Sufficient conditions for uniform asymptotic stability and input-to-state stability using high-order control barrier functions

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***Abstract*—Control barrier functions (CBFs) ensure safety of controlled dynamical systems by enforcing forward invariance of safe subsets of the state space. First-order CBFs are applicable for systems where the control input appears in the first time derivative of the controlled output. High-order CBFs (HOCBFs) extend the notion of CBFs to systems of any order, following a procedure reminiscent of the recursive design of a control Lyapunov function in backstepping. Asymptotic stability of compact safe sets for Lipschitz continuous HOCBF-based controllers has recently been reported in literature. In this paper, we extend this result by establishing sufficient conditions for uniform asymptotic stability of closed, but not necessarily compact, safe sets. Moreover, we show that uniform asymptotic stability holds for differential inclusions that correspond to allowing the control input to take on arbitrary values that satisfy the HOCBF-induced input constraints. This result circumvents the need to establish continuity properties of optimization-based safeguarding control laws. Sufficient conditions for input-to-state stability are also established, by constructing a vector comparison system from the worst-case evolution of the HOCBF along the disturbed versions of the aforementioned differential inclusions. The theoretical results are illustrated by two case studies.**

***Index Terms*—Control barrier functions, input-to-state stability, uniform asymptotic stability.**

I. Introduction

Many control problems may be separated into two objectives: a mission objective and a safety objective. For instance, in control of autonomous vehicles, the mission objective may be to reach a destination, while the safety objective may be to avoid collisions. Designing an explicit control law that achieves both objectives simultaneously is often challenging. A preferred solution is then to solve the two objectives separately, and subsequently synthesize the two control strategies.

In [1], the authors proposed solving the safety objective by designing a safety controller that enforces forward invariance of the subzero level set of a scalar function, combined with a sigmoid function to activate the safety controller when approaching the boundary of the safe set. The safety controller was constructed using a version of Sontag's formula [2], thus merging the ideas of barrier functions [3] and control Lyapunov functions (CLFs) [4]. Another notion of CBFs, later referred to as reciprocal CBFs, was introduced in [5], where the authors recognized that CBFs give rise to a state-dependent set of inputs that guarantee safety. The CBF may then be synthesized with any nominal control law by solving an optimization problem that finds the safe input that is closest to the nominal control input (by some appropriate measure).

The most prevalent form of CBFs today was first introduced in [6] under the name zeroing CBFs, and later popularized by [7]. Consider an affine control system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^n, \qquad u \in U \subset \mathbb{R}^m, \tag{1}$$

a scalar function $B : \mathbb{R}^n \to \mathbb{R}$, and an extended class-$\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}$ (see Definition 1). Let $U_B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be the set-valued mapping defined by

$$U_B(x) := \{u \in U : L_f B(x) + L_g B(x)u \le -\alpha(B(x))\}. \tag{2}$$

where $L_f B$ and $L_g B$ are the Lie derivatives of B along f and g, respectively. The subzero level set $K := \{x \in \mathbb{R}^n : B(x) \le 0\}$, referred to as the safe set, is forward invariant for any control law satisfying $u(x) \in U_B(x)$, for all x.

In fact, the CLF-like inequality used in the definition of U_B suggests that $u(x) \in U_B(x)$ achieves asymptotic stability of K. Since, in general, K is a noncompact set, additional assumptions on B are required to conclude the stronger property of *uniform* asymptotic stability [8]. Uniform asymptotic stability is desired, since it is often associated with robustness properties, as pointed out in [9] in the context of CBFs. A natural extension of the results of [9] is the notion of input-to-state safety [10], [11], as an analogue to the well-known notion of input-to-state stability [12]. Contrary to CLFs, CBFs additionally constrain the solutions when in the interior of the safe set. To be precise, the convergence rate to the boundary

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of K is restricted, when approaching from the interior of K. This is key to the performance of CBF-based controllers and, if designed correctly, ensures that the commanded control inputs are feasible.

Guaranteed forward invariance of the safe set in the presence of bounded disturbances is achieved using robust CBF formulations [13]–[16]. In essence, robust CBFs shrink the admissible input set, by adding a (possibly state-dependent) penalty term to the inequality used in the definition of U_B. Since robust CBFs must account for worst-case disturbances, the resulting controller may be overly conservative. The adaptive CBF formulations proposed in [17]–[21] also guarantee robust forward invariance, but use adaptation and learning to reduce conservatism as the system evolves.

First-order CBFs, as presented above, are suitable for safety constraints of relative degree one systems, i.e., when the control input appears in the first derivative of B along the solutions of (1). Several authors have proposed extensions of CBFs to systems of higher relative degree, following procedures reminiscent of backstepping [22]. Notable contributions include exponential CBFs [23] and high-order CBFs (HOCBFs) [24], [25], where the latter is a generalization of the former. In [26], it was shown that Lipschitz continuous HOCBF-based controllers achieve asymptotic stability of compact safe sets, by appealing to the solutions of an induced comparison system. Similar results as [26] were also reported in [27], for the special case of second-order CBFs. A passivity-based method for establishing asymptotic stability of noncompact safe sets with compact boundaries was reported in [28], using a relaxed first-order CBF formulation that does not require strict decrease of the CBF for solutions evolving outside the safe set.

In the discussion so far, we have implicitly assumed that maximal solutions are complete. In the remainder, this somewhat restrictive assumption is circumvented by making use of the notions of forward pre-invariance, pre-asymptotic stability and input-to-state pre-stability, as the counterpart to forward invariance, asymptotic stability, and input-to-state stability, respectively, for systems with non-complete maximal solutions [29].

This paper extends the theory of HOCBFs, by establishing sufficient conditions for uniform pre-asymptotic stability and small-input input-to-state pre-stability, for systems with closed, but not necessarily compact, safe sets. The result on input-to-state pre-stability makes use of a vector comparison system constructed from the worst-case evolution of the HOCBF along the disturbed system. Unlike [26], [27], the proofs presented herein do not require Lipschitz continuity, or even continuity, of the feedback function, thus enabling a larger class of controllers. This result is achieved by representing the system (1), or the disturbed version of (1), as a differential inclusion, with u constrained to $U_B(x)$. Additionally, we show that U_B is convex-valued and outer semicontinuous, and use the aforementioned regularity properties to show that, for any locally bounded function $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(x) \in U_B(x)$, the resulting closed-loop system is nominally robust, in the sense that the stability properties provided by CBFs are retained under Krasovskii-regularization. CBF-induced differential inclusions were also studied in [30], where it was shown that the safe set is forward invariant for any $u(x) \in U_B(x)$, under the assumption that U_B is a locally Lipschitz set-valued mapping.

The theoretical contributions are illustrated by two case studies. In the first case study we consider a vehicle with linear kinematics, and obstacle avoidance with respect to circular obstacle domains, using the CBF design of [31]. We show that the safe set is uniformly pre-asymptotically stable, but the corresponding system with additive disturbances is not small-input input-to-state pre-stable with respect to the safe set, thus contradicting the second claim of [9, Proposition 5][1]. In the second case study we use a vehicle with unicycle kinematics [7], [25], [32], [33] to construct a system that is pre-asymptotically stable, but not uniformly pre-asymptotically stable.

The remainder of this paper is organized as follows. Necessary preliminaries are reviewed in Section II. Section III establishes some key properties of systems constructed from CBFs. Sufficient conditions for uniform pre-asymptotic stability and input-to-state pre-stability using HOCBFs are established in Section IV and V, respectively. The two case studies are presented in Section VI. Finally, Section VII concludes the paper.

II. Preliminaries

A. Notation

$\mathbb{R}$ is the set of real numbers and $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$ are the set of non-negative and non-positive numbers, respectively, while $\mathbb{R}^n_{\leq 0}$ is the set of vectors $x \in \mathbb{R}^n$ with non-positive entries, i.e.,

$$\mathbb{R}^n_{\leq 0} := \{x \in \mathbb{R}^n : x_i \leq 0 \quad \forall i \in \{1, \dots, n\}\}. \tag{3}$$

The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted $|x|$, while $|x|_K := \inf_{y \in K} |x - y|$ is the Euclidean distance from the point x to the set $K \subset \mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, we define the inner product $\langle x, y\rangle := x^\top y$. For two column vectors $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, we occasionally use the compact notation $(x_1, x_2) := [x_1^\top \; x_2^\top]^\top \in \mathbb{R}^{n_1+n_2}$.

For a scalar function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f := \left(\frac{\partial f}{\partial x}\right)^\top$ is a column vector. When convenient we use the Lie derivative notation: $L_f B(x) := \langle \nabla B(x), f(x)\rangle$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field and $B : \mathbb{R}^n \to \mathbb{R}$ is a scalar function. For a set X, ∂X is the boundary. The $i \times j$ zero matrix is denoted $0_{i\times j}$, while the $i \times i$ identity matrix is denoted $I_{i\times i}$. For two vectors $x, y \in \mathbb{R}^n$, the inequality $x \leq y$ is read componentwise, i.e., $x_i \leq y_i, \forall i \in \{1, \dots, n\}$. For a mapping $t \mapsto x(t)$, $\|x\|_\infty := \sup_{t\geq 0} |x(t)|$ is the $\mathcal{L}_\infty$ norm. Finally, $\dot{x}$ is the time derivative of x.

[1]For the system $\dot{x} = f(x)$, let $x \mapsto B(x)$ satisfy $L_f B(x) \leq -\alpha(B(x))$, where α is an extended class-$\mathcal{K}$ function. [9, Proposition 5] claims that there exist $\bar{w} > 0$ and class-$\mathcal{K}$ function γ such that the set $\{x : B(x) \leq \gamma(\|w\|_\infty)\}$ is asymptotically stable for the disturbed system $\dot{x} = f(x) + w$, for any w such that $\|w\|_\infty \leq \bar{w}$.

B. Mathematical preliminaries

Definition 1 (Class-$\mathcal{K}$ functions): A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$. A class-$\mathcal{K}$ function belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ is an extended class-$\mathcal{K}$ function ($\alpha \in \mathcal{K}_e$) if it is strictly increasing and $\alpha(0) = 0$. □

Definition 2 (Class-$\mathcal{KL}$ functions): A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if, for each fixed s, $\beta(\cdot, s) \in \mathcal{K}$, and for each fixed r, $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to 0$. □

Definition 3 (Outer semicontinuity of set-valued mappings): A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous at x if

$$\limsup_{y \to x} F(y) \subset F(x). \tag{4}$$

F is outer semicontinuous relative to a set $X \subset \mathbb{R}^n$, if the mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $F(x)$ for $x \in X$ and $\emptyset$ for $x \notin X$ is outer semicontinuous at each $x \in X$. □

Definition 4 (Local boundedness of set-valued mappings): A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally bounded at $x \in \mathbb{R}^n$ if there exists a neighborhood $\mathcal{U}$ of x such that $F(\mathcal{U}) \subset \mathbb{R}^n$ is bounded. F is locally bounded on a set $X \subset \mathbb{R}^n$, if the mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $F(x)$ for $x \in X$ and $\emptyset$ for $x \notin X$ is locally bounded at each $x \in X$. □

Definition 3 is adapted from [34, Definition 5.4] and [29, Definition 5.9], while Definition 4 is adapted from [29, Definition 5.14]. A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous relative to X if and only if the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in X, y \in F(x)\}$ is closed in $X \times \mathbb{R}^n$ [29, Lemma 5.10]. If F is locally bounded and closed valued, then outer semicontinuity is equivalent to the notion of upper semicontinuity used in e.g. [35]. Any continuous function is outer semicontinuous when evaluated as a set-valued mapping.

Definition 5 (Upper Dini derivative): The upper Dini derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$D^+ f(x) := \limsup_{h \to 0^+} \frac{f(x + h) - f(x)}{h}. \tag{5}$$

□

The upper Dini derivative is sometimes referred to as the upper right-hand derivative, see e.g. [36].

C. Constrained differential inclusions

A constrained differential inclusion is a system of the form

$$\dot{x} \in F(x), \qquad x \in X \subset \mathbb{R}^n, \tag{6}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. See [37]–[39] for an extensive study on barrier functions applied to differential inclusions. Differential inclusions are a special case of hybrid inclusions. For this reason, we refer to the hybrid dynamical systems literature, in particular [29], [40], for further study on the modeling framework adopted herein. A solution x to (6), see [29, Definition 2.6], is defined on a time domain denoted $\operatorname{dom} x$. Solutions that cannot be extended are said to be maximal, while solutions that exist on an unbounded time domain, $\operatorname{dom} x = [0, \infty)$, are said to be complete.

Assumption 1 (Adapted from [29, Assumption 6.5]): X is a closed set. $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded on X. For each $x \in X$, $F(x)$ is a nonempty convex set.

By [29, Theorem 6.30], a system that satisfies Assumption 1 is well-posed [29, Definition 6.27]. For a well-posed system, solutions depend semi-continuously on initial conditions and vanishing perturbations. Additionally, maximal solutions are either complete, escape to infinity in finite time, or exist on a compact time domain $\operatorname{dom} x = [t_0, t_1]$, with $t_0 \leq t_1$ and $x(t_1) \in \partial X$. The last situation occurs if $F(x(t_1)) \cap T_X(x(t_1)) = \emptyset$, where $T_X(x)$ is the tangent cone of X at x.

System (6) allows us to consider all solutions to system (1) with input constrained to $U_B(x)$, where the equivalence can be made explicit by defining $F(x) := \{f(x) + g(x)u : u \in U_B(x)\}$. Restricting the solutions to a subset X accounts for the fact that U_B may be empty for some $x \in \mathbb{R}^n$. Another direction in which differential inclusions are convenient is when evaluating robustness of solutions for discontinuous differential equations. That is, solutions to the system

$$\dot{x} = f(x), \qquad x \in X, \tag{7}$$

subject to arbitrarily small state perturbations, are captured by (6) with F taken as

$$F(x) := \bigcap_{\delta > 0} \overline{\operatorname{con}} f((x + \delta\mathbb{B}) \cap X), \tag{8}$$

where $\overline{\operatorname{con}}$ denotes the closed convex hull. F defined in (8) is referred to as the Krasovskii regularization of f [41], and the solutions to the system

$$\dot{x} \in \bigcap_{\delta > 0} \overline{\operatorname{con}} f((x + \delta\mathbb{B}) \cap X), \qquad x \in X, \tag{9}$$

are the generalized Krasovskii solutions of (7) [29, Definition 4.2]. Note that, in the case of continuous f and closed X, the Krasovskii regularization of f simply becomes $F(x) = \{f(x)\}$ for $x \in X$ and $F(x) = \emptyset$ for $x \notin X$. If X is closed and f is locally bounded, then the Krasovskii regulated system (9) is well-posed. In general, system (9) does not inherit the stability properties of the differential equation (7) when f is discontinuous, as illustrated by Example 1 below.

Example 1: Consider system (7) with $X = \mathbb{R}$, and let $f(x) := 1$ for $x > 0$ and $f(x) := 0$ for $x \leq 0$. Since $x_0 \in \mathbb{R}_{\leq 0} \implies x(t) = x_0 \ \forall t \geq 0$, $\mathbb{R}_{\leq 0}$ is forward invariant for (7). The Krasovskii regularization of f yields $F(x) = \{f(x)\}$ for $x \neq 0$, and $F(0) = [0, 1]$. Since $x(t) = t$ is one possible solution to (9) starting from $x_0 = 0$, $\mathbb{R}_{\leq 0}$ is not forward invariant for the regularized system. □

D. Stability definitions

The main purpose of CBFs is to render safe sets forward invariant. Additionally rendering the safe set uniformly asymptotically stable achieves robustness towards bounded temporary disturbances.

Definition 6 (Forward pre-invariance [42, Definition 1]): Let $K \subset X$ be a closed set. K is forward pre-invariant for (6) if, for each $x_0 \in K$, each maximal solution x to (6) starting from $x_0 \in K$, satisfies $x(t) \in K$ for all $t \in \operatorname{dom} x$. The set K is forward invariant if it is forward pre-invariant and all maximal solutions starting in K are complete. □

Definition 7 (UGpAS): Let $K \subset X$ be a closed set. K is uniformly globally pre-asymptotically stable (UGpAS) for (6) if there exists $\beta \in \mathcal{KL}$ such that, for each $x_0 \in X$, all solutions x to (6) starting from x_0 satisfy

$$|x(t)|_K \leq \beta(|x_0|_K, t) \qquad \forall t \in \operatorname{dom} x. \tag{10}$$

The set K is uniformly globally asymptotically stable (UGAS) if it is UGpAS and all maximal solutions are complete. □
See [29, Definition 3.6] for an equivalent definition of UGpAS. For completeness we also provide a definition of global pre-asymptotic stability, modified from [40, Definition 3.1].

Definition 8 (GpAS): Let $K \subset X$ be a closed set. K is stable for (6) if for each $\epsilon > 0$ there exists $\delta > 0$ such that all solutions x to (6) with $|x_0|_K \leq \delta$ satisfies $|x(t)|_K \leq \epsilon$ for all $t \in \operatorname{dom} x$. K is pre-attractive for (6) if, for all solutions x to (6), $\operatorname{dom} x \mapsto |x(t)|_K$ is bounded, and if x is complete then $\lim_{t\to\infty} |x(t)|_K = 0$. K is globally pre-asymptotically stable if it is stable and globally pre-attractive. K is globally asymptotically stable if it is globally pre-asymptotically stable and all maximal solutions are complete. □
Forward pre-invariance, as defined above, is sometimes referred to as strong forward pre-invariance. Observe that a set K can be forward invariant and UGpAS, since forward invariance requires only completeness of solutions starting in K. Suppose Assumption 1 is satisfied. Then we can make the following assertions: if K is forward pre-invariant and $\partial K \cap \partial X = \emptyset$, then maximal solutions starting in K are either complete or escape to infinity in finite time inside the set K; if additionally K is compact, then K is forward invariant. Consider the system

$$\dot{x} \in F(x, w), \qquad x \in X, \tag{11}$$

where $F : \mathbb{R}^n \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, and $t \mapsto w(t) \in \mathbb{R}^p$ is a disturbance input.

Definition 9 (ISpS): Let $K \subset X$ be a closed set. The system (11) is small-input input-to-state pre-stable (small-input ISpS) with respect to K if there exists $\bar{w} > 0$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}_\infty$ such that, for each $x_0 \in X$, all solutions x to (11) starting from x_0 satisfy, $\forall t \in \operatorname{dom} x$,

$$||w||_\infty \leq \bar{w} \implies |x(t)|_K \leq \beta(|x_0|_K, t) + \gamma(||w||_\infty). \tag{12}$$

The system (11) is small-input input-to-state stable (small-input ISS) with respect to K if it is small-input ISpS with respect to K and all maximal solutions are complete. □

Clearly, small-input ISpS implies UGpAS. Small-input ISpS is strengthened to ISpS if (12) holds for arbitrarily large $\bar{w}$.

Remark 1: The stability properties of a (possibly compact) set K for the time-varying system $\dot{x} = f(x, t)$ are equivalent to the stability properties of the noncompact set $K \times \mathbb{R}_{\geq 0}$ for the time-invariant system $\dot{\xi} = \hat{f}(\xi) := (f(x, \tau), 1)$ with state $\xi := (x, \tau)$. Since the results of this paper apply to general closed sets, the results may also be applied to systems with time-varying dynamics or time-varying safety constraints, by augmenting the state with $\tau \in \mathbb{R}_{\geq 0}$ satisfying $\dot{\tau} = 1$, $\tau(0) = t_0$, and replacing the explicit time dependence with an implicit time dependence using τ.

E. Comparison systems

To show UGpAS and ISpS for systems constructed from HOCBFs, we will appeal to a vector comparison principle, following a similar idea as the proof of [26, Proposition 3]. See [43] for scalar comparison systems applied to first-order barrier functions. Extending the comparison principle to vector systems requires that the right-hand side of the differential equation is quasimonotone nondecreasing.

Definition 10 (Quasimonotone nondecreasing): A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasimonotone nondecreasing if, for two vectors $x, y \in \mathbb{R}^n$ with $x \leq y$, the implication

$$x_i = y_i \implies f_i(x) \leq f_i(y) \tag{13}$$

holds for all $i \in \{1, \ldots, n\}$. □
An intuition for the quasimonotone nondecreasing property is given in [26, below Definition 3].

To relax the often-used Lipschitz assumption on the extended class-$\mathcal{K}$ function α in (2), we make use of the notion of max-solutions for systems with possibly non-unique solutions.

Definition 11 (Max-solution [44, Definition 1.6.1]): Let x be a solution to the system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{14}$$

with initial condition x_0, existing on a time interval $[0, t_1)$. The solution x is said to be a max-solution if, for any solution y to (14) with initial condition $y_0 = x_0$, the inequality $x(t) \geq y(t)$ holds for all $t \in [0, t_1)$. □

Let f be continuous and quasimonotone nondecreasing. Then, for each initial condition $x_0 \in \mathbb{R}^n$, the system (14) admits a max-solution [44, theorems 1.6.1 and 1.6.2].

Remark 2: Max-solutions, as defined in Definition 11, are usually referred to as *maximal solutions*. However, we reserve the term *maximal solutions* for solutions that cannot be extended, in accordance with [29, Definition 2.7].

The following lemma is modified from [44, Corollary 1.7.1], which is a special case of [44, Theorem 1.7.1]. See also [45, Theorem 1.3.1].

Lemma 1 (Vector comparison lemma): For system (14), let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and quasimonotone nondecreasing. Let x denote the max-solution to (14) with initial condition x_0, existing on a time interval $[0, t_1)$. Let $t \mapsto y(t)$ be a continuous function that satisfies

$$y_0 \leq x_0, \qquad D^+ y(t) \leq f(y(t)) \qquad \forall t \in [0, t_1). \tag{15}$$

Then $y(t) \leq x(t)$ for all $t \in [0, t_1)$.

Proof: See [44, Corollary 1.7.1]. ■

III. CBFs and Safeguarding Control Laws

Two notions of CBFs are studied in [6]: reciprocal CBFs and zeroing CBFs. Reciprocal CBFs diverge to infinity at the boundary of the safe set, whereas zeroing CBFs attain the value zero at the boundary of the safe set. In this paper we consider zeroing CBFs, with the convention that the safe set is the *subzero* level set of the associated CBF. Note that some CBF literature use the opposite convention, i.e., the safe set is the *superzero* level set of the associated CBF.

A. Definition and basic properties

We formally define CBFs for affine control systems of the form (1), where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and U are assumed to satisfy the following assumption:

Assumption 2: f and g are continuous functions. U is a closed convex set.

Definition 12 (CBF): Let $X \subset \mathbb{R}^n$ be a closed set, and let $B : X \to \mathbb{R}$ be a continuously differentiable function that defines the set

$$K := \{x \in X : B(x) \le 0\}. \tag{16}$$

B is a CBF with respect to K for (1) if there exists $\alpha \in \mathcal{K}_e$ such that

$$\inf_{u\in U} \Big[L_f B(x) + L_g B(x)u\Big] \le -\alpha(B(x)) \qquad \forall x \in X. \tag{17}$$

□

Since X is closed and B is continuous, K is a closed set. A CBF B and the associated $\alpha \in \mathcal{K}_e$ define the mapping $U_B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by (2) for $x \in X$ and being arbitrary for $x \notin X$.

Lemma 2: For the system (1), let $B : X \to \mathbb{R}$ be a CBF defining the set K, and let α be such that (17) is satisfied. If Assumption 2 is satisfied then $U_B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined in (2) is outer semicontinuous on X, and for each $x \in X$, $U_B(x)$ is convex and nonempty.

Proof: Since f and g are continuous, and B is continuously differentiable, $L_f B$ and $L_g B$ are continuous. Since U is closed, $U_B(x)$ is closed for each x. Together with continuity of α and the fact that U is closed, the set

$$\{(x, u) \in X \times U : L_f B(x) + L_g B(x)u \le -\alpha(B(x))\} \tag{18}$$

is closed. Then, by [29, Lemma 5.10], U_B is outer semicontinuous on X. We proceed with showing that, for each $x \in X$, $U_B(x)$ is convex if U is convex. Let $u', u'' \in U_B(x) \subset U$. By convexity of U, $\tau u' + (1-\tau)u'' \in U$, $\forall \tau \in [0, 1]$. Using $L_g B(x)u \le -L_f B(x) - \alpha(B(x))$ for $u \in \{u', u''\}$, we obtain, $\forall \tau \in [0, 1]$,

$$\begin{aligned}&\tau L_g B(x)u' + (1-\tau)L_g B(x)u'' \\ &\le -\tau(L_f B(x) + \alpha(B(x))) - (1-\tau)(L_f B(x) + \alpha(B(x))) \\ &\le -L_f B(x) - \alpha(B(x)),\end{aligned} \tag{19}$$

which implies $\tau u' + (1-\tau)u'' \in U_B(x)$. ■

Convexity of $U_B(x)$ is a consequence of the fact that we define CBFs for systems that are affine in the control input. The convexity and outer semicontinuity properties stated in Lemma 2 are used in the proof of Proposition 1 below, which establishes nominal robustness for discontinuous safeguarding control laws designed using CBFs. An example of a pathology related to nonconvexity is studied in [46], where a CBF-like function was used to identify safe headings of a ship with respect to a stationary object, resulting in a nonconvex set of safe headings. The system in [46] did not admit any continuous safeguarding control law for the ship heading, whereas the proposed discontinuous control law failed to ensure safety in the presence of arbitrarily small disturbances. The pathology was overcome by using a robust hybrid feedback control law.

Given an input mapping $U_B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we define the system

$$\dot{x} \in F_B(x), \qquad x \in X, \tag{20}$$

with $F_B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ taken as

$$F_B(x) := \{f(x) + g(x)u : u \in U_B(x)\} \qquad \forall x \in X. \tag{21}$$

If Assumption 2 is satisfied, then F_B is outer semicontinuous relative to X. This follows directly from continuity of f and g, and the properties of U_B stated in Lemma 2. If, additionally, U is a bounded set then F_B is locally bounded and system (20)-(21) satisfies Assumption 1.

B. Uniform asymptotic stability of the safe set

Forward pre-invariance of K for the system (20)-(21) is shown in [33]. It also follows immediately from the definition of a CBF that K is pre-asymptotically stable. Moreover, the function $V : X \to \mathbb{R}_{\ge 0}$ defined by

$$V(x) := \max\{0, B(x)\}^2 \tag{22}$$

is a Lyapunov function for the set K [9]. Sufficient conditions on V for UGpAS of K are given in Theorem 1.

Theorem 1: For the system (1), let $B : X \to \mathbb{R}$ be a CBF defining the set K. If there exist $\underline{\gamma}, \overline{\gamma} \in \mathcal{K}_\infty$ such that V defined in (22) satisfies

$$\underline{\gamma}(|x|_K) \le V(x) \le \overline{\gamma}(|x|_K) \qquad \forall x \in X, \tag{23}$$

then K is UGpAS for the system (20)-(21).

Proof: The gradient of V is continuous and given by

$$\nabla V(x) := \begin{cases} 2B(x)\nabla B(x) & x \in (X \setminus K) \cup \partial K, \\ 0 & x \in K. \end{cases} \tag{24}$$

Using (22)-(24) and the implication $u \in U_B(x) \implies$

$$\langle \nabla B(x), f(x) + g(x)u\rangle \le -\alpha(B(x)), \tag{25}$$

we obtain, $\forall x \in X$ and $\forall \eta \in F_B(x)$,

$$\begin{aligned}\langle \nabla V(x), \eta\rangle &\le -2\sqrt{V(x)}\alpha\left(\sqrt{V(x)}\right) \\ &\le -2\sqrt{\underline{\gamma}(|x|_K)}\alpha\left(\sqrt{\underline{\gamma}(|x|_K)}\right).\end{aligned} \tag{26}$$

UGpAS of K follows from [29, Theorem 3.18]. ■

Remark 3: Theorem 1 differs from the results of [9] in two main aspects: 1) Theorem 1 states UGpAS, while [9, Proposition 4] states asymptotic stability under an implicit assumption of forward completeness; 2) Theorem 1 is stated for differential inclusions and does not include a Lipschitz assumption on the control input.

C. Safeguarding control laws

Safeguarding control laws, i.e., control laws that ensure safety of the system (1) with respect to the set K, may be obtained by making a selection from U_B. We study the stability and nominal robustness properties of the resulting closed-loop systems, without imposing any continuity-assumptions on the safeguarding control laws.

Definition 13 (Safeguarding control law): A locally bounded function $\kappa_B : X \to U$ is a safeguarding control law if

$$\kappa_B(x) \in U_B(x) \qquad \forall x \in X. \tag{27}$$

□

Perhaps counterintuitively, the existence of a CBF does not guarantee the existence of a safeguarding control law. In particular, there may not exist a locally bounded function satisfying (27), even when U_B is outer semicontinuous, nonempty, and convex-valued. A counterexample is given next:

Example 2 (No safeguarding control law): Consider the system (1) with $x, u \in \mathbb{R}$, $f(x) := 0$, and $g(x) := x^2$. With $B(x) := x$ and $\alpha(\phi) := \phi$, U_B in (2) may be written as $U_B(x) := \{u \in \mathbb{R} : x^2 u \leq -x\}$. In particular, $U_B(x) := (-\infty, -1/x]$ for $x \neq 0$, and $U_B(0) = \mathbb{R}$. By Lemma 2, U_B is outer semicontinuous. To realize this, it is useful to observe that $U_B(x) \subset U_B(0)$, for all x. Since $\partial U_B(x) = -1/x$ approaches $-\infty$ as $x \to 0^+$, there cannot exist any locally bounded function κ_B satisfying (27). If we instead select $\alpha(\phi) := \phi^3$, then $U_B(x) = \{u : x^2 u \leq -x^3\}$, and $\kappa_B(x) := -x$ is a safeguarding control law. □

The pathology of Example 2 is related to the fact that the control authority vanishes at the boundary of the safe set. In the following, we will assume that there exists a safeguarding control law, since this is of the most practical interest. Clearly, if κ_B satisfies (27), then the closed-loop system

$$\dot{x} = f_{cl}(x) := f(x) + g(x)\kappa_B(x), \qquad x \in X, \tag{28}$$

inherits the stability properties of the differential inclusion (20)-(21). Less obvious is the fact that the system

$$\dot{x} \in F_{cl}(x) := \bigcap_{\delta > 0} \overline{\text{con}} f_{cl}((x + \delta\mathbb{B}) \cap X), \qquad x \in X, \tag{29}$$

also inherits the stability properties of (20)-(21) even for discontinuous safeguarding control laws. This is stated in Proposition 1 below. Additionally, if Assumption 2 is satisfied, and κ_B is a safeguarding control law, then the Krasovskii regularization of f_{cl} satisfies Assumption 1, and the system (29) is well-posed.

Proposition 1: For system (1), let $B : X \to \mathbb{R}$ be a CBF defining the set K. Suppose Assumption 2 is satisfied, let $\kappa_B : \mathbb{R}^n \to U$ be a locally bounded function that satisfies (27), and let (29) be the Krasovskii regularization of (28). Then (29) satisfies Assumption 1. If additionally K is UGpAS for (20)-(21), then K is UGpAS for (29).

Proof: By assumption, κ_B is locally bounded, and f and g are continouos. Then f_{cl} is locally bounded. Outer semicontinuity and local boundedness of F_{cl} follows from [29, Lemma 5.16]. Convexity of $F_{cl}(x)$ follows from the fact that the intersection of convex sets is convex. Noting that Definition 12 requires X to be closed, Assumption 1 is satisfied.

The second statement holds if $F_{cl}(x) \subset F_B(x)$, $\forall x \in X$. Define

$$\bar{\kappa}_B(x) := \bigcap_{\delta > 0} \overline{\text{con}} \kappa_B((x + \delta\mathbb{B}) \cap X). \tag{30}$$

Using continuity of f and g, F_{cl} satisfies

$$F_{cl}(x) = \{f(x) + g(x)u : u \in \bar{\kappa}_B(x)\}. \tag{31}$$

By Lemma 2, U_B is outer semicontinuous and convex-valued, which implies, $\forall x \in X$,

$$\bar{\kappa}_B(x) \subset \bigcap_{\delta > 0} \overline{\text{con}} U_B((x + \delta\mathbb{B}) \cap X) = U_B(x) \tag{32}$$

which in turn implies $F_{cl}(x) \subset F_B(x)$. ■

The statement of Proposition 1 is useful in view of optimization-based controllers, since establishing continuity properties of such controllers is challenging due to the lack of closed-form solutions [47]. A popular CBF-based controller is given by the quadratic program

$$\kappa_B(x) := \arg \min_{u \in U_B(x)} (u - \kappa(x))^\top P(u - \kappa(x)), \tag{33}$$

where $\kappa : \mathbb{R}^n \to U$ is a nominal control law, and $P \in \mathbb{R}^{m \times m}$ is a positive definite symmetric cost matrix. Lipschitz continuity of (33), for the special case of $U = \mathbb{R}^m$ and $L_g B(x) \neq 0, \forall x \in X$, was shown in [9], provided that f, g, α, and κ are Lipschitz continuous. However, if $U_B(x)$ is replaced by the intersection of several safe input sets, each enforcing a safety constraint, then Lipschitz continuity may fail even when the conditions stated in the previous sentence holds [27], [47].

IV. High-order safety constraints

Safety constraints for systems with higher relative degree may be enforced by backstepping CBF-like functions until the control input appears [23], [48].

Definition 14 (Relative degree): Let $X \subset \mathbb{R}^n$. A continuous and sufficiently differentiable function $B_1 : X \to \mathbb{R}$ has relative degree r with respect to the system (1), if the following conditions are satisfied:

- $L_g L_f^{r-i} B_1(x) = 0$ for all $x \in X$ and for all $i \in \{2, \dots, r\}$.
- $L_g L_f^{r-1} B_1(x) \neq 0$ for some $x \in X$.

□

A. High-order CBFs

Suppose $B_1 : X \to \mathbb{R}$ has relative degree r with respect to system (1). For $i \in \{2, \dots, r\}$, recursively define the functions $B_i : X \to \mathbb{R}$ as

$$B_i(x) := L_f B_{i-1}(x) + \alpha_{i-1}(B_{i-1}(x)), \tag{34}$$

with α_{i-1} taken as sufficiently differentiable class-$\mathcal{K}_e$ functions. Let each B_i define a set

$$K_i := \{x \in X : B_i(x) \leq 0\}. \tag{35}$$

Enforcing

$$L_f B_r(x) + L_g B_r(x) u \leq -\alpha_r(B_r(x)), \tag{36}$$

with $\alpha_r \in \mathcal{K}_e$, achieves forward pre-invariance of the intersection $K_1 \cap \ldots \cap K_r$. However, the set K_1 is not forward pre-invariant, since solutions starting in $\partial K_1 \cap (X \setminus K_2)$ will leave the set K_1. To see this, observe that $x_0 \in \partial K_1 \cap (X \setminus K_2)$ implies $B_1(x_0) = 0$ and $\dot{B}_1(x_0) = L_f B_1(x_0) > 0$. The above solution for backstepping CBFs was first proposed in [24] (see also [25]). While Definition 14 does not require $L_g L_f^{r-1} B(x) \neq 0$ for all $x \in X$, sufficient control authority is required to enforce (36).

We collect B_i, for $i \in \{1, \ldots, r\}$, in the vector function $B : X \to \mathbb{R}^r$, i.e.,

$$B(x) := \begin{bmatrix} B_1(x) & \ldots & B_r(x) \end{bmatrix}^\top. \tag{37}$$

Moreover, we define the safe set as

$$K := \{x \in X : B(x) \leq 0\} = \bigcap_{i=1}^{r} K_i. \tag{38}$$

Definition 15 (HOCBF): Let $X \subset \mathbb{R}^n$ be a closed set, and let $B_1 : X \to \mathbb{R}$ have relative degree r with respect to system (1). For $i \in \{2, \ldots, r\}$, let $B_i : X \to \mathbb{R}$ be recursively defined in (34), with $\alpha_{i-1} \in \mathcal{K}_e$. Let $B : X \to \mathbb{R}^r$ defined in (37) be a continuously differentiable function that defines the set K in (38). B is an HOCBF with respect to K for the system (1) if there exists $\alpha_r \in \mathcal{K}_e$ such that

$$\inf_{u \in U} \left[L_f B_r(x) + L_g B_r(x) u \right] \leq -\alpha_r(B_r(x)) \qquad \forall x \in X. \tag{39}$$

□

Similar to CBFs, each HOCBF gives rise to a safe input set given by

$$U_B(x) := \{u \in U : L_f B_r(x) + L_g B_r(x) u \leq -\alpha_r(B_r(x))\} \tag{40}$$

for all $x \in X$. If B is an HOCBF then K is closed and forward pre-invariant for the system (20)-(21) with U_B defined in (40) (see e.g. [33] for proof). Observe that the results of Section III generalize to HOCBFs. That is, Lemma 2 and Proposition 1 hold with U_B, F_B, and K defined by HOCBFs. Additionally, Theorem 1 can be used to establish UGpAS of the set K_r.

In Definition 15, the vector function $B : X \to \mathbb{R}^r$ is referred to as an HOCBF. This differs from the definition given in [24], [25], where it is the scalar function B_1 that is referred to as the HOCBF. Referring to B as the HOCBF highlights the fact that it is K, and not K_1, that is rendered forward pre-invariant. Additionally, the stability properties of K are determined by B, and not by B_1 alone.

B. Induced lower-dimensional comparison system

An HOCBF induces a comparison system of the form [26]

$$\dot{z} = \Gamma(z), \tag{41}$$

with state $z \in \mathbb{R}^r$ and $\Gamma : \mathbb{R}^r \to \mathbb{R}^r$ given by

$$\Gamma(z) := \begin{bmatrix} -\alpha_1(z_1) + z_2 & -\alpha_2(z_2) + z_3 & \ldots & -\alpha_r(z_r) \end{bmatrix}^\top. \tag{42}$$

System (41)-(42) has a cascaded structure, and Γ is quasi-monotone nondecreasing. UGAS of the origin of (41)-(42) was shown in [26], assuming Γ is Lipschitz. Suppose x is a complete solution to (20)-(21) with U_B in (40). Then, by appealing to the comparison result of Lemma 1, UGAS of the origin for (41)-(42) implies $\lim_{t\to\infty} B(x(t)) \leq 0$. This translates to pre-attractivity of K, but is insufficient to conclude stability. Proposition 2 below states UGAS of the noncompact set $\mathbb{R}^r_{\leq 0}$ for (41)-(42), and will be used to show UGpAS of K for (20)-(21),(40).

Proposition 2: If the functions α_i, $i \in \{1, \ldots, r\}$ belong to class-$\mathcal{K}_e$, then the set $\mathbb{R}^r_{\leq 0}$ is UGAS for (41)-(42).

Proof: For each $i \in \{2, \ldots, r\}$, define $\underline{z}_i := [z_i \ \ldots \ z_r]^\top \in \mathbb{R}^{r-i+1}$ and $\underline{\Gamma}_i(\underline{z}_i) := [-\alpha_i(z_i) + z_{i+1} \ \ldots \ -\alpha_r(z_r)]^\top$, and consider the system

$$\dot{z}_{i-1} = -\alpha_{i-1}(z_{i-1}) + z_i \tag{43a}$$

$$\dot{\underline{z}}_i = \underline{\Gamma}_i(\underline{z}_i). \tag{43b}$$

Suppose $\mathbb{R}^{r-i+1}_{\leq 0}$ is UGAS for (43b), and note that $z_i \leq |\underline{z}_i|_{\mathbb{R}^{r-i+1}_{\leq 0}}$. Then, by applying Proposition 4 in Appendix A, the set $\mathbb{R}^{r-i+2}_{\leq 0}$ is UGAS for the system (43). By inspection, $\mathbb{R}_{\leq 0}$ is UGAS for the system (43b) with $i = r$. Using induction, $\mathbb{R}^r_{\leq 0}$ is UGAS for (41). ■

C. Sufficient conditions for UGpAS using HOCBFs

Asymptotic stability of K was shown in [26], for any Lipschitz continuous control law $\kappa_B : \mathbb{R}^n \to U$ satisfying (27), under the assumption of forward completeness, and assuming that K is compact. Theorem 2 below extends this result, and provides sufficient conditions for UGpAS of (not necessarily compact) K. To this end, we replace $V : X \to \mathbb{R}_{\geq 0}$ in (22) with

$$V(x) := \sum_{i=1}^{r} \max\{0, B_i(x)\}^2. \tag{44}$$

V in (44) is positive definite with respect to the set K, but not necessarily decreasing along the solutions of (20)-(21),(40).

Theorem 2: Let $B : X \to \mathbb{R}^r$ be an HOCBF for system (1) defining the set K. If there exist $\underline{\gamma}, \overline{\gamma} \in \mathcal{K}_\infty$, such that $V : X \to \mathbb{R}_{\geq 0}$ defined in (44) satisfies

$$\underline{\gamma}(|x|_K) \leq V(x) \leq \overline{\gamma}(|x|_K) \qquad \forall x \in X, \tag{45}$$

then K is UGpAS for system (20)-(21), with U_B defined in (40).

Proof: By Proposition 2, $\mathbb{R}^r_{\leq 0}$ is UGAS for the comparison system (41)-(42). The remainder of the proof can be constructed following similar steps as the proof of Theorem 3 below. ■

Theorem 1 may be viewed as a special case of Theorem 2, since the latter theorem also applies to first-order CBFs.

V. Robustness of HOCBFs

We study the robustness properties of HOCBF-based systems that are affine in the disturbance input, i.e., systems of the form

$$\dot{x} \in F_{B,w}(x, w), \qquad x \in X, \tag{46}$$

with $F_{B,w} : \mathbb{R}^n \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ taken as

$$F_{B,w}(x,w) := \{f(x) + g(x)u + h(x)w : u \in U_B(x)\} \quad (47)$$

for $x \in X$. Here, $h : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is a known function, $t \mapsto w(t) \in \mathbb{R}^p$ is an unknown disturbance, and U_B is defined by an HOCBF for the undisturbed system (1). System (40),(46)-(47) includes additive disturbances and input-disturbances as special cases, obtained by setting $h(x) := I_{n\times n}$ and $h(x) := g(x)$, respectively.

A. Comparison system with disturbances

For the system (1), let $B : X \to \mathbb{R}^r$ be an HOCBF for the set K. To assess robustness properties of the disturbed system (40),(46)-(47), we construct a comparison system from the worst-case evolution of B along (40),(46)-(47). For each $i \in \{1, ..., r\}$, let $\Delta_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be given by

$$\Delta_i(\phi) := \sup_{\{x : B_i(x) = \phi\}} |L_h B_i(x)|. \quad (48)$$

Recall that if B is an HOCBF of order r, then $L_g B_i(x) = 0$ for all $i \in \{1, \ldots, r-1\}$. The evolution of B along (40),(46)-(47) satisfies, $\forall \eta \in F_{B,w}(x, w)$,

$$\begin{aligned} \langle \nabla B_i(x), \eta \rangle &= -\alpha_i(B_i(x)) + B_{i+1}(x) + L_h B_i(x) w \\ &\leq -\alpha_i(B_i(x)) + B_{i+1}(x) + |L_h B_i(x)||w| \\ &\leq -\alpha_i(B_i(x)) + B_{i+1}(x) + \Delta_i(B_i(x))|w|, \end{aligned} \quad (49)$$

for $i \in \{1, ..., r-1\}$, and

$$\begin{aligned} \langle \nabla B_r(x), \eta \rangle &\leq -\alpha_r(B_r(x)) + L_h B_r(x) w \\ &\leq -\alpha_r(B_r(x)) + |L_h B_r(x)||w| \\ &\leq -\alpha_r(B_r(x)) + \Delta_r(B_r(x))|w|. \end{aligned} \quad (50)$$

Using (49)-(50), we construct a comparison system

$$\dot{z} = \Gamma(z) + \Delta(z)\mu, \quad (51)$$

with Γ defined in (42), $\mu \in \mathbb{R}_{\geq 0}$ taking the role of $|w|$, and disturbance gain $\Delta : \mathbb{R}^r \to \mathbb{R}^r_{\geq 0}$ given by

$$\Delta(z) := \begin{bmatrix} \Delta_1(z_1) & \ldots & \Delta_r(z_r) \end{bmatrix}^\top. \quad (52)$$

While the gradient of a smooth CLF is zero at the boundary of the desired set, the gradient of an HOCBF is typically nonzero, and may even be unbounded, at the boundary of the safe set. Depending on h, this implies that, for some $i \in \{1, \ldots r\}$, $\Delta_i(0)$ may be nonzero, or even $\Delta_i(0) = \infty$ in some cases. If $\Delta_i(0) = \infty$ for some $i \in \{1, \ldots, r\}$, then no robustness properties can be concluded from the solutions of the comparison system (51) with Γ in (42) and Δ in (48),(52).

Proposition 3 below states that, if Δ is globally bounded, then (42) is small-input ISS with respect to $\mathbb{R}^r_{\leq 0}$. Additional investigations are required when Δ is locally, but not globally, bounded.

Proposition 3: If, for each $i \in \{1, \ldots, r\}$, the functions α_i belong to class-$\mathcal{K}_e$, and there exists $c_i \geq 0$ such that $\Delta_i(z_i) \leq c_i$ for all $z_i \in \mathbb{R}$, then the system (51),(42),(48),(52), is small-input ISS with respect to $\mathbb{R}^r_{\leq 0}$.

Proof: Let $\underline{z}_i := [z_i \ \ldots \ z_r]^\top \in \mathbb{R}^{r-i+1}$ and $\underline{\Gamma}_i(\underline{z}_i)$ be defined as in Proposition 2. For each $i \in \{2, \ldots, r\}$, let c_i satisfy $\sup_{z_i \in \mathbb{R}} \Delta_i(z_i) \leq c_i$, define $\underline{c} := [c_i \ c_{i+1} \ \ldots \ c_r]$, and consider the system, $\mu \in \mathbb{R}_{\geq 0}$,

$$\dot{z}_{i-1} = -\alpha_{i-1}(z_{i-1}) + z_i + c_{i-1}\mu \quad (53a)$$

$$\dot{\underline{z}}_i = \underline{\Gamma}_i(\underline{z}_i) + \underline{c}_i \mu. \quad (53b)$$

Suppose (53b) is small-input ISS with respect to $\mathbb{R}^{r-i+1}_{\leq 0}$, and note that $z_i \leq |\underline{z}_i|_{\mathbb{R}^{r-1+1}_{\leq 0}}$. Then, by applying Proposition 5 in Appendix A, (53) is small-input ISS with respect to $\mathbb{R}^{r-i+2}_{\leq 0}$. By inspection, (53b) with $i = r$ is small-input ISS with respect to $\mathbb{R}_{\leq 0}$. Using induction, (53) with $i = 2$ is small-input ISS with respect to $\mathbb{R}^r_{\leq 0}$. Since $\Delta_i(z_i) \leq c_i$, for all $z_i \in \mathbb{R}$, and for all i, the system (51),(42),(48),(52) is small-input ISS with respect to $\mathbb{R}^r_{\leq 0}$. ■

B. Sufficient conditions for ISpS using HOCBFs

Theorem 3 below states sufficient conditions for small-input ISpS of system (40),(46)-(47) with respect to K. The theorem does not require the conditions of Proposition 3 to be satisfied, but rather includes small-input ISS for the comparison system as an assumption.

Theorem 3: Let $B : X \to \mathbb{R}^r$ be an HOCBF for the system (1) defining the set K. Let Γ, Δ be defined in (42) and (48),(52), respectively. Suppose the following conditions are satisfied:

- There exist $\underline{\gamma}, \overline{\gamma} \in \mathcal{K}_\infty$ such that $V : X \to \mathbb{R}$ defined in (44) satisfies

$$\underline{\gamma}(|x|_K) \leq V(x) \leq \overline{\gamma}(|x|_K) \qquad \forall x \in X. \quad (54)$$

- The system (42),(48),(52),(51) is small-input ISS with respect to $\mathbb{R}^r_{\leq 0}$.

Then the system (40),(46)-(47) is small-input ISS with respect to K.

Proof: For the comparison system (51), define $v : \mathbb{R}^r \to \mathbb{R}$ as

$$v(z) := \sum_{i=1}^{r} \max\{0, z_i\}^2, \quad (55)$$

which is positive definite with respect to $\mathbb{R}^r_{\leq 0}$. Using small-input ISS of (51), let $\bar{\mu} > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ be such that the solutions of (51) satisfy, $\forall z_0 \in \mathbb{R}^r$, $\forall t \geq 0$,

$$\|\mu\|_\infty \leq \bar{\mu} \implies v(z(t)) \leq \beta(v(z_0), t) + \gamma(\|\mu(t)\|_\infty). \quad (56)$$

In the following, let x be any maximal solution to (40),(46)-(47) with initial condition $x_0 \in \mathbb{R}^n$, and let z be the max-solution to (51) with initial condition $z_0 = B(x_0)$, and input $\mu(t) = \|w\|_\infty$ for all $t \geq 0$. The upper Dini derivative of $t \mapsto B(x(t))$ satisfies, $\forall t \in \operatorname{dom} x$,

$$D^+ B(x(t)) \leq \Gamma(B(x(t)) + \Delta(B(x(t))\|w\|_\infty. \quad (57)$$

Note that Γ and Δ are quasimonotone nondecreasing. Then, for any fixed c, the function $\Gamma(\cdot) + \Delta(\cdot)c$ is quasimonotone nondecreasing. It follows from Lemma 1, and using (57), that

$$B(x(t)) \leq z(t) \qquad \forall t \in \operatorname{dom} x. \quad (58)$$

Let $\underline{\gamma}, \overline{\gamma} \in \mathcal{K}_\infty$ satisfy (54). Using (56) and (58), and the definitions of V and v, we obtain, for all $t \in \operatorname{dom} x$,

$$\begin{aligned} &\|w\|_\infty \leq \bar{\mu} \implies \\ &\quad V(x(t)) \leq v(z(t)) \leq \beta(v(z_0), t) + \gamma(\|w\|_\infty) \\ &\qquad\qquad = \beta(V(x_0), t) + \gamma(\|w\|_\infty) \\ &\qquad\qquad \leq \beta(\overline{\gamma}(|x_0|_K), t) + \gamma(\|w\|_\infty). \end{aligned} \tag{59}$$

In particular, using (54) and (59), and $\gamma(a+b) \leq \gamma(2a)+\gamma(2b)$ for any $\gamma \in \mathcal{K}_\infty$, solutions to (40),(46)-(47) satisfy

$$\begin{aligned} &\|w\|_\infty \leq \bar{\mu} \implies \\ &|x(t)|_K \leq \underline{\gamma}^{-1}(2\beta(\overline{\gamma}(|x_0|_K), t)) + \underline{\gamma}^{-1}(2\gamma(\|w\|_\infty)) \end{aligned} \tag{60}$$

for all $t \in \operatorname{dom} x$. ■

Theorem 3 can be strengthened to ISpS if the comparison system is ISS.

VI. Case studies

We investigate the stability and robustness properties of CBF designs for safe motion control of two vehicles. The first case study illustrates that, in general, UGpAS does not imply small-input ISpS with respect to additive disturbances, while the second case study illustrates that, in general, global pre-asymptotic stability does not imply UGpAS. Before continuing, we define the unit circle and the 90-degree rotation matrix as

$$\mathbb{S}^1 := \{x \in \mathbb{R}^2 : x^\top x = 1\}, \quad S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{61}$$

respectively.

A. Case study 1: Linear kinematics

Let $x := (p, v) \in \mathbb{R}^2 \times \mathbb{R}^2$ denote the position and velocity of a point-mass vehicle, define

$$f_{lin}(x) := \begin{bmatrix} v \\ 0 \end{bmatrix}, \qquad g_{lin}(x) := \begin{bmatrix} 0_{2\times 2} \\ I_{2\times 2} \end{bmatrix}, \tag{62}$$

and consider the system

$$\dot{x} = f_{lin}(x) + g_{lin}(x)u, \tag{63}$$

with control input $u \in \mathbb{R}^2$. Here, subscript lin refers to the linear kinematics.

1) CBF design and nominal stability: For ease of exposition, we consider a stationary disk-shaped obstacle domain centered at the origin, with unit radius. Let $0 < \epsilon \ll 1$ and define

$$X := \{x \in \mathbb{R}^4 : |p| \geq \epsilon\}. \tag{64}$$

Let $B_1, B_2 : X \to \mathbb{R}$ be given by

$$B_1(x) := 1 - |p|, \qquad B_2(x) := L_{f_{lin}} B_1(x) + B_1(x). \tag{65}$$

Using $L_{g_{lin}} B_1(x) = 0$ and

$$L_{f_{lin}} B_1(x) = \frac{-p^\top v}{|p|}, \tag{66}$$

$$L_{f_{lin}} B_2(x) = \frac{v^\top S p p^\top S v}{|p|^3} + L_{f_{lin}} B_1(x), \tag{67}$$

$$L_{g_{lin}} B_2(x) = \frac{-p^\top}{|p|}, \tag{68}$$

the function $B(\cdot) := \begin{bmatrix} B_1(\cdot) & B_2(\cdot) \end{bmatrix}^\top$ is a second-order CBF for the set $K := K_1 \cap K_2$. For $x \in X$, let $U_{B,lin} : \mathbb{R}^4 \rightrightarrows \mathbb{R}^2$ be given by

$$\begin{aligned} U_{B,lin}(x) := \{&u \in \mathbb{R}^2 : \\ &L_{f_{lin}} B_2(x) + L_{g_{lin}} B_2(x) u \leq -B_2(x)\}. \end{aligned} \tag{69}$$

Noting that

$$|x|_K \to \infty \iff \sum_{i=1}^{2} \max\{0, B_i(x)\} \to \infty, \tag{70}$$

the conditions of Theorem 2 are satisfied, and K is UGpAS for the system

$$\dot{x} \in \{f_{lin}(x) + g_{lin}(x)u : u \in U_{B,lin}(x)\}, \qquad x \in X. \tag{71}$$

2) Robustness towards additive disturbances: We first consider the effect of additive disturbances acting on the system (71), i.e., robustness properties of the system

$$\dot{x} \in \{f_{lin}(x) + g_{lin}(x)u + w : u \in U_{B,lin}(x)\}, \quad x \in X, \tag{72}$$

with disturbance input $w \in \mathbb{R}^4$. For any fixed p,

$$|\nabla B_2(x)| = \left| \begin{bmatrix} \frac{v^\top S p p^\top S}{|p|^3} + \frac{-p^\top}{|p|} & \frac{-p^\top}{|p|} \end{bmatrix}^\top \right| \tag{73}$$

may grow unbounded as $|v|$ grows unbounded. Additionally, $|v|$ cannot be bounded for any fixed ϕ such that $B_2(x) = \phi$. As a result, Theorem 3 cannot be used to conclude small-input ISpS of (72) with respect to K. Indeed, selecting the safeguarding control law

$$\begin{aligned} \kappa_{B,lin}(x) &:= \arg\min_{u \in U_{B,lin}(x)} |u - \kappa(x)|, \\ \kappa(x) &:= -(v - Sp) + Sv, \end{aligned} \tag{74}$$

we can show that (72) is not small-input ISpS with respect to K. Here, κ is a nominal control law which aims to steer the vehicle in circles around the origin with velocity $v = Sp$. Since $\kappa_{B,lin}(x) \in U_{B,lin}(x)$ for all $x \in X$, the unique solutions to the system

$$\dot{x} = f_{lin}(x) + g_{lin}(x)\kappa_{B,lin}(x) + w, \qquad x \in X, \tag{75}$$

are also solutions to (72). We simulate the response of (75) with the bounded worst-case perturbation $t \mapsto w(t)$ given by

$$w(t) = \frac{\nabla B_2(x(t))}{|\nabla B_2(x(t))|} \bar{w} \qquad \forall t \in \operatorname{dom} x, \tag{76}$$

where $\bar{w} > 0$ is a small positive number. Initial conditions are selected as $p_0 = (1, 0)$ and $v_0 = (0, v_2(0))$, with positive and increasing values of $v_2(0)$. The resulting trajectories are provided in Figure 1. Observe that increasing values of $v_2(0)$ results in decreasing values of $\inf_{t \in \operatorname{dom} x} |p(t)|$. This implies that, for any fixed $\bar{w} > 0$ and $\delta \in (\epsilon, 1)$, there exist $v_2(0)$ such that $\inf_{t \in \operatorname{dom} x} |p(t)| \leq \delta$. Since $x_0 := (p_0, v_0) \in \partial K$, for all $v_2(0) \in \mathbb{R}$, and $|x(t)|_K \geq |x(t)|_{K_1} \geq 1 - |p(t)|$, there cannot exist $\gamma \in \mathcal{K}_\infty$ and $\bar{w} > 0$ such that, for all $t \in \operatorname{dom} x$,

$$|x_0|_K = 0 \text{ and } \|w\|_\infty \leq \bar{w} \implies |x(t)|_K \leq \gamma(\|w\|_\infty). \tag{77}$$

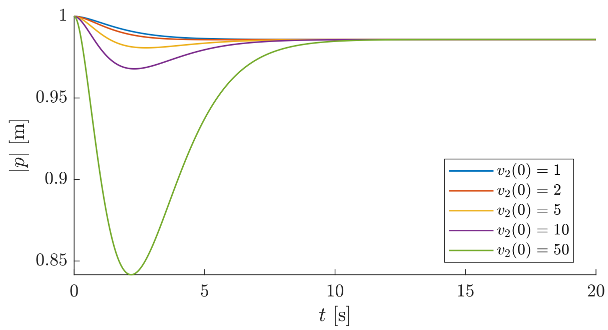

Fig. 1. Evolution of $|\boldsymbol{p}|$ for the linear kinematics model with additive disturbance; system (74)-(75) with $\boldsymbol{f_{lin}, g_{lin}}$ in (63), $\boldsymbol{U_{B,lin}}$ generated by (65)-(69), and $\boldsymbol{w}$ in (76) with small disturbance magnitude $\boldsymbol{\bar{w} = 0.01}$. Solutions are initialized at $\boldsymbol{x_0 := (p_0, v_0)}$, with $\boldsymbol{p_0 = (1, 0)}$ and $\boldsymbol{v_0 = (0, v_2(0))}$. Since $\boldsymbol{x_0 \in K}$ and $\boldsymbol{\inf_{t \in \mathrm{dom}\, x} |p(t)|}$ decreases as $\boldsymbol{v_2(0)}$ increases, the system is not ISpS with respect to $\boldsymbol{K}$.

3) Robustness towards input disturbances: We now study the robustness properties of the system

$$\dot{x} \in \{f_{lin}(x) + g_{lin}(x)(u + w) : u \in U_{B,lin}(x)\}, \qquad x \in X, \tag{78}$$

with input disturbance $w \in \mathbb{R}^2$. Using $|L_{g_{lin}} B_1(x)| = 0$ and

$$|L_{g_{lin}} B_2(x)| = \left| \frac{-p^\top}{|p|} \right| = 1 \qquad \forall x \in X, \tag{79}$$

we obtain the comparison system

$$\dot{z}_1 = -z_1 + z_2, \qquad \dot{z}_2 = -z_2 + \mu. \tag{80}$$

Clearly, (80) is ISS with respect to $\mathbb{R}^2_{\leq 0}$. Small-input ISpS of system (78) with respect to K follows from Theorem 3 and (70). We verify this result by simulating the response of the system

$$\dot{x} = f_{lin}(x) + g_{lin}(x)(\kappa_{B,lin}(x) + w), \qquad x \in X, \tag{81}$$

with the bounded worst-case perturbation

$$w(t) = \frac{L_g B_2(x(t))^\top}{|L_g B_2(x(t))|} \bar{w} \qquad \forall t \in \mathrm{dom}\, x. \tag{82}$$

The resulting trajectories are provided in Figure 2. The initial conditions correspond to those used in Section VI-A.2, where we recall that $x_0 \in \partial K$ for all trajectories. Contrary to the system with additive disturbances, see Figure 1, $\inf_{t \in \mathrm{dom}\, x} |p(t)|$ is independent of $v_2(0)$, as expected from the ISpS property of the system with respect to K, and the fact that all solutions start from ∂K.

B. Case study 2: Unicycle kinematics

We adopt the unit circle representation of orientation [32], [33]. Let $p \in \mathbb{R}^2$, $v \in \mathbb{R}$, and $z \in \mathbb{S}^1$ be the position, forward speed, and orientation vector of a vehicle, respectively, define $x := (p, v, z) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{S}^1$,

$$f_{uni}(x) := \begin{bmatrix} vz \\ 0 \\ 0_{2\times1} \end{bmatrix}, \qquad g_{uni}(x) := \begin{bmatrix} 0_{2\times1} & 0_{2\times1} \\ 1 & 0 \\ 0_{2\times1} & Sz \end{bmatrix}, \tag{83}$$

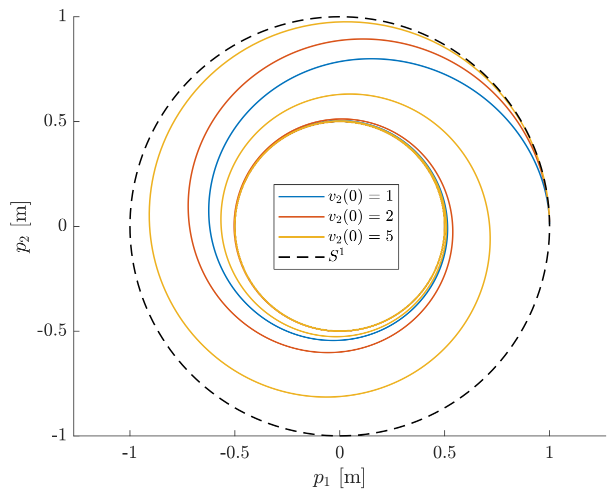

Fig. 2. Trajectory of solutions to the linear kinematics model with input disturbances; system (81) with $\boldsymbol{f_{lin}, g_{lin},}$ in (63), $\boldsymbol{\kappa_{B,lin}}$ in (74) with $\boldsymbol{U_{B,lin}}$ generated by (65)-(69), and $\boldsymbol{w}$ in (82) with magnitude $\boldsymbol{\bar{w} = 0.5}$. The initial conditions are stated in the caption of Figure 1.

and consider the system

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{z} \end{bmatrix} = f_{uni}(x) + g_{uni}(x)u, \tag{84}$$

with control input $u \in \mathbb{R}^2$. Here, subscript uni refers to unicycle kinematics.

1) CBF for obstacle avoidance: We first repeat the CBF design of Section VI-A.1. Let $B_1, B_2 : X \to \mathbb{R}$ be given by

$$B_1(x) := 1 - |p|, \qquad B_2(x) := L_{f_{uni}} B_1(x) + B_1(x). \tag{85}$$

where X will be specified shortly. Trivial calculations yield $L_{g_{uni}} B_1(x) = 0$, and

$$L_{f_{uni}} B_1(x) = \frac{-p^\top vz}{|p|}, \tag{86}$$

$$L_{f_{uni}} B_2(x) = \frac{-(p^\top Svz)^2}{|p|^3} + L_{f_{uni}} B_1(x), \tag{87}$$

$$L_{g_{uni}} B_2(x) = \frac{-p^\top}{|p|} \begin{bmatrix} z & vSz \end{bmatrix}. \tag{88}$$

The first thing to note is that $v = 0$ and $z = \pm Sp/|p|$ implies $L_{g_{uni}} B_2(x) = 0$, i.e., the control authority vanishes if the vehicle is at rest and oriented perpendicular to the vector towards the origin. Since, additionally, $v = 0 \implies L_{f_{uni}} B_1(x) = L_{f_{uni}} B_2(x) = 0$, there does not exist $\alpha_2 \in \mathcal{K}_e$ such that

$$\inf_{u \in \mathbb{R}^2} \{L_{f_{uni}} B_2(x) + L_{g_{uni}} B_2(x) u\} \leq -\alpha_2(B_2(x)) \tag{89}$$

is satisfied for all $x \in \{x \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{S}^1 : |p| > 0\}$. To maintain control authority for all $x \in X$, we select

$$X := \{x \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{S}^1 : |p| \geq \epsilon, \ v \geq v_{min}\}, \tag{90}$$

for some $v_{min} > 0$, and $\epsilon \in (0, 1)$. This approach of excluding points of vanishing control from the state space is enabled by the modeling framework adopted herein, and allows us to study the stability properties before completing the system. If v drops below v_{min}, we may assume that the vehicle switches to a different low-speed controller. A more elegant solution to the problem of vanishing control authority, that does not require maintaining a positive forward speed, is obtained by using a hybrid CBF formulation [33]. Another possible approach, not further explored, is to adopt the relaxed CBF formulation of [28] to obtain a nonempty safe input set at the points of vanishing control.

For all $(p, v, z) \in X$, the following implication holds:

$$(p, v, z) \in K_1 \setminus K_2 \implies (p, v, -z) \in K_1 \cap K_2. \tag{91}$$

In other words, for any $x \in K_1 \setminus K_2$, the safe set K can be reached simply by reversing the orientation. Using (91), it may be verified that, $\forall x \in X$,

$$\begin{aligned} |x|_K &\leq |x|_{K_1} + |x|_{K_2} \\ &\leq |x|_{K_1} + |z - (-z)| \leq |x|_{K_1} + 2. \end{aligned} \tag{92}$$

Meanwhile, for any fixed p, $B_2(x)$ can be made arbitrarily large by selecting $z = -p/|p|$, and v sufficiently large. As a result, there does not exist $\overline{\gamma} \in \mathcal{K}_\infty$ such that

$$\sum_{i=1}^{2} \max\{0, B_i(x)\}^2 \leq \overline{\gamma}(|x|_K) \qquad \forall x \in X. \tag{93}$$

The conditions of Theorem 2 are not satisfied, and Theorem 2 cannot be used to conclude UGpAS of K for the system

$$\begin{aligned} &\dot{x} \in \{f_{uni}(x) + g_{uni}(x)u : u \in U_{B,uni}(x)\}, \qquad x \in X, \\ &U_{B,uni}(x) := \{u \in \mathbb{R}^2 : \\ &\qquad L_{f_{uni}} B_2(x) + L_{g_{uni}} B_2(x)u \leq -B_2(x)\}. \end{aligned} \tag{94}$$

Of course, this does not mean that K is not UGpAS for (94) with B_1, B_2 in (85). Note that $|x|_{K_1} \leq (1-\epsilon)$ for all $x \in X$. Together with (92), this implies $|x|_K \leq 3 - \epsilon$ for all $x \in X$. Since K is globally pre-asymptotically stable, and the distance from x to K is globally bounded, we conjecture that K is in fact UGpAS for (94) with B_1, B_2 in (85). Further investigations into the stability properties of K for this system are outside the scope of this paper.

2) Reversed safety constraint: It is, however, easy to construct a system where K is globally pre-asymptotically stable, but not UGpAS, simply by reversing the safety constraint. That is, we design a CBF to maintain the vehicle inside the unit disk, rather than outside it. Redefine $B_1, B_2 : X \to \mathbb{R}$ as

$$B_1(x) := |p| - 1, \qquad B_2(x) := L_{f_{uni}} B_1(x) + B_1(x). \tag{95}$$

By similar arguments as above, Theorem 2 cannot be used to conclude UGpAS of K for the system (94)-(95). However, with the reversed safety constraint, $|x|_{K_1}$, and consequently $|x|_K$, cannot be bounded. To see that K is not UGpAS for (94) with B_1, B_2 in (95), define the nominal control law [32]

$$\kappa(x) := \begin{bmatrix} -(v-1) \\ \frac{-\tilde{z}_2}{\sqrt{1-0.95\tilde{z}_1^2}} \end{bmatrix}, \qquad \tilde{z} := \frac{1}{|p|}\begin{bmatrix} p & Sp \end{bmatrix}^\top z, \tag{96}$$

the safeguarding control law

$$\kappa_{B_{uni}}(x) := \arg \min_{u \in U_{B_{uni}}(x)} |u - \kappa(x)|, \tag{97}$$

and note that the unique solutions to the system

$$\dot{x} = f_{uni}(x) + g_{uni}(x)\kappa_{B_{uni}}(x), \qquad x \in X, \tag{98}$$

are also solutions to (94). Consider the solutions to (98) starting from $x_0 := (p_0, v_0, z_0)$ with $|p_0| := 1$, $z_0 := p_0$, and $v_0 \geq v_{min} > 0$. In other words, vehicle positioned at the unit circle, and oriented away from the origin with a positive velocity, such that $x_0 \in \partial K_1 \setminus K_2$. For these initial conditions, $\sup_{t \in \text{dom}\, x} |p(t)|$ can be made arbitrarily large by selecting v_0 sufficiently large; see Figure 3. Note that (91)-(92) also hold for B_1, B_2 in (95). Using $|x_0| \in K_1 \implies |x_0|_K \leq 2$, and $|x(t)|_K \geq |x(t)|_{K_1} \geq |p(t)| - 1$, there cannot exist $\beta \in \mathcal{KL}$ such that

$$|x(t)|_K \leq \beta(|x_0|_K, t) \qquad \forall t \in \text{dom}\, x. \tag{99}$$

Remark 4: An equivalent unicycle model, valid for non-zero forward speeds, may be expressed as

$$\dot{p} = \nu, \qquad \dot{\nu} = \frac{\nu}{|\nu|} u_1 + S\nu u_2, \tag{100}$$

with state $(p, \nu) \in \{(p, \nu) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\nu| > 0\}$. This suggests that the lack of UGpAS of K for the system (94)-(95) is due to a pathology inherited from the chosen unicycle model, and the measure of distance from $x := (p, v, z)$ to K.

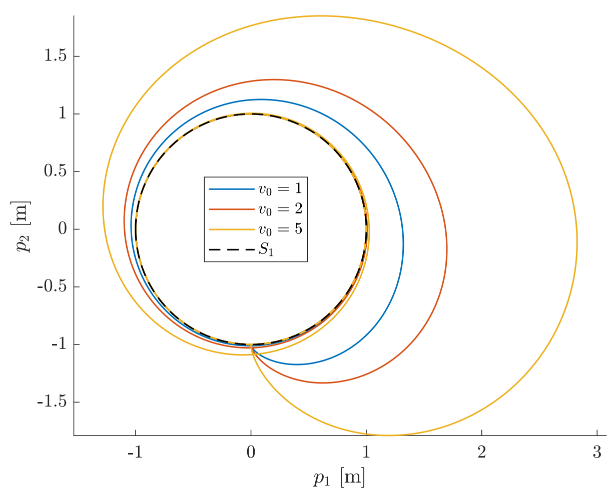

Fig. 3. Trajectory of solutions to the unicycle model with reversed safety constraint; system (98) with f_{lin}, g_{lin}, in (83), $\kappa_{B,uni}$ in (96)-(97) with $U_{B,uni}$ generated by (95),(94). Initial conditions $x_0 := (p_0, v_0, z_0)$ with $p_0 := (0, -1)$, $z_0 := (0, -1)$, and varying $v_0 \geq v_{min}$. Since $|x_0|_K \leq 2$ for all $v_0 \geq v_{min}$, and $\sup_{t \in \text{dom}\, x} |p(t)|$ can be made arbitrarily large by selecting v_0 sufficiently large, K is not UGpAS.

VII. Concluding Remarks

The main theoretical results of this paper are theorems 2 and 3, which state sufficient conditions for UGpAS and small-input ISpS, respectively, for systems constructed using HOCBFs. Moreover, it was shown that for CBF-based safeguarding control laws, the Krasovskii regularization of the closed-loop system (28) inherits the stability properties of (28). This result enables the use of a wider class of control systems. Equally important, it removes the need of verifying continuity properties of optimization-based safeguarding control laws. Additionally, the use of CBFs for systems with non-complete maximal solutions was enabled by adopting the notion of pre-asymptotic stability and pre-stability used in e.g. [29].

The importance of the theoretical results was illustrated by two case studies. In the first case study, we investigated stability and robustness properties for a CBF-based obstacle avoidance design for a vehicle with linear kinematics. It was shown that the safe set was UGpAS, but the corresponding system with additive disturbances was not small-input ISpS with respect to the safe set. This result shows that for noncompact safe sets, UGpAS does not imply small-input ISpS with respect to additive disturbances. In the second case study, we first investigated the stability properties of the CBF-based obstacle avoidance design with unicycle kinematics. For this system, the question of UGpAS of the safe set was inconclusive. Next, we showed by reversing the safety constraint that, in general, global pre-asymptotic stability does not imply UGpAS.

Current research by the authors is focused on hybrid HOCBF formulations for continuous-time systems, continuing the work in [32]. Other interesting directions for future work include extending the notion of robust CBFs and input-to-state safety CBFs to systems with high-order safety constraints. To this end, the construction of the comparison system used as basis for Theorem 3 may be useful.

APPENDIX

Proposition 4 below is used in the proof of Proposition 2. Let n_2 be a positive integer, define $n := 1 + n_2$ and $z := \begin{bmatrix} z_1 \ z_2^\top \end{bmatrix}^\top \in \mathbb{R}^n$, and consider the system

$$\dot{z}_1 = -\alpha_1(z_1) + h_2(z_2) \tag{101a}$$
$$\dot{z}_2 = f_2(z_2) \tag{101b}$$

where $\alpha_1 \in \mathcal{K}_e$, and h_2 and f_2 are continuous. We say that h_2 has property P if there exists $\gamma \in \mathcal{K}_\infty$ such that

$$h_2(z_2) \le \gamma\left(|z_2|_{\mathbb{R}^{n_2}_{\le 0}}\right) \qquad \forall z_2 \in \mathbb{R}^{n_2}. \tag{102}$$

Proposition 4: If $\alpha_1 \in \mathcal{K}_e$, h_2 has property P, and the set $\mathbb{R}^{n_2}_{\le 0}$ is UGAS for (101b) then the set $\mathbb{R}^n_{\le 0}$ is UGAS for (101).

Proof: Using UGAS of $\mathbb{R}^{n_2}_{\le 0}$ for (101b), let $\beta_2 \in \mathcal{KL}$ be such that the solutions of (101b) satisfy

$$|z_2(t)|_{\mathbb{R}^{n_2}_{\le 0}} \le \beta_2(|z_2(s)|_{\mathbb{R}^{n_2}_{\le 0}}, t-s) \qquad \forall t \ge s \ge 0. \tag{103}$$

Let $\gamma \in \mathcal{K}_\infty$ satisfy (102). Define the function

$$V_1(z_1) := \int_0^{z_1} \alpha_1(|r|_{\mathbb{R}_{\le 0}})dr, \tag{104}$$

and note that there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}\left(|z_1|_{\mathbb{R}_{\le 0}}\right) \le V_1(z_1) \le \overline{\alpha}\left(|z_1|_{\mathbb{R}_{\le 0}}\right) \qquad \forall z_1 \in \mathbb{R}. \tag{105}$$

Also note that, for all $z_1 \in \mathbb{R}$, we have

$$\begin{aligned}
&\langle \nabla V_1(z_1), -\alpha_1(z_1) + h_2(z_2)\rangle \\
&\qquad = \alpha_1\left(|z_1|_{\mathbb{R}_{\le 0}}\right)\left(-\alpha_1(z_1) + h_2(z_2)\right) \\
&\qquad \le \alpha_1\left(|z_1|_{\mathbb{R}_{\le 0}}\right)\left(-\alpha_1(|z_1|_{\mathbb{R}_{\le 0}}) + \gamma\left(|z_2|_{\mathbb{R}^{n_2}_{\le 0}}\right)\right) \\
&\qquad \le -\frac{1}{2}\alpha_1^2\left(|z_1|_{\mathbb{R}_{\le 0}}\right) + \frac{1}{2}\gamma^2\left(|z_2|_{\mathbb{R}^{n_2}_{\le 0}}\right).
\end{aligned} \tag{106}$$

Let $c > 0$ be such that

$$\gamma(c) < \frac{1}{2}\lim_{r\to\infty} \alpha_1(r). \tag{107}$$

Let $\delta \in \mathcal{K}_\infty$ satisfy $\beta_2(r, \delta(r)) \le c$ for all r, and define $t_c := \delta(|z(0)|_{\mathbb{R}^n_{\le 0}})$. It follows from integrating (106) from 0 to t_c and using (103) that, for all $t \in [0, t_c]$,

$$V_1(z_1(t)) \le V_1(z_1(0)) + t_c\frac{1}{2}\gamma^2\left(\beta_2\left(|z_2(0)|_{\mathbb{R}^{n_2}_{\le 0}}, 0\right)\right). \tag{108}$$

In particular, using (105) and the definition of t_c, there exists $\kappa_1 \in \mathcal{K}_\infty$ such that

$$|z_1(t)|_{\mathbb{R}_{\le 0}} \le \kappa_1\left(|z(0)|_{\mathbb{R}^n_{\le 0}}\right) \qquad \forall t \in [0, t_c]. \tag{109}$$

Let $\sigma \in \mathcal{K}_\infty$ be such that

$$r \ge 0,\ e \in [0, c],\ r \ge \sigma(e) \quad -\frac{1}{2}\alpha_1^2(r) + \frac{1}{2}\gamma^2(e) \le 0. \tag{110}$$

Using (110) and the definition of δ, there exist $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ such that, for all $t \ge s \ge t_c$,

$$V_1(z_1(t)) \le \beta_1(V_1(z_1(s)), t-s) + \gamma_1\left(\sup_{\tau\in[s,t]} |z_2(\tau)|_{\mathbb{R}^{n_2}_{\le 0}}\right). \tag{111}$$

Using (105), and (111) there exist $\hat{\beta}_1 \in \mathcal{KL}$ and $\hat{\gamma}_1 \in \mathcal{K}_\infty$ such that, for all $t \ge s \ge t_c$,

$$|z_1(t)|_{\mathbb{R}_{\le 0}} \le \hat{\beta}_1(|z_1(s)|_{\mathbb{R}_{\le 0}}, t-s) + \hat{\gamma}_1\left(\sup_{\tau\in[s,t]} |z_2(\tau)|_{\mathbb{R}^{n_2}_{\le 0}}\right). \tag{112}$$

Insert $s = (t + t_c)/2$ into (112), to obtain, for all $t \ge t_c$,

$$\begin{aligned}
|z_1(t)|_{\mathbb{R}_{\le 0}} \le& \hat{\beta}_1\left(\left|z_1\left(\frac{t+t_c}{2}\right)\right|_{\mathbb{R}_{\le 0}}, \frac{t-t_c}{2}\right) \\
&+ \hat{\gamma}_1\left(\sup_{\tau\in[(t+t_c)/2,t]} |z_2(\tau)|_{\mathbb{R}^{n_2}_{\le 0}}\right).
\end{aligned} \tag{113}$$

Use (103) with $s = t_c$ and t replaced by $(t + t_c)/2$ to obtain

$$\sup_{\tau\in[(t+t_c)/2,t]} |z_2(\tau)|_{\mathbb{R}^{n_2}_{\le 0}} \le \beta_2\left(|z_2(t_c)|_{\mathbb{R}^{n_2}_{\le 0}}, \frac{t-t_c}{2}\right). \tag{114}$$

Using (113)-(114), it follows that, for all $t \ge t_c$,

$$\begin{aligned}
|z_1(t)|_{\mathbb{R}_{\le 0}} \le& \hat{\beta}_1\left(\left|z_1\left(\frac{t+t_c}{2}\right)\right|_{\mathbb{R}_{\le 0}}, \frac{t-t_c}{2}\right) \\
&+ \hat{\gamma}_1\left(\beta_2\left(|z_2(t_c)|_{\mathbb{R}^{n_2}_{\le 0}}, \frac{t-t_c}{2}\right)\right).
\end{aligned} \tag{115}$$

Using (112) with $s = t_c$ and t replaced by $(t + t_c)/2$, and also using (108), we obtain, for all $t \geq t_c$,

$$\left| z_1\left(\frac{t+t_c}{2}\right)\right|_{\mathbb{R}_{\leq 0}} \leq \hat{\beta}_1\left(\kappa_1(|z(0)|_{\mathbb{R}^n_{\leq 0}}), \frac{t-t_c}{2}\right) + \hat{\gamma}_1\left(\beta_2(|z_2(0)|_{\mathbb{R}^{n_2}_{\leq 0}}, 0)\right). \tag{116}$$

Inserting (116) into (113), the solutions of (101a) satisfy, for all $t \geq t_c$,

$$|z_1(t)|_{\mathbb{R}_{\leq 0}} \leq \bar{\beta}_1(|z(0)|_{\mathbb{R}^n_{\leq 0}}, t - t_c), \tag{117}$$

with $\bar{\beta}_1 \in \mathcal{KL}$ given by

$$\bar{\beta}_1(r, s) := \hat{\beta}_1\left(\hat{\beta}_1\left(\kappa_1(r), \frac{s}{2}\right) + \hat{\gamma}_1(\beta_2(r, 0)), \frac{s}{2}\right) + \hat{\gamma}_1\left(\beta_2\left(r, \frac{s}{2}\right)\right). \tag{118}$$

For all $r \geq 0$, and for all $s \geq \delta(r)$, let $\tilde{\beta}_1 \in \mathcal{KL}$ satisfy

$$\tilde{\beta}_1(r, \delta(r)) \geq \kappa_1(r), \quad \tilde{\beta}_1(r, s) \geq \bar{\beta}_1(r, s - \delta(r)). \tag{119}$$

Using (103) and (117)-(119), and the definition of t_c, the solutions to (101) satisfy, for all $t \geq 0$,

$$|z(t)|_{\mathbb{R}^n_{\leq 0}} \leq \tilde{\beta}_1(|z(0)|_{\mathbb{R}^n_{\leq 0}}, t) + \beta_2(|z(0)|_{\mathbb{R}^n_{\leq 0}}, t). \tag{120}$$

■

The steps used to obtain (117)-(118) from (103) and (112) borrow from [36, Lemma 4.7] and [22, Lemma C.4].

Proposition 5 below is used in the proof of Proposition 3. Let $d \in \mathbb{R}, d_2 \in \mathbb{R}^{n_2}$, $d := \begin{bmatrix} d_1 \ d_2^\top \end{bmatrix}^\top \in \mathbb{R}^n$, and consider the system

$$\dot{z}_1 = -\alpha_1(z_1) + h_2(z_2) + d_1 \tag{121a}$$
$$\dot{z}_2 = f_2(z_2, d_2) \tag{121b}$$

where $\alpha_1 \in \mathcal{K}_e$, and h_2 and f_2 are continuous.

Proposition 5: If $\alpha_1 \in \mathcal{K}_e$, h_2 has property P, and the set $\mathbb{R}^{n_2}_{\leq 0}$ is small-input ISS for (121b) then the set $\mathbb{R}^n_{\leq 0}$ is small-input ISS for (121).

Proof: Using small-input ISS of $\mathbb{R}^{n_2}_{\leq 0}$ for (121b), let $\beta_2 \in \mathcal{KL}$, $\bar{d}_2 \in \mathbb{R}$, and $\gamma_2 \in \mathcal{K}_\infty$ be such that the solutions of (121b) satisfy, for all $t \geq s$,

$$||d_2||_\infty \leq \bar{d}_2 \implies |z_2(t)|_{\mathbb{R}^{n_2}_{\leq 0}} \leq \beta_2(|z_2(s)|_{\mathbb{R}^{n_2}_{\leq 0}}, t - s) + \gamma_2(||d_2||\infty). \tag{122}$$

Let V_1 be defined as in (104), and observe that

$$\begin{aligned} &\langle \nabla V_1(z_1), -\alpha_1(z_1) + h(z_2) + d_1 \rangle \\ =&\alpha_1(|z_1|_{\mathbb{R}_{\leq 0}})(-\alpha_1(z_1) + h(z_2) + d_1) \\ \leq&\alpha_1(|z_1|_{\mathbb{R}_{\leq 0}})(-\alpha_1(|z_1|_{\mathbb{R}_{\leq 0}}) + \gamma(|z_2|_{\mathbb{R}^n_{\leq 0}}) + d_1) \\ \leq& -\frac{1}{2}\alpha_1(z_1)^2 + \frac{1}{2}\left(\gamma(|z_2|) + d_1\right)^2 \\ \leq& -\frac{1}{2}\alpha_1(|z_1|_{\mathbb{R}_{\leq 0}})^2 + \gamma^2(|z_2|_{\mathbb{R}_{\leq 0}}) + d_1^2. \end{aligned} \tag{123}$$

Let $c > 0$ satisfy

$$\gamma(2c) < \frac{1}{4}\lim_{r\to\infty} \alpha_1(r). \tag{124}$$

Let $\delta \in \mathcal{K}_\infty$ satisfy $\beta_2(r, \delta(r)) \leq c$ for all r, and define $t_c := \delta(|z(0)|_{\mathbb{R}^2_{\leq 0}})$. It follows from integrating (123) from 0 to t_c, and using (122), that

$$\begin{aligned} &V_1(z_1(t)) \leq V_1(z_1(0)) \\ &+ \int_0^{t_c} -\frac{1}{2}\alpha_1^2(|z_1(\tau)|_{\mathbb{R}_{\leq 0}}) + \gamma^2(|z_2(\tau)|_{\mathbb{R}^{n_2}_{\leq 0}}) + ||d_1||^2_\infty d\tau \\ &\leq V_1(z_1(0)) + t_c\gamma^2(2\beta_2(|z_2(0)|, 0)) \\ &+ \int_0^{t_c} -\frac{1}{4}\alpha_1^2(|z_1(\tau)|) + \gamma^2(2||d||_\infty) + ||d||^2_\infty d\tau. \end{aligned} \tag{125}$$

Let $\bar{d} > 0$ satisfy $\bar{d} \leq \bar{d}_2$, and

$$\gamma(2\bar{d}) + \bar{d} < \frac{1}{4}\lim_{r\to\infty} \alpha_1(r). \tag{126}$$

Using (105), and the definition of t_c, there exists $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ such that, for all $t \in [0, t_c]$,

$$||d||_\infty \leq \bar{d} \implies |z_1(0)|_{\mathbb{R}_{\leq 0}} \leq \kappa_1(|z(0)|) + \kappa_2(||d||_\infty). \tag{127}$$

Let $\sigma \in \mathcal{K}_\infty$ be such that

$$r \geq 0, \ e \in [0, c], \ r \geq \sigma(e) \quad -\frac{1}{4}\alpha_1^2(r) + \gamma^2(2e) \leq 0. \tag{128}$$

Using (126), (128), and the definition of δ, there exist $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ such that, for all $t \geq s \geq t_c$

$$\begin{aligned} &||d||_\infty \leq \bar{d} \implies \\ &V_1(z_1(t)) \leq \beta_1(V_1(z_1(s)), t - s) + \gamma_1\left(\sup_{\tau\in[s,t]} |z_2(\tau)|_{\mathbb{R}^{n_2}_{\leq 0}}\right) \\ &\qquad + \gamma_1(||d||_\infty). \end{aligned} \tag{129}$$

Using (105), and (129) there exist $\hat{\beta}_1 \in \mathcal{KL}$ and $\hat{\gamma}_1 \in \mathcal{K}_\infty$ such that, for all $t \geq s \geq t_c$,

$$\begin{aligned} &||d||_\infty \leq \bar{d} \implies \\ &|z_1(t)|_{\mathbb{R}_{\leq 0}} \leq \hat{\beta}_1(|z_1(s)|_{\mathbb{R}_{\leq 0}}, t - s) + \hat{\gamma}_1\left(\sup_{\tau\in[s,t]} |z_2(\tau)|_{\mathbb{R}^{n_2}_{\leq 0}}\right) \\ &\qquad + \hat{\gamma}_1(||d||_\infty). \end{aligned} \tag{130}$$

The remainder of the proof follows similar steps as (113)-(120) in the proof of Proposition 3, using (130) in the place of (112), adding $\gamma_2(||d||_\infty)$ to the right-hand side of the bound in (114), and with the assumption $||d||_\infty \leq \bar{d}$ inserted where applicable. ■

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