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ORIGINAL ARTICLE

Investment in two alternative projects with multiple switches and the exit option

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Abstract

This paper uses an analytical framework to examine a firm's investment and switching strategy under uncertainty. The context is the possibility to launch and operate two distinct projects, one at a time, with exposure to a stochastic exogenous price. We allow for multiple switches between the two projects, along with abandonment options from each. These possibilities fundamentally influence the operational strategy. We show that under some conditions, a dichotomous waiting region may arise at the investment stage. In this case we have an inaction region, for a range of prices in a certain bounded interval, where the firm does not invest and waits to have more information about the price evolution. This region vanishes for a high level of uncertainty. Additionally, the firm may operate with a negative instantaneous profit. We prove that investment in this region is never optimal. Numerical examples enable comparative statics, while extension to allow for time-to-build is included.

Keywords Investment analysis · Decision analysis · Finance · Dynamic programming/optimal control

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1 Introduction

In this paper we study the investment strategy of a firm in two alternative projects (which we also call alternative modes), under price uncertainty. The projects differ one from another in terms of size and running payoff: one being more profitable when the market conditions are favourable but leading to larger losses in times of crisis (the larger scale project), whereas the other project, with smaller scale, leads to smaller profits and smaller losses.

Once the investment takes place, the firm may still switch from one project to the other. Therefore, the firm may adjust itself to the conditions of the market. Moreover, we assume that the firm may decide to exit the market, and this decision is possible in both projects.

In this setting, the firm needs to decide not only when the investment occurs, but also in which project the firm will first invest. Once the firm is in the market, it may switch from one project to the other, and, if the market conditions turn out to be unfavourable, the firm may still decide whether to change project or to exit the market. Hence, we consider the whole life of the firm, starting from the investment decision, the switching times and, finally, the exit time. Both investment and exit are one time decisions, whereas the switching strategy leads to a sequence of switching times.

The flexibility provided by the option to switch between the two projects is particularly relevant when the firm has to face volatile markets or imminent social, economic or financial crisis. In the 2008 crisis, many firms felt the need to adjust their production processes in order to face declining markets and to avoid large losses. During the pandemic crisis caused by SARS-COV-2, the ability of the firms to switch from one project to another gained even more relevance, as companies were scrambling to mobilise responses. For instance, in China many companies decided to reallocate employees to new and valuable activities instead of considering the layoff strategy. After the peak of the crisis has passed, these companies needed to readjust to the new situation, preparing for a faster recovery.

More recently, under pressure from investors and consumers, many Western companies have started to unwind their investments, close stores and pause sales in Russia. Some, after at first taking temporary measures, have revised their plans and decided to exit the country completely. For example, British American Tobacco decided to exit its Russian business, whereas Philip Morris suspended planned investments and reduced manufacturing in Russia.[1]

Taking into account the possible actions of the firm, the optimal strategy involves the characterisation of the following sequential decisions:

(i) Investment decision: the firm decides in which project the initial investment takes place and when it occurs;
(ii) Switching decisions: the firm chooses when to switch between the two alternative projects;
(iii) Exit decision: the firm decides when and from which project the firm leaves the market.

In Fig. 1, we present a the state transition diagram, representing the firm's possible actions—investment, switching between projects, and exiting the market—including the costs associated with each decision.

Besides characterising the optimal decisions along the life of the firm, we also provide an extensive comparative static, which shows the influence of the parameters in the optimal strategy. Additionally, we discuss the impact of the switching and exit options in the investment decision.

Finally, we also consider the case that there is a time lag between the investment decision and the time that the investment is active—the so-called *time-to-build*. When we consider

[1] New York Times, 14 October 2022.

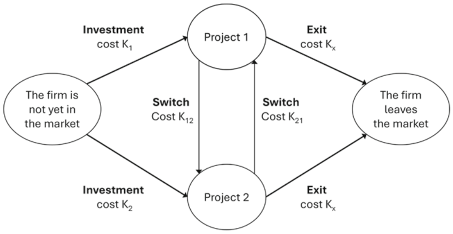

Fig. 1 State transition diagram, including the investment, switching and exit decisions

time-to-build, it may happen that the firm starts production in the hysteresis region, and, therefore, we are able to check the robustness of this region with changes in the parameter values.

This paper is organised as follows: in Sect. 2, we discuss the related literature, and, in particular, we position our paper. In Sect. 3, we introduce the investment model, solution for which is then presented and discussed in Sect. 4. In Sect. 5, we study (illustrate numerically) the effects of the parameters in the optimal strategies. In Sect. 6, we discuss the effect of the time-to-build in the investment strategy. Finally, Sect. 7 concludes the paper.

2 Related work

The model presented in this paper relies on the real options framework, for which there is a large and rapidly growing number of publications. The topic of sequential investment decisions is of great interest in Real Options and agglomerates the contributions of many researchers. We focus our attention in related literature regarding investment in alternative projects. One of the simplest models regarding investment in alternative projects was addressed by Dixit [6], which is an adaptation of the single project case studied by McDonald and Siegel [16]. In Dixit [6], the choice of the project is irreversible and switching between projects is not possible. According to Dixit [6], when the firm has the option to invest in one of N alternative projects, each project should be evaluated separately, and then the project with the highest option value is chosen. Thus, the author presents the following investment rule: (i) when the initial price is small enough, the decision-maker will invest in the project with the largest option value as soon as the price reaches the smallest threshold; (ii) when the initial price is larger than the mentioned threshold, it is optimal to invest in the project with large net present value. Hence the investment decision is a trigger strategy.

Later, Décamps et al. [4] showed that the investment rule proposed by Dixit [6] was not completely correct. Using $N = 2$, the authors proved that the optimal investment rule may be dichotomous, meaning that there is an inaction region between two disconnected investment regions, where it is optimal to wait before making an investment. Additionally, for firms that hold the option to switch from a smaller scale to a larger scale project, the existence of such an inaction region can persist even if the uncertainty of the output price increases. This result is interesting *per se*, as it suggests that larger volatility does not necessarily lead to the adoption of larger projects when there is the option to increase the scale of operations. As the authors note, this result does not hold true if switching between projects is not allowed. The existence of this inaction region and its behaviour with respect to volatility is in contrast with standard results from real options, which highlights its relevance.

Others have also studied similar investment problems. Nishihara and Ohyama [18] extended the analysis of investment in mutually exclusive projects to the framework of competition. Bobtcheff and Villeneuve [3] studied investments in two mutually exclusive projects with two sources of uncertainty. Siddiqui and Fleten [20] used the model of Décamps et al. [4] for analysing the investment decision in alternative technologies in the electricity industry. Dias [5] described a set of real options models to evaluate investments in petroleum exploration and production under market and technical uncertainties. The author presented, in particular, a model concerning the selection of mutually exclusive alternatives under uncertainty.

Our paper is a natural extension of Décamps et al. [4]. In contrast with Décamps et al. [4], we allow for multiple switches between the two projects (the larger and smaller projects), along with abandonment options from each. As we introduce more possibilities into our model, it is relevant to address the following research questions:

RQ1 : How do multiple switching options and the possibility to exit the market affect the investment region? In particular, are there situations where the investment strategy is not a trigger strategy, i.e. where the inaction region is still present?

RQ2 : Under which conditions is the investment threshold monotonic with respect to the underlying parameters?

The answers to these questions are quite interesting. In particular, with respect to RQ1, we show that, as happens in Décamps et al. [4], a dichotomous investment region defined in terms of the underlying uncertainty may be optimal. In this case, we have an inaction region, for a range of prices in a certain bounded interval, where the firm does not invest and waits to have more information on the price evolution. Our findings show that the inaction region exists for small values of the volatility, but when one increases the volatility, the inaction region tends to disappear, which does not occur in Décamps et al. [4]. Regarding the behaviour of the investment strategy, related with RQ2, we show that monotony of the investment threshold does not hold for all sets of parameters. Indeed, increasing the drift of the price process does not necessarily imply a decrease of the investment threshold.

Embedded in the investment model we have a switching problem, since the firm may decide to switch from one project to another. These models have been studied by different authors. Duckworth and Zervos [9] characterised the optimal strategy when the exit option is not available. Zervos et al. [22] presented a full characterisation of the optimal strategy for a firm that is currently on the market and can choose between two production models or to exit the market. In one of the modes, the firm decides to mothball production. An inaction region is found by the authors although no economical characterisation is provided. Guerra et al. [10] analysed the economical meaning of such region and found that it cannot be attained through a continuous decrease of the price. The authors showed that firms producing in subsidised markets may end up producing at a loss in such a region, due to the retraction of the support scheme. Contrary to these authors, we assume that, in the smaller scale project, we may still have a positive profit. Nevertheless, such a region (which we call the *hysteresis region*) may exist, and, hence, this leads to the next research question, RQ3:

RQ3 : What are the optimal switching strategies and how do the parameters influence these strategies? May the firm choose optimally to produce at a loss rather than to switch to another mode?

Regarding the firm's optimal switching strategy after investment, our findings are aligned with the ones in Guerra et al. [10]. In our set up, we assume the existence of two alternative projects where the firm is producing with a possible positive profit (and it is not necessarily in a mothball state in one of the two projects). We fully characterise the situations where the firm may decide to stay with the largest project even with a loss, waiting to have more information before deciding if it leaves the market or if it switches to the smaller project.

In particular, we show that when the drift term is negative and small and/or the volatility is small, this region does not exist. Moreover, increasing the switching costs increases the size of the hysteresis region, up to certain values, after which this region no longer exists. Additionally, this region is never attained due to a continuous decrease of the revenue for a firm already producing.

In many real problems, once the investment decision is taken, it will require a certain time in order to actually start producing, in particular in large scale (infrastructure) construction projects such as transportation infrastructure projects, power generating plant and aerospace and pharmaceutical investments. There are several authors that have studied the impact of time-to-build in different features of investment. We refer to Majd and Pindyck [15], Bar-Ilan and Strange [1], Milne and Whalley [17], Bar-Ilan et al. [2], and Nunes and Pimentel [19] and references therein regarding the effect of time-to-build. Since the market conditions evolve during an investment lag, the profitability of the investment may also change. Hence, we consider an extension to our model, by introducing time-to-build in the investment decision. With this feature, we want to answer the following research question, RQ4:

RQ4: What is the impact of the time-to-build in the investment decision when one considers two mutually exclusive projects? And how likely is it that the firm actually starts production in the hysteresis region?

When compared with the main conclusions of Bar-Ilan and Strange [1], our results are not completely aligned with those presented by the authors. In our case, we recover the monotonicity of the investment threshold as a function of the volatility for small and large values of times-to-build, contrary to the effect described in Bar-Ilan and Strange [1], where an increase of the uncertainty results in a non-monotonic investment threshold. Additionally, the behaviour of the investment threshold with the volatility of the price may differ for different investment lags and different abandonment costs. On the other hand, and similar to Bar-Ilan and Strange [1], our results show that the effect of uncertainty becomes weaker when the time lag increases.

We provide numerical insights, regarding the sojourn time in the hysteresis region and how likely it is that the firm will end up producing with negative profits. Contrary to the original model, in this extension the firm may start producing in the hysteresis region. In that case our results show that the expected sojourn time there decreases with increasing volatility but the probability of resuming production (and switching to the other project) rather than exiting the market increases. This result is not in line with the result in Guerra et al. [10], which can be explained by the fact that, in our model, the profit is positive, whereas in the model of Guerra et al. [10], there are only running costs once the firm enters into mothballing.

3 Model

We consider a monopolistic firm that has the opportunity to invest in one of two alternative projects, whose price evolves stochastically over time. We denote by P_t the price at time t and we assume that $\{P_t, t \geq 0\}$ is a geometric Brownian motion (gBm), with drift μ and volatility $\sigma > 0$. We let r denote the risk rate, and we assume that $r > \mu$.

The instantaneous profit of project i is given by:

$$\pi_i(p) = \alpha_i p - \beta_i, \quad i = 1, 2, \quad \alpha_i, \beta_i \in \mathbf{R}. \tag{1}$$

The coefficients β_i can be interpreted as the instantaneous costs of production for project $i = 1, 2$, and hence $\beta_i > 0$. Moreover, the coefficients α_i can be seen as the quantity produced.
We assume the following ordering in the production parameters:

$$\alpha_1 > \alpha_2 \geq 0, \quad \beta_1 > \beta_2.$$

We note that although we assume that the profit is a linear function of the price, this assumption is not very restrictive, as we can also consider isoelastic functions. In fact, since the power of a gBm is still a gBM with different drift and diffusion parameters, the results can be generalised for instantaneous profit functions as $\pi_i(p) = \alpha_i p^\gamma - \beta_i$.

As Décamps et al. [4], we will use the terminology of large and small scale projects. Then, we call project 1 the large scale project and project 2 the small scale project. Along the text, and in order to keep explanation simpler, we may refer to the large scale project as mode 1 or project 1. In opposition, we may designate the smaller project by mode 2 or project 2. For $p < \beta_i/\alpha_i$, the profit is negative and, thus, it may be optimal to leave the market. Hence we consider that the firm has the option to leave the market from both modes.

Since the firm may produce in one of the two possible modes and can abandon the market, we introduce the process: $\{Z_t, t \geq 0\}$, with $Z_t \in \{1, 2, ex\}$, where $Z_t = i$ means that the firm is operating in mode i, with $i = 1, 2$, and $Z_t = ex$ means that the firm has abandoned the market. The state ex is absorbing. For instance, a realisation of the process such that $Z_s = ex$ and $Z_t \in \{1, 2\}$, for $t > s$, is not admissible as state ex is absorbing. A *strategy* for the switching and exit decisions is then a realisation of the stochastic process $\{Z_t, t \geq 0\}$. We let $\mathcal{S}$ denote the set of all admissible strategies for the switching and exit decisions.

Considering the transition between projects, we denote the time when the j^{th} transition from state a to state b occurs by $T_j^{a,b}$, with $a, b \in \{1, 2\}$. Following Zervos et al. [22], these times can be defined recursively by:

$$T_1^{a,b} = \inf\{t > 0 : Z_{t^-} = a, Z_t = b\} \quad \text{and} \quad T_{j+1}^{a,b} = \inf\{t > T_j^{a,b} : Z_{t^-} = a, Z_t = b\}, \tag{2}$$

with $a \in \{1, 2\}$, $b \in \{1, 2, ex\}$, and $j \in \mathbb{N}$. The exit times are defined by

$$\tau_1 = \inf\{T_j^{1,ex} < \infty\}, \quad \tau_2 = \inf\{T_j^{2,ex} < \infty\}, \quad \tau = \inf\{\tau_1, \tau_2\}. \tag{3}$$

Switching from one project to another one implies a cost payment and leaving the market generates a cost or a salvage value. We let K_i denote the investment cost of the firm when it enters the market in mode i, K_{ij} denote the transition cost from mode i to mode j, and K_x represent the net divestment value, which can be positive or negative. For ease of terminology, we also refer to this value as *exit cost*. We assume that $rK_x - \beta_2 < 0$, so that exit from the smaller scale project for small values of the price is always optimal. Since $\beta_1 > \beta_2$, exit from the larger scale project may also be optimal.

Assuming that the firm can invest in the project at any time $\zeta \geq 0$, the investment problem can be formalised as follows:

$$\begin{aligned} W(p) = & \sup_{\zeta > 0,\, Z \in \mathcal{S}} \mathbb{E}_{z,p} \\ & \left[\int_\zeta^\infty e^{-rt} \left(\pi_1(P_t)\mathcal{I}_{\{Z_t=1\}} + \pi_2(P_t)\mathcal{I}_{\{Z_t=2\}} \right) dt - e^{-r\zeta} \left(K_1 \mathcal{I}_{\{Z_\zeta=1\}} + K_2 \mathcal{I}_{\{Z_\zeta=2\}} \right) \right. \\ & \left. - K_{12} \sum_{j=1}^\infty e^{-rT_j^{12}} \mathcal{I}_{\{T_j^{12}<\infty\}} - K_{21} \sum_{j=1}^\infty e^{-rT_j^{21}} \mathcal{I}_{\{T_j^{21}<\infty\}} - K_x e^{-r\tau} \mathcal{I}_{\{\tau<\infty\}} \right], \end{aligned} \tag{4}$$

where $\mathcal{I}_A$ represents the indicator function,[2] and $E_{z,p}[\ldots]$ is the expected value conditional to the information that initial values are $Z_0 = z$ and $P_0 = p$. This is a switching model combined with discretionary stopping, in which we have to find the investment moment, the optimal moments to switch between production modes, and the exit times as well.

Let

$$V(z,p) = \sup_{Z\in\mathcal{S}} \mathbb{E}_{z,p}\left[\int_0^\infty e^{-rt}\left(\pi_1(P_t)\mathcal{I}_{\{Z_t=1\}} + \pi_2(P_t)\mathcal{I}_{\{Z_t=2\}}\right)dt \right.$$

$$\left. -K_{12}\sum_{j=1}^{\infty} e^{-rT_j^{12}}\mathcal{I}_{\{T_j^{12}<\infty\}} - K_{21}\sum_{j=1}^{\infty} e^{-rT_j^{21}}\mathcal{I}_{\{T_j^{21}<\infty\}} - K_x e^{-r\tau}\mathcal{I}_{\{\tau<\infty\}}\right]. \quad (5)$$

Then, using the strong Markov property of the gBM, it follows that

$$W(p) = \sup_{\zeta\geq 0} E_p\left[e^{-r\zeta}u(P_\zeta)\right], \quad (6)$$

with

$$u(p) = \max\left(V(1,p) - K_1, V(2,p) - K_2\right). \quad (7)$$

To simplify the notation, in the rest of the paper, we use $v_1(p) = V(1,p)$ and $v_2(p) = V(2,p)$.

Thus, in order to solve the investment problem, we need to solve first (5), which corresponds to the optimal switching strategy. Both problems will be discussed in the following section.

4 Optimal strategy

As motivated in the previous section, we need to start by deriving the optimal switching strategy before solving the investment problem.

4.1 Switching problem

In order to solve the switching problem defined in 5, we start by providing the corresponding Hamilton-Jacobi-Bellman (HJB) equations. As we have two projects, we have two HJB equations, and each HJB equation has three members. Whenever the firm is producing in mode i, it has the following options: [1] it continues producing in that mode, [2] it switches to the other production process, or [3] it exits the market. Therefore, the associated HJB equations are coupled and are of the following form:

$$\max\{\overset{[1]}{(\mathcal{L}v_1)(p) - rv_1(p) + \pi_1(p)}, \overset{[2]}{v_2(p) - v_1(p) - K_{12}}, \overset{[3]}{-v_1(p) - K_x}\} = 0, \quad (8)$$

$$\max\{(\mathcal{L}v_2)(p) - rv_2(p) + \pi_2(p), v_1(p) - v_2(p) - K_{21}, -v_2(p) - K_x\} = 0, \quad (9)$$

where $\mathcal{L}v_i = \mu x v_i' + \frac{\sigma^2}{2}x^2 v_i''$, with v_i' and v_i'' being, respectively, the first and second derivative of v_i, with $i = 1, 2$. To simplify the explanation, we number the different decisions (producing in the same mode, switching to the other mode, and exit), using [1], [2] and [3], respectively. The HJB equations naturally divide the space into several 'action' regions, depending on

[2] That is equal to 1 if A holds true and 0 otherwise.

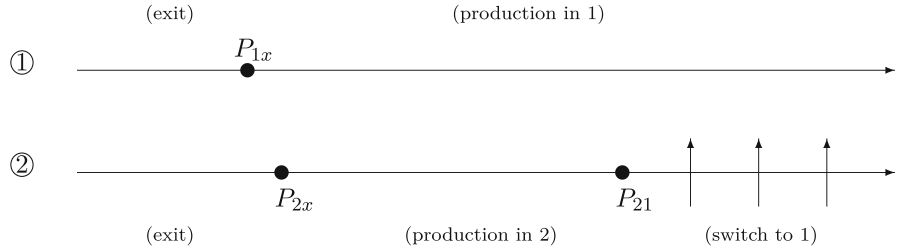

Fig. 2 No-downgrading strategy: if the firm is in the small scale project and the current price increases above P_{21}, the firm switch to the large project. If the price goes below P_{2x}, it leaves the market. If the firm is in the large scale project, it leaves the market as soon as the price goes below P_{1x}

where each of the parcels of the above equations is equal to zero. The theoretical framework for this problem is presented in Zervos [21].

To find the solution to the HJB Eqs. (8) and (9), we start by noticing that the ordinary differential equations that hold in region [1] (in both HJB equations) are Cauchy-Euler equations, and their solutions are as follows:

$$E_i p^{d_1} + C_i p^{d_2} + \frac{\alpha_i}{r-\mu} p - \frac{\beta_i}{r}, \tag{10}$$

where $d_1 < 0$ and $d_2 > 1$ solve the characteristic equation $\frac{\sigma^2}{2} d^2 + (\mu - \frac{\sigma^2}{2})d - r = 0$, and E_i and C_i are constants such that the smooth-pasting conditions hold. The value function for region [3] is the value of exiting and, therefore, in this region the solution is trivial and equal to $-K_x$. Finally, in region [2], the firm should optimally change from the large scale project to the smaller scale project or vice versa. Thus, the value function for a firm that is actually in project i, with a price that belongs to region [2], is given by $v_i = v_j - K_{ij}$, with $i, j = 1, 2$ and $i \neq j$.

Depending on the set of parameters chosen, we may have different optimal strategies. Since we are considering that $r K_x - \beta_i < 0$, leaving the market will be optimal for some values of the price. Next, we present the two optimal strategies for a firm that is already producing. The thresholds that appear in both strategies have the following meaning: P_{ix} represents the exit threshold from project i, meaning that if the firm is producing with project i and the price is equal or below P_{ix}, then it is optimal for the firm to exit the market; P_{ij} is the switching threshold from project i to project j, meaning that if the firm is producing with the large scale project and the price is lower or equal to P_{12} then it is optimal to switch to the smaller scale project, and if the firm is producing in project 2 and the price is higher or equal to P_{21} then it is optimal to switch to the other project. We use up-arrows and down-arrows to highlight the regions in which switching is optimal. Finally, the threshold P_h appears only in the second strategy depicted and limits an inaction region where the firm continues in production although the instantaneous profit may be negative.

No-downgrading strategy: in this case, once the firm produces with the larger project, it will never be optimal to switch to the smaller one. On the contrary, if the firm starts producing in the smaller project, then it will be eventually optimal to switch to the larger scale one, for large values of the price. In both cases, it can be optimal to exit the market for small values of price. This strategy is depicted in Fig. 2, where in the horizontal axe we have the price at the current time, P_t.

Hysteresis strategy: in contrast to the previous case, it may be optimal to switch from project 1 to project 2, and the other way around. At a first glance, we would expect an

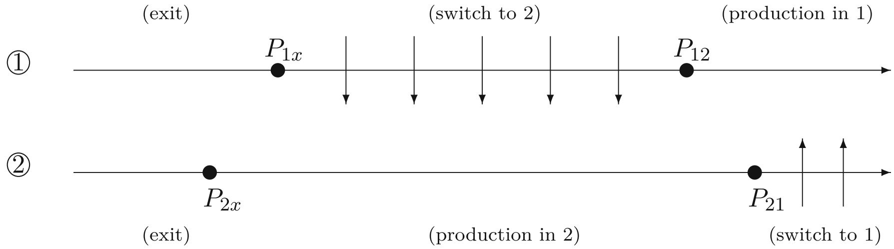

Fig. 3 Downgrading without hysteresis (not optimal strategy)

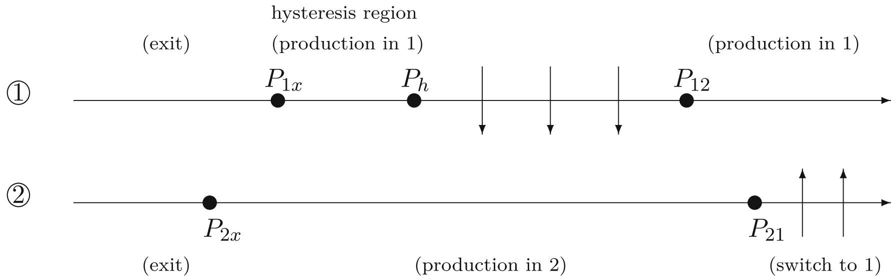

Fig. 4 Hysteresis strategy: if the firm is in the large scale project, it will switch to the smaller one as soon as the price goes below P_{12}. If the firm is in the small scale project, it will exit the market if the price hits the value P_{2x} or will switch to the large project if it reaches the value P_{21}

optimal strategy as the one depicted in Fig. 3. However, exit is an irreversible decision and it can be shown that the strategy in Fig. 3 is not optimal. In fact, if a firm is producing, the larger scale project and the price decreases, then it is optimal to switch to the other project, as in this mode the firm is hedging against larger losses. In case the price is really low, the firm is producing at a loss and therefore the option to exit becomes attractive. At this point, it may not be optimal to exit the market nor to switch to the smaller scale project, since the firm pays (or receives) exactly the same in case it leaves the market either out of project 1 or project 2 (K_x), and there is a cost for switching from the larger to the smaller project. Thus, it is better for the firm to wait before deciding either to switch (in case the price increases) or to exit (in case the price decreases even more), which leads to the existence of an hysteresis region. We note that in Guerra et al. [10], the authors also find such a region, where in their case the firm may be in operating state or in mothballing (where there are only costs and no revenue).

In Fig. 4, we depict this strategy, where the hysteresis region corresponds to prices between P_{1x} and P_h.

We note that a firm will never enter the hysteresis region due to a continuous movement of the revenue.

The optimality of the strategies depicted in Figs. 2 and 4 depends on the relationship between the involved parameters. Next, we present a set of conditions that will be critical for the optimality of each one of these two strategies. The same kind of conditions can be found in Zervos et al. [22] and Guerra et al. [10].

Set of Conditions 1 *The following conditions hold true:*

$$(i)\ \beta_2 + rK_{12} < \beta_1,\ (ii)\pi_1(\delta) - \pi_2(\delta) < 0,\ (iii)K_{21} < K_{21}^{\dagger},\ \textit{and}\ (iv)\ K_{12} < K_{12}^{\dagger} \tag{11}$$

where

$$\delta = \frac{(\beta_1 - rK_x)(d_2 - 1)}{\alpha_1 d_2}, \tag{12}$$

and $K_{12}^{\dagger}$ *and* $K_{21}^{\dagger}$ *are defined in Appendix A.1.3.*

As we will state in Proposition 1: (a) δ equals the exit threshold in the no-downgrading strategy, and (b) under the previous conditions, the firm should adopt the hysteresis strategy. All the conditions in the Set of Conditions 1 point to the fact that for certain initial prices it is optimal to switch to project 2. Condition (i) states that the perpetual cost of staying in the larger project is larger than switching to the smaller project and then staying in production forever in such a project. Condition (ii) means that the instantaneous profit in the larger project is smaller than in the smaller project at the exit threshold. Then, producing with prices near δ is more profitable in project 2 than in project 1. Finally, in conditions (iii) and (iv), we can interpret $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ as the fair prices for each one of the investments given the current market conditions. Thus, switching to project 2 may be optimal only when it is cheap to switch between project 1 and 2, and *vice-versa*.

In the next proposition, we present the conditions for each one of the switching strategies to be optimal as well as the corresponding value functions v_1 and v_2. The proof of the optimality of the functions v_1 and v_2 follows the lines of the proofs provided by Zervos et al. [22]. All the parameters and thresholds are defined in Appendix A.

Proposition 1 *Consider the switching problem (5). Then, if the Set Conditions 1 holds, the firm should follow the hysteresis strategy, depicted in Fig. 4, and the corresponding value functions are:*

$$v_1(p) = \begin{cases} -K_x & p < P_{1x} \\ Ep^{d_1} + Fp^{d_2} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & P_{1x} \le p < P_h \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} - K_{12} & P_h \le p < P_{12} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & P_{12} \le p \end{cases}, \tag{13}$$

$$v_2(p) = \begin{cases} -K_x & p < P_{2x} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} & P_{2x} \le p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K_{21} & P_{21} \le p \end{cases}. \tag{14}$$

If the Set of Conditions 1 does not hold, then the firm should follow the no -downgrading strategy, depicted in Fig. 2, and the value functions v_1 and v_2 are given by the following equations:

$$v_1(p) = \begin{cases} -K_x, & p < P_{1x} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}, & p \ge P_{1x} \end{cases}, \tag{15}$$

$$v_2(p) = \begin{cases} -K_x, & p < P_{2x} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r}, & P_{2x} \le p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K_{21}, & p \ge P_{21} \end{cases}. \tag{16}$$

Additionally, the exit thresholds are such that

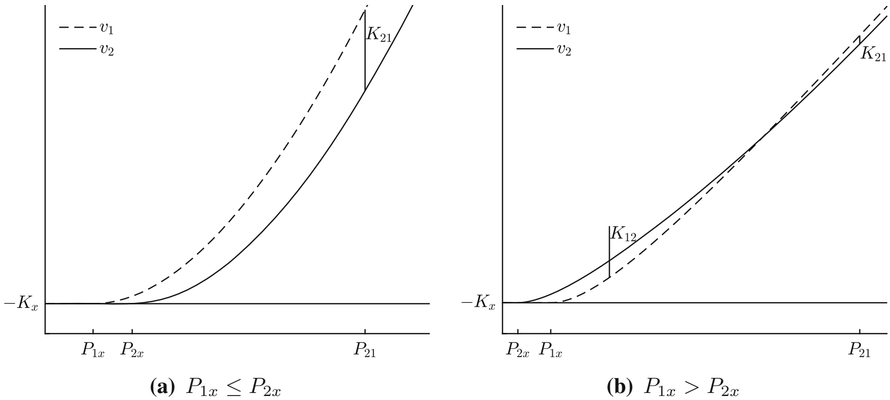

Fig. 5 Illustrations of the value functions v_i, with $i = 1, 2$, for the switching problem, as a function of the current price p in the horizontal axis, when the firm should implement the no- downgrading strategy. In both figures, P_{ix} is the exit threshold from project i and P_{21} is the threshold that triggers a switch from project 2 to project 1. K_{ij} is the switching cost from project i to project j

(a) $P_{1x} = \delta$;
(b) $P_{1x} \leq P_{2x}$, *if* $\pi_1(\delta) \geq \pi_2(\delta)$ *or* $\left(\pi_1(\delta) - \pi_2(\delta) < 0 \text{ and } K_{21} \geq K_{21}^{\dagger}\right)$
(c) $P_{1x} > P_{2x}$, *if* $\left(\pi_1(\delta) - \pi_2(\delta) < 0 \text{ and } K_{21} < K_{21}^{\dagger}\right)$

The parameters and thresholds for both strategies are provided in Appendix A.

The results presented in Proposition 1 have to be interpreted as follows: given that the current price is p, and that the firm is producing in mode $i \in \{1, 2\}$, then its value is given by $v_i(p)$. The expression for v_i depends solely on the Set Conditions 1 being satisfied (and in that case the hysteresis strategy is the strategy that should be followed by the firm) or not (and, hence, the firm should implement the no-downgrading strategy).

We note that in the no-downgrading strategy, v_1 represents a standard exit problem, whereas for the derivation of the value function v_2, one takes into account that the firm, while in project 2, has two options: the option to exit and the option to switch to the other project. This implies that the value function in the continuation region is composed of three terms: one corresponding to the exit option, another to the switching option and, finally, the value of producing in mode 2. In the hysteresis strategy, the value functions v_1 and v_2 are more evolved, as there are more options (and regions) available for the firm to choose.

In Figs. 5 and 6 we provide an illustration of the value functions provided in Proposition 1. Figure 5 illustrates the case where it is never optimal to switch from mode 1 to mode 2. The main difference between both panels is the relationship between the two exit thresholds. In panel (a) $P_{1x} \leq P_{2x}$ and, consequently, v_1 dominates v_2. Thus, for any value of $K_{12} > K_{12}^{\dagger}$, switching from 1 to 2 will never be optimal. When the firm is in mode 2, as long as the difference between v_1 and v_2 is smaller than the switching cost K_{21}, the firm will keep on producing in mode 2. Then, at P_{21}, $v_1(P_{21}) - v_2(P_{21}) = K_{21}$, and thus the firm switches to mode 1.

In the panel (b) we have that $P_{1x} > P_{2x}$ and, consequently, the dominance of v_1 over v_2 does not occur. But switching from mode 1 to mode 2, even when v_2 is larger than v_1, does not overpay the switching cost K_{12}, and, for this reason, the no-downgrading strategy is optimal. As in panel (a), when the firm is in production mode 2 and the process hits the

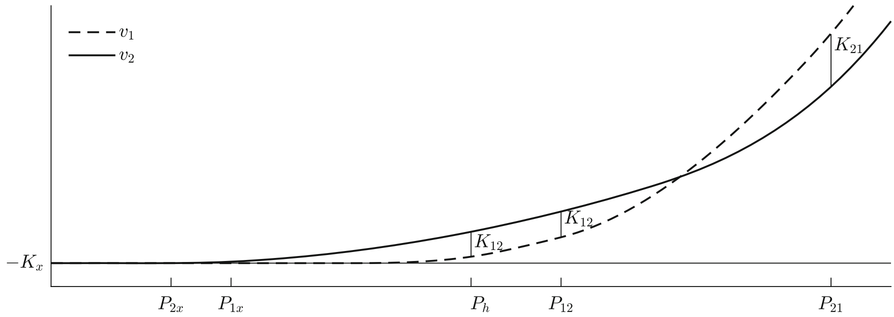

Fig. 6 Illustration of the value functions v_i, with $i = 1, 2$, for the switching problem, as a function of the current price p in the horizontal axis, when the firm should implement the hysteresis strategy. P_{ix} is the exit threshold from project i, P_{ij} is the switching threshold from project i to project j and P_h is the hysteresis threshold. K_{ij} is the switching cost from project i to project j

value P_{21}, switching to the mode 1 is optimal because the additional gain pays the switching cost K_{21}.

In Fig. 6, we present an illustration of the value functions v_1 and v_2 in which the hysteresis strategy is optimal. When the value of producing in project 2 is larger than the value of producing in project 1, the firm may want to switch to the smaller scale project. This decision may happen at the levels of price P_h and P_{12} because at those points $v_2(P_h) - v_1(P_h) = v_2(P_{12}) - v_1(P_{12}) = K_{12}$. For values of $p \in (P_h, P_{12})$, we have that $v_2(p) - v_1(p) > K_{12}$, which means that the additional gain from producing in project 2 compensates the switching cost.

When the price is less than P_h, switching from project 1 to project 2 is not optimal, as the profit from the smaller project does not compensate the switching cost. So the firm keeps producing in the hysteresis region, where the profit will be negative, waiting to decide if it should leave in case the price continues decreasing or if should switch to project 2, in case the price increases.

We finalise this section noticing that the exit option has a determinant effect on the existence of the hysteresis region. Duckworth and Zervos [9] and Ly Vath and Pham [14] showed that such region does not exist when the exit option is absent from the model.

4.2 Investment problem

With the solution to the switching problem (5), we are now in position to solve the investment problem (6). We start by assuming that the following set of conditions holds true:

Set of Conditions 2 (i) $K_1 > -K_x$ *and* $K_2 > -K_x$
(ii) $K_1 < K_2 + K_{21}$ *and* $K_2 < K_1 + K_{12}$

Condition (i) means that it is never optimal for the firm to invest and exit at the same time, and Condition (ii) means that it is more costly to enter in the market with project 1 (resp., 2) and to switch immediately to the project 2 (resp., 1) than to enter directly with project 2 (resp., 1).

Upon investment, the firm will pay the investment cost K_1 or K_2, depending in which project the firm invests, and from that point on it will receive the value obtained from the optimal switching strategy. We call the sum of these two values the *investment reward*, which is given in Proposition 2.

The structure of the function u is important in order to guess the shape of the waiting and investment regions. In the following proposition, we define the function u in light of the value functions v_1 and v_2. Its proof is provided in B.1.

Proposition 2 *The investment reward, hereby denoted by u, is given by:*

- *If* $K_1 \geq K_2$, *then*

$$u(p) = \begin{cases} v_2(p) - K_2, & p < z \\ v_1(p) - K_1, & p \geq z \end{cases}, \tag{17}$$

where

– $z \in (P_{1x}, P_{21})$, *if the no-downgrading strategy is optimal.*
– $z \in (P_{12}, P_{21})$, *if the hysteresis strategy is optimal.*

- *If* $K_1 < K_2$ *and* $P_{1x} > P_{2x}$, *then*

$$u(p) = \begin{cases} v_1(p) - K_1, & p < z_1 \\ v_2(p) - K_2, & z_1 \leq p < z_2 \\ v_1(p) - K_1, & p \geq z_2 \end{cases}, \tag{18}$$

where

– $z_1 \in (P_{2x}, P_{1x})$ *and* $z_2 \in (P_{1x}, P_{21})$, *if the no-downgrading strategy is optimal.*
– $z_1 \in (P_{2x}, P_h)$ *and* $z_2 \in (P_{12}, P_{21})$ $z \in (P_{12}, P_{21})$, *if the hysteresis strategy is optimal.*

- *Otherwise*

$$u(p) = v_1(p) - K_1, \tag{19}$$

where v_1 *and* v_2 *are given by* (15) *and* (16), *in the no-downgrading case, and by* (13) *and* (14), *in the hysteresis case.*

Proposition 2 shows that the investment reward is highly dependent on both the investment cost and the structure of the optimal switching strategy. One can conclude that for large values of the price p, the perpetual value of investment in project 1 is larger than in project 2, regardless of the investment cost in each project. For small values of the price p, the perpetual value of investment is not straightforward, since we can find situations where $v_2(p) - K_2$ dominates $v_1(p) - K_1$, and *vice-versa*.

The properties of the investment reward u will impact significantly on the shape of the investment region, and in the values of the investment value, as W is such that the following equation holds:

$$\max\{(\mathcal{L}W)(p) - rW(p), -W(p) - u(p)\} = 0. \tag{20}$$

Based on the shape of u, one can guess that investment in the smaller project may be optimal for small values of the price p. Thus one expects the following strategies:

Connected investment region: The firm waits for larger prices and then invests in the more profitable project, which is project 1. In this case, the value function for the investment problem (6), hereby denoted by W_1, is as follows:

$$W_1(p) = \begin{cases} B_2 p^{d_2}, & p < \gamma_3 \\ v_1(p) - K_1, & p > \gamma_3 \end{cases}. \tag{21}$$

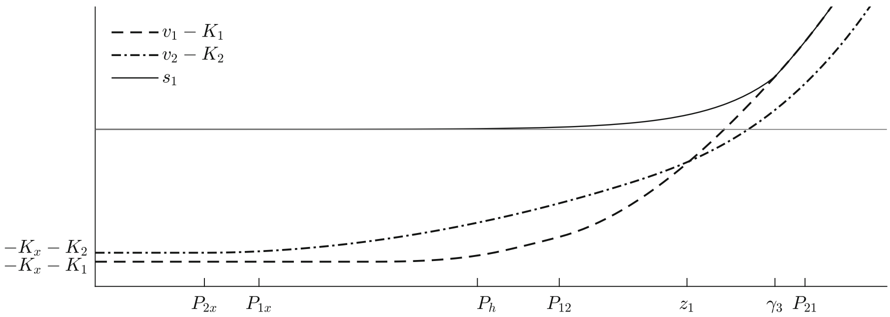

Fig. 7 Illustration of the value functions for the investment problem in the connected case, W_1, as a function of the current price p in the horizontal axis. We also include the thresholds: P_{ix} is the exit threshold from project i, P_h is the hysteresis threshold and P_{ij} is the switching threshold from project i to project j. $v_i - K_i$ is the value in case we invest in project i, with $i = 1, 2$

In Fig. 7 we illustrate the behaviour of W_1 as a function of the price p, including the relevant thresholds for the switching problem.

For $p > z_1$, the value of the firm producing in mode 1 is larger than its value in mode 2, and therefore the investment will occur in this mode. The investment threshold, γ_3, is the value of p for which the value of investment in mode 1 is equal to the value of waiting. Thus, for values larger than this threshold, the value of investment is larger than the option to differ. We note that, in the case depicted in Fig. 7, there is no complete dominance of $v_1(p) - K_1$ over $v_2(p) - K_2$. However, if $v_1(p) - K_1$ dominates $v_2(p) - K_2$, which happens when $K_1 < K_2$ and $P_{1x} \leq P_{2x}$, the optimal strategy is a threshold one, and the value function is still given by W_1 as in (21). A threshold investment decision in alternative projects was already presented by Dixit [6].

Non-connected investment region: The firm invests for moderate values of the price ($p \in (\gamma_1, \gamma_2)$), and in that case it invests in the project 2. But when the price is around z_1, a point of intersection between the two curves, then it may be optimal to wait for larger values of the price ($p \in (\gamma_3, \infty)$) and then invest in the project 1. Therefore, in this case, the value function for the investment problem (4.2) denoted by W_2 and is given by:

$$W_2(p) = \begin{cases} B_1 p^{d_2}, & p < \gamma_1 \\ v_2 - K_2, & \gamma_1 < p < \gamma_2 \\ A_2 p^{d_1} + B_2 p^{d_2}, & \gamma_2 < p < \gamma_3 \\ v_1 - K_1, & p \geq \gamma_3 \end{cases}. \tag{22}$$

In Fig. 8 we plot W_2. One can observe that, for values of $p < z_1$, the value being in project 2 is larger than in project 1. Therefore, as $z_1 > \gamma_1$, γ_1 triggers the investment in project 2. Finally for values of p larger than γ_3, the value of investment in the large scale project is larger than the value of investment in the small scale project. Thus, γ_3 triggers investment in project 1. One may notice that investment in project 1 occurs for values of $p \in (P_{12}, P_{21})$, which means that investment in the hysteresis region is never optimal, as we state in Proposition 3. Finally, we note that in this case the optimal strategy is not a threshold type. A disconnected investment region was also found in Décamps et al. [4]. Mathematically, the inaction region found between the two investment regions is explained by the fact that u has an upward kink. Investing in such region is never optimal because there is always a solution to the equation

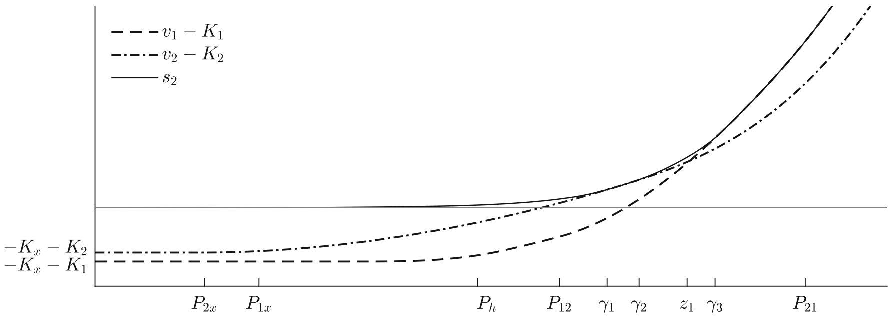

Fig. 8 Illustration of the value functions for the investment problem in the non-connected case, W_2, as a function of the current price p in the horizontal axis. We also include the thresholds: P_{ix} is the exit threshold from project i, P_h is the hysteresis threshold and P_{ij} is the switching threshold from project i to project j. $v_i - K_i$ is the value in case we invest in project i, with $i = 1, 2$

$rs(p) - (\mathcal{L}s)(p) = 0$ that is larger than u and pastes conveniently u. The same phenomenon is described by Décamps et al. [4].

In Figs. 7 and 8 we consider that $v_1(p) - K_1$ crosses $v_2(p) - K_2$ only once. However, as one can see in Proposition 2, we may have situations where $v_1(p) - K_1$ crosses twice $v_2(p) - K_2$. Even in this case, the optimal strategy is given by W_1 or W_2, depending on the set parameters.

In the next proposition, we show that the hysteresis region is never reached through investment. The proof can be found in Appendix B.2.

Proposition 3 *Investment in the hysteresis region is never optimal.*

An immediate consequence of this proposition is the fact that a firm never exits from the larger scale project. In fact, it is always optimal to switch to project 2. Once in project 2, the firm produces while it waits to decide whether to exit the market (in case the price decreases) or to switch to the larger project (in case the price increases).

Finally, in the next proposition we present the conditions for W_1 or W_2 to be the solution of the investment problem (6), which depend mainly on the relationship between costs and prices.

Proposition 4 *Let W be the value function associated with the investment problem* (6). *Then the following happens:*

- *If $K_1 \geq K_2$ or ($K_1 < K_2$ and $P_{1x} > P_{2x}$) then*

(a) *$W(p) = W_2(p)$, if $K_2 < K_2^+$ and $K_1 > K_1^-$.*
The optimal strategy is as follows: if $p \in (\gamma_3, \infty)$, then it is optimal to invest immediately in project 1, the larger project, where $\gamma_3 \in (P_{12}, P_{21})$; if $p \in (\gamma_1, \gamma_2)$, then it is optimal to invest immediately in project 2, the smaller project, where $\gamma_1 \in (P_{2x}, P_{21})$ and $\gamma_2 \in (\gamma_1, \gamma_3)$; otherwise, the firm waits.

(b) *$W(p) = W_1(p)$, if $K_2 \geq K_2^+$ or $K_1 \leq K_1^-$.*
The optimal strategy is as follows: if $p \in (\gamma_3, \infty)$, then it is optimal to invest in project 1, where $\gamma_3 \in (\max(P_{1x}, P_{12}), \infty)$; otherwise the firm waits.

- *If $K_1 < K_2$ and $P_{1x} \leq P_{2x}$, then $W(p) = W_1(p)$.*
The optimal strategy is as follows: if $p \in (\gamma_3, \infty)$, then it is optimal to invest in project 1, where $\gamma_3 \in (P_{1x}, \infty)$; otherwise the firm waits.

Table 1 Values for the diffusion parameters, interest rate and exit cost used along the numerical examples

$\mu = 0$	$\sigma = 0.2$	$r = 5\%$	$K_x = -1$	$\alpha_1 = \beta_1 = 1$	$\alpha_2 = \beta_2 = 0.5$	$K_{12} = 0.1$	$K_{21} = 0.3$

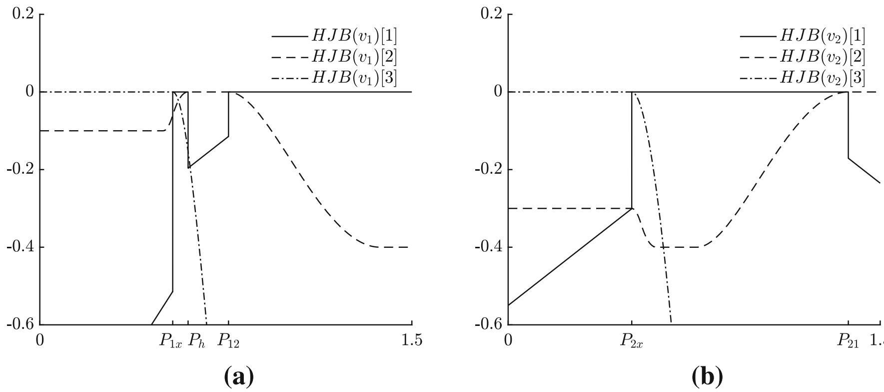

Fig. 9 Numerical illustration of the verification of HJB equations (8)–(9), as a function of the price, p, in the horizontal axis, which depends on the thresholds (P_{ix} is the exit threshold from project i, P_h is the hysteresis threshold and P_{ij} is the switching threshold from project i to project j)

The bounds K_1^- and K_2^+ are such that K_1^- does not depend on K_1 and K_2^+ does not depend on K_1 and K_2. The constants A_2 and B_2, the thresholds γ_1, γ_2 and γ, and the bounds K_1^- and K_2^+ are defined in Appendix A.2.

5 Results and sensitivity analysis

In this section, we assess the impact of the parameters in the optimal decisions, analysing (i) the behaviour of the triggers associated with the investment and switching decisions, and (ii) which strategy (no-downgrading or hysteresis) the firm should adopt. We also show how our set-up, namely the existence of multiple switching opportunities and the exit option, impacts the investment strategy. In the model of Décamps et al. [4], one of the projects generates a higher output flow and, thus, switching from that project to the other is never optimal. In our case, depending on the parameters, multiple switches may happen, notably in the hysteresis strategy, depicted in Fig. 3. Therefore, at the end of this section, we will be able to answer the RQ1-RQ3.

Due to the mathematical complexity of the expressions for the triggers presented in the above subsections, this analysis will be presented numerically only. The parameters for the base case are the ones presented in Table 1. In the subsequent Tables, the values in bold correspond to this base case.

5.1 The effects of the parameters in the switching strategy

Before we move to the comparative statics, in Fig. 9 we show the numerical verification of the HJB Eqs. (8)–(9) for the baseline parameters. In Fig. 9, panel (a) (resp., panel (b)), we have three different lines: the solid, dashed and dash-dotted lines that represent, respectively,

Table 2 Thresholds and bounds for the switching strategy with changing μ, the rest of the parameters are as in Table 1

μ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	$K_{21}^{\dagger}$	$K_{12}^{\dagger}$	Strategy
−0.250	0.973			1.019	1.712	0.001		ND
−0.150	0.931			0.965	1.565	0.014		ND
−0.030	0.729			0.694	1.397	2.625	0.037	ND
−0.010	0.619	0.706	0.771	0.575	1.389	9.725	0.161	Hyst
0.000	**0.536**	**0.598**	**0.761**	**0.498**	**1.372**	**21.547**	**0.318**	**Hyst**
0.010	0.440	0.483	0.750	0.409	1.355	59.330	0.613	Hyst
0.025	0.274	0.296	0.735	0.256	1.330		1.588	Hyst
0.030	0.215	0.232	0.729	0.202	1.322		2.167	Hyst

the first, second and third term of the HJB Eq. (8) (resp., (9)). The HJB equation is satisfied if (i) all the terms are not positive; and (ii) only one term of the HJB equation is equal to zero. As we have already discussed, these terms are related with the production, switching and abandonment regions respectively. Thus, such regions can be identified observing the range of prices that make each one of the terms of the HJB equation equals to zero. For instance, the solid line in panel (a) is equal to zero when the current price p is such that $p \in (P_{1x}, P_h) \cup (P_{12,+\infty})$, which means that such a region is the continuation region for a firm that is currently in project 1. Additionally, these plots guarantee that for the parameters in Table 1, the solution we provide verifies the HJB equations (8) and (9). This verification give us the guarantee that the thresholds for the baseline case are correct. The same kind of verification has been performed for all different sets of values of the parameters that we present in the following subsections.

5.1.1 Comparative statics with respect to μ and σ

Next, we study the impact of μ and σ on the relevant thresholds, as well as the optimal strategy.

Table 2 presents the behaviour of the thresholds with changing μ, while keeping other parameters constant, and equal to the values of the base case presented in Table 1. The last column of the table indicates the optimal strategy that the firm should follow, with **ND** denoting the no-downgrading strategy and **Hyst** denoting the hysteresis strategy. We have included in Table 2 all the thresholds (note that P_h and P_{12} are not applicable for the ND case and the missing values for the bounds $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ are irrelevant). The base case, for which $\mu = 0$, corresponds to the line with bold values.

The information regarding $K_{12}^{\dagger}$ and $K_{21}^{\dagger}$ allows us to split the situations in which the no-downgrading strategy and hysteresis strategy are optimal. In fact, whenever $K_{21} \geq K_{21}^{\dagger}$ or $K_{12} \geq K_{12}^{\dagger}$, the no-downgrading strategy is optimal. In this case, the cost of switching between projects is too expensive and therefore the exit decision is preferable when compared with switching to the smaller project. On the contrary, in case $K_{21} < K_{21}^{\dagger}$ and $K_{12} < K_{12}^{\dagger}$, switching from mode 1 to mode 2 is a feasible option as the associated costs are sufficiently low. In Table 2, $\beta_2 + rK_{12} < \beta_1$ and $\pi_1(\delta) - \pi_2(\delta) < 0$, thus we start by checking condition iii) in the set of conditions 1 (where only $K_{21}^{\dagger}$ has to be computed). In case condition iii) fails, then we check condition iv).

Table 3 Thresholds and bounds for the switching strategy with changing σ, the rest of the parameters are as in Table 1

σ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	$K_{21}^{\dagger}$	$K_{12}^{\dagger}$	Strategy
0.025	0.970			0.982	1.094	0.033	≈ 0	ND
0.050	0.897			0.891	1.135	0.545	0.005	ND
0.090	0.791			0.762	1.207	2.541	0.084	ND
0.100	0.765	0.834	0.839	0.732	1.225	3.290	0.106	Hyst
0.200	**0.536**	**0.598**	**0.761**	**0.498**	**1.372**	**21.547**	**0.318**	**Hyst**
0.250	0.451	0.511	0.730	0.414	1.442	48.797	0.405	Hyst
0.500	0.209	0.257	0.619	0.181	1.791		0.646	Hyst

Table 2 suggests that increasing the drift lowers the exit thresholds in both projects, which means that if the expectations about the price of the product increase, then the firm is more willing to stay in the market. Moreover, the switching thresholds, P_{12} and P_{21}, decrease with μ. Then, the firm stays in production in the current project for smaller values of the price before deciding to switch. We also observe from the results of Table 2 that both $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ increase with μ, and therefore the set of Conditions 1 becomes less feasible. This means that for small (and negative) values of μ, the firm should optimally follow the no-downgrading strategy since it becomes non-profitable to change from the larger project to the smaller one, being preferable to exit if the price decreases. On the contrary, when μ is large enough, the firm is less willing to exit the market since it expects to attain large prices in the future. Thus, both switching from mode 1 to mode 2 and waiting in the hysteresis region may be optimal decisions. Finally, we remark that the amplitude of the hysteresis region decreases with μ. Since the firm expects larger future prices, the hysteresis region becomes almost useless.

Regarding the influence of the volatility parameter, the numerical results are presented in Table 3. We can observe that the firm should follow the hysteresis strategy when the uncertainty is large. This happens because both $K_{21}^{\dagger}$ and $K_{12}^{\dagger}$ increase with σ, and, consequently, the set of Conditions 1 is verified. Larger uncertainty means that the firm may wish to wait (with eventually negative profits), in the expectation that the future expected profits will increase and cover the losses accumulated during a hysteresis period. We can observe that the exit threshold decreases with the volatility in both projects. And also, as the market becomes less predictable, the firm is more willing to switch, i.e. it accommodates to the uncertainty using the switching option.

Before we finish this section, we note that in all the scenarios presented, a firm producing in the hysteresis region is producing at a loss since its instantaneous profit is negative. Indeed, one can easily verify that $\pi_1(P_h) = \alpha_1 P_h - \beta_1$ varies between $(-0.768, -0.294)$ (resp., $(-0.743, -0.166)$), for $\mu \in (-0.01, 0.03)$ (resp., $\sigma \in (0.1, 0.5)$). It is also interesting to note that when either μ or σ increase, the instantaneous loss of the firm producing in the hysteresis region increases.

The analysis developed so far allows us to answer RQ3. In fact, both the hysteresis and no-downgrading strategies can be optimal depending on the parameters chosen. The hysteresis strategy becomes optimal when μ and σ increase. Additionally, our findings regarding the behaviour of the optimal thresholds with μ and σ are similar to the ones obtained in Guerra et al. [10]. Finally, one can also observe that the firm can produce optimally at a loss if the it has the expectation that such losses will be recovered with an increase of prices.

Table 4 Thresholds for the optimal strategy with changing K_{12}, the rest of the parameters are as in Table 1

K_{12}	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	Strategy
0.001	0.502	0.506	0.785	0.498	1.338	Hyst
0.050	0.525	0.565	0.773	0.498	1.355	Hyst
0.100	**0.536**	**0.598**	**0.761**	**0.498**	**1.372**	**Hyst**
0.315	0.563	0.716	0.718	0.500	1.430	Hyst
0.318	0.564	0.717	0.718	0.500	1.431	Hyst
0.318	0.564			0.500	1.431	ND
10.000	0.564			0.500	1.431	ND

Table 5 Thresholds for the optimal strategy with changing K_{21}, the rest of the parameters are as in Table 1

K_{21}	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	Strategy
0.300	**0.536**	**0.598**	**0.761**	**0.498**	**1.372**	**Hyst**
1.000	0.545	0.609	0.700	0.507	1.615	Hyst
1.300	0.548	0.613	0.685	0.509	1.701	Hyst
1.400	0.564			0.510	1.742	ND
5.000	0.564			0.531	2.558	ND

5.1.2 Comparative statics with respect to the switching costs K_{21} and K_{12}

We analyse the influence of the switching costs in the optimal decision. The numerical results are presented in Table 4 (for K_{12}) and Table 5 (for K_{21}).

Observing Tables 3 and 4, we can conclude that changing either K_{12} or K_{21} leads to the same type of behaviour in the optimal strategy and thresholds. The exit thresholds in both projects increase when the switching costs increase, meaning that the firm is more likely to exit the market as the switching costs are larger. As soon as the no-downgrading strategy is optimal, one can observe that (i) K_{12} does not affect the optimal strategy because it is never optimal to switch to project 2, and (ii) K_{21} does not affect P_{1x} only.

The hysteresis threshold, P_h, and the size of the hysteresis region increase with the switching costs, meaning that the firm stays more time in such a region as the cost of switching is larger. Also, as the cost increases, the threshold P_{21} increases and P_{12} decreases, which means that switching becomes less attractive

5.2 The effect of the parameters in the investment strategy

In this section, we illustrate with numerical examples the results derived in the previous section. It is important to note that when we change μ and σ, we also change the solution of the underlying switching problem (5). We will only choose costs that satisfy the Set of Conditions 2, reducing ourselves to the types of solutions: W_1 and W_2, as described in Proposition 4.

5.2.1 Comparative statics with respect to σ

In this section, we analyse the effect of the volatility in the investment strategy. In order to facilitate the numerical analysis in this case, we slightly change the baseline parameters. Now we consider that $\alpha_2 = 0.6$, $K_{12} = 0.25$, $K_{21} = 0.5$, and the remaining parameters are

Table 6 Thresholds and bounds for the optimal investment strategies with changing σ, the rest of the parameters are as in section 5.2.1

σ	K_1^-	K_2^+	γ_1	γ_2	z_2	γ_3	Strategy	Inv
0.10	1.540	4.596	1.195	1.438	1.458	1.476	Hyst	W_2
0.15	1.732	4.030	1.323	1.495	1.527	1.556	Hyst	W_2
0.20	1.839	3.540	1.461	1.549	1.594	1.634	Hyst	W_2
0.24	1.892	3.197	1.578	1.593	1.649	1.696	Hyst	W_2
0.25	1.902	3.118			1.662	1.715	Hyst	W_1
0.30	1.942	2.761			1.730	1.856	Hyst	W_1

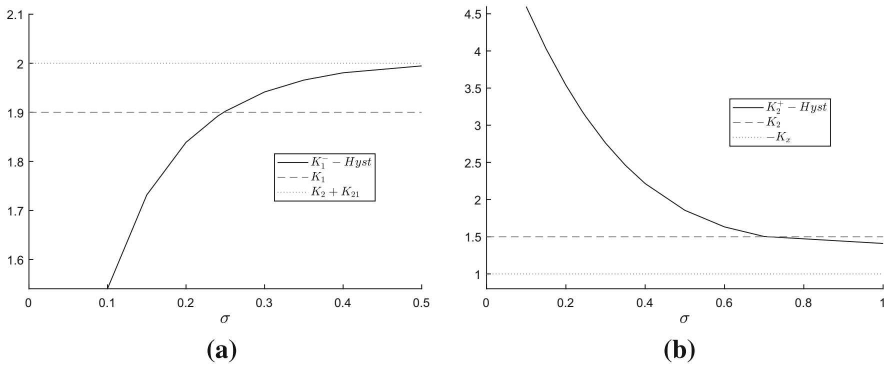

Fig. 10 The bounds for the optimal investment strategies with changing σ, as in Table 6. In **a**, K_1^- is represented and K_2^+ is represented in **b**

as in Table 1. The investment costs are set as: $K_1 = 1.9$ and $K_2 = 1.5$. Since the values for the switching thresholds for this sensitivity analysis are not in Sect. 5.1, we present them in Appendix C.2.

The numerical results are presented in Table 6. In order to help in the explanation of the results, we also present Fig. 10, where we plot the behaviour of the bounds K_1^- and K_2^+, which play a major role in the decision regarding in which project the investment takes place, as a function of the volatility. If $K_1 \leq K_1^-$ or $K_2 \geq K_2^+$, then the investment should take place in the larger scale project, if the initial price is greater than γ_3. On the other hand, if $K_2 \in (K_2^-, K_2^+)$ and $K_1 \in (K_1^-, K_1^+)$, investment in the small scale project may be optimal. We note that K_1 and K_2 are fixed, $K_1^+ = K_2 + K_{21}$, and $K_2^- = -K_x$ do not depend on σ, whereas the bound K_1^- and K_2^+ depend on the volatility.

In Fig. 10, one can see that K_1^- increases, with the volatility, to K_2+K_{21}, and K_2^+ decreases to $-K_x$. Since $K_1 < K_2 + K_{21} = \lim_{\sigma \nearrow \infty} K_1^-(\sigma)$, and $K_2 > -K_x = \lim_{\sigma \nearrow \infty} K_2^+(\sigma)$, investment in project 2 is never optimal, if one considers $\sigma \nearrow +\infty$. We note that the parameter K_1 does not impact on the value of the thresholds K_1^- and K_2^+, and K_2 does not influence the value of the threshold K_2^+.

However, it can still happen that investment in the small scale project is optimal, even for large values of volatility, depending on the investment cost. For instance, if one assumes $K_1 = 1.99$ instead of $K_1 = 1.9$, which is the value considered for our illustration, the optimal strategy will be W_2 (and not W_1, as in our case).

The results of Table 6 confirm that the strategy W_2 is more likely to be optimal for small values of volatility. In this case the firm is willing to invest in project 2 for low values of the revenue ($p \in (\gamma_1, \gamma_2)$). If the volatility increases, then the firm is sceptical about investing, and, therefore, waits for large values of revenue. In this case, only investment in project 1 is optimal. This is highlighted by the fact that the size of the region (γ_1, γ_2) is decreasing with σ. In our illustration, one can see that investing in project 2 is no longer optimal for values of volatility larger than 0.24, which happens because K_1 becomes smaller than K_1^-. Additionally, γ_3 increases with σ, which confirms the usual effect of increasing volatility increases the investment trigger.

These results can be compared with the ones from Dixit [6] and Décamps et al. [4]. Dixit [6] noticed that larger uncertainty in the price process leads to a threshold investment strategy, meaning that it is never optimal to invest in the smaller scale project. The firm can only invest in the smaller project if the initial price is larger than the investment threshold γ_3. Décamps et al. [4], showed that the optimal investment region may be dichotomous. Additionally, the authors found that the existence of the inaction region persists when the volatility of the price process increases, if switching from the smaller to the larger project is allowed. This means that investing in the smaller project may be optimal for high values of volatility. However, the behaviour described by the later authors is no longer verified when we consider the possibility of multiple switches between the two projects and the existence of an abandonment option. Some authors, such as Kwon [13] and Hagspiel et al. [11]) found that the optimal investment time can be affected by the fact, upon investment, the firm acquires an option to exit. Then one may ask about the influence of the exit option in the optimal investment strategy when the volatility increases. In Fig. 11, we can see that the results of Décamps et al. [4] for a model with a single switch are no longer verified when we consider the exit option. This allows us to conclude that the existence of an exit option makes the investment in the smaller project not optimal for large values of volatility. We can then answer RQ1 by stating that an inaction region where it is not optimal to invest may exist, i.e. the optimal investment strategy may be dichotomous. This happens mainly when the volatility is small. When the volatility increases, investment in project 2 is no longer optimal because firms prefer to leave the market if prices decrease.

5.2.2 Comparative statics with respect to μ

In this section, we consider the parameters in Table 1, and use the investment costs $K_1 = 1.3$ and $K_2 = 1.05$. To facilitate the analysis, we will use a range of values for μ also considered in Table 2. The results are shown in Table 7. The line filled in bold corresponds to the base case. As explained before, the value of the bounds K_1^- and K_2^+ for each specific μ are relevant to decide the optimal investment rule that the firm should follow. We recall that the strategy W_2 is optimal when $K_1 > K_1^-$ and $K_2 < K_2^+$; otherwise, the firm should follow W_1 and invest in the large scale project as soon as the initial price becomes larger than the threshold γ_3.

We can conclude that when we increase the drift, the firm prefers to invest directly in project 1 instead of investing in project 2. This is because, as μ increases, the firm expects to attain large values of profit sooner, and, consequently, it is more profitable to produce in the larger project. Furthermore, switching between the two projects is then optimal for larger values of the drift (and, in that case, the firm follows the hysteresis strategy); whereas, for small and negative values, the firm stays producing with the large scale project, in the the no-downgrading strategy until it eventually exits the market.

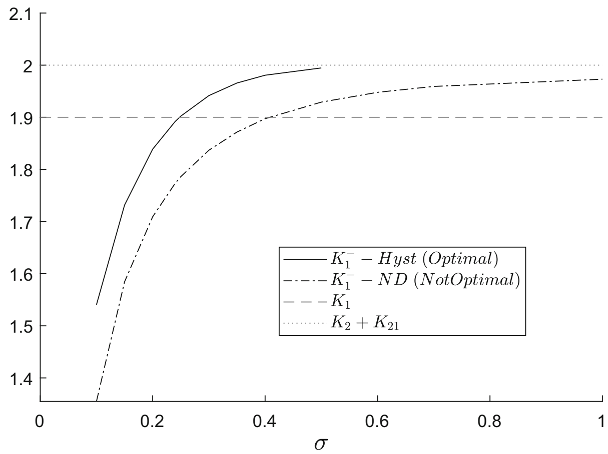

Fig. 11 Comparison between the values of the bound K_1^- for the (optimal) "hysteresis strategy" and the (not optimal) "no-downgrading strategy" with changing σ, considering the other parameters as in Table 6

Table 7 Thresholds for the optimal investment strategies with changing μ. The investment costs are $K_1 = 1.3$ and $K_2 = 1.05$

μ	K_1^-	K_2^+	γ_1	γ_2	γ_3	$\gamma_3 - \max(P_{1x}, P_{12})$	Strategy	Inv
−0.250	1.235	1.218	1.381	1.510	1.584	0.610	ND	W_2
−0.150	1.303	1.190			1.450	0.519	ND	W_1
−0.100	1.338	1.141			1.418	0.532	ND	W_1
−0.030		0.734			1.464	0.735	ND	W_1
0.000		**− 0.063**			**1.533**	**0.772**	**Hyst**	$\mathbf{W_1}$
0.010					1.545	0.795	Hyst	W_1
0.025					1.534	0.800	Hyst	W_1
0.030					1.521	0.791	Hyst	W_1

The most unexpected result shown in Table 7 is the non-monotonic behaviour of the investment threshold γ_3 in project 1. We note that it starts to decrease with μ, then increases, and then decreases again. Such a non-monotonic behaviour is also present in the distance between the investment threshold and the exit/switching thresholds (P_{1x} and P_{12}). Since this behaviour is unexpected, we provide a numerical verification of the HJB equations for the levels of μ where the inversion of the behaviour happens, in Appendix C.1. This ensures that the numerical solution to the optimal stopping problems is correct.

This behaviour is justified by the existence of two effects: on the one hand, the firm is willing to enter the market for smaller initial prices because it expects to attain larger values of profits sooner. But, on the other hand, the existence of the exit option restricts the admissible range of initial prices for which is optimal to invest. For a possible explanation for these effects, we analyse the equations that lead to the investment and the exit thresholds and how they depend on μ. We consider the following two equations:

$$(d_2 - d_1)A\gamma_3^{d_1} + (d_2 - 1)\frac{\alpha_1}{r-\mu}\gamma_3 - d_2\left(\frac{\beta_1}{r} + K_1\right) = 0 \tag{23}$$

Table 8 Thresholds for the optimal investment strategies with changing μ for the small value of $\sigma = 0.03$, the rest of parameters are set as in a Table 7

μ	K_1^-	K_2^+	γ_1	γ_2	z_2	γ_3	Strategy	Inv
−0.0050	1.000	5.200	0.973	1.349	1.354	1.359	Hyst	W_2
−0.0025	1.000	5.381	0.975	1.323	1.329	1.334	Hyst	W_2
0.0000	1.000	5.537	0.977	1.298	1.304	1.309	Hyst	W_2
0.0025	1.000	5.648	0.976	1.273	1.279	1.284	Hyst	W_2
0.0050	1.000	5.710	0.971	1.249	1.255	1.261	Hyst	W_2
0.0100	1.000	5.746	0.957	1.207	1.213	1.219	Hyst	W_2

$$(d_2 - d_1) A P_{1x}^{d_1} + (d_2 - 1) \frac{\alpha_1}{r - \mu} P_{1x} - d_2 \left(\frac{\beta_1}{r} - K_x \right) = 0, \tag{24}$$

where the first equation is derived in Appendix (A.2.2) and the second one is obtained from (31)–(32), for the no-downgrading strategy[3]. By a simple inspection of (23) and (24) one may think that the behaviour of γ_3 is the same as P_{1x}, as the equations are quite similar. However, whereas P_{1x} can be found explicit (see Eq. (12)) and one can prove analytically that it is decreasing with μ, γ_3 can only be found implicitly by solving (23). This means that γ_3 depends on A and P_{1x}, and both depend on μ. From the previous equations, we can obtain an implicit equation for γ_3:

$$\gamma_3 = \left(\frac{(d_2 - 1) \frac{\alpha_1}{r-\mu} \gamma_3 - d_2 \left(\frac{\beta_1}{r} + K_1 \right)}{(d_2 - 1) \frac{\alpha_1}{r-\mu} P_{1x} - d_2 \left(\frac{\beta_1}{r} - K_x \right)} \right)^{\frac{1}{d_1}} P_{1x},$$

which is a highly non-linear equation in μ (because d_1, d_2 and P_{1x} depend on μ), and shows that the variation of γ_3 depends on the relative position of γ_3 and P_{1x}.

The results from Table 7 shows that, for our set of parameters, the strategy W_1 is almost always optimal. Then, we also consider values for the parameters such that the optimal strategy is W_2. In this case, the firm invests in the smaller scale project if $p \in (\gamma_1, \gamma_2)$ and invests in the large scale if $p > \gamma_3$. Those results are displayed in Table 8. As the optimal switching strategy is the hysteresis one, the firm will never exit the market from the large scale project. As a consequence, γ_3, the investment threshold in project 1 becomes monotonic and decreasing, as it is usual in standard investment problems, whereas γ_1, the investment threshold that is closer to the exit threshold, becomes non-monotonic. This analysis answers RQ2 and makes clear the impact of the exit option: the firm is generally willing to invest for smaller initial prices, when μ increases; however, this may change when the investment threshold is too close to the exit threshold. Otherwise, a sudden decrease in the profit would lead to an exit decision, which would not be optimal because there are fixed costs involved.

5.2.3 Comparative statics with respect to the exit cost K_x

The role of the exit option in the investment decision has been studied in real options models with different features (see for instance Duckworth and Zervos [8], Kwon [13], Hagspiel et al. [11]). In this section, we also analyse the impact of the exit option in the investment strategy.

[3] For the hysteresis strategy, the reasoning would be similar, but using the appropriate equations.

Table 9 Thresholds for the optimal investment and switching strategies with changing exit cost K_x

K_x	P_{1x}	P_h	P_{2x}	K_1^-	K_2^+	γ_1	γ_2	γ_3	Strat	Inv
−1.0	0.518	0.600	0.455	1.839	3.540	1.461	1.549	1.634	Hyst	W_2
0.0	0.481	0.550	0.417	1.891	3.253	1.535	1.549	1.634	Hyst	W_2
0.2	0.473	0.541	0.409	1.899	3.199	1.548	1.549	1.634	Hyst	W_2
0.4	0.466	0.531	0.401	1.907	3.146			1.641	Hyst	W_1
1.0	0.442	0.501	0.378	1.926	2.995			1.663	Hyst	W_1

In fact, the exit option becomes less valuable when the exit cost increases. If $K_x \geq \beta_2/r$ then leaving the market is not optimal. For this purpose, we analyse the behaviour of the investment thresholds by increasing the value of the exit cost K_x.

We consider the parameters as in Sect. 5.2.1, fixing $\sigma = 0.2$. The investment and switching thresholds are summarised in Table 9. We note that the quantities P_{12}, z_2 and P_{21} do not change with K_x. Their values are: $P_{12} = 0.889$, $z_2 = 1.594$ and $P_{21} = 1.857$. We can conclude that investing in project 2 is not optimal when the abandonment cost is large. Thus, the strategy W_2 is optimal only for small values of K_x.

We find that increasing the exit cost increases the investment threshold. On the one hand, when investment in project 2 is optimal then γ_1 increases, but the remaining thresholds γ_2 and γ_3 do not change. This means that the size of the investment region in project 2 decreases. The timing to invest in mode 1 remains unchanged. On the other hand, when investment in mode 2 is never optimal, then γ_3 increases K_x. This means that investment in the production mode 1 is postponed.

So far, we have seen that investment never occurs in the hysteresis region. Then, in the next section, we will analyse the possibility of being producing in the hysteresis region when we consider the time-to-build feature. We assume that investment is not instantaneous, meaning that investment will only be effective after a certain lag period, also known as time-to-build. This analysis will give us insights to answer RQ4.

6 Extension: investment with time-to-build

In this section, we assume that the firm invests in the market, but it only starts producing n units of time after the investment. Thus, if investment takes place at time τ, it will only be effective at time $\tau + n$, when production will start. Hence, we include a *time-to-build* feature in the problem and we study how this will impact the investment strategy, notably in terms of the relevance of the hysteresis region found in the optimal switching strategy.

In this case, the investment problem can be written as follows:

$$\tilde{W}_n(p) = \sup_{\tau \geq 0} E_p \left[\max \left(e^{-r(\tau+n)} v_1(P_{\tau+n}) - e^{-r\tau} K_1, e^{-r(\tau+n)} v_2(P_{\tau+n}) - e^{-r\tau} K_2 \right) \right] \tag{25}$$

where we use the notation $\tilde{W}_n$ to emphasise the dependence of the decision on the time-to-build, which we assume to be known and equal to n. Using the strong Markov property and the law of iterated expectations, one can rewrite (25) as

$$\tilde{W}_n(p) = \sup_{\tau \geq 0} E_p \left[e^{-r\tau} u_n(P_\tau) \right] \tag{26}$$

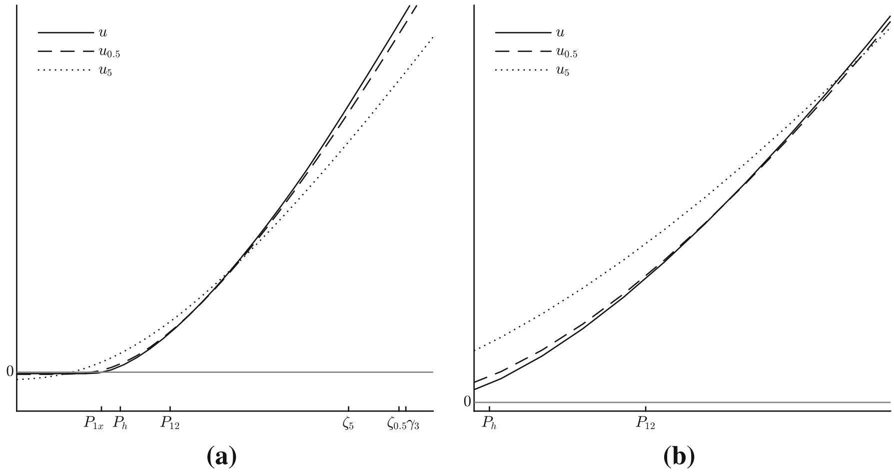

Fig. 12 **a** Plot of u, $u_{0.5}$ and u_5 for the baseline case, as a function of the price, p, in the horizontal axis. P_{ix} is the exit threshold from project i, P_h is the hysteresis threshold and P_{ij} is the switching threshold from project i to project j, with $i \neq j = 1, 2$. **b** Zoom of the figure for small values of revenue

where

$$u_n(p) = E_p \left[\max \left(e^{-rn} v_1(P_n) - K_1, e^{-rn} v_2(P_n) - K_2 \right) \right]. \quad (27)$$

The function u_n represents the perpetual value of the firm after investment, assuming that after investment the firm acts optimally according with the switching strategy. This perpetual value is itself an expected value, as the revenue at the moment that the firm starts operation is a random variable ($P_{\tau+n}$ is not known at time τ).

As we have seen in Sect. 5, the value functions v_1 and v_2 have different branches, representing the value of the firm operating in mode 1 and 2 for different values of the revenue. Therefore, in the computation of the expected value (27), one needs to take into account the probability that after n periods of time, the revenue will be in one of the branches that define the value functions v_1 and v_2. This means, in particular, that the investment strategy defined in (26) is unable to identify in which production mode the firm should optimally invest. Also, as a result of the expectation operator, the function u is smooth enough so that the investment strategy is a threshold strategy. For notation purposes, we let ζ_n denote the investment threshold when the time-to-build is equal to n. Although we will present a more detailed analysis in Sect. 6.1, we can already answer the first part of RQ4, by saying that the existence of time-to-build in the model results in investment strategies that are connected and, consequently, there is no inaction region in the optimal solution.

In Fig. 12, one can see that the function u_n is getting smoother as n increases. Additionally, $u_n > u$ for small values of revenue, but for large values of revenue $u_n < u$. This is explained by the fact that when P_τ is small there is a strictly positive probability that the revenue increases during the n periods of time, which increases the expected value of $\max \left(e^{-rn} v_1(P_n) - K_1, e^{-rn} v_2(P_n) - K_2 \right)$. A similar argument can be used when P_τ is large.

We can also observe in Fig. 12 that the investment threshold decreases when the time-to-build increases. This result seems to be contrary to the standard results in real options because, when we increase the time-to-build, we increase the uncertainty and, consequently, one could expect a larger investment threshold. However, the effect of the time-to-build on

Table 10 Approximate values for thresholds ζ_n, and the probability of attaining the hysteresis region after n periods of time, for different values of the drift μ. The remaining parameters are as in Table 1

n	$\mu = -0.01$		$\mu = 0$		$\mu = 0.01$	
	ζ_n	$Prob$	ζ_n	$Prob$	ζ_n	$Prob$
0.5	1.501	8.557×10^{-8}	1.519	3.421×10^{-11}	1.535	3.502×10^{-16}
1	1.493	0.150×10^{-3}	1.508	2.803×10^{-6}	1.510	8.912×10^{-8}
3	1.349	0.030	1.418	0.006	1.423	0.002

the investment threshold is not so straightforward because, as explained above, the expected value of the investment increases, at least for small values of price. Additionally, in our case, the firm can invest in one of the two projects: it invests in project 1 for large values of revenue and in project 2 for smaller values of revenue. Thus, as the perpetual value of investment in this case takes into account the probability of having either larger or smaller revenues in $\tau + n$, the threshold decreases with n. A similar behaviour was already reported by Bar-Ilan and Strange [1]. The authors of the previous paper consider the option to invest in a single project followed by an exit option and find a similar reasoning for the investment threshold behaviour. This result is potentiated by the exit option, since it bounds the possible losses, which makes uncertainty less harmful for the firm. The opposite behaviour of the investment threshold is also described when authors do not consider the exit option (see, for instance, Proposition 3 in Nunes and Pimentel [19]).

6.1 Sojourn in the hysteresis region

When we consider the time-to-build, it may happen that the firm starts production in the hysteresis region, in case it exists. As stated in Proposition 3, we proved that investment in the hysteresis region is never optimal. Moreover, from Fig. 4, it is clear that the hysteresis region is never attained due to a continuous decrease of the revenue. Hence, we present a numerical study concerning the investment threshold and the probability of entering the hysteresis region at the moment that production begins, as a function of the time-to-build, the drift and the volatility of the price process.

We consider the values of the parameters in Table 1 and, consequently, the Set of Conditions 1 holds true, meaning that the hysteresis region exists. For this section, the numerical examples were computed using the Monte-Carlo simulations for Eq. (27). Thus, although the thresholds can slightly change from simulation to simulation, they allow us to confidently describe their qualitative behaviour in the sensitivity analysis.

Based on Table 10, we can conclude that the investment threshold increases with μ. In this case, the investment threshold increases because the firm wants to ensure that invests in mode 1. This result is opposite to the one found by Nunes and Pimentel [19], where the threshold decreases with μ. This is because in the latter paper the authors consider a single project. We can also observe that the probability that the firm starts producing in the hysteresis region increases with the time-to-build and decreases with μ.

A similar analysis can be done varying the volatility (see Table 11). We can see that the investment threshold is monotonically increasing with the volatility for all values of n. Some authors, like Bar-Ilan and Strange [1] and Nunes and Pimentel [19], found that the monotony of the investment threshold with changing the volatility depends on the size of the time-to-build n. Such a behaviour was not found in our simulations. Our numerical findings suggest

Table 11 Approximate values for thresholds ζ_n, and the probability of attaining the hysteresis region after n periods of time, for different values of σ. The remaining parameters are as in Table 1

n	$\sigma = 0.1$		$\sigma = 0.2$		$\sigma = 0.25$	
	ζ_n	$Prob$	ζ_n	$Prob$	ζ_n	$Prob$
0.5	1.298	2.479×10^{-10}	1.519	3.421×10^{-11}	1.625	5.336×10^{-11}
1	1.290	7.989×10^{-6}	1.508	2.803×10^{-6}	1.614	3.487×10^{-6}
3	1.247	0.010	1.418	0.006	1.486	0.007

that the investment threshold increases with the uncertainty, and this holds for all time lags that we have considered (ranging from 0.5 up to 3). Therefore, the firm invests in a larger threshold in order to avoid the negative losses. On the other hand, and similarly to Bar-Ilan and Strange [1], we see that uncertainty has a smaller effect on the investment thresholds when the size of the investment lags increase. Moreover, we can easily see that changing the volatility does not result in a significant change in the probability that the firm starts producing in the hysteresis region.

These results answer the second part of RQ4 by showing that the probability of investing in the hysteresis region is small, especially for small values of n. Nevertheless, this event may happen, particularly when the drift of the price is negative and the volatility is small, and, consequently, the hysteresis region should be taken into account in the analysis of the investment strategy. Moreover, it is also relevant to assess: (i) the probability that the firm will resume production at a positive profit, and (ii) the (expected) sojourn time in this region.

In order to study these two points, we consider the parameters as the ones set in Table 1, and assume that the current value of the price process, p, is one of the following values:

$$p_1 = P_{1x} + 0.15h, \quad p_2 = P_{1x} + 0.5h, \quad p_3 = P_{1x} + 0.85h, \quad h = \frac{P_h - P_{1x}}{2}. \tag{28}$$

Thus, p_1 is a point close to the exit threshold, p_2 is half-way in the hysteresis region, and p_3 is close to the threshold P_h, where the firm leaves the hysteresis region and start producing with a larger revenue in project 2. In order to study (i), we note that since the revenue follows a geometric Brownian motion, such probability has the following expression

$$Pr_i = Pr\left\{\tau_{P_{1x}} > \tau_{P_h} | P_0 = p_i\right\} = \begin{cases} \dfrac{\left(\frac{p_i}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}} - 1}{\left(\frac{P_h}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}} - 1}, & \frac{\sigma^2}{2} - \mu > 0 \\ \dfrac{1 - \left(\frac{p_i}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}}}{1 - \left(\frac{P_h}{P_{1x}}\right)^{1-\frac{2\mu}{\sigma^2}}}, & \frac{\sigma^2}{2} - \mu < 0 \end{cases}, \tag{29}$$

for $i = 1, 2, 3$, where $\tau_{P_{1x}}$ is the exit time (from project 1) and τ_{P_h} is the time at which the firm leaves the hysteresis region, and switches to project 2. To analyse point (ii), we use the following expression for the expected time that the firm stays in the hysteresis region:

$$Est_i = E[\min\{\tau_{P_{1x}}, \tau_{P_h}\}] = \begin{cases} \frac{1}{\frac{\sigma^2}{2} - \mu} \left[\log \frac{p_i}{P_{1x}} - \log \frac{P_{1x}}{P_h} Pr_i\right], & \frac{\sigma^2}{2} - \mu > 0 \\ \infty, & \frac{\sigma^2}{2} - \mu < 0 \end{cases}. \tag{30}$$

The expressions for these quantities can be found, for instance, in Section of 15.3.6 Karlin and Taylor [12].

Table 12 Impact of the sojourn in hysteresis as a function of μ when $\sigma = 0.20$

μ	Pr_1	Pr_2	Pr_3	Est_1	Est_2	Est_3
−0.010	0.146	0.492	0.846	0.056	0.107	0.053
0.000	**0.150**	**0.500**	**0.850**	**0.039**	**0.075**	**0.037**
0.010	0.153	0.506	0.853	0.029	0.055	0.028

Table 13 Impact of the sojourn in hysteresis as a function of σ when $\mu = 0.01$

σ	P_{1x}	P_h	Pr_1	Pr_2	Pr_3	Est_1	Est_2	Est_3
0.200	0.440	0.483	0.153	0.506	0.853	0.029	0.055	0.028
0.300	0.310	0.352	0.152	0.504	0.852	0.024	0.045	0.022
0.500	0.167	0.204	0.151	0.502	0.851	0.021	0.039	0.010

In Table 12, we study (i) and (ii) as functions of the diffusion parameter μ. We can see that all the probabilities of leaving the hysteresis region by resuming production in mode 2 increase with μ. On the contrary, the expected sojourn time in the hysteresis region decreases because the size of the hysteresis region is also decreasing. Furthermore, for fixed μ, $Pr_1 < Pr_2 < Pr_3$ and $Est_1 > Est_2 > Est_3$. Since we are considering values of the process closer to the threshold P_h, it becomes more likely to leave the hysteresis region by hitting this bound than by exiting the market. Additionally, the expected sojourn time in hysteresis decreases when the price p gets closer to P_h. These results are not surprising, because as we increase the drift, it is more likely that the revenue increases, and, therefore, the firm will leave the hysteresis region earlier and will start producing in project 2.

In Table 13, we analyse (i) and (ii) as functions of the volatility. We also add information regarding the thresholds P_{1x}, P_h, which allows us to understand the results better. As we saw in Sect. 5.1.1, the two thresholds and the size of the hysteresis region decrease with the volatility. The probabilities that the firm leaves the hysteresis region by switching to project 2 slightly decrease with the volatility. Additionally, the expected time in this region decreases. This suggests that the firm takes a decision of leaving the hysteresis region sooner with increasing volatility. Looking at the probabilities, it is likely that the firm leaves this region by abandoning the market, which is an interesting result, especially in view of the decreasing thresholds.

7 Conclusions

In this paper, we study the investment problem of a firm that, upon investment, may switch between two projects, one being larger than the other. In both alternative projects there is the option to exit. Following the usual approach, we address the problem in a backward way, solving first the switching problem and then the investment problem. Finally, we analyse the impact of the time-to-build in our model. Allowing for multiple switches and exit from both projects raised the research questions RQ1-RQ4, defined in Sect. 2. In a nutshell, the most relevant conclusions of the paper are in fact the answers to these questions.

Regarding RQ1, we found that, similarly to Décamps et al. [4], for some values of the parameters, the investment region may be disconnected, meaning that there is an interval of small values of the price (γ_1, γ_2), where the firm invests in the smaller scale project, and for large values of the price ($p > \gamma_3$), it invests in the large scale project. However, for prices in the set $(0, \gamma_1) \cup (\gamma_2, \gamma_3)$, the firm does not invest and waits to have more information. But, contrary to Décamps et al. [4], this inaction region vanishes for large values of the volatility and, in these cases, we have a trigger investment strategy. Our analysis shows that this is mainly due to the exit option that is not included in the model of Décamps et al. [4].

RQ2 concerns the behaviour of the investment threshold with the uncertainty parameters. We show that the exit option influences the behaviour of the smallest investment threshold (γ_1 or γ_3, depending on the choice of parameters), which is non-monotonic with the drift of the price process. On the other hand, the three investment thresholds are increasing with the volatility.

RQ3 addresses the optimality of the switching strategy. We show that there are only two optimal switching strategies, the no-downgrading and the hysteresis strategies, being optimal one or the other depending on the parameters. When the hysteresis strategy is optimal, the firm may decide to produce at a loss rather than exit or switching project. This inaction region, which we name as hysteresis region, serves as buffer for more irreversible or expensive actions. This region is never attained through a continuous movement of the price, but it can be reached in projects that start producing after a given time lag.

Finally, in RQ4, the impact of the time-to-build in the investment model is discussed. By adding the possibility of time-to-build, we are able to understand the role of the hysteresis region better, as this region cannot be attained by a continuous change in the price process after the investment takes place. We show that it is more likely that the firm will start production in the hysteresis region for large values of time-to-build, negative values of the drift and small values of the volatility of the price process. We also observed that the expected sojourn time in the hysteresis region decreases with the drift and with the volatility. In this analysis, we obtained some interesting results concerning the investment thresholds. Namely, we show that the investment threshold is increasing with the drift and the volatility. These results are different from the ones presented in Bar-Ilan and Strange [1]. Since there are many features that are different between our model and the one presented in Bar-Ilan and Strange [1], we relegate further analysis for future work. We highlight that, in the aforementioned paper, the authors consider a single project and the investment cost is spent when the construction process (time-to-build) is finished (and not when the investment decision is made). This is a fundamental difference between the two models because the projects become more expensive when the time-to-build increases, if the firm has to spend the investment cost immediately after the investment decision.

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A Parameters and thresholds

In this appendix, we present the smooth pasting conditions for the optimisation problems (5) and (6), which allows to derive the constant terms and the thresholds.

A.1 Switching problem (5)

As discussed in Sect. 5, according to the relationship between the parameters, there are two optimal strategies. A technical analysis of a similar switching problem can be seen in Zervos et al. [22].

A.1.1 No-downgrading strategy

For the no-downgrading strategy, the smooth-fit conditions applied to the thresholds P_{1x}, P_{2x} and P_{21} are

$$0 = AP_{1x}^{d_1} + \frac{\alpha_1}{r-\mu}P_{1x} - \frac{\beta_1}{r} + K_x \tag{31}$$

$$0 = d_1AP_{1x}^{d_1} + \frac{\alpha_1}{r-\mu}P_{1x} \tag{32}$$

$$0 = CP_{2x}^{d_1} + DP_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} - \frac{\beta_2}{r} + K_x \tag{33}$$

$$0 = Cd_1P_{2x}^{d_1} + Dd_2P_{2x}^{d_2} + \frac{\alpha_2}{r-\mu}P_{2x} \tag{34}$$

$$0 = (C-A)P_{21}^{d_1} + DP_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} - \frac{\beta_2-\beta_1}{r} + K_{21} \tag{35}$$

$$0 = (C-A)d_1P_{21}^{d_1} + Dd_2P_{21}^{d_2} + \frac{\alpha_2-\alpha_1}{r-\mu}P_{21} \tag{36}$$

Solving equations (31)–(32) allow us to get

$$A = -\frac{\alpha_1}{d_1(r-\mu)}P_{1x}^{1-d_1} \tag{37}$$

$$P_{1x} = \frac{d_2-1}{\alpha_1 d_2}(\beta_1 - rK_x) = \delta. \tag{38}$$

Computing Eq. (34), minus d_1 multiplied by Eq. (33), leads to:

$$D = -\frac{d_1P_{2x}^{-d_2}}{(d_2-d_1)r}\left[\frac{-d_2\alpha_2}{d_2-1}P_{2x} + (\beta_2 - rK_x)\right]. \tag{39}$$

Performing (34) minus d_2 multiplied by (33)

$$C = -\frac{d_2P_{2x}^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1\alpha_2}{d_1-1}P_{2x} + (-\beta_2 + rK_x)\right]. \tag{40}$$

Calculating [(36)−d_1(35)]

$$D = -\frac{d_1P_{21}^{-d_2}}{(d_2-d_1)r}\left[\frac{d_2(\alpha_1-\alpha_2)}{d_2-1}P_{21} + (\beta_2 - \beta_1 - rK_{21})\right]. \tag{41}$$

Similarly, Eq. (36) and Eq. (35) multiplied d_1 can be simplified allowing us to get

$$C - A = -\frac{d_2 P_{21}^{-d_1}}{(d_2 - d_1)r}\left[-\frac{d_1(\alpha_1 - \alpha_2)}{d_1 - 1}P_{21} - (\beta_2 - \beta_1 - rK_{21})\right]. \tag{42}$$

Taking into account that the parameter D is given by the expressions (39),(41) and C by the expressions (37),(40), (42), we can find expressions for boundary points P_{21} and P_{2x}. In fact, $P_{21} > P_{2x}$ satisfy the system of equations

$$G_1(P_{21}, P_{2x}) := P_{21}^{-d_1}\left[\frac{(\alpha_2 - \alpha_1)(1 - d_2)}{r - \mu}P_{21} + d_2\left(\frac{\beta_2 - \beta_1}{r} - K_{21}\right)\right] - A(d_1 - d_2) - P_{2x}^{-d_1}\left[\frac{\alpha_2(1 - d_2)}{r - \mu}P_{2x} + d_2\left(\frac{\beta_2}{r} - K_x\right)\right] = 0 \tag{43}$$

$$G_2(P_{21}, P_{2x}) := P_{2x}^{-d_2}\left[\frac{\alpha_2(1 - d_1)}{r - \mu}P_{2,ex} + d_1\left(\frac{\beta_2}{r} - K_x\right)\right] - P_{21}^{-d_2}\left[\frac{(\alpha_2 - \alpha_1)(1 - d_1)}{r - \mu}P_{21} + d_1\left(\frac{\beta_2 - \beta_1}{r} - K_{21}\right)\right] = 0. \tag{44}$$

A.1.2 Hysteresis strategy

Applying the smooth-fit conditions to the thresholds P_{2x}, P_{1x}, P_h, P_{12} and P_{21} we get the following system of equations:

$$0 = EP_{1x}^{d_1} + FP_{1x}^{d_2} + \frac{\alpha_1}{r - \mu}P_{1x} - \frac{\beta_1}{r} + K_x \tag{45}$$

$$0 = d_1 EP_{1x}^{d_1} + d_2 FP_{1x}^{d_2} + \frac{\alpha_1}{r - \mu}P_{1x} \tag{46}$$

$$0 = (E - C)P_h^{d_1} + (F - D)P_h^{d_2} + \frac{\alpha_1 - \alpha_2}{r - \mu}P_h - \frac{\beta_1 - \beta_2}{r} + K_{12} \tag{47}$$

$$0 = d_1(E - C)P_h^{d_1} + d_2(F - D)P_h^{d_2} + \frac{\alpha_1 - \alpha_2}{r - \mu}P_h \tag{48}$$

$$0 = (C - A)P_{12}^{d_1} + DP_{12}^{d_2} + \frac{\alpha_2 - \alpha_1}{r - \mu}P_{12} - \frac{\beta_2 - \beta_1}{r} - K_{12} \tag{49}$$

$$0 = d_1(C - A)P_{12}^{d_1} + d_2 DP_{12}^{d_2} + \frac{\alpha_2 - \alpha_1}{r - \mu}P_{12} \tag{50}$$

$$0 = CP_{2x}^{d_1} + DP_{2x}^{d_2} + \frac{\alpha_2}{r - \mu}P_{2x} - \frac{\beta_2}{r} + K_x \tag{51}$$

$$0 = d_1 CP_{2x}^{d_1} + d_2 DP_{2x}^{d_2} + \frac{\alpha_2}{r - \mu}P_{2x} \tag{52}$$

$$0 = (C - A)P_{21}^{d_1} + DP_{21}^{d_2} + \frac{\alpha_2 - \alpha_1}{r - \mu}P_{21} - \frac{\beta_2 - \beta_1}{r} + K_{21} \tag{53}$$

$$0 = d_1(C - A)P_{21}^{d_1} + d_2 DP_{21}^{d_2} + \frac{\alpha_2 - \alpha_1}{r - \mu}P_{21} \tag{54}$$

Taking into account the relationships:

$$r = -\frac{\sigma^2}{2}d_1 d_2, \quad \mu = \frac{\sigma^2}{2}(1 - d_1 - d_2) \quad \text{and} \quad r - \mu = -\frac{\sigma^2}{2}(1 - d_1)(1 - d_2)$$

we can obtain

$$\frac{r(d_2-1)}{d_2(r-\mu)} = \frac{d_1}{d_1-1} \qquad \frac{r(d_1-1)}{d_1(r-\mu)} = \frac{d_2}{d_2-1}.$$

Computing d_1 (49) − (50) we get

$$D = -\frac{d_1 P_{12}^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{12} + (\beta_1-\beta_2-rK_{12})\right]. \tag{55}$$

Simplifying the Eqs. (50) and d_2 multiplied by (49), we obtain

$$C - A = -\frac{d_2 P_{12}^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1(\alpha_2-\alpha_1)}{d_1-1}P_{12} + (\beta_1-\beta_2-rK_{12})\right]. \tag{56}$$

Analysing Eqs. (53), (54), we get

$$D = -\frac{d_1 P_{21}^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{21} + (\beta_1-\beta_2+rK_{21})\right] \tag{57}$$

$$C - A = -\frac{d_2 P_{21}^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1(\alpha_2-\alpha_1)}{d_1-1}P_{21} + (\beta_1-\beta_2+rK_{21})\right]. \tag{58}$$

Since we have two expressions for D and to $C - A$, we are able to define the equations that allow us to compute the thresholds P_{21} and P_{12}

$$\begin{aligned}
&P_{21}^{-d_1}\left[\frac{d_1(\alpha_2-\alpha_1)}{d_1-1}P_{21} + (\beta_1-\beta_2+rK_{21})\right] \\
&- P_{12}^{-d_1}\left[\frac{d_1(\alpha_2-\alpha_1)}{d_1-1}P_{12} + (\beta_1-\beta_2-rK_{12})\right] = 0 \\
&P_{21}^{-d_2}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{21} + (\beta_1-\beta_2+rK_{21})\right] \\
&- P_{12}^{-d_2}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{12} + (\beta_1-\beta_2-rK_{12})\right] = 0.
\end{aligned}$$

Taking into account Eqs. (45), (46) as well as (45) and (46), we obtain

$$F = -\frac{d_1 P_{1x}^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_1}{d_2-1}P_{1x} + (-\beta_1+rK_x)\right] \tag{59}$$

$$F - D = -\frac{d_1 P_h^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2(\alpha_1-\alpha_2)}{d_2-1}P_h + (-\beta_1+\beta_2+rK_{12})\right] \tag{60}$$

Combining Eqs. (59)–(60) with (55)

$$\begin{aligned}
0 = {} & P_{1x}^{-d_2}\left[\frac{d_2\alpha_1}{d_2-1}P_{1x} + (-\beta_1+rK_x)\right] - P_h^{-d_2}\left[\frac{d_2(\alpha_1-\alpha_2)}{d_2-1}P_h + (-\beta_1+\beta_2+rK_{12})\right] \\
& - P_{12}^{-d_2}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{12} + (\beta_1-\beta_2-rK_{12})\right]
\end{aligned} \tag{61}$$

A different expression for D can be obtained solving the equations (51) and (52):

$$D = -\frac{d_1 P_{2x}^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_2}{d_2-1}P_{2x} + (-\beta_2+rK_x)\right]. \tag{62}$$

Combining (62) using (55) multiplied by -1, we get

$$P_{2x}^{-d_2}\left[\frac{d_2\alpha_2}{d_2-1}P_{2x}+(-\beta_2+rK_x)\right]-P_{12}^{-d_2}\left[\frac{d_2(\alpha_2-\alpha_1)}{d_2-1}P_{12}+(\beta_1-\beta_2-rK_{12})\right]=0. \tag{63}$$

Solving Eqs. (45), (46), and (51), (52), we obtain

$$E=-\frac{d_2P_{1x}^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1\alpha_1}{d_1-1}P_{1x}+(-\beta_1+rK_x)\right] \tag{64}$$

$$C=-\frac{d_2P_{2x}^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1\alpha_2}{d_1-1}P_{2x}+(-\beta_2+rK_x)\right] \tag{65}$$

From (47) and (48), we can compute

$$E-C=-\frac{d_2P_h^{-d_1}}{(d_2-d_1)r}\left[\frac{d_1(\alpha_1-\alpha_2)}{d_1-1}P_h+(-\beta_1+\beta_2+rK_{12})\right]. \tag{66}$$

Therefore,

$$P_{1x}^{-d_1}\left[\frac{d_1\alpha_1}{d_1-1}P_{1x}+(-\beta_1+rK_x)\right]-P_{2x}^{-d_1}\left[\frac{d_1\alpha_2}{d_1-1}P_{2x}+(-\beta_2+rK_x)\right]$$
$$-P_h^{-d_1}\left[\frac{d_1(\alpha_1-\alpha_2)}{d_1-1}P_h+(-\beta_1+\beta_2+rK_{12})\right]=0 \tag{67}$$

Equations (61)–(63)–(67) allow us to recover thresholds P_{1x}, P_{2x} and P_h.

A.1.3 Constants $K_{12}^{\dagger}$ and $K_{21}^{\dagger}$

Looking at the definition of the functions G_1 and G_2 defined in (43) and (44), we know that these functions depend on K_{21}. To highlight such a dependence, we write $G_1(P_{21}, P_{2x}) \equiv G_1(P_{21}, P_{2x}; K_{21})$ and $G_2(P_{21}, P_{2x}) \equiv G_2(P_{21}, P_{2x}; K_{21})$. Then, we proceed as Zervos et al. [22], to find the bounds $K_{12}^{\dagger}$ and $K_{21}^{\dagger}$.

The $K_{21}^{\dagger}$ is such that there is a unique solution $(x, y, k) = (x, y, K_{21}^{\dagger})$, with $y > x$, to the system of equations

$$G_1(x, y, k)=0, \quad G_2(x, y, k)=0, \quad G_1(\delta, y, k)=0, \tag{68}$$

where δ is defined in (38). The bound for K_{12} is

$$K_{12}^{\dagger}=-K_{21}+\frac{\hat{x}^{d_2}}{r}\left[P_{21}^{-d}\left(\frac{(\alpha_1-\alpha_2)d_2}{d_2-1}P_{21}-(\beta_1-\beta_2+rK_{21})\right)\right.$$
$$\left.-\hat{x}^{-d}\left(\frac{(\alpha_1-\alpha_2)d_2}{d_2-1}\hat{x}-(\beta_1-\beta_2+rK_{21})\right)\right], \tag{69}$$

where $\hat{x} \in [P_{2x}, P_{21}]$ is a solution to:

$$(\alpha_2-\alpha_1)x\left[\frac{d_1}{d_1-1}-\frac{d_2}{d_2-1}\right]+x^{d_1}P_{21}^{-d_1}\left[\frac{(\alpha_1-\alpha_2)d_1}{d_1-1}P_{21}-(\beta_1-\beta_2+rK_{21})\right]$$
$$-x^{d_2}P_{21}^{-d_2}\left[\frac{(\alpha_1-\alpha_2)d_2}{d_2-1}P_{21}-(\beta_1-\beta_2+rK_{21})\right]=0. \tag{70}$$

Note that $K_{21}^{\dagger}$ is independent of K_{12} and K_{21}, but $K_{12}^{\dagger}$ depends on K_{21}.

A.2 Investment problem (6)

In this section, we will present the smooth-pasting conditions to find the parameters and thresholds associated to the Investment Problem defined in Eq. (6).

A.2.1 $K_2^+ \geq K_2$ and $K_1 \geq K_1^-$

We start by noticing that the function W_2 can be written as

$$u(p) = \begin{cases} B_1 p^{d_2} & p \in [0, \gamma_1) \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu} p - \frac{\beta_2}{r} - K_2 & p \in [\gamma_1, \gamma_2] \\ A_1 p^{d_1} + B_2 p^{d_2} & p \in (\gamma_2, \gamma_3) \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r} - K_1 & p \in [\gamma_3, +\infty). \end{cases} \tag{71}$$

Using the smooth-fit conditions, the parameters B_1, A_1 and A_2, and the thresholds γ_1, γ_2 and γ_3 satisfy the following equations:

$$0 = (B_1 - D)\gamma_1^{d_2} - C\gamma_1^{d_1} - \frac{\alpha_2}{r-\mu}\gamma_1 + \frac{\beta_2}{r} + K_2$$
$$0 = (B_1 - D)d_2\gamma_1^{d_2} - Cd_1\gamma_1^{d_1} - \frac{\alpha_2}{r-\mu}\gamma_1$$
$$0 = (C - A_1)\gamma_2^{d_1} + (D - B_2)\gamma_2^{d_2} + \frac{\alpha_2}{r-\mu}\gamma_2 - \frac{\beta_2}{r} - K_2$$
$$0 = (C - A_1)d_1\gamma_2^{d_1} + (D - B_2)d_2\gamma_2^{d_2} + \frac{\alpha_2}{r-\mu}\gamma_2$$
$$0 = (A - A_1)\gamma_3^{d_1} - B_2\gamma_3^{d_2} + \frac{\alpha_1}{r-\mu}\gamma_3 - \frac{\beta_1}{r} - K_1$$
$$0 = (A - A_1)d_1\gamma_3^{d_1} - B_2 d_2\gamma_3^{d_2} + \frac{\alpha_1}{r-\mu}\gamma_3$$

Solving these equations we can get the following expressions:

$$B_1 = D + \frac{d_1\gamma_1^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_1 - (\beta_2 + rK_2)\right]$$
$$B_2 = D + \frac{d_1\gamma_2^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_2 - (\beta_2 + rK_2)\right]$$
$$= \frac{d_1\gamma_3^{-d_2}}{(d_1-d_2)r}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma_3 - (\beta_1 + rK_1)\right]$$
$$A_1 = C - \frac{d_2\gamma_2^{-d_1}}{(d_1-d_2)r}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma_2 - (\beta_2 + rK_2)\right]$$
$$= A - \frac{d_2\gamma_3^{-d_1}}{(d_1-d_2)r}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma_3 - (\beta_1 + rK_1)\right].$$

The threshold γ_1 is a solution to the following equation:

$$\phi(\gamma_1, K_2) := C(d_1 - d_2)\gamma_1^{d_1} + (1 - d_2)\frac{\alpha_2}{r-\mu}\gamma_1 + d_2\left(\frac{\beta_2}{r} + K_2\right) = 0$$

and the thresholds γ_2 and γ_3 are a solution to the system of equations

$$D(d_1-d_2)r + d_1\gamma_2^{-d_2}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_2 - (\beta_2 + rK_2)\right]$$
$$- d_1\gamma_3^{-d_2}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma_3 - (\beta_1 + rK_1)\right] = 0$$
$$(C-A)(d_1-d_2)r - d_2\gamma_2^{-d_1}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma_2 - (\beta_2 + rK_2)\right]$$
$$+ d_2\gamma_3^{-d_1}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma_3 - (\beta_1 + rK_1)\right] = 0.$$

A.2.2 $K_2^+ < K_2$ or $K_1 < K_1^-$

Since $z_2 < \gamma_3 < P_{2,1}$, using the smooth-pasting conditions, we get the following expressions

$$B_2\gamma^{d_2} = A\gamma^{d_1} + \frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r} - K_1$$
$$d_2B_2\gamma^{d_2} = d_1A\gamma^{d_1} + \frac{\alpha_1}{r-\mu}\gamma$$

We conclude, that

$$B_1 = -\frac{d_1\gamma^{-d_2}}{(d_2-d_1)r}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma - (\beta_1 + rK_1)\right] \tag{72}$$

and γ_3 satisfies the equation

$$(d_2-d_1)A\gamma_3^{d_1} + (d_2-1)\frac{\alpha_1}{r-\mu}\gamma_3 - d_2\left(\frac{\beta_1}{r} + K_1\right) = 0. \tag{73}$$

A.2.3 The bounds K_1^+, K_1^-, K_2^+ and K_2^-

Let us assume that the value function is given by W_2. Then, for values of revenue $p \in (\gamma_1, \gamma_2)$ the firm invests in the project 2. It is straightforward that the investment is not optimal if $v_2^*(p) - K_2 < 0$. Thus, $\gamma_1 > \hat{p}$, where $v_2^*(\hat{p}) - K_2 = 0$. Furthermore, since $K_2 > -K_x$, it is not optimal to invest in project 2 for values of $p \leq P_{2x}$. On the other hand, since γ_1 triggers the investment in project 2, $\gamma_1 < z_2$, where is defined in Proposition 2. Additionally, since $K_2 + K_{21} > K_1$, it will be never optimal to invest in project 2 for values of revenue larger than P_{21}. Thus $\gamma_1 < P_{21}$.

Let K_2^- and K_2^+ be respectively the upper and lower bounds for K_2. Thus, K_2^+ is obtained as solution to the equation

$$K_2^+ = \max\{K_2 : \phi(\gamma_1, K_2) = 0, \gamma_1 \in [P_{2x}, P_{21}]\}, \tag{74}$$

and K_2^- is obtained as solution to the equation

$$K_2^- = \min\{K_2 : \phi(\gamma_1, K_2) = 0, \gamma_1 \in [P_{2x}, P_{21}]\}, \tag{75}$$

where ϕ is defined in Sect. A.2.1. These two equations have to be solved in K_2 because all the remaining parameters are fixed. The parameter K_2^- can be explicitly computed because the pair $(\gamma_1, K_2) = (P_{2x}, -K_x)$ solves the equation $\phi(\gamma_1, K_2) = 0$. Since we are imposing that $K_2 > -K_x$, then $K_2^- = -K_x$.

Fix now $K_2 \in (K_2^-, K_2^+)$, then γ_1 can be obtained following Sect. A.2.1. Considering the structure of W_2, one has that $0 < \gamma_1 < \gamma_2 < \gamma_3$. Thus, following the same line of reasoning, one can obtain K_1^- and K_1^+ fixing $\gamma_2 = \gamma_1$ and $\gamma_2 = \gamma_3$ and solving the system of equations obtained at the end of Sect. A.2.1, in K_1.

Considering $\gamma_2 = \gamma_3 = \gamma$, we get

$$D(d_1 - d_2)r + d_1\gamma^{-d_2}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma - (\beta_2 + rK_2)\right]$$
$$- d_1\gamma^{-d_2}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma - (\beta_1 + rK_1)\right] = 0$$
$$(C - A)(d_1 - d_2)r - d_2\gamma^{-d_1}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma - (\beta_2 + rK_2)\right]$$
$$+ d_2\gamma^{-d_1}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma - (\beta_1 + rK_1)\right] = 0.$$

One can easily see that $(\gamma, K_1) = (P_{21}, K_2 + K_{21})$ solves the system of equations. As we are imposing the condition $K_1 < K_2 + K_{21}$, this implies that $K_1^+ = K_2 + K_{21}$.

Fix now $\gamma_2 = \gamma_1$. Then, the lower bound can be found solving the system

$$D(d_1 - d_2)r + d_1\gamma_1^{-d_2}\left[\frac{d_2\alpha_2}{(d_2-1)}\gamma_1 - (\beta_2 + rK_2)\right]$$
$$- d_1\gamma_3^{-d_2}\left[\frac{d_2\alpha_1}{(d_2-1)}\gamma_3 - (\beta_1 + rK_1)\right] = 0$$
$$(C - A)(d_1 - d_2)r - d_2\gamma_1^{-d_1}\left[\frac{d_1\alpha_2}{(d_1-1)}\gamma_1 - (\beta_2 + rK_2)\right]$$
$$+ d_2\gamma_3^{-d_1}\left[\frac{d_1\alpha_1}{(d_1-1)}\gamma_3 - (\beta_1 + rK_1)\right] = 0,$$

in γ_3 and K_1.

B Proofs

B.1 Proof of Proposition 2

We start by deriving $u^* = \max(v_1(p) - K_1, v_2(p) - K_2)$, with v_1 and v_2 defined in Proposition 1.

Let us assume, without loss of generality, that $K_1 = 0$ and $K_2 = 0$. Then, it is straightforward that

$$v_1(p) = v_2(p) \Leftrightarrow p \in (0, P_{2x}) \cup (P_{12}, P_{21}). \tag{76}$$

For $x \in (P_{2x}, P_{1x})$, it is also trivial that $v_2(p) > v_1(p)$. Taking into account that

$$v_2(P_{1x}) > v_1(P_{1x}) = -K_x \quad \text{and} \quad v_2(P_h) > v_1(P_h) = v_2(P_h) - K_{12}, \tag{77}$$

and the monotony of v_2 and v_1 we can conclude that $v_2(p) > v_1(p)$ for $p \in (P_{1x}, P_h)$. Finally, for $p \in (P_h, P_{12})$, $v_2(p) > v_1(p) = v_2(p) - K_{12}$.

In fact, due to the continuity of v_1 and v_2, we can conclude that there is a unique point $z_2 \in (P_{12}, P_{21})$ such that $v_1(z_2) = v_2(z_2)$. Additionally, from the convexity of v_2 in (P_{12}, P_{21}) we get that $v_2(p) > v_1(p)$ for $p < z_2$ and $v_2(p) < v_1(p)$ for $p > z_2$.

Given the continuity of $v_1 - K_1$ in K_1, we know that $z_2(K_1)$ is increasing and

$$lim_{K_1 \to K_2 + K_{21}} z_2(K_1) = P_{21}. \tag{78}$$

Additionally, when we consider $0 < K_1 < K_{12}$ then $v_1(p) - K_1 < v_2(p)$ for $p \in (0, z_2)$ and $v_1(p) - K_1 > v_2(p)$ when $p > z_2$. One can easily check that the analysis made is still true when we start by considering $K_1 = K_2 > 0$.

To finalise this part of the proof, one need to check what happens when we decrease K_1 taking into account that $K_2 < K_1 + K_{12}$.

Let us consider the limit case $K_2 = K_1 + K_{12}$. In this case, $v_2(p) - K_2 < v_1(p) - K_1$ for $p \in (0, P_{2x})$. Additionally, $v_2(p) - K_2 < v_1(p) - K_1$ for $p \in (P_{2x}, z_1)$ with $z_1 \in (P_{2x}, P_h)$, $v_2(p) - K_2 > v_1(p) - K_1$ for $p \in (z_1, P_h)$, $v_2(p) - K_2 = v_1(p) - K_1$ for $p \in (P_h, P_{12})$ and $v_2(p) - K_2 < v_1(p) - K_1$ when $p > P_{12}$. Therefore, in light of the continuity of $v_2 - K_2$ in k_2,, we get the result when we consider $K_1 < K_2 < K_1 + K_{12}$.

B.2 Proof of Proposition 3

Let us assume the following scenario: the set of initial parameters is such that the optimal switching strategy is the hysteresis strategy and the value function is in Proposition 1 and there is $z_1 > P_{1x}$ such that the investment threshold γ is such that $\gamma \in (P_{1x}, P_h)$. From standard real options analysis (see for instance Dixit and Pindyck [7]) we know that the smooth paste condition are given by

$$\begin{cases} B_0\gamma^{d_2} = E\gamma^{d_1} + F\gamma^{d_2} + \frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r} - K_1 \\ d_2 B_0 \gamma^{d_2 - 1} = E d_1 \gamma^{d_1 - 1} + F d_2 \gamma^{d_2 - 1} + \frac{\alpha_1}{r-\mu} \end{cases}. \tag{79}$$

One may notice that, by definition of the threshold P_{1x}, the pair $(\gamma, B_0) = (P_{1x}, 0)$ is a solution to the system (79) for $K_1 = -K_x$. Given the analysis made for the switching problem, it is known that there is a unique solution $(P_{2x}, P_{1x}, P_h, P_{12}, P_{21})$ such that $P_{2x} < P_{1x} < P_h < P_{12} < P_{21}$. Therefore, fixing E and F as defined (45) and (46), the arguments above allow us to conclude that

$$\begin{cases} 0 = E\gamma^{d_1} + F\gamma^{d_2} + \frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r} + K_x \\ 0 = E d_1 \gamma^{d_1 - 1} + F d_2 \gamma^{d_2 - 1} + \frac{\alpha_1}{r-\mu} \end{cases}.$$

has a unique solution, P_{1x}, for $0 < \gamma < P_h$.

To get our conclusions, we analyse a perturbed version of system (79), considering $K_1 = -K_x + \varepsilon$. Multiplying the first equation of system (79) by d_2 and the second one by γ, the system can be reduced to a single equation

$$m(\gamma) := (d_2 - d_1)E\gamma^{d_1} + (d_2 - 1)\frac{\alpha_1}{r-\mu}\gamma - \frac{\beta_1}{r}d_2 + K_x d_2 - \varepsilon d_2$$

These equations have two solutions. To prove this statement, we may notice that

$$\lim_{\gamma \to 0^+} m(\gamma) = \lim_{\gamma \to +\infty} m(\gamma) = +\infty \quad \text{and} \quad m''(\gamma) = (d_2 - d_1)d_1(d_1 - 1)E\gamma^{d_1 - 2} > 0.$$

Additionally, choosing $\varepsilon = 0$, we know from Proposition 1 that there is at least one solution to that equation, which is P_{1x}. Additionally, if there is a second one is greater than P_h. This can be proved noticing that

$$m(P_h) = (d_2 - d_1)E P_h^{d_1} + (d_2 - 1)\frac{\alpha_1}{r-\mu}P_h - \frac{\beta_1}{r}d_2 + K_x d_2 - \varepsilon d_2$$

$$= (d_2 - d_1) E P_h^{d_1} + (d_2 - 1) \frac{\alpha_2}{r - \mu} P_h + \left(K_x - \frac{\beta_2}{r} - K_{12} \right) d_2 - \varepsilon d_2$$

the second equality following in light of the smooth paste conditions presented in Appendix A.1.2. Given the expression for E presented in Appendix A.1.2, we get the following

$$\begin{aligned} m(P_h) &= - \left(\frac{P_h}{P_{2x}} \right)^{d_1} \left[(d_2 - 1) \frac{\alpha_2}{r - \mu} P_h + \left(K_x - \frac{\beta_2}{r} \right) d_2 \right] + (d_2 - 1) \frac{\alpha_2}{r - \mu} P_h \\ &\quad + \left(K_x - \frac{\beta_2}{r} - K_{12} \right) d_2 - \varepsilon d_2 \\ &= \underbrace{\left(1 - \left(\frac{P_h}{P_{2x}} \right)^{d_1} \right)}_{>0} \underbrace{\left[(d_2 - 1) \frac{\alpha_2}{r - \mu} P_h + \left(K_x - \frac{\beta_2}{r} \right) d_2 \right]}_{<0} - (\varepsilon + K_{12})\, d_2 < 0 \end{aligned}$$

The sign of the second term follows from the fact that $E > 0$ because it is the value of an option.

Due to the continuity of $m(\gamma; \varepsilon) \equiv m(\gamma)$ on ε, for any $\varepsilon > 0$ there are two solutions, one that is smaller than P_{1x} and a second one that is greater than P_h. Both hypotheses contradict the possibility of investment in the hysteresis region.

C Additional figures and tables

C.1 Numerical Verification of the HJB equations in Sect. 5.2.2

In Table 6, we illustrate the behaviour of the optimal investment strategy with changing μ. We find that γ_3 is not monotonic with μ. To verify that the behaviour is not a consequence of a numerical error, we present the numerical verification of the HJB equations for the following values of μ: $\mu = -0.030$ and $\mu = 0.025$ (these are the values of μ where the monotony of γ_3 changes). For each value of μ, we present three plots since we have to compute the value functions v_1, v_2 and W. As the HJB equations are written as the maximum between three terms for the switching problem and two terms for the investment problem, all these terms must be non-positive and at least one of them must be equal to zero. Figure 13 shows the verification plots for $\mu = -0.03$ and Fig. 14 shows the verification plots for $\mu = 0.025$.

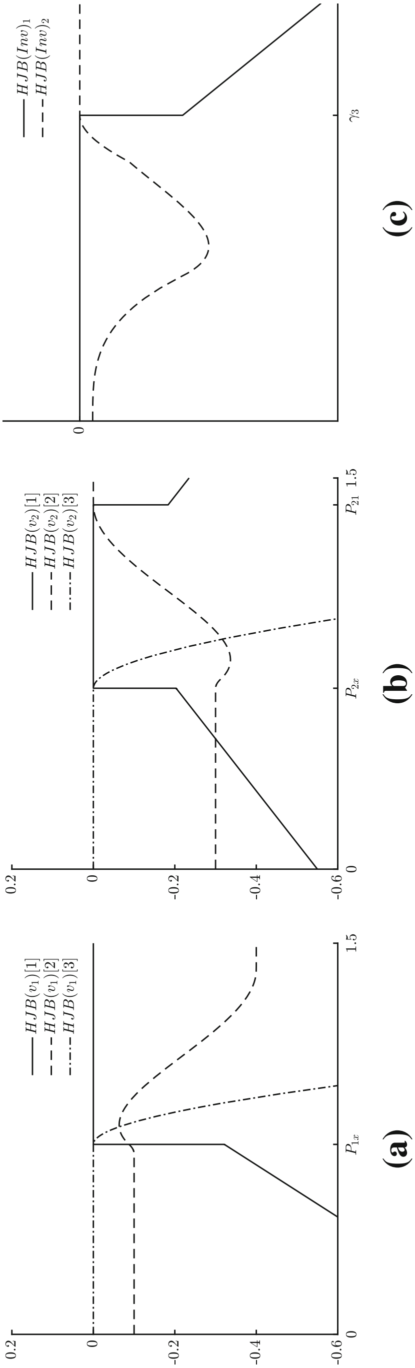

Fig. 13 $\mu = -0.030$, HJB verification for v_1 (**a**), for v_2 (**b**) and for the investment problem (**c**)

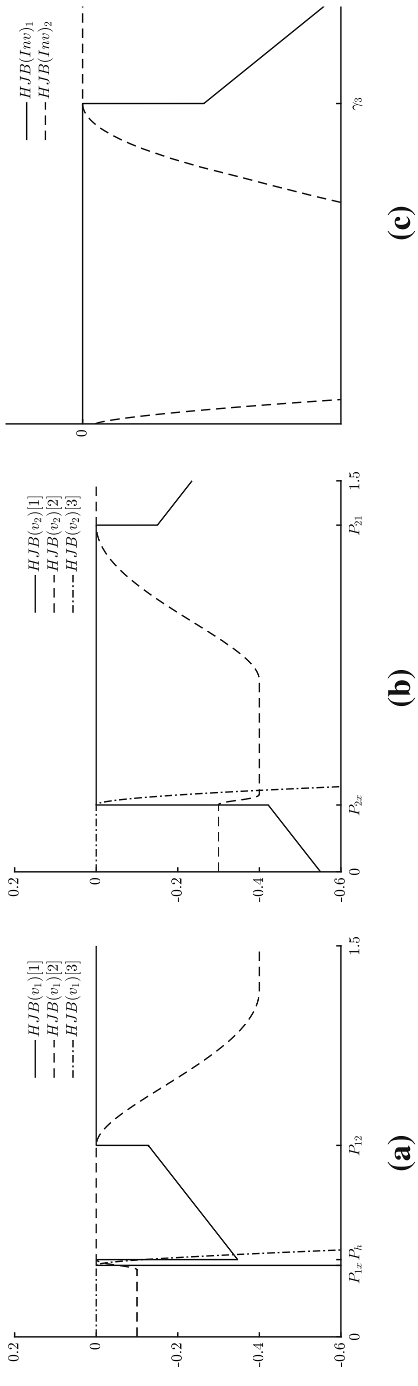

Fig. 14 Case: $\mu = 0.025$, HJB verification for v_1 (**a**), for v_2 (**b**) and for the investment problem (**c**)

C.2 Switching parameters of Sect. 5.2.1

In Table 14, we present the switching thresholds regarding the set of parameters used in Sect. 5.2.1. For the reader convenience, we provide here the parameters considered: $\mu = 0$, $r = 0.05$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 0.6$, $\beta_2 = 0.5$, $K_x = -1$, $K_{12} = 0.25$, $K_{21} = 0.5$.

Table 14 Switching thresholds for the illustration in Sect. 5.2.1

σ	P_{1x}	P_h	P_{12}	P_{2x}	P_{21}	Strategy
0.1000	0.7198	0.7782	1.0017	0.6536	1.6142	Hyst
0.1500	0.6107	0.6841	0.9400	0.5451	1.7371	Hyst
0.2000	0.5183	0.5999	0.8894	0.4550	1.8566	Hyst
0.2400	0.4556	0.5407	0.8546	0.3947	1.9514	Hyst
0.2500	0.4413	0.5270	0.8465	0.3811	1.9752	Hyst
0.3000	0.3775	0.4649	0.8091	0.3208	2.0942	Hyst

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