



Orbital stabilization of point-to-point maneuvers in underactuated mechanical systems[☆]

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ABSTRACT

The task of inducing, via continuous static state-feedback control, an asymptotically stable heteroclinic orbit in a nonlinear control system is considered in this paper. The main motivation comes from the problem of ensuring convergence to a so-called point-to-point maneuver in an underactuated mechanical system. Namely, to a smooth curve in its state-control space which is consistent with the system dynamics and connects two (linearly) stabilizable equilibrium points. The proposed method uses a particular parameterization, together with a state projection onto the maneuver as to combine two linearization techniques for this purpose: the Jacobian linearization at the equilibria on the boundaries and a transverse linearization along the orbit. This allows for the computation of stabilizing control gains offline by solving a semidefinite programming problem. The resulting nonlinear controller, which simultaneously asymptotically stabilizes both the orbit and the final equilibrium, is time-invariant, locally Lipschitz continuous, requires no switching, and has a familiar feedforward plus feedback-like structure. The method is also complemented by synchronization function-based arguments for planning such maneuvers for mechanical systems with one degree of underactuation. Numerical simulations of the non-prehensile manipulation task of a ball rolling between two points upon the “butterfly” robot demonstrates the efficacy of the synthesis.

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1. Introduction

A *point-to-point* (PtP) motion is perhaps the most fundamental of all motions in robotics: Starting from rest at a certain configuration (point), the task is to steer the system to rest at a different goal configuration. Often it can also be beneficial, or even necessary, to know a specific predetermined motion which smoothly connects the two configurations, in the form of a curve in the state-control space which is consistent with the system dynamics—a *maneuver* (Hauser & Hindman, 1995). For instance, this ensures that the controls remain within the admissible range along the nominal motion, and that neither any kinematic- nor dynamic constraints are violated along it. Knowledge of a maneuver is also especially important for an *underactuated mechanical system* (UMS) (Liu & Yu, 2013; Spong, 1998). Indeed, as an UMS has fewer independent controls (actuators) than degrees of freedom, any feasible motion must necessarily comply

with the dynamic constraints which arise due to the system's underactuation (Shiriaev, Perram, & Canudas-de Wit, 2005).

Planning such (open-loop) PtP maneuvers in an UMS, e.g. a swing-up motion of a pendulum-type system with several passive degrees of freedom, is of course a nontrivial task in itself. Suppose, however, that such a maneuver has been found. Then the next step is to design a stabilizing feedback for it. For non-feedback-linearizable systems (i.e. the vast majority) this is also a nontrivial task. The challenge again lies in the lack of actuation, which may severely limit the possible actions the controller can take. This can make reference tracking controllers less suited for this purpose, as they, often unnecessarily so, are tasked with tracking one specific trajectory (among infinitely many) along the maneuver.

For tasks which do not require a specific *timing* of the motion, one can instead design an *orbitally stabilizing feedback*: a time-invariant state-feedback controller which (asymptotically) stabilizes the set of all the states along the maneuver—its *orbit*. For a PtP maneuver, such a feedback controller is therefore equivalent to inducing an asymptotically stable *heteroclinic orbit* in the resulting autonomous closed-loop system. Namely, an invariant, one-dimensional manifold which (smoothly) connects the initial and final equilibrium points. There are some clear advantageous to such an approach: First, all solutions initialized upon the orbit asymptotically converge to the final equilibrium along the maneuver, with the behavior when evolving along it known a

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priori. Second, by invoking a reduction principle (El-Hawwary & Maggiore, 2013), the final equilibrium's asymptotic stability is ensured by the orbit's asymptotic stability. Third, the closed-loop system is time invariant.

In regard to the problem of designing such feedback, the maneuver regulation approach proposed in Hauser and Hindman (1995) is of particular interest. There, the task of stabilizing—via static state feedback—non-vanishing (i.e. equilibrium-free) orbits of feedback-linearizable systems was considered; with the approach later extended to a class of non-minimum phase systems in normal form in Al-Hiddabi and McClamroch (2002). The key idea in these papers is to convert a linear tracking controller into a controller stabilizing the orbit of a known maneuver. This is achieved by using a projection of the system states onto the maneuver, a *projection operator* as we will refer to it here, to recover the corresponding “time” to be used in the controller, thus eliminating its time dependence. The former tracking error therefore instead becomes a *transverse error*—a weighted measure of the distance from the current state to the maneuver's orbit.

It has long been known for non-trivial orbits (e.g. periodic ones) that strict contraction in the directions transverse to it is equivalent to its asymptotic stability (Borg, 1960; Hartman & Olech, 1962; Hauser & Chung, 1994; Manchester & Slotine, 2014; Urabe, 1967; Zubov, 1999). Moreover, this contraction can be determined from a specific linearization of the system dynamics along the nominal orbit (Leonov, Ponomarenko, & Smirnova, 1995), a so-called *transverse linearization* (Hauser & Chung, 1994; Manchester, 2011; Sætre & Shiriaev, 2020; Shiriaev, Freidovich, & Gusev, 2010). Since this contraction occurs on transverse hypersurfaces, only the linearization of a set of transverse coordinates of dimension one less than the dimension of the state space needs to be stabilized; a fact which has been readily used to stabilize periodic orbits in UMSs (Shiriaev et al., 2010; Surov, Shiriaev, Freidovich, Gusev, & Paramonov, 2015).

For the purpose we consider in this paper, namely the design of a continuous (orbitally) stabilizing feedback controller for PtP motions with a known maneuver, one must also take into consideration the equilibria located at the boundaries of the motion. On the one hand, this directly excludes regular transverse coordinates-based methods such as Manchester (2011) and Shiriaev et al. (2010, 2005), which would then require some form of control switching and/or orbit jumping à la those in La Hera, Freidovich, Shiriaev, and Mettin (2009) and Sellami, Mamedov, and Khusainov (2020). The ideas proposed by Hauser and Hindman (1995) in regard to maneuver regulation, on the other hand, can be modified as to also handle the equilibria, but suffers from other shortcomings: 1) the choice of projection operator is strictly determined by the tracking controller, thus excluding simpler operators, e.g., operators only depending on the configuration variables; while most importantly, 2) the requirement of a feedback-linearizable system and constant feedback gains greatly limits its applicability to stabilize (not necessarily PtP) motions of both UMSs and nonlinear dynamical systems in general.

Contributions. The main contribution of this paper is an approach that extends the applicability of the ideas in Hauser and Hindman (1995) to a larger class of dynamical systems, as well as to different types of behaviors, including point-to-point (PtP) maneuvers. The main novelty in our approach lies in the use of a specific parameterization of the maneuver, together with an operator providing a projection onto it. Roughly speaking, this allows us to merge the transverse linearization with the regular Jacobian linearization at the boundary equilibria. This, in turn, allows us to derive a (locally Lipschitz) Lyapunov function candidate for the nominal orbit as a whole. Specifically, the paper's main contributions are:

- (1) Sufficient conditions ensuring that a (locally Lipschitz continuous) feedback controller orbitally stabilizes a known PtP maneuver of a nonlinear control-affine system; see Theorem 10 in Section 4.
- (2) A constructive procedure allowing for the design of such feedback by solving a semidefinite programming problem; see Proposition 14 in Section 4.
- (3) A synchronization function-based method for planning PtP maneuvers for a class of underactuated mechanical systems; see Theorem 17 in Section 5.
- (4) Arguments facilitating the generation of orbitally stable PtP motions of a ball rolling between any two points upon the frame of the “butterfly” robot; see Proposition 20 in Section 6

Note also that, with only minor modifications, these statements can also be used to generate and orbitally stabilize (hybrid) periodic motions, or to ensure contraction toward a non-vanishing motion defined on a finite time interval.

All proofs are given in the Appendix. A statement is ended by □ if its proof is not provided.

Notation. I_n denotes the $n \times n$ identity matrix and $0_{n \times m}$ an $n \times m$ matrix of zeros, with $0_n = 0_{n \times n}$. For $S \subset \mathbb{R}^n$, $\text{int}(S)$ denotes its interior and $\text{cl}(S)$ its closure. For $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^T x}$. For some $\epsilon > 0$ and $x \in \mathbb{R}^n$ we denote $B_\epsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \epsilon\}$. For column vectors x and y , $\text{col}(x, y) := [x^T, y^T]^T$ is used. For $x, y \in \mathbb{R}^n$ we denote $\mathfrak{L}(x, y) = \{x + (y - x)\iota, \iota \in [0, 1]\}$. If $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^1 , then $Dh : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times n}$ denotes its Jacobian matrix, and if $m = 1$ then $D^2h : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ denotes its Hessian matrix. If $s \mapsto h(s)$ is differentiable at $s \in S \subseteq \mathbb{R}$, then $h'(s) = \frac{d}{ds}h(s)$. $\|\sigma(x)\| = O(\|x\|^k)$ if there exists $c > 0$ such that $\|\sigma(x)\| \leq c\|x\|^k$ as $\|x\| \rightarrow 0$. $\mathbb{M}_{>0}^n$ (resp. $\mathbb{M}_{\geq 0}^n$) denotes the set of all real, symmetric, positive (resp. semi-) definite $n \times n$ matrices, such that $R > 0_n$ if $R \in \mathbb{M}_{>0}^n$.

2. Problem formulation

Consider a nonlinear control-affine system

$$\dot{x} = f(x) + B(x)u \quad (1)$$

with state $x \in \mathbb{R}^n$ and with ($m \leq n$) controls $u \in \mathbb{R}^m$. It is assumed that both $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the columns of the full-rank matrix function $B : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, denoted $b_i(\cdot)$, are twice continuously differentiable (C^2).

Let the pair $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ correspond to an equilibrium of (1), i.e., $f(x_e) + B(x_e)u_e \equiv 0_{n \times 1}$. If we denote

$$A(x, u) := Df(x) + \sum_{i=1}^m Db_i(x)u_i, \quad (2)$$

then the (forced) equilibrium point, x_e , is said to be linearly stabilizable if there exists some $K \in \mathbb{R}^{m \times n}$ such that $A(x_e, u_e) + B(x_e)K$ is Hurwitz (stable). That is, the full-state feedback $u = u_e + K(x - x_e)$ then renders x_e an exponentially stable equilibrium of (1).

We will assume knowledge of a point-to-point (PtP) maneuver connecting two separate linearly-stabilizable equilibrium points of (1). Specifically, we assume that a so-called s -parameterization of the maneuver is known:

Definition 1. Let (x_α, u_α) and (x_ω, u_ω) , $x_\alpha \neq x_\omega$, be linearly-stabilizable equilibrium points of (1). For $S := [s_\alpha, s_\omega] \subset \mathbb{R}$, $s_\alpha < s_\omega$, the triplet of functions

$$x_\star : S \rightarrow \mathbb{R}^n, \quad u_\star : S \rightarrow \mathbb{R}^m, \quad \text{and} \quad \rho : S \rightarrow \mathbb{R}_{\geq 0}, \quad (3)$$

constitute an **s -parameterization** of the PtP maneuver

$$\mathcal{M} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : x = x_\star(s), u = u_\star(s), s \in S\}$$

of (1), whose boundaries are (x_α, u_α) and (x_ω, u_ω) , if

- P1** $x_*(\cdot)$ is of class C^2 and traces out a non-self-intersecting curve, while $u_*(\cdot)$ and $\rho(\cdot)$ are C^1 ;¹
- P2** $(x_*(s_i), u_*(s_i)) = (x_i, u_i)$ for both $i \in \{\alpha, \omega\}$;
- P3** $\rho(s_\alpha) = \rho(s_\omega) \equiv 0$, while $\rho(s) > 0$ for all $s \in \text{int}(S)$;
- P4** $\|\mathcal{F}(s)\| > 0$ for all $s \in S$, where $\mathcal{F}(s) := x'_*(s)$;
- P5** $\mathcal{F}(s)\rho(s) = f(x_*(s)) + B(x_*(s))u_*(s)$ for all $s \in S$.

The definition requires some further comments. Given an s -parameterized maneuver \mathcal{M} , we denote by

$$\mathcal{O} := \{x \in \mathbb{R}^n : x = x_*(s), s \in S\} \tag{4}$$

its corresponding orbit (i.e. its projection upon state space). Due to the properties of \mathcal{M} stated in Definition 1, one may in fact consider \mathcal{O} to consist of a (forced) heteroclinic orbit of (1) and its limit points: by P1, \mathcal{O} is a C^2 -smooth, one-dimensional embedded submanifold of \mathbb{R}^n ; by P2, its boundaries correspond to two separate (forced) equilibrium points of (1); whereas by P3, P4 and P5, it is a controlled invariant set of (1) that contains no (forced) equilibrium points on its interior.

Here the latter point can be verified by viewing the curve parameter $s = s(t)$ as a solution to

$$\dot{s} = \rho(s). \tag{5}$$

Since $\dot{x}_*(s(t)) = x'_*(s(t))\dot{s}(t) = \mathcal{F}(s(t))\rho(s(t))$ by the chain rule, one finds, by inserting this into the left-hand side of the expression in P5, that \mathcal{M} is consistent with the dynamics (1). Thus, whereas $\|\dot{x}_*(s(t))\| \equiv 0$ for $s(t) \in \{s_\alpha, s_\omega\}$, the key aspect of an s -parameterization is that the regularity condition P4 holds for $x_*(\cdot)$, as $\rho(\cdot)$ instead vanishes at the boundaries.² This property allows for a compact representation of the motion, something which is clearly seen from the nominal state curve's arc length: $\int_{-\infty}^{\infty} \|\dot{x}_*(s(\tau))\| d\tau = \int_{s_\alpha}^{s_\omega} \|\mathcal{F}(\sigma)\| d\sigma$. It is also vital to the approach we suggest, as it allows one to construct a well-defined projection onto the maneuver.

For \mathcal{O} as defined in (4), denote $\text{dist}(\mathcal{O}, x) := \inf_{y \in \mathcal{O}} \|x - y\|$. We aim to solve the following problem in this paper:

Problem 2 (Orbital Stabilization). For (1), construct a control law $u = k(x)$, with $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ locally Lipschitz in a neighborhood of \mathcal{O} and satisfying $k(x_*(s)) \equiv u_*(s)$ for all $s \in S$, such that \mathcal{O} is an asymptotically stable set of the closed-loop system. Namely, for every $\epsilon > 0$, there is a $\delta > 0$, such that for any solution $x(\cdot)$ of the closed-loop system satisfying $\text{dist}(\mathcal{O}, x(t_0)) < \delta$, it is implied that $\text{dist}(\mathcal{O}, x(t)) < \epsilon$ for all $t \geq t_0$ (stability), and that $\text{dist}(\mathcal{O}, x(t)) \rightarrow 0$ as $t \rightarrow \infty$ (attractivity).

Note that the asymptotic stability of \mathcal{O} is equivalent to the asymptotic orbital stability of all the solutions upon it Hahn et al. (1967), Leonov et al. (1995), Urabe (1967) and Zubov (1999). Thus Problem 2 is a so-called orbital stabilization problem, which can be stated for any type of orbit (equilibrium points, (hybrid) periodic orbits, etc.). Moreover, as we here consider a heteroclinic orbit on which all solutions converge to x_ω , a solution to Problem 2 also implies the (local) asymptotic stability of x_ω by the reduction principle in El-Hawwary and Maggiore (2013).

3. Preliminaries

3.1. Projection operators

A key part of our approach is a projection onto the set \mathcal{O} defined in (4). We define such projection operators in terms of a specific s -parameterization (see Definition 1) next.

¹ While the notion of an s -parameterization can be relaxed in regard to the smoothness of the triplet (x_*, u_*, ρ) , we will, for simplicity's sake, require that P1 holds in this paper.

² As $\rho(\cdot)$ is required to be C^1 and $\rho(s_\alpha) = \rho(s_\omega) \equiv 0$, the rate at which $s(\cdot)$ convergence to s_ω (resp. s_α) in positive (resp. negative) time from within S can be at most exponential, which corresponds to $\rho'(s_\omega) < 0$ (resp. $\rho'(s_\alpha) > 0$).

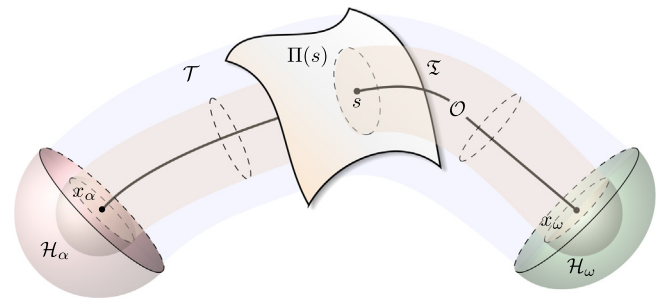


Fig. 1. Illustration of the moving Poincaré section $\Pi(s)$, defined in (6), “traveling” along the orbit \mathcal{O} whose boundaries are x_α and x_ω . The gradient of the projection operator is assumed to be nonzero and well defined within the blue-shaded tubular neighborhood \mathcal{T} . Within the darkly shaded hemispheres \mathcal{H}_α and \mathcal{H}_ω , on the other hand, the gradient vanishes as the projection operator projects the states onto the respective equilibrium therein. The aim of this paper is to guarantee the existence of a positively invariant neighborhood \mathfrak{X} , within which all solutions converge to \mathcal{O} . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Definition 3 (Projection Operators for PtP Maneuvers). Let $\mathfrak{X} \subset \mathbb{R}^n$ denote a simply-connected neighborhood of \mathcal{O} , whose interior can be partitioned into three subsets, denoted \mathcal{H}_α , \mathcal{T} and \mathcal{H}_ω (i.e., $\text{cl}(\mathfrak{X}) = \text{cl}(\mathcal{H}_\alpha \cup \mathcal{T} \cup \mathcal{H}_\omega)$), which are such that

- \mathcal{T} is a tubular neighborhood of \mathcal{O} ;
- $\mathcal{B}_\epsilon(x_\alpha) \setminus \mathcal{T} \subset \text{cl}(\mathcal{H}_\alpha)$ and $\mathcal{B}_\epsilon(x_\omega) \setminus \mathcal{T} \subset \text{cl}(\mathcal{H}_\omega)$, for some $\epsilon > 0$;
- $\text{cl}(\mathcal{T}) \cap \text{cl}(\mathcal{H}_i) \neq \emptyset \forall i \in \{\alpha, \omega\}$, and $\text{cl}(\mathcal{H}_\alpha) \cap \text{cl}(\mathcal{H}_\omega) = \emptyset$.

A map $p : \mathfrak{X} \rightarrow S$ is said to be a **projection operator** for \mathcal{O} if it Lipschitz continuous and well defined within its domain \mathfrak{X} , as well as satisfies

- C1** $p(x_*(s)) \equiv s$ for all $s \in S$;
- C2** $p(\mathcal{H}_\alpha) \equiv s_\alpha$, $p(\mathcal{H}_\omega) \equiv s_\omega$, and $p(\text{int}(\mathcal{T})) \in \text{int}(S)$;
- C3** $p(\cdot)$ is C^r , $r \geq 2$, within \mathcal{H}_α , \mathcal{T} and \mathcal{H}_ω .³

In order to provide some intuition behind the need for the conditions stated in Definition 3, we define the set

$$\Pi(s) := \{x \in \mathfrak{X} : p(x) = s\}. \tag{6}$$

As is illustrated in Fig. 1, for some $s \in \text{int}(S)$, this set traces out a hypersurface, a so-called moving Poincaré section (Leonov, 2006; Shiriaev et al., 2010), whose tangent space at $x_*(s)$ is orthogonal to the transpose of

$$\mathcal{P}(s) := Dp(x_*(s)). \tag{7}$$

By Condition C1, it follows that $\mathcal{P}(s)\mathcal{F}(s) \equiv 1$ for all $s \in S$, from which, in turn, one can deduce that the surface $\Pi(s)$ is locally transverse to $\mathcal{F}(s)$. The tubular neighborhood \mathcal{T} in the definition (consider the blue-shaded tube in Fig. 1) is therefore guaranteed to everywhere have a nonzero radius as \mathcal{O} does not have any self-intersections (see P1 in Definition 1). It can be taken as any connected subset of $\bigcup_{s \in \text{int}(S)} \Pi(s)$ such that the surfaces $\Pi(s_1) \cap \mathcal{T}$ and $\Pi(s_2) \cap \mathcal{T}$ are locally disjoint for any $s_1, s_2 \in \text{int}(S)$, $s_1 \neq s_2$. Thus $Dp(x)$ is nonzero, bounded and of class C^{r-1} for any x within \mathcal{T} .

Conditions C2 and C3, on the other hand, guarantee the existence of the two open half-ball-like regions, \mathcal{H}_α and \mathcal{H}_ω , contained in $\Pi(s_\alpha)$ and $\Pi(s_\omega)$, respectively (see the darkly shaded semi-ellipsoids in Fig. 1). As a consequence, $\|Dp(x)\| \equiv 0$ for all $x \in \mathcal{H}_\alpha \cup \mathcal{H}_\omega$, and hence $p(\cdot)$ is C^2 almost everywhere within \mathfrak{X} , except at $\mathfrak{X}_\alpha := \lim_{s \rightarrow s_\alpha^+} \Pi(s)$ and $\mathfrak{X}_\omega := \lim_{s \rightarrow s_\omega^-} \Pi(s)$, which

³ While one can generally relax the condition in C3 to $r \geq 1$, we require $r \geq 2$ for the approach we suggest in this paper.

correspond to the intersections of the boundary of \mathcal{T} with the boundaries of \mathcal{H}_α and \mathcal{H}_ω , respectively.

The following statements shows that one can obtain projection operators satisfying Definition 3, which are similar to those in Hauser and Hindman (1995).

Proposition 4. Given a PtP maneuver as by Definition 1, let the smooth matrix-valued function $\Lambda : \mathcal{S} \rightarrow \mathbb{M}_{\geq 0}^n$ be such that $\mathfrak{h}(s) := \Lambda(s)\mathcal{F}(s)$ is of class C^2 on \mathcal{S} , and $\mathcal{F}^\top(s)\Lambda(s)\mathcal{F}(s) > 0$ holds for all $s \in \mathcal{S}$. Then there is an $\epsilon > 0$ and a neighborhood \mathfrak{X} of \mathcal{O} , such that

$$p(x) = \underset{\substack{s \in \mathcal{S} \\ \mathfrak{X}(x_*(s), x) \subset \mathcal{B}_\epsilon(x_*(s))}}{\operatorname{arg\,min}} \left[(x - x_*(s))^\top \Lambda(s) (x - x_*(s)) \right] \quad (8)$$

is a projection operator for \mathcal{O} (see Definition 3), with $\mathcal{B}_\epsilon(x_*(s)) \subset \mathfrak{X}$ for all $s \in \mathcal{S}$. Moreover,

$$\mathcal{P}(s) := Dp(x_*(s)) = \frac{\mathcal{F}^\top(s)\Lambda(s)}{\mathcal{F}^\top(s)\Lambda(s)\mathcal{F}(s)} \quad (9)$$

holds for its Jacobian matrix $Dp(\cdot)$ evaluated inside \mathcal{T} .

Note that, in order to effectively compute such operators, knowledge of the hypersurfaces \mathfrak{X}_α and \mathfrak{X}_ω can be used to locally partition \mathfrak{X} into its respective subsets (see Definition 3), with (8) then generally having to be solved numerically when x is in \mathcal{T} (see also Hauser & Hindman, 1995). Notice also that $\Lambda(\cdot)$ is not required to be positive definite nor constant; indeed, for certain maneuvers, this may allow one to use operators depending only on a few state variables and which can be directly evaluated rather than found numerically (cf. Example 12 and Section 6.3).

3.2. Implicit representation of the orbit

Given a projection operator $p : \mathfrak{X} \rightarrow \mathcal{S}$ as by Definition 3, we denote by $x_p(x) := (x_* \circ p)(x)$ the corresponding projection onto \mathcal{O} , and define the following function:

$$e(x) := x - x_p(x). \quad (10)$$

From the properties of $x_*(\cdot)$ and $p(\cdot)$ (see Definitions 1 and 3, respectively), it follows that $e = e(x)$ is well defined for $x \in \mathfrak{X}$, locally Lipschitz in a neighborhood of \mathcal{O} , and therefore twice continuously differentiable everywhere therein except at the two hypersurfaces \mathfrak{X}_α and \mathfrak{X}_ω on the orbit's boundaries. Most importantly, however, is the fact that the zero-level set of this function corresponds to the nominal orbit \mathcal{O} which we aim to stabilize, while, locally, its magnitude is nonzero away from it. Our goal will therefore be to design a control law which guarantees the existence of a positively invariant neighborhood \mathfrak{T} of \mathcal{O} (see Fig. 1) within which e converges to zero.

With this goal in mind, observe from the definition of a projection operator (Definition 3) that one may interpret $e(x)$ differently depending on where in \mathfrak{X} the current state is located. Indeed, consider the open sets $(\mathcal{H}_\alpha, \mathcal{H}_\omega)$ and the tube \mathcal{T} introduced in Section 3.1. Clearly, whenever $x \in \mathcal{H}_i$ for a fixed $i \in \{\alpha, \omega\}$, one has $e = x - x_i$ as $p(x) \equiv s_i$, and thus $De(x) = I_n$ therein. For $x \in \mathcal{T}$, on the other hand, the function $e(\cdot)$ forms an excessive set of so-called *transverse coordinates* (Sætre & Shiriaev, 2020). This can be observed from its Jacobian matrix evaluated along the orbit, which inside of \mathcal{T} is given by

$$\mathcal{E}_\perp(s) := De(x_*(s)) = I_n - \mathcal{F}(s)\mathcal{P}(s) \quad (11)$$

with \mathcal{P} defined in (7). Since $\mathcal{P}(s)\mathcal{F}(s) \equiv 1$, the matrix $\mathcal{E}_\perp(s)$ can be used to project any vector $x \in \mathbb{R}^n$ upon the hyperplane orthogonal to $\mathcal{P}^\top(s)$. As it will appear throughout this paper, we recall some of its properties:

Lemma 5 (Sætre & Shiriaev, 2020). For all $s \in \mathcal{S}$, the matrix function $\mathcal{E}_\perp : \mathcal{S} \rightarrow \mathbb{R}^{n \times n}$ defined in (11) is a projection matrix, i.e. $\mathcal{E}_\perp^2(s) = \mathcal{E}_\perp(s)$; its rank is $n - 1$; while $\mathcal{P}(s)$ and $\mathcal{F}(s)$ span its left- and right annihilator spaces, respectively. \square

3.3. Merging two types of linearizations

To stabilize the zero-level set of the function $e = e(x)$, we will consider a control law of the following form:

$$u = u_*(p(x)) + K(p(x))e. \quad (12)$$

Here $u_* : \mathcal{S} \rightarrow \mathbb{R}^m$ is the known function corresponding to the control curve of the s -parameterized maneuver (see Definition 1) and $K : \mathcal{S} \rightarrow \mathbb{R}^{m \times n}$ is smooth (i.e. of class C^∞). Note that, due to $p(\cdot)$ being locally Lipschitz in \mathfrak{X} , the (local) existence and uniqueness of a solution $x(t)$ to (1) is guaranteed if u is taken according to (12), as the right-hand side of (1) is then locally Lipschitz continuous in a neighborhood of \mathcal{O} .

Whenever $Dp(\cdot)$ is well defined, we have by the chain rule that the time derivative of e under (12) is given by

$$\dot{e} = De(x)(f(x) + B(x)[u_*(p) + K(p)e]), \quad (13)$$

where $p = p(x)$. With the aim of providing conditions ensuring that a control law of the form (12) is a solution to Problem 2, we state the following lemma, which we later will use to derive the first-order approximation of the right-hand side of (13) with respect to e .

Lemma 6. Any C^2 function $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$, satisfying $\sigma(y) = 0$ for all $y \in \mathcal{O}$, can be equivalently rewritten as

$$\sigma(x) = D\sigma(x_p(x))e(x) + O(\|e(x)\|^2) \quad (14)$$

for almost all x in a neighborhood $\hat{\mathfrak{X}} \subseteq \mathfrak{X}$ of \mathcal{O} .

For $A(\cdot)$ as in (2), let $A_{cl}(s) := A_s(s) + B_s(s)K(s)$ with

$$A_s(s) := A(x_*(s), u_*(s)) \quad \text{and} \quad B_s(s) := B(x_*(s)). \quad (15)$$

We may then use Lemma 6 to state the following.

Proposition 7. For some projection operator $p : \mathfrak{X} \rightarrow \mathcal{S}$ as by Definition 3, consider the closed-loop system (1) under the (locally Lipschitz) control law (12). There then exists a neighborhood $\mathcal{N}(\mathcal{O})$ of \mathcal{O} , such that the time derivative of $e = e(x)$, defined in (10), can be written in the following forms within three specific subsets of $\mathcal{N}(\mathcal{O})$: (i) If $x(t) \in \mathcal{H}_i \cap \mathcal{N}(\mathcal{O})$ with $i \in \{\alpha, \omega\}$ fixed, then

$$\dot{e} = A_{cl}(s_i)e + O(\|e\|^2); \quad (16)$$

(ii) If $x(t) \in \mathcal{T} \cap \mathcal{N}(\mathcal{O})$, then

$$\dot{e} = [\mathcal{E}_\perp(p)A_{cl}(p) - \mathcal{F}(p)\mathcal{P}'(p)\rho(p)]\mathcal{E}_\perp(p)e + O(\|e\|^2), \quad (17)$$

where $p = p(x)$ and $\mathcal{P}'(s) = \mathcal{F}^\top(s)D^2p(x_*(s))$.

Consider the linear, time-invariant system

$$\dot{y} = A_{cl}(s_i)y, \quad y \in \mathbb{R}^n, \quad (18)$$

for some fixed $i \in \{\alpha, \omega\}$. It corresponds to the first-order approximation system of (16). It is also equivalent to the Jacobian linearization of (1) under the linear control law $u = u_*(s_i) + K(s_i)(x - x_i)$ about the respective equilibrium point. The first-order approximation system of (17) along \mathcal{O} , on the other hand, is equivalent to the following system of differential-algebraic equations:

$$\dot{z} = [\mathcal{E}_\perp(s)A_{cl}(s) - \mathcal{F}(s)\mathcal{P}'(s)\rho(s)]z, \quad (19a)$$

$$0 = \mathcal{P}(s)z, \quad (19b)$$

where $z \in \mathbb{R}^n$, $s = s(t)$ solves (5), and with the condition $0 = \mathcal{P}(s)z$ obtained directly from (A.1) using Lemma 5 (see also Leonov et al., 1995, Sec. 4 or Sætre & Shiriaev, 2020, Thm. 7 for alternative derivations of (19)). Note that (19) is different

to the first-order variational system of (1) about $x_*(s(t))$, which instead is given by

$$\dot{\chi} = [A_{cl}(s) + B_s(s)(u'_*(s) - K(s)\mathcal{F}(s)) \mathcal{P}(s)] \chi. \quad (20)$$

The solutions to (19) and (20) are however related through $z(t) = \mathcal{E}_\perp(s(t))\chi(t)$. Hence, by recalling the properties of $\mathcal{E}_\perp(\cdot)$ (see Lemma 5), it follows that (19) captures the transverse components of the variational system (20), and is therefore referred to as a *transverse linearization*.

It is well known (see, e.g., Khalil (2002, Theorem 4.6)) that the origin of (18) is exponentially stable at both s_α, s_ω if, and only if, for any $Q_\alpha, Q_\omega \in \mathbb{M}_{>0}^n$, there exist $R_\alpha, R_\omega \in \mathbb{M}_{>0}^n$ satisfying a pair of algebraic Lyapunov equations (ALEs):

$$A_{cl}^\top(s_\alpha)R_\alpha + R_\alpha A_{cl}(s_\alpha) = -Q_\alpha, \quad (21a)$$

$$A_{cl}^\top(s_\omega)R_\omega + R_\omega A_{cl}(s_\omega) = -Q_\omega. \quad (21b)$$

A similar statement can also be readily obtained for (19) by either a slight reformulation of Theorem 1 in Sætre, Shiriaev, Pchelkin, and Chemori (2020) or from the stronger statements found in Leonov (1990) and Leonov et al. (1995) (see, respectively, Theorem 5.1 and Theorem 1 therein).

Lemma 8. Suppose there exist C^1 -smooth matrix-valued functions $R, Q_\perp : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$ such that the projected Lyapunov differential equation (PrjLDE)

$$\mathcal{E}_\perp^\top [A_{cl}^\top \mathcal{E}_\perp^\top R + R \mathcal{E}_\perp A_{cl} + Q_\perp + \rho(R' - (\mathcal{P}')^\top \mathcal{F}^\top R - R \mathcal{F} \mathcal{P}')] \mathcal{E}_\perp = 0_n \quad (22)$$

is satisfied for all $s \in \mathcal{S}$ (here the s -arguments of the functions have been omitted for brevity). Then the time derivative of the scalar function $V_\perp = z^\top R(s(t))z$, with $z = z(t)$ governed by (19), is $\dot{V}_\perp = -z^\top Q_\perp(s(t))z$. \square

Note here that by (19b) we have $z^\top R(s)z = z^\top R_\perp(s)z$ where $R_\perp(s) := \mathcal{E}_\perp^\top(s)R(s)\mathcal{E}_\perp(s)$. Due to the fact that $\mathcal{E}_\perp^2(s) = \mathcal{E}_\perp(s)$, this motivates the following:

Proposition 9. Let the C^1 function $\rho : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ satisfy $\rho(s_\alpha) = \rho(s_\omega) \equiv 0$, $\rho'(s_\alpha) > 0$, and $\rho(s) > 0$ for all $s \in \text{int}(\mathcal{S})$. Then there exists a C^1 solution $R : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$ to (22) for some smooth $Q_\perp : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$ if, and only if, there exists a unique C^1 solution $R_\perp : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$ to

$$\mathcal{E}_\perp^\top(s) [A_{cl}^\top(s)R_\perp(s) + R_\perp(s)A_{cl}(s) + \rho(s)R_\perp'(s) + Q_\perp(s)] \mathcal{E}_\perp(s) = 0_n \quad (23)$$

satisfying $R_\perp(s) = \mathcal{E}_\perp^\top(s)R_\perp(s)\mathcal{E}_\perp(s)$ for all $s \in \mathcal{S}$.

4. Main results

We now provide conditions ensuring that a control law of the form (12) is a solution to Problem 2.

Theorem 10. Given a projection operator $p(\cdot)$ as by Definition 3, consider the closed-loop system (1) under the (locally Lipschitz) control law (12). If there exists a C^1 -smooth matrix function $R : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$ such that

1. for some $Q_\alpha, Q_\omega \in \mathbb{M}_{>0}^n$, $R_\alpha = R(s_\alpha)$ and $R_\omega = R(s_\omega)$ satisfy the ALEs (21);
2. for some smooth $Q_\perp : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$, $R_\perp(s) := \mathcal{E}_\perp^\top(s)R(s)\mathcal{E}_\perp(s)$ satisfies (23) for all $s \in \mathcal{S}$;

then

- (a) the final equilibrium, x_ω , is asymptotically stable;

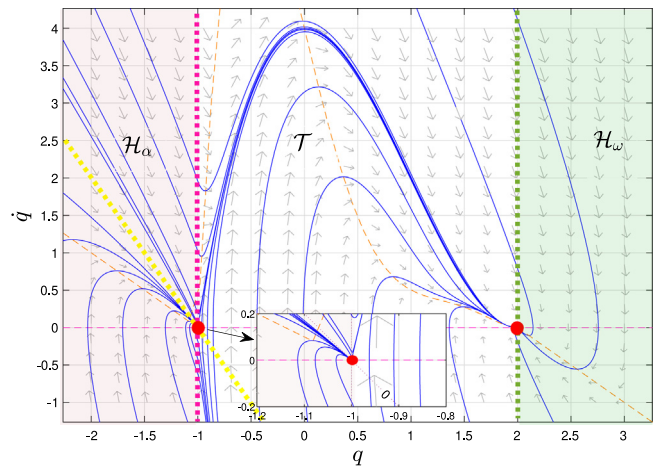


Fig. 2. Phase portrait of $\dot{q} = u$, with u corresponding to Example 12 for $q_\alpha = -1$, $q_\omega = 2$, $\kappa = 1$, $p(x) = \text{sat}_{q_\alpha}^{q_\omega}(q)$ and $k_1 = k_2 = 4$. The level curve $\dot{q} + 2(q+1) = 0$ crossing $(q_\alpha, 0)$ is illustrated by the yellow, dotted line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- (b) the one-dimensional manifold \mathcal{O} , defined in (4), is invariant and exponentially stable;
- (c) there exists a pair of numbers, $\mu, \nu \in \mathbb{R}_{>0}$, such that the time derivative of the locally Lipschitz continuous function $V(x) = e^\top(x)R(p(x))e(x)$ satisfies $\dot{V}(x) \leq -\mu V(x)$ for almost all x in $\mathcal{T} = \{x \in \mathbb{R}^n : V(x) < \nu\}$.

Remark 11. Under a control law (12) satisfying the conditions in Theorem 10, any solution of (1) initialized in vicinity of \mathcal{O} will converge either directly to the initial equilibrium x_α , which is rendered partially unstable (a “saddle”), or onto $\mathcal{O} \setminus \{x_\alpha\}$ and then onward to x_ω . This implies that the system’s states can get “trapped” if they enter the region of attraction of x_α . Indeed, they will then converge toward x_α at an exponential rate, but never enter into the tube \mathcal{T} from within which they can converge to x_ω . This issue can be resolved by some ad hoc modification to the controller (12). For example, one can limit the codomain of the projection operator used in (12). For an operator of the form (8), this would correspond to $p(x) = \arg \min_{s \in [s_\alpha + \epsilon, s_\omega]}(\cdot)$ for some sufficiently small $\epsilon > 0$. A similar alternative is to let $\epsilon \in [0, \epsilon_M]$ be a bounded dynamic variable, e.g. $\dot{\epsilon} = \lambda_\epsilon \cdot \text{sign}(\delta_\epsilon - \|x - x_\alpha\|)$ for small $\epsilon_M, \delta_\epsilon, \lambda_\epsilon > 0$, although the control law will then no longer be truly static in a neighborhood of x_α .

Before we move on to showing how such a feedback can be constructed, we will apply the method to a simple fully-actuated, one-degree-of-freedom system as to highlight the effect of the projection operator upon the resulting feedback controller.

Example 12. Consider the double integrator

$$\ddot{q} = u, \quad q(t), u(t) \in \mathbb{R},$$

with state vector $x = \text{col}(q, \dot{q})$. Starting from rest at q_α , the task is to drive the system to rest at $q_\omega (> q_\alpha)$ along the curve $x_*(s) = \text{col}(s, \rho(s))$. Here $s \in \mathcal{S} := [q_\alpha, q_\omega]$ and $\rho(s) := \kappa(s - q_\alpha)(q_\omega - s)^2$ for some constant $\kappa > 0$. As $\|\mathcal{F}(s)\|^2 = 1 + (\rho'(s))^2 \geq 1$, this is an s -parameterization as by Definition 1.

Suppose $p(\cdot)$ is a projection operator in line with Definition 3 (we will provide some candidates for this operator shortly). Using $p = p(x)$, we define $e_1 := q - p$, $e_2 := \dot{q} - \rho(p)$ and $u_*(p) := \rho'(p)\rho(p)$, such that $u = u_*(p) - k_1 e_1 - k_2 e_2$ is of the form of (12). Let us therefore check when this feedback, corresponding to

a constant $K = [-k_1, -k_2]$, satisfies the conditions in [Theorem 10](#) for a given $p(\cdot)$.

Let $k_1, k_2 > 0$ such that $A_{cl} := \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$ is Hurwitz, and denote by $R \in \mathbb{M}_{>0}^2$ the unique solution to $A_{cl}^T R + R A_{cl} = -2I_2$, which corresponds to the ALEs (21). We may then consider the (locally Lipschitz) Lyapunov function candidate $V = 2^{-1} e^T R e$, with $e = \text{col}(e_1, e_2)$, whose zero-level set evidently corresponds to the desired orbit. Within the interiors of $\Pi(q_\alpha)$ and $\Pi(q_\omega)$, with $\Pi(\cdot)$ defined in (6), we have $\dot{V} = -\|e\|^2$ since $\|Dp\| = 0$ therein. To determine the stability of the orbit as a whole, we therefore need to check that we also have contraction within some tubular neighborhood contained in \mathcal{T} for the chosen projection operator. We consider two different such operators next.

By taking inspiration from [Hauser and Hindman \(1995\)](#), let us first consider the projection operator corresponding to taking $\Lambda = R$ in (8). Using (9), we then observe that $\mathcal{E}_\perp^T(s) R \mathcal{F}(s) = 0_{1 \times 2}$ for all $s \in \mathcal{S}$. Hence (23) is everywhere satisfied for this R_\perp and $Q = I_2$ (see [Hauser and Hindman \(1995\)](#) for further details). Moreover, we have $\dot{V} = -\|e\|^2$ for all x such that $p(x) \in (q_\alpha, q_\omega)$.

Consider now instead the operator obtained by taking $\Lambda = \text{diag}(1, 0)$ in (8). This is equivalent to $p(x) = \text{sat}_{q_\alpha}^{q_\omega}(q)$, where $\text{sat}_a^b(q) = \max(a, \min(q, b))$ is the saturation function (an example of using this operator is shown in [Fig. 2](#)). Clearly then $e_1 \equiv 0$ for $q \in [q_\alpha, q_\omega]$, while it can be shown that $e_2 = -(k_2 + \rho'(p))e_2$. Thus, the time derivative of the above Lyapunov function candidate satisfies

$$\dot{V} = -R_{22}(k_2 + \rho'(p))e_2^2 = -R_{22}(k_2 + \rho'(p))\|e\|^2$$

whenever $q \in (q_\alpha, q_\omega)$, with $R_{22} > 0$ the bottom-right element of R . We may therefore ensure that V will be strictly decreasing everywhere inside the tube (except, of course, on the nominal orbit) by taking, e.g., $k_2 > \sup_{s \in \mathcal{S}} |\rho'(s)|$. This is nevertheless in contrast to the previous operator (i.e. $\Lambda = R$) where $k_2 > 0$ could be taken arbitrarily small and still ensure contraction, thus highlighting the dependence of the feedback $K(\cdot)$ upon the choice of $p(\cdot)$.

As shown in this example, it is trivial to combine [Theorem 10](#) with a specific projection operator as in [Hauser and Hindman \(1995\)](#):

Corollary 13. *If there exist C^1 -smooth matrix-valued functions $R, Q : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$ satisfying, for all $s \in \mathcal{S}$,*

$$\rho(s)R'(s) + A_{cl}^T(s)R(s) + R(s)A_{cl}(s) = -Q(s), \quad (24)$$

then $R(s)$ satisfies the conditions in [Theorem 10](#) provided that $x_\star(\cdot)$ is C^3 and $p(\cdot)$ is taken as in [Proposition 4](#) with $\Lambda(s) = R(s)$. \square

[Corollary 13](#) shows the possibility of finding a feedback matrix $K(\cdot)$ that solves [Problem 2](#) by solving a differential Riccati equation. However, it also forces one to use a particular projection operator (see [Proposition 4](#)), which generally requires one to solve an optimization problem at each iteration. Meanwhile, [Example 12](#) showed that it also can be possible to find projection operators which are very simple and can be computed directly. This motivates a method which allows one to attempt to find a solution for any choice of projection operator. To this end, let $B_\perp(s) := \mathcal{E}_\perp(s)B_s(s)$ and

$$A_\perp(s) := \mathcal{E}_\perp(s)A_s(s) - \rho(s)\mathcal{F}(s)\mathcal{F}^T(s)D^2p(x_\star(s))\mathcal{E}_\perp(s).$$

Inspired by linear matrix inequality (LMI) approaches such as that in [Bernussou, Peres, and Geromel \(1989\)](#), the following statement provides one such method.

Proposition 14. *Given a projection operator $p(\cdot)$ in the sense of [Definition 3](#), suppose that for a strictly positive, smooth function*

$\lambda : \mathcal{S} \rightarrow \mathbb{R}_{>0}$, there exists a pair of smooth matrix-valued functions $Y : \mathcal{S} \rightarrow \mathbb{R}^{m \times n}$ and $W : \mathcal{S} \rightarrow \mathbb{M}_{>0}^n$, which for all $s \in \mathcal{S}$ satisfy the matrix inequality

$$\begin{aligned} & \rho(s)W'(s) - W(s)A_\perp^T(s) - A_\perp(s)W(s) - Y^T(s)B_\perp^T(s) - B_\perp(s)Y(s) \\ & - \lambda(s)[\mathcal{E}_\perp(s)W(s) + W(s)\mathcal{E}_\perp^T(s)] \geq 0_n. \end{aligned} \quad (25)$$

Further suppose that for some $K_\alpha, K_\omega \in \mathbb{R}^{m \times n}$ which are such that $(A_s(s_\alpha) + B_s(s_\alpha)K_\alpha)$ and $(A_s(s_\omega) + B_s(s_\omega)K_\omega)$ are both Hurwitz, the following two identities hold:

$$K_\alpha W(s_\alpha) = Y(s_\alpha) \quad \text{and} \quad K_\omega W(s_\omega) = Y(s_\omega). \quad (26)$$

Then by taking $K(s) = Y(s)W^{-1}(s)$ in (12) the matrix function $R(s) = W^{-1}(s)$ satisfies all the requirements stated in [Theorem 10](#).

In order to find a solution pair (W, Y) to [Proposition 14](#), one can use some transcription method as to discretize the differential LMI (25) into a finite set of LMIs. One can then attempt to find an approximate solution using semidefinite programming (SDP). In regard to handling the constant stabilizing matrices K_α and K_ω in the resulting SDP formulation, there are two main options:

(1) Add, for both $s \in \{s_\alpha, s_\omega\}$, the LMI constraints

$$W(s)A_s^T(s) + A_s(s)W(s) + Y^T(s)B_s(s) + B_s(s)Y(s) < 0_n;$$

(2) Add the equality constraints (26), in which some stabilizing matrices K_α and K_ω have already been found.

In case of the latter option, one can for example use LQR: Take, for both $i \in \{\alpha, \omega\}$, $K_i = -\Gamma_i^{-1}B_s^T(s_i)R_i$, where $R_i \in \mathbb{M}_{>0}^m$ solves the algebraic Riccati equation

$$A_s^T(s_i)R_i + R_i A_s(s_i) - R_i B_s(s_i) \Gamma_i^{-1} B_s^T(s_i) R_i = -Q_i \quad (27)$$

given some $\Gamma_i \in \mathbb{M}_{>0}^m$ and $Q_i \in \mathbb{M}_{>0}^m$.

5. Planning point-to-point maneuvers of underactuated mechanical systems

Consider now the following task: Find an s -parameterized PtP maneuver (see [Definition 1](#)) of an underactuated mechanical systems with n_q degrees of freedom, one degree of underactuation, and equations of motion

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q, \dot{q})\dot{q} + \mathbf{G}(q) = \mathbf{B}_u u. \quad (28)$$

Here $q = \text{col}(q_1, \dots, q_{n_q}) \in \mathbb{R}^{n_q}$ are generalized coordinates, $\dot{q} \in \mathbb{R}^{n_q}$ the corresponding generalized velocities, $x = \text{col}(q, \dot{q})$ denotes the $n = 2n_q$ states, while $u \in \mathbb{R}^m$ is a vector of $m = n_q - 1$ control inputs; $\mathbf{M}(\cdot) \in \mathbb{M}_{>0}^{n_q}$ is the (smooth) inertia matrix; the constant matrix $\mathbf{B}_u \in \mathbb{R}^{n_q \times m}$ has full rank; $\mathbf{C}(\cdot, \cdot)$ corresponds to Coriolis and centrifugal forces, which we in this paper write as $\mathbf{C}(q, \dot{q}) = \mathbf{C}_1(q, \dot{q}) + \mathbf{C}_2(q, \dot{q})$ with $\mathbf{C}_1(q, \dot{q}) := \sum_{i=1}^{n_q} \frac{\partial \mathbf{M}(q)}{\partial q_i} \dot{q}_i$ and $\mathbf{C}_2(q, \dot{q}) := -\frac{1}{2} \left[\frac{\partial \mathbf{M}(q)}{\partial q_1} \dot{q}, \dots, \frac{\partial \mathbf{M}(q)}{\partial q_{n_q}} \dot{q} \right]^T$; while $\mathbf{G}(\cdot) \in \mathbb{R}^{n_q}$ is the (smooth) gradient of the system's potential energy.

For a pair of points (configurations) q_α and q_ω , $q_\alpha \neq q_\omega$, suppose there exist $u_\alpha, u_\omega \in \mathbb{R}^m$ such that $\mathbf{G}(q_\alpha) \equiv \mathbf{B}_u u_\alpha$ and $\mathbf{G}(q_\omega) \equiv \mathbf{B}_u u_\omega$. The task we want solve in this section can then be more accurately formulated:

Problem 15. For $x_\alpha = \text{col}(q_\alpha, 0_{n_q \times 1})$ and $x_\omega = \text{col}(q_\omega, 0_{n_q \times 1})$, find for the system (28) an s -parameterized PtP maneuver connecting x_α and x_ω , i.e., a triplet (x_\star, u_\star, ρ) of the form (3) satisfying [Definition 1](#).

To solve this problem, we propose a procedure inspired by the approach in [Shiriaev et al. \(2005\)](#).

5.1. Synchronization function–based orbit generation

Since (28) is a second-order system, the state curve $x_* : \mathcal{S} \rightarrow \mathcal{O}$ (see Definition 1) can be written on the form

$$x_*(s) := \text{col}(\Phi(s), \Phi'(s)\rho(s)). \quad (29)$$

Here $\Phi(s) = \text{col}(\phi_1(s), \dots, \phi_{n_q}(s))$ is a vector-valued function, which we will assume is smooth, that traces out a curve in the configuration space of the system. As one may consider the generalized coordinates as being synchronized when confined to this curve, we will refer to the smooth, scalar functions $\phi_i(\cdot)$ as *synchronization functions*.⁴ Moreover, the scalar function $\rho : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ may now, in addition to governing the dynamics of the curve parameter s (see (5)), also be considered as to set the speed at which the curve formed by $\Phi(\cdot)$ is traversed.

Let us now derive condition upon the functions $\Phi(\cdot)$, $\rho(\cdot)$ and $u_*(s)$ such that they together provide as solution to Problem 15. In this regard, we first note that Property P4 in Definition 1, i.e. $\|\mathcal{F}(s)\| > 0$, is equivalent to

$$\|\Phi'(s)\|^2 + \|\Phi''(s)\rho(s) + \Phi'(s)\rho'(s)\|^2 > 0. \quad (30)$$

Next we note that Property P2 obviously requires that $\Phi(s_\alpha) = q_\alpha$ and $\Phi(s_\omega) = q_\omega$. Furthermore, to ensure consistency with the dynamics of (28), corresponding to Property P5, it is clear that the functions $\Phi(\cdot)$, $\rho(\cdot)$ and $u_*(s)$ must satisfy the following equality for all $s \in \mathcal{S}$:

$$\mathfrak{A}(s)\rho'(s)\rho(s) + \mathfrak{B}(s)\rho^2(s) + \mathfrak{C}(s) = \mathbf{B}_u u_*(s). \quad (31)$$

Here $\mathfrak{A}(s) := \mathbf{M}(\Phi(s))\Phi'(s)$, $\mathfrak{B}(s) := \mathbf{M}(\Phi(s))\Phi''(s) + \mathbf{C}(\Phi(s), \Phi'(s))\Phi'(s)$, and $\mathfrak{C}(s) := \mathbf{G}(\Phi(s))$. Due to the assumption that \mathbf{B}_u has full rank, we can multiply (31) from the left by any of its left inverses $\mathbf{B}_u^\dagger \in \mathbb{R}^{m \times n_q}$ i.e. $\mathbf{B}_u^\dagger \mathbf{B}_u = \mathbf{I}_m$, to obtain

$$u_*(s) = \mathbf{B}_u^\dagger [\mathfrak{A}(s)\rho'(s)\rho(s) + \mathfrak{B}(s)\rho^2(s) + \mathfrak{C}(s)]. \quad (32)$$

Hence, if $\Phi : \mathcal{S} \rightarrow \mathbb{R}^{n_q}$ and $\rho : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ are known, then the corresponding $u_*(\cdot)$ can be found from (32).

From the above it is clear that if the system (28) was fully actuated, i.e. $m = n_q$, and therefore $\mathbf{B}_u^\dagger = \mathbf{B}_u^{-1}$, then Property P5 would immediately be satisfied simply by taking $u_*(\cdot)$ according to (32) for any combination of $\Phi(\cdot)$ and $\rho(\cdot)$ (see Example 12). This is, however, not the case for the underactuated systems we consider, as $\mathbf{B}_u \in \mathbb{R}^{n_q \times n_q - 1}$ has a family of full-rank left annihilators. Denote by $\mathbf{B}_u^\perp \in \mathbb{R}^{1 \times n_q}$ such an annihilator, i.e. $\mathbf{B}_u^\perp \mathbf{B}_u = \mathbf{0}_{1 \times m}$. Multiplying (31) from the left by \mathbf{B}_u^\perp , one then finds that $\Phi(\cdot)$ and $\rho(\cdot)$ must satisfy

$$\alpha(s)\rho'(s)\rho(s) + \beta(s)\rho^2(s) + \gamma(s) = 0 \quad (33)$$

for all $s \in \mathcal{S}$, where $\alpha(s) := \mathbf{B}_u^\perp \mathfrak{A}(s)$, $\beta(s) := \mathbf{B}_u^\perp \mathfrak{B}(s)$ and $\gamma(s) := \mathbf{B}_u^\perp \mathfrak{C}(s)$.

Our suggested approach for solving Problem 15 can now roughly be described as follows: For a particular choice of a smooth $\Phi(\cdot)$, try to find some $\rho(\cdot)$ satisfying (33) and Property P3 in Definition 1, i.e. $\rho(s_\alpha) = \rho(s_\omega) \equiv 0$ and $\rho(s) > 0$ for all $s \in \text{int}(\mathcal{S})$. If a (satisfactory) solution $\rho(\cdot)$ is found, then the corresponding unique $u_*(\cdot)$ is in turn found directly from (32).

In order to help us find such a function $\rho(\cdot)$, we will utilize the fact that a solution $s = s(t)$ to $\dot{s} = \rho(s)$ must then also be a solution to the second-order differential equation (cf. (33))

$$\alpha(s)\ddot{s} + \beta(s)\dot{s}^2 + \gamma(s) = 0. \quad (34)$$

⁴ If one replaces s with a known function of only the generalized coordinates, i.e. $\theta = \theta(q)$, then the relations $\phi_i(\theta)$ have commonly been referred to as *virtual (holonomic) constraints* (see, e.g., Shiriaev et al. (2005)). This terminology is somewhat misleading for the purpose we consider in this paper, however, and we therefore use the more fitting notion of synchronization functions.

We will refer to (34) as the *reduced dynamics* associated with the synchronization functions $\Phi(\cdot)$. Next we briefly review some key properties of this equation, originally derived in Shiriaev et al. (2005), Shiriaev, Robertsson, Perram, and Sandberg (2006).

5.2. Properties of the reduced dynamics

The following is a (weaker) reformulation of Theorem 3 in Shiriaev et al. (2006), and thus stated without proof.

Lemma 16. *Let $s_e \in \mathcal{S}$ be an equilibrium point of (34), i.e. $\gamma(s_e) \equiv 0$, satisfying $\alpha(s_e) \neq 0$, and denote*

$$v(s) := \gamma'(s)/\alpha(s). \quad (35)$$

Then the equilibrium point s_e is a center if $v(s_e) > 0$, while it is a saddle if $v(s_e) < 0$. \square

Here the conditions for a saddle equilibrium follows directly from the Hartman–Grobman theorem (see also Hahn et al., 1967, Sec. 20), whereas the condition for a center equilibrium point, on the other hand, can be attained by noticing that the solutions of (33) form certain level curves. More precisely, let $\rho(\cdot) \geq 0$ solve (33), and note that $\beta(s) := \alpha'(s) + \hat{\beta}(s)$ with $\hat{\beta}(s) := \mathbf{B}_u^\perp \mathbf{C}_2(\Phi(s), \Phi'(s))\Phi'(s)$. Then

$$\frac{1}{2}\alpha(s) \exp\left(\int_{s_r}^s \frac{2\hat{\beta}(\eta)}{\alpha(\eta)} d\eta\right) =: \frac{1}{2}\alpha(s)\Psi(s_r, s) \quad (36)$$

is an integrating factor of (33) for any $s_r \in \mathcal{S}$. By Shiriaev et al. (2005, Thm. 1), if $s = s(t) \in \mathcal{S}$ is simultaneously a solution to (34) and to $\dot{s} = \rho(s)$, with $\rho : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ strictly positive on $\text{int}(\mathcal{S})$, then for any pair of points $s_1, s_2 \in \mathcal{S}$:

$$\alpha^2(s_2)\rho^2(s_2) - \Psi(s_2, s_1) \left[\alpha^2(s_1)\rho^2(s_1) - 2 \int_{s_1}^{s_2} \Psi(s_1, \tau)\alpha(\tau)\gamma(\tau)d\tau \right] = 0. \quad (37)$$

Note that for certain systems, $\hat{\beta}(s) \equiv 0 \forall s \in \mathcal{S}$, and hence $\Psi \equiv 1$. This property, which can make it significantly easier to check if (37) is satisfied, holds for all systems whose inertia matrix $\mathbf{M}(\cdot)$ is constant, and for any system where the passive joint is the first in a kinematic chain, such as underactuated systems of Class-I according to the classification of Olfati-Saber (2001).

5.3. Conditions for the existence of a PtP maneuver

We will now demonstrate how one can use the properties of the reduced dynamics in order to obtain a solution to Problem 15. In this regard, recall the definitions of $v(\cdot)$ and $\Psi(\cdot)$ given in (35) and (36), respectively.

Theorem 17. *Let the smooth vector-valued function $\Phi : \mathcal{S} \rightarrow \mathbb{R}^{n_q}$ be such that $\Phi(s_\alpha) = q_\alpha$, $\Phi(s_\omega) = q_\omega$, $\|\Phi'(s_\alpha)\| \neq 0$, $\|\Phi'(s_\omega)\| \neq 0$, $v(s_\alpha) \leq 0$ and $v(s_\omega) \leq 0$. Further suppose that the following conditions hold: $\alpha(s) \neq 0$ for all $s \in \mathcal{S}$; there exists a single point $s_e \in \text{int}(\mathcal{S})$ satisfying $\gamma(s_e) \equiv 0$, for which $v(s_e) > 0$; and*

$$\int_{s_\alpha}^{s_\omega} \Psi(s_\alpha, \tau)\alpha(\tau)\gamma(\tau)d\tau \equiv 0. \quad (38)$$

Then there exists a unique, bounded, smooth function $\rho : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ satisfying (33), such that the triplet (x_, u_*, ρ) , with $x_*(\cdot)$ given by (29) and $u_*(\cdot)$ by (32), is a solution to Problem 15. That is, they constitute an s -parameterized point-to-point maneuver of (28) as by Definition 1.*

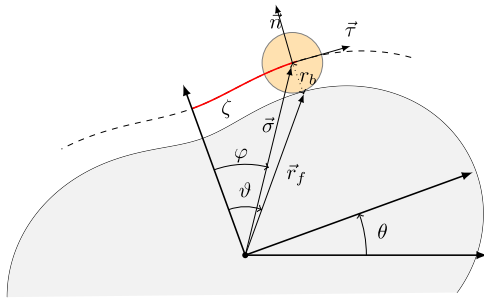


Fig. 3. The coordinate convention used in Section 6.1, with frame having the form of the “butterfly” robot.

Remark 18. As $\|\mathbf{G}(q_\alpha) - \mathbf{B}_u u_\alpha\| = \|\mathbf{G}(q_\omega) - \mathbf{B}_u u_\omega\| = 0$, a solution to Theorem 17 implies $\gamma(\hat{s}) \equiv 0$ for $\hat{s} \in \{s_\alpha, s_\omega\}$. Hence (33) is then trivially true at $\hat{s} \in \{s_\alpha, s_\omega\}$, while from its derivative with respect to s ,

$$\alpha \rho'' \rho + \alpha (\rho')^2 + (3\alpha' + 2\hat{\beta}) \rho' \rho + (\alpha'' + \hat{\beta}') \rho^2 + \gamma' = 0,$$

one finds that $(\rho'(\hat{s}))^2 = -\gamma'(\hat{s})/\alpha(\hat{s})$. Thus, for s_α and s_ω to be hyperbolic (saddle) equilibrium points of (34), and consequently $\rho'(s_\alpha) > 0$ and $\rho'(s_\omega) < 0$, it is further required that $v(s_\alpha) < 0$ and $v(s_\omega) < 0$. From this, one can deduce that the function $\gamma(s)/\alpha(s)$ then must change its sign an odd number of times over the open interval (s_α, s_ω) . Considering only one sign change, the necessary existence of a point $s_e \in \text{int}(S)$ for which $\gamma(s_e) = 0$ and $v(s_e) > 0$ (i.e. a center) is evident.

Remark 19. Due to the requirement of a center on $\text{int}(S)$, Theorem 17 cannot be used to construct an s -parameterized PtP maneuver between two adjacent equilibria for systems where the equilibria of (34) are fixed. In light of Remark 18, one can in such cases instead attempt to use an alternative set of conditions which are based on $\alpha(s)$ changing its sign once over $\text{int}(S)$ instead of $\gamma(s)$. Such conditions can be obtained from Theorem 1 in Surov, Gusev, and Shiriaev (2018), and correspond to replacing the conditions in the second sentence in Theorem 17 with the following: $v(s_\alpha) < 0$ and $v(s_\omega) < 0$; $\gamma(s) > 0$ for all $s \in \text{int}(S)$; and there exists a single point $s_s \in \text{int} S$ satisfying $\alpha(s_s) \equiv 0$ and $\hat{\beta}(s_s) < -\frac{3}{2}\alpha'(s_s) < 0$. Roughly speaking, these conditions ensure that the point $(s, \dot{s}) = (s_s, \sqrt{-\gamma(s_s)/\beta(s_s)})$ is finite-time attractive (resp. repellent) for all solutions of (34) within a neighborhood lying to the left (resp. right) of this point in the upper (s, \dot{s}) -plane.

6. Application to non-prehensile manipulation

We will now apply both the motion planning method proposed in Section 5 and the feedback design approach outlined in Section 4 as to solve the following non-prehensile manipulation (Ruggiero, Lippiello, & Siciliano, 2018) problem: Generate an asymptotically orbitally stable PtP motion corresponding to a ball rolling between any two points upon an actuated planar frame. We begin by describing the system model and provide some necessary assumptions.

6.1. System description and mathematical model

Consider a ball of (effective) radius r_b which is rolling without slipping upon the boundary of an actuated frame; see Fig. 3. The edge of the frame is traced out by the polar coordinates $(\vartheta, r_f(\vartheta))$, with $\vartheta \in \mathcal{I} \subseteq \mathbb{S}^1$ and where the scalar function $r_f: \mathcal{I} \rightarrow \mathbb{R}_{>0}$ is smooth. This representation can be used to describe several well-known nonlinear systems, including the ball-and-beam (Hauser,

Sastry, & Kokotovic, 1992), $r_f(\vartheta) = \frac{\text{const.}}{\cos(\vartheta)}$; the disk-on-disk (Ryu, Ruggiero, & Lynch, 2013), $r_f(\vartheta) = \text{const.}$; as well as the so-called “butterfly” robot (Lynch, Shiroma, Arai, & Tanie, 1998), whose frame, as in Surov et al. (2015), can be of the form

$$r_f(\vartheta) = a - b \cos(2\vartheta), \quad a, b \in \mathbb{R}_{>0}. \quad (39)$$

We will make the following assumptions, whose validity must be checked for any found motion of the system:

- A1. The ball’s center traces out a smooth curve when it traverses the frame⁵;
- A2. The ball is always in contact with the frame;
- A3. The ball always rolls without slipping.

Let θ and φ be defined as shown in Fig. 3, and take $q = \text{col}(\theta, \varphi)$. Then, in light of the above assumptions, the system matrices corresponding to (28) are given by

$$\mathbf{M}(q) = \begin{bmatrix} J_f + J_b + m \|\vec{\sigma}\|^2 & -(m\vec{\sigma} \cdot \vec{n} + \frac{J_b}{r_b})\zeta \\ -(m\vec{\sigma} \cdot \vec{n} + \frac{J_b}{r_b})\zeta' & (\frac{J_b}{r_b} + m)\zeta'^2 \end{bmatrix},$$

$$\mathbf{C}(q, \dot{q}) = \begin{bmatrix} c_{11}\dot{\varphi} & c_{11}\dot{\theta} - c_{12}\dot{\varphi} \\ -c_{11}\dot{\theta} & (\frac{J_b}{r_b} + m)\zeta'\zeta''\dot{\varphi} \end{bmatrix}, \quad \mathbf{B}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{G}(q) = \text{col}\left(m\vec{g} \cdot \left(\frac{d}{d\theta} \text{Rot}(\theta)\right)\vec{\sigma}\right), m\vec{g} \cdot (\text{Rot}(\theta)\vec{\tau}\zeta')$$

where $c_{11} := m\zeta'\vec{\sigma} \cdot \vec{\tau}$, $c_{12} := (m\vec{\sigma} \cdot \vec{n} + \frac{J_b}{r_b})\zeta'' + c_{11}\kappa\zeta'$ and $\vec{g} = \text{col}(0, g)$. See Surov et al. (2015) for a more detailed description of the system parameters and variables, albeit with a slightly different notation.

6.2. Maneuver design

We will now utilize the procedure outlined in Section 5 to plan PtP maneuvers for such systems. For this purpose, let $\psi(\varphi)$ denote the tangential angle of the polar curve at φ . Namely, the angle such that the unit tangent vector $\vec{\tau}$ at φ can be written as $\vec{\tau} = \text{col}(\cos(\psi), \sin(\psi))$; or equivalently, the angle such that $\frac{\partial\psi}{\partial\zeta} = \kappa$ where ζ is the arc length and $\kappa = \kappa(\varphi)$ is the signed curvature of the curve traced out by the ball. Hence ψ is trivial for systems with constant curvature, e.g., $\psi \equiv 0$ for the ball-and-beam system and $\psi = -\varphi$ for the disk-on-disk.

With this in mind, consider

$$\Phi(s) = \text{col}(\Theta(s) - \psi(s), s), \quad s \in S \subseteq \mathbb{S}, \quad (40)$$

for some smooth, scalar function $\Theta(\cdot)$. Simply put, if one takes $\Theta = 0$, then the synchronization function (40) aligns $\vec{\tau}$ with the fixed horizontal axis (see Fig. 3), such that the ball can be considered as to be rolling on a horizontal surface. The function $\Theta(\cdot)$ can therefore be used to slow down or speed up the rolling motion by altering the “slope” upon which the ball rolls.

For this choice of $\Phi(\cdot)$, the functions $\alpha(\cdot)$ and $\gamma(\cdot)$ in (33) are given by $\gamma(s) = mg\zeta'\sin(\Theta(s))$ and

$$\alpha(s) = \left(\frac{J_b}{R} \left(\kappa + \frac{1}{R}\right) + m(1 + \vec{\sigma} \cdot \vec{k})\right)\zeta'^2 - \left(m\vec{\sigma} \cdot \vec{n} + \frac{J_b}{R}\right)\zeta'\Theta'.$$

From this and Lemma 16, the following can be deduced:

Proposition 20. A point $s_e \in S$, for which $\alpha(s_e) \neq 0$, is an equilibrium point of (34) if $\Theta(s_e) \equiv 0$. Moreover, it is a center if $\Theta'(s_e)/\alpha(s_e) > 0$, or a saddle if $\Theta'(s_e)/\alpha(s_e) < 0$. \square

⁵ Mathematically, this is equivalent to $r_b\kappa_f(\vartheta) < 1 \forall \vartheta \in \mathcal{I}$, where $\kappa_f(\vartheta)$ is the signed curvature of the planar curve at ϑ .

Table 1
Parameter values of the “butterfly” robot (BR).

m [kg]	r_b [m]	J_b [kg m ²]	J_f [kg m ²]	g [m s ⁻²]
3.0×10^{-3}	1.09×10^{-2}	5.8×10^{-7}	8.9×10^{-4}	9.81

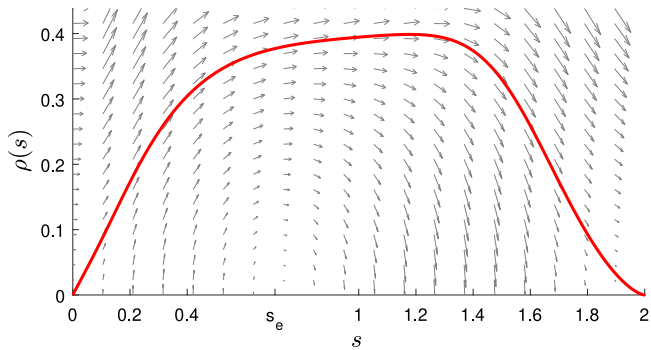


Fig. 4. Phase portrait of (34), with the red curve the solution of (33) satisfying P3 in Definition 1.

One can therefore choose the equilibrium points of (34) freely through the choice of Θ . In light of the discussion in Section 5.3, this in turn can be utilized to find a solution satisfying the conditions in Theorem 17. More specifically, let Θ be taken such that $\alpha(s) \neq 0$ on S , $\Theta'(s_\alpha)/\alpha(s_\alpha) \leq 0$ and $\Theta'(s_\omega)/\alpha(s_\omega) \leq 0$, as well as $\Theta(s_e) = 0$ and $\Theta'(s_e)/\alpha(s_e) > 0$ for some $s_e \in \text{int}(S)$. Then Condition (38) corresponds to the existence of a separatrix connecting s_α and s_ω , for which the corresponding function $\rho : S \rightarrow \mathbb{R}_{\geq 0}$ can be found from (37). We utilize this procedure in the following example.

6.3. Simulation example: The “butterfly” robot

Consider the “butterfly” robot (BR) illustrated in Fig. 3. Its shape is described by (39) with $a = 1.14 \times 10^{-1}$ and $b = 3.9 \times 10^{-2}$, while the values of the system parameters are given in Table 1. The task we will consider is to maneuver the ball from $\varphi_\alpha = 0$ rad to $\varphi_\omega = 2$ rad.

Motion planning. In light of Proposition 20, consider the synchronization functions (40) with $\Theta(s) = k(s - s_\alpha)(s - s_e)(s_\omega - s)^2$, where $s_\alpha = 0$, $s_e \approx 0.707$, $s_\omega = 2$, and $k = 0.01$. The corresponding unique (positive) solution to (33), found using (37) and satisfying Property P3 in Definition 1, is shown in red in Fig. 4. The corresponding nominal control input found from (32) can be seen in Fig. 5, where it is measured relative to the right vertical axis.

Projection operator. We took $\Lambda = \text{diag}(0, 1, 0, 0)$ in (8) with $S := [s_\alpha, s_\omega]$, which is equivalent to $p(x) = \text{sat}_{s_\alpha}^{s_\omega}(\varphi) = \max(s_\alpha, \min(\varphi, s_\omega))$.

Control design. Since the Jacobian linearization is linearly controllable at both $x_\alpha = x_*(s_\alpha)$ and $x_\omega = x_*(s_\omega)$, we computed a pair of constant LQR-based feedback matrices $K_\alpha, K_\omega \in \mathbb{R}^{m \times n}$ by solving the algebraic Riccati Eqs. (27) using the CARE command in MATLAB, with $\Gamma_\alpha = \Gamma_\omega = 10^5$ and $Q_\alpha = Q_\omega = I_4$. Note that the magnitude of Γ_α and Γ_ω here simply reflects the small parameter values (see Table 1). We then took $\lambda = 0.5$, and formulated a semidefinite programming (SDP) problem following Proposition 14 with the equality constraints (26). In order to discretize the differential LMI (25) into a finite number of LMIs, we took the elements of the matrix functions W and Y as sixth-order Beziér polynomials, and took (25) evaluated at 200 evenly spaced points as LMI constraints in the SDP. The resulting SDP was then

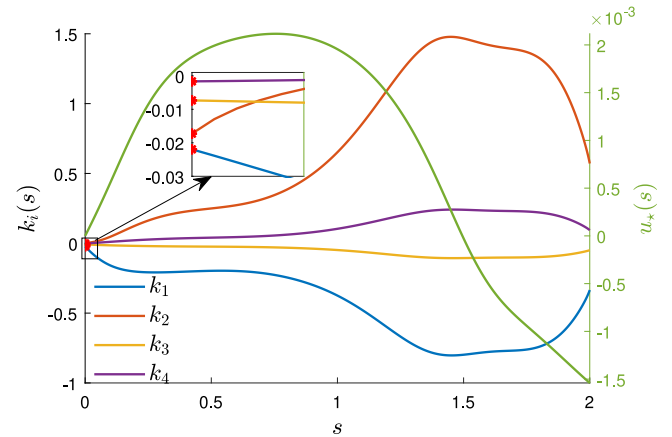


Fig. 5. Found elements of $K(s) = [k_1(s), k_2(s), k_3(s), k_4(s)]$ (left axis) and the nominal control input $u_*(s)$ (right axis).

solved using the YALMIP toolbox for MATLAB (Löfberg, 2004) together with the SDPT3 solver (Tütüncü, Toh, & Todd, 2003). Fig. 5 shows the elements of the obtained $K(s) = Y(s)W^{-1}(s) \in \mathbb{R}^{1 \times 4}$.

Implementation. Following the discussion of Remark 11, the projection operator was implemented as $p(x) = \text{sat}_{s_\alpha + \epsilon}^{s_\omega}(\varphi)$, where the dynamic variable $\epsilon \in [0, \epsilon_M]$ was governed by $\dot{\epsilon} = \epsilon_M \text{sign}(\epsilon_M - \|x - x_\alpha\|)$ with $\epsilon_M = 10^{-3}$ (similar results were obtained with a constant $\epsilon = \epsilon_M$). Since exact measurements of all the states were assumed to be given, the implementation of the controller (12) is straightforward: **Step 1:** Given x , compute $p = p(x)$; **Step 2:** Compute $u_*(p)$, $K(p)$ and $x_*(p)$ (e.g. using splines or lookup tables); **Step 3:** Take $u = u_*(p) + K(p)e$ with $e = x - x_*(p)$.

Simulation results. The response of the system when starting with the initial conditions $x(0) = x_\alpha + \text{col}(0.1, -0.3, 0, 0)$ is shown in Fig. 6, with some snapshots of the system’s configuration shown in Fig. 7. As the states are initially within the half-ball corresponding to x_α , it can be seen that the controller first brings the states close to x_α , after which they then follow the nominal orbit to x_ω . Notice also that Assumption A2 holds, as the normal force F_n between the ball and the frame is everywhere positive.

To test the sensitivity of the closed-loop system to noise and perturbations, we simulated the system with the same initial conditions, but with a small amount of white noise added to the measurements passed to the controller, with the actual mass of the ball, m_b , being 10% larger than that assumed, as well as with the matched disturbance $10^{-4} \sin(t)$ added to the right-hand side of (28). The resulting system response is shown in Fig. 8.

Fig. 9 shows the system response for $x(0) = x_\omega + \text{col}(0.1, 0.1, 0, 0)$. Interestingly, these initial conditions do not lie in the region of attraction of the linear feedback $u = u_*(s_\omega) + K(s_\omega)(x - x_\omega)$. Notice also that φ becomes less than 2 rad just before $t = 1$ s, at which the gradient of the projection operator has a discontinuity. It can be seen that the smoothness of the control signal is violated at this time instant, but it is clear from the highlighted rectangle that Lipschitz continuity is still preserved.

7. Discussion

Is this Orbital Stabilization? The main focus of this paper has been upon the stabilization of the set \mathcal{O} (see (4)) corresponding to an assumed-to-be-known maneuver \mathcal{M} . Even though this set consists of a heteroclinic orbit and its limit points, it may not be

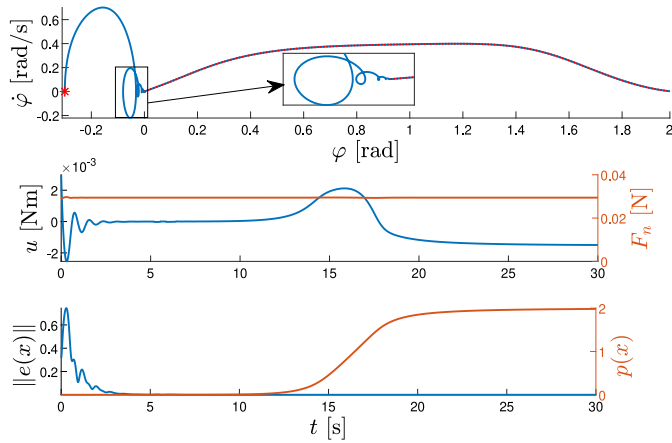


Fig. 6. Response of the BR system initialized close to x_α .

immediately clear that this form of set-stabilizing feedback can be referred to as an *orbitally stabilizing* feedback. We however believe such a classification is not only justified, but that it is in fact an important one to make. To illustrate this point, consider the *orbital stabilization problem* (see [Problem 2](#)). As previously stated, it is equivalent to ensuring the asymptotic orbital stability ([Hahn et al., 1967](#); [Leonov et al., 1995](#); [Urabe, 1967](#); [Zubov, 1999](#)) of the desired motion. It therefore incorporates the problem of stabilizing several important behaviors, including those corresponding to equilibria (trivial orbits), limit cycles (periodic orbits) and PtP maneuvers (heteroclinic orbits). This motivates developing general-purpose methods which can be used to control and stabilize these types of maneuvers (and more). Take, for instance, the method we have proposed in this paper: In the case of trivial orbits, [Theorem 10](#) and [Proposition 14](#) condenses down to a standard linear feedback stabilizing the Jacobian linearization and to the satisfaction of an algebraic Lyapunov equation; whereas for nontrivial periodic orbits, a control law of the form (12) satisfying (22), e.g. found by solving the then periodic differential LMI (25), will exponentially stabilize the desired orbit.

Rate of convergence. A major (practical) limitation of the proposed scheme is the slow convergence away from the initial equilibrium. In light of this issue, a possible ad hoc modification was proposed in [Remark 11](#) as to ensure that the state do not remain too long about x_α . The suggested modifications were, roughly speaking, based on removing the initial equilibrium and instead starting part way along the maneuver, either by removing it altogether (static approach) or gradually moving away from it (dynamic approach). As an alternative way of handling this issue, especially the slow convergence away from the initial equilibrium, one can instead consider maneuvers where x_α is finite-time repellent with respect to \mathcal{O} . If also x_ω is finite-time attractive, then we refer to it as a *finite-time* PtP maneuver. For such a maneuver, $\rho(\cdot)$ can of course no longer be Lipschitz about s_α and/or s_ω . For instance, taking $\rho(s) = \kappa|s - q_\alpha|^{n_\alpha}|q_\omega - s|^{n_\omega}$ for any $n_\alpha, n_\omega \in (0.5, 1)$ in [Example 12](#) corresponds to a finite-time maneuver. Note, however, that for $p(x) = \text{sat}_{q_\alpha}^{q_\omega}(q)$, the orbitally stabilizing feedback then cannot be Lipschitz about x_ω , as $\rho'(s) \rightarrow -\infty$ when $s \rightarrow s_\omega$. Note also that for such a maneuver to exist in the solution space of an underactuated mechanical system, the reduced dynamics (34) must have certain type of singular point at the respective boundaries. Take, for example, $s\ddot{s} + (1-a)\dot{s}^2 - bs(s-c) = 0$ with $a > 3/2$ and $b, c > 0$. It has a heteroclinic orbit connecting $s_\alpha = 0$ and $s_\omega = c$. Here s_α is not only an equilibrium point, but also a singular point of the type considered in [Surov et al. \(2018\)](#), making it finite-time repellent with respect to the orbit.

8. Conclusion

We have introduced a method for inducing, via locally Lipschitz-continuous static state-feedback control, an asymptotically stable heteroclinic orbit in a nonlinear control system. Our suggested approach used a particular parameterization of a known point-to-point maneuver, together with a so-called projection operator, as to merge a Jacobian linearization with a transverse linearization for the purpose of control design. Moreover, a possible way of constructing such a feedback by solving a semidefinite programming problem was suggested, while statements which may be used to plan such maneuvers for mechanical systems with one degree of underactuation using synchronization functions were provided.

It was demonstrated that the approach could be used to solve the challenging nonprehensile manipulation problem of rolling a ball, in a stable manner, between any two points upon a smooth actuated planar frame. This provided a general solution applicable to a number of well-known nonlinear systems, including the ball-and-beam, the disk-on-disk and the “butterfly” robot. The approach was successfully demonstrated on the latter system in numerical simulations.

Acknowledgments

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Appendix

A.1. Proof of [Proposition 4](#)

We need to show that all the conditions in [Definition 3](#) are satisfied. To this end, we begin by differentiating the terms inside the brackets in (8) with respect to s , from which we obtain the function

$$z(x, s) := (x - x_*(s))^T [A'(s)(x - x_*(s)) - 2\Lambda(s)\mathcal{F}(s)].$$

Since $\frac{\partial}{\partial s} z(x, s)|_{x=x_*(s)} = 2\mathcal{F}^T(s)\Lambda(s)\mathcal{F}(s) > 0$ and $z(x_*(s), s) \equiv 0$, [Condition C1](#) is implied. Moreover, by noting from [Property P1](#) in [Definition 1](#) that the curve $x_*(\cdot)$ has bounded curvature and is not self-intersecting, the implicit function theorem ([Berger, 1977](#), Thm. 3.1.10) ensures that there exists, in a certain vicinity of each point on \mathcal{O} , a unique function $p(x)$ satisfying $z(x, p(x)) \equiv 0$, which in turn implies that $p(x)$ solves (8). Thus, for $\mathfrak{X} \subset \mathbb{R}^n$ a sufficiently small neighborhood of \mathcal{O} , the requirement $\mathfrak{L}(x_*(p(x)), x) \subset \mathcal{B}_\epsilon(x_*(s)) \subset \mathfrak{X}$ ensures the uniqueness of a solution to (8) within $\mathcal{T} := \{x \in \mathfrak{X} : z(x, p(x)) = 0\}$. Moreover, if $x \in \mathcal{T}$, then ([Berger, 1977](#), Cor. 3.1.11)

$$Dp(x) = \frac{\mathcal{F}^T \Lambda - e^T \Lambda'}{\mathcal{F}^T \Lambda \mathcal{F} + e^T [\frac{1}{2} \Lambda' e - 2 \Lambda' \mathcal{F} - \Lambda \mathcal{F}']}$$

for such a solution $p = p(x)$, with $e := x - x_*(p)$, and where we have omitted the p -arguments to shorten the notation, i.e. $\mathcal{F} = \mathcal{F}(p)$ etc. Hence $Dp(\cdot)$ is nonzero and \mathcal{C}^1 (as $\Lambda \mathcal{F}'$ is) within $\mathcal{T} \subset \mathbb{R}^n$, with $\mathcal{P}(s) := Dp(x_*(s))$ given by (9) therein.

What remains is therefore to show the parts of [C2](#) and [C3](#) in [Definition 3](#) relating to the sets \mathcal{H}_α and \mathcal{H}_ω also hold. Let us assume these sets exist. Due to the expression for $Dp(\cdot)$ above, which is valid within \mathcal{T} , together with $\mathcal{P}\mathcal{F} = 1$, it follows that sufficiently close to x_ω the states will leave \mathcal{T} and enter \mathcal{H}_ω if they go in the direction $\mathcal{F}(s_\omega)$ when on $\mathfrak{X}_\omega := \text{cl}(\mathcal{T}) \cap \text{cl}(\mathcal{H}_\omega)$. Take \mathfrak{X} such that any $x \in \mathcal{H}_\omega$ can be written as $x = x_\omega + c\mathcal{F}(s_\omega)$ for some

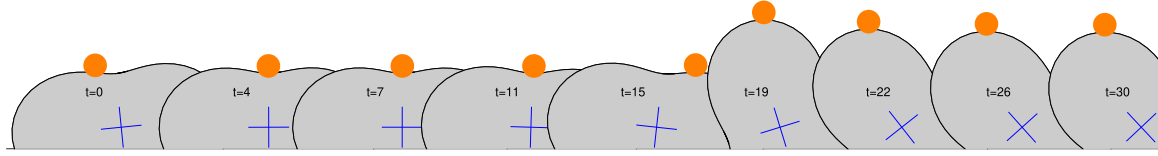


Fig. 7. Snapshots of the configuration of the “butterfly” robot system corresponding to the response shown in Fig. 6.

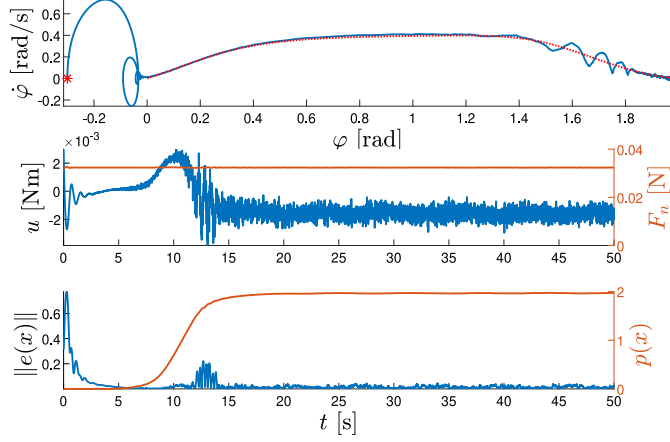


Fig. 8. Response of the BR system initialized close to x_α , with white noise added to all the state measurements, with the mass of the ball increased by 10% and subject to a small matched disturbance.

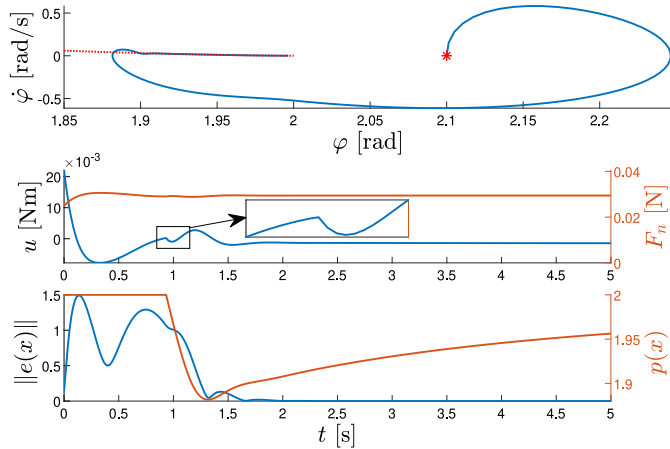


Fig. 9. Response of the BR system initialized close to x_ω .

$\chi_\omega \in \mathfrak{X}_\omega$ and $c > 0$. Since $z(\chi_\omega, s_\omega) = 0$, one can, for $\|x - x_\omega\|$ sufficiently small, always find a Lagrange multiplier $\mu_\omega > 0$ associated with the inequality constraint $s_\omega - s \geq 0$, such that $z(x, s_\omega) + \mu_\omega = 0$. Thus s_ω is a minimizer by the Karush–Kuhn–Tucker conditions. Moreover, due to the constraint $\mathcal{L}(x_\star(s), x) \subset \mathcal{B}_\epsilon(x_\star(s)) \subset \mathfrak{X}$ and the condition $\mathcal{F}^\top \Lambda \mathcal{F} > 0$, we can always take both \mathfrak{X} and \mathcal{H}_ω to be sufficiently small as to guarantee that s_ω is the unique minimizer of (8) for all $x \in \mathcal{H}_\omega$. Using the same arguments about $\mathfrak{X}_\alpha := \text{cl}(\mathcal{H}_\alpha) \cap \text{cl}(\mathcal{T})$, the existence of \mathcal{H}_α and \mathcal{H}_ω in Condition C2 is therefore implied, and the requirements of C3 are met.

A.2. Proof of Lemma 6

According to Taylor’s theorem (see, e.g., Berger (1977, Thm. 2.1.33)), $\sigma(x) = D\sigma(y)(x - y) + O(\|x - y\|^2)$ holds for all x in some neighborhood of a fixed $y \in \mathcal{O}$. Due to the properties of

a projection operator (see Definition 3), there is a neighborhood $\hat{\mathfrak{X}} \subseteq \mathfrak{X}$ of \mathcal{O} , such that $\mathcal{L}(x_p(x), x) \subset \hat{\mathfrak{X}}$ for all $x \in \hat{\mathfrak{X}}$. Hence, for any $x \in \hat{\mathfrak{X}}$, we may take $y = x_p(x)$ to obtain (14). Due to $p(\cdot)$ being at least C^1 within \mathcal{H}_α , \mathcal{T} and \mathcal{H}_ω , the validity of (14) is ensured almost everywhere within $\hat{\mathfrak{X}}$.

A.3. Proof of Proposition 7

Recall that $De(x) = I_n$ whenever x is within either \mathcal{H}_α or \mathcal{H}_ω . By computing the Jacobian matrix of the right-hand side of (13) and using (14), we therefore readily obtain (16). In order to also show that (17) is valid within \mathcal{T} , we note that (14) must also be valid for the function $e(\cdot)$ itself within the interior of \mathcal{T} , as $p \in C^2$ therein. Let $p = p(x)$ and recall that $\mathcal{E}_\perp^2 = \mathcal{E}_\perp$ (see Lemma 5). Applying (14) to each element of e , and then multiplying from the left by $\mathcal{E}_\perp(p)$, one finds that

$$e(x) = \mathcal{E}_\perp(p)e(x) + \mathcal{F}(p)l(x) \quad (\text{A.1})$$

must hold for $x \in \mathcal{T}$, with $l : \mathbb{R}^n \rightarrow \mathbb{R}$ some C^2 function satisfying $\|l(x)\| = O(\|e\|^2)$. Using the fact that $De(x) = I_n - \mathcal{F}(p(x))Dp(x)$ whenever $x \in \mathcal{T}$, the Jacobian matrix of the right-hand side of (13) can also be computed inside \mathcal{T} . By writing it in the form (14) and using (A.1), one obtains (17).

The above still applies even if there are points such that $\mathcal{L}(x_p(x), x)$ does not remain in a given subset of \mathfrak{X} , regardless of how small $\mathcal{N}(\mathcal{O})$ is taken. Indeed, within \mathcal{H}_i one can use the equivalence between the right-hand side of (13) with the function obtained by fixing $p = s_i$. Moreover, Property P1 in Definition 1 ensures that one can always find a function which is C^2 -smooth in $\mathcal{N}(\mathcal{O})$ and equivalent to the right-hand side of (13) for all x in $\mathcal{N}(\mathcal{O}) \cap \mathcal{T}$. Specifically, there exists an $\epsilon > 0$ such that one can extend the maneuver at its boundaries in the appropriate direction along $\mathcal{F}(s_i)$ and $u'_s(s_i)$ for $|s - s_i| < \epsilon$. An appropriate projection onto this extended maneuver, which is equivalent to p in \mathcal{T} and which is C^2 -smooth in the whole of $\mathcal{N}(\mathcal{O})$, can then be constructed and used to define the aforementioned function.

A.4. Proof of Proposition 9

In the following, we will sometimes omit the s -arguments as to shorten the notation. Given a solution $R(s)$ to (22), let $R_\perp := \mathcal{E}_\perp^\top R \mathcal{E}_\perp$. Clearly $R_\perp = \mathcal{E}_\perp^\top R_\perp \mathcal{E}_\perp$ then holds by Lemma 5. Differentiating $R_\perp(s)$ with respect to s yields $R'_\perp = \left(\frac{d}{ds} \mathcal{E}_\perp^\top\right) R \mathcal{E}_\perp + \mathcal{E}_\perp^\top R' \mathcal{E}_\perp + \mathcal{E}_\perp^\top R \left(\frac{d}{ds} \mathcal{E}_\perp\right)$. By then using that $\left(\frac{d}{ds} \mathcal{E}_\perp\right) \mathcal{E}_\perp = -\mathcal{F} \mathcal{F}^\top D^2 p(x_\star)$, one finds, by inserting the above expression for R'_\perp into (23), that (23) holds if $R(s)$ satisfies (22).

To show that the converse holds as well, let $R_\perp(s) = \mathcal{E}_\perp^\top(s) R_\perp(s) \mathcal{E}_\perp(s)$ solve (22). Taking then $R(s) := R_\perp(s) + h_R(s) \mathcal{P}^\top(s) \mathcal{P}(s)$, with $h_R : S \rightarrow \mathbb{R}_{>0}$ an arbitrary smooth function, one can easily show, using the properties stated in Lemma 5, that $R(s)$ satisfies (22).

What remains is therefore to show that a solution $R_\perp(s) = \mathcal{E}_\perp^\top(s) R_\perp(s) \mathcal{E}_\perp(s)$ to (22) is unique. In this regard, first note that by Lemma 5 and the relation $\mathcal{P} \mathcal{F} \equiv 1$, we can always find some $\omega^\top, \mathcal{J} : S \rightarrow \mathbb{R}^{n \times n-1}$ which are sufficiently smooth and satisfy $\omega(s) \mathcal{F}(s) \equiv 0_{n-1 \times 1}$, $\mathcal{P}(s) \mathcal{J}(s) \equiv 0_{1 \times n-1}$ and $\omega(s) \mathcal{J}(s) \equiv I_{n-1}$ for all

$s \in S$. In particular, we will here take \mathcal{J} satisfying $\dot{\mathcal{J}} = -\mathcal{F}\dot{p}\mathcal{J}$. This allows us to write $\mathcal{E}_\perp(s) = \Omega(s)E\Omega^{-1}(s)$ in which $E := \text{diag}(0, I_{n-1})$, $\Omega(s) := [\mathcal{P}(s), \mathcal{J}(s)]$ and $\Omega^{-1}(s) = [\mathcal{P}^\top(s), \omega^\top(s)]^\top$. We can then equivalently rewrite (22) as

$$E\Omega^\top[\dot{R} + A_{cl}^\top(\Omega^{-1})^\top E\Omega^\top \hat{R}_\perp + \hat{R}_\perp \Omega E\Omega^{-1} A_{cl} - \dot{p}^\top \mathcal{F}^\top R - R\mathcal{F}\dot{p} + Q] \Omega E = 0_n,$$

with $\dot{p} = \mathcal{F}^\top D^2 p(x_*) \rho$. It can further be shown that the parts of this equation which are not trivially zero correspond to the following matrix differential equation:

$$A^\top \mathcal{R}_\perp + \mathcal{R}_\perp A + \mathcal{J}^\top [\dot{R} - \dot{p}^\top \mathcal{F}^\top R - R\mathcal{F}\dot{p}] \mathcal{J} + Q_\perp = 0_n,$$

where $A(s) := \omega(s)A_{cl}(s)\mathcal{J}(s)$, while the matrix functions $\mathcal{R}_\perp(s) := \mathcal{J}^\top(s)R(s)\mathcal{J}(s)$ and $Q_\perp(s) := \mathcal{J}^\top(s)Q(s)\mathcal{J}(s)$ evidently are both C^1 -smooth, symmetric and positive definite. Since $\dot{\mathcal{J}} = -\mathcal{F}\dot{p}\mathcal{J}$, we have $\dot{\mathcal{R}}_\perp = \mathcal{J}^\top [\dot{R} - \dot{p}^\top \mathcal{F}^\top R - R\mathcal{F}\dot{p}] \mathcal{J}$. We can therefore rewrite the above equation as

$$\mathcal{R}'_\perp(s)\rho(s) = -A^\top(s)\mathcal{R}_\perp(s) - \mathcal{R}_\perp(s)A(s) - Q_\perp(s). \tag{A.2}$$

In order to show uniqueness, we use hypotheses that $\rho(s_\alpha) = 0$ and $\rho'(s_\alpha) > 0$. Hence, due to both $\mathcal{R}_\perp(s_\alpha)$ and $Q_\perp(s_\alpha)$ being members of $\mathbb{M}_{>0}^{n-1}$ and satisfying the algebraic Lyapunov Eq. (A.2) for $s = s_\alpha$, it follows that the matrix $A(s_\alpha) := \omega(s_\alpha)A_{cl}(s_\alpha)\mathcal{J}(s_\alpha)$ must necessarily be Hurwitz, which in turn implies that $\mathcal{R}_\perp(s_\alpha)$ is unique (Khalil, 2002, Theorem 4.6). Since the right-hand side of (A.2) is continuously differentiable, it then has a unique solution $\mathcal{R}_\perp(s)$ satisfying $\mathcal{R}_\perp(s(t_\alpha)) = \mathcal{R}_\perp(s_\alpha)$. Consequently $\hat{R}_\perp(s) = \mathcal{E}_\perp^\top(s)R(s)\mathcal{E}_\perp(s) = \omega^\top(s)\mathcal{J}^\top(s)R(s)\mathcal{J}(s)\omega(s) = \omega^\top(s)\mathcal{R}_\perp(s)\omega(s)$ is also unique. This concludes the proof.

A.5. Proof of Theorem 10

By the reduction principle stated in Corollary 11 in El-Hawwary and Maggiore (2013), (b) \implies (a). Moreover, the forward invariance property stated in (b) holds due to the existence of the maneuver (see Property P5 in Definition 1) and the properties of projection operators (see Definition 3). We therefore claim that (c) \implies (b). Indeed, first note that V is differentiable everywhere in \mathfrak{X} except at the hypersurfaces (having zero Lebesgue measure; cf. Rademacher's theorem Clarke, Ledyaev, Stern, & Wolenski, 2008) $\mathfrak{X}_\alpha := \lim_{s \rightarrow s_\alpha^+} \Pi(s)$ and $\mathfrak{X}_\omega := \lim_{s \rightarrow s_\omega^-} \Pi(s)$ (see (6) for the definition of Π). Denote $v(t) = V(x(t))$ and consider the upper-right (Dini) derivative $v^+(t)$ of $v(t)$, defined by $v^+(t) := \limsup_{h \rightarrow 0^+} \frac{1}{h} [v(t+h) - v(t)]$. At $x = x(t)$, this is equivalent to (see Yoshizawa (1975))

$$V^+(x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(x + hf_{cl}(x)) - V(x)].$$

Here $f_{cl}(x) := f(x) + B(x)[u_*(p) + K(p)e]$ corresponds to the right-hand side of the autonomous closed-loop system, which we recall is locally Lipschitz and thus guaranteeing (local) existence and uniqueness of solutions. It is known (Clarke et al., 2008) that the following holds:

$$V^+(x) \leq \limsup_{y \rightarrow x} \{DV(y)f_{cl}(x) : y \notin \mathfrak{X}_\alpha \cup \mathfrak{X}_\omega\}.$$

Hence (c) implies $v^+(t) \leq -\mu \cdot v(t)$ holding for all $t \geq t_0$ if the system is initialized within some neighborhood \mathfrak{T} at time t_0 . Thus (c) \implies (b) follows from the comparison lemma (see, e.g., Khalil (2002) and Yoshizawa (1975)).

What remains is therefore to show that the theorem's hypotheses imply (c). Under the assumption that V is differentiable at some x in \mathfrak{X} , one finds, using the shorthand notation $p = p(x)$, that its time derivative is

$$\dot{V} = \dot{e}^\top R(p)e + e^\top [R'(p)Dp(x)\dot{x}]e + e^\top R(p)\dot{e}. \tag{A.3}$$

Hence, whenever x is within the interior of either of the sets $\text{int } \mathcal{H}_i$, $i \in \{\alpha, \omega\}$, where $\|Dp(x)\| = 0$, one has by (16) and the ALEs (21) that the following holds therein:

$$\dot{V} = -e^\top Q_e e + O(\|e\|^3). \tag{A.4}$$

Whenever x is in \mathcal{T} , one instead has $Dp(x)\dot{x} = \rho(p(x)) + O(\|e\|)$ (this follows from the first-order Taylor expansion about $x_*(p(x))$ and by using (5)). Thus by (17), (A.1) and (A.3) we obtain, for $x \in \mathcal{T}$:

$$\dot{V} = e^\top \mathcal{E}_\perp^\top [A_{cl}^\top \mathcal{E}_\perp^\top R + R\mathcal{E}_\perp A_{cl} + \rho[R' - (\mathcal{P}')^\top \mathcal{F}^\top R - R\mathcal{F}\mathcal{P}']] \mathcal{E}_\perp e + O(\|e\|^3).$$

Since a solution $R_\perp(s)$ to (23) implies a solution to (22) (see Proposition 9), we thus obtain, using also (A.1), that

$$\dot{V} = -e^\top Q_\perp(p)e + O(\|e\|^3). \tag{A.5}$$

Thus, by (A.4) and (A.5), there exists some constant $\mu > 0$ such that the differential inequality $\dot{V} \leq -\mu V$ holds almost everywhere (or everywhere if one considers $V^+(x)$) within a neighborhood \mathfrak{T} of \mathcal{O} where $\|e\|$ is sufficiently small. This concludes the proof.

A.6. Proof of Proposition 14

Let us first demonstrate that the ALEs (21) are satisfied. To this end, we note that the constant matrix $A_{cl}(s_\alpha) = A_s(s_\alpha) + B_s(s_\alpha)Y(s_\alpha)W^{-1}(s_\alpha)$ is Hurwitz. Thus by Theorem 1 in Bernussou et al. (1989), there exists a matrix $\hat{Q}_\alpha \in \mathbb{M}_{>0}^n$ such that $\text{sym}[A_s(s_\alpha)W(s_\alpha) + B_s(s_\alpha)Y(s_\alpha)] = -\hat{Q}_\alpha$, where $\text{sym}[A] = A + A^\top$. For $R(s_\alpha) = W^{-1}(s_\alpha)$ we may therefore take $Q_\alpha = W^{-1}(s_\alpha)\hat{Q}_\alpha W^{-1}(s_\alpha)$ in (21a). The exact same arguments can be used for the point s_ω .

Let us now show that a matrix function $W(\cdot)$ solving the differential LMI (25) is equivalent to a solution $R(\cdot)$ to (22) (and therefore also a solution R_\perp to (23)). For this purpose, recall that for any smooth nonsingular matrix function $W : S \rightarrow \mathbb{R}^{n \times n}$ one has $\frac{d}{ds}W^{-1}(s) = -W^{-1}(s)[\frac{d}{ds}W(s)]W^{-1}(s)$. Thus taking $R(s) := W^{-1}(s)$ and dropping the s -argument to keep the notation short, we obtain the following from (25): $\rho R' \leq -\text{sym}[RA_\perp + RB_\perp K + \lambda RE_\perp]$. Multiplying from the left by \mathcal{E}_\perp^\top and by \mathcal{E}_\perp from the right, this can be written as $\mathcal{E}_\perp^\top \text{sym}[R\mathcal{E}_\perp A_{cl} + \lambda R + \rho(2^{-1}R' - R\mathcal{F}\mathcal{F}^\top D^2 p(x_*))] \mathcal{E}_\perp \leq 0_n$. Hence, as $R = W^{-1} \in \mathbb{M}_{>0}^n$ and λ is strictly positive, there must exist a C^1 -smooth matrix-valued function $Q_\perp : S \rightarrow \mathbb{M}_{>0}^n$ such that R solves the PrjLDE (22).

A.7. Proof of Theorem 17

The boundary conditions imposed on $\Phi(\cdot)$ are obvious, whereas those on $\Phi'(\cdot)$ are obtained directly from (30) by setting $\rho(s_\alpha) = \rho(s_\omega) = 0$. The condition $\alpha(s) \neq 0$ ensures the uniqueness and smoothness of the solutions to (34). Moreover, it implies that the integrating factor (36) is both nonzero and bounded on S , such that (33) \iff (37) on S by the fundamental theorem of calculus. By taking $s_1 = s_\alpha$ and $s_2 = s$ in (37), the function $\rho(s) = \sqrt{-2 \frac{\Psi(s, s_\alpha)}{\alpha^2(s)} \int_{s_\alpha}^s \Psi(s_\alpha, \tau) \alpha(\tau) \gamma(\tau) d\tau}$ can be obtained. Clearly it is smooth, bounded and satisfies (33) on S , while $\rho(s_\omega) = 0$ due to (38). To show that it is also real and strictly positive on $\text{int } S$, it suffices to note that the smooth function $\Upsilon(s) := \gamma(s)/\alpha(s)$ (which has the same sign as $\alpha(s)\gamma(s)$) is strictly negative on (s_α, s_e) and strictly positive on (s_e, s_ω) as $\Upsilon(\bar{s}) = 0$ and $\Upsilon'(\bar{s}) = v(\bar{s}) \forall \bar{s} \in \{s_\alpha, s_e, s_\omega\}$. Thus the term inside the square root is strictly positive on (s_α, s_e) . Since $\gamma(s) \neq 0$ for all $\text{int}(S) \setminus \{s_e\}$, and $v(s_e) > 0$, the terms inside the square root, and therefore also the function ρ , must remain strictly positive on (s_e, s_ω) .

References

- Al-Hiddabi, S. A., & McClamroch, N. H. (2002). Tracking and maneuver regulation control for nonlinear nonminimum phase systems: Application to flight control. *IEEE Transactions on Control Systems Technology*, 10(6), 780–792.
- Berger, M. S. (1977). *Nonlinearity and functional analysis: lectures on nonlinear problems in mathematical analysis*, Vol. 74. Academic Press.
- Bernussou, J., Peres, P., & Geromel, J. (1989). A linear programming oriented procedure for quadratic stabilization of uncertain systems. *Systems & Control Letters*, 13(1), 65–72.
- Borg, G. (1960). *A condition for the existence of orbitally stable solutions of dynamical systems*. Elander.
- Clarke, F. H., Ledyaev, Y. S., Stern, R. J., & Wolenski, P. R. (2008). *Nonsmooth analysis and control theory*, Vol. 178. Springer Science & Business Media.
- El-Hawwary, M. I., & Maggiore, M. (2013). Reduction theorems for stability of closed sets with application to backstepping control design. *Automatica*, 49(1), 214–222.
- Hahn, W. (1967). *Stability of motion*, Vol. 138. Springer.
- Hartman, P., & Olech, C. (1962). On global asymptotic stability of solutions of differential equations. *Transactions of the American Mathematical Society*, 104(1), 154–178.
- Hauser, J., & Chung, C. C. (1994). Converse Lyapunov functions for exponentially stable periodic orbits. *Systems & Control Letters*, 23(1), 27–34.
- Hauser, J., & Hindman, R. (1995). Maneuver regulation from trajectory tracking: Feedback linearizable systems. *IFAC Proceedings Volumes*, 28(14), 595–600.
- Hauser, J., Sastry, S., & Kokotovic, P. (1992). Nonlinear control via approximate input-output linearization: The ball and beam example. *IEEE Transactions on Automatic Control*, 37(3), 392–398.
- Khalil, H. K. (2002). *Nonlinear systems* (3rd ed.). Prentice hall Upper Saddle River, NJ.
- La Hera, P. X., Freidovich, L. B., Shiriaev, A. S., & Mettin, U. (2009). New approach for swinging up the Furuta pendulum: Theory and experiments. *Mechatronics*, 19(8), 1240–1250.
- Leonov, G. (1990). The orbital stability of the trajectories of dynamic systems. *Journal of Applied Mathematics and Mechanics*, 54(4), 425–429.
- Leonov, G. A. (2006). Generalization of the Andronov–Vitt theorem. *Regular and Chaotic Dynamics*, 11(2), 281–289.
- Leonov, G., Ponomarenko, D., & Smirnova, V. (1995). Local instability and localization of attractors. From stochastic generator to Chua's systems. *Acta Applicandae Mathematica*, 40(3), 179–243.
- Liu, Y., & Yu, H. (2013). A survey of underactuated mechanical systems. *IET Control Theory & Applications*, 7(7), 921–935.
- Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *2004 IEEE int. conf. on robotics and automation* (pp. 284–289). IEEE.
- Lynch, K. M., Shiroma, N., Arai, H., & Tanie, K. (1998). The roles of shape and motion in dynamic manipulation: The butterfly example. In *1998 IEEE int. conf. on robotics and automation*, Vol. 3 (pp. 1958–1963). IEEE.
- Manchester, I. R. (2011). Transverse dynamics and regions of stability for nonlinear hybrid limit cycles. *IFAC Proceedings Volumes*, 44(1), 6285–6290.
- Manchester, I. R., & Slotine, J.-J. E. (2014). Transverse contraction criteria for existence, stability, and robustness of a limit cycle. *Systems & Control Letters*, 63, 32–38.
- Olfati-Saber, R. (2001). *Nonlinear control of underactuated mechanical systems with application to robotics and aerospace vehicles* (Ph.D. thesis), Massachusetts Institute of Technology.
- Ruggiero, F., Lippiello, V., & Siciliano, B. (2018). Nonprehensile dynamic manipulation: A survey. *IEEE Robotics and Automation Letters*, 3(3), 1711–1718.
- Ryu, J.-C., Ruggiero, F., & Lynch, K. M. (2013). Control of nonprehensile rolling manipulation: Balancing a disk on a disk. *IEEE Transactions on Robotics*, 29(5), 1152–1161.
- Sætre, C. F., & Shiriaev, A. S. (2020). On excessive transverse coordinates for orbital stabilization of periodic motions. In *IFAC world congress 2020* (pp. 9250–9255). Elsevier.
- Sætre, C. F., Shiriaev, A. S., Pchelkin, S., & Chemori, A. (2020). Excessive transverse coordinates for orbital stabilization of (underactuated) mechanical systems. In *European control conf.* (pp. 895–900). IEEE.
- Sellami, S., Mamedov, S., & Khusainov, R. (2020). A ROS-based swing up control and stabilization of the Pendubot using virtual holonomic constraints. In *2020 int. conf. nonlinearity, information and robotics* (pp. 1–5). IEEE.
- Shiriaev, A. S., Freidovich, L. B., & Gusev, S. V. (2010). Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE Transactions on Automatic Control*, 55(4), 893–906.
- Shiriaev, A. S., Perram, J. W., & Canudas-de Wit, C. (2005). Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on Automatic Control*, 50(8), 1164–1176.
- Shiriaev, A. S., Robertsson, A., Perram, J., & Sandberg, A. (2006). Periodic motion planning for virtually constrained Euler–Lagrange systems. *Systems & Control Letters*, 55(11), 900–907.
- Spong, M. W. (1998). Underactuated mechanical systems. In *Control problems in robotics and automation* (pp. 135–150). Springer.
- Surov, M., Gusev, S., & Shiriaev, A. S. (2018). New results on trajectory planning for underactuated mechanical systems with singularities in dynamics of a motion generator. In *2018 conf. on decision and control (CDC)* (pp. 6900–6905). IEEE.
- Surov, M., Shiriaev, A. S., Freidovich, L., Gusev, S., & Paramonov, L. (2015). Case study in non-prehensile manipulation: planning and orbital stabilization of one-directional rollings for the “Butterfly” robot. In *2015 int. conf. on robotics and automation (ICRA)* (pp. 1484–1489). IEEE.
- Tütüncü, R. H., Toh, K.-C., & Todd, M. J. (2003). Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical Programming*, 95(2), 189–217.
- Urabe, M. (1967). *Nonlinear autonomous oscillations: analytical theory*, Vol. 34. Academic Press.
- Yoshizawa, T. (1975). *Stability theory and the existence of periodic solutions and almost periodic solutions*, Vol. 14. New York, NY: Springer-Verlag.
- Zubov, V. I. (1999). *Theory of oscillations*, Vol. 4. World Scientific.



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