in each variable, since these can be expressed as taking certain colimits in Cat to limits. It remains only to observe that for $n = 0$, the simplicial space $\iota F_{\mathrm{lc}}([m])$ is indeed constant: the unique map $[m] \to [0]$ is the localisation $[m] \to |[m]| \simeq *$ and so we know that the map $\iota F_{\mathrm{lc}}([0]) \to \iota F_{\mathrm{lc}}([m])$ is an equivalence; since $[0]$ is terminal in Δ, the diagram is then necessarily constant. □

Remark 3.1.4. If we regard a simplicial ∞-category $X : \Delta^{\mathrm{op}} \to \mathrm{Cat}$ that satisfies the Segal condition as a *double ∞-category* whose objects are the objects of X_0, horizontal morphisms are the morphisms in X_0, vertical morphisms are the objects of X_1, and squares are the morphisms in X_1, then the construction of Proposition 3.1.3 can be interpreted as extracting an $(\infty, 2)$-category from the double ∞-category $[n] \mapsto F([n])$ by forgetting the non-invertible vertical morphisms. Such a construction can be performed more generally, but the conditions in Lemma 3.1.3 seem required to ensure the resulting 2-fold Segal space is complete.

Returning to our specific situation, for any ∞-category A, we have natural equivalences

$$\mathrm{Map}_{\mathrm{CAT}}\big(A, \mathrm{Cocart}^{\mathrm{lax}}(B \times S)\big) \simeq \iota\mathrm{CrvOrtho}(A^{\mathrm{op}}, B \times S) \simeq \mathrm{Map}_{\mathrm{CAT}}\big(B \times S, \mathrm{Cart}^{\mathrm{opl}}(A)\big)$$

by Corollary 2.3.4. By the Yoneda lemma, this implies that for all B, the functor

$$\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}, \quad S \mapsto \mathrm{Cocart}^{\mathrm{lax}}(B \times S)$$

preserves limits. Moreover, on underlying ∞-groupoids, we have equivalences

$$\iota\,\mathrm{Cocart}^{\mathrm{lax}}(B\times S)\simeq \iota\,\mathrm{Cocart}(B\times S)\simeq \mathrm{Map}(B\times S,\mathrm{Cat})\simeq \mathrm{Map}(S,\mathrm{Fun}(B,\mathrm{Cat})),$$

so that the functor $\iota\,\mathrm{Cocart}^{\mathrm{lax}}(B\times|S|)\to\iota\,\mathrm{Cocart}^{\mathrm{lax}}(B\times S)$ corresponds to the functor

$$\mathrm{Map}(|S|,\mathrm{Fun}(B,\mathrm{Cat}))\to\mathrm{Map}(S,\mathrm{Fun}(B,\mathrm{Cat}))$$

given by composition with $S\to|S|$; this is therefore a monomorphism by the universal property of the localisation $|S|$, which says that $\mathrm{Fun}(|S|,X)\to\mathrm{Fun}(S,X)$ is fully faithful with image those functors that take all morphisms in S to equivalences.

Let us denote by

$$\mathrm{Cocart}^{\mathrm{lax}}_S(B\times S)\subseteq\mathrm{Cocart}^{\mathrm{lax}}(B\times S)$$

the full subcategory of cocartesian fibrations which are locally constant on S, that is, those obtained by pulling back a cocartesian fibration over $B\times|S|$, or equivalently those whose straightening to a functor $B\times S\to\mathrm{Cat}$ factors through the localisation to $B\times|S|$. Applying Proposition 3.1.3, we then have the following.

Corollary 3.1.5. *The functor*

$$\mathrm{Cat}^{\mathrm{op}}\longrightarrow\mathrm{CAT};\qquad S\longmapsto\mathrm{Cocart}^{\mathrm{lax}}_S(B\times S)$$

preserves limits, and the bisimplicial space

$$([m],[n])\longmapsto\mathrm{Map}_{\mathrm{CAT}}\Big([m],\mathrm{Cocart}^{\mathrm{lax}}_{[n]}(B\times[n])\Big)\tag{3.1.6}$$

is a complete 2-fold Segal ∞-groupoid. □

The same assertion holds if we instead take $\mathrm{Cocart}^{\mathrm{lax,R}}_S(B\times S)$, $\mathrm{Cart}^{\mathrm{opl}}_S(B\times S)$ or $\mathrm{Cart}^{\mathrm{opl,L}}_S(B\times S)$, which are all defined analogously.

Definition 3.1.7. Let B be a small ∞-category. We define $\mathbf{Cocart}^{\mathrm{lax}}(B)$ to be the $(\infty,2)$-category associated to the complete 2-fold Segal ∞-groupoid (3.1.6). Likewise, we define the $(\infty,2)$-category $\mathbf{Cart}^{\mathrm{opl}}(B)$ to be the $(\infty,2)$-category associated to the 2-fold complete Segal ∞-groupoid

$$([m],[n])\longmapsto\mathrm{Map}_{\mathrm{CAT}}\Big([m],\mathrm{Cart}^{\mathrm{opl}}_{[n]}(B\times[n])\Big)\tag{3.1.8}$$

We define the $(\infty,2)$-categories $\mathbf{Cocart}^{\mathrm{lax,R}}(B)$ and $\mathbf{Cart}^{\mathrm{opl,L}}(B)$ similarly.

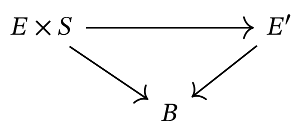
In the special case where $B=*$, the equivalent 2-fold complete Segal spaces

$$\mathbf{Cocart}^{\mathrm{lax}}(*)\simeq\mathbf{Cart}^{\mathrm{opl}}(*)$$

provide a model for the $(\infty,2)$-category $\mathbf{Cat}$ of ∞-categories (this is proved more precisely in Section 5.3). Consequently, we can identify

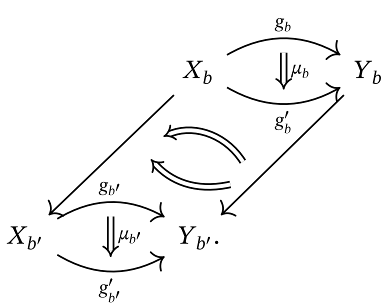
$$\mathbf{Cocart}^{\mathrm{lax,R}}(*) \simeq \mathbf{Cat}^{\mathrm{R}} \quad \text{and} \quad \mathbf{Cart}^{\mathrm{opl,L}}(*) \simeq \mathbf{Cat}^{\mathrm{L}}.$$

Observation 3.1.9. If the ∞-category S has contractible realisation (i.e. $|S| \simeq *$), then the objects of $\mathrm{Cocart}^{\mathrm{lax}}_S(B \times S)$ are by definition the cocartesian fibrations over $B \times S$ that are pulled back along the projection $B \times S \to B$, that is, those of the form $E \times S \to B \times S$ for a cocartesian fibration $E \to B$. A morphism between two such objects can then be identified with a commutative triangle



for cocartesian fibrations $E, E' \to B$. Note that this applies in particular for $S = [n]$. In particular, a 2-morphism in $\mathbf{Cocart}^{\mathrm{lax}}(B)$ is simply a natural transformation μ over B of maps g, g' between cocartesian fibrations $X \to B$ and $Y \to B$.

This we can view as a family of natural transformations $\mu_b : g_b \longrightarrow g'_b$ that commutes with the lax structure maps, in the sense that for each $b \longrightarrow b'$, there is a commuting diagram



Depicting this diagram cubically, it can also be viewed as a lax natural transformation between two functors $B \times [1] \longrightarrow \mathrm{Cat}$ that are constant along the interval. Note that $\mathbf{Cocart}^{\mathrm{lax,R}}(B) \subseteq \mathbf{Cocart}^{\mathrm{lax}}(B)$ is the 1-full sub-2-category whose morphisms are lax natural transformations consisting of right adjoints.

Remark 3.1.10. Note that for any two ∞-categories B and S, taking opposite ∞-categories defines an equivalence

$$(-)^{\mathrm{op}} : \ \mathrm{Cocart}^{\mathrm{lax}}_S(B \times S) \xrightarrow{\sim} \mathrm{Cart}^{\mathrm{opl}}_{S^{\mathrm{op}}}(B^{\mathrm{op}} \times S^{\mathrm{op}}).$$

Using this, one deduces that taking opposite ∞-categories defines an equivalence of $(\infty,2)$-categories, where in the target, the 2-morphisms are reversed

$$(-)^{\mathrm{op}} : \mathbf{Cocart}^{\mathrm{lax}}(B) \xrightarrow{\sim} \mathbf{Cart}^{\mathrm{opl}}(B^{\mathrm{op}})^{2-\mathrm{op}}.$$

We now come to our main technical result, Theorem B.

Theorem 3.1.11. *Let B be an ∞-category. Then there is a natural equivalence of $(\infty, 2)$-categories*

$$\mathrm{Adj}\colon \mathbf{Cocart}^{\mathrm{lax,R}}(B) \xrightarrow{\sim} \mathbf{Cart}^{\mathrm{opl,L}}(B^{\mathrm{op}})^{(1,2)-\mathrm{op}} \tag{3.1.12}$$

sending each cocartesian fibration to the cartesian fibration classifying the same functor $B \longrightarrow \mathrm{Cat}$. Here in the target, the directions of 1- and 2-morphisms are changed, as indicated.

In particular, for $B = *$, this produces an equivalence

$$\mathbf{Cat}^{\mathrm{R}} \simeq (\mathbf{Cat}^{\mathrm{L}})^{(1,2)-\mathrm{op}}.$$

For the proof, recall first that a functor $g\colon C \longrightarrow D$ between ∞-categories is a right adjoint if the corresponding cartesian fibration $p\colon X \longrightarrow [1]$ is a cocartesian fibration as well, see [21, Section 5.2.2]. Dually, a functor $f\colon D \longrightarrow C$ is a left adjoint if the corresponding cocartesian fibration is a cartesian fibration as well. In other words, one can encode adjunctions by functors $p\colon X \longrightarrow [1]$ that are simultaneously cartesian and cocartesian fibrations; the two adjoint functors can be extracted from this by (co)cartesian straightening.

We now extend this statement by showing that a functor between two B-parametrised categories (in the form of cocartesian fibrations over B) is a parametrised right adjoint if and only if the corresponding curved orthofibration over $[1] \times B$ is also a Gray fibration, when considered over $B \times [1]$, and similarly in the dual situation. More generally:

Lemma 3.1.13. *Let $p = (p_1, p_2)\colon X \longrightarrow A \times B$ be a functor. Then the following conditions are equivalent:*

1. *p is a curved orthofibration and the functor $A^{\mathrm{op}} \longrightarrow \mathrm{Cocart}^{\mathrm{lax}}(B)$ classifying p via Corollary 2.3.4 takes values in the wide subcategory $\mathrm{Cocart}^{\mathrm{lax,R}}(B)$,*
2. *p is a curved orthofibration whose restriction p_l to $A \times \iota(B)$ is a cocartesian fibration as well,*
3. *$\overline{p} = (p_2, p_1)\colon X \longrightarrow B \times A$ is a Gray fibration and the functor $B \to \mathrm{Cocart}^{\mathrm{lax}}(A)$ classifying B takes values in the full subcategory $\mathrm{Bicart}^{(\mathrm{op})\mathrm{lax}}(A)$,*
4. *$\overline{p} = (p_2, p_1)\colon X \longrightarrow B \times A$ is a Gray fibration whose restriction to $\iota(B) \times A$ is a cartesian fibration as well.*

Dually, a curved orthofibration $q = (q_1, q_2)\colon Y \longrightarrow B \times A$ classifies a functor $A \longrightarrow \mathrm{Cart}^{\mathrm{opl,L}}(B)$ via Corollary 2.3.4 if and only if q_r is a cartesian fibration as well, or equivalently if and only if $(q_2, q_1)\colon Y \longrightarrow A \times B$ is an op-Gray fibration, which is then automatically classified by a functor $A^{\mathrm{op}} \to \mathrm{Bicart}^{(\mathrm{op})\mathrm{lax}}(B)$.

Proof. For the equivalence between (1) and (2), we claim that one can check both conditions fibrewise in B. Namely, for (1), this follows by the naturality of the straightening equivalence (Corollary 2.3.4) in B, and for (2), this is an immediate consequence of Corollary 2.2.7. So, it suffices to prove that the two conditions are equivalent for $B = \{*\}$, where the assertion becomes that a cartesian fibration classifies a diagram of ∞-categories and right adjoints if and only it is also a cocartesian fibration, which is [23, Proposition 4.7.4.17]. The equivalence between (2) and (4) follows from characterisation (3) of curved orthofibrations in Proposition 2.3.3, and finally, the equivalence between (3) and (4) is part of Corollary 2.2.7. □

Let us write $\mathrm{M}(A,B)$ for the ∞-groupoid of functors $p\colon X \longrightarrow A\times B$ satisfying the equivalent conditions of Lemma 3.1.13, so that there are natural inclusions of path components

$$\iota\mathrm{Gray}(B,A) \longleftarrow M(A,B) \hookrightarrow \iota\mathrm{CrvOrtho}(A,B)$$

where the left inclusion sends $p=(p_1,p_2)$ to (p_2,p_1). Likewise, let us write $N(A,B)$ for the ∞-groupoid of functors $q\colon Y \longrightarrow B\times A$ satisfying the equivalent opposite conditions of Lemma 3.1.13, so that there are natural inclusions of path components

$$\iota\mathrm{OpGray}(A,B) \longleftarrow N(A,B) \hookrightarrow \iota\mathrm{CrvOrtho}(B,A).$$

More generally, let us write $M_S(A,B\times S)\subseteq M(A,B\times S)$ and $N_S(A,B\times S)\subseteq N(A,B\times S)$ for the natural subspaces spanned by fibrations $p\colon X\longrightarrow A\times(B\times S)$ such that each $X_{a,b}\longrightarrow S$ is locally constant, that is, the associated functor factors through $|S|$.

Corollary 3.1.14. *For any ∞-category B, unstraightening over $[m]$ provides natural equivalence of 2-fold complete Segal spaces*

$$\mathrm{Un}^{\mathrm{ct}}\colon \mathrm{Map}_{\mathrm{CAT}}\Big([m],\mathrm{Cocart}^{\mathrm{lax,R}}_{[n]}(B\times[n])\Big) \xrightarrow{\simeq} M_{[n]}\big([m]^{\mathrm{op}},B\times[n]\big)$$
$$\mathrm{Un}^{\mathrm{cc}}\colon \mathrm{Map}_{\mathrm{CAT}}\Big([m],\mathrm{Cart}^{\mathrm{opl,L}}_{[n]}(B\times[n])\Big) \xrightarrow{\simeq} N_{[n]}\big([m],B\times[n]\big).$$

Proof. Apply Lemma 3.1.13 and use that local constancy along $[n]$ can be checked when $[m]=*$, in which case the unstraightening functors are equivalent to the identity. □

Lemma 3.1.15. *The dualisation functor from Theorem 2.5.1 with respect to $B\times S$*

$$\mathrm{D}^{\mathrm{ct}}\colon \mathrm{Gray}(B\times S,A)\xrightarrow{\sim}\mathrm{CrvOrtho}((B\times S)^{\mathrm{op}},A)$$

restricts to an equivalence of ∞-groupoids

$$\mathrm{D}^{\mathrm{ct}}\colon \mathrm{M}_S(A,B\times S)\xrightarrow{\sim}\mathrm{N}_{S^{\mathrm{op}}}(A,(B\times S)^{\mathrm{op}}).$$

Proof. When $A=*$, dualisation over $B\times S$ simply sends cocartesian fibrations to their dual cartesian fibrations. This preserves local constancy in S and by naturality in A, one sees that dualisation preserves those objects that restrict to locally constant fibrations over $\{b\}\times S\times A$.

By the addenda of Theorem 2.5.1, for $B\times S=*$, the dualisation equivalence restricts to a natural self-equivalence of the ∞-category of cocartesian fibrations over A that is equivalent to the identity. By naturality in $B\times S$, one therefore sees that the dualisation preserves those objects that restrict for each $x\in B\times S$ to a bicartesian fibration over $\{x\}\times A$, as required. □

Proof of Theorem 3.1.11. Corollary 3.1.14 and Lemma 3.1.15 yield a natural equivalence

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAT}}\big(A^{\mathrm{op}},\mathrm{Cocart}^{\mathrm{lax,R}}_S(B\times S)\big) & \xrightarrow{\mathrm{Un}^{\mathrm{ct}}} & M_S(A,B\times S) \\ \sim\downarrow & & \downarrow \mathrm{D}^{\mathrm{ct}} \\ \mathrm{Map}_{\mathrm{CAT}}\big(A,\mathrm{Cart}^{\mathrm{opl,L}}_{S^{\mathrm{op}}}(B^{\mathrm{op}}\times S^{\mathrm{op}})\big) & \xleftarrow{\mathrm{Str}^{\mathrm{cc}}} & N_{S^{\mathrm{op}}}(A^{\mathrm{op}},B^{\mathrm{op}}\times S^{\mathrm{op}}) \end{array} \tag{3.1.16}$$

Taking A and S to be simplices, one obtains the desired equivalence between 2-fold Segal spaces $\mathbf{Cocart}^{\mathrm{lax,R}}(B) \simeq \mathbf{Cart}^{\mathrm{opl,L}}(B^{\mathrm{op}})^{(1,2)-\mathrm{op}}$. □

Example 3.1.17. A *two-variable adjunction* consists of functors $F : B \times C \to D$ and $G : B^{\mathrm{op}} \times D \to C$, together with a natural equivalence

$$\mathrm{Map}_D(F(b,c),d) \simeq \mathrm{Map}_C(c, G(b,d));$$

the prototypical example is the tensor-hom adjunction in a (left-)closed monoidal ∞-category. This is a special case of our parametrised adjunctions: It follows from the Yoneda lemma that given F, the functor G is uniquely determined and exists if and only if $F(b,-)$ is a left adjoint for all $b \in B$. We can then view F as a parametrised left adjoint

$$\begin{array}{ccc} B \times C & \xrightarrow{(\mathrm{pr}_1, F)} & B \times D \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & B. & \end{array}$$

Since the dual cocartesian fibration to $\mathrm{pr}_1 : B \times C \to B$ is the projection $B^{\mathrm{op}} \times C \to B^{\mathrm{op}}$, Theorem 3.1.11 produces a parametrised right adjoint in the form

$$\begin{array}{ccc} B^{\mathrm{op}} \times C & \xleftarrow{(\mathrm{pr}_1, G)} & B^{\mathrm{op}} \times D \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & B^{\mathrm{op}}. & \end{array}$$

At the moment, we only know that $G(b,-)$ is right adjoint to $F(b,-)$ for each b, but we will verify in Corollary 3.3.16 below that G indeed gives the expected natural equivalence on mapping ∞-groupoids. We will apply this fibrational description of two-variable adjunctions to analyse the monoidal properties of the internal mapping functor in Corollary 3.4.10.

3.2 | Identifying mates

Our goal in this subsection is to describe the effect on morphisms of the equivalence from Theorem 3.1.11 in terms of mates or Beck–Chevalley transformations, see Proposition 3.2.7 below.

In order to do this, let us first recollect how one can obtain the unit and counit of the adjunction from a bicartesian fibration $p : X \longrightarrow [1]$, using the following general construction:

Construction 3.2.1. Let $p : X \longrightarrow [1]$ be a cocartesian fibration and I any ∞-category. By [21, Proposition 3.1.2.1], post-composition with p determines a cocartesian fibration

$$p^I : \mathrm{Fun}(I, X) \to \mathrm{Fun}(I, [1]),$$

with cocartesian morphisms those natural transformations that are given by p-cocartesian morphisms at each object of I. Given a functor $\phi : I \to X$, its *cocartesian transport functor* $\phi_{\mathrm{cc}} : I \times [1] \to X$ is the diagram corresponding to the essentially unique p^I-cocartesian morphism with domain ϕ covering the map $p\phi \Rightarrow \mathrm{const}_1$ in $\mathrm{Fun}(I, [1])$. Alternatively, it is the unique diagram whose restriction to $I \times \{0\}$ is given by ϕ such that each $\phi_{\mathrm{cc}}(i)\colon [1] \longrightarrow X$ is p-cocartesian over $p(\phi(i) \leqslant 1)$.

Dually, for a cartesian fibration $p\colon X \longrightarrow [1]$ and a functor $\psi\colon I \longrightarrow X$, one can form the *cartesian transport functor* $\psi_{\mathrm{ct}}\colon I \times [1] \to X$ of ψ.

Example 3.2.2. Let $p\colon X \longrightarrow [1]$ be a cocartesian fibration classifying a functor $f\colon D \longrightarrow C$. Taking ϕ to be the fibre inclusion $i_0\colon D \simeq X_0 \hookrightarrow X$, one obtains a diagram $i_{0,\mathrm{cc}}\colon D \times [1] \to X$. The restriction to $D \times \{1\}$ gives a functor $D \longrightarrow X_1 \simeq C$ naturally equivalent to f (as a consequence of [21, Lemma 5.2.1.4]) and for each $d \in D$, the arrow $i_{0,\mathrm{cc}}(d)\colon d \rightarrowtail f(d)$ is p-cocartesian.

Construction 3.2.3. Let $p\colon X \longrightarrow [1]$ be a cartesian and cocartesian fibration classifying an adjoint pair $f\colon D \leftrightarrows C\ : g$. Applying Example 3.2.2 and taking the cartesian transport functor of the resulting diagram $i_{0,\mathrm{cc}}\colon D \times [1] \to X$ yields a functor

$$(i_{0,\mathrm{cc}})_{\mathrm{ct}}\colon D \times [1] \times [1] \to X,$$

which takes $y \in D$ to the square

$$\begin{array}{ccc} y & \xrightarrow{\eta_d} & gf(y) \\ \Vert & & \downarrow \\ y & \rightarrowtail & f(y). \end{array}$$

The functor $(i_{0,\mathrm{cc}})_{\mathrm{ct}}(-,-,0)$ factors through the fibre $D \simeq X_0$, and encodes the unit transformation $\eta\colon \mathrm{id}_D \Rightarrow gf$ of the adjunction classified by p. The above square shows that for a fixed object y, the unit $\eta_y\colon y \longrightarrow gf(y)$ is obtained by taking a cocartesian arrow $y \longrightarrow f(y)$ and factoring it as a fibrewise map followed by a cartesian map.

Dually, starting with the cartesian transport of the fibre inclusion $C \hookrightarrow X$ and then taking the cocartesian transport gives $(i_{1,\mathrm{ct}})_{\mathrm{cc}}\colon C \times [1] \times [1] \to X$ whose restriction to $C \times [1] \times \{1\}$ encodes the counit transformation $\epsilon\colon fg \Rightarrow \mathrm{id}_C$ of the adjunction.

To understand the behaviour of a B-parametrised right adjoint, let us start by showing that a map in $\mathrm{Cocart}^{\mathrm{lax}}(B)$ can roughly be viewed as a lax natural transformation; this picture will be made more precise in Section 5.

Construction 3.2.4. Let $g\colon C \longrightarrow D$ be a morphism in $\mathrm{Cocart}^{\mathrm{lax}}(B)$ and $\beta\colon b \longrightarrow b'$ one in B. Then g determines a natural transformation of the form

$$\begin{array}{ccc} C(b) & \xrightarrow{g} & D(b) \\ {\scriptstyle \beta_!}\downarrow & \swArrow{\rho_\beta} & \downarrow{\scriptstyle \beta_!} \\ C(b') & \xrightarrow[g]{} & D(b'). \end{array} \tag{3.2.5}$$

We will refer to this as the *β-component* of g. To see this, note that for each object $x \in C(b)$, the image of the cocartesian lift $\tilde{\beta}\colon x \longrightarrow \beta_! x$ under g factors uniquely as

$$g(\tilde{\beta})\colon g(x) \rightarrowtail \beta_! g(x) \xrightarrow{\rho_\beta(x)} g(\beta_! x).$$

Alternatively, Example 2.3.5 shows that $\rho_\beta(x)$ can also be obtained as the interpolating edge associated to x in the curved orthofibration $p : X \longrightarrow [1]^{\mathrm{op}} \times B$ classifying g.

To organise these interpolating morphisms $\rho_\beta(x)$ into a natural transformation, one can use a similar manoeuvre as in Construction 3.2.1 and consider the diagram

$$\begin{array}{ccc} \mathrm{Fun}(C(b), C) & \xrightarrow{\quad g_* \quad} & \mathrm{Fun}(C(b), D) \\ & \searrow \quad \swarrow & \\ & \mathrm{Fun}(C(b), B). & \end{array}$$

whose vertical maps are cocartesian fibrations. Applying the previous construction to the map $\beta :\ \mathrm{const}_b \Rightarrow \mathrm{const}_{b'}$ in the base and the fibre inclusion $C(b) \hookrightarrow C$ covering its domain const_b, we obtain the desired natural transformation $\rho_\beta :\ \beta_! g \Rightarrow g\beta_!$. This restricts to the interpolating maps $\rho_\beta(x)$ defined above because g_*-cocartesian arrows are given pointwise by g-cocartesian arrows (see [21, Proposition 3.1.2.1]).

When $g :\ C \longrightarrow D$ is a B-parametrised right adjoint, the lax commuting square (3.2.5) gives rise to a natural transformation between the fibrewise left adjoints and the change-of-fibre functors $\beta_!$.

Definition 3.2.6. Consider a lax commuting square of the form (3.2.5) such that the horizontal functors are part of adjunctions $f :\ D(b) \leftrightarrows C(b) :\ g$ and $f :\ D(b') \leftrightarrows C(b') :\ g$. Then the *Beck–Chevalley transformation* associated to ρ_β is the composition

$$f\beta_! \xRightarrow{f\beta_!\eta} f\beta_! gf \xRightarrow{f\rho_\beta f} fg\beta_! f \xRightarrow{\epsilon\beta_! f} \beta_! f.$$

We are now ready to describe the effect of the equivalence Adj from Theorem 3.1.11 on morphisms. To this end, let $g :\ C \longrightarrow D$ be a morphism in $\mathrm{Cocart}^{\mathrm{lax,R}}(B)$ and let $f = \mathrm{Adj}(g)$ be the induced morphism in $\mathrm{Cart}^{\mathrm{opl,L}}(B^{\mathrm{op}})^{\mathrm{op}}$. Construction 3.2.4 and the dual analysis for maps in $\mathrm{Cart}^{\mathrm{opl}}(B)$ show that for each $\beta :\ b \longrightarrow b'$ in B, the maps g and f give rise to lax commuting squares of the form

$$\begin{array}{ccc} C(b) & \xrightarrow{\ g\ } & D(b) \\ {\scriptstyle \beta_!}\downarrow & \overset{\rho_\beta}{\Swarrow} & \downarrow{\scriptstyle \beta_!} \\ C(b') & \xrightarrow[\ g\ ]{} & D(b') \end{array} \qquad\qquad \begin{array}{ccc} C(b) & \xleftarrow{\ f\ } & D(b) \\ {\scriptstyle (\beta^{\mathrm{op}})^*}\downarrow & \overset{\lambda_\beta}{\Nwarrow} & \downarrow{\scriptstyle (\beta^{\mathrm{op}})^*} \\ C(b') & \xleftarrow[\ f\ ]{} & D(b'). \end{array}$$

Note that in these diagrams, the vertical change-of-fibre functors are equivalent. The transformation λ_β is given by the Beck–Chevalley transformation associated to ρ_β, more precisely:

Proposition 3.2.7. *Let $g :\ C \longrightarrow D$ be a map in $\mathrm{Cocart}^{\mathrm{lax},R}(B)$, and $\beta :\ b \to b'$ a morphism in B, so that the β-component of g is given by (3.2.5). Then regarding β as a morphism in B^{op}, the β-component of $f = \mathrm{Adj}(g)$ is given by the Beck–Chevalley transformation associated to ρ_β, that is, λ_β is equivalent to the composition*

$$f\beta_! \xRightarrow{f\beta_!\eta} f\beta_! gf \xRightarrow{f\rho_\beta f} fg\beta_! f \xRightarrow{\epsilon\beta_! f} \beta_! f.$$

Proof. The equivalence $\mathrm{Adj}\colon\ \mathrm{Cocart}^{\mathrm{lax,R}}(B) \longrightarrow \mathrm{Cart}^{\mathrm{opl,L}}(B)$ is given at the level of morphisms by (3.1.16) for $A = [1]^{\mathrm{op}}$. In other words, consider a map $g\colon\ C \longrightarrow D$ in $\mathrm{Cocart}^{\mathrm{lax,R}}(B)$ and let $p = (p_1, p_2)\colon\ X \longrightarrow [1] \times B$ be the corresponding curved orthofibration, as in Lemma 3.1.13. Then the map $f = \mathrm{Adj}(g)\colon\ D \longrightarrow C$ in $\mathrm{Cart}(B^{\mathrm{op}})$ is the straightening of the curved orthofibration which is *dual* (relative to B) to the Gray fibration $\overline{p} = (p_2, p_1)\colon\ X \longrightarrow B \times [1]$.

Let us now fix $\beta\colon\ b \longrightarrow b'$ in B. For $x \in C(b) \simeq X_{1,b}$, Construction 3.2.4 and Example 2.3.5 identify $\rho_\beta(x)$ with the corresponding p-interpolating morphism $\beta_! g(x) \longrightarrow g\beta_!(x)$ in X, *where p is considered as a curved orthofibration*. On the other hand, for $y \in D(b) \simeq X_{0,b}$, (the dual of) Construction 3.2.4, Definition 2.4.8 and Proposition 2.5.5 show that $\lambda_\beta(y)\colon\ f\beta_! y \longrightarrow \beta_! f y$ is given by the associated $\overline{p}$-interpolating morphism, *where $\overline{p}$ is considered as a Gray fibration*.

To relate λ_β and ρ_β, take $y \in D(b)$ and consider the following diagram in X

$$\begin{array}{ccccc} y & \xrightarrow{\ \eta\ } & gf(y) & \twoheadrightarrow & f(y) \\ \big\downarrow\!\!\rightarrowtail & & \big\downarrow\!\!\rightarrowtail & & \big\downarrow\!\!\rightarrowtail \\ & & \beta_! gf(y) & & \\ & \nearrow^{\beta_!\eta} & \big\downarrow{\scriptstyle \rho_\beta f} & & \\ \beta_!(y) & \longrightarrow & g\beta_! f(y) & \twoheadrightarrow & \beta_! f(y), \end{array} \tag{3.2.8}$$

which we build in steps as follows; first we obtain the outer square as the essentially unique one in which the two vertical arrows are p-cocartesian (as always denoted as $\rightarrowtail$) and the map $y \longrightarrow f(y)$ is locally p-cocartesian. We then factor the two horizontal maps into a fibrewise map, followed by a p-cartesian morphism (denoted as $\twoheadrightarrow$). Finally, we factor the induced map $gf(y) \longrightarrow g\beta_! f(y)$ into a cocartesian map followed by a fibrewise one. Note that the right rectangle is then precisely the Q-diagram exhibiting $\rho_\beta(f(y))\colon\ \beta_! gf(y) \longrightarrow g\beta_! f(y)$ as a p-interpolating edge (Construction 2.3.5).

The resulting morphism $y \longrightarrow gf(y)$ in the top row is the unit of the adjoint pair (f, g) (at the fibre over b) and the map $\beta_!\eta$ exists since the left vertical map is p-cocartesian. Now note that the maps $\beta_!\eta$ and $\rho_\beta f$ are both contained in the fibre $X_{0,b'} \simeq D(b')$, so that choosing locally p-cocartesian lifts over $(0, b') \to (1, b')$ yields a commuting diagram

$$\begin{array}{ccccccc} \beta_!(y) & \xrightarrow{\ \beta_!\eta\ } & \beta_! gf(y) & \xrightarrow{\ \rho_\beta f\ } & g\beta_! f(y) & & \\ \big\downarrow\!\!\rightarrowtail & & \big\downarrow\!\!\rightarrowtail & & \big\downarrow\!\!\rightarrowtail & \searrow\!\!\!\!\twoheadrightarrow & \\ f\beta_!(y) & \xrightarrow[\ f\beta_!\eta\ ]{} & f\beta_! gf(y) & \xrightarrow[\ f\rho_\beta f\ ]{} & fg\beta_! f(y) & \xrightarrow[\ \epsilon\ ]{} & \beta_! f(y). \end{array}$$

Here the top lives in $X_{0,b'}$ and the bottom in $X_{1,b'}$. Pasting this diagram below (3.2.8), the resulting outer diagram determines a Q'-diagram in X (Definition 2.4.8) that exhibits the bottom composite $f\beta_!(y) \longrightarrow \beta_! f(y)$ as the $\overline{p}$-interpolating arrow associated to y. In particular, the bottom composite is equivalent to $\lambda_\beta(y)$.

To obtain the identification as natural transformations, we proceed as in Construction 3.2.4, replacing $g\colon\ C \longrightarrow D$ by $g\colon\ \mathrm{Fun}(D(b), C) \longrightarrow \mathrm{Fun}(D(b), D)$ and applying the above argument to the case where y is replaced by the fibre inclusion $\iota\colon\ D(b) \hookrightarrow D$, viewed as an object of $\mathrm{Fun}(D(b), D)$. Since the equivalences in Corollary 3.1.14 and Lemma 3.1.15 commute with taking functor categories (since by adjunction, they commute with products), it follows that λ_β is naturally equivalent to the Beck–Chevalley transformation associated to ρ_β, as asserted. □

3.3 | Parametrised correspondences

Our goal in this section is to derive a characterisation of parametrised adjoints, which we defined in terms of their associated fibrations in §3.1, by means of a natural equivalence analogous to the usual equivalence

$$\mathrm{Map}_C(f(d), c) \simeq \mathrm{Map}_D(d, g(c))$$

on mapping ∞-groupoids associated to an adjunction $f \dashv g$.

To motivate the form this will take, let us first observe that we can phrase the preceding condition in terms of left fibrations: f is left adjoint to g if there is an equivalence

$$(f^{\mathrm{op}} \times \mathrm{id}_D)^* \, \mathrm{Tw}^{\ell}(D) \simeq (\mathrm{id}_C \times g)^* \, \mathrm{Tw}^{\ell}(C) \tag{3.3.1}$$

of left fibrations over $C^{\mathrm{op}} \times D$, since the twisted arrow ∞-categories are the left fibrations for the mapping ∞-groupoid functors, and precomposition corresponds to pullback of left fibrations. We will prove a parametrised analogue of Equation (3.3.1); to state this, we first need some notation:

Notation 3.3.2. To simplify a number of formulae, we use $(-)^{\vee}$ to denote the cocartesian fibration dual to a cartesian fibration in this subsection.

Now suppose $p : E \to B$ is a cocartesian fibration, corresponding to a functor $F : B \to \mathrm{Cat}$. The natural transformation

$$\mathrm{Tw}^{\ell}(F(-)) \to F(-)^{\mathrm{op}} \times F(-)$$

then corresponds to a commutative triangle

$$\begin{array}{ccc} \mathrm{Tw}^{\ell}_B(E) & \longrightarrow & (E^{\mathrm{op}})^{\vee} \times_B E \\ & \searrow \quad \swarrow & \\ & B, & \end{array}$$

since $(E^{\mathrm{op}})^{\vee} \to B$ is the cocartesian fibration classified by $F(-)^{\mathrm{op}}$. Here $\mathrm{Tw}^{\ell}_B(E) \to (E^{\mathrm{op}})^{\vee} \times_B E$ is a left fibration by the dual of [21, Proposition 2.4.2.11] and the observation that a locally cocartesian fibration with ∞-groupoid fibres is automatically a left fibration.

Theorem 3.3.3. *If $g : C \to D$ is a B-parametrised right adjoint, with parametrised left adjoint $f : D^{\vee} \to C^{\vee}$, then there is an equivalence*

$$(f^{\mathrm{op}} \times_B \mathrm{id})^* \, \mathrm{Tw}^{\ell}_B(C) \simeq (\mathrm{id} \times_B g)^* \, \mathrm{Tw}^{\ell}_B(D)$$

of left fibrations over $(D^{\mathrm{op}})^{\vee} \times_B C$.

Before we embark upon the proof of Theorem 3.3.3, left us first observe that if $E \to [1]$ is the bicartesian fibration corresponding to an adjunction, then we can also phrase Equation (3.3.1) in terms of the *correspondence* associated to this functor, in the following sense.

Definition 3.3.4. A *correspondence* is a left fibration $X \to A^{\mathrm{op}} \times B$. We define the ∞-category Corr of correspondences by the pullback

$$\begin{array}{ccc} \mathrm{Corr} & \longrightarrow & \mathrm{LFib} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Cat} \times \mathrm{Cat} & \xrightarrow{(-)^{\mathrm{op}} \times (-)} & \mathrm{Cat}, \end{array}$$

where here LFib denotes the full subcategory of Ar(Cat) spanned by the left fibrations.

We use the following result from [27], see also [1]:

Theorem 3.3.5 (Stevenson). *There is an equivalence*

$$\mathrm{corr} \colon \mathrm{Cat}/[1] \xrightarrow{\sim} \mathrm{Corr},$$

over $\mathrm{Cat} \times \mathrm{Cat}$*, where the functor* $\mathrm{Cat}/[1] \to \mathrm{Cat} \times \mathrm{Cat}$ *is given by taking fibres over* 0 *and* 1*. The value of the functor* corr *on* $f \colon E \to [1]$ *is defined by the natural pullback square*

$$\begin{array}{ccc} \mathrm{corr}(E) & \longrightarrow & \mathrm{Tw}^{\ell}(E) \\ \downarrow & & \downarrow \\ E_0^{\mathrm{op}} \times E_1 & \longrightarrow & E^{\mathrm{op}} \times E, \end{array}$$

where E_0 and E_1 are the fibres of f over 0 and 1, respectively.

It is easy to check that if $E \to [1]$ is in fact a cocartesian fibration, corresponding to a functor $f \colon E_0 \to E_1$, then there is a pullback square

$$\begin{array}{ccc} \mathrm{corr}(E) & \longrightarrow & \mathrm{Tw}^{\ell}(E_1) \\ \downarrow & & \downarrow \\ E_0^{\mathrm{op}} \times E_1 & \xrightarrow{f^{\mathrm{op}} \times \mathrm{id}} & E_1^{\mathrm{op}} \times E_1, \end{array}$$

while if it is a cartesian fibration, corresponding to $g \colon E_1 \to E_0$, then we have a pullback

$$\begin{array}{ccc} \mathrm{corr}(E) & \longrightarrow & \mathrm{Tw}^{\ell}(E_0) \\ \downarrow & & \downarrow \\ E_0^{\mathrm{op}} \times E_1 & \xrightarrow{\mathrm{id} \times g} & E_0^{\mathrm{op}} \times E_0. \end{array}$$

Combining these squares, we get the equivalence Equation (3.3.1) when $E \to [1]$ corresponds to an adjunction. We now want to develop a parametrised version of this story.

Definition 3.3.6. A *B-parametrised correspondence* is a left fibration $X \to (E_0^{\mathrm{op}})^{\vee} \times_B E_1$ for cocartesian fibrations $E_0, E_1 \to B$. We define the ∞-category $\mathrm{Corr}(B)$ thereof by the pullback square

$$\begin{array}{ccc} \mathrm{Corr}(B) & \longrightarrow & \mathrm{LFib} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Cocart}(B) \times \mathrm{Cocart}(B) & \xrightarrow{(-^{\mathrm{op}})^\vee \times_B (-)} & \mathrm{Cat}. \end{array}$$

Using [21, Proposition 2.4.2.11] once again, we find that a B-parametrised correspondence is equivalently given by the data of two cocartesian fibrations $E_0 \to B$, $E_1 \to B$ and a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & (E_0^{\mathrm{op}})^\vee \times_B E_1 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

between cocartesian fibrations, such that f preserves cocartesian edges and f_a is a left fibration for every $b \in B$. Straightening this data in the base B, we get that $\mathrm{Corr}(B)$ is equivalently given by the following pullback:

$$\begin{array}{ccc} \mathrm{Corr}(B) & \longrightarrow & \mathrm{Fun}(B, \mathrm{LFib}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1^*} \\ \mathrm{Fun}(B, \mathrm{Cat}) \times \mathrm{Fun}(B, \mathrm{Cat}) & \xrightarrow{(-)^{\mathrm{op}} \times (-)} & \mathrm{Fun}(B, \mathrm{Cat}). \end{array}$$

Because $\mathrm{Fun}(B, -)$ preserves pullbacks, this implies that $\mathrm{Corr}(B)$ is equivalent to $\mathrm{Fun}(B, \mathrm{Corr})$.

Corollary 3.3.7. *There is an equivalence*

$$\mathrm{corr}_B : \ \mathrm{RCocart}([1], B) \xrightarrow{\sim} \mathrm{Corr}(B)$$

over $\mathrm{Cocart}(B) \times \mathrm{Cocart}(B)$*, where the functor* $\mathrm{RCocart}([1], B) \to \mathrm{Cocart}(B) \times \mathrm{Cocart}(B)$ *is given by taking fibres over 0 and 1. Given* $f : E \to [1] \times B$ *in* $\mathrm{RCocart}([1], B)$*, its value* $\mathrm{corr}_B(E)$ *is defined by the natural pullback square*

$$\begin{array}{ccc} \mathrm{corr}_B(E) & \longrightarrow & \mathrm{Tw}_B^{\ell}(E) \\ \downarrow & & \downarrow \\ (E_0^{\mathrm{op}})^\vee \times_B E_1 & \longrightarrow & (E^{\mathrm{op}})^\vee \times_B E, \end{array}$$

where $(E^{\mathrm{op}})^\vee \in \mathrm{RCocart}([1], B)$ *is obtained by dualising in the second variable.*

Proof. Unstraightening over B, we have the square

$$\begin{array}{ccc} \mathrm{RCocart}([1], B) & \xrightarrow{\mathrm{corr}_B} & \mathrm{Corr}(B) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathrm{Fun}(B, \mathrm{Cat}/[1]) & \xrightarrow{\mathrm{Fun}(B, \mathrm{corr})} & \mathrm{Fun}(B, \mathrm{Corr}), \end{array}$$

so the claim follows from Theorem 3.3.5. ☐

Proposition 3.3.8. *Suppose $p : E \to [1] \times B$ is a curved orthofibration, corresponding to a functor $g : E_1 \to E_0$ over B. Then there is a pullback square*

$$\begin{array}{ccc} \mathrm{corr}_B(E) & \longrightarrow & \mathrm{Tw}^{\ell}_B(E_0) \\ \downarrow & & \downarrow \\ (E_0^{\mathrm{op}})^{\vee} \times_B E_1 & \xrightarrow{\mathrm{id}\times_B g} & (E_0^{\mathrm{op}})^{\vee} \times_B E_0. \end{array}$$

In order to prove this, we first make some fibrational observations:

Proposition 3.3.9. *Consider a commutative square of ∞-categories*

$$\begin{array}{ccc} E & \xrightarrow{g} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y, \end{array}$$

where p and q are cocartesian fibrations and g takes p-cocartesian morphisms to q-cocartesian ones. For $x \in X$, let $g_x : E_x \to F_{fx}$ be the restriction of g to the fibres over x. If a morphism $\phi : e' \to e$ in E_x is g_x-cartesian, then ϕ is also g-cartesian.

Proof. For $e'' \in E$ over $x'' \in X$, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}_E(e'', e') & & \longrightarrow & & \mathrm{Map}_E(e'', e) \\ & \searrow & & \swarrow & \\ \downarrow & & \mathrm{Map}_X(x'', x) & & \downarrow \\ \mathrm{Map}_F(ge'', ge') & & \longrightarrow & & \mathrm{Map}_F(ge'', ge) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Map}_Y(fx'', fx), & & \end{array}$$

where we want to show that the back square is cartesian. It suffices to check that we have a cartesian square on the fibres over any $\xi \in \mathrm{Map}_X(x'', x)$, but since p and q are cocartesian fibrations and g preserves cocartesian morphisms, we can identify this square as

$$\begin{array}{ccc} \mathrm{Map}_{E_x}(\xi_! e'', e') & \longrightarrow & \mathrm{Map}_{E_x}(\xi_! e'', e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{F_{fx}}(f(\xi)_! ge'', ge') & \longrightarrow & \mathrm{Map}_{F_{fx}}(f(\xi)_! ge'', ge), \end{array}$$

which is cartesian by the assumption that ϕ is g_x-cartesian. ☐

Corollary 3.3.10. *For any functor $p: E \to B$, a morphism from $x' \to y'$ to $x \to y$ in $\mathrm{Tw}^\ell(E)$ of the form*

$$\begin{array}{ccc} x' & \longleftarrow & x \\ \downarrow & & \downarrow \\ y' & \longrightarrow & y \end{array}$$

is $\mathrm{Tw}^\ell(p)$-cartesian if $x \to x'$ is p-cocartesian and $y' \to y$ is p-cartesian. In particular, if $p: E \to B$ is a cartesian fibration, then $\mathrm{Tw}^\ell(E) \to \mathrm{Tw}^\ell(B)$ has cartesian lifts of morphisms of the form

$$\begin{array}{ccc} a & =\!= & a \\ \downarrow & & \downarrow \\ b & \xrightarrow{g} & b' \end{array}$$

in $\mathrm{Tw}^\ell(B)$.

Proof. To find that a morphism in $\mathrm{Tw}^\ell(E)$, in which $x \to x'$ is an equivalence and $y' \to y$ is p-cartesian, is $\mathrm{Tw}^\ell(p)$-cartesian, apply Proposition 3.3.9 to the square

$$\begin{array}{ccc} \mathrm{Tw}^\ell(E) & \xrightarrow{\mathrm{Tw}^\ell(p)} & \mathrm{Tw}^\ell(B) \\ \downarrow & & \downarrow \\ E^{\mathrm{op}} & \xrightarrow{p^{\mathrm{op}}} & B^{\mathrm{op}}, \end{array}$$

noting that on fibres over $x \in E$ with $b = p(x)$, we have the functor $E_{x/} \to B_{b/}$, where a morphism of the specified form is cartesian. Dually, applying Proposition 3.3.9 to the square

$$\begin{array}{ccc} \mathrm{Tw}^\ell(E) & \xrightarrow{\mathrm{Tw}^\ell(p)} & \mathrm{Tw}^\ell(B) \\ \downarrow & & \downarrow \\ E & \xrightarrow{p} & B, \end{array}$$

we find that a morphism in $\mathrm{Tw}^\ell(E)$ with $y' \to y$ an equivalence and $x \to x'$ p-cocartesian is $\mathrm{Tw}^\ell(p)$-cartesian. Since cartesian morphisms are closed under composition, the general case follows. □

Let $p: E \to A \times B$ be a curved orthofibration; recall from Proposition 2.3.3 that p can then be interpreted as a map of cocartesian fibrations over B. Applying $\mathrm{Tw}^\ell_B(-)$ to it gives the diagram

$$\begin{array}{ccc} \mathrm{Tw}^\ell_B(E) & \longrightarrow & \mathrm{Tw}^\ell(A) \times B \\ \downarrow & & \downarrow \\ (E^{\mathrm{op}})^\vee \times_B E & \longrightarrow & A^{\mathrm{op}} \times A \times B, \end{array}$$

where we use that $\mathrm{Tw}^\ell_B(A \times B) \simeq \mathrm{Tw}^\ell(A) \times B$. Applying Corollary 3.3.10 fibrewise and appealing to Proposition 3.3.9 again, we get the following.

Corollary 3.3.11. *Suppose $p : E \to A \times B$ is a curved orthofibration. Then*

$$\mathrm{Tw}^{\ell}_{B}(E) \to \mathrm{Tw}^{\ell}(A)$$

has cartesian lifts of morphisms in $\mathrm{Tw}^{\ell}(A)$ *of the form*

$$\begin{array}{ccc} a & = & a \\ \downarrow & & \downarrow \\ a'' & \xrightarrow{g} & a', \end{array}$$

given by the cartesian morphisms for $\mathrm{Tw}^{\ell}(E_b) \to \mathrm{Tw}^{\ell}(A)$ *described above.*

Observation 3.3.12. In particular, for any $a \in A$, the projection

$$\mathrm{Tw}^{\ell}_{B}(E) \times_{\mathrm{Tw}^{\ell}(A)} A_{a/} \to A_{a/}$$

is a cartesian fibration, and $\mathrm{Tw}^{\ell}_{B}(E) \times_{\mathrm{Tw}^{\ell}(A)} A_{a/} \to A_{a/} \times B$ is a curved orthofibration.

Notation 3.3.13. Given $E \to [1] \times B$ in $\mathrm{RCocart}([1], B)$, we have pullback squares

$$\begin{array}{ccc} \mathrm{Tw}^{\ell}_{B}(E)|_0 & \longrightarrow & \mathrm{Tw}^{\ell}_{B}(E) \\ \downarrow & & \downarrow \\ (E_0^{\mathrm{op}})^{\vee} \times_B E & \longrightarrow & (E^{\mathrm{op}})^{\vee} \times_B E, \end{array} \qquad \begin{array}{ccc} \mathrm{Tw}^{\ell}_{B}(E)|_1 & \longrightarrow & \mathrm{Tw}^{\ell}_{B}(E) \\ \downarrow & & \downarrow \\ (E^{\mathrm{op}})^{\vee} \times_B E_1 & \longrightarrow & (E^{\mathrm{op}})^{\vee} \times_B E. \end{array}$$

Proof of Proposition 3.3.8. Applying Observation 3.3.12 to $E \to [1] \times B$ and $0 \in [1]$, we see that $\mathrm{Tw}^{\ell}_{B}(E)|_0 \to [1]_{0/} \cong [1]$ is a cartesian fibration. Moreover, from the description of the cartesian morphisms, we see that

$$\mathrm{Tw}^{\ell}_{B}(E)|_0 \to (E_0^{\mathrm{op}})^{\vee} \times_B E$$

is a morphism between cartesian fibrations to [1] that preserves cartesian morphisms.

Taking fibres for $\mathrm{Tw}^{\ell}_{B}(E)|_0 \to (E_0^{\mathrm{op}})^{\vee} \times_B E$ over 0 and 1, we get the following diagram where both squares are cartesian:

$$\begin{array}{ccccc} \mathrm{Tw}^{\ell}_{B}(E_0) & \longrightarrow & \mathrm{Tw}^{\ell}_{B}(E)|_0 & \longleftarrow & \mathrm{corr}_B(E) \\ \downarrow & & \downarrow & & \downarrow \\ (E_0^{\mathrm{op}})^{\vee} \times_B E_0 & \longrightarrow & (E_0^{\mathrm{op}})^{\vee} \times_B E & \longleftarrow & (E_0^{\mathrm{op}})^{\vee} \times_B E_1. \end{array}$$

Taking the cartesian transport over $[0] \to [1]$, we therefore get a commutative square

$$\begin{array}{ccc} \mathrm{corr}_B(E) & \longrightarrow & \mathrm{Tw}^{\ell}_{B}(E_0) \\ \downarrow & & \downarrow \\ (E_0^{\mathrm{op}})^{\vee} \times_B E_1 & \xrightarrow{\mathrm{id} \times_B g} & (E_0^{\mathrm{op}})^{\vee} \times_B E_0. \end{array}$$

It remains to show that this square is cartesian, which we can check on fibres since the vertical maps are both left fibrations. To do this, we can first restrict to fibres over $b \in B$, where we have the square

$$\begin{array}{ccc} \mathrm{corr}(E_b) & \longrightarrow & \mathrm{Tw}^{\ell}(E_{0,b}) \\ \downarrow & & \downarrow \\ E_{0,b}^{\mathrm{op}} \times E_{1,b} & \xrightarrow{\mathrm{id}\times g_b} & E_{0,b}^{\mathrm{op}} \times E_{0,b}, \end{array}$$

which is cartesian because g_b is a right adjoint. □

Observation 3.3.14. If $p : E \to [1] \times B$ is in $\mathrm{RCocart}([1], B)$, then we also have $(p^{\mathrm{op}})^{\vee} : (E^{\mathrm{op}})^{\vee} \to [1]^{\mathrm{op}} \times B$ in $\mathrm{RCocart}([1]^{\mathrm{op}}, B)$. Since there is a natural equivalence $\mathrm{Tw}^{\ell}(C) \simeq \mathrm{Tw}^{\ell}(C^{\mathrm{op}})$ over the permutation $C^{\mathrm{op}} \times C \simeq C \times C^{\mathrm{op}}$, we get for a cocartesian fibration $X \to B$ a natural equivalence

$$\mathrm{Tw}_B^{\ell}(X) \simeq \mathrm{Tw}_B^{\ell}((X^{\mathrm{op}})^{\vee})$$

over $(X^{\mathrm{op}})^{\vee} \times_B X \simeq X \times_B (X^{\mathrm{op}})^{\vee}$, and hence also

$$\mathrm{corr}_B(E) \simeq \mathrm{corr}_B((E^{\mathrm{op}})^{\vee})$$

over $(E_0^{\mathrm{op}})^{\vee} \times_B E_1 \simeq E_1 \times_B (E_0^{\mathrm{op}})^{\vee}$.

Combining this with Proposition 3.3.8, we get the following dual version thereof:

Corollary 3.3.15. *Suppose $p : E \to [1] \times B$ is in $\mathrm{RCocart}([1], B)$. If $(p^{\mathrm{op}})^{\vee} : (E^{\mathrm{op}})^{\vee} \to [1]^{\mathrm{op}} \times B$ is a curved orthofibration, corresponding to a functor $f : E_0^{\vee} \to E_1^{\vee}$ over B^{op}, then there is a pullback square*

$$\begin{array}{ccc} \mathrm{corr}_B(E) & \longrightarrow & \mathrm{Tw}_B^{\ell}(E_1) \\ \downarrow & & \downarrow \\ (E_0^{\mathrm{op}})^{\vee} \times_B E_1 & \xrightarrow{f^{\mathrm{op}}\times_B \mathrm{id}} & (E_1^{\mathrm{op}})^{\vee} \times_B E_1. \end{array}$$

Proof of Theorem 3.3.3. Suppose that g corresponds to the curved orthofibration $p : E \to [1] \times B$. Then $p^{\vee} : E^{\vee} \to [1] \times B^{\mathrm{op}}$ is a curved orthofibration over $B^{\mathrm{op}} \times [1]$ whose cocartesian unstraightening over $[1]$ gives f. Hence $(p^{\mathrm{op}})^{\vee} : (E^{\mathrm{op}})^{\vee} \to [1]^{\mathrm{op}} \times B$ is the curved orthofibration for f^{op}. Combining Proposition 3.3.8 and Corollary 3.3.15, we thus get equivalences

$$(\mathrm{id} \times_B g)^* \mathrm{Tw}_B^{\ell}(D) \xrightarrow{\sim} \mathrm{corr}_B(E) \xleftarrow{\sim} (f^{\mathrm{op}} \times_B \mathrm{id})^* \mathrm{Tw}_B^{\ell}(C)$$

of left fibrations over $(D^{\mathrm{op}})^{\vee} \times_B C$. □

Specialising this to the case of projections, we get the following.

Corollary 3.3.16. *Let $F : X \times B \to Y$ be a functor such that $F(-, b)$ is a left adjoint for all $b \in B$, and let $G : Y \times B^{\mathrm{op}} \to X$ be the functor corresponding to the B^{op}-parametrised right adjoint of F, regarded as a functor $X \times B \to Y \times B$ over B. Then there is an equivalence*

$$(\mathrm{id} \times G)^* \operatorname{Tw}^{\ell}(X) \simeq (F^{\mathrm{op}} \times \mathrm{id})^* \operatorname{Tw}^{\ell}(Y)$$

of left fibrations over $X^{\mathrm{op}} \times Y \times B$, and hence a natural equivalence of mapping spaces

$$\operatorname{Map}_X(x, G(y, b)) \simeq \operatorname{Map}_Y(F(x, b), y).$$

3.4 | Lax monoidal adjunctions

Recall that an ∞-operad O is a map of ∞-categories $p : O \longrightarrow \mathrm{Fin}_*$ to the 1-category of pointed finite sets, satisfying the following conditions:

(1) O has all p-cocartesian lifts for inert morphisms in Fin_* (i.e. those maps which are bijections away from the basepoint).
(2) Let $x \in O$ be an object with $p(x) = \langle n \rangle$ and let $\rho^i_x : x \longrightarrow x_i$ be a p-cocartesian lift of the unique inert map $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$ which sends i to 1. For every $f : \langle m \rangle \longrightarrow \langle n \rangle$ and $y \in O_{\langle m \rangle}$, post-composition with the ρ^i_x induce an equivalence

$$\operatorname{Map}^f_O(y, x) \xrightarrow{\ \sim\ } \textstyle\prod_i \operatorname{Map}^{\rho^i \circ f}_O(y, x_i).$$

(3) For every tuple $(x_1, \dots, x_n)$ of objects in $O_{\langle 1 \rangle}$, there exists an $x \in O_{\langle n \rangle}$ together with p-cocartesian lifts $\rho^i_x : x \longrightarrow x_i$.

A morphism in O is called *inert* if it is the cocartesian lift of an inert map in Fin_*.

If O is an ∞-operad, then an *O-monoid* in an ∞-category X with finite products is a functor $M : O \to X$ satisfying the *Segal condition*: for every $x \in O_{\langle n \rangle}$ with inert maps $\rho^i_x : x \longrightarrow x_i$, the morphism $M(x) \to \prod_i M(x_i)$ is an equivalence. An *O-monoidal ∞-category* is a cocartesian fibration corresponding to an O-monoid in Cat (or equivalently an ∞-operad with a map to O that is a cocartesian fibration).

Definition 3.4.1. The $(\infty, 2)$-category $\mathbf{MonCat}^{\mathrm{lax}}_O$ of O-monoidal ∞-categories and *lax O-monoidal functors* between them is given by the 1-full sub-2-category of $\mathbf{Cocart}^{\mathrm{lax}}(O)$ whose:

(1) objects are O-monoidal ∞-categories,
(2) morphisms are functors $C^{\otimes} \longrightarrow D^{\otimes}$ over O that preserve the cocartesian morphisms lying over inert morphisms in O.

By definition, the underlying ∞-category of $\mathbf{MonCat}^{\mathrm{lax}}_O$ is the full subcategory of ∞-operads over O whose objects are the O-monoidal ∞-categories. A *strong* O-monoidal functor corresponds to a morphism $C^{\otimes} \longrightarrow D^{\otimes}$ that preserves all cocartesian edges.

Example 3.4.2. Let us explicitly mention the special case $O = \mathrm{Fin}_*$, where $\mathbf{MonCat}^{\mathrm{lax}}_O$ has objects symmetric monoidal ∞-categories, 1-morphisms lax symmetric monoidal functors, and 2-morphisms symmetric monoidal transformations. In particular, Theorem A from the introduction is a statement about morphism categories therein (and in the oplax analogue defined below).

Definition 3.4.3. The $(\infty, 2)$-category $\mathbf{MonCat}_{O}^{\mathrm{opl}}$ of O-monoidal ∞-categories and *oplax O-monoidal functors* between them is the sub-2-category of $\mathbf{Cart}^{\mathrm{opl}}(O^{\mathrm{op}})$ whose:

(1) objects are cartesian fibrations $C_{\otimes} \to O^{\mathrm{op}}$ corresponding to O-monoids,
(2) morphisms are functors $C_{\otimes} \longrightarrow D_{\otimes}$ that preserve cartesian morphisms lying over inert morphisms in O^{op}.

Note, in particular, that the objects in $\mathbf{MonCat}_{O}^{\mathrm{opl}}$ are a priori not O-monoidal ∞-categories: one has to take the cocartesian fibration over O dual to a cartesian fibration over O^{op} to get an O-monoidal ∞-category in the usual sense. The following lemma thus simply asserts that essentially *by definition*, an oplax O-monoidal functor is a lax O-monoidal functor between the opposite categories.

Lemma 3.4.4. *Taking opposite categories defines an equivalence of $(\infty, 2)$-categories*

$$(-)^{\mathrm{op}} : \mathbf{MonCat}_{O}^{\mathrm{opl}} \longrightarrow \left(\mathbf{MonCat}_{O}^{\mathrm{lax}}\right)^{2-\mathrm{op}}.$$

Proof. It suffices to verify that the equivalence of Remark 3.1.10 identifies the relevant sub-2-categories. Given a cartesian fibration $C_{\otimes} \longrightarrow O^{\mathrm{op}}$, let us write $C^{\mathrm{op},\otimes} \longrightarrow O$ for the opposite cocartesian fibration. The Segal map

$$(\rho_i^*)_i : C_{\otimes}(x) \longrightarrow \prod_i C_{\otimes}(x_i)$$

is then the *opposite* of the Segal map $(\rho_{i,!})_i : C^{\mathrm{op},\otimes}(x) \longrightarrow \prod_i C^{\mathrm{op},\otimes}(x_i)$, so that one is an equivalence if and only if the other is. Finally, a functor preserving cartesian lifts of inert morphisms is sent to the functor between opposite categories, which preserves cocartesian lifts of inert morphisms. □

For example, for $O = (\mathrm{Fin}_*)_{\mathrm{inert}}$, the trivial operad, $\mathbf{MonCat}_{O}^{\mathrm{opl}}$ is a 2-fold Segal space model for the $(\infty, 2)$-category $\mathbf{Cat}$ (by the discussion in Section 5.2). In this case, we find, unsurprisingly, that taking opposite categories defines an equivalence $\mathbf{Cat} \simeq \mathbf{Cat}^{2-\mathrm{op}}$.

Lemma 3.4.5. *Let $g : C^{\otimes} \longrightarrow D^{\otimes}$ be a lax O-monoidal functor, that is, a morphism in* $\mathbf{MonCat}_{O}^{\mathrm{lax}}$. *Then the following two conditions are equivalent:*

(1) *For every $x \in O_{\langle 1 \rangle}$, the induced map on fibres $g : C^{\otimes}(x) \longrightarrow D^{\otimes}(x)$ is a right adjoint.*
(2) *For every $x \in O$, the induced map on fibres $g : C^{\otimes}(x) \longrightarrow D^{\otimes}(x)$ is a right adjoint.*

Proof. This follows from the fact that for each $x \in O$, there is a commuting square

$$\begin{array}{ccc} C^{\otimes}(x) & \xrightarrow{g_x} & D^{\otimes}(x) \\ {\scriptstyle (\rho^i_!)_i}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle (\rho^i_!)_i} \\ \prod_i C^{\otimes}(x_i) & \xrightarrow[(g_{x_i})]{} & \prod_i D^{\otimes}(x_i) \end{array}$$

where $\rho^i : x \longrightarrow x_i$ are the canonical inert maps decomposing x into its components $x_i \in O_{\langle 1 \rangle}$. □

Definition 3.4.6. A lax O-monoidal functor $g\colon C^{\otimes} \longrightarrow D^{\otimes}$ is a *lax O-monoidal right adjoint* if it satisfies the equivalent conditions of Lemma 3.4.5.

Likewise, an oplax O-monoidal functor $f\colon C_{\otimes} \longrightarrow D_{\otimes}$ is called an *oplax O-monoidal left adjoint* if it induces left adjoint functors between the fibres over each $x \in O^{\mathrm{op}}$ (equivalently, all $x \in O^{\mathrm{op}}_{\langle 1\rangle}$).

Theorem 3.4.7. *For each ∞-operad O, there is a natural equivalence of $(\infty, 2)$-categories*

$$\mathrm{Adj}\colon \mathbf{MonCat}_O^{\mathrm{lax,R}} \xrightarrow{\ \sim\ } \left(\mathbf{MonCat}_O^{\mathrm{opl,L}}\right)^{(1,2)-\mathrm{op}}$$

between the 1*-full sub-2-categories whose morphisms are lax O-monoidal right adjoints and oplax O-monoidal left adjoints.*

Proof. It suffices to show that the equivalence of Theorem 3.1.11 identifies the two relevant sub-2-categories

$$\begin{array}{ccc}
\mathbf{MonCat}_O^{\mathrm{lax,R}} & \overset{\sim}{\dashrightarrow} & \left(\mathbf{MonCat}_O^{\mathrm{opl,L}}\right)^{(1,2)-\mathrm{op}} \\
\big\downarrow & & \big\downarrow \\
\mathbf{Cocart}^{\mathrm{lax,R}}(O) & \underset{\sim}{\xrightarrow{\mathrm{Adj}}} & \left(\mathbf{Cart}^{\mathrm{opl,L}}(O^{\mathrm{op}})\right)^{(1,2)-\mathrm{op}}
\end{array}.$$

At the level of objects, note that the functor Adj sends a cocartesian fibration over O to the cartesian fibration over O^{op} classifying the same functor $O \longrightarrow \mathrm{Cat}$. In particular, a cocartesian fibration over O satisfies the Segal conditions if and only if its image under Adj does.

It remains to verify that the functor Adj sends a map $g\colon C^{\otimes} \longrightarrow D^{\otimes}$ that preserves cocartesian lifts of inert maps to a functor $F\colon D_{\otimes} \longrightarrow C_{\otimes}$ of cartesian fibrations over O^{op} that preserves cartesian lifts of inert maps (for the reverse implication, reverse the roles of f and g in the next argument). By Proposition 3.2.7, this comes down to the following assertion: for any inert map $\beta\colon x \longrightarrow x'$ in O, the lax O-monoidal functor g defines the commuting left square

$$\begin{array}{ccc}
C^{\otimes}(x) & \xrightarrow{g_x} & D^{\otimes}(x) \\
{\scriptstyle \beta_!}\big\downarrow & & \big\downarrow{\scriptstyle \beta_!} \\
C^{\otimes}(x') & \xrightarrow[g_{x'}]{} & D^{\otimes}(x')
\end{array}
\qquad\qquad
\begin{array}{ccc}
C^{\otimes}(x) & \xleftarrow{f_x} & D^{\otimes}(x) \\
{\scriptstyle \beta_!}\big\downarrow & \Leftarrow & \big\downarrow{\scriptstyle \beta_!} \\
C^{\otimes}(x') & \xleftarrow[f_{x'}]{} & D^{\otimes}(x')
\end{array}$$

and we have to verify that the associated Beck–Chevalley transformation on the right is an equivalence. Using the Segal condition, these squares can be identified with

$$\begin{array}{ccc}
\prod_{i\in I} C^{\otimes}(x_i) & \xrightarrow{(g_{x_i})} & \prod_{i\in I} D^{\otimes}(x_i) \\
{\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}} \\
\prod_{j\in J} C^{\otimes}(x_j) & \xrightarrow[(g_{x_j})]{} & \prod_{j\in J} D^{\otimes}(x_j)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\prod_{i\in I} C^{\otimes}(x_i) & \xleftarrow{(f_{x_i})} & \prod_{i\in I} D^{\otimes}(x_i) \\
{\scriptstyle \mathrm{pr}}\big\downarrow & \Leftarrow & \big\downarrow{\scriptstyle \mathrm{pr}} \\
\prod_{j\in J} C^{\otimes}(x_j) & \xleftarrow[(f_{x_j})]{} & \prod_{j\in J} D^{\otimes}(x_j)
\end{array}$$

where the vertical functors are projections associated to an inclusion of finite sets $J \subseteq I$. But for such projections, the Beck–Chevalley transformation is always an equivalence (since the unit and counit maps can be computed in each factor). □

By considering morphism categories in the statement of Theorem 3.4.7, we find the following.

Corollary 3.4.8. *Given an ∞-operad $\mathcal{O}$ and two $\mathcal{O}$-monoidal ∞-categories $\mathcal{C}$ and $\mathcal{D}$, taking adjoints gives a canonical equivalence between the ∞-category of oplax $\mathcal{O}$-monoidal left adjoint functors $\mathcal{C} \to \mathcal{D}$ and the opposite of the ∞-category of lax $\mathcal{O}$-monoidal right adjoint functors.*

As another application of our machinery, we have the following result. Recall the ∞-operad $\mathbb{RM}\mathrm{od}$, defined in [23, Section 4.2.1], which encodes the data of an $\mathbb{E}_1$-algebra equipped with a right module.

Proposition 3.4.9. *Let $\mathcal{O}$ be an ∞-operad and let $\mathcal{C}$ be an $(\mathbb{E}_1 \otimes \mathcal{O})$-monoidal ∞-category, and $\mathcal{D}$ a right $\mathcal{O}$-monoidal module over it, that is, the pair $(\mathcal{C}, \mathcal{D})$ is equipped with the structure of an $\mathbb{RM}\mathrm{od} \otimes \mathcal{O}$-algebra. Suppose further that the action of $\mathcal{C}$ on $\mathcal{D}$ is colourwise closed, that is, the action $- \otimes c\colon \mathcal{D}(x) \to \mathcal{D}(x)$ admits a right adjoint $[c, -]\colon \mathcal{D}(x) \to \mathcal{D}(x)$ for every colour $x \in \mathcal{O}_{\langle 1 \rangle}$ and $c \in \mathcal{C}(x)$.*

Then the mapping object functors $[-,-]\colon \mathcal{D}(x)^{\mathrm{op}} \times \mathcal{D}(x) \to \mathcal{C}(x)$ admit a canonical lax $\mathcal{O}$-monoidal refinement.

Observing that every $\mathbb{E}_1 \otimes \mathcal{O}$-monoidal ∞-category acts on itself $\mathcal{O}$-monoidally (via the map $\mathbb{RM}\mathrm{od} \to \mathbb{E}_1$ from [23, Remark 4.2.1.5]) and that $\mathbb{E}_1 \otimes \mathbb{E}_n \simeq \mathbb{E}_{n+1}$ for $1 \leqslant n \leqslant \infty$, see [23, Theorem 5.1.2.2], we obtain the following.

Corollary 3.4.10. *Let $\mathcal{O}$ be an ∞-operad and $\mathcal{C}$ an $(\mathbb{E}_1 \otimes \mathcal{O})$-monoidal ∞-category such that the $\mathbb{E}_1$-monoidal ∞-categories $\mathcal{C}(x)$ are right closed for every colour $x \in \mathcal{O}_{\langle 1 \rangle}$. Then the mapping object functors $[-,-]\colon \mathcal{C}(x)^{\mathrm{op}} \times \mathcal{C}(x) \to \mathcal{C}(x)$ admit a canonical lax $\mathcal{O}$-monoidal refinement. In particular, if $\mathcal{C}$ is a closed $\mathbb{E}_{n+1}$-monoidal ∞-category for some $1 \leqslant n \leqslant \infty$, then $[-,-]\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ carries a canonical lax $\mathbb{E}_n$-monoidal refinement.* □

For $n = \infty$, such a lax symmetric monoidal refinement was first established by Hinich in [18, Section A.5] using different means; his construction most certainly agrees with ours, but let us refrain from attempting a formal comparison in this paper.

Proof of Proposition 3.4.9. Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, (\mathcal{D}^{\mathrm{op}})^{\otimes} \to \mathcal{O}$ be the cocartesian fibrations of operads witnessing the $\mathcal{O}$-monoidal structures of $\mathcal{C}, \mathcal{D}$ and $\mathcal{D}^{\mathrm{op}}$, respectively.

We follow the same strategy as in Example 3.1.17, and so wish to construct the morphism

$$\begin{array}{ccc} (\mathcal{D}^{\mathrm{op}})^{\otimes} \times_{\mathcal{O}} \mathcal{C}^{\otimes} & \xleftarrow{(\mathrm{pr}_1, [-,-])} & (\mathcal{D}^{\mathrm{op}})^{\otimes} \times_{\mathcal{O}} \mathcal{D}^{\otimes} \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & (\mathcal{D}^{\mathrm{op}})^{\otimes} & \end{array}$$

in $\mathbf{MonCat}^{\mathrm{lax,R}}_{(\mathcal{D}^{\mathrm{op}})^{\otimes}}$ from its counterpart in $\mathbf{MonCat}^{\mathrm{opl,L}}_{\mathcal{D}^{\otimes}}$ using Theorem 3.4.7. By definition of the tensor product of operads, we can regard $(\mathcal{C}, \mathcal{D})$ as an $\mathbb{RM}\mathrm{od}$-algebra in $\mathbf{MonCat}_{\mathcal{O}}$, that is, an

$\mathbb{E}_1$-algebra in $\mathbf{MonCat}_O$ equipped with a right module. So, in particular, the action

$$\otimes : D \times C \longrightarrow D$$

is itself a (strongly) O-monoidal functor. Applying cartesian unstraightening, we obtain a map

$$\mu : D_{\otimes} \times_O C_{\otimes} \longrightarrow D_{\otimes},$$

with which we form

$$\begin{array}{ccc} D_{\otimes} \times_O C_{\otimes} & \xrightarrow{(\mathrm{pr}_1, \mu)} & D_{\otimes} \times_O D_{\otimes} \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & D_{\otimes}. & \end{array}$$

By Lemma 3.4.5, this indeed defines a morphism in $\mathbf{MonCat}^{\mathrm{opl,L}}_{D_{\otimes}}$, which dualises as desired by Corollary 3.3.16. □

4 | PARAMETRISED UNITS AND COUNITS

Consider two symmetric monoidal ∞-categories $C^{\otimes}$ and $D^{\otimes}$ and a lax symmetric monoidal right adjoint $g : C^{\otimes} \to D^{\otimes}$, with left adjoint f. Given any finite collection of objects $\{y_i\}$ in D, we have a canonical comparison map $\bigotimes y_i \to g(\bigotimes f(y_i))$ given by

$$\bigotimes y_i \xrightarrow{\eta} gf\big(\bigotimes y_i\big) \xrightarrow{g(\mu)} g\big(\bigotimes f(y_i)\big),$$

where η is the unit of the adjunction $f \dashv g$ and μ is given by the oplax monoidal structure of f. This is the prototypical example of the parametrised unit morphism that we will consider in this section.

The goal of §4.1 is to make explicit the functoriality of these maps, and in §4.2, we similarly produce a functor extracting adjoint morphisms in a parametrised adjunction.

4.1 | Parametrised (co)units

Let us consider a parametrised left adjoint f over B with parametrised right adjoint g from Theorem 3.1.11

$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ & \searrow \quad \swarrow & \\ & B & \end{array} \qquad \begin{array}{ccc} C^{\vee} & \xrightarrow{g} & D^{\vee} \\ & \searrow \quad \swarrow & \\ & B^{\mathrm{op}}; & \end{array}$$

here and in the remainder of this section, we will again use $(-)^{\vee}$ to denote the cocartesian fibration dual to a cartesian fibration to ease notation.

Given any edge $\beta : b \to b'$ in B and any $y \in D_{b'}$, the natural map $\lambda_\beta(y) : f_b\beta^*(y) \longrightarrow \beta^* f_{b'}(y)$ (dual to Construction 3.2.4) is adjoint to a map

$$\eta_\beta(y) : \beta^*(y) \longrightarrow g_b f_b \beta^*(y) \xrightarrow{g_b\lambda_\beta(y)} g_b\beta^* f_{b'}(y).$$

One can also obtain $\eta_\beta(y)$ from the Beck–Chevalley transformation (Definition 3.2.6) as

$$\eta_\beta(y) : \beta^*(y) \longrightarrow \beta^* g_{b'} f_{b'}(y) \xrightarrow{\rho_\beta(f_{b'}(y))} g_b\beta^* f_{b'}(y).$$

The goal of this section is to describe the functoriality of this unit morphism $\eta_\beta(y)$ in β and y. To motivate the functoriality in β, let us consider the following.

Example 4.1.1. Let $\beta : b \longrightarrow b'$, $\gamma : b' \longrightarrow b''$ and $y \in D_{b''}$. Then we claim that

$$\eta_{\gamma\beta}(y) \simeq g_b\beta^*(\lambda_\gamma(y))\circ\eta_\beta(\gamma^*(y))$$

and dually that

$$\eta_{\beta\alpha}(z) \simeq \rho_\alpha(\beta^* f_{b''}(z))\circ\alpha^*\eta_\beta(z)$$

for any $\alpha : a \to b$ in B and $z \in D_{b'}$. To see the first identification, consider the diagram

$$\begin{array}{ccccc} f_b\beta^*\gamma^*(y) & \xrightarrow{\lambda_\beta(\gamma^*(y))} & \beta^* f_{b'}\gamma^*(y) & \longrightarrow\!\!\!\!\!\rightarrow & f_{b'}\gamma^*(y) & & \\ & & \downarrow{\scriptstyle \beta^*\lambda_\gamma(y)} & & \downarrow{\scriptstyle \lambda_\gamma(y)} & & \\ & & \beta^*\gamma^* f_{b''}(y) & \longrightarrow\!\!\!\!\!\rightarrow & \gamma^* f_{b''}(y) & \longrightarrow\!\!\!\!\!\rightarrow & f_{b''}(y) \end{array}$$

in C. The top row factors the image under f of the cartesian arrow $\beta^*\gamma^*(y) \longrightarrow \gamma^*(y)$ into $\lambda_\beta(\gamma^*(y))$, followed by a cartesian morphism (cf. Construction 3.2.4). Notice that the total composite along the top is the image under f of the cartesian arrow $\beta\gamma^*(y) \longrightarrow y$, and that following the bottom gives a factorisation of this into a fibrewise followed by a cartesian morphism. Therefore, we conclude that $\lambda_{\gamma\beta}(y) \simeq \beta^*\lambda_\gamma(y)\circ\lambda_\beta(\gamma^*(y))$. Applying g_b and precomposing with the unit of the adjoint pair (f_b, g_b) gives the claim.

The second identification arises from a dual analysis using the description of η_α in terms of the mate ρ_α.

Example 4.1.1 indicates how the arrow η_β depends on β via both pre- and postcomposition. Our goal will now be to make this precise by proving the following:

Theorem 4.1.2. *Let $f : D \longrightarrow C$ be a parametrised left adjoint over B with parametrised right adjoint g. Then there are canonical diagrams*

$$\begin{array}{ccc} D \times_B \mathrm{Tw}^\ell(B) & \xrightarrow{\eta} & \mathrm{Ar}(D^\vee) \\ \downarrow & & \downarrow \\ B^{\mathrm{op}} & \xrightarrow{\mathrm{const}} & \mathrm{Ar}(B^{\mathrm{op}}) \end{array} \qquad \begin{array}{ccc} C^\vee \times_{B^{\mathrm{op}}} \mathrm{Tw}^r(B^{\mathrm{op}}) & \xrightarrow{\epsilon} & \mathrm{Ar}(C) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\mathrm{const}} & \mathrm{Ar}(B) \end{array}$$

whose restrictions to $D \times_B \{\mathrm{id}_b\}$ and $C^\vee \times_{B^{\mathrm{op}}} \{\mathrm{id}_b\}$ for some b in B are equivalent to the unit and counit of the adjoint pair (f_b, g_b) respectively.

After some preliminaries, we will produce the functors η and ϵ in Construction 4.1.7 below. We will refer to these functors as the *parametrised unit* and *counit*, respectively.

To prepare the construction, let us write

$$p = (p_1, p_2)\colon X \longrightarrow B \times [1], \qquad q = (q_1, q_2)\colon X^\vee \longrightarrow [1] \times B^{\mathrm{op}}$$

for the curved orthofibrations classified by $f\colon D \longrightarrow C$ and $g\colon C^\vee \longrightarrow D^\vee$, respectively. Recall from Theorem 3.1.11 that $\overline{q} = (q_2, q_1)$ is the Gray fibration dual to the curved orthofibration p. In particular,

$$X_0 \simeq D, \qquad X_1 \simeq C, \qquad X_0^\vee \simeq D^\vee \quad \text{and} \quad X_1^\vee \simeq C^\vee.$$

Naively, one could try to imitate Construction 3.2.3 and construct the unit map using a cocartesian transport of the fibre inclusion $X_0 \hookrightarrow X$, followed by a cartesian transport. More precisely, since p_2 is a cocartesian fibration, we can form the cocartesian transport (Construction 3.2.1)

$$i_{0,\mathrm{cc}}\colon X_0 \times [1] \longrightarrow X$$

along p_2 of the fibre inclusion $i_0\colon X_0 \hookrightarrow X$; this takes $y \in X_{0,b} \simeq D_b$ to the cocartesian morphism $y \longrightarrow f_b(y)$. Dually, q_1 is a cartesian fibration and we can form the cartesian transport

$$j_{1,\mathrm{ct}}\colon X_1^\vee \times [1] \longrightarrow X^\vee$$

along q_1 of the fibre inclusion $j_1\colon X_1^\vee \hookrightarrow X^\vee$; this takes $x \in X_{b,1}^\vee \simeq C_b^\vee$ to the cartesian morphism $g_b(x) \longrightarrow x$.

To construct the unit as in Construction 3.2.3, we would now like take the cartesian transport of $i_{0,\mathrm{cc}}$ (and dually the cocartesian transport of $j_{1,\mathrm{ct}}$ for the counit). This can be done *fibrewise* over $b \in B$, but for a global construction, we will need to replace $p\colon X \longrightarrow B \times [1]$ by its dual $q\colon X^\vee \longrightarrow [1] \times B$, which is a cartesian fibration over [1]. Here we run into a problem; however, $i_{0,\mathrm{cc}}$ is a functor between curved orthofibrations which generally does *not* preserve cartesian arrows in the B-direction and hence does not induce a map between the dual fibrations. Indeed, for $\beta\colon b \longrightarrow b'$ and a cartesian morphism $\tilde{\beta}\colon \beta^* y \longrightarrow y$ in X_0, the image of the cartesian arrow $(\tilde{\beta}, 1)$ in $X_0 \times 1$ under $i_{0,\mathrm{cc}}$ is

$$f_b(\beta^* y) \longrightarrow f_{b'}(y).$$

This is cartesian for all β and y if and only if $f\colon C \longrightarrow D$ preserves cartesian morphisms over B.

To deal with this issue (and the dual issue for the counit map), we will first extend the functor $i_{0,\mathrm{cc}}$ to the *free* cartesian fibration on $X_0 \times [1]$ and then dualise over B. Let us therefore briefly recall the description of free fibrations from [8, Section 4].

Notation 4.1.3. Given a functor $\phi\colon E \to B$, we write

$$F_B^{\mathrm{cc}}(E) := E \times_B \mathrm{Ar}(B) \longrightarrow B; \quad (e, \phi(e) \to b') \longmapsto b',$$

where the pullback is formed along evaluation at 0 and the map to B is given by evaluation at 1. Dually, define

$$F_B^{\mathrm{ct}}(E) := E \times_B \mathrm{Ar}(B) \longrightarrow B; \quad (e, b \to \phi(e)) \longmapsto b,$$

where the pullback is formed along evaluation at 1 and the map to B is given by evaluation at 0.

We will need the following result from [8, Theorem 4.5].

Theorem 4.1.4. *The natural maps*

$$E \longrightarrow F_B^{\mathrm{ct}}(E), \qquad E \longrightarrow F_B^{\mathrm{cc}}(E)$$

over B induced by the constant diagram functor $B \longrightarrow \mathrm{Ar}(B)$, exhibit $F_B^{\mathrm{ct}}(E)$ and $F_B^{\mathrm{cc}}(E)$ as the free cartesian and cocartesian fibrations on $\phi : E \longrightarrow B$, respectively. In other words, the functors

$$F_B^{\mathrm{ct}} : \mathrm{Cat}_{/B} \longrightarrow \mathrm{Cart}(B), \qquad F_B^{\mathrm{cc}} : \mathrm{Cat}_{/B} \longrightarrow \mathrm{Cocart}(B)$$

are left adjoint to the forgetful functors $\mathrm{Cart}(B) \longrightarrow \mathrm{Cat}_{/B}$ *and* $\mathrm{Cocart}(B) \longrightarrow \mathrm{Cat}_{/B}$. □

Remark 4.1.5. Consider a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle p} & \\ & B, & \end{array}$$

where p is a cartesian fibration. We can extend this uniquely to a diagram

$$\begin{array}{ccccc} & & \overset{f}{\frown} & & \\ E & \longrightarrow & F_B^{\mathrm{ct}}(E) & \xrightarrow{\;\overline{f}\;} & E' \\ & {\scriptstyle\phi}\searrow & \downarrow & \swarrow{\scriptstyle p} & \\ & & B, & & \end{array}$$

where $\overline{f}$ preserves cartesian morphisms. Informally, the functor $\overline{f}$ is given by

$$\left(e, b \xrightarrow{\beta} \phi(e)\right) \longmapsto \beta^* f(e),$$

where $\beta^* f(e) \to f(e)$ is a cartesian morphism in E' over β.

Viewing $X_0 \times [1]$ as an ∞-category over B via the functor

$$X_0 \times [1] \longrightarrow X_0 \longrightarrow B,$$

we can then extend $i_{0,\mathrm{cc}}$ to the free cartesian fibration over B as in Remark 4.1.5. Similarly, we can extend $j_{1,\mathrm{ct}}$ to the free cocartesian fibration over B^{op}, giving

$$\bar{\imath}_{0,\mathrm{cc}} : F_B^{\mathrm{ct}}\big(X_0 \times [1]\big) \simeq F_B^{\mathrm{ct}}(X_0) \times [1] \longrightarrow X,$$

$$\bar{\jmath}_{1,\mathrm{ct}} : F_{B^{\mathrm{op}}}^{\mathrm{cc}}\big(X_1^\vee \times [1]\big) \simeq F_{B^{\mathrm{op}}}^{\mathrm{cc}}(X_1^\vee) \times [1] \longrightarrow X^\vee.$$

These can also be viewed as functors into arrow ∞-categories, informally given by

$$F_B^{\mathrm{ct}}(X_0) \longrightarrow \mathrm{Ar}(X); \quad \big(y \in X_{b',0}, b \xrightarrow{\beta} b'\big) \longmapsto \big(\beta^* y \to \beta^* f_{b'} y\big),$$

$$F_{B^{\mathrm{op}}}^{\mathrm{cc}}(X_1^\vee) \longrightarrow \mathrm{Ar}(X^\vee); \quad \big(x \in X_{b,1}^\vee, b \xrightarrow{\beta} b'\big) \longmapsto \big(\beta_!^{\mathrm{op}} g_b x \to \beta_!^{\mathrm{op}} x\big).$$

By construction, the functors $\bar{\imath}_{0,\mathrm{cc}}$ and $\bar{\jmath}_{1,\mathrm{ct}}$ preserve cartesian morphisms over B and cocartesian morphisms over B^{op}, respectively. Therefore, they induce functors between the dual (co)cartesian fibrations, and we obtain functors

$$\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}}) : \mathrm{D}^{\mathrm{cc}}\big(F_B^{\mathrm{ct}}(X_0) \times [1]\big) \simeq \mathrm{D}^{\mathrm{cc}}\big(F_B^{\mathrm{ct}}(X_0)\big) \times [1] \longrightarrow X^\vee,$$

$$\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}}) : \mathrm{D}^{\mathrm{ct}}\big(F_{B^{\mathrm{op}}}^{\mathrm{cc}}(X_1^\vee) \times [1]\big) \simeq \mathrm{D}^{\mathrm{ct}}\big(F_{B^{\mathrm{op}}}^{\mathrm{cc}}(X_1^\vee)\big) \times [1] \longrightarrow X.$$

The first equivalence in the two lines follows from the fact that the dual of a constant cocartesian fibration is a constant cartesian fibration. We now note that the domains of the functors $\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}})$ and $\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}})$ admit further simplification, by means of the following extension of the duality between arrow and twisted arrow ∞-categories from [13, Lemma 3.1.3].

Lemma 4.1.6. *For $\phi : E \longrightarrow B$, the duals of the free fibrations on ϕ can be identified as*

$$\mathrm{D}^{\mathrm{cc}}\big(F_B^{\mathrm{ct}}(E)\big) \simeq E \times_B \mathrm{Tw}^{\ell}(B) \to B^{\mathrm{op}},$$

$$\mathrm{D}^{\mathrm{ct}}\big(F_B^{\mathrm{cc}}(E)\big) \simeq E \times_B \mathrm{Tw}^{r}(B) \to B^{\mathrm{op}}$$

naturally in ϕ. □

We now have all the ingredients to construct the parametrised unit and counit from Theorem 4.1.2.

Construction 4.1.7. As before, let $p = (p_1, p_2) : X \longrightarrow B \times [1]$ and $q = (q_1, q_2) : X^\vee \longrightarrow [1] \times B^{\mathrm{op}}$ be the orthofibrations classified by f and g. Lemma 4.1.6 now implies that the functors $\bar{\imath}_{0,\mathrm{cc}}$ and $\bar{\jmath}_{1,\mathrm{ct}}$ have duals

$$\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}}) : X_0 \times_B \mathrm{Tw}^{\ell}(B) \times [1] \longrightarrow X^\vee,$$

$$\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}}) : X_1^\vee \times_{B^{\mathrm{op}}} \mathrm{Tw}^{r}(B^{\mathrm{op}}) \times [1] \longrightarrow X$$

which preserve cocartesian and cartesian morphisms over B, respectively. Now we can form the cartesian transport of $\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}})$ via q_1 and the cocartesian transport of $\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}})$ via p_2, respectively. This gives functors

$$(\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}}))_{\mathrm{ct}} : X_0 \times_B \mathrm{Tw}^{\ell}(B) \times [1] \times [1] \longrightarrow X^{\vee},$$
$$(\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}}))_{\mathrm{cc}} : X_1^{\vee} \times_{B^{\mathrm{op}}} \mathrm{Tw}^{r}(B^{\mathrm{op}}) \times [1] \times [1] \longrightarrow X.$$

We can informally describe these functors as follows: the value of $(\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}}))_{\mathrm{ct}}$ at an object $(y \in X_{b',0}, \beta : b \to b')$ is the square

$$\begin{array}{ccc} \beta_!^{\mathrm{op}}(y) & \longrightarrow & g_b \beta_!^{\mathrm{op}} f_{b'} y \\ \| & & \downarrow \\ \beta_!^{\mathrm{op}}(y) & \longrightarrow & \beta_!^{\mathrm{op}} f_{b'} y \end{array}$$

in $X^{\vee}$. Note that the top horizontal arrow takes values in $X_0^{\vee} \simeq D^{\vee}$. Dually, the value of $(\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}}))_{\mathrm{cc}}$ at $(x \in X_{b',1}^{\vee}, \beta : b \to b')$ is the square

$$\begin{array}{ccc} \beta_!^{*} g_{b'} x & \longrightarrow & \beta^{*} x \\ \downarrow & & \| \\ f_b \beta^{*} g_{b'} x & \longrightarrow & \beta^{*} x, \end{array}$$

whose bottom arrow is contained in $X_1 \simeq C$. We then obtain the desired parametrised unit and counit maps as the restrictions

$$\eta := (\mathrm{D}^{\mathrm{cc}}(\bar{\imath}_{0,\mathrm{cc}}))_{\mathrm{ct}}|_{X_0 \times_B \mathrm{Tw}^{\ell}(B) \times [1] \times \{0\}} : X_0 \times_B \mathrm{Tw}^{\ell}(B) \times [1] \longrightarrow X_0^{\vee}$$
$$\epsilon := (\mathrm{D}^{\mathrm{ct}}(\bar{\jmath}_{1,\mathrm{ct}}))_{\mathrm{cc}}|_{X_1^{\vee} \times_{B^{\mathrm{op}}} \mathrm{Tw}^{r}(B^{\mathrm{op}}) \times [1] \times \{1\}} : X_1^{\vee} \times_{B^{\mathrm{op}}} \mathrm{Tw}^{r}(B^{\mathrm{op}}) \times [1] \longrightarrow X_1.$$

Note that this construction is natural in X (and hence in $f : C \longrightarrow D$) and is compatible with base change along $B' \longrightarrow B$. In the case where $B = *$ is a point, the free fibration and dualisation functors are naturally equivalent to the identity [3, 28] and the above construction reduces to the construction of the (co)unit from Construction 3.2.3.

4.2 | Passing to adjoint morphisms

Next, we consider the functoriality of passing to the adjoint of a morphism in the parametrised setting. We first sketch a construction in the non-parametrised case, which will have the benefit of generalising readily. Given an adjunction

$$f : D \rightleftarrows C : g,$$

the unit transformation η fits in a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\eta} & \mathrm{Ar}(D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ C & \xrightarrow{g} & D. \end{array}$$

Here ev_1 is a cocartesian fibration, so we can extend η to the free cocartesian fibration on f, giving a commuting square

$$\begin{array}{ccc} F_C^{\mathrm{cc}}(D) & \xrightarrow{\overline{\eta}} & \mathrm{Ar}(D) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ C & \xrightarrow{g} & D. \end{array}$$

Unwinding definitions, we find that $\overline{\eta}$ takes $(d, f(d) \xrightarrow{\phi} c)$ to the composite $d \to gf(d) \xrightarrow{g(\phi)} g(c)$, that is, to the morphism adjoint to ϕ. We now give a parametrised version of this construction.

Construction 4.2.1. We keep the notation of Theorem 4.1.2 and let $f : D \longrightarrow C$ be a parametrised left adjoint over B, with right adjoint $g : C^\vee \longrightarrow D^\vee$. The parametrised unit η fits in a commutative square

$$\begin{array}{ccc} D \times_B \mathrm{Tw}^\ell(B) & \xrightarrow{\eta} & \mathrm{Ar}(D^\vee) \\ {\scriptstyle \mathrm{D}^{\mathrm{cc}}(\overline{f})}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ C^\vee & \xrightarrow{g} & D^\vee, \end{array}$$

where $\mathrm{D}^{\mathrm{cc}}(\overline{f})$ is obtained by first extending $f : D \to C$ to $\overline{f} : F_B^{\mathrm{ct}}(D) \to C$ and then dualising over B. At the level of objects, $\mathrm{D}^{\mathrm{cc}}(\overline{f})$ is therefore given by

$$\left(y \in D_{b'}, \beta : b \to b'\right) \longmapsto \beta^* f_{b'}(y).$$

Now we can extend η over the free cocartesian fibration on $\mathrm{D}^{\mathrm{cc}}(\overline{f})$, giving a commutative square

$$\begin{array}{ccc} F_{C^\vee}^{\mathrm{cc}}(D \times_B \mathrm{Tw}^\ell(B)) & \xrightarrow{\overline{\eta}} & \mathrm{Ar}(D^\vee) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ C^\vee & \xrightarrow{g} & D^\vee. \end{array} \tag{4.2.2}$$

Here $\overline{\eta}$ is given by the assignment

$$\left(y \in D_{b'}, b \xrightarrow{\beta} b', \beta^* f_{b'}(y) \xrightarrow{\phi} x\right) \longmapsto \left(\beta^* y \xrightarrow{\eta_\beta(y)} g_b \beta^* f_{b'} y \xrightarrow{g(\phi)} g_{b''} x\right),$$

where $x \in C^{\vee}_{b''}$. We can also pass to the dual cartesian fibrations, which gives a commutative square

$$\begin{array}{ccc} (D \times_B \mathrm{Tw}^{\ell}(B)) \times_{C^\vee} \mathrm{Tw}^r(C^\vee) & \xrightarrow{\bar{\eta}^\vee} & \mathrm{Tw}^r(D^\vee) \\ \downarrow & & \downarrow \\ (C^{\mathrm{op}})^\vee & \xrightarrow{g^{\mathrm{op}}} & (D^{\mathrm{op}})^\vee. \end{array} \tag{4.2.3}$$

Construction 4.2.1 encodes the functoriality of passing to the adjoint morphism in the generic case of a parametrised adjunction. However, if the parametrised adjunction has a particularly simple form, then the functoriality can be improved significantly.

Example 4.2.4. Recall from Example 3.1.17 that given a functor $f : D \times B \to C$ such that for each $b \in B$, $f(-, b) : D \to C$ is a left adjoint, the diagram

$$\begin{array}{ccc} D \times B & \xrightarrow{(f, \mathrm{id}_B)} & C \times B \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & B & \end{array}$$

is an example of a parametrised left adjoint. In this case, the parametrised unit from Theorem 4.1.2 is a functor

$$\eta : D \times \mathrm{Tw}^{\ell}(B) \times [1] \to D.$$

To an object $(y, b \xrightarrow{\beta} b')$, this assigns the map $y \to g(f(y, b'), b)$ adjoint to $f(y, b) \to f(y, b')$. To a morphism

$$\left(y_0 \to y_1, \quad \begin{array}{ccc} b_0 & \longleftarrow & b_1 \\ \downarrow & & \downarrow \\ b'_0 & \longrightarrow & b'_1 \end{array} \right)$$

it assigns the square

$$\begin{array}{ccc} y_0 & \longrightarrow & y_1 \\ \downarrow & & \downarrow \\ g(f(y_0, b'_0), b_0) & \longrightarrow & g(f(y, b'_1), b_1). \end{array}$$

Now we consider the commutative square (4.2.3) from Construction 4.2.1; in our special case, this simplifies to

$$\begin{array}{ccc} (D \times \mathrm{Tw}^{\ell}(B)) \times_{C \times B^{\mathrm{op}}} \mathrm{Tw}^r(C \times B^{\mathrm{op}}) & \longrightarrow & \mathrm{Tw}^r(D) \\ \downarrow & & \downarrow \\ C^{\mathrm{op}} \times B & \xrightarrow{g^{\mathrm{op}}} & D^{\mathrm{op}}. \end{array}$$

An object of $(D \times \mathrm{Tw}^{\ell}(B)) \times_{C \times B^{\mathrm{op}}} \mathrm{Tw}^{r}(C \times B^{\mathrm{op}})$ can be described as a list

$$(y, b \to b', f(y, b') \to x, b'' \to b),$$

and the top horizontal functor takes this to the composite

$$y \to g\big(f(y, b'), b\big) \to g(x, b) \to g(x, b'')$$

in $\mathrm{Tw}^{r}(D)$. For a morphism,

$$\left(\begin{array}{ccccccccc} y_0 & & b_0 & \longrightarrow & b_0' & f(y_0, b_0') & \longrightarrow & x_0 & b_0'' & \longrightarrow & b_0 \\ \downarrow & & \uparrow & & \downarrow & \downarrow & & \uparrow & \downarrow & & \uparrow \\ y_1, & & b_1 & \longrightarrow & b_1', & f(y_1, b_1') & \longrightarrow & x_1, & b_1'' & \longrightarrow & b_1 \end{array}\right)$$

we get in $\mathrm{Tw}^{r}(D)$ a morphism

$$\begin{array}{ccc} y_0 & \longrightarrow & y_1 \\ \downarrow & & \downarrow \\ g(x_0, b_0'') & \longleftarrow & g(x_1, b_1''). \end{array}$$

We note that this is an equivalence if the maps $y_0 \to y_1$, $x_1 \to x_0$ and $b_0'' \to b_1''$ are equivalences. This means our functor factors through the localisation of the ∞-category

$$(D \times \mathrm{Tw}^{\ell}(B)) \times_{C \times B^{\mathrm{op}}} \mathrm{Tw}^{r}(C \times B^{\mathrm{op}})$$

at these morphisms. Our final goal in this section is to identify this localisation, for which we first recall a result of Hinich.

Proposition 4.2.5 (Hinich). *Let $p : E \to B$ be a cocartesian fibration. Suppose for all $b \in B$, we have a collection W_b of morphisms in E_b such that for $\beta : b \to b'$ in B, the functor $\beta_! : E_b \to E_{b'}$ induced by p takes W_b into $W_{b'}$. Then we can form the cocartesian fibration $E' \to B$ corresponding to the functor $b \mapsto \mathrm{E}_b[W_b^{-1}]$. The canonical morphism of cocartesian fibrations $E \to E'$ exhibits E' as the localisation of E at the collection of morphisms $x \xrightarrow{\phi} x'$ such that $p(\phi)$ is an equivalence and $p(\phi)_! x \to x'$ is in $W_{p(x')}$.*

Proof. This is a special case of [19, Proposition 2.1.4] (or more precisely, of the stronger result that is actually proved in [19, Section 2.2]). See also [25, Proposition A.14] for a generalisation, as well as a more invariant proof. □

This allows us to prove the following.

Corollary 4.2.6. *Suppose $p : E \to B$ is a cocartesian fibration; then the identity map of E induces (via the free cocartesian fibration) a morphism of cocartesian fibrations $F_B^{\mathrm{cc}}(E) = E \times_B \mathrm{Ar}(B) \to E$;*

passing to the dual cartesian fibrations, we get a morphism of cartesian fibrations

$$E \times_B \mathrm{Tw}^r(B) \xrightarrow{\Phi} E^\vee$$

over B^{op}. *For any functor* $A \to B^{\mathrm{op}}$, *the induced morphism of cartesian fibrations*

$$E \times_B \mathrm{Tw}^r(B) \times_{B^{\mathrm{op}}} A \xrightarrow{\Phi'} E^\vee \times_{B^{\mathrm{op}}} A$$

exhibits $E^\vee \times_{B^{\mathrm{op}}} A$ *as a localisation.*

Proof. Suppose first that $A \to B$ is the identity. At the fibre over $b \in B^{\mathrm{op}}$, we get the functor

$$E \times_B B_{/b} \to E_b$$

taking $(x \in E_{b'}, b' \xrightarrow{\beta} b)$ to $\beta_! x$. This has a fully faithful right adjoint (taking $x \in E_b$ to (x, id_b)); hence, it is the localisation at the class W_b of morphisms $(x \xrightarrow{\phi} y, b' \xrightarrow{\gamma} b'' \xrightarrow{\beta} b)$ such that $\beta_! \gamma_! x \to \beta_! y$ is an equivalence. For $\beta : b \to b' \in B$, the functor induced by the cartesian fibration over B^{op}, $E \times_B B_{/b} \to E \times_B B_{/b'}$ is given by composition with β, and hence takes W_b to $W_{b'}$. The result then follows from (the dual of) Proposition 4.2.5. Finally, note that the conclusion of Proposition 4.2.5 is preserved under base change along any functor $A \to B$, and therefore, the general result follows. □

Taking p to be the identity of B, we obtain the following special case.

Corollary 4.2.7. *For any* ∞*-category* B*, the projection*

$$\mathrm{Tw}^r(B) \to B^{\mathrm{op}}$$

is a localisation, as is the functor

$$A \times_{B^{\mathrm{op}}} \mathrm{Tw}^r(B) \to A$$

for any functor $A \to B^{\mathrm{op}}$. □

Returning to the B-indexed family of left adjoints $f : D \times B \longrightarrow C$ from Example 4.2.4, we see that the functor

$$(D \times \mathrm{Tw}^\ell(B)) \times_{C \times B^{\mathrm{op}}} \mathrm{Tw}^r(C \times B^{\mathrm{op}}) \to \mathrm{Tw}^r(D)$$

obtained from the parametrised unit factors through $(D \times B) \times_C \mathrm{Tw}^r(C)$. We have thus proved the following.

Corollary 4.2.8. *Let* $f : D \times B \longrightarrow C$ *be a functor such that each* $f_b : D \longrightarrow C$ *is a left adjoint. Then there is a functor*

$$(D \times B) \times_C \mathrm{Tw}^r(C) \to \mathrm{Tw}^r(D),$$

which takes $(y, b, f(y, b) \to x)$ *to the adjoint map* $y \to g(x, b)$. □

Restricting to the fibre over $x \in C$, we see in particular:

Corollary 4.2.9. *In the situation of Corollary 4.2.8, for every $x \in C$, there is a natural map*

$$(D \times B) \times_C C_{/x} \to \mathrm{Tw}^r(D)$$

sending $(y, b, f(y, b) \to x)$ to the adjoint map $y \to g(b, x)$. □

Example 4.2.10. Let C be a closed symmetric monoidal ∞-category, with the tensor product viewed as a C-parametrised left adjoint as in Example 3.1.17. From Corollary 4.2.8, we obtain a functor

$$(C \times C) \times_C \mathrm{Tw}^r(C) \to \mathrm{Tw}^r(C),$$

taking $(x, y, x \otimes y \to z)$ to the adjoint map $x \to [y, z]$. Fixing $z \in C$, this specialises as in Corollary 4.2.9 to a natural functor

$$(C \times C) \times_C C_{/z} \to \mathrm{Tw}^r(C),$$

which sends $(x, y, x \otimes y \to z)$ to the adjoint morphism $x \to [y, z]$.

5 | LAX NATURAL TRANSFORMATIONS AND THE CALCULUS OF MATES

The goal of this final section is to prove Theorem E, that is, to produce straightening equivalences

$$\mathbf{Cocart}^{\mathrm{lax}}(B) \simeq \mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat}) \quad \text{and} \quad \mathbf{Cart}^{\mathrm{opl}}(B) \simeq \mathbf{Fun}^{\mathrm{opl}}(B^{\mathrm{op}}, \mathbf{Cat}),$$

where the right-hand $(\infty, 2)$-categories consist of functors $B \to \mathbf{Cat}$ as objects, (op)lax natural transformations as morphisms and modifications between these as 2-morphisms. Following [11], we define these ∞-categories as right adjoints to the (oplax) Gray tensor product for $(\infty, 2)$-categories constructed by Gagna, Lanari and Harpaz in [7], so that there are equivalences

$$\mathrm{Map}_{\mathrm{Cat}_2}(\mathbf{A} \boxtimes B, \mathbf{Cat}) \simeq \mathrm{Map}_{\mathrm{Cat}_2}\big(\mathbf{A}, \mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})\big),$$

$$\mathrm{Map}_{\mathrm{Cat}_2}(B \boxtimes \mathbf{A}, \mathbf{Cat}) \simeq \mathrm{Map}_{\mathrm{Cat}_2}\big(\mathbf{A}, \mathbf{Fun}^{\mathrm{opl}}(B, \mathbf{Cat})\big).$$

Their Gray tensor product is defined using Lurie's scaled simplicial sets from [22] as a model for $(\infty, 2)$-categories, and so, we begin with a short review of these in §5.1. In §5.2, we then show that Lurie's straightening equivalence for locally cocartesian fibrations restricts to an equivalence

$$\mathrm{Fun}(A \boxtimes B, \mathbf{Cat}) \simeq \mathrm{Gray}(A, B),$$

from which we then deduce Theorem E and Corollary F in §5.3.

5.1 | Scaled simplicial sets as a model for $(\infty, 2)$-categories

We start by recalling a few definitions.

Definition 5.1.1. A *marked simplicial set* is a pair (X, T) with X a simplicial set and $T \subseteq X_1$ a set of 1-simplices that contains the degenerate ones. Let sSet^+ denote the category of marked simplicial sets.

By [21, Theorem 3.1.5.1], the category sSet^+ has a model structure Quillen equivalent to the Joyal model structure on sSet, whose fibrant objects are precisely quasi-categories marked by their equivalences. We also write Cat^+_Δ for the category of *marked simplicial categories*, that is, categories enriched in marked simplicial sets.

Definition 5.1.2. A *scaled simplicial set* is a pair (X, S) with X a simplicial set and $S \subseteq X_2$ a set of 2-simplices that contains the degenerate ones. As usual, we will write $X^\sharp = (X, X_2)$ for X with the maximal scaling. Let $\mathrm{sSet}^{\mathrm{sc}}$ denote the category of scaled simplicial sets, with the morphisms being maps of simplicial sets that preserve the scalings.

We write $N^{\mathrm{sc}} : \mathrm{Cat}^+_\Delta \longrightarrow \mathrm{sSet}^{\mathrm{sc}}$ for the scaled nerve, which takes a marked simplicial category $\mathbf{C}$ to the coherent nerve $N\mathbf{C}$ of its underlying simplicial category, scaled by the set of 2-simplices $\Delta^2 \longrightarrow N\mathbf{C}$ corresponding to functors of simplicial categories $F : \mathfrak{C}(\Delta^2) \longrightarrow \mathbf{C}$ such that the edge $\Delta^1 = \mathfrak{C}(\Delta^2)(0,2) \longrightarrow \mathbf{C}(F(0), F(2))$ is marked; here $\mathfrak{C}$ denotes the path category functor, left adjoint to the coherent nerve N. Its upgrade to a left adjoint of N^{sc} we denote by $\mathfrak{C}^{\mathrm{sc}}$.

The following is [22, Theorem 4.2.7]:

Theorem 5.1.3 (Lurie). *There is a model structure on* $\mathrm{sSet}^{\mathrm{sc}}$ *where the cofibrations are the monomorphisms and the weak equivalences are the maps f such that $\mathfrak{C}^{\mathrm{sc}} f$ is a Dwyer–Kan equivalence of marked simplicial categories. Moreover, the adjunction $\mathfrak{C}^{\mathrm{sc}} \vdash N^{\mathrm{sc}}$ is a Quillen equivalence where* Cat^+_Δ *carries the marked Bergner model structure.* □

Remark 5.1.4. An explicit description of the fibrant objects in $\mathrm{sSet}^{\mathrm{sc}}$ in terms of lifting properties has been obtained by Gagna, Harpaz and Lanari in [6].

It is then a consequence of the main results of [22] that the underlying ∞-category of $\mathrm{sSet}^{\mathrm{sc}}$ is equivalent to Cat_2. Given this, a particularly simple description of the equivalence follows from work of Barwick and Schommer-Pries [4]: Let us write Θ_2 for the full subcategory of the (ordinary) category of strict 2-categories spanned by the strict 2-categories

$$[m]\big([n_1], \dots, [n_m]\big) = \quad 0 \underset{n_1}{\overset{0}{\underset{\Downarrow}{\overset{\Downarrow}{\cdots\cdots}}}} 1 \underset{n_2}{\overset{0}{\underset{\Downarrow}{\overset{\Downarrow}{\cdots\cdots}}}} \cdots \underset{n_m}{\overset{0}{\underset{\Downarrow}{\overset{\Downarrow}{\cdots\cdots}}}} m.$$

Since these have only identities as invertible k-morphisms, we obtain a full subcategory inclusion $\Theta_2 \hookrightarrow \mathrm{sSet}^{\mathrm{sc}}$ by viewing these 2-categories as marked simplicial categories with only degenerate edges marked and then applying the scaled nerve. Now consider the functor

$$\delta_2 : \Delta \times \Delta \longrightarrow \Theta_2 \hookrightarrow \mathrm{sSet}^{\mathrm{sc}} \quad \big([m], [n]\big) \longmapsto [m]([n], \dots, [n]). \tag{5.1.5}$$

It now follows from the main results of [4] that the derived mapping ∞-groupoids

$$\mathrm{Map}^h_{\mathrm{sSet}^{\mathrm{sc}}}(\delta_2(-,-), (X, S)) : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathrm{Gpd}.$$

form a complete 2-fold Segal ∞-groupoid, and that this assignment induces an equivalence between the ∞-category associated to the model structure on $\mathrm{sSet}^{\mathrm{sc}}$ from Theorem 5.1.3 and the

∞-category of complete 2-fold Segal spaces. In other words: scaled simplicial sets are a model for $(\infty, 2)$-categories.

Definition 5.1.6. We write $\mathbf{Cat}^{\mathrm{sc}}$ for the large scaled simplicial set $N^{\mathrm{sc}}(\mathrm{sSet}^{+,\circ})$, where the category $\mathrm{sSet}^{+,\circ}$ of fibrant marked simplicial sets is regarded as enriched in itself via its internal Hom.

In this section, we will use the $(\infty, 2)$-category associated to the scaled simplicial set $\mathbf{Cat}^{\mathrm{sc}}$ as our preferred model for the $(\infty, 2)$-category $\mathbf{Cat}$ of ∞-categories.

We now recall the definition of the (oplax) Gray tensor product in terms of scaled simplicial sets, as given in [7].

Definition 5.1.7. If (X, S) and (Y, T) are scaled simplicial sets, we define their *oplax Gray tensor product*

$$(X, S) \boxtimes (Y, T) = (X \times Y, S \boxtimes T)$$

to be the scaled simplicial set with underlying simplicial set $X \times Y$, with scaling $S \boxtimes T$ consisting of the 2-simplices of the forms:

- $(s_1\alpha, \tau)$ with $\alpha \in X_1, \tau \in T$,
- $(\sigma, s_0\beta)$ with $\sigma \in S, \beta \in Y_1$.

For simplicial sets X and Y, we will abbreviate $X^\sharp \boxtimes Y^\sharp$ to just $X \boxtimes Y$.

From [7, Theorem 2.14], we quote the following.

Theorem 5.1.8 (Gagna, Harpaz and Lanari). *The oplax Gray tensor product*

$$\boxtimes : \mathrm{sSet}^{\mathrm{sc}} \times \mathrm{sSet}^{\mathrm{sc}} \longrightarrow \mathrm{sSet}^{\mathrm{sc}}$$

is a left Quillen bifunctor. □

It follows that the oplax Gray tensor product induces a functor on the level of ∞-categories

$$- \boxtimes - : \mathrm{Cat}_2 \times \mathrm{Cat}_2 \longrightarrow \mathrm{Cat}_2,$$

which preserves colimits in each variable. As the name suggests, this is supposed to be thought of as a homotopy-coherent refinement of the standard oplax Gray tensor product for strict 2-categories [10]. This is supported by the following.

Proposition 5.1.9. *For any $m, n \geqslant 0$, there is a natural isomorphism between the oplax Gray tensor product $[m] \boxtimes [n]$ from Theorem 5.1.8 and the standard oplax Gray tensor product $[m] \boxtimes_{\mathrm{st}} [n]$ of $[m]$ and $[n]$, computed in strict 2-categories and depicted informally as*

$$\begin{array}{ccccccc}
00 & \longrightarrow & 10 & \longrightarrow & 20 & \dashrightarrow & m0 \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
01 & \longrightarrow & 11 & \longrightarrow & 21 & \dashrightarrow & m1 \\
\vdots & \swarrow & \vdots & \swarrow & \vdots & \swarrow & \vdots \\
0n & \longrightarrow & 1n & \longrightarrow & 2n & \dashrightarrow & mn.
\end{array} \tag{5.1.10}$$

Proof. Note that $\Delta[n]^\sharp$ is a scaled simplicial set model for $[n]$, viewed as an $(\infty, 2)$-category. The oplax Gray tensor product from Theorem 5.1.8 can then be modelled by the marked simplicial category $\mathfrak{C}^{\mathrm{sc}}(\Delta[m]^\sharp \boxtimes \Delta[m]^\sharp)$. Forgetting the marking, this simplicial category is the Boardman–Vogt resolution of $[m] \times [n]$ (cf. [24, Proposition 6.3.3]). Consequently, it can be identified with the simplicial category whose objects are tuples $x = (x^0, x^1)$ with $0 \leqslant x^0 \leqslant m$ and $0 \leqslant x^1 \leqslant n$, and where

$$\mathrm{Map}_{\mathfrak{C}^{\mathrm{sc}}(\Delta[m]^\sharp \boxtimes \Delta[n]^\sharp)}(x, y) = \mathrm{N}\big(\mathrm{Ch}_{x,y}\big)$$

is the nerve of the poset $\mathrm{Ch}_{x,y}$ of nondegenerate chains $\sigma = [x = x_0 < x_1 < \cdots < x_t = y]$ in $[m] \times [n]$ starting at x and ending at y, ordered by subchain inclusions. Composition is given by concatenation of chains. Furthermore, a subchain inclusion $\sigma' \subseteq \sigma$ is marked if it is obtained by removing one x_i from σ, such that either $x_i^0 = x_{i+1}^0$ or $x_{i-1}^1 = x_i^1$.

On the other hand, $[m] \boxtimes_{\mathrm{st}} [n]$ can be described as the following strict 2-category [11]: its objects are tuples $x = (x^0, x^1)$ with $0 \leqslant x^0 \leqslant m$ and $0 \leqslant x^1 \leqslant n$ and

$$\mathrm{Map}_{[m] \boxtimes_{\mathrm{st}} [n]}(x, y) = \mathrm{MaxCh}_{x,y}$$

is the poset whose objects are maximal nondegenerate chains from x to y, with order generated by

$$\downarrow_{\longrightarrow} \;\leq\; {}^{\longrightarrow}\!\downarrow$$

in the picture (5.1.10). Composition is concatenation of such chains. For each tuple x and y, we will specify a map of posets $\max :\ \mathrm{Ch}_{x,y} \longrightarrow \mathrm{MaxCh}_{x,y}$ as follows: for any chain σ from x to y in the grid (5.1.10), let $\sigma \subseteq \max(\sigma)$ be the unique maximal chain extending σ that is maximal with respect to the partial ordering on MaxCh: this means that every arrow in the chain σ going r steps right and d steps down is replaced by the maximal chain first going r steps right and then d steps down. One easily verifies that max is a map of posets, which sends every marked arrow in $\mathrm{Ch}_{x,y}$ to the identity. Furthermore, it is compatible with concatenation of chains. We therefore obtain a natural map

$$\phi :\ \mathfrak{C}^{\mathrm{sc}}(\Delta[m]^\sharp \boxtimes \Delta[m]^\sharp) \longrightarrow [m] \boxtimes_{\mathrm{st}} [n]$$

where we view $[m] \boxtimes_{\mathrm{st}} [n]$ as a marked simplicial category by taking nerves of mapping categories and marking equivalences (which in this case are just identities). To see that this is an equivalence, it remains to verify that $\max :\ \mathrm{Ch}_{x,y} \longrightarrow \mathrm{MaxCh}_{x,y}$ exhibits $\mathrm{MaxCh}_{x,y}$ as the localisation of $\mathrm{Ch}_{x,y}$ at the marked arrows. To see this, observe that the functor max is a cocartesian fibration. For each maximal chain τ, the inverse image of τ has a maximal element (τ itself) and for every other σ in the inverse image, the inclusion of chains $\sigma \subseteq \tau$ is a composite of marked arrows. It follows that the fibres of max have contractible realisation, so ϕ is an equivalence as desired. □

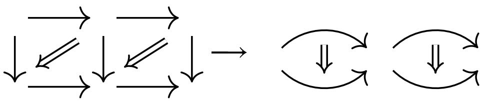
We will need the following observation about the functor δ_2.

Lemma 5.1.11. *For each $[m], [n] \in \Delta$, there is a natural map of $(\infty, 2)$-categories*

$$[m] \boxtimes [n] \longrightarrow \delta_2([m], [n]) = [m]([n], \ldots, [n]),$$

which exhibits the codomain as the localisation of $[m] \boxtimes [n]$ at all 1-morphisms contained in some $\{i\} \boxtimes [n]$.

Proof. Since $[m] \boxtimes [n]$ and $\delta_2([m],[n]) = [m]([n], \dots, [n])$ are both gaunt 2-categories (i.e. the only invertible 2-morphisms are the identities), the desired natural map $[n] \boxtimes [m] \longrightarrow [m]([n], \dots, [n])$ is simply the evident map of strict 2-categories that collapses all $\{i\} \boxtimes [n]$ to the ith vertex in $[m]([n], \dots, [n])$. For instance, for $[m] = [2]$ and $[n] = [1]$, it is given pictorially by the map collapsing the vertical 1-morphisms



To see that this is a localisation, note that both the domain and codomain are functors $\Delta \times \Delta \longrightarrow \mathrm{Cat}_2$ satisfying the co-Segal conditions; it therefore suffices to show this when $[n]$ and $[m]$ are 0 or 1, where the result is easily verified. □

5.2 | Scaled unstraightening of Gray fibrations

Let us now recall Lurie's straightening theorem for locally cocartesian fibrations over scaled simplicial sets.

Proposition 5.2.1. *If* $\mathbf{C}$ *is a marked simplicial category, then the marked simplicial category* $\mathrm{Fun}^+(\mathbf{C}, \mathrm{sSet}^+)^\circ$ *of fibrant–cofibrant objects in the projective model structure on the enriched functor category* $\mathrm{Fun}^+(\mathbf{C}, \mathrm{sSet}^+)$ *is weakly equivalent to* $\mathbf{Fun}^{\mathrm{sc}}(N^{\mathrm{sc}}(\mathbf{C}), \mathbf{Cat}^{\mathrm{sc}})$*, where* $\mathbf{Fun}^{\mathrm{sc}}(-,-)$ *denotes the internal Hom in scaled simplicial sets. In other words, the projective model structure on* $\mathrm{Fun}^+(\mathbf{C}, \mathrm{sSet}^+)$ *describes the* $(\infty, 2)$*-category of functors from* $\mathbf{C}$ *to* $\mathbf{Cat}$.

Proof. This follows from [21, Proposition A.3.4.13] since sSet^+ is an excellent model category by [21, Example A.3.2.22]. □

Definition 5.2.2. If (X, S) is a scaled simplicial set and $p : E \longrightarrow X$ is a locally cocartesian inner fibration, then we say that p is cocartesian over S if for every $\sigma : [2] \longrightarrow X$ in S, the base change $\sigma^* E \longrightarrow [2]$ is a cocartesian inner fibration.

Theorem 5.2.3 (Lurie). *Let* (X, S) *be a scaled simplicial set. Then there is a left proper combinatorial marked simplicial model structure on the slice category* $\mathrm{sSet}^+/X^\sharp$ *(where* $X^\sharp$ *denotes* X *with all 1-simplices marked) such that the cofibrations are the monomorphisms, and an object* $(E, T) \xrightarrow{p} X^\sharp$ *is fibrant if and only if*

1. *the underlying map of simplicial sets* $p : E \longrightarrow X$ *is a locally cocartesian inner fibration,*
2. T *is precisely the set of locally* p*-cocartesian edges in* E*,*
3. *the locally cocartesian inner fibration* p *is cocartesian over* S.

We write $\mathrm{sSet}^+_{(X,S)}$ *for* $\mathrm{sSet}^+/X^\sharp$ *equipped with this model structure.*

Proof. As a simplicial model category, this is a special case of [22, Theorem 3.2.6], applied to the categorical pattern $(X, X_1, S, \emptyset)$. The marked simplicial enrichment follows from [22, Remark 3.2.26]. □

Theorem 5.2.4 (Lurie). *If (X, S) is a scaled simplicial set, then there is a marked simplicial Quillen equivalence*

$$\mathrm{Str}^{\mathrm{sc}}_{(X,S)} : \mathrm{sSet}^+_{(X,S)} \rightleftarrows \mathrm{Fun}^+(\mathfrak{C}^{\mathrm{sc}}(X,S), \mathrm{sSet}^+) : \mathrm{Un}^{\mathrm{sc}}_{(X,S)}$$

where $\mathrm{Fun}^+(\mathfrak{C}^{\mathrm{sc}}(X,S), \mathrm{sSet}^+)$ is equipped with the projective model structure.

Proof. As an (unenriched) Quillen equivalence, this follows from [22, Theorem 3.8.1]. The compatibility with the simplicial enrichment is discussed in [22, Remark 3.8.2], and the same argument clearly extends to show that this is a marked simplicial adjunction. □

This marked simplicial Quillen equivalence induces a weak equivalence between the underlying (fibrant) marked simplicial categories of fibrant–cofibrant objects, that is, an equivalence of $(\infty, 2)$-categories. Combining this with Proposition 5.2.1, we get the following.

Corollary 5.2.5. *Given any scaled simplicial set (X, S), there is an equivalence of fibrant scaled simplicial sets*

$$\mathbf{Fun}^{\mathrm{sc}}((X,S), \mathbf{Cat}^{\mathrm{sc}}) \simeq \mathrm{N}^{\mathrm{sc}}\Big((\mathrm{sSet}^+_{(X,S)})^\circ\Big).$$

Remark 5.2.6. The categories $\mathrm{sSet}^+_{(X,S)}$ only depend pseudonaturally on (X, S), and therefore, the equivalence in Corollary 5.2.5 is not literally natural at the point-set level, but this can be dealt with in the same way as in the proof of the analogous statement for the usual unstraightening equivalence in [8, Corollary A.32].

Specialising to the case of a Gray tensor product of two scaled simplicial sets, we obtain the following.

Proposition 5.2.7. *Straightening for locally cocartesian fibrations gives an equivalence between maps of scaled simplicial sets $(X, S) \boxtimes (Y, T) \longrightarrow \mathbf{Cat}^{\mathrm{sc}}$ and locally cocartesian inner fibrations $E \longrightarrow X \times Y$ such that*

(1) *for $x \in X$, the restriction $E_x \longrightarrow Y$ is cocartesian over T,*
(2) *for $y \in Y$, the restriction $E_y \longrightarrow X$ is cocartesian over S,*
(3) *for 1-simplices $\alpha : x \longrightarrow x'$ in X, $\beta : y \longrightarrow y'$ in Y, p is cocartesian over the 2-simplex $(s_1\alpha, s_0\beta)$.*

Remark 5.2.8. Condition (3) can be rephrased as follows: for any $e \in E_{x,y}$, if $e \longrightarrow (\alpha, \mathrm{id}_y)_! e$ is a locally cocartesian morphism over (α, id_y), and $(\beta, \mathrm{id}_y)_! e \longrightarrow (\mathrm{id}_{x'}, \beta)_!(\alpha, \mathrm{id}_y)_! e$ is a locally cocartesian morphism over $(\mathrm{id}_{x'}, \beta)$, then the composite $e \longrightarrow (\mathrm{id}_{x'}, \beta)_!(\alpha, \mathrm{id}_y)_! e$ is locally cocartesian over (α, β).

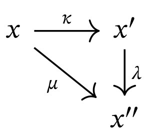
Proof of Proposition 5.2.7. Corollary 5.2.5 implies that there is a natural equivalence of ∞-categories

$$\mathbf{Fun}^{\mathrm{sc}}((X,S) \boxtimes (Y,T), \mathbf{Cat}^{\mathrm{sc}}) \simeq \mathrm{N}^{\mathrm{sc}}\Big((\mathrm{sSet}^+)^\circ_{(X,S)\boxtimes(Y,T)}\Big).$$

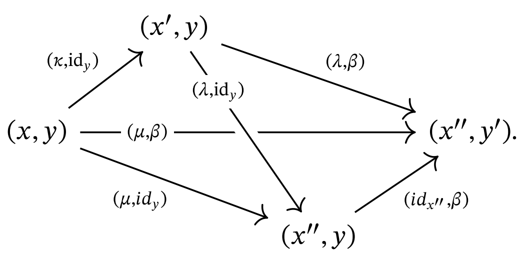
Next, one can apply the equivalence $X \boxtimes Y \simeq (X \times Y, T_{-})$ of [7, Proposition 2.10] to obtain an equivalence $(\mathrm{sSet}^+)^{\circ}_{(X,S)\boxtimes(Y,T)} \simeq (\mathrm{sSet}^+)^{\circ}_{(X\times Y,T_{-})}$. The objects of the right-hand side are exactly those of the proposition. However, one can also show directly that the fibrant objects of $(\mathrm{sSet}^+)_{(X,S)\boxtimes(Y,T)}$ are precisely the locally cocartesian fibrations satisfying conditions (1)–(3) above. We will do this to keep the treatment self-contained. By definition, the fibrant objects are locally cocartesian fibrations $p : E \longrightarrow X \times Y$ such that for $(\sigma, \tau) \in (S \times T)_{\mathrm{opl}}$, the pullback $(\sigma, \tau)^* E \longrightarrow [2]$ is a cocartesian fibration. On the other hand, conditions (1)–(3) assert that E is cocartesian over the subset of 2-simplices $(S \times T)' \subseteq (S \times T)_{\mathrm{opl}}$ given as follows:

- $(s_0^2 x, \tau)$ with $x \in X_0$ and $\tau \in T$.
- $(\sigma, s_0^2 y)$ with $\sigma \in S$ and $y \in Y_0$.
- $(s_1 \alpha, s_0 \beta)$ with $\alpha \in X_1, \beta \in Y_1$.

We claim that this already implies that p is cocartesian over every 2-simplex in $(S \times T)_{\mathrm{opl}}$. Indeed, let us show that p is cocartesian over $(\sigma, s_0\beta)$, for $\sigma \in S$ of the form



and $\beta : y \longrightarrow y'$ in Y_1; the case of a 2-simplex $(s_1\alpha, \tau)$ will follow from a similar argument. Consider the 3-simplex $\xi = (s_2\sigma, s_0^2\beta)$, which may be depicted as:



Note that $d_2\xi = (\sigma, s_0\beta)$, while the faces $d_0\xi = (s_1\lambda, s_0\beta)$, $d_1\xi = (s_1\mu, s_0\beta)$ and $d_3\xi = (\sigma, s_0^2 y)$ are all in $(S \times T)'$. If $p : E \longrightarrow X \times Y$ is cocartesian over $(S \times T)'$, a locally cocartesian arrow $e \longrightarrow (\mu, \beta)_! e$ can therefore be identified, in turn, with the following composites of locally cocartesian arrows:

- $e \to (\mu, \mathrm{id}_y)_! e \to (\mathrm{id}_{x''}, \beta)_! (\mu, \mathrm{id}_y)_! e$, since p is cocartesian over $d_1\xi$,
- $e \to (\kappa, \mathrm{id}_y)_! e \to (\lambda, \mathrm{id}_y)_! (\kappa, \mathrm{id}_y)_! e \to (\mathrm{id}_{x''}, \beta)_! (c, \mathrm{id}_y)_! e$, since p is cocartesian over $d_3\xi$,
- $e \to (\kappa, \mathrm{id}_y)_! e \to (\lambda, \beta)_! (\kappa, \mathrm{id}_y)_! e$, since p is cocartesian over $d_0\xi$.

The last assertion means precisely that p is cocartesian over $d_2\xi = (\sigma, s_0\beta)$, as desired. □

Remark 5.2.9. The previous result also follows by combining [7, Proposition 2.10] with Lurie's scaled unstraightening (Theorem 5.2.4).

Specialising to Gray tensor products of ∞-categories, we obtain the following.

Corollary 5.2.10. *Let A and B be ∞-categories. Then there is a natural equivalence of ∞-categories*

$$\mathrm{Fun}(A \boxtimes B, \mathbf{Cat}) \simeq \mathrm{Gray}(A, B).$$

Proof. Combine Lemma 2.4.6 and Proposition 5.2.7. □

Remark 5.2.11. Similarly to (2) of Remark 2.5.7, it is not a priori clear to us that the equivalence constructed in Corollary 5.2.10 restricts to the usual straightening equivalence

$$\mathrm{Fun}(A \times B, \mathrm{Cat}) \simeq \mathrm{Cocart}(A \times B)\ :$$

Besides this direct equivalence, one can use the two inclusions

$$\mathrm{Cocart}(A \times B) \subseteq \mathrm{RCocart}(A, B), \mathrm{LCocart}(A, B)$$

and apply straightening in one factor after the other to obtain two more equivalences

$$\mathrm{Cocart}(A \times B) \simeq \mathrm{Fun}(B, \mathrm{Cocart}(A)) \simeq \mathrm{Fun}(A \times B, \mathrm{Cat})$$

and

$$\mathrm{Cocart}(A \times B) \simeq \mathrm{Fun}(A, \mathrm{Cocart}(B)) \simeq \mathrm{Fun}(A \times B, \mathrm{Cat}),$$

and by construction, the equivalence from Corollary 5.2.10 restricts to the latter of these. Again, it will follow from the uniqueness results of [14] that these three equivalences agree.

5.3 | Unstraightening of lax natural transformations and the calculus of mates

As an application of the scaled unstraightening for Gray fibrations provided by Corollary 5.2.10, we will now prove the main theorem of this section.

Theorem 5.3.1. *There are equivalences of* $(\infty, 2)$*-categories*

$$\mathbf{Cocart}^{\mathrm{lax}}(B) \simeq \mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat}) \quad \textit{and} \quad \mathbf{Cart}^{\mathrm{opl}}(B) \simeq \mathbf{Fun}^{\mathrm{opl}}(B^{\mathrm{op}}, \mathbf{Cat}),$$

which are natural in B.

In particular, this implies that the $(\infty, 2)$-categories $\mathbf{Cocart}^{\mathrm{lax}}(*)$ and $\mathbf{Cart}^{\mathrm{opl}}(*)$ are both equivalent to $\mathbf{Cat}$, as mentioned already after Definition 3.1.7.

Remark 5.3.2. In fact, it follows from [14] that the natural equivalence of Theorem 5.3.1 is essentially unique.

Proof. Let us start with the lax case, the oplax case being similar.

Lax case. Let B be an ∞-category and consider the $(\infty, 2)$-category $\mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})$ determined by the natural equivalence of ∞-groupoids

$$\mathrm{Map}_{\mathrm{Cat}_2}\left(\mathbf{A}, \mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})\right) \simeq \mathrm{Map}_{\mathrm{Cat}_2}(\mathbf{A} \boxtimes B, \mathbf{Cat}).$$

In terms of Segal ∞-groupoids, $\mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})$ is then described by the bisimplicial ∞-groupoid

$$N_{\Delta\times\Delta}\big(\mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})\big) : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathrm{Gpd}; \quad ([m],[n]) \longmapsto \mathrm{Map}_{\mathrm{Cat}_2}\Big(\delta_2([m],[n]) \boxtimes B, \mathbf{Cat}\Big),$$

where δ_2 is the functor (5.1.5). On the other hand, the $(\infty,2)$-category $\mathbf{Cocart}^{\mathrm{lax}}(B)$ was defined as the complete Segal ∞-groupoid whose value on $([m],[n])$ is given by the ∞-groupoid of functors (of ∞-categories) $[m] \longrightarrow \mathrm{Cocart}^{\mathrm{lax}}_{[n]}(B \times [n])$. Our goal will be to prove that there is a natural equivalence between these two bisimplicial ∞-groupoids.

To see this, let us first consider the following two natural subgroupoid inclusions:

$$\mathrm{Map}_{\mathrm{Cat}_2}\Big(\delta_2([m],[n]) \boxtimes B, \mathbf{Cat}\Big) \hookrightarrow \mathrm{Map}_{\mathrm{Cat}_2}\Big(([m] \boxtimes [n]) \boxtimes B, \mathbf{Cat}\Big) \xhookrightarrow{\mathrm{Un}^{\mathrm{sc}}} \iota\big(\mathrm{Cat}/[m] \times [n] \times B\big)$$

$$\mathrm{Map}_{\mathrm{Cat}}\big([m], \mathrm{Cocart}^{\mathrm{lax}}_{[n]}(B \times [n])\big) \xhookrightarrow{\mathrm{Un}^{\mathrm{cc}}} \iota\mathrm{Gray}([m], B \times [n]) \hookrightarrow \iota\big(\mathrm{Cat}/[m] \times B \times [n]\big).$$

The first inclusion uses the localisation $[m] \boxtimes [n] \longrightarrow \delta_2([m],[n])$ from Lemma 5.1.11. It suffices to verify that upon reversing the factors of B and $[n]$, both of these inclusions determine the same subgroupoid of functors $p : E \longrightarrow B \times [m] \times [n]$. Unraveling the definitions, the image of the first map is the subgroupoid of functors p with the following properties:

(a) for each $j \in [n], b \in B$, the restriction $E_{j,b} \longrightarrow [m]$ is a cocartesian fibration,
(b) for each $i \in [m], b \in B$, the restriction $E_{i,b} \longrightarrow [n]$ is a cocartesian fibration,
(c) for each $i \in [m], j \in [n]$, the restriction $E_{i,j} \longrightarrow B$ is a cocartesian fibration,
(d) for arrows $\kappa : i \longrightarrow i'$, $\phi : j \longrightarrow j'$, $\beta : b \longrightarrow b'$ in $[m], [n]$ and B, respectively, p is cocartesian over the 2-simplices $(s_1\kappa, s_1\phi, s_0\beta)$ and $(s_1\kappa, s_0\phi, s_0^2\beta)$,
(e) for any $i \in [m]$, the restriction $E_i \longrightarrow [n] \times B$ arises as the base change of a Gray fibration over $[0] \times B$; in particular, it is a cocartesian fibration, so this already implies (b) and (c).

The first four conditions describe the image of $\mathrm{Un}^{\mathrm{sc}}$, by a 2-fold application of Proposition 5.2.7. The fifth condition follows from Corollary 5.2.10 and Lemma 5.1.11, together with the fact that the Gray tensor product preserves colimits in each variable, so that

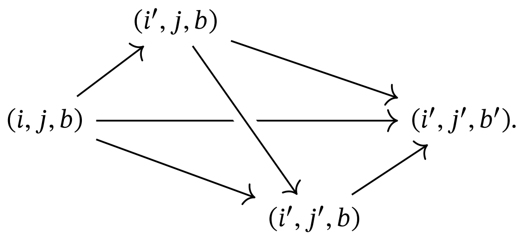
$$\begin{aligned}\delta_2([m],[n]) \boxtimes A &\simeq \left([m] \boxtimes [n] \coprod_{\iota[m]\boxtimes[n]} \iota[m] \boxtimes [0]\right) \boxtimes A \\ &\simeq ([m] \boxtimes [n]) \boxtimes A \coprod_{(\iota[m]\boxtimes[n])\boxtimes A} (\iota[m] \boxtimes [0]) \boxtimes A.\end{aligned}$$

On the other hand, unraveling Definition 3.1.7 shows that the image of the second map consists, after permuting B and $[n]$, of functors p with the following properties:

(a') for each $j \in [n]$ and $b \in B$, the restriction $E_{j,b} \longrightarrow [m]$ is a cocartesian fibration,
(b') for each $i \in [m]$, the restriction $E_i \longrightarrow [n] \times B$ is a cocartesian fibration,
(c') for each $\kappa : i \longrightarrow i'$, $\phi : j \longrightarrow j'$ and $\beta : b \longrightarrow b'$ in $[m], [n]$ and B, respectively, p is cocartesian over $(s_1\kappa, s_0\phi, s_0\beta)$,
(d') for each $i \in [m]$, the restriction $E_i \longrightarrow [n] \times B$ is a cocartesian fibration which arises as the base change of a cocartesian fibration over $[0] \times B$ (in particular, this implies (b')).

Indeed, by Lemma 2.4.6, the first three conditions are equivalent to p being an object of $\mathrm{Gray}([m], B \times [n])$. Condition (d') is then equivalent to the straightening of this Gray fibration over $[m]$ taking values in the full subcategory $\mathrm{Cocart}^{\mathrm{lax}}_{[n]}(B \times [n]) \subseteq \mathrm{Cocart}^{\mathrm{lax}}(B \times [n])$.

Evidently, condition (e) is equivalent to (d') and conditions (a) and (a') coincide. It therefore remains to show that (d) and (c') are equivalent. Let us fix $\kappa : i \longrightarrow i'$, $\phi : j \longrightarrow j'$ and $\beta : b \longrightarrow b'$ as above and consider the 3-simplex $\xi = (s_2 s_1(\kappa), s_1 s_0(\phi), s_0^2(\beta))$ given by



Note that p is cocartesian over $d_0\xi$ by condition (e) (or (d')). Assuming condition (d), we furthermore have that p is cocartesian over $d_3\xi$ and $d_1\xi$. The argument from Proposition 5.2.7 shows that p is cocartesian over $d_2\xi$, which is precisely condition (c'). Conversely, condition (c') implies that p is cocartesian over $d_2\xi$ and $d_3\xi$. An argument similar to that of Proposition 5.2.7 then shows that p is cocartesian over $d_1\xi$ as well, so that (d) follows.

Oplax case. Likewise, the $(\infty, 2)$-category $\mathbf{Fun}^{\mathrm{opl}}(B^{\mathrm{op}}, \mathbf{Cat})$ corresponds to the 2-fold Segal ∞-groupoid

$$([m],[n]) \longmapsto \mathrm{Map}_{\mathrm{Cat}_2}\Big(B^{\mathrm{op}} \boxtimes \delta_2([m],[n]), \mathbf{Cat}\Big).$$

Lemma 5.1.11 identifies this mapping ∞-groupoid with the ∞-groupoid of maps $B^{\mathrm{op}} \boxtimes [m] \boxtimes [n] \longrightarrow \mathbf{Cat}$ whose restriction to each $B^{\mathrm{op}} \boxtimes \{i\} \boxtimes [n]$ arises from $B^{\mathrm{op}} \longrightarrow \mathbf{Cat}$.

Under scaled unstraightening, this mapping ∞-groupoid is identified with a certain ∞-groupoid of locally cocartesian fibrations $p : E \longrightarrow B^{\mathrm{op}} \times [m] \times [n]$. Unraveling the definitions as in the lax case, one sees that this is the ∞-groupoid of those functors p such that:

- denoting by pr_B the projection onto B^{op}, we have that p defines a map $\mathrm{pr}_B \circ p \longrightarrow \mathrm{pr}_B$ in $\mathrm{Cocart}(B^{\mathrm{op}})$,
- for each $b \in B^{\mathrm{op}}$, the map $E_b \longrightarrow [m] \times [n]$ is a Gray fibration,
- for each $i \in [m]$, the Gray fibration $E_i \longrightarrow B^{\mathrm{op}} \times [n]$ arises as the base change of a cocartesian fibration over $B^{\mathrm{op}} \times [0]$.

Dualising over B^{op}, that is, applying cocartesian unstraightening and cartesian straightening over B, this is identified with the ∞-groupoid of maps $q : F \longrightarrow B \times [m] \times [n]$ such that

- q defines a map $\mathrm{pr}_B \circ q \longrightarrow \mathrm{pr}_B$ in $\mathrm{Cart}(B)$,
- for each $b \in B$, the restriction $F_b \longrightarrow [m] \times [n]$ is a Gray fibration,
- for each $i \in [m]$, the curved orthofibration $F_i \longrightarrow B \times [n]$ arises as the base change of a cartesian fibration over $B \times [0]$.

Permuting $[m]$ and $[n]$, this is equivalent to q defining an element in $\iota\mathrm{CrvOrtho}(B \times [n], [m])$ such that for all $i \in [m]$, the map $F_i \longrightarrow B \times [n]$ arises as the base change of a cartesian fibration over $B \times [0]$. Under straightening over $[m]$, this is precisely the ∞-groupoid of (m, n)-simplices of $\mathbf{Cart}^{\mathrm{opl}}(B)$ (see Definition 3.1.7). □

Observation 5.3.3. Since the equivalence

$$\mathbf{Fun}^{\mathrm{lax}}(B,\mathbf{Cat}) \xrightarrow{\sim} \mathbf{Cocart}^{\mathrm{lax}}(B)$$

of Theorem 5.3.1 is by construction given on objects by the unstraightening functor, for functors $F, G : B \to \mathbf{Cat}$, we obtain an equivalence

$$\mathrm{Nat}^{\mathrm{lax}}(F,G) \simeq \mathrm{Fun}_{/B}(\mathrm{Un}^{\mathrm{cc}}(F), \mathrm{Un}^{\mathrm{cc}}(G)),$$

depending 2-functorially on $F, G \in \mathbf{Fun}^{\mathrm{lax}}(B,\mathbf{Cat})$, where the left-hand side denotes the mapping ∞-category in $\mathbf{Fun}^{\mathrm{lax}}(B,\mathbf{Cat})$. Taking G to be the constant functor with value $X \in \mathbf{Cat}$, we get an equivalence

$$\mathrm{Nat}^{\mathrm{lax}}(F,\mathrm{const}_X) \simeq \mathrm{Fun}(\mathrm{Un}^{\mathrm{cc}}(F), X),$$

depending 1-functorially on $X \in \mathrm{Cat}$, since we have a natural equivalence $\mathrm{Un}^{\mathrm{cc}}(\mathrm{const}_X) \simeq X \times B$. Let $\theta : F \to \mathrm{const}_{\mathrm{Un}^{\mathrm{cc}}(F)}$ be the lax natural transformation corresponding to the identity map under the above natural equivalence and observe that this induces a natural transformation

$$\mathrm{Fun}(\mathrm{Un}^{\mathrm{cc}}(F), -) \xrightarrow{\mathrm{const}} \mathrm{Nat}^{\mathrm{lax}}(\mathrm{const}_{\mathrm{Un}^{\mathrm{cc}}(F)}, \mathrm{const}_{(-)}) \xrightarrow{\theta^*} \mathrm{Nat}^{\mathrm{lax}}(F, \mathrm{const}_{(-)})$$

between functors of $(\infty,2)$-categories $\mathbf{Cat} \longrightarrow \mathbf{Cat}$. This natural transformation is a natural equivalence, since the underlying natural transformation between functors of $(\infty,1)$-categories $\mathrm{Cat} \longrightarrow \mathrm{Cat}$ is an equivalence by definition of θ.

In other words, $\mathrm{Un}^{\mathrm{cc}}(F)$ has the universal property of the lax colimit of F: it corepresents the functor

$$\mathrm{Nat}^{\mathrm{lax}}(F,\mathrm{const}_{(-)}) : \mathbf{Cat} \to \mathbf{Cat}.$$

Similarly, the cartesian unstraightening of $F : B \to \mathrm{Cat}$ is the oplax colimit: it satisfies

$$\mathrm{Nat}^{\mathrm{opl}}(F,\mathrm{const}_X) \simeq \mathrm{Fun}(\mathrm{Un}^{\mathrm{ct}}(F), X).$$

Such a characterisation of the unstraightening was first established in [8], where the authors defined lax (co)limits for functors $F : B \to \mathrm{Cat}$ as certain weighted (co)limits. As an application of Theorem 5.3.1, we can therefore deduce that their lax colimits really have the desired universal property expressed above.

Definition 5.3.4. For an ∞-category B, let us write $\mathbf{Fun}^{\mathrm{lax,R}}(B,\mathbf{Cat}) \subseteq \mathbf{Fun}^{\mathrm{lax}}(B,\mathbf{Cat})$ for the 1-full sub-2-category spanned by those lax natural transformations sending each object in B to a right adjoint. Likewise, let $\mathbf{Fun}^{\mathrm{opl,L}}(B,\mathbf{Cat}) \subseteq \mathbf{Fun}^{\mathrm{opl}}(B,\mathbf{Cat})$ for the 1-full sub-2-category spanned by those oplax natural transformations with values in left adjoints.

Combining Theorems 3.1.11 and 5.3.1, we obtain the following.

Theorem 5.3.5. *Let B be an ∞-category. Then there is an equivalence*

$$\mathrm{Adj} : \mathbf{Fun}^{\mathrm{lax,R}}(B,\mathbf{Cat}) \xrightarrow{\sim} \left(\mathbf{Fun}^{\mathrm{opl,L}}(B,\mathbf{Cat})\right)^{(1,2)-\mathrm{op}}$$

sending each lax natural transformation $F \Rightarrow G$ with values in right adjoints to the corresponding oplax natural transformation $G \Rightarrow F$ with values in left adjoints.

Proof. Unravelling the proof of Theorem 5.3.1, one sees that the equivalence $\mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat}) \simeq \mathbf{Cocart}^{\mathrm{lax}}(B)$ is given at the level of objects by the usual unstraightening from [21] (which in this special agrees with the locally cocartesian unstraightening from [22]). In particular, this equivalence identifies the 1-full sub-2-category $\mathbf{Fun}^{\mathrm{lax,R}}(B, \mathbf{Cat})$ with $\mathbf{Cocart}^{\mathrm{lax,R}}(B)$, and similarly for the oplax case. The result then follows from Theorem 5.3.1. □

Denoting by $\mathbf{LFun}^{\mathrm{lax}}(B, \mathbf{Cat})$ the full sub-2-category of $\mathbf{Fun}^{\mathrm{lax}}(B, \mathbf{Cat})$ spanned by all functors taking values in left adjoints and similarly for $\mathbf{RFun}^{\mathrm{opl}}(B, \mathbf{Cat})$ we also find the following.

Theorem 5.3.6. *Let B be an ∞-category. Then there is an equivalence*

$$\mathbf{RFun}^{\mathrm{opl}}(B, \mathbf{Cat}) \xrightarrow{\sim} \mathbf{LFun}^{\mathrm{lax}}(B^{\mathrm{op}}, \mathbf{Cat})$$

sending each diagram $B \to \mathrm{Cat}$ with values in right adjoints to the corresponding diagram $B^{\mathrm{op}} \to \mathrm{Cat}$ of left adjoints.

Proof. Simply observe that both sides are equivalent to $\mathbf{Bicart}^{\mathrm{(op)lax}}(B)$ via the unstraightening equivalence from Theorem 5.3.6. □

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