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On the spectrum of the double-layer operator on locally-dilation-invariant Lipschitz domains

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Abstract

We say that Γ, the boundary of a bounded Lipschitz domain, is locally dilation invariant if, at each $x \in \Gamma$, Γ is either locally C^1 or locally coincides (in some coordinate system centred at x) with a Lipschitz graph Γ_x such that $\Gamma_x = \alpha_x \Gamma_x$, for some $\alpha_x \in (0, 1)$. In this paper we study, for such Γ, the essential spectrum of D_Γ, the double-layer (or Neumann–Poincaré) operator of potential theory, on $L^2(\Gamma)$. We show, via localisation and Floquet–Bloch-type arguments, that this essential spectrum is the union of the spectra of related continuous families of operators K_t, for $t \in [-\pi, \pi]$; moreover, each K_t is compact if Γ is C^1 except at finitely many points. For the 2D case where, additionally, Γ is piecewise analytic, we construct convergent sequences of approximations to the essential spectrum of D_Γ; each approximation is the union of the eigenvalues of finitely many finite matrices arising from Nyström-method approximations to the operators K_t. Through error estimates with explicit constants, we also construct functionals that determine whether any particular locally-dilation-invariant piecewise-analytic Γ

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satisfies the well-known spectral radius conjecture, that the essential spectral radius of D_Γ on $L^2(\Gamma)$ is $< 1/2$ for all Lipschitz Γ. We illustrate this theory with examples; for each we show that the essential spectral radius *is* $< 1/2$, providing additional support for the conjecture. We also, via new results on the invariance of the essential spectral radius under locally-conformal $C^{1,\beta}$ diffeomorphisms, show that the spectral radius conjecture holds for all Lipschitz curvilinear polyhedra.

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1 Introduction

Given a bounded Lipschitz domain[1] $\Omega_- \subset \mathbb{R}^d, d \geq 2$, with boundary Γ and outward-pointing unit normal vector n, the interior and exterior Dirichlet and Neumann problems for Laplace's equation (posed in Ω_- and in $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$, respectively), can be reformulated as boundary integral equations involving the operators

$$D_\Gamma \pm \tfrac{1}{2}I \quad \text{and} \quad D_\Gamma' \pm \tfrac{1}{2}I \tag{1.1}$$

(see, e.g., [64], [46, Sects. 5.9, 5.15.1]), where the *double-layer (or Neumann or Neumann-Poincaré) operator* D_Γ and the *adjoint double-layer operator* D_Γ' are defined by

$$D_\Gamma \phi(x) = \int_\Gamma \frac{\partial \Phi(x, y)}{\partial n(y)} \phi(y) \, ds(y) \quad \text{and} \quad D_\Gamma' \phi(x) = \int_\Gamma \frac{\partial \Phi(x, y)}{\partial n(x)} \phi(y) \, ds(y), \tag{1.2}$$

for $\phi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$, with the integrals understood, in general, as Cauchy principal values. Here $\Phi(x, y)$ is the fundamental solution for Laplace's equation, defined by[2]

$$\Phi(x, y) := \frac{1}{2\pi} \log\left(\frac{1}{|x - y|}\right), \quad d = 2, \quad := \frac{1}{(d-2)c_d} \frac{1}{|x - y|^{d-2}}, \quad d \geq 3, \tag{1.3}$$

where c_d is the surface measure of the unit sphere in $\mathbb{R}^d$. Explicitly,

$$D_\Gamma \phi(x) = \frac{1}{c_d} \int_\Gamma \frac{(x - y) \cdot n(y)}{|x - y|^d} \phi(y) \, ds(y) \quad \text{and}$$

$$D_\Gamma' \phi(x) = \frac{1}{c_d} \int_\Gamma \frac{(y - x) \cdot n(x)}{|x - y|^d} \phi(y) \, ds(y), \tag{1.4}$$

for $\phi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$.

Complementing (1.1),

$$D_\Gamma' - \lambda I, \tag{1.5}$$

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[1] For us, as, e.g., in [47], "domain" will just mean "non-empty open set"; a domain need not be connected.

[2] Our sign convention and normalisation are those of many authors (e.g. [41, 62]), but other authors (e.g. [34]), use a fundamental solution that is the negative of ours.

with $\lambda \in \mathbb{C} \setminus \{-\frac{1}{2}, \frac{1}{2}\}$, arises as the operator in the boundary integral equation reformulation of transmission problems in electrostatics, where the Laplace equation $\Delta u = 0$ holds in Ω_- and Ω_+ and the trace of u or its normal derivative jumps across Γ (see, e.g., [46, Sect. 5.12]). In this context the case $|\lambda| \geq 1/2$, especially with λ real, is classically of interest (e.g., [46, Sect. 5.12]); more recently the case where λ is complex with $|\lambda| < 1/2$ has been studied intensively as a model of quasi-static electromagnetic plasmonic problems (e.g., [1, 15, 27, 59]).

Motivated by these physical applications, and by questions in harmonic analysis, there has been long-standing interest in the computation of the spectrum and essential spectrum[3] of D_Γ as an operator on a variety of function spaces, especially for non-smooth domains (e.g., [1, 12, 18, 27, 34, 38, 39, 48, 52, 53]). The largest part of this literature is concerned specifically with the 2D/3D cases where Γ is a (curvilinear) polygon (e.g., [10, 49, 50, 60, 61]) or polyhedron (e.g., [15, 16, 20, 26, 27, 48, 50, 56]). In this paper we will study and compute the essential spectrum of D_Γ as an operator on $L^2(\Gamma)$ for a substantially larger class of boundaries, namely for the case where the boundary Γ is *locally dilation invariant* in the sense of Definition 1.3 below. In 2D (3D) this class includes polygons (polyhedra) but it also admits much wilder boundary behaviour (e.g., Fig. 1) as we discuss next in Sect. 1.1.

1.1 The spectral radius conjecture and the main question we address

Given a bounded linear operator $T \colon Y \to Y$ on a Banach space Y, we define its *spectral radius*, $\rho(T; Y)$, and its *essential spectral radius*, $\rho_{\mathrm{ess}}(T; Y)$, by

$$\rho(T; Y) := \sup_{\lambda \in \sigma(T; Y)} |\lambda| \quad \text{and} \quad \rho_{\mathrm{ess}}(T; Y) := \sup_{\lambda \in \sigma_{\mathrm{ess}}(T; Y)} |\lambda|, \qquad (1.6)$$

abbreviating $\rho(T; Y)$ and $\rho_{\mathrm{ess}}(T; Y)$ by $\rho(T)$ and $\rho_{\mathrm{ess}}(T)$, respectively, when the space Y is clear from the context. The analysis and computation we will carry out are motivated by the so-called *spectral radius conjecture*.

This conjecture, in the explicit 1994 formulation of Kenig [34], is as follows, where $L_0^2(\Gamma) := \{\phi \in L^2(\Gamma) : \int_\Gamma \phi \, \mathrm{d}s = 0\}$.

Conjecture 1.1 If Γ is the boundary of a bounded Lipschitz domain Ω_- and is connected, the spectral radius of D'_Γ on $L_0^2(\Gamma)$ is $< \frac{1}{2}$, i.e. $\rho(D'_\Gamma; L_0^2(\Gamma)) < \frac{1}{2}$.

In Sect. 2 we will discuss the following alternative formulation of the conjecture which makes sense regardless of the connectivity of Γ, and show its equivalence with Conjecture 1.1.

Conjecture 1.2 If Γ is the boundary of a bounded Lipschitz domain Ω_-, the essential spectral radius of D_Γ on $L^2(\Gamma)$ is $< \frac{1}{2}$, i.e. $\rho_{\mathrm{ess}}(D_\Gamma; L^2(\Gamma)) < \frac{1}{2}$.

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[3] Given a Banach space Y and a bounded linear operator $T \colon Y \to Y$ we denote the spectrum of T, the set of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not invertible, by $\sigma(T; Y)$, and the essential spectrum, the set of λ for which $T - \lambda T$ is not Fredholm, by $\sigma_{\mathrm{ess}}(T; Y)$, abbreviating these by $\sigma(T)$ and $\sigma_{\mathrm{ess}}(T)$ where the Banach space Y is clear from the context.
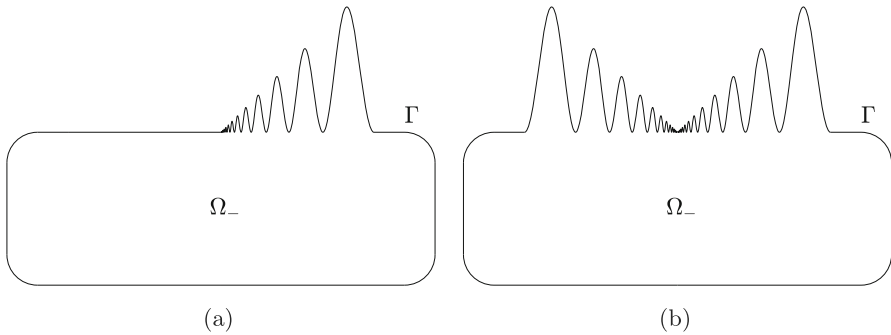
Fig. 1 Examples of bounded Lipschitz domains Ω_- with boundaries $\Gamma \in \mathscr{D}_{\mathcal{A}} \subset \mathscr{D}$ that are piecewise analytic locally dilation invariant

The spectral radius conjecture is very well studied, owing to its intrinsic interest in harmonic analysis, its possible relevance for computation,[4] and its immediate role within electrostatics, as well as in interpretations of electrodynamical problems [26]. It originated in the setting of continuous functions in $\overline{\Omega_-}$, dating all the way back to Neumann in the late 1800s, who treated convex domains, and to Radon [55], who famously analyzed curves of bounded rotation. In a tour de force, Král [38, 39] completely characterized when the essential norm of D_Γ is $< 1/2$. This result was extended into higher dimensions by Burago and Maz'ya [8] and Netuka [51]. For polyhedra in 3D, the essential norm can be $> 1/2$. Nevertheless, Rathsfeld [56] (and see [57]) and Grachev and Maz'ya [20] independently proved the spectral radius conjecture in the continuous setting holds for general polyhedra. These results were extended to locally conformal deformations of polyhedra by Medková [45]. Even when specialised to the continuous setting, the history is vast, and we refer to Wendland [65] for an in-depth survey.

As modern harmonic analysis developed, the natural setting for the double-layer potential shifted toward $L^2(\Gamma)$. We make particular mention of the demonstrations of L^2-boundedness of the Cauchy integral due to Calderón [9] and Coifman, Mcintosh, and Meyer [13], and Verchota's [64] application of these results to study invertibility of $D_\Gamma' \pm \frac{1}{2}I$ on $L^2(\Gamma)$ and $L_0^2(\Gamma)$ when Γ is connected. Since then, a flurry of activity and findings in this area have provided support for Kenig's conjecture, though a complete proof has proved elusive.

Indeed, to the best of our knowledge, Conjecture 1.1 has been established (only) in the following cases: (a) Ω_- is convex [18] (and see [12] for extensions to locally convex domains); (b) Ω_- has small Lipschitz character[5] [48], a case which includes all C^1 domains [19]; (c) ε-regular Semmes–Kenig–Toro domains Ω_- for sufficiently small $\varepsilon > 0$ [30], including in particular all domains whose gradient has vanishing mean oscillation [28]; (d) Ω_- is a polygon or curvilinear polygon in 2D [60, 61], or a Lipschitz polyhedron in 3D [16].

Note that polygonal and polyhedral boundaries Γ are locally invariant under all dilations: at every point $x \in \Gamma$, Γ locally coincides with a graph Γ_x such that (in

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[4] Notably, if the spectral radius conjecture holds then the Neumann series for $(D_\Gamma' \pm \frac{1}{2}I)\phi = g$, equivalently the Neumann iteration $\pm \frac{1}{2}\phi^{(n)} = g - D_\Gamma'\phi^{(n-1)}$, $n \in \mathbb{N}$, converges in $L_0^2(\Gamma)$.

[5] See, e.g., [11, Definition 3.1] for the definition of the Lipschitz character of a Lipschitz domain.

some local coordinate system centred at x) $\Gamma_x = \alpha\Gamma_x$ *for all* $\alpha > 0$. In this paper we will investigate domains where, locally, the dilation invariance $\Gamma_x = \alpha_x\Gamma_x$ only holds *for one* $\alpha_x \in (0, 1)$ (in which case we say that Γ is *locally dilation invariant at x*). Precisely our focus will be on the following class of boundaries.

Definition 1.3 (*Locally dilation invariant*) Given Γ, the boundary of a Lipschitz domain Ω_-, we say that $\Gamma \in \mathscr{D}$, the set of *locally-dilation-invariant* boundaries if, at every $x \in \Gamma$, Γ is either locally C^1 or locally coincides (in some coordinate system centred at x) with a Lipschitz graph Γ_x which is dilation invariant with respect to some $\alpha_x \in (0, 1)$, i.e. $\Gamma_x = \alpha_x\Gamma_x$.

Note that, already in 2D, $\mathscr{D}$ is a hugely larger class of domains than that of the curvilinear polygons. Indeed, for a Jordan curve $\Gamma \subset \mathbb{R}^2$ and $x \in \Gamma$, let

$$v_\Gamma(x) = \int_0^{2\pi} |\{x + re^{i\theta} \in \Gamma : r > 0\}| \frac{d\theta}{2\pi},$$

where $|\cdot|$ in this equation denotes the counting measure of a set. Then $D_\Gamma\phi$ is uniformly continuous on Ω_- for every $\phi \in C(\Gamma)$ if and only if

$$\sup_{x\in\Gamma} v_\Gamma(x) < \infty, \tag{1.7}$$

see [7, 36, 37]. If Γ is Lipschitz and locally dilation invariant at x, then it is clear that $v_\Gamma(x) = \infty$, unless Γ coincides with two line segments around x. It follows that the only curves $\Gamma \in \mathscr{D}$ satisfying (1.7) are curvilinear polygons. That is, except for curvilinear polygons, our curves exhibit such wild boundary behaviour that it is not possible to consider D_Γ in the setting of continuous boundary data.

In the context of studying the double-layer and related operators, the class of domains $\mathscr{D}$ seems to have been first considered in [11], where the essential numerical range[6] of D_Γ was studied and, in 2D and 3D, examples of boundaries $\Gamma \in \mathscr{D}$ with large Lipschitz character were constructed such that $D_\Gamma: L^2(\Gamma) \to L^2(\Gamma)$ has arbitrarily large *essential numerical radius*,

$$w_{\mathrm{ess}}(D_\Gamma) := \sup_{z\in W_{\mathrm{ess}}(D_\Gamma)} |z|,$$

and so also arbitrarily large *essential norm*,

$$\|D_\Gamma\|_{\mathrm{ess}} := \inf_{K\,\mathrm{compact}} \|D_\Gamma - K\|,$$

since, for any bounded operator T on a Hilbert space [22, Sect. 1.3],

$$\tfrac{1}{2}\|T\|_{\mathrm{ess}} \le w_{\mathrm{ess}}(T) \le \|T\|_{\mathrm{ess}}. \tag{1.8}$$

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[6] Recall that, for a bounded linear operator $T: H \to H$ on a Hilbert space H, the numerical range of T is $W(T) := \{\langle T\phi, \phi\rangle : \|\phi\| = 1\}$ and its essential numerical range is $W_{\mathrm{ess}}(T) := \bigcap_{K\,\mathrm{compact}} \overline{W(T + K)}$.

The 2D examples in [11] (see, e.g., [11, Fig. 3] and cf. Fig. 1) with arbitrarily large $w_{ess}(D_\Gamma)$ are necessarily examples for which (1.7) fails to hold, since for curvilinear polygons on $L^2(\Gamma)$ it is well-known that

$$\|D_\Gamma\|_{ess} = w_{ess}(D_\Gamma) = \rho(D_\Gamma; L^2(\Gamma)) < \tfrac{1}{2}$$

(see [10, 49, 60, 61] for $\|D_\Gamma\|_{ess}$ and $\rho(D_\Gamma; L^2(\Gamma))$); equality for $w_{ess}(D_\Gamma)$ follows by (1.8) and since $W_{ess}(D_\Gamma) \supset \sigma_{ess}(D_\Gamma; L^2(\Gamma))$ [6]).

A natural question, prompted by the examples from Chandler-Wilde and Spence [11] with $w_{ess}(D_\Gamma)$ arbitrarily large and the spectral radius conjecture, is the following:

Given that there exist $\Gamma \in \mathscr{D}$ with $w_{ess}(D_\Gamma) \gg \tfrac{1}{2}$, in particular such examples in 2D, is there a $\Gamma \in \mathscr{D}$, in particular an example in 2D, with $\rho_{ess}(D_\Gamma; L^2(\Gamma)) \geq \tfrac{1}{2}$?

Of course, a positive answer would provide a counterexample to Conjecture 1.2 and hence to the original spectral radius conjecture, Conjecture 1.1. The aim of this paper is to address this question, through mathematical analysis and computational methods supported by numerical analysis error estimates where constants are made explicit. These will enable us to estimate $\rho_{ess}(D_\Gamma; L^2(\Gamma))$ for a large class of $\Gamma \in \mathscr{D}$ sufficiently accurately to determine whether or not $\rho_{ess}(D_\Gamma; L^2(\Gamma)) \geq \tfrac{1}{2}$.

1.2 Our main results and their significance

The first step in our analysis is Theorem 5.2 in Sect. 5.1, the localisation result (cf. [11, 15, 16, 48]) that, for $\Gamma \in \mathscr{D}$ (and for any dimension $d \geq 2$), there exists a finite set $F \subset \Gamma$ such that

$$\sigma_{ess}(D_\Gamma; L^2(\Gamma)) = \bigcup_{x \in F} \sigma_{ess}(D_{\Gamma_x}; L^2(\Gamma_x)). \tag{1.9}$$

In the above formula Γ_x is defined as above if Γ is locally dilation invariant at x, while if Γ is locally C^1 at x then Γ_x is the graph of a C^1 compactly supported function so that [19] D_{Γ_x} is compact and $\sigma_{ess}(D_{\Gamma_x}; L^2(\Gamma_x)) = \{0\}$.

The localisation (1.9) reduces the computation of $\sigma_{ess}(D_\Gamma; L^2(\Gamma))$ to that of $\sigma_{ess}(D_{\Gamma_x}; L^2(\Gamma_x))$ for finitely many x for which $\alpha_x \Gamma_x = \Gamma_x$, for some $\alpha_x \in (0, 1)$. Computation of $\sigma_{ess}(D_{\Gamma_x}; L^2(\Gamma_x))$ for such x is our focus in Sect. 4 where we study and compute spectral properties of D_Γ in the case that $\Gamma \subset \mathbb{R}^d$ is a *dilation invariant* Lipschitz graph, meaning that there exists an $\alpha \in (0, 1)$ such that $\alpha \Gamma = \Gamma$ (an example is Fig. 3). If $\Gamma_0 \subset \Gamma \setminus \{0\}$ is a particular relatively closed and bounded Lipschitz subgraph of Γ such that

$$\Gamma = \bigcup_{j \in \mathbb{Z}} \alpha^j \Gamma_0, \tag{1.10}$$

and such that $(\alpha^j \Gamma_0) \cap \Gamma_0$ has zero surface measure for $j \neq 0$, we show that D_Γ can be written as a discrete ℓ_1 convolution operator whose entries are bounded linear operators on $L^2(\Gamma_0)$ related to the discretization (1.10) of Γ. This allows us to decompose D_Γ, by a Floquet–Bloch transform, into a continuous family $(K_t)_{t \in [-\pi, \pi]}$ of operators

$K_t: L^2(\Gamma_0) \to L^2(\Gamma_0)$, and, by standard results for such convolutions (see [54, Theorem 2.3.25]), to characterise the essential spectrum of D_Γ as (Theorem 4.4)

$$\sigma_{\text{ess}}(D_\Gamma; L^2(\Gamma)) = \sigma(D_\Gamma; L^2(\Gamma)) = \bigcup_{t \in [-\pi, \pi]} \sigma(K_t; L^2(\Gamma_0)). \qquad (1.11)$$

This characterisation is particularly useful when, apart from a singularity at 0, Γ is C^1 (see, e.g., Fig. 3), for then (Corollary 4.6) each K_t is compact, and so has a more easily computed discrete spectrum. The characterisation (1.11) holds for every Lipschitz dilation invariant graph in any dimension; in Sect. 4.3 we specialise to the 2D case where Γ, except at 0, is the graph of a real analytic function (see, e.g., Fig. 3); we denote by $\mathcal{A}$ this subset of the 2D dilation invariant graphs. We show, for each $t \in [-\pi, \pi]$, that K_t is unitarily equivalent to $\tilde{K}_t$, a 2×2 matrix of integral operators on $L^2(0, 1)$ that have real analytic kernels. Further, the range of each of these integral operators is a space of 1-quasi-periodic real analytic functions. As a consequence,

$$\sigma(K_t; L^2(\Gamma_0)) = \sigma(\tilde{K}_t; (C[0, 1])^2),$$

and this latter spectrum can be computed by approximating $\tilde{K}_t$ by a $2N \times 2N$ matrix $\tilde{A}_{t,N}$ obtained by a simple midpoint-rule based Nyström discretization. As is well-known (see [41, 63], Theorem 4.11), the midpoint rule is exponentially convergent for periodic analytic functions, so that it follows from Nyström-method spectral estimates for integral operators with continuous kernels [3] that, for each t, the eigenvalues of $A_{t,N}$ converge at an exponential rate to those of $\tilde{K}_t$ as $N \to \infty$.

This leads (see Theorem 4.19) to a Nyström approximation, $\sigma^N(D_\Gamma)$, for $\sigma(D_\Gamma) = \sigma_{\text{ess}}(D_\Gamma)$, which is $\{0\}$ plus the union of the eigenvalues of finitely many $2N \times 2N$ matrices. Our first main result is to show, as Theorem 4.19, that $\sigma^N(D_\Gamma) \to \sigma(D_\Gamma)$ in the Hausdorff metric as $N \to \infty$; that this convergence is achievable is somewhat surprising given that D_Γ is neither compact nor self-adjoint. Our second, and more substantial result (Theorem 4.21) is that we develop a fully discrete algorithm to test whether, as an operator on $L^2(\Gamma)$, $\rho_{\text{ess}}(D_\Gamma) < \frac{1}{2}$. Precisely, we construct (see Remark 4.22), for each $c > 0$, a nonlinear functional $S_c : \mathcal{A} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ with the properties that: a) the functional can be computed in finitely many arithmetic operations and finitely many evaluations of elementary functions, given finitely many sampled values of f and its first and second derivatives for real arguments, plus bounds on the analytic continuation of f to a neighbourhood of the real line that depends on c; b) $\rho_{\text{ess}}(D_\Gamma) < \frac{1}{2}$ if, for some $c > 0$ and $m, M, N \in \mathbb{N}$, it holds that:

(i) $\rho(A^M_{t_k,N}) < \frac{1}{2}$, for $k = 1, \ldots, m$, where $t_k := (k - 1/2)\pi/m$ and $A^M_{t_k,N}$ is a specific approximation to $A_{t_k,N}$ depending on the parameter M;
(ii) $S_c(\Gamma, m, M, N) < 0$.

Conversely, if $\rho_{\text{ess}}(D_\Gamma) < \frac{1}{2}$, then, for all sufficiently small $c > 0$, (i) and (ii) hold for all sufficiently large $N, M, m \in \mathbb{N}$.

In Sect. 6 we bring these results together to address our question at the end of Sect. 1.1. We restrict attention to the following 2D class of domains; this class

includes polygons and piecewise-analytic curvilinear polygons, but also wilder boundary behaviour as illustrated in Fig. 1.

Definition 1.4 (*Piecewise analytic locally dilation invariant*) In the case $d = 2$ we say that $\Gamma \in \mathscr{D}_\mathcal{A}$, the set of *piecewise-analytic locally-dilation-invariant* boundaries, if $\Gamma \in \mathscr{D}$ and Γ is locally analytic (i.e., is locally the graph of a real-analytic function) at all but finitely many $x \in \Gamma$.

Given $\Gamma \in \mathscr{D}_\mathcal{A}$, if $F \subset \Gamma$ is the finite set of points at which Γ is not locally analytic and is locally dilation invariant, it is easy to see that $\Gamma_x \in \mathcal{A}$ for $x \in F$ and that (1.9) holds for this set F. Thus (see Theorem 6.1)

$$\Sigma^N(D_\Gamma) := \bigcup_{x \in F} \sigma^N(D_{\Gamma_x}) \to \sigma_{\mathrm{ess}}(D_\Gamma) \tag{1.12}$$

in the Hausdorff metric as $N \to \infty$, and note that $\Sigma^N(D_\Gamma)$ is the union of the eigenvalues of finitely many $2N \times 2N$ matrices. Further, as we discuss in Sect. 6, it follows that $\rho_{\mathrm{ess}}(D_\Gamma) < \frac{1}{2}$ if these eigenvalues lie within the disc of radius $\frac{1}{2}$, i.e. if

$$R_N(D_\Gamma) := \max\{|z| : z \in \Sigma^N(D_\Gamma)\} < \tfrac{1}{2}, \tag{1.13}$$

and if also, for some $c > 0$,

$$\mathscr{S}_c(\Gamma, N) := \max_{x \in F} S_c(\Gamma_x, N, N, N) < 0, \tag{1.14}$$

where $\mathscr{S}_c(\Gamma, N)$ can be computed in finitely many arithmetic operations plus finitely many evaluations of elementary functions, given inputs describing each Γ_x as discussed above. Conversely, if $\rho_{\mathrm{ess}}(D_\Gamma) < \frac{1}{2}$ then, for all sufficiently small $c > 0$ and all sufficiently large N, $R_N(D_\Gamma) < \frac{1}{2}$ and $\mathscr{S}_c(\Gamma, N) < 0$. Thus, given inputs describing Γ, our fully discrete algorithms enable us to test, for individual $\Gamma \in \mathscr{D}_\mathcal{A}$, the validity of Conjecture 1.2, i.e. whether or not $\rho_{\mathrm{ess}}(D_\Gamma) < \frac{1}{2}$, through computation of the eigenvalues of finitely many finite matrices, plus finitely many additional arithmetic operations.

In Sect. 5 we prove the localisation result (1.9). We also, in the spirit of Medková's study in the continuous setting [43, 45], consider the stability of Conjecture 1.2 under locally conformal deformations. We prove that the spectral radius conjecture is independent of such deformations, under the assumption that the absolute value of the kernel of the double-layer potential also defines a bounded operator on $L^2(\Gamma)$. While this additional hypothesis may fail to include the wildest of boundaries, it applies to many domains from $\mathscr{D}$. For example, as we discuss in Sect. 5.2, it is satisfied by any dilation invariant Lipschitz graph Γ whose generating set Γ_0 is polygonal or polyhedral, in particular it holds if Γ is a polyhedron and, in 2D, if $\Gamma \in \mathscr{D}_\mathcal{A}$. As one consequence (Corollary 5.7), Conjecture 1.2 holds for Lipschitz curvilinear polyhedra, because [16] it holds for polyhedra; as another (Corollary 5.11), if it holds for $\Gamma \in \mathscr{D}_\mathcal{A}$, then it holds for any locally conformal deformation of Γ.

To illustrate the above results, and use them to test Conjecture 1.2 and address our main question from Sect. 1.1, we include a range of numerical examples in Sect. 6, for piecewise-analytic Lipschitz graphs $\Gamma \in \mathcal{A}$, and for piecewise-analytic locally-dilation-invariant $\Gamma \in \mathscr{D}_{\mathcal{A}}$ that are the boundaries of bounded Lipschitz domains. For each example we plot an approximation ($\sigma^N(D_\Gamma)$ or $\Sigma^N(D_\Gamma)$, as appropriate) to $\sigma_{\mathrm{ess}}(D_\Gamma)$. Moreover, we employ the algorithms described above to provide convincing numerical evidence that $\rho_{\mathrm{ess}}(D_\Gamma; L^2(\Gamma)) < \frac{1}{2}$ in every case we examine, including cases where $w_{\mathrm{ess}}(D_\Gamma)$ is significantly $> \frac{1}{2}$. These results are evidence that the spectral radius conjecture holds for the class of 2D domains $\mathscr{D}_{\mathcal{A}}$; we emphasise again that this conjecture has not been studied previously for boundaries in this class, except for the special case of curvilinear polygons.

Let us briefly summarise the remainder of the paper. In Sect. 2 we prove the equivalence of Conjectures 1.1 and 1.2. In Sect. 3, as a key step to our main results, we derive bounds for the spectral radii of general compact operators in Sect. 3.1 (e.g., Corollary 3.5), specialising to the case of integral operators approximated by the Nyström method in Sect. 3.2 (e.g., Theorem 3.8). In Sect. 4 we prove the results noted above on the essential spectrum and essential spectral radius of D_Γ in the case when Γ is a dilation-invariant Lipschitz graph, with particular focus (Sects. 4.2–4.4) on the 2D piecewise-analytic case. In Sect. 5 we prove our localisation results. In Sect. 6 we bring the earlier results together, in particular to study the spectral radius conjecture for $\Gamma \in \mathscr{D}_{\mathcal{A}}$, and we illustrate our theory by numerical examples.

2 Formulations of the spectral radius conjecture

Our results are related to the spectral radius conjecture of Kenig [34, Problem 3.2.12], that $\rho(D_\Gamma'; L_0^2(\Gamma)) < \frac{1}{2}$ if Γ is the boundary of a bounded Lipschitz domain. Conjecture 1.1, stated in the introduction, is a version of this conjecture that avoids difficulties when Γ is not connected[7]. In this section we show the equivalence between Conjecture 1.1 and Conjecture 1.2, also stated in the introduction. Conjecture 1.2 concerns the essential spectrum rather than the spectrum and makes sense whatever the connectedness of Γ, Ω_-, and Ω_+.

Of course, because D_Γ' is the adjoint of D_Γ (as an operator on $L^2(\Gamma)$), and so shares the same essential spectrum, a statement equivalent to Conjecture 1.2 is that $\rho_{\mathrm{ess}}(D_\Gamma'; L^2(\Gamma)) < \frac{1}{2}$. Further, as bounded Lipschitz domains have only finitely many boundary components, it is clear that Conjecture 1.2 is true if it is true whenever Γ is connected.[8] Thus the equivalence of Conjectures 1.1 and 1.2, i.e. that Conjecture 1.1 holds (as claimed, whenever Γ is connected) if and only if Conjecture 1.2 holds (as claimed, whatever the topology of Γ), is implied by the following lemma.

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[7] By results of Mitrea [47, Theorem 4.1] if Ω_- or Ω_+ are not connected, $\{\frac{1}{2}, -\frac{1}{2}\} \cap \sigma(D_\Gamma'; L_0^2(\Gamma)) \neq \emptyset$. Indeed, $D_\Gamma' \pm \frac{1}{2}I$ are Fredholm of index zero on $L^2(\Gamma)$ by part (2) of [47, Theorem 4.1], but by (4) of the same theorem, and since $D_\Gamma(1) = -\frac{1}{2}$ so that $D_\Gamma'(L_0^2(\Gamma)) \subset L_0^2(\Gamma)$ [see (2.1) below], the codimension of $D_\Gamma' - \frac{1}{2}I$ on $L_0^2(\Gamma)$ is ≥ 1 if Ω_+ is not connected; that of $D_\Gamma' + \frac{1}{2}I$ is ≥ 1 if Ω_- is not connected.

[8] Note that if Γ_1 and Γ_2 are separate components of Γ, so that there is a positive distance between Γ_1 and Γ_2, the double-layer operator from $L^2(\Gamma_1)$ to $L^2(\Gamma_2)$ has a kernel that is bounded, so is a Hilbert-Schmidt operator and hence compact.

Lemma 2.1 *Assume that* Γ *is the boundary of a bounded Lipschitz domain* Ω_- *and is connected. Then* $\rho(D'_\Gamma; L^2_0(\Gamma)) < \frac{1}{2}$ *if and only if* $\rho_{ess}(D'_\Gamma; L^2(\Gamma)) < \frac{1}{2}$.

The main step in the proof of this lemma is the following theorem stated in [18].[9] As noted in [18], the proof of this result is that in Kellogg's classical book for the case of smooth boundaries (see [33, Chapter IX, Sect. 11]), which carries over to the Lipschitz case.

Theorem 2.2 *[Theorem 1.1 in [18]]* *If* Γ *is the boundary of a bounded Lipschitz domain* Ω_-, *the eigenvalues of* D'_Γ, *as an operator on* $L^2(\Gamma)$, *are real and lie in* $[-\frac{1}{2}, \frac{1}{2}]$.

Proof of Lemma 2.1 Let $\phi \in L^2(\Gamma)$ and let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2(\Gamma)$. We first observe that, since $D_\Gamma(1) = -\frac{1}{2}$, we have

$$\langle D'_\Gamma \phi, 1 \rangle = \langle \phi, D_\Gamma(1) \rangle = -\frac{1}{2}\langle \phi, 1 \rangle, \tag{2.1}$$

so $D'_\Gamma(L^2_0(\Gamma)) \subseteq L^2_0(\Gamma)$.

Now assume that $\lambda \in \mathbb{C}$ and $D'_\Gamma - \lambda I$ is invertible as an operator on $L^2_0(\Gamma)$. Let P be orthogonal projection from $L^2(\Gamma)$ onto the constants, so that $Q := I - P$ is projection onto $L^2_0(\Gamma)$. Then $(D'_\Gamma - \lambda)Q + P$ is invertible as an operator on $L^2(\Gamma)$. It follows that

$$D'_\Gamma - \lambda I = ((D'_\Gamma - \lambda)Q + P) + ((D'_\Gamma - \lambda)P - P) \tag{2.2}$$

is Fredholm as an operator on $L^2(\Gamma)$. This implies that $\sigma_{ess}(D'_\Gamma; L^2(\Gamma)) \subset \sigma(D'_\Gamma; L^2_0(\Gamma))$, which settles one direction.

Conversely, assume that $\rho_{ess}(D'_\Gamma; L^2(\Gamma)) < \frac{1}{2}$. Let $\lambda \in \mathbb{C}$ with $|\lambda| \geq \frac{1}{2}$. Then $D'_\Gamma - \lambda I$ is Fredholm of index 0 on $L^2(\Gamma)$, so [see (2.2)] $(D'_\Gamma - \lambda)Q + P$ is Fredholm of index zero on $L^2(\Gamma)$, so that $D'_\Gamma - \lambda I$ is Fredholm of index zero on $L^2_0(\Gamma)$. Thus $\lambda \in \sigma(D'_\Gamma; L^2_0(\Gamma))$ if and only if λ is an eigenvalue of D'_Γ. Hence, if $\lambda \notin [-\frac{1}{2}, \frac{1}{2}]$, $D'_\Gamma - \lambda I$ is invertible on both $L^2(\Gamma)$ and $L^2_0(\Gamma)$ by Theorem 2.2 (as $L^2_0(\Gamma) \subseteq L^2(\Gamma)$, every eigenvalue on $L^2_0(\Gamma)$ is also an eigenvalue on $L^2(\Gamma)$). But also, if $\lambda = \pm\frac{1}{2}$, since Γ is connected, Verchota's results [64] show that $D'_\Gamma \pm \frac{1}{2}I$ is invertible on $L^2_0(\Gamma)$. Thus $D'_\Gamma - \lambda I$ is invertible on $L^2_0(\Gamma)$ for $|\lambda| \geq \frac{1}{2}$, so that, since the spectrum is closed, $\rho(D'_\Gamma; L^2_0(\Gamma)) < \frac{1}{2}$.   □

3 Approximation of the spectral radius for compact operators

In this section we recall in Sect. 3.1 results from operator approximation theory in Banach spaces related to the spectra of compact operators, and derive what appear to

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[9] In [18, Theorem 1.1] slightly more is claimed, that the eigenvalues of D'_Γ lie in $[-\frac{1}{2}, \frac{1}{2})$. This is more than is claimed in [33, Chapter IX, Sect. 11], and in fact $\frac{1}{2}$ is an eigenvalue if Ω_+ is not connected; see [47, Theorem 4.1].

be new general criteria for $\rho(T) < \rho_0$ when T is compact and $\rho_0 > 0$ (Lemma 3.2, Corollary 3.5). This leads, in Sect. 3.2, to results relating to the approximation of integral operators with continuous kernels by the Nyström method that will be key for the arguments in Sect. 4.2 and Sect. 4.3. Notably, Theorem 3.8 provides criteria for $\rho(T) < \rho_0$ when T is an integral operator with a continuous kernel that requires the computation only of the spectral radius of a finite matrix plus the norms of finitely many finite matrix resolvents.

3.1 Operator approximation results

We recall first two standard results on the approximation of operators in $\mathcal{L}(Y)$, the space of bounded linear operators on a Banach space Y. The first is the basic per-turbation estimate that, if $S, T \in \mathcal{L}(Y)$ and S is invertible, then T is invertible if $\|S - T\| \, \|S^{-1}\| < 1$, with

$$\left\| T^{-1} \right\|^{-1} \geq \left\| S^{-1} \right\|^{-1} - \|S - T\|. \tag{3.1}$$

If $S \in \mathcal{L}(Y)$ is invertible, then $\|S^{-1}\|^{-1} = \inf_{\|\phi\|=1} \|S\phi\|$, this sometimes called the *lower norm* of S (see, e.g., [42, Lemma 2.35]). The second estimate is as follows:

Lemma 3.1 (Theorem 4.7.7 of [23]) *Let Y be a Banach space, $S \in \mathcal{L}(Y)$ and $\lambda \notin \sigma(S) \cup \{0\}$. If $T \in \mathcal{L}(Y)$ is a compact operator that satisfies*

$$\|(T - S)T\| < |\lambda| \left\| (S - \lambda I)^{-1} \right\|^{-1},$$

then $T - \lambda I$ is invertible.

The following result is a consequence of the above estimates and the maximum principle applied to the resolvent. Here $\mathbb{T} = \{z : |z| = 1\}$ is the unit circle in the complex plane, so that $\rho_0 \mathbb{T}$ is the circle of radius ρ_0.

Lemma 3.2 *Let Y be a Banach space, $S, \widehat{S}, T \in \mathcal{L}(Y)$, $\rho_0 > 0$, $F \subset \rho_0 \mathbb{T}$, and suppose that: T is compact; $\rho(\widehat{S}) < \rho_0$; for every $\lambda \in \rho_0 \mathbb{T}$ there exists $\mu \in F$ such that*

$$\|(T - S)T\| < \rho_0 \left(\left\| (\widehat{S} - \mu I)^{-1} \right\|^{-1} - \|S - \widehat{S}\| - |\lambda - \mu| \right). \tag{3.2}$$

Then $\rho(T) < \rho_0$.

The idea is to choose F to be a finite set,[10] S a finite rank approximation to T, and $\widehat{S}$ a numerical approximation to S, in which case one can show $\rho(T) < \rho_0$ by computing $\rho(\widehat{S})$ and $\|(\widehat{S} - \mu I)^{-1}\|$ for finitely many μ. Taking $F = \rho_0 \mathbb{T}_n$, where $\mathbb{T}_n$ is the nth roots of unity, we obtain:

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[10] We will also apply this lemma later in the case that $F = \rho_0 \mathbb{T}$, when (3.2) reduces to the condition that, for every $\lambda \in \rho_0 \mathbb{T}$, $\|(T - S)T\| < \rho_0(\|(\widehat{S} - \lambda I)^{-1}\|^{-1} - \|S - \widehat{S}\|)$.

Corollary 3.3 *Let Y be a Banach space, $S, \widehat{S}, T \in \mathcal{L}(Y)$, $\rho_0 > 0$, $n \in \mathbb{N}$, and suppose that T is compact, $\rho(\widehat{S}) < \rho_0$, and*

$$\|(T - S)T\| < \rho_0 \left(\|(\widehat{S} - \mu I)^{-1}\|^{-1} - \|S - \widehat{S}\| - 2\rho_0 \sin\left(\frac{\pi}{2n}\right) \right), \quad \mu \in \rho_0 \mathbb{T}_n. \tag{3.3}$$

Then $\rho(T) < \rho_0$.

Proof of Lemma 3.2 Suppose that the conditions of the lemma are satisfied. Then $\rho(\widehat{S}) < \rho_0$ and, for every $\lambda \in \rho_0 \mathbb{T}$, there exists $\mu \in F$ such that (3.2) holds. It follows from (3.1) that

$$\|(T - S)T\| < \rho_0 \left(\|(\widehat{S} - \lambda I)^{-1}\|^{-1} - \|S - \widehat{S}\| \right), \quad \lambda \in \rho_0 \mathbb{T}. \tag{3.4}$$

The resolvent map $\lambda \mapsto (\widehat{S} - \lambda I)^{-1}$ is analytic on $|\lambda| > \rho(\widehat{S})$, which set contains all λ with $|\lambda| \geq \rho_0$. Thus, by the maximum principle, $\|(\widehat{S} - \lambda I)^{-1}\|$ attains its maximum in $|\lambda| \geq \rho_0$ on $\rho_0 \mathbb{T}$. Thus (3.4) in fact holds for all λ with $|\lambda| \geq \rho_0$, so that $S - \lambda I$ is invertible for all such λ and, by (3.1),

$$\|(T - S)T\| < \rho_0 \left\| (S - \lambda I)^{-1} \right\|^{-1}, \quad \text{if } |\lambda| \geq \rho_0. \tag{3.5}$$

Since T is compact, the result follows from Lemma 3.1. $\square$

When Lemma 3.2 is used for computation with F finite, it is desirable to minimise the cardinality of F since $\|(\widehat{S} - \mu I)^{-1}\|$ has to be computed for every $\mu \in F$. One can choose $F = \rho_0 \mathbb{T}_n$, with points uniformly distributed on $\rho_0 \mathbb{T}$, as in Corollary 3.3, but n needs to be at least large enough so that $\|(\widehat{S} - \mu I)^{-1}\|^{-1} > 2\rho_0 \sin(\pi/(2n))$, for every $\mu \in F$. In many applications, including in Sects. 4.2 and 4.3, $\|(\widehat{S} - \mu I)^{-1}\|$ varies significantly as μ moves around $\rho_0 \mathbb{T}$ and it is more efficient to vary the spacing of the points in F approximately in proportion to $\|(\widehat{S} - \mu I)^{-1}\|^{-1}$. The adaptive algorithm described in the following lemma, which we will see implemented in Fig. 4b below, approximately achieves this.

Lemma 3.4 *Let Y be a Banach space, $\widehat{S} \in \mathcal{L}(Y)$, and $\rho_0 > 0$. Suppose that $\rho(\widehat{S}) < \rho_0$, and recursively define μ_ℓ, for $\ell \in \mathbb{N}$, by $\mu_1 := \rho_0$, and by*

$$\nu_\ell := \left\| (\widehat{S} - \mu_\ell I)^{-1} \right\|^{-1} \quad \text{and} \quad \mu_{\ell+1} := \mu_\ell e^{i \frac{\nu_\ell}{2\rho_0}}, \quad \text{for } \ell \in \mathbb{N}. \tag{3.6}$$

Further, set $n_* \in \mathbb{N}$ to be the smallest integer such that $\sum_{\ell=1}^{n_*} \nu_\ell \geq 4\pi\rho_0$, and set $F := \{\mu_1, \mu_2, \ldots, \mu_{n_*+1}\}$. Then, for every $\lambda \in \rho_0 \mathbb{T}$ there exists $\mu \in F$ such that

$$\left\| (\widehat{S} - \lambda I)^{-1} \right\|^{-1} \geq \left\| (\widehat{S} - \mu I)^{-1} \right\|^{-1} - |\lambda - \mu| \geq R_* := \min_{\ell=1,\ldots,n_*} \left(\frac{\nu_\ell}{4} + \frac{\nu_{\ell+1}}{2} \right). \tag{3.7}$$

Proof Arguing as in the proof of Lemma 3.2, $\|(\widehat{S} - \lambda I)^{-1}\|^{-1}$ is bounded below by some $c > 0$ on $\rho_0 \mathbb{T}$, so that n_* is well-defined with $n_* < 1 + 4\pi\rho_0/c$. For $\ell = 1, \ldots, n_* + 1$, $\mu_\ell = \rho_0 e^{i\theta_\ell}$, with $\theta_1 = 0$, $\theta_{n_*+1} \geq 2\pi$, and $\theta_{\ell+1} - \theta_\ell = v_\ell/(2\rho_0)$. Thus, if $\lambda \in \rho_0 \mathbb{T}$, then

$$\lambda = \mu_\ell \exp(i s v_\ell/(2\rho_0)) = \mu_{\ell+1} \exp(i (s - 1) v_\ell/(2\rho_0)),$$

for some $\ell \in \{1, \ldots, n_*\}$ and some $s \in [0, 1]$. Since $|e^{it} - 1| = |\int_0^t e^{iu}\, du| \leq |t|$, for $t \in \mathbb{R}$, it follows, using (3.1), that

$$\|(\widehat{S} - \lambda I)^{-1}\|^{-1} \geq v_\ell - |\lambda - \mu_\ell| \geq v_\ell - \frac{s}{2} v_\ell = \left(1 - \frac{s}{2}\right) v_\ell \quad \text{and}$$

$$\|(\widehat{S} - \lambda I)^{-1}\|^{-1} \geq v_{\ell+1} - |\lambda - \mu_{\ell+1}| \geq v_{\ell+1} - \frac{1-s}{2} v_\ell.$$

In particular, $v_{\ell+1} \geq v_\ell/2$ and $v_\ell \geq v_{\ell+1} - v_\ell/2$, so that $s^* := 3/2 - v_{\ell+1}/v_\ell \in [0, 1]$. But, for $0 \leq s \leq s^*$, $(1 - s/2)v_\ell \geq (1 - s^*/2)v_\ell = v_\ell/4 + v_{\ell+1}/2$, while, for $s^* \leq s \leq 1$, $v_{\ell+1} - (1 - s)v_\ell/2 \geq v_{\ell+1} - (1 - s^*)v_\ell/2 = v_\ell/4 + v_{\ell+1}/2$, and the bound (3.7) follows. $\qquad\square$

The following corollary is immediate from the above lemma and Lemma 3.2.

Corollary 3.5 *Let Y be a Banach space, $S, \widehat{S}, T \in \mathcal{L}(Y)$, $\rho_0 > 0$, and suppose that T is compact, $\rho(\widehat{S}) < \rho_0$, and*

$$\|(T - S)T\| < \rho_0 \left(R_* - \|S - \widehat{S}\|\right), \tag{3.8}$$

where R_* is as defined in Lemma 3.4. Then $\rho(T) < \rho_0$.

We will apply the above results in the case when $S = T_N$, where $(T_N)_{N \in \mathbb{N}}$ is a *collectively compact*[11] sequence of operators converging strongly to T (we write $T_N \to T$ for strong convergence). A standard, simple but important result (e.g., [2, Cor. 1.9], [41, Theorem 10.10]) is that

$$(T_N)_{N \in \mathbb{N}} \text{ collectively compact}, \quad T_N \to T \quad \Rightarrow \quad \|(T - T_N)T\| \to 0. \tag{3.9}$$

A consequence of (3.9) is Theorem 3.6 below, which follows from [2, Theorem 4.8] and [2, Theorem 4.16] (or see [3]). This gives conditions on operators T_N and T that ensure convergence of $\sigma(T_N)$ to $\sigma(T)$ in the standard Hausdorff metric $d_H(\cdot, \cdot)$ (see, e.g., [24, Sect. 3.1.2]) on the set $\mathbb{C}^C$ of compact subsets of $\mathbb{C}$. Given a sequence $(A_N)_{N \in \mathbb{N}} \subset \mathbb{C}^C$ and $A \in \mathbb{C}^C$ we will write $A_N \overset{\text{H}}{\to} A$ if $d_H(A, B) \to 0$, i.e. if A_N converges to A in the Hausdorff metric. We recall (e.g., [24, Proposition 3.6]) that $A_N \overset{\text{H}}{\to} A$ if and only if $(A_N)_{N \in \mathbb{N}}$ is uniformly bounded and $\liminf A_N = \limsup A_N = A$, where $\liminf A_N$ is the set of limits of sequences (z_N) such that $z_N \in A_N$ for each N, while $\limsup A_N$ is the set of partial limits of such sequences.

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[11] Recall, e.g., [2], [41, Sect. 10.3] that a set $S \subset \mathcal{L}(Y)$ is said to be *collectively compact* if $\{T\phi : T \in S, \phi \in Y, \|\phi\| \leq 1\}$ is relatively compact.

Theorem 3.6 [2] *Let Y be a Banach space and $T \in \mathcal{L}(Y)$ a compact operator. Further, assume that $(T_N)_{N \in \mathbb{N}} \subset \mathcal{L}(Y)$ is collectively compact and converging strongly to T. Then $\sigma(T_N) \overset{\mathrm{H}}{\to} \sigma(T)$.*

3.2 The Nyström method

In this section, with a view to applications in Sects. 4.2 and 4.3, we apply the results of Sect. 3.1 to the case where $Y = C(X)$, for some compact $X \subset \mathbb{R}^{d-1}$, and where the operators are integral operators that we approximate by the Nyström method. So suppose $d \geq 2$, let $X \subset \mathbb{R}^{d-1}$ be a compact set of positive $((d-1)$-dimensional) Lebesgue measure, and $K \in \mathcal{L}(C(X))$ an integral operator with continuous kernel $K(\cdot, \cdot)$, so that K is compact. Thus, for $\phi \in C(X)$ and $x \in X$,

$$K\phi(x) = \int_X K(x, y)\phi(y)\, dy = J(K(x, \cdot)\phi), \quad \text{where}$$

$$J(\psi) := \int_X \psi(y)\, dy, \quad \text{for } \psi \in C(X). \tag{3.10}$$

In the Nyström method we approximate K by replacing the integration functional $J : C(X) \to \mathbb{C}$ by a sequence of numerical quadrature rules. For each $N \in \mathbb{N}$ we choose points $x_{q,N} \in X$ and weights[12] $\omega_{q,N} \geq 0$, for $q = 1, \ldots, N$, and define $J_N : C(X) \to \mathbb{C}$ by

$$J_N(\psi) := \sum_{q=1}^{N} \omega_{q,N} \psi(x_{q,N}), \qquad \psi \in C(X), \tag{3.11}$$

and a Nyström approximation $K_N \in \mathcal{L}(C(X))$ to K by

$$K_N\phi(x) := J_N(K(x, \cdot)\phi)) = \sum_{q=1}^{N} \omega_{q,N} K(x, x_{q,N})\phi(x_{q,N}), \quad x \in X, \ \phi \in C(X). \tag{3.12}$$

We will assume that the sequence of quadrature rules is convergent, by which we mean that

$$J_N \to J, \quad \text{i.e.} \quad J_N\psi \to J\psi, \quad \text{for all } \psi \in C(X). \tag{3.13}$$

This implies (e.g., [2, Proposition 2.1, 2.2]) that $K_N \to K$ and that the sequence $(K_N)_{N \in \mathbb{N}} \subset \mathcal{L}(C(X))$ is collectively compact, so that (3.9) holds and Theorem 3.6 is applicable. We will also assume that, for each N, $J_N\phi = J\phi$ if $\phi \in C(X)$ is constant, i.e. that

$$\sum_{q=1}^{N} \omega_{q,N} = |X|, \tag{3.14}$$

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[12] We assume, for simplicity, that the weights $\omega_{q,N}$ are positive, but the theory below applies, with minor changes, to the case of general real or complex weights.

where $|X|$ denotes the Lebesgue measure of X.

Define $\widehat{K}_N : \mathbb{C}^N \to C(X)$ by

$$\widehat{K}_N v(x) := \sum_{q=1}^{N} \omega_{q,N} K(x, x_{q,N}) v_q,$$

for $x \in X$ and $v = (v_1, ..., v_N) \in \mathbb{C}^N$. Moreover, let $P_N : C(X) \to \mathbb{C}^N$ be defined by $P_N \phi(q) := \phi(x_{q,N})$, for $q = 1, ..., N$, $\phi \in C(X)$, and define $A_N : \mathbb{C}^N \to \mathbb{C}^N$ by $A_N := P_N \widehat{K}_N$, so that the matrix entries of A_N are given by

$$A_N(p, q) = \omega_{q,N} K(x_{p,N}, x_{q,N}), \quad p, q = 1, \ldots, N. \qquad (3.15)$$

In the following we will use $\| \cdot \|_\infty$ to denote all of: i) the standard supremum norm on $C(X)$; ii) the standard infinity norm on $\mathbb{C}^N$; iii) the induced operator norm of an operator on $C(X)$; iv) the infinity norm of a square matrix.

Lemma 3.7 *The following inequalities and equalities hold for all $N \in \mathbb{N}$:*

(i)

$$\max_{1 \leq p \leq N} \sum_{q=1}^{N} \omega_{q,N} |K(x_{p,N}, x_{q,N})| = \|A_N\|_\infty \leq \|K_N\|_\infty$$

$$= \max_{x \in X} \sum_{q=1}^{N} \omega_{q,N} |K(x, x_{q,N})|;$$

(ii) $\sigma(K_N) = \{0\} \cup \sigma(A_N)$;
(iii) *for $\lambda \in \mathbb{C} \setminus \sigma(K_N)$,*

$$\max \left(|\lambda|^{-1}, \left\| (A_N - \lambda I)^{-1} \right\|_\infty \right) \leq \left\| (K_N - \lambda I)^{-1} \right\|_\infty$$

$$\leq |\lambda|^{-1} \left(1 + \|K_N\|_\infty \left\| (A_N - \lambda I)^{-1} \right\|_\infty \right).$$

Proof The first equality in (i) is the standard explicit formula for the infinity norm of a matrix. The last equality is proved similarly, and then (i) is clear (or see [23, Lemma 4.7.17]). That the spectra of K_N and A_N coincide on $\mathbb{C} \setminus \{0\}$ is standard (e.g., [23, Lemma 4.7.18]), and $0 \in \sigma(K_N)$ since K_N is compact and $C(X)$ is infinite-dimensional, so (ii) holds. Part (iii) is a combination of Hackbusch [23, Lemma 4.7.18] and Anselone [2, Proposition 2.3], plus the facts that $0 \in \sigma(K_N)$ and (e.g., [14, Th. 1.2.10]) $\|(K_N - \lambda I)^{-1}\| \geq (\text{dist}(\lambda, \sigma(K_N)))^{-1}$. $\qquad \square$

Note that, by Lemma 3.7(i) and (3.14),

$$\|A_N\|_\infty \leq \|K_N\|_\infty \leq K_{\max}|X|, \quad \text{where} \quad K_{\max} := \max_{x,y \in X} |K(x, y)|. \qquad (3.16)$$

Commonly, for computational efficiency or otherwise, and this is the case in Sects. 4.2 and 4.3, we approximate $K(\cdot, \cdot)$ by another continuous kernel $K^\dagger(\cdot, \cdot)$ in (3.12) and (3.15). Let $K_N^\dagger \in \mathcal{L}(C(X))$ denote the operator defined by the right hand side of (3.12) with K replaced by $K^\dagger$, and similarly $A_N^\dagger$ denote the matrix (3.15) with K replaced by $K^\dagger$. Lemma 3.7(i) and (3.14) imply that

$$\|A_N - A_N^\dagger\|_\infty \leq \|K_N - K_N^\dagger\|_\infty \leq e^\dagger |X|, \quad \text{where} \quad e^\dagger := \max_{x,y \in X} |K(x,y) - K^\dagger(x,y)|. \tag{3.17}$$

The following theorem (cf. Corollary 3.5) follows in large part from Lemmas 3.2, 3.4, and 3.7.

Theorem 3.8 *Suppose that $\rho_0 > 0$, $N \in \mathbb{N}$, and $\rho(A_N^\dagger) < \rho_0$, and recursively define μ_ℓ, for $\ell \in \mathbb{N}$, by $\mu_1 := \rho_0$, and by*

$$\nu_\ell := \left\| (A_N^\dagger - \mu_\ell I)^{-1} \right\|_\infty^{-1} \quad \text{and} \quad \mu_{\ell+1} := \mu_\ell e^{i \frac{\nu_\ell}{2\rho_0}}, \quad \text{for } \ell \in \mathbb{N}. \tag{3.18}$$

Further, let $n_N \in \mathbb{N}$ be the smallest integer such that $\sum_{\ell=1}^{n_N} \nu_\ell \geq 4\pi\rho_0$, and let

$$R_N := \min_{\ell=1,\ldots,n_N} \left(\frac{\nu_\ell}{4} + \frac{\nu_{\ell+1}}{2} \right). \tag{3.19}$$

If

$$\|(K - K_N)K\|_\infty < \rho_0 \left(\rho_0 \left(1 + \|K_N^\dagger\|_\infty R_N^{-1} \right)^{-1} - \|K_N - K_N^\dagger\|_\infty \right), \tag{3.20}$$

or $\|A_N - A_N^\dagger\|_\infty < R_N$ and

$$\|(K - K_N)K\|_\infty < \rho_0^2 \left(1 + \|K_N\|_\infty (R_N - \|A_N - A_N^\dagger\|_\infty)^{-1} \right)^{-1}, \tag{3.21}$$

then $\rho(K) < \rho_0$. Conversely, if $\rho(K) < \rho_0$, provided $e^\dagger$ defined by (3.17) is sufficiently small, (3.20), (3.21), and $\rho(A_N^\dagger) < \rho_0$ hold for all sufficiently large N.

Proof If $\rho_0 > 0$ and $\rho(A_N^\dagger) < \rho_0$ then, by Lemma 3.4, $\|(A_N^\dagger - \lambda I)^{-1}\|_\infty \leq R_N^{-1}$ for $\lambda \in \rho_0 \mathbb{T}$, so that $\|(K_N^\dagger - \lambda I)^{-1}\|_\infty \leq r_0^{-1}(1 + \|K_N^\dagger\|_\infty R_N^{-1})$, by Lemma 3.7(iii). Thus $\rho(K) < \rho_0$ if (3.20) holds, by Lemma 3.2 applied with $Y = C(X)$, $T = K$, $S = K_N$, $\widehat{S} = K_N^\dagger$, and $F = \rho_0 \mathbb{T}$. Further, for $\lambda \in \rho_0 \mathbb{T}$, the first of the above bounds and (3.1) implies that $R_N - \|A_N - A_N^\dagger\|_\infty \leq \|(A_N - \lambda I)^{-1}\|_\infty^{-1}$, so that, by Lemma 3.7(iii), $\|(K_N - \lambda I)^{-1}\|_\infty \leq r_0^{-1}(1 + \|K_N\|_\infty (R_N - \|A_N - A_N^\dagger\|_\infty)^{-1})$. Thus $\rho(K) < \rho_0$ if (3.21) holds, by Lemma 3.2 applied with $Y = C(X)$, $T = K$, $\widehat{S} = S = K_N$, and $F = \rho_0 \mathbb{T}$.

To see the converse, note that, by Theorem 3.6, $\sigma(K_N) \overset{\text{H}}{\to} \sigma(K)$. Indeed [2, Theorem 4.7], there exists $N_0 \in \mathbb{N}$ such that $\|(K_N - \lambda I)^{-1}\|_\infty$ is bounded uniformly in

λ and N for $N \geq N_0$ and $|\lambda| \geq \rho_0$, which implies, by (3.1) and (3.17), that the same holds for $\|(K_N^\dagger - \lambda I)^{-1}\|_\infty$ if $e^\dagger$ is sufficiently small. This implies, by Lemma 3.7(iii), that the same holds for $\|(A_N^\dagger - \lambda I)^{-1}\|_\infty$. Noting Lemma 3.7(ii), it follows that there exists $c > 0$ such that, for all sufficiently large N, $\rho(A_N^\dagger) = \rho(K_N^\dagger) < \rho_0$ and $R_N \geq c$. Thus, and applying (3.16) and (3.17), we see that, for some $c^* > 0$, the right hand sides of (3.20) and (3.21) are $\geq c^*$ for all sufficiently large N if $e^\dagger$ is sufficiently small. But also $\|(K - K_N)K\| \to 0$ by (3.9), so that (3.20) holds for all sufficiently large N. □

Remark 3.9 (*Computational cost as N increases*) The argument in the above proof makes clear that, if $\rho(K) < \rho_0$ and $e^\dagger$ is sufficiently small, then, for some $N_0 \in \mathbb{N}$, $\|(A_N^\dagger - \lambda I)^{-1}\|_\infty$ is bounded uniformly in λ and N for $\lambda \in \rho_0 \mathbb{T}$ and $N \geq N_0$. This in turn implies that, for some $n_{\max} \in \mathbb{N}$, $n_N \leq n_{\max}$ for $N \geq N_0$. Thus the computational cost of evaluation of R_N given by (3.19) is $O(N^3)$, the cost of inverting $n_N \leq n_{\max}$ order N matrices by classical direct methods.

Remark 3.10 (*Comparison of* (3.20) *and* (3.21)) Let RHS$_1$ and RHS$_2$ denote the right hand sides of (3.20) and (3.21), respectively. If $K_N^\dagger = K_N$ (so $A_N^\dagger = A_N$), then RHS$_2 =$ RHS$_1$. If $K_N^\dagger \neq K_N$ with $\|A_N - A_N^\dagger\|_\infty < R_N$, then, where $\mathcal{D} := \rho_0^{-2}(R_N + \|K_N\|_\infty - \|A_N - A_N^\dagger\|_\infty)(\text{RHS}_2 - \text{RHS}_1)$,

$$
\mathcal{D} = \rho_0^{-1}\|K_N - K_N^\dagger\|_\infty(\|K_N\|_\infty - \rho_0 + R_N - \|A_N - A_N^\dagger\|_\infty)
$$
$$
+ \frac{\|K_N^\dagger\|_\infty(\|K_N - K_N^\dagger\|_\infty - \|A_N - A_N^\dagger\|_\infty) + R_N(\|K_N - K_N^\dagger\|_\infty + \|K_N^\dagger\|_\infty - \|K_N\|_\infty)}{R_N + \|K_N^\dagger\|_\infty},
$$

so that RHS$_2 -$ RHS$_1 > \rho_0\|K_N - K_N^\dagger\|_\infty(\|K_N\|_\infty - \rho_0)/(R_N + \|K_N\|_\infty - \|A_N - A_N^\dagger\|_\infty)$, recalling (3.17). Thus (3.20) implies (3.21) if $\|A_N - A_N^\dagger\|_\infty < R_N$ and $\rho_0 \leq \|K_N\|_\infty$. Note that [2, Theorem 2.13] $\|K_N\|_\infty \to \|K\|$ as $N \to \infty$ and it is $\rho_0 < \|K\|_\infty$ for which Theorem 3.8 is arguably of most interest, as $\rho(K) \leq \|K\|_\infty$, and we may be able to estimate $\|K\|_\infty$ sharply by other methods.

Remark 3.11 (*The matrix case*) In Sect. 4.3 we will apply the above results, in particular Theorem 3.8, in a case where K is a 2×2 matrix of integral operators on $C(X)$ with continuous kernels, and K_N is its Nyström method approximation defined by approximating each integral operator in the 2×2 matrix as in (3.12). The matrix A_N is then a $2N \times 2N$ matrix consisting of four $N \times N$ blocks each defined as in (3.15). Parts (ii) and (iii) of Lemma 3.7 apply in this case, as does (i) in a straightforwardly modified form, in particular we still have that $\|A_N\|_\infty \leq \|K_N\|_\infty$. (Here $\|A_N\|_\infty$ is the usual infinity norm of the matrix A_N and $\|K_N\|_\infty$ is the norm of K_N as an operator on $(C(X))^2$, which we equip with the norm defined by $\|(\phi_1, \phi_2)\|_\infty := \max\{\|\phi_1\|_\infty, \|\phi_2\|_\infty\}$, for $(\phi_1, \phi_2) \in C(X)^2$). Thus Theorem 3.8, which depends on Lemma 3.7(ii) and (iii) and the general Banach space results of Sect. 3.1, still applies. One way to see the validity of Lemma 3.7(ii) and (iii) in this matrix case is to argue as follows. Choose $x^* \in \mathbb{R}^{d-1}$ such that $X' := X + x^*$ does not intersect X. K is equivalent, through an obvious isometric isomorphism, to a matrix operator K' on $C(X) \times C(X')$, which is in turn

equivalent, through another obvious isometric isomorphism, to a single integral opera-tor $\widetilde{K}$ on $C(X \cup X')$. Lemma 3.7 applies to $\widetilde{K}$ and its Nyström method approximation, so that (ii) and (iii) of this lemma [and (i) in modified form] apply to K.

4 The double-layer operator on dilation invariant graphs

Let $A \subseteq \mathbb{R}^{d-1}$ be an open cone, $f \colon A \to \mathbb{R}$ a Lipschitz continuous function, and consider the graph $\Gamma = \{(x, f(x)) : x \in A\} \subset \mathbb{R}^d$ in the case that $\alpha \Gamma = \Gamma$, for some $\alpha \in (0, 1)$; that is, in the case that

$$f(\alpha x) = \alpha f(x), \quad x \in A. \tag{4.1}$$

We will term such graphs *dilation invariant*.

In this section, the largest of the paper, we study the spectrum and essential spec-trum of the double-layer (DL) operator $D_\Gamma : L^2(\Gamma) \to L^2(\Gamma)$ given by (1.2) on such graphs. (The case $A = \mathbb{R}^{d-1}$ is of particular interest for later applications.) In Sect. 4.1 we show, for general dimension $d \geq 2$, that (as operators on $L^2(\Gamma)$) $\sigma(D_\Gamma) = \sigma_{\mathrm{ess}}(D_\Gamma)$ (and, similarly, that $W(D_\Gamma) = W_{\mathrm{ess}}(D_\Gamma)$) and, by Floquet–Bloch-transform arguments, that $\sigma(D_\Gamma)$ is the union of the spectra of a family of operators $K_t : L^2(\Gamma_0) \to L^2(\Gamma_0)$, for $t \in [-\pi, \pi]$, where Γ_0 is a particular relatively closed and bounded subset of Γ. Moreover, helpful for the later computation of $\sigma(K_t)$, each K_t is compact, and so has a discrete spectrum, if $f \in C^1(A \setminus \{0\})$ (Corollary 4.6).

In the remaining subsections, Sects. 4.2–4.4, we focus on the 2D case, considering the Nyström approximation of spectral properties of K_t, combining the general results of Sect. 3.2 with explicit estimates for the particular operators K_t. The case $A = \mathbb{R}$, with f real-analytic on $\mathbb{R} \setminus \{0\}$, is treated in Sect. 4.3. It is this case that is relevant, via, e.g., (1.12), to the computation of $\sigma(D_\Gamma)$ and the spectral radius conjecture when Γ is the boundary of a bounded Lipschitz domain. But this case is rather complex; the operator K_t is studied by reducing it to a 2×2 operator matrix, corresponding to the split of $\mathbb{R} \setminus \{0\}$ into the two half-axes $(-\infty, 0)$ and $(0, \infty)$. To get the main ideas across, and prove many of the results we need in a simpler setting, we first study, in Sect. 4.2, the easier case $A = (0, \infty)$ with f real-analytic. Sections 4.2 and 4.3 are concerned with computation of the spectrum and spectral radius of D_Γ. In Sect. 4.4, related to the question at the end of Sect. 1.1, we also compute lower bounds for $W_{\mathrm{ess}}(D_\Gamma)$, under the same assumptions on Γ as in Sects. 4.2 and 4.3.

4.1 Floquet–Bloch transform results

Let $V_\alpha \colon L^2(\Gamma) \to L^2(\Gamma)$ be dilation by α, that is,

$$V_\alpha \phi(x) = \alpha^{(d-1)/2} \phi(\alpha x), \quad x \in \Gamma. \tag{4.2}$$

V_α is unitary and commutes with D_Γ: noting that $n(y) = n(\alpha y)$, for $y \in \Gamma$, we see that, for all $\phi \in L^2(\Gamma)$ and almost every $x \in \Gamma$,

$$D_\Gamma V_\alpha \phi(x) = \frac{1}{c_d} \int_\Gamma \frac{(x-y) \cdot n(y)}{|x-y|^d} \alpha^{(d-1)/2} \phi(\alpha y) \, ds(y)$$

$$= \frac{\alpha^{(d-1)/2}}{c_d} \int_\Gamma \frac{(x - \alpha^{-1}y) \cdot n(\alpha^{-1}y)}{|x - \alpha^{-1}y|^d} \phi(y) \alpha^{-(d-1)} ds(y)$$

$$= \frac{\alpha^{(d-1)/2}}{c_d} \int_\Gamma \frac{(\alpha x - y) \cdot n(y)}{|\alpha x - y|^d} \phi(y) \, ds(y)$$

$$= V_\alpha D_\Gamma \phi(x).$$

This already implies that the spectrum and the essential spectrum of D_Γ coincide, as implied by the following simple proposition (cf. [11, Lemma 2.7]).

Proposition 4.1 *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. If T commutes with a sequence of unitary operators $(U_j)_{j \in \mathbb{N}}$ that converges weakly to 0, then $\sigma(T) = \sigma_{\mathrm{ess}}(T)$.*

Proof Assume there exists $\phi \in H \setminus \{0\}$ such that $T\phi = 0$. As T and U_j commute, also $TU_j\phi = 0$ for all $j \in \mathbb{N}$. In particular, $\{U_j\phi : j \in \mathbb{N}\} \subseteq \ker(T)$. As the operators U_j are unitary and $U_j \to 0$ weakly, the sequence $(U_j\phi)_{j\in\mathbb{N}}$ cannot have a convergent subsequence. Hence, $\ker(T)$ is either trivial or infinite-dimensional. Similarly, $\ker(T')$ is either trivial or infinite-dimensional. Thus, if T is Fredholm, it is invertible. Considering $T - \lambda I$ instead of T yields the result. □

Corollary 4.2 *Let $\Gamma \subset \mathbb{R}^d$ be a dilation invariant graph. Then $\sigma(D_\Gamma) = \sigma_{\mathrm{ess}}(D_\Gamma)$, $W(D_\Gamma) = W_{\mathrm{ess}}(D_\Gamma)$, and $\|D_\Gamma\|_{\mathrm{ess}} = \|D_\Gamma\|$.*

Proof Assume that $\phi, \psi \in L^2(\Gamma)$ have compact support and $0 \notin \mathrm{supp}\,\phi \cup \mathrm{supp}\,\psi$. Then

$$\left\langle V_\alpha^j \phi, \psi \right\rangle = \int_\Gamma \alpha^{j(d-1)/2} \phi(\alpha^j x) \overline{\psi(x)} \, ds(x).$$

If $|j|$ is sufficiently large, the integrand vanishes. Because compactly supported functions are dense in $L^2(\Gamma)$, it follows that $V_\alpha^j \to 0$ weakly as $|j| \to \infty$. The equality of spectrum and essential spectrum follows from Proposition 4.1. The results for the numerical range and norm follow from Chandler-Wilde and Spence [11, Lemma 2.7]. □

To make use of standard Floquet–Bloch/Fourier transform results, it is convenient to view D_Γ as a discrete convolution operator. For $j \in \mathbb{Z}$ let $\Gamma_j := \{(\tilde{x}, f(\tilde{x})) \in \Gamma : |\tilde{x}| \in [\alpha^{j+1}, \alpha^j]\}$. We can identify $L^2(\Gamma_j)$ with a closed subspace of $L^2(\Gamma)$ by extending by 0. Let $P_j : L^2(\Gamma) \to L^2(\Gamma_j)$ denote orthogonal projection. Clearly,

$$P_j \phi(x) = \begin{cases} \phi(x) & \text{if } x \in \Gamma_j, \\ 0 & \text{otherwise,} \end{cases}$$

and we note that $P_j V_\alpha = V_\alpha P_{j+1}$, so that also $P_j V_\alpha^k = V_\alpha^k P_{j+k}$, $k \in \mathbb{Z}$. Let

$$D_j := V_\alpha^j P_j D_\Gamma|_{L^2(\Gamma_0)} : L^2(\Gamma_0) \to L^2(\Gamma_0),$$

so that

$$D_j \phi(x) = \alpha^{j(d-1)/2} D_\Gamma \phi(\alpha^j x), \quad x \in \Gamma_0, \quad \phi \in L^2(\Gamma_0). \tag{4.3}$$

Let $\mathcal{G} : L^2(\Gamma) \to \ell^2(\mathbb{Z}, L^2(\Gamma_0))$ be the unitary operator.[13] $\phi \mapsto (V_\alpha^j P_j \phi)_{j \in \mathbb{Z}}$, and let $\widetilde{D}_\Gamma := \mathcal{G} D_\Gamma \mathcal{G}^{-1}$. It is a straightforward calculation to see that the action of $\widetilde{D}_\Gamma$ is that of a discrete convolution: for $\psi = (\psi_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}, L^2(\Gamma_0))$, $\widetilde{D}_\Gamma \psi = ((\widetilde{D}_\Gamma \psi)_m)_{m \in \mathbb{Z}}$ where

$$(\widetilde{D}_\Gamma \psi)_m = \sum_{n \in \mathbb{Z}} D_{m-n} \psi_n, \quad m \in \mathbb{Z}. \tag{4.4}$$

The series in the above definition converges absolutely; indeed $\widetilde{D}_\Gamma$ is an operator in the so-called *Wiener algebra*, i.e. $\sum_{j \in \mathbb{Z}} \|D_j\| < \infty$ (e.g., [42, Defintion 1.43]), by the following estimate.

Proposition 4.3 *The operators D_j are Hilbert-Schmidt for $|j| \geq 2$ and satisfy the Hilbert-Schmidt norm estimate*

$$\|D_j\|_{\mathrm{HS}} \leq \frac{1}{c_d} \frac{|\Gamma_0|}{(\alpha - \alpha^{|j|})^{d-1}} \alpha^{|j|(d-1)/2}, \quad |j| \geq 2, \tag{4.5}$$

where $|\Gamma_0|$ denotes the surface measure of Γ_0.

Proof For $j \geq 2$ we have

$$\int_{\Gamma_j} \int_{\Gamma_0} \left| \frac{(x-y) \cdot n(y)}{|x-y|^d} \right|^2 ds(y) \, ds(x) \leq \int_{\Gamma_j} \int_{\Gamma_0} \frac{1}{|x-y|^{2d-2}} \, ds(y) \, ds(x)$$

and $|x - y| \geq \alpha - \alpha^j$. Hence

$$\int_{\Gamma_j} \int_{\Gamma_0} \left| \frac{(x-y) \cdot n(y)}{|x-y|^d} \right|^2 ds(y) \, ds(x) \leq \frac{|\Gamma_j||\Gamma_0|}{(\alpha - \alpha^j)^{2d-2}} = \frac{|\Gamma_0|^2 \alpha^{j(d-1)}}{(\alpha - \alpha^j)^{2d-2}}.$$

Similarly, for $j \leq -2$,

$$\int_{\Gamma_j} \int_{\Gamma_0} \left| \frac{(x-y) \cdot n(y)}{|x-y|^d} \right|^2 ds(y) \, ds(x) \leq \frac{|\Gamma_0|^2 \alpha^{j(d-1)}}{(\alpha^{j+1} - 1)^{2d-2}} = \frac{|\Gamma_0|^2 \alpha^{-j(d-1)}}{(\alpha - \alpha^{-j})^{2d-2}}.$$

Thus, recalling the standard characterisation of the Hilbert-Schmidt norm of integral operators (e.g., [32, Ex. 11.11]), $P_j D_\Gamma|_{L^2(\Gamma_0)}$ is Hilbert-Schmidt, with Hilbert-Schmidt norm bounded by the right hand side of (4.5). As unitary operators preserve Hilbert-Schmidt norms, it follows that D_j is Hilbert-Schmidt and that (4.5) holds. □

---

[13] This is a discretization operator in the sense, e.g., of Lindner [42, Sect. 1.2.3].

Given a Hilbert space H let $\mathcal{F} : \ell^2(\mathbb{Z}, H) \to L^2([-\pi, \pi], H)$ denote the operator, often termed in the general Hilbert space case, e.g., [5], a *Floquet–Bloch transform*, that constructs a Fourier series from coefficients, given by

$$(\mathcal{F}\psi)(t) = (2\pi)^{-1/2} \sum_{j \in \mathbb{Z}} e^{ijt} \psi_n, \quad t \in [-\pi, \pi],$$

for $\psi = (\psi_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}, H)$. It is standard that this is a unitary operator (e.g., [14, Proof of Theorem 4.4.9]) that diagonalises discrete convolutions (see, e.g., [14, Theorem 4.49] for the case when H is finite-dimensional, [54, Theorem 2.3.25] for the general case). In the case $H = L^2(\Gamma_0)$, defining $\widehat{D} : L^2([-\pi, \pi], H) \to L^2([-\pi, \pi], H)$ by $\widehat{D} := \mathcal{F}\widetilde{D}_\Gamma \mathcal{F}^{-1}$, straightforward calculations yield that

$$\langle \widehat{D}\phi, \psi \rangle_{L^2([-\pi,\pi], L^2(\Gamma_0))} = \int_{-\pi}^{\pi} \langle K_t\phi(t), \psi(t) \rangle \, dt, \quad \phi, \psi \in L^2([-\pi, \pi], L^2(\Gamma_0)),$$
(4.6)

where

$$K_t := \sum_{j=-\infty}^{\infty} e^{ijt} D_j, \quad t \in \mathbb{R}.$$
(4.7)

(The bounds of Proposition 4.10 imply that K_t is well-defined by (4.7) and depends continuously on t; indeed the mapping $t \mapsto K_t$ is C^∞.) The following characterisation follows immediately from (4.6), the continuity of $t \mapsto K_t$, and Corollary 4.2; note that conv denotes the closed convex hull.

Theorem 4.4 *We have*

$$\|D_\Gamma\|_{\text{ess}} = \|D_\Gamma\| = \max_{t \in [-\pi,\pi]} \|K_t\|,$$

$$W_{\text{ess}}(D_\Gamma) = \overline{W(D_\Gamma)} = \text{conv}\left(\bigcup_{t \in [-\pi,\pi]} W(K_t) \right),$$

$$\sigma_{\text{ess}}(D_\Gamma) = \sigma(D_\Gamma) = \bigcup_{t \in [-\pi,\pi]} \sigma(K_t).$$

Proof That the essential spectrum, numerical range, and norm coincide with their non-essential counterparts is Corollary 4.2. Since D_Γ and $\widehat{D}$ are unitarily equivalent, they have the same spectrum, numerical range, and norm. The result thus follows from (4.6); see [54, Theorem 2.3.25] for the case of the spectrum; the argument for the norm and numerical range are similar. □

Remark 4.5 (*Symmetry of K_t*) Where $K_t(\cdot, \cdot)$ denotes the kernel of K_t, $K_{-t}(\cdot, \cdot) = \overline{K_t(\cdot, \cdot)}$, for $t \in [-\pi, \pi]$. Thus $\|K_{-t}\| = \|K_t\|$, $W(K_{-t}) = \{\bar{z} : z \in W(K_t)\}$, and $\sigma(K_{-t}) = \{\bar{z} : z \in \sigma(K_t)\}$, for $t \in [-\pi, \pi]$, so that, where $w(K_t) := \sup_{z \in W(K_t)} |z|$ is the *numerical radius* of K_t,

$$\|D_\Gamma\|_{\text{ess}} = \max_{t \in [0,\pi]} \|K_t\|, \quad w_{\text{ess}}(D_\Gamma) = \max_{t \in [0,\pi]} w(K_t), \quad \rho_{\text{ess}}(D_\Gamma) = \max_{t \in [0,\pi]} \rho(K_t).$$

Our focus in the next subsections will be 2D cases where f is analytic on $A \setminus \{0\}$. In such cases, indeed whenever Γ is locally C^1 away from 0, K_t is compact for all t as a consequence of standard results on the double-layer operator on C^1 domains [19] and Proposition 4.10.

Corollary 4.6 *Suppose that* $f \in C^1(A \setminus \{0\})$ *so that* Γ *is locally* C^1 *at each* $x \in \Gamma \setminus \{0\}$. *Then* K_t *is compact for every* $t \in \mathbb{R}$.

Proof Let $Q = P_{-1} + P_0 + P_1$. Then $QD_\Gamma|_{L^2(\Gamma_{-1} \cup \Gamma_0 \cup \Gamma_1)}$ is the DL operator on $\Gamma_{-1} \cup \Gamma_0 \cup \Gamma_1$. It thus follows from [19, Theorem 1.2] that $QD_\Gamma|_{L^2(\Gamma_{-1} \cup \Gamma_0 \cup \Gamma_1)}$ is compact. As, for $j = -1, 0, 1$,

$$D_j = V_\alpha^j P_j D_\Gamma|_{L^2(\Gamma_0)} = V_\alpha^j P_j Q D_\Gamma|_{L^2(\Gamma_0)},$$

D_j is compact for $j = -1, 0, 1$. The compactness of K_t thus follows from Proposition 4.3. □

4.2 The 2D case: one-sided infinite graphs

We continue to assume that Γ is a dilation invariant graph, as defined at the start of Sect. 4, but specialise now to the case where $d = 2$ and the cone is the half-axis $A = \mathbb{R}_+ := (0, \infty)$. Thus $\Gamma = \{(x, f(x)) : x \in \mathbb{R}_+\}$ and, for some $\alpha \in (0, 1)$, $f(\alpha x) = \alpha f(x)$, $x \in \mathbb{R}_+$. Our starting point is Corollary 4.4 which expresses $\sigma(D_\Gamma)$ as the union of the spectra of the operators $K_t, t \in [-\pi, \pi]$. Recall that these operators are compact if $f \in C^1(\mathbb{R}_+)$. Our goal is to apply the Nyström method and the results of Sect. 3.2 to compute spectral properties of K_t and hence of D_Γ in the case that $f \in \mathcal{A}(\mathbb{R}_+)$, the space of functions $\mathbb{R}_+ \to \mathbb{R}$ that are real analytic (a prototypical example is Fig. 2). Our standing assumption through this subsection is that

$$\Gamma = \{(x, f(x)) : x \in \mathbb{R}_+\} \quad \text{where} \quad f \in \mathcal{A}(\mathbb{R}_+) \quad \text{and, for some}$$
$$\alpha \in (0, 1), \ f(\alpha x) = \alpha f(x), \quad x \in \mathbb{R}_+. \tag{4.8}$$

Notably, via approximations of the spectrum of K_t for each t, we will obtain (see Theorem 4.13) a Nyström approximation $\sigma^N(D_\Gamma)$, for $\sigma(D_\Gamma) = \sigma_{\text{ess}}(D_\Gamma)$, which is $\{0\}$ plus the union over finitely many $t \in [-\pi, \pi]$ of the spectra of $N \times N$ matrices $A_{t,N}^M$, where each $A_{t,N}^M$ is obtained via Nyström discretisation of a unitary transformation, $\tilde{K}_t$, of K_t. Our first main result, Theorem 4.13, is to show that $\sigma^N(D_\Gamma) \xrightarrow{\text{H}} \sigma(D_\Gamma)$ as $N \to \infty$. Our other, more substantial result (Theorem 4.15) is to develop a fully discrete algorithm to test whether the spectral radius conjecture holds for Γ, i.e. to test whether $\sigma_{\text{ess}}(D_\Gamma) = \sigma(D_\Gamma) < \frac{1}{2}$. This algorithm, which derives from Theorem 3.8, requires the computation only of the spectral radii of finitely many finite matrices plus the norms of finitely many finite matrix resolvents.

Before we begin our analysis we note the following equivalences to (4.8) that will play a key role in our calculations. Here, and throughout, the notations

$$\Sigma_c := \{z \in \mathbb{C} : \text{Im } z \in (-c, c)\}, \quad \Sigma_0 := \mathbb{R}, \tag{4.9}$$

Fig. 2 Graph of $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) := x \sin^2(\pi \log_\alpha(x))$, for $\alpha = \frac{3}{4}$
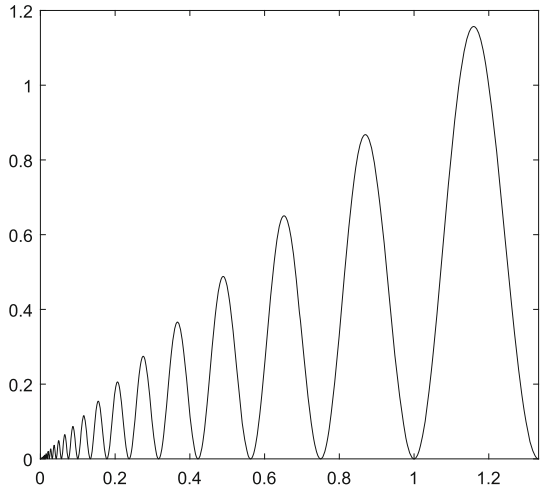
for $c > 0$ will be convenient.

Lemma 4.7 *Given $f : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in (0, 1)$, define $g : \mathbb{R} \to \mathbb{R}$ by*

$$g(x) := \alpha^{-x} f(\alpha^x), \quad x \in \mathbb{R}, \quad \textit{so that} \quad f(x) = x g(\log_\alpha x), \quad x \in \mathbb{R}_+. \quad (4.10)$$

Then the following are equivalent:

(i) *f is real analytic on $\mathbb{R}_+$ and $f(\alpha x) = \alpha f(x)$, $x > 0$;*
(ii) *$g : \mathbb{R} \to \mathbb{R}$ is real analytic and $g(x + 1) = g(x)$, $x \in \mathbb{R}$;*
(iii) *for some $c > 0$, g has an analytic extension to Σ_c that satisfies $g(z + 1) = g(z)$, for $z \in \Sigma_c$, and g and its derivatives g' and g'' are bounded in Σ_c.*

Note that if f satisfies our standing assumption (4.8), then, defining g by (4.10),

$$f'(x) = g(\log_\alpha x) + \frac{g'(\log_\alpha x)}{\log \alpha}, \quad f''(x) = \frac{g'(\log_\alpha x)}{x \log \alpha} + \frac{g''(\log_\alpha x)}{x \log^2 \alpha}, \quad x > 0. \quad (4.11)$$

It follows from (4.10), the first of (4.11), and the equivalence of i) and iii), that f is Lipschitz continuous on $[0, \infty)$ if we set $f(0) := 0$.

To make use of the results from Sect. 3.2 it is convenient to make a change of variables so that we work with integral operators on $[0, 1]$ rather than Γ_0. Introducing the unitary transformation $U : L^2(\Gamma_0) \to L^2(0, 1)$ given by

$$U\phi(s) := \phi(\alpha^s, f(\alpha^s))(1 + f'(\alpha^s)^2)^{1/4} \alpha^{s/2} |\log \alpha|^{1/2}, \quad s \in [0, 1], \quad \phi \in L^2(\Gamma_0), \quad (4.12)$$

define

$$\tilde{K}_t := U K_t U^{-1} = \sum_{j=-\infty}^{\infty} e^{ijt} U D_j U^{-1}, \quad t \in \mathbb{R}. \quad (4.13)$$

Straightforward computations, starting from (4.13), (4.3), and (1.4), give that

$$\tilde{K}_t \phi(x) = \int_0^1 \tilde{K}_t(x, y) \phi(y) \, dy, \quad x \in [0, 1], \quad t \in \mathbb{R}, \tag{4.14}$$

where, for $t \in \mathbb{R}$, and $x, y \in \mathbb{R}$ with $x - y \notin \mathbb{Z}$,

$$\tilde{K}_t(x, y) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{ijt} \frac{p_j(x, y)}{1 + q_j(x, y)^2} \left(\frac{1 + f'(\alpha^x)^2}{1 + f'(\alpha^y)^2} \right)^{1/4} \alpha^{\frac{x+y+j}{2}} |\log \alpha|, \tag{4.15}$$

with

$$p_j(x, y) := \frac{(\alpha^y - \alpha^{x+j}) f'(\alpha^y) + f(\alpha^{x+j}) - f(\alpha^y)}{(\alpha^{x+j} - \alpha^y)^2},$$

$$q_j(x, y) := \frac{f(\alpha^{x+j}) - f(\alpha^y)}{\alpha^{x+j} - \alpha^y}. \tag{4.16}$$

By Taylor's theorem applied to $F(t) := f((1 - t)\alpha^y + t\alpha^{x+j})$, we see that, for the same range of x and y,

$$q_j(x, y) = \int_0^1 f'((1 - t)\alpha^y + t\alpha^{x+j}) \, dt, \tag{4.17}$$

$$p_j(x, y) = \int_0^1 f''((1 - t)\alpha^y + t\alpha^{x+j})(1 - t) \, dt. \tag{4.18}$$

Using (4.17) and (4.18) to extend the definitions of $q_j(x, y)$ and $p_j(x, y)$ to $\{(x, y) \in \mathbb{R}^2 : x - y \in \mathbb{Z}\}$,[14] we see that, for each $t \in \mathbb{R}$, each term in the sum (4.15) is continuous on $\mathbb{R}^2$. Further, using (4.11) and the equivalence of (i) and (iii) in Lemma 4.7, it is easy to see that $p_j(x, y) = O(1)$ as $j \to \infty$, $= O(\alpha^{-j})$ as $j \to -\infty$, uniformly for x and y in compact subsets of $\mathbb{R}$, so that the series (4.15) converges absolutely and uniformly on compact subsets, so that $\tilde{K}_t(\cdot, \cdot) \in C(\mathbb{R}^2)$. Further, $f(\alpha x) = \alpha f(x)$ implies that $\tilde{K}_t(x + 1, y) = e^{-it} \tilde{K}_t(x, y)$ and $\tilde{K}_t(x, y + 1) = e^{it} \tilde{K}_t(x, y)$, $x, y \in \mathbb{R}$. Note that $\tilde{K}_t(\cdot, \cdot) \in C(\mathbb{R}^2)$ implies that $\tilde{K}_t : L^2(0, 1) \to C[0, 1]$, that $\tilde{K}_t$ is compact as an operator on both $C[0, 1]$ and $L^2(0, 1)$, and that the spectrum of $\tilde{K}_t$ is the same on $C[0, 1]$ as on $L^2(0, 1)$, so that

$$\sigma(K_t; L^2(\Gamma_0)) = \sigma(\tilde{K}_t; L^2(0, 1)) = \sigma(\tilde{K}_t; C[0, 1]), \quad t \in \mathbb{R}. \tag{4.19}$$

We will use the above equivalence, and the Nyström method results from Sect. 3.2, to compute the spectrum and spectral radius of K_t, in particular by applying Theorem 3.8. As a step towards estimating the left hand side of (3.20) in the case when $K = \tilde{K}_t$ and K_N is its Nyström method approximation, we first show that, under our standing

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[14] Precisely, for $j \in \mathbb{Z}$, $p_j(x, y)$ and $q_j(x, y)$ are given explicitly by (4.16) for $y - x \neq j$, while (4.17) and (4.18) imply that $p_j(x, y) = \frac{1}{2} f''(\alpha^y)$, $q_j(x, y) = f'(\alpha^y)$, for $y - x = j$.

assumption (4.8), $\tilde{K}_t(x, \cdot)$ and $\tilde{K}_t(\cdot, y)$ can be extended to bounded analytic functions on Σ_c for some $c > 0$. We will use the notation $\|\cdot\|_c$ for the sup norm on Σ_c and, for functions $h \colon \Sigma_c \times \Sigma_d \to \mathbb{C}$, the notation

$$\|h\|_{c,d} := \sup_{\substack{x \in \Sigma_c \\ y \in \Sigma_d}} |h(x, y)|.$$

In an extension of these notations, for $h \colon \Sigma_c \times \Sigma_c \to \mathbb{C}$ we define

$$\|h\|_{c,0} := \sup_{\substack{x \in \Sigma_c \\ y \in \mathbb{R}}} |h(x, y)| \quad \text{and} \quad \|h\|_{0,c} := \sup_{\substack{x \in \mathbb{R} \\ y \in \Sigma_c}} |h(x, y)|,$$

and, for functions $h \colon \Sigma_c \to \mathbb{C}$, we set $\|h\|_0 := \|h\|_\infty := \sup_{x \in \mathbb{R}} |h(x)|$, the ordinary sup norm on $\mathbb{R}$.

Let us now estimate the norms $\|\tilde{K}_t\|_{c,0}$ and $\|\tilde{K}_t\|_{0,c}$ under our standing assumption (4.8). Since g is real-valued on $\mathbb{R}$, it follows, using Lemma 2.1, that (4.21) holds for all sufficiently small $c > 0$.

Proposition 4.8 *Given* (4.8)*, define* g *by* (4.10) *so that, by Lemma* 4.7*,* g *has an analytic continuation to* Σ_c*, for some* $c > 0$*, such that* g*,* g'*, and* g'' *are bounded on* Σ_c *and* $g(z + 1) = g(z)$*,* $z \in \Sigma_c$*. Let* $\mathfrak{I}_c := \|\operatorname{Im} g\|_c + \|\operatorname{Im} g'\|_c / |\log \alpha|$*, and set*

$$\mathfrak{F}_d := \|g\|_d + \|g'\|_d / |\log \alpha| \quad \text{and} \quad \mathfrak{G}_d := \|g'\|_d + \|g''\|_d / |\log \alpha|, \quad \text{for } d = 0, c. \tag{4.20}$$

If

$$c \leq \arccos(\alpha) / |\log \alpha| \quad \text{and} \quad \mathfrak{I}_c < 1, \tag{4.21}$$

then $\tilde{K}_t(x, \cdot)$ *and* $\tilde{K}_t(\cdot, y)$ *extend to bounded analytic functions on* Σ_c *for all* $x, y, t \in \mathbb{R}$*, and*

$$\|\tilde{K}_t\|_{c,0} \leq \frac{\left(1 + \mathfrak{F}_c^2\right)^{1/4}}{\pi \left(1 - \mathfrak{I}_c^2\right)} \left[\mathfrak{G}_c \frac{1 + \alpha^{1/2} + \alpha^{-1/2}}{4\alpha^2} + |\log \alpha| (\mathfrak{F}_c + \mathfrak{F}_0) \sum_{j=2}^{\infty} \frac{\alpha^{j/2}}{\alpha - \alpha^j}\right], \tag{4.22}$$

$$\|\tilde{K}_t\|_{0,c} \leq \frac{\left(1 + \mathfrak{F}_0^2\right)^{1/4}}{\pi \left(1 - \mathfrak{I}_c^2\right)^{5/4}} \left[\mathfrak{G}_c \frac{1 + \alpha^{1/2} + \alpha^{-1/2}}{4\alpha^2} + 2|\log \alpha| \mathfrak{F}_c \sum_{j=2}^{\infty} \frac{\alpha^{j/2}}{\alpha - \alpha^j}\right]. \tag{4.23}$$

Moreover, $\mathbb{R} \to C([0, 1] \times [0, 1])$*,* $t \mapsto \tilde{K}_t(\cdot, \cdot)$ *is continuous.*

Remark 4.9 [*Bound on the sum in* (4.22) *and* (4.23)] For fixed $\alpha \in (0, 1)$, let $F(x) := \alpha^{x/2} / (\alpha - \alpha^x)$, for $x > 0$, so that the jth term in the sum in (4.22) and (4.23) is $F(j)$.

Since F is decreasing on $(1, \infty)$ we have that, for $n = 2, 3, \ldots$ and $0 < \alpha < 1$,

$$\sum_{j=n+1}^{\infty} \frac{\alpha^{j/2}}{\alpha - \alpha^j} \le \mathcal{B}_n(\alpha) \quad \text{so that} \tag{4.24}$$

$$\sum_{j=2}^{\infty} \frac{\alpha^{j/2}}{\alpha - \alpha^j} \le \mathcal{B}_n^*(\alpha) := \sum_{j=2}^{n} F(j) + \mathcal{B}_n(\alpha), \tag{4.25}$$

where

$$\mathcal{B}_n(\alpha) := \int_n^{\infty} F(x)\, dx = \frac{1}{2\alpha^{1/2}} \int_n^{\infty} \frac{dx}{\sinh(|\log \alpha|(x-1)/2)}$$
$$= \frac{\log(\tanh((n-1)|\log\alpha|/4))}{\alpha^{1/2} \log \alpha}.$$

We note, for later reference, that

$$\mathcal{B}_n(\alpha) \sim 2\alpha^{n/2-1}/|\log \alpha|, \quad \text{as} \quad n \to \infty. \tag{4.26}$$

Proof of Proposition 4.8 Using (4.11) it follows that

$$f(\alpha^x) = \alpha^x g(x), \quad f'(\alpha^x) = g(x) + \frac{g'(x)}{\log \alpha},$$
$$f''(\alpha^x) = \frac{g'(x)}{\alpha^x \log \alpha} + \frac{g''(x)}{\alpha^x \log^2 \alpha}, \quad x \in \mathbb{R}. \tag{4.27}$$

Note that the first of the bounds (4.21) implies that $c|\log \alpha| < \pi/2$, so that the assumptions we have made on g mean that (4.10) provides an analytic extension of f from $\mathbb{R}_+$ to the sector of the complex plane

$$G_c := \{re^{i\theta} : r > 0, |\theta| < c|\log \alpha|\} = \{\alpha^z : z \in \Sigma_c\},$$

and the Eqs. (4.11) hold for all x in this sector. Further, with this extension, (4.27) holds for all $x \in \Sigma_c$. Since (4.11) holds for all $x \in G_c$, and noting that G_c is convex, we see that the integrals (4.17) and (4.18) are well-defined for all $x, y \in \Sigma_c$, $j \in \mathbb{Z}$, and provide analytic continuations of p_j and q_j to $\Sigma_c \times \Sigma_c$. Further, by the uniqueness of analytic continuation, the equations (4.16) hold for all $x, y \in \Sigma_c$, with $x + j \neq y$.

To complete the proof we will demonstrate that (4.15) provides, for each $x, y \in \mathbb{R}$, analytic continuations of $\tilde{K}_t(\cdot, y)$ and $\tilde{K}_t(x, \cdot)$ from $\mathbb{R}$ to Σ_c which satisfy the bounds (4.22) and (4.23), by showing that each term in (4.15) is well-defined (i.e., that $f'(\alpha^y)^2 \neq -1$, for $y \in \Sigma_c$, and $1 + (q_j(x, y))^2 \neq 0$ for $x, y \in \Sigma_c$), and that the series (4.15) converges absolutely and uniformly for $(x, y) \in \mathbb{R} \times \Sigma_c$ and $(x, y) \in \Sigma_c \times \mathbb{R}$.

Using (4.27) we see that

$$\sup_{z \in G_c} |\operatorname{Im} f'(z)| = \sup_{y \in \Sigma_c} |\operatorname{Im} f'(\alpha^y)| \le \Im_c,$$

so that

$$\inf_{y \in \Sigma_c} \left| 1 + f'(\alpha^y)^2 \right| \geq 1 - \sup_{y \in \Sigma_c} \left| \mathrm{Im}\, f'(\alpha^y) \right|^2 \geq 1 - \mathfrak{I}_c^2, \tag{4.28}$$

and, using (4.17),

$$\inf_{x, y \in \Sigma_c} \left| 1 + q_j(x, y)^2 \right| \geq 1 - \sup_{z \in G_c} \left| \mathrm{Im}\, f'(z) \right|^2 \geq 1 - \mathfrak{I}_c^2. \tag{4.29}$$

Thus, where $T_j(x, y)$ denotes the j-th term in the sum (4.15), we see that $T_j(x, y)$ is well-defined and analytic for $x, y \in \Sigma_c$, $j \in \mathbb{Z}$. Moreover, we have $T_j(x + 1, y) = e^{-it} T_{j+1}(x, y)$ and it follows from $f(\alpha x) = \alpha f(x)$ that $T_j(x, y+1) = e^{it} T_{j-1}(x, y)$, for $x, y \in \Sigma_c$, $j \in \mathbb{Z}$. Thus, to prove absolute and uniform convergence of (4.15) and the bounds (4.22) and (4.23), it suffices to restrict consideration to $x, y \in \Xi_c := \{z \in \mathbb{C} : \mathrm{Re}\, z \in [0, 1],\ \mathrm{Im}\, z \in (-c, c)\}$.

We see that $\sup_{x \in \Xi_c} |\alpha^{x/2}| \leq 1$ and, using (4.27), that

$$\sup_{x \in [0,1]} \left| 1 + f'(\alpha^x)^2 \right| \leq 1 + \mathfrak{F}_0^2, \quad \sup_{x \in \Xi_c} \left| 1 + f'(\alpha^x)^2 \right| \leq 1 + \mathfrak{F}_c^2.$$

To obtain a bound on $|T_j(x, y)|$ for $(x, y) \in [0, 1] \times \Xi_c$, and for $(x, y) \in \Xi_c \times [0, 1]$, it remains to bound $p_j(x, y)$. Let

$$G_c^* := \{ re^{i\theta} : \alpha^2 \leq r \leq \alpha^{-1}, |\theta| < c |\log \alpha| \} = \{ \alpha^z : \mathrm{Re}\, z \in [-1, 2],\ \mathrm{Im}\, z \in (-c, c) \}.$$

It is clear that $(1 - t)\alpha^y + t\alpha^{x+j} \in G_c^*$ for every $t \in [0, 1]$, $|j| \leq 1$, $x \in \Xi_c$, and $y \in [0, 1]$ if and only if G_c^* is star-shaped with respect to every point in $[\alpha, 1]$, which holds if and only if c satisfies the first of the bounds (4.21). Likewise, $(1 - t)\alpha^y + t\alpha^{x+j} \in G_c^*$ for every $t \in [0, 1]$, $|j| \leq 1$, $x \in [0, 1]$, and $y \in \Xi_c$ if and only if the first of (4.21) holds. Thus, if (4.21) holds, $|j| \leq 1$, and either $(x, y) \in [0, 1] \times \Xi_c$ or $(x, y) \in \Xi_c \times [0, 1]$, it follows from (4.11) and (4.18) that

$$|p_j(x, y)| \leq \frac{1}{2} \sup_{z \in G_c^*} |f''(z)| \leq \frac{\mathfrak{G}_c}{2\alpha^2 |\log \alpha|}.$$

On the other hand, if $|j| \geq 2$ and $x, y \in \Xi_c$, then it follows from (4.16) (which we have observed above holds for all $x, y \in \Sigma_c$ with $x + j \neq y$) that

$$|p_j(x, y)| \leq \frac{|f'(\alpha^y)| + |q_j(x, y)|}{|\alpha^{x+j} - \alpha^y|}.$$

Further, for $j \in \mathbb{Z}$,

$$|q_j(x, y)| \leq \sup_{z \in G_c} |f'(z)| \leq \mathfrak{F}_c, \quad x, y \in \Xi_c,$$

while $|f'(\alpha^y)| \leq \mathfrak{F}_0$, for $y \in [0, 1]$, $\leq \mathfrak{F}_c$, for $y \in \Xi_c$. Moreover, for $|j| \geq 2$, $x, y \in \Xi_c$,

$$|\alpha^{x-j} - \alpha^y| \geq \left|\alpha^{\operatorname{Re} x - j} - \alpha^{\operatorname{Re} y}\right| \geq \begin{cases} \alpha - \alpha^j, & j \geq 2, \\ \alpha^{j+1} - 1, & j \leq -2. \end{cases}$$

Putting these bounds together we see that, for $|j| \leq 1$,

$$\sup_{x \in \Xi_c, y \in [0,1]} |T_j(x, y)| \leq \frac{\mathfrak{G}_c \left(1 + \mathfrak{F}_c^2\right)^{1/4} \alpha^{j/2}}{2\alpha^2 \left(1 - \mathfrak{I}_c^2\right)},$$

$$\sup_{x \in [0,1], y \in \Xi_c} |T_j(x, y)| \leq \frac{\mathfrak{G}_c \left(1 + \mathfrak{F}_0^2\right)^{1/4} \alpha^{j/2}}{2\alpha^2 \left(1 - \mathfrak{I}_c^2\right)^{5/4}},$$

while, for $|j| \geq 2$,

$$\sup_{x \in \Xi_c, y \in [0,1]} |T_j(x, y)| \leq \frac{|\log \alpha|(\mathfrak{F}_0 + \mathfrak{F}_c) \left(1 + \mathfrak{F}_c^2\right)^{1/4} \alpha^{|j|/2}}{\left(1 - \mathfrak{I}_c^2\right) (\alpha - \alpha^{|j|})},$$

$$\sup_{x \in [0,1], y \in \Xi_c} |T_j(x, y)| \leq \frac{2|\log \alpha| \mathfrak{F}_c \left(1 + \mathfrak{F}_c^2\right)^{1/4} \alpha^{|j|/2}}{\left(1 - \mathfrak{I}_c^2\right)^{5/4} (\alpha - \alpha^{|j|})}.$$

From these bounds on $|T_j(x, y)|$ it is clear that the series (4.15), with p_j and q_j given by (4.17) and (4.18), converges absolutely and uniformly for $(x, y) \in [0, 1] \times \Xi_c$ and $(x, y) \in \Xi_c \times [0, 1]$, so that $\tilde{K}_t(x, \cdot)$ and $\tilde{K}_t(\cdot, y)$ are analytic in Σ_c, for $x, y \in \mathbb{R}$, as required. The convergence of the series is also uniform in t, which implies the continuity of $t \mapsto \tilde{K}_t(\cdot, \cdot)$. Further, the above bounds on $|T_j(x, y)|$ imply the bounds (4.22) and (4.23). $\qquad\square$

Recalling (4.19) and that the kernel of $\tilde{K}_t$ is continuous, we will approximate the spectrum and spectral radius of K_t, for $t \in \mathbb{R}$, by approximating the spectrum and spectral radius of $\tilde{K}_t$, considered as an operator on $C[0, 1]$, using the Nyström method results of Sect. 3.2. Define $\tilde{K}_{t,N} : C[0, 1] \to C[0, 1]$, for $N \in \mathbb{N}$ and $t \in \mathbb{R}$, by [cf. (3.12)]

$$\begin{aligned} \tilde{K}_{t,N}\phi(x) &:= J_N(\tilde{K}_t(x, \cdot)\phi) \\ &= \frac{1}{N} \sum_{q=1}^{N} \tilde{K}_t(x, x_{q,N})\phi(x_{q,N}), \quad x \in [0, 1], \quad \phi \in C[0, 1], \quad (4.30) \end{aligned}$$

where

$$J_N\psi(x) := \sum_{q=1}^{N} \omega_{q,N}\phi(x_{q,N}), \quad \psi \in C[0, 1], \quad N \in \mathbb{N},$$

with $x_{q,N} := \frac{1}{N}(q - \frac{1}{2})$, $\omega_{q,N} := N^{-1}$, for $q = 1, \ldots, N$, $N \in \mathbb{N}$. Note that $J_N \psi$ is just the N-point midpoint rule approximation to $J\psi := \int_0^1 \psi(x)\,dx$, and that this numerical quadrature rule sequence satisfies (3.13) and (3.14), so that, for each $t \in \mathbb{R}$, $\tilde{K}_{t,N} \to \tilde{K}_t$ as $N \to \infty$ and $(\tilde{K}_{t,N})_{N\in\mathbb{N}}$ is collectively compact. This implies, from the general result (3.9), that $\|(\tilde{K}_t - \tilde{K}_{t,N})\tilde{K}_t\|_\infty \to 0$ as $N \to \infty$. In our case this convergence is exponential (cf. [41, Example 12.11], [63, Sect. 19]).

Proposition 4.10 *Let f and c be as in Proposition 4.8. Then, for every $t \in \mathbb{R}$ and $N \in \mathbb{N}$,*

$$\left\|(\tilde{K}_t - \tilde{K}_{t,N})\tilde{K}_t\right\|_\infty \le \frac{2e^{2\pi c}\|\tilde{K}_t\|_{0,c}\|\tilde{K}_t\|_{c,0}}{e^{2\pi Nc} - 1}.$$

To prove this proposition we will use the following classical result.

Theorem 4.11 (Theorem 9.28 in [40]) *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a 1-periodic function that can be extended to a bounded analytic function on Σ_c for some $c > 0$. Then*

$$|J\psi - J_N\psi| = \left|\int_0^1 \psi(x)\,dx - \frac{1}{N}\sum_{q=1}^N \psi(x_{q,N})\right| \le \frac{2\|\psi\|_c}{e^{2\pi Nc} - 1}, \quad N \in \mathbb{N}.$$

Proof. It is clear from the definitions of $\tilde{K}_t$ and $\tilde{K}_{t,N}$ that it is enough to consider the case $t \in [-\pi, \pi]$. Assuming $t \in [-\pi, \pi]$, let $M_{g_t} : C[0,1] \to C[0,1]$ be multiplication by $g_t(x) := e^{ixt}$. Then M_{g_t} is a surjective isometry with $M_{g_t}^{-1} = M_{g_{-t}}$. Let $\tilde{L}_t := M_{g_t}\tilde{K}_t M_{g_t}^{-1}$ and $\tilde{L}_{t,N} := M_{g_t}\tilde{K}_{t,N}M_{g_t}^{-1}$. Then $\tilde{L}_t$ is an integral operator with kernel $\tilde{L}_t(x,y) = e^{i(x-y)t}\tilde{K}_t(x,y)$. As $\tilde{K}_t(x+1,y) = e^{-it}\tilde{K}_t(x,y)$ and $\tilde{K}_t(x,y+1) = e^{it}\tilde{K}_t(x,y)$, we get $\tilde{L}_t(x+1,y) = \tilde{L}_t(x,y) = \tilde{L}_t(x,y+1)$ for all $x, y \in \mathbb{R}$. Moreover, $\|\tilde{L}_t\|_{0,c} \le e^{\pi c}\|\tilde{K}_t\|_{0,c}$ and $\|\tilde{L}_t\|_{c,0} \le e^{\pi c}\|\tilde{K}_t\|_{c,0}$. For $\phi \in C[0,1]$,

$$\psi(x) := \int_0^1 \tilde{L}_t(x,y)\phi(y)\,dy, \quad x \in \Sigma_c,$$

is an analytic and 1-periodic extension of $\tilde{L}_t\phi$ from $[0,1]$ to Σ_c. Thus also $y \mapsto \tilde{L}_t(x,y)\psi(y)$ is analytic and 1-periodic for every $x \in \mathbb{R}$. By Theorem 4.11 we obtain that, for $x \in \mathbb{R}$,

$$\begin{aligned}
|(\tilde{L}_t - \tilde{L}_{t,N})\tilde{L}_t\phi(x)| &= |\tilde{L}_t\psi(x) - \tilde{L}_{t,N}\psi(x)| \\
&= |J(\tilde{L}_t(x,\cdot)\psi)) - J_N(\tilde{L}_t(x,\cdot)\psi))| \\
&\le \frac{2C_t(x)}{e^{2\pi Nc} - 1},
\end{aligned}$$

where $C_t(x) := \sup_{y \in \Sigma_c} |\tilde{L}_t(x, y)\psi(y)|$. As $|\psi(y)| \leq \sup_{z \in [0,1]} |\tilde{L}_t(y, z)| \, \|\phi\|_\infty$, for $y \in \Sigma_c$, we get

$$C_t(x) \leq \|\tilde{L}_t\|_{0,c}\|\tilde{L}_t\|_{c,0} \, \|\phi\|_\infty \leq e^{2\pi c}\|\tilde{K}_t\|_{0,c}\|\tilde{K}_t\|_{c,0} \, \|\phi\|_\infty,$$

for all $x \in [0, 1]$ and $\phi \in C[0, 1]$. Therefore,

$$\left\|(\tilde{K}_t - \tilde{K}_{t,N})\tilde{K}_t\right\|_\infty = \left\|(\tilde{L}_t - \tilde{L}_{t,N})\tilde{L}_t\right\|_\infty \leq \frac{2e^{2\pi c}\|\tilde{K}_t\|_{0,c}\|\tilde{K}_t\|_{c,0}}{e^{2\pi Nc} - 1}. \quad \square$$

Let $A_{t,N}$ be the $N \times N$-matrix defined by [cf. (3.15)]

$$A_{t,N}(p, q) := \omega_{q,N}\tilde{K}_t(x_{p,N}, x_{q,N}) = \frac{1}{N}\tilde{K}_t(x_{p,N}, x_{q,N}), \quad p, q = 1, \ldots, N. \tag{4.31}$$

$\tilde{K}_t(\cdot, \cdot)$ is defined by the series (4.15) which we truncate to evaluate numerically. For $M \in \mathbb{N}$ let

$$\tilde{K}_t^M(x, y) := \frac{1}{2\pi} \sum_{j=-M}^{M} e^{ijt} \frac{p_j(x, y)}{1 + q_j(x, y)^2} \left(\frac{1 + f'(\alpha^x)^2}{1 + f'(\alpha^y)^2}\right)^{1/4} \alpha^{\frac{x+y+j}{2}} |\log \alpha|, \quad x, y \in [0, 1], \tag{4.32}$$

where p_j and q_j are defined by (4.16), and consider the matrices $A_{t,N}^M$ given by $A_{t,N}^M(p, q) := \frac{1}{N}\tilde{K}_t^M(x_{p,N}, x_{q,N})$, $p, q = 1, \ldots, N$, so that $A_{t,N}^M$ is an approximation to $A_{t,N}$ obtained by using finitely many terms in the series defining $\tilde{K}_t(\cdot, \cdot)$. Similarly, define $\tilde{K}_{t,N}^M$ by (4.30) with $\tilde{K}_{t,N}$ and $\tilde{K}_t(\cdot, \cdot)$ replaced by $\tilde{K}_{t,N}^M$ and $\tilde{K}_t^M(\cdot, \cdot)$, respectively. For $M, N \in \mathbb{N}$, with $M \geq 2$, and $t \in \mathbb{R}$, by (3.17),

$$\left\|A_{t,N} - A_{t,N}^M\right\|_\infty \leq \|\tilde{K}_{t,N} - \tilde{K}_{t,N}^M\|_\infty \leq \sup_{x,y \in [0,1]} |\tilde{K}_t(x, y) - \tilde{K}_t^M(x, y)| \leq C_1(M), \tag{4.33}$$

where, using the notations of (4.20) and Remark 4.9,

$$C_1(M) := \frac{2|\log \alpha|\mathfrak{F}_0}{\pi} \left(1 + \mathfrak{F}_0^2\right)^{1/4} \mathcal{B}_M(\alpha); \tag{4.34}$$

this bound (4.33)–(4.34) [cf. (4.22)] is obtained as in Proposition 4.8 (set $c = 0$, only take the terms with $|j| \geq M + 1$, and note that $\mathfrak{I}_0 = 0$), and by using (4.24). Notice that, by (4.26), $C_1(M) = O(\alpha^{M/2})$ as $M \to \infty$.

Our aim now is to estimate $\rho(D_\Gamma; L^2(\Gamma))$ and $\sigma(D_\Gamma; L^2(\Gamma))$ by computing $\rho(A_{t,N}^M)$ and $\sigma(A_{t,N}^M)$ for only finitely many t. For this purpose bounds on the Lipschitz constants of $A_{t,N}^M$ and $\tilde{K}_{t,N}$ as functions of t will be helpful. It follows from (4.32) that, for $M, N \in \mathbb{N}$, $t \in \mathbb{R}$, and $p, q = 1, \ldots, N$,

$$\left| \frac{\partial}{\partial t} A_{t,N}^M(p,q) \right| \leq B_N^M(p,q)$$

$$:= \frac{1}{2\pi N} \sum_{j=-M}^{M} \frac{|j| \, |p_j(x_{p,N}, x_{q,N})|}{1 + q_j(x_{p,N}, x_{q,N})^2}$$

$$\left(\frac{1 + f'(\alpha^{x_{p,N}})^2}{1 + f'(\alpha^{x_{q,N}})^2} \right)^{1/4} \alpha^{\frac{x_{p,N} + x_{q,N} + j}{2}} \, |\log \alpha| , \qquad (4.35)$$

so that, for $s, t \in \mathbb{R}$, $M, N \in \mathbb{N}$,

$$\left\| A_{t,N}^M - A_{s,N}^M \right\|_\infty \leq |s - t| \left\| B_N^M \right\|_\infty . \qquad (4.36)$$

Note that, for all M and N,

$$\|B_N^M\|_\infty \leq C_2 := \frac{1}{2\pi} \sup_{x,y \in [0,1]} \sum_{j=-\infty}^{\infty} |j| \, |p_j(x,y)| \left(1 + \|f'\|_\infty^2 \right)^{1/4} \alpha^{\frac{x+y+j}{2}} \, |\log \alpha| , \qquad (4.37)$$

which is finite by the bounds on p_j in the proof of Proposition 4.8. Similarly, from (4.15), for $t \in \mathbb{R}$, $N \in \mathbb{N}$, $x, y \in [0, 1]$,

$$\left| \frac{\partial}{\partial t} \tilde{K}_{t,N}(x,y) \right| \leq C_2 \quad \text{so that} \quad \|\tilde{K}_{t,N} - \tilde{K}_{s,N}\|_\infty \leq |s - t| C_2, \qquad (4.38)$$

for $s, t \in \mathbb{R}$ and $N \in \mathbb{N}$, by (3.17). Further, by (3.16), (4.23) with $c = 0$, and (4.25) with[15] $n = 10$,

$$\|\tilde{K}_{t,N}\|_\infty \leq C_3 := \frac{\left(1 + \mathfrak{F}_0^2\right)^{1/4}}{\pi} \left[\mathfrak{G}_0 \frac{1 + \alpha^{1/2} + \alpha^{-1/2}}{4\alpha^2} + 2|\log \alpha| \, \mathfrak{F}_0 \, \mathcal{B}_{10}^*(\alpha) \right], \qquad (4.39)$$

for $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Provided c is such that the conditions of Lemma 4.7(iii) and (4.21) hold, we have also, by Propositions 4.8 and 4.10 and (4.25), that,

$$\left\| (\tilde{K}_t - \tilde{K}_{t,N}) \tilde{K}_t \right\|_\infty \leq \frac{C_4}{e^{2\pi Nc} - 1}, \quad t \in \mathbb{R}, \quad N \in \mathbb{N}, \qquad (4.40)$$

where $C_4 := 2e^{2\pi c} C_5 C_6$, and C_5 and C_6 denote the right hand sides of (4.22) and (4.23), respectively, with the sum replaced by its upper bound $\mathcal{B}_{10}^*(\alpha)$.

As noted above Proposition 4.10, $\{\tilde{K}_{t,N} : N \in \mathbb{N}\}$ is collectively compact for each $t \in \mathbb{R}$. We will need, in the proof of Theorem 4.13, the following stronger statement.

Lemma 4.12 *The sets* $\{\tilde{K}_{t,N} : t \in \mathbb{R}, \ N \in \mathbb{N}\}$ *and* $\{\tilde{K}_{t,N}^M : t \in \mathbb{R}, \ M, N \in \mathbb{N}\}$ *are collectively compact.*

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[15] Computations indicate that $\mathcal{B}_{10}^*(\alpha)$ exceeds the left hand side of (4.25) by not more than 1.4% for $\alpha \in (0, 1)$.

Proof Where c is such that the conditions of Lemma 4.7(iii) and (4.21) hold, Proposition 4.8 implies that, for $t \in \mathbb{R}$, $N \in \mathbb{N}$, and $\phi \in C[0, 1]$ with $\|\phi\|_\infty \leq 1$, $\tilde{K}_{t,N}\phi$ has an analytic continuation from $[0, 1]$ to Σ_c. Where $\mathcal{F}$ denotes this set of analytic continuations, it follows from (4.22) that $\mathcal{F}$ is uniformly bounded on Σ_c, so that (e.g., [58, Theorem 14.6]) $\mathcal{F}$ is normal, so that $\{\tilde{K}_{t,N}\phi : t \in \mathbb{R}, N \in \mathbb{N}, \|\phi\|_\infty \leq 1\}$ is relatively compact, i.e., $\{\tilde{K}_{t,N} : t \in \mathbb{R}, N \in \mathbb{N}\}$ is collectively compact. The same argument applies to the family $\tilde{K}_{t,N}^M$, on noting, by inspection of the proof of Proposition 4.8, that, under the same conditions on c, $\tilde{K}_t^M(\cdot, y)$ extends to an analytic function on Σ_c bounded by the right hand side of (4.22), for all $t, y \in \mathbb{R}$, $M \in \mathbb{N}$.   □

In the following result, which holds for every Γ that satisfies our standing assumption (4.8), we propose an approximation for the spectrum of D_Γ as the union of the spectra of finitely many finite matrices, and show that this approximation converges in the Hausdorff metric. In this theorem $\sigma(\tilde{K}_{t,N}^M)$ denotes the spectrum of $\tilde{K}_{t,N}^M$ either on $L^2(0, 1)$ or on $C[0, 1]$ [cf. (4.19)], which coincides with $\{0\} \cup \sigma(A_{t,N}^M)$ by Lemma 3.7(ii), and $\sigma(D_\Gamma)$ denotes the spectrum of D_Γ on $L^2(\Gamma)$, which coincides with the essential spectrum by Corollary 4.2.

Theorem 4.13 *Choose sequences* $(m_N)_{N\in\mathbb{N}}, (M_N)_{N\in\mathbb{N}} \subset \mathbb{N}$ *such that* $m_N, M_N \to \infty$ *as* $N \to \infty$, *and for each* N, *let* $T_N := \{\pm(k - 1/2)\pi/m_N : k = 1, \ldots, m_N\}$. *Then*

$$\sigma^N(D_\Gamma) := \bigcup_{t\in T_N} \sigma(\tilde{K}_{t,N}^{M_N}) = \{0\} \cup \bigcup_{t\in T_N} \sigma(A_{t,N}^{M_N}) \overset{\text{H}}{\to} \sigma(D_\Gamma) = \sigma_{\text{ess}}(D_\Gamma) \quad (4.41)$$

as $N \to \infty$.

Proof Set $\Lambda := \sigma(D_\Gamma)$, so that $\Lambda = \bigcup_{t\in[-\pi,\pi]} \sigma(\tilde{K}_t)$ by Theorem 4.4 and (4.19), and set $\Lambda_N := \sigma^N(D_\Gamma)$. We first observe that $(\Lambda_N)_{N\in\mathbb{N}}$ is uniformly bounded as $\{\|\tilde{K}_{t,N}^{M_N}\|_\infty : N \in \mathbb{N}, t \in T_N\}$ is bounded, by (4.33) and (4.39). Thus, as noted above Theorem 3.6, to prove that $\Lambda_N \overset{\text{H}}{\to} \Lambda$ it is enough to show that $\liminf \Lambda_N = \limsup \Lambda_N = \Lambda$.

Next, we note that if the sequences $(\mathfrak{M}_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, and $(t_k)_{k\in\mathbb{N}} \subset [-\pi, \pi]$ satisfy $N_k \to \infty$, $\mathfrak{M}_k \to \infty$, $t_k \to t \in [-\pi, \pi]$, as $k \to \infty$, then the sequence $(\tilde{K}_{t_k,N_k}^{\mathfrak{M}_k})_{k\in\mathbb{N}}$ is collectively compact by Lemma 4.12, and converges strongly to $\tilde{K}_t$ by (4.38) and (4.33), and since $\tilde{K}_{t,N_k} \to \tilde{K}_t$, as noted above Proposition 4.10. Thus also $\sigma(\tilde{K}_{t_k,N_k}^{\mathfrak{M}_k}) \overset{\text{H}}{\to} \sigma(\tilde{K}_t)$, by Theorem 3.6.

For every $t \in [-\pi, \pi]$ we can choose a sequence $(t_N)_{N\in\mathbb{N}}$ such that $t_N \in T_N$ for each N and $t_N \to t$. Then, by the observation just made, $\Lambda_N \supset \sigma(\tilde{K}_{t_N,N}^{M_N}) \overset{\text{H}}{\to} \sigma(\tilde{K}_t)$ as $N \to \infty$, so that $\liminf \Lambda_N \supset \liminf \sigma(\tilde{K}_{t_N,N}^{M_N}) = \sigma(\tilde{K}_t)$. Thus $\liminf \Lambda_N \supset \Lambda$.

If $\lambda \in \limsup \Lambda_N$ then there exists a sequence $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \to \lambda$ and, for each k, $\lambda_k \in \Lambda_{N_k}$, so that $\lambda_k \in \sigma(\tilde{K}_{t_k,N_k}^{\mathfrak{M}_k})$, for some $t_k \in T_{N_k}$, where $\mathfrak{M}_k := M_{N_k}$. By passing to a subsequence if necessary, we may assume that $t_k \to t \in [-\pi, \pi]$, so that $\sigma(\tilde{K}_{t_k,N_k}^{\mathfrak{M}_k}) \overset{\text{H}}{\to} \sigma(\tilde{K}_t)$, in particular

$\liminf \sigma(\tilde{K}^{\mathfrak{M}_k}_{t_k, N_k}) = \sigma(\tilde{K}_t)$, so that $\lambda \in \sigma(\tilde{K}_t) \subset \Lambda$. Thus $\limsup \Lambda_N \subset \Lambda \subset \liminf \Lambda_N$. But also $\liminf \Lambda_N \subset \limsup \Lambda_N$, so the proof is complete. $\qquad\square$

Remark 4.14 (*Reduced computation expression for $\sigma^N(D_\Gamma)$*) Note that, by Remark 4.5 and Lemma 3.7(ii), $\sigma^N(D_\Gamma)$, given by (4.41), can be written as

$$\sigma^N(D_\Gamma) = \{0\} \cup \{\lambda, \bar{\lambda} : \lambda \in \sigma(A^{M_N}_{t,N}), \ t \in T_N, \ t > 0\}.$$

Our second main result, obtained by applying Theorem 3.8,[16] provides a criterion, given $\rho_0 > 0$, for $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$. Note that this criterion requires computation of spectral quantities only for the $N \times N$ matrices $A^M_{t,N}$ for finitely many $t \in [0, \pi]$.

Theorem 4.15 *Under our standing assumption* (4.8), *define g by* (4.10) *so that, by Lemma 4.7, g has an analytic continuation to Σ_c, for some $c > 0$, such that g, g', and g'' are bounded on Σ_c, and, without loss of generality, assume that* (4.21) *holds. Suppose that $\rho_0 > 0$, $m, M, N \in \mathbb{N}$, with $M \geq 2$. For $k = 1, \ldots, m$, set $t_k := (k - 1/2)\pi/m$, suppose that $\rho(A^M_{t_k,N}) < \rho_0$ for $k = 1, \ldots, m$, and recursively define $\mu_{k,\ell}$, for $\ell = 1, 2, \ldots$, by $\mu_{k,1} := \rho_0$, and by*

$$v_{k,\ell} := \left\| (A^M_{t_k,N} - \mu_{k,\ell} I)^{-1} \right\|^{-1}_\infty \quad \text{and} \quad \mu_{k,\ell+1} := \mu_{k,\ell} e^{i\frac{v_{k,\ell}}{2\rho_0}}, \quad \text{for } \ell \in \mathbb{N}. \quad (4.42)$$

Further, for $k = 1, \ldots, m$, let n_k denote the smallest integer such that $\sum_{\ell=1}^{n_k} v_{k,\ell} \geq 4\pi\rho_0$, and let

$$R_{m,M,N} := \min_{\substack{k=1,\ldots,m \\ \ell=1,\ldots,n_k}} \left(\frac{v_{k,\ell}}{4} + \frac{v_{k,\ell+1}}{2} \right).$$

If

$$\mathcal{L}_c(f, N) := \frac{C_4}{e^{2\pi Nc} - 1} < \mathcal{R}_c(f, m, M, N)$$

$$:= r_0^2 \left(1 + C_3 \left(R_{m,M,N} - C_1(M) - \frac{\pi}{2m} \|B^M_N\|_\infty \right)^{-1} \right)^{-1}, \quad (4.43)$$

then $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$. Conversely, if $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$, then, provided m and M are sufficiently large, there exists $N_0 \in \mathbb{N}$ such that (4.43) *holds and $\rho(A^M_{t,N}) < \rho_0$ for $t \in [0, \pi]$ and all $N \geq N_0$.*

Proof By Theorem 4.4, Remark 4.5, and (4.19), to show that $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$, it is enough to check that $\rho(\tilde{K}_t; C[0, 1]) < \rho_0$ for every $t \in [0, \pi]$. So pick $t \in [0, \pi]$. Then $|t - t_k| \leq \pi/(2m)$, for some $k \in \{1, \ldots, m\}$. To conclude that $\rho(\tilde{K}_t; C[0, 1]) <$

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[16] To obtain (4.43), motivated by Remark 3.10 our starting point is (3.21) rather than (3.20), since our interest will be to apply Theorem 4.15 in cases where $\rho_0 < \|D_\Gamma\|$.

ρ_0 we apply Theorem 3.8 with $K = \tilde{K}_t$, $K_N = \tilde{K}_{t,N}$, $A_N = A_{t,N}$, and $A_N^\dagger = A_{t_k,N}^M$, noting that, with these definitions, $\|K_N\|_\infty = \|\tilde{K}_{t,N}\|_\infty \leq C_3$,

$$\|A_N - A_N^\dagger\|_\infty \leq \|A_{t,N} - A_{t,N}^M\|_\infty + \|A_{t,N}^M - A_{t_k,N}^M\|_\infty \leq C_1(M) + \frac{\pi}{2m}\|B_N^M\|_\infty,$$

by (4.33) and (4.36), and $\|(K - K_N)K\|_\infty \leq C_4/(e^{2\pi Nc} - 1)$, by (4.40).

Conversely, if $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$ then, by Theorem 4.4 and (4.19), $\rho(\tilde{K}_t; C[0, 1]) < \rho_0$ for $t \in [0, \pi]$. Arguing as in the proof of Theorem 3.8, this implies that for every $t \in [0, \pi]$ there exists $N_0(t)$ such that $\|(\tilde{K}_{t,N} - \lambda I)^{-1}\|$ is bounded uniformly in λ and N for $N \geq N_0(t)$ and $|\lambda| \geq \rho_0$. This, combined with the estimate (4.38) and a standard compactness argument, implies that, for some $N^* \in \mathbb{N}$, and $c^* > 0$, $\|(\tilde{K}_{t,N} - \lambda I)^{-1}\|^{-1} \geq c^*$ for all $N \geq N^*$, $|\lambda| \geq \rho_0$, and $t \in [0, \pi]$. It follows, by (4.33) and (3.1), and recalling Lemma 3.7(ii), that, for all $t \in [0, \pi]$ and all sufficiently large M and N, $\rho(A_{t,N}^M) = \rho(\tilde{K}_{t,N}^M; C[0, 1]) < \rho_0$ and $\|(\tilde{K}_{t,N}^M - \lambda I)^{-1}\|^{-1} \geq c^*/2$, for $|\lambda| \geq \rho_0$. Thus, and by Lemma 3.7(iii), $R_{m,N,M} \geq 3c^*/8$, for all $m \in \mathbb{N}$ and all sufficiently large M and N. It follows that (4.43) holds, for all sufficiently large m, M, and N, since (4.34) and (4.26), and that $\|B_N^M\|_\infty$ is bounded independently of M and N by (4.37), imply that the right hand side of (4.43) is positive and bounded away from zero. $\square$

4.3 The 2D case: two-sided infinite graphs

We now extend the results of the previous subsection, for the case when $d = 2$ and $A = \mathbb{R}_+$, to the more involved case[17] where $d = 2$ and $A = \mathbb{R}$. Our goals and methods are those of Sect. 4.2, but, where $\dot{\mathbb{R}} := \mathbb{R}\backslash\{0\}$, we assume now that $f \in \mathcal{A}(\dot{\mathbb{R}})$, the space of functions $\mathbb{R} \to \mathbb{R}$ that are real analytic on $\dot{\mathbb{R}}$ with $f(0) = 0$ (a prototypical example is Fig. 3). Thus our standing assumption through this subsection is that

$$\Gamma = \{(x, f(x)) : x \in \mathbb{R}\} \quad \text{where} \quad f \in \mathcal{A}(\dot{\mathbb{R}}) \quad \text{and, for some}$$
$$\alpha \in (0, 1), \quad f(\alpha x) = \alpha f(x), \quad x \in \mathbb{R}. \tag{4.44}$$

Define $f_\pm : \mathbb{R}_+ \to \mathbb{R}$ by $f_\pm(x) := f(\pm x)$, $x \in \mathbb{R}_+$, and $g_\pm : \mathbb{R} \to \mathbb{R}$ [cf. (4.10)] by

$$g_\pm(x) := \alpha^{-x} f_\pm(\alpha^x), \quad x \in \mathbb{R}, \quad \text{so that} \quad f_\pm(x) = x g_\pm(\log_\alpha x), \quad x \in \mathbb{R}_+. \tag{4.45}$$

Note that the assumption $f \in \mathcal{A}(\dot{\mathbb{R}})$ is equivalent to the assumption that $f_\pm \in \mathcal{A}(\mathbb{R}_+)$ and $f(0) = 0$. Note also that Lemma 4.7 applies with f and g replaced by $f_\pm$ and $g_\pm$. One simple consequence of these observations [see the discussion below (4.11)] is that (4.44) implies $f \in C^{0,1}(\mathbb{R})$.

As in the one-sided case, our starting point is the formula for $\sigma(D_\Gamma)$ in Corollary 4.4 in terms of $\sigma(K_t)$, for $t \in [-\pi, \pi]$. Again, K_t is compact for every $t \in \mathbb{R}$ by Corol-

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[17] As discussed in the introduction and at the beginning of Sect. 4, it is this case which is relevant to the spectral radius conjecture.
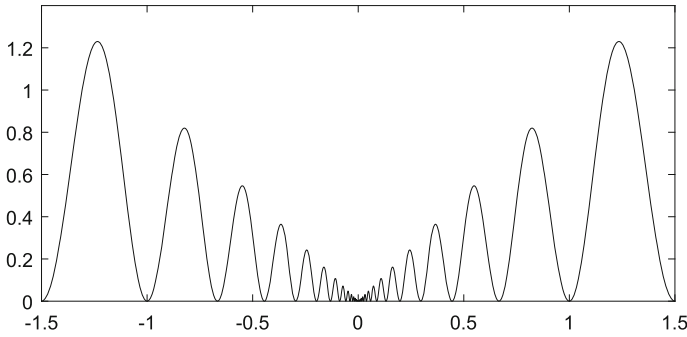
Fig. 3 Graph of $f : \mathbb{R} \to \mathbb{R}$, $f(x) := |x| \sin^2(\pi \log_\alpha |x|)$, in the case $\alpha = \frac{2}{3}$

lary 4.6, and, following the pattern of Sect. 4.2, our goal is to compute $\sigma(K_t)$ and $\rho(K_t)$ by the results of Sect. 3.2. As in Sect. 4.2, to make use of these results it is convenient to work with integral operators on $[0, 1]$ rather than Γ_0. Reflecting that Γ_0 has two components, $\Gamma_0^+ := \{(x, f(x)) \in \Gamma_0 : x > 0\}$ and $\Gamma_0^- := \{(x, f(x)) \in \Gamma_0 : x < 0\}$, it is natural to work in this case with a unitary operator $U : L^2(\Gamma_0) \to (L^2(0, 1))^2$, defined by $U\psi := (U_+ P_+ \psi, U_- P_- \psi)^T$, $\psi \in L^2(\Gamma_0)$, where $P_\pm$ is restriction to $\Gamma_0^\pm$, i.e. $P_\pm \psi := \psi|_{\Gamma_0^\pm}$, $\psi \in L^2(\Gamma_0)$, and $U_\pm : L^2(\Gamma_0^\pm) \to L^2(0, 1)$ is the unitary operator defined by [cf. (4.12)]

$$U_\pm \phi(x) := \phi(\pm \alpha^x, f_\pm(\alpha^x))(1 + f'_\pm(\alpha^x)^2)^{1/4} \alpha^{x/2} |\log \alpha|^{1/2},$$
$$x \in [0, 1], \quad \phi \in L^2(\Gamma_0^\pm).$$

With U as given above we define an operator $\tilde{K}_t$ on $(L^2(0, 1))^2$, which is unitarily equivalent to K_t, by (4.13). It is easy to see [cf. (4.14)] that

$$\tilde{K}_t = \begin{pmatrix} \tilde{K}_t^- & \tilde{L}_t^- \\ \tilde{L}_t^+ & \tilde{K}_t^+ \end{pmatrix}, \quad t \in \mathbb{R}, \tag{4.46}$$

where the entries of $\tilde{K}_t$ are integral operators on $L^2(0, 1)$ with continuous kernels. The kernels of $\tilde{K}_t^\pm$ are $\tilde{K}_t^\pm(\cdot, \cdot)$, where [cf. (4.15)]

$$\tilde{K}_t^\pm(x, y) := \frac{1}{2\pi} \sum_{j=-\infty}^\infty e^{ijt} \frac{p_j^\pm(x, y)}{1 + q_j^\pm(x, y)^2} \left(\frac{1 + f'_\pm(\alpha^x)^2}{1 + f'_\pm(\alpha^y)^2} \right)^{1/4} \alpha^{\frac{x+y+j}{2}} |\log \alpha|, \quad x, y \in \mathbb{R}, \tag{4.47}$$

and $p_j^\pm$ and $q_j^\pm$ are defined by (4.16), (4.17), and (4.18), but with f replaced by $f_\pm$. The kernels of $\tilde{L}_t^\pm$ are $\tilde{L}_t^\pm(\cdot, \cdot)$, where, for $x, y \in \mathbb{R}$,

$$\tilde{L}_t^\pm(x, y) := \frac{1}{2\pi} \sum_{j=-\infty}^\infty e^{ijt} \frac{\tilde{p}_j^\pm(x, y)}{1 + \tilde{q}_j^\pm(x, y)^2} \left(\frac{1 + f'_\pm(\alpha^x)^2}{1 + f'_\mp(\alpha^y)^2} \right)^{1/4} \alpha^{\frac{x+y+j}{2}} |\log \alpha| \tag{4.48}$$

and

$$\tilde{p}_j^{\pm}(x, y) := \frac{(\alpha^y + \alpha^{x+j})f'_{\mp}(\alpha^y) + f_{\pm}(\alpha^{x+j}) - f_{\mp}(\alpha^y)}{(\alpha^{x+j} + \alpha^y)^2},$$

$$\tilde{q}_j^{\pm}(x, y) := \frac{f_{\pm}(\alpha^{x+j}) - f_{\mp}(\alpha^y)}{\alpha^{x+j} + \alpha^y}. \tag{4.49}$$

Analogously to (4.19), we have that

$$\sigma(K_t; L^2(\Gamma_0)) = \sigma(\tilde{K}_t; (L^2(0, 1))^2) = \sigma(\tilde{K}_t; (C[0, 1])^2), \quad t \in \mathbb{R}. \tag{4.50}$$

As in the one-sided case [see the discussion around (4.30)], to estimate spectral properties of $\tilde{K}_t$ as an operator on $(C[0, 1])^2$ we approximate $\tilde{K}_t^{\pm}$ by finite rank operators $\tilde{K}_{t,N}^{\pm}$ given by

$$\tilde{K}_{t,N}^{\pm}\phi(x) := \frac{1}{N}\sum_{n=1}^{N} \tilde{K}_t^{\pm}(x, x_{n,N})\phi(x_{n,N}), \quad \phi \in C[0, 1], \quad x \in [0, 1], \quad N \in \mathbb{N}.$$

Similarly, we approximate $\tilde{L}_t^{\pm}$ by $\tilde{L}_{t,N}^{\pm}$, leading to finite rank approximations $\tilde{K}_{t,N}$, $N \in \mathbb{N}$, to $\tilde{K}_t$, given by (4.46) with $\tilde{K}_t^{\pm}$ and $\tilde{L}_t^{\pm}$ replaced by $\tilde{K}_{t,N}^{\pm}$ and $\tilde{L}_{t,N}^{\pm}$. Arguing as below (4.30), we have that $\tilde{K}_{t,N} \to \tilde{K}_t$ and $(\tilde{K}_{t,N})_{N\in\mathbb{N}}$ is collectively compact, so that $\|(\tilde{K}_t - \tilde{K}_{t,N})\tilde{K}_t\|_{\infty} \to 0$ as $N \to \infty$, where $\|\cdot\|_{\infty}$ here denotes the operator norm of an operator on $(C[0, 1])^2$ equipped with the norm $\|\cdot\|_{\infty}$ defined by $\|\phi\|_{\infty} := \max(\|\phi_+\|_{\infty}, \|\phi_-\|_{\infty})$, for $\phi = (\phi_+, \phi_-)^T \in (C[0, 1])^2$ Indeed, we have the following analogue of Proposition 4.10, in which our other norm notations are as defined above Proposition 4.8.

Proposition 4.16 *Let $t \in \mathbb{R}$ and $c > 0$ and assume that, for every $x, y \in \mathbb{R}$, $\tilde{K}_t^{\pm}(x, \cdot)$, $\tilde{L}_t^{\pm}(x, \cdot)$, $\tilde{K}_t^{\pm}(\cdot, y)$ and $\tilde{L}_t^{\pm}(\cdot, y)$ have analytic continuations from $\mathbb{R}$ to Σ_c that are bounded in Σ_c. Then $\|(\tilde{K}_t - \tilde{K}_{t,N})\tilde{K}_t\|_{\infty} \leq 2e^{2\pi c}C^*/(e^{2\pi Nc} - 1)$, where*

$$C^* := \max \Big\{ \|\tilde{K}_t^-\|_{0,c}\|\tilde{K}_t^-\|_{c,0} + \|\tilde{L}_t^-\|_{0,c}\|\tilde{L}_t^+\|_{c,0} + \|\tilde{K}_t^-\|_{0,c}\|\tilde{L}_t^-\|_{c,0} + \|\tilde{L}_t^-\|_{0,c}\|\tilde{K}_t^+\|_{c,0},$$

$$\|\tilde{L}_t^+\|_{0,c}\|\tilde{K}_t^-\|_{c,0} + \|\tilde{K}_t^+\|_{0,c}\|\tilde{L}_t^+\|_{c,0} + \|\tilde{L}_t^+\|_{0,c}\|\tilde{L}_t^-\|_{c,0} + \|\tilde{K}_t^+\|_{0,c}\|\tilde{K}_t^+\|_{c,0} \Big\}. \tag{4.51}$$

Proof Notice first that

$$(\tilde{K}_t - K_t^N)\tilde{K}_t = \begin{pmatrix} (\tilde{K}_t^- - K_{t,N}^-)\tilde{K}_t^- + (\tilde{L}_t^- - L_{t,N}^-)\tilde{L}_t^+ & (\tilde{K}_t^- - K_{t,N}^-)\tilde{L}_t^- + (\tilde{L}_t^- - L_{t,N}^-)\tilde{K}_t^+ \\ (\tilde{L}_t^+ - L_{t,N}^+)\tilde{K}_t^- + (\tilde{K}_t^+ - K_{t,N}^+)\tilde{L}_t^+ & (\tilde{L}_t^+ - L_{t,N}^+)\tilde{L}_t^- + (\tilde{K}_t^+ - K_{t,N}^+)\tilde{K}_t^+ \end{pmatrix},$$

and denote the entries of this matrix by $A_{j,k}$, $j, k = 1, 2$. For each of these terms we obtain an estimate similar to that in Proposition 4.10, by arguing as in the proof of that

proposition. Since also

$$\|(\tilde{K}_t - K_t^N)\tilde{K}_t\|_\infty = \max_{j=1,2} \sum_{k=1}^{2} \|A_{j,k}\|_\infty,$$

the result follows. $\qquad\square$

Let us now estimate, under our standing assumption (4.44), the norms of the kernels that appear in Proposition 4.16 [cf. Proposition 4.8].

Proposition 4.17 *Given* (4.44), *define* $g_\pm$ *by* (4.45) *so that, by Lemma* 4.7, $g_\pm$ *have analytic continuations to* Σ_c, *for some* $c > 0$, *such that* $g_\pm$, $g'_\pm$, *and* $g''_\pm$ *are bounded on* Σ_c *and* $g_\pm(z+1) = g_\pm(z)$, $z \in \Sigma_c$. *Let*

$$\mathfrak{I}_{c,\pm} := \|\operatorname{Im} g_\pm\|_c + \|\operatorname{Im} g'_\pm\|_c/|\log\alpha| \quad \text{and}$$
$$\mathfrak{K}_{c,\pm} := \alpha^{-1}|\log\alpha| \max(\|g_\pm\|_c, \|g_\mp\|_0), \qquad (4.52)$$

and set

$$\mathfrak{F}_{d,\pm} := \|g_\pm\|_d + \|g'_\pm\|_d/|\log\alpha| \quad \text{and}$$
$$\mathfrak{G}_{d,\pm} := \|g'_\pm\|_d + \|g''_\pm\|_d/|\log\alpha|, \quad \text{for } d = 0, c. \qquad (4.53)$$

If

$$c \le \arccos(\alpha)/|\log\alpha|, \quad \mathfrak{I}_{c,\pm} < 1, \quad \text{and} \quad \|\operatorname{Im} g_\pm\|_c + \alpha c \mathfrak{K}_{c,\pm} < \alpha^2, \qquad (4.54)$$

then $\tilde{K}_t^\pm(x, \cdot)$, $\tilde{L}_t^\pm(x, \cdot)$, $\tilde{K}_t^\pm(\cdot, y)$, *and* $\tilde{L}_t^\pm(\cdot, y)$ *extend to bounded analytic functions on* Σ_c *for all* $x, y, t \in \mathbb{R}$, *and*

$$\|\tilde{K}_t^\pm\|_{c,0} \le \frac{\left(1 + \mathfrak{F}_{c,\pm}^2\right)^{1/4}}{\pi\left(1 - \mathfrak{I}_{c,\pm}^2\right)}\left[\mathfrak{G}_{c,\pm}\frac{1 + \alpha^{1/2} + \alpha^{-1/2}}{4\alpha^2} + |\log\alpha|(\mathfrak{F}_{c,\pm} + \mathfrak{F}_{0,\pm})\sum_{j=2}^{\infty}\frac{\alpha^{j/2}}{\alpha - \alpha^j}\right],$$

$$\|\tilde{K}_t^\pm\|_{0,c} \le \frac{\left(1 + \mathfrak{F}_{0,\pm}^2\right)^{1/4}}{\pi\left(1 - \mathfrak{I}_{c,\pm}^2\right)^{5/4}}\left[\mathfrak{G}_{c,\pm}\frac{1 + \alpha^{1/2} + \alpha^{-1/2}}{4\alpha^2} + 2|\log\alpha|\,\mathfrak{F}_{c,\pm}\sum_{j=2}^{\infty}\frac{\alpha^{j/2}}{\alpha - \alpha^j}\right],$$

$$\|\tilde{L}_t^\pm\|_{c,0} \le \frac{1}{2\pi}\frac{|\log\alpha|\,\mathfrak{F}_{0,\mp} + \mathfrak{K}_{c,\pm}}{1 - \alpha^{-4}\left(\|\operatorname{Im} g_\pm\|_c + \alpha c \mathfrak{K}_{c,\pm}\right)^2}\left(1 + \mathfrak{F}_{c,\pm}^2\right)^{1/4}\sum_{j=-\infty}^{\infty}\frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2}, \qquad (4.55)$$

$$\|\tilde{L}_t^\pm\|_{0,c} \le \frac{1}{2\pi}\frac{|\log\alpha|\,\mathfrak{F}_{c,\mp} + \mathfrak{K}_{c,\mp}}{1 - \alpha^{-4}\left(\|\operatorname{Im} g_\mp\|_c + \alpha c \mathfrak{K}_{c,\mp}\right)^2}\left(\frac{1 + \mathfrak{F}_{0,\pm}^2}{1 - \mathfrak{I}_{c,\mp}^2}\right)^{1/4}\sum_{j=-\infty}^{\infty}\frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2}. \qquad (4.56)$$

Moreover, the mappings $\mathbb{R} \to C([0,1] \times [0,1])$, $t \mapsto \tilde{K}_t^\pm(\cdot, \cdot)$ *and* $t \mapsto \tilde{L}_t^\pm(\cdot, \cdot)$, *are continuous.*

Remark 4.18 [*Bound on the sum in* (4.55) *and* (4.56)] For fixed $\alpha \in (0, 1)$, let $F(x) := (\alpha^{x/2} + \alpha^{-x/2})^{-1}$, for $x \in \mathbb{R}$, so that the jth term in the sum in (4.55) and (4.56) is $\alpha^{-2} F(j)$, and note that F is even. Arguing as in Remark 4.9, since F is decreasing on $[0, \infty)$,

$$\sum_{j=n+1}^{\infty} \frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2} = \alpha^{-2} \sum_{j=n+1}^{\infty} F(j) \leq C_n(\alpha) := \alpha^{-2} \int_n^{\infty} F(x) \, dx = \frac{2 \arctan(\alpha^{n/2})}{\alpha^2 |\log \alpha|},$$
(4.57)

for $0 < \alpha < 1$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Thus, for $0 < \alpha < 1$ and $n \in \mathbb{N}_0$,

$$\sum_{j=-\infty}^{\infty} \frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2} \leq C_n^*(\alpha) := \frac{1}{2\alpha^2} + 2\alpha^{-2} \sum_{j=1}^{n} F(j) + 2C_n(\alpha) \quad \text{and} \tag{4.58}$$

$$\left| \sum_{j=-\infty}^{\infty} \frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2} - \sum_{j=-n}^{n} \frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2} \right| \leq 2C_n(\alpha). \tag{4.59}$$

Proof of Proposition 4.17 The results for $\tilde{K}_t^{\pm}(\cdot, \cdot)$ follow immediately from Proposition 4.8, applied with f replaced by $f_{\pm}$. It remains to consider the off-diagonal entries $\tilde{L}_t^{\pm}$. We give the detail for $\tilde{L}_t^{+}$; the results for $\tilde{L}_t^{-}$ follow by the same argument with the roles of f_+ and f_- reversed.

We extend the definition of $\tilde{L}_t^{+}$ via (4.48) to all $x, y \in \Sigma_c$. The upcoming computations will show that this is well-defined and the estimates (4.55) and (4.56) hold. Denote by $T_j(x, y)$ the jth term in the sum (4.48), that is,

$$T_j(x, y) := e^{ijt} \frac{\tilde{p}_j^+(x, y)}{1 + \tilde{q}_j^+(x, y)^2} \left(\frac{1 + f_+'(\alpha^x)^2}{1 + f_-'(\alpha^y)^2} \right)^{1/4} \alpha^{\frac{x+y+j}{2}} |\log \alpha|, \quad x, y \in \Sigma_c.$$
(4.60)

We clearly have $T_j(x + 1, y) = e^{-it} T_{j+1}(x, y)$ and the dilation invariance, $f(\alpha x) = \alpha f(x)$, implies that $T_j(x, y+1) = e^{it} T_{j-1}(x, y)$ for $x, y \in \Sigma_c, j \in \mathbb{Z}$. Thus, to prove the uniform convergence of the series (4.48) and the bounds (4.55) and (4.56), it suffices to restrict consideration to $x, y \in \Xi_c = \{z \in \mathbb{C} : \operatorname{Re} z \in [0, 1], \operatorname{Im} z \in (-c, c)\}$. First notice that, for $x, y \in \Xi_c$,

$$\left| \alpha^{x+j} + \alpha^y \right| \geq \alpha^{\operatorname{Re} x+j} \cos(\log \alpha \operatorname{Im} x) + \alpha^{\operatorname{Re} y} \cos(\log \alpha \operatorname{Im} y)$$
$$\geq \alpha^{\operatorname{Re} x+j+1} + \alpha^{\operatorname{Re} y+1} \geq \alpha^{j+2} + \alpha^2, \tag{4.61}$$

by the first of (4.54), which implies that $\cos(t \log \alpha) \geq \alpha$, for $-c \leq t \leq c$. Hence, for $x, y \in \Xi_c$ and $j \in \mathbb{Z}$, recalling (4.45),

$$\left| \tilde{q}_j^+(x, y) \right| = \left| \frac{\alpha^{x+j} g_+(x) - \alpha^y g_-(y)}{\alpha^{x+j} + \alpha^y} \right| \leq \frac{\alpha^{\operatorname{Re} x+j} |g_+(x)| + \alpha^{\operatorname{Re} y} |g_-(y)|}{\alpha^{\operatorname{Re} x+j+1} + \alpha^{\operatorname{Re} y+1}}.$$

Using the estimate

$$(a + b)/(c + d) \leq \max \{a/c, b/d\}, \tag{4.62}$$

which holds for all $a, b \geq 0, c, d > 0$, it follows that

$$\sup_{x \in \Xi_c, y \in [0,1]} \left| \tilde{q}_j^+(x, y) \right| \leq \mathfrak{K}_{c,+}/|\log \alpha| \quad \text{and}$$

$$\sup_{x \in [0,1], y \in \Xi_c} \left| \tilde{q}_j^+(x, y) \right| \leq \mathfrak{K}_{c,-}/|\log \alpha|, \quad j \in \mathbb{Z}.$$

Noting that (4.27) holds with f and g replaced with $f_\pm$ and $g_\pm$, we have also that $|f_\pm'(\alpha^y)| \leq \mathfrak{F}_{0,\pm}$, for $y \in [0, 1]$, while $|f_\pm'(\alpha^y)| \leq \mathfrak{F}_{c,\pm}$, for $y \in \Xi_c$. Thus

$$\sup_{x \in \Xi_c, y \in [0,1]} \left| \tilde{p}_j^+(x, y) \right| \leq \sup_{x \in \Xi_c, y \in [0,1]} \frac{|f_-'(\alpha^y)| + \left| \tilde{q}_j^+(x, y) \right|}{|\alpha^{x+j} + \alpha^y|} \leq \frac{\mathfrak{F}_{0,-} + \mathfrak{K}_{c,+}/|\log \alpha|}{\alpha^{j+2} + \alpha^2} \quad \text{and}$$

$$\sup_{x \in [0,1], y \in \Xi_c} \left| \tilde{p}_j^+(x, y) \right| \leq \frac{\mathfrak{F}_{c,-} + \mathfrak{K}_{c,-}/|\log \alpha|}{\alpha^{j+2} + \alpha^2}.$$

Moreover, for $x, y \in \Xi_c$,

$$\left| \text{Im} \, \tilde{q}_j^+(x, y) \right| = \left| \text{Im} \left(\frac{\alpha^{x+j} g_+(x) - \alpha^y g_-(y)}{\alpha^{x+j} + \alpha^y} \right) \right|$$

$$= \frac{1}{|\alpha^{x+j} + \alpha^y|^2} \left| \text{Im} \left(\alpha^{2(\text{Re} \, x+j)} g_+(x) - \alpha^{\bar{x}+j+y} g_-(y) + \alpha^{x+j+\bar{y}} g_+(x) - \alpha^{2 \, \text{Re} \, y} g_-(y) \right) \right|.$$

Thus, using (4.61), noting that $|\text{Im}(\alpha^{it} z)| \leq |\text{Im}(z)| + |\sin(t \log \alpha)||z| \leq |\text{Im}(z)| + |t \log \alpha||z|$, for $t \in \mathbb{R}$ and $z \in \mathbb{C}$, and using (4.62) again to obtain the last two inequalities, we see that for $x, y \in \Xi_c$, where $r = \alpha^{\text{Re} \, x+j}$ and $s = \alpha^{\text{Re} \, y}$,

$$\left| \text{Im} \, \tilde{q}_j^+(x, y) \right| \leq \frac{1}{\alpha^2(r+s)^2} \left(r^2 | \text{Im} \, g_+(x) | + s^2 | \text{Im} \, g_-(y) | \right.$$

$$\left. + rs \left(| \text{Im} \, g_-(y) | + | \text{Im} \, g_+(x) | + c | \log \alpha | (|g_-(y)| + |g_+(x)|) \right) \right)$$

$$\leq \frac{r | \text{Im} \, g_+(x) | + s | \text{Im} \, g_-(y) | + c | \log \alpha | \max\{s|g_-(y)|, r|g_+(x)|\}}{\alpha^2(r+s)}$$

$$\leq \alpha^{-2} \left(\max\{| \text{Im} \, g_+(x) |, | \text{Im} \, g_-(y) |\} + c | \log \alpha | \max\{|g_-(y)|, |g_+(x)|\} \right).$$

Thus

$$\sup_{x \in \Xi_c, y \in [0,1]} \left| \text{Im} \, \tilde{q}_j^+(x, y) \right| \leq \alpha^{-2} \left(\| \text{Im} \, g_+ \|_c + \alpha c \mathfrak{K}_{c,+} \right) \quad \text{and}$$

$$\sup_{x \in [0,1], y \in \Xi_c} \left| \text{Im} \, \tilde{q}_j^+(x, y) \right| \leq \alpha^{-2} \left(\| \text{Im} \, g_- \|_c + \alpha c \mathfrak{K}_{c,-} \right)$$

so that

$$\inf_{x \in \Xi_c, y \in [0,1]} \left| 1 + \tilde{q}_j^+(x, y)^2 \right| \geq 1 - \sup_{x \in \Xi_c, y \in [0,1]} \left| \text{Im} \, \tilde{q}_j^+(x, y) \right|^2$$

$$\geq 1 - \alpha^{-4} \left(\| \operatorname{Im} g_+ \|_c + \alpha c \Re_{c,+} \right)^2 \quad \text{and}$$

$$\inf_{x \in [0,1], y \in \Xi_c} \left| 1 + \tilde{q}_j^+(x,y)^2 \right| \geq 1 - \alpha^{-4} \left(\| \operatorname{Im} g_- \|_c + \alpha c \Re_{c,-} \right)^2.$$

Combining these estimates, and using (4.28) with f replaced by f_-, we get

$$\sup_{x \in \Xi_c, y \in [0,1]} \left| T_j(x,y) \right| \leq \frac{|\log \alpha| \mathfrak{F}_{0,-} + \Re_{c,+}}{1 - \alpha^{-4} \left(\| \operatorname{Im} g_+ \|_c + \alpha c \Re_{c,+} \right)^2} \left(1 + \mathfrak{F}_{c,+}^2 \right)^{1/4} \frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2} \quad \text{and}$$

$$\sup_{x \in [0,1], y \in \Xi_c} \left| T_j(x,y) \right| \leq \frac{|\log \alpha| \mathfrak{F}_{c,-} + \Re_{c,-}}{1 - \alpha^{-4} \left(\| \operatorname{Im} g_- \|_c + \alpha c \Re_{c,-} \right)^2} \left(\frac{1 + \mathfrak{F}_{0,+}^2}{1 - \mathfrak{I}_{c,-}^2} \right)^{1/4} \frac{\alpha^{j/2}}{\alpha^{j+2} + \alpha^2}.$$

The estimates (4.55) and (4.56) and the other results for $\tilde{L}_t^+$ follow. □

Let $\tilde{B}_{t,N}^\pm$ and $\tilde{C}_{t,N}^\pm$ be the $N \times N$ matrices defined by the right hand side of (4.31) with $\tilde{K}_t(\cdot,\cdot)$ replaced by $\tilde{K}_t^\pm(\cdot,\cdot)$ and $\tilde{L}_t^\pm(\cdot,\cdot)$, respectively, and define the $2N \times 2N$ matrix $A_{t,N}$ by

$$A_{t,N} := \begin{pmatrix} B_{t,N}^- & C_{t,N}^- \\ C_{t,N}^+ & B_{t,N}^+ \end{pmatrix}, \quad t \in \mathbb{R}, \quad N \in \mathbb{N}.$$

Recalling Remark 3.11, the matrix $A_{t,N}$ and the operator $K_{t,N}$ are related by Lemma 3.7(ii) and (iii). As in Sect. 4.2, our goal is to estimate the spectrum and spectral radius of $K_{t,N}$ via computing the spectra of approximations to $A_{t,N}$ for some fixed N and finitely many $t \in [-\pi, \pi]$.

To define these approximations, proceeding analogously to Sect. 4.2, for $M \in \mathbb{N}$ let $\tilde{K}_t^{\pm,M}(\cdot,\cdot)$ and $\tilde{L}_t^{\pm,M}(\cdot,\cdot)$ be approximations to $\tilde{K}_t^\pm(\cdot,\cdot)$ and $\tilde{L}_t^\pm(\cdot,\cdot)$, respectively, given by (4.47) and (4.48) but with the infinite series replaced by finite sums from $j = -M$ to M [cf. (4.32)]. Let $B_{t,N}^{\pm,M}$ and $C_{t,N}^{\pm,M}$ be the corresponding approximations to the matrices $B_{t,N}^\pm$ and $C_{t,N}^\pm$, so that

$$B_{t,N}^{\pm,M}(p,q) = \frac{1}{N} \tilde{K}_t^{\pm,M}(x_{p,N}, x_{q,N}),$$

$$C_{t,N}^{\pm,M}(p,q) = \frac{1}{N} \tilde{L}_t^{\pm,M}(x_{p,N}, x_{q,N}), \quad p, q = 1, \ldots, N,$$

and let

$$A_{t,N}^M := \begin{pmatrix} B_{t,N}^{-,M} & C_{t,N}^{-,M} \\ C_{t,N}^{+,M} & B_{t,N}^{+,M} \end{pmatrix}, \quad t \in \mathbb{R}, \quad M, N \in \mathbb{N}. \tag{4.63}$$

Similarly, define the operators $\tilde{K}_{t,N}^{\pm,M}$ and $\tilde{L}_{t,N}^{\pm,M}$ as we defined $\tilde{K}_{t,N}^\pm$ and $\tilde{L}_{t,N}^\pm$ above Proposition 4.16, but replacing the kernels $\tilde{K}_t^\pm(\cdot,\cdot)$ and $\tilde{L}_t^\pm(\cdot,\cdot)$ in their definitions

by the respective approximations $\tilde{K}_t^{\pm,M}(\cdot,\cdot)$ and $\tilde{L}_t^{\pm,M}(\cdot,\cdot)$, and let

$$\tilde{K}_{t,N}^M = \begin{pmatrix} \tilde{K}_{t,N}^{-,M} & \tilde{L}_{t,N}^{-,M} \\ \tilde{L}_{t,N}^{+,M} & \tilde{K}_{t,N}^{+,M} \end{pmatrix}, \quad t \in \mathbb{R}, \quad M, N \in \mathbb{N}. \tag{4.64}$$

Then, similarly to (4.33), we have that, for $t \in \mathbb{R}$ and $M, N \in \mathbb{N}$ with $M \geq 2$,

$$\|A_{t,N} - A_{t,N}^M\|_\infty \leq \|\tilde{K}_{t,N} - \tilde{K}_{t,N}^M\|_\infty \leq C_1(M), \tag{4.65}$$

where, using the notations of Remarks 4.9 and 4.18 and Proposition 4.17, and setting $\mathfrak{K}_0 := \mathfrak{K}_{0,+} = \mathfrak{K}_{0,-}$,

$$C_1(M) := \frac{1}{\pi} \max \Big\{ \left(1 + \mathfrak{F}_{0,+}^2\right)^{1/4} \left(2|\log\alpha|\mathfrak{F}_{0,+}\mathcal{B}_M(\alpha) + (|\log\alpha|\mathfrak{F}_{0,-} + \mathfrak{K}_0)\mathcal{C}_M(\alpha)\right),$$
$$\left(1 + \mathfrak{F}_{0,-}^2\right)^{1/4} \left(2|\log\alpha|\mathfrak{F}_{0,-}\mathcal{B}_M(\alpha) + (|\log\alpha|\mathfrak{F}_{0,+} + \mathfrak{K}_0)\mathcal{C}_M(\alpha)\right) \Big\}. \tag{4.66}$$

Note that by (4.26) and (4.57), $C_1(M) = O(\alpha^{M/2})$ as $M \to \infty$.

Arguing as in (4.35), we have also that, for $M, N \in \mathbb{N}, t \in \mathbb{R}, p, q = 1, \ldots, N$,

$$\left| \frac{\partial}{\partial t} B_{t,N}^{\pm,M}(p,q) \right| \leq B_N^{\pm,M}(p,q) \quad \text{and} \quad \left| \frac{\partial}{\partial t} C_{t,N}^{\pm,M}(p,q) \right| \leq C_N^{\pm,M}(p,q), \tag{4.67}$$

where $B_N^{\pm,M}(p,q)$ is defined by (4.35) but with $p_j, q_j,$ and f replaced by $p_j^\pm, q_j^\pm$ and $f_\pm$, respectively [compare (4.15) and (4.47)]. Similarly, $C_N^{\pm,M}(p,q)$ is defined by the right hand side of (4.35) with p_j and q_j replaced by $\tilde{p}_j^\pm$ and $\tilde{q}_j^\pm$, respectively, and with f replaced by $f_\pm$ in the numerator, by $f_\mp$ in the denominator [compare (4.15) and (4.48)]. Thus, where

$$B_N^M := \begin{pmatrix} B_N^{-,M} & C_N^{-,M} \\ C_N^{+,M} & B_N^{+,M} \end{pmatrix}, \quad M, N \in \mathbb{N}, \tag{4.68}$$

Eq. (4.36) holds (with the above definitions of $A_{t,N}, A_{t,N}^M$ and B_N^M) for all $s, t \in \mathbb{R}$, $M, N \in \mathbb{N}$. Note also, arguing as in (4.37) and (4.38), that $\|B_N^M\|_\infty$ is bounded uniformly for $M, N \in \mathbb{N}$, and that $\|\tilde{K}_{t,N} - \tilde{K}_{s,N}\|_\infty = O(|s-t|)$ as $|s-t| \to 0$, uniformly for $s, t \in \mathbb{R}, N \in \mathbb{N}$. Further, similarly to (4.39) and (4.66), $\|\tilde{K}_{t,N}\|_\infty \leq C_3$ for $t \in \mathbb{R}$ and $N \in \mathbb{N}$, where[18]

$$C_3 := \frac{1}{2\pi} \max \Big\{ \left(1 + \mathfrak{F}_{0,+}^2\right)^{1/4} \left(\mathfrak{G}_{0,+}\mathfrak{R}(\alpha) + 4|\log\alpha|\mathfrak{F}_{0,+}\mathcal{B}_{10}^*(\alpha) + (|\log\alpha|\mathfrak{F}_{0,-} + \mathfrak{K}_0)\mathcal{C}_{10}^*(\alpha)\right),$$
$$\left(1 + \mathfrak{F}_{0,-}^2\right)^{1/4} \left(\mathfrak{G}_{0,-}\mathfrak{R}(\alpha) + 4|\log\alpha|\mathfrak{F}_{0,-}\mathcal{B}_{10}^*(\alpha) + (|\log\alpha|\mathfrak{F}_{0,+} + \mathfrak{K}_0)\mathcal{C}_{10}^*(\alpha)\right) \Big\}, \tag{4.69}$$

---

[18] Computations indicate that $\mathcal{C}_{10}^*(\alpha)$ exceeds the left hand side of (4.58) by not more than 2.1% for $\alpha \in (0, 1)$.

$\Re(\alpha) := (1 + \alpha^{1/2} + \alpha^{-1/2})/(2\alpha^2)$, $\mathcal{B}_{10}^*$ and $\mathcal{C}_{10}^*$ are defined by (4.25) and (4.58), and the other notations are defined in Proposition 4.17. Provided c is such that the conditions of Proposition 4.17 hold, we have also [cf. (4.40)], by Propositions 4.16 and 4.17, that (4.40) holds with

$$C_4 := 2e^{2\pi c}\widetilde{C}^*, \tag{4.70}$$

where $\widetilde{C}^*$ is defined by the right hand side of (4.51), but with the norms on that right hand side replaced by the upper bounds in Proposition 4.17. Additionally, we replace the infinite sums in these upper bounds by the bounds $\mathcal{B}_{10}^*(\alpha)$ and $\mathcal{C}_{10}^*(\alpha)$.

The following result, which holds for every Γ that satisfies our standing assumption (4.44), is identical to Theorem 4.13, except that $\tilde{K}_{t,N}^{M_N}$ and $A_{t,N}^{M_N}$ are defined here by (4.64) and (4.63), respectively. In this theorem $\sigma(\tilde{K}_{t,N}^M)$ denotes the spectrum of $\tilde{K}_{t,N}^M$ either on $(L^2(0, 1))^2$ or on $(C[0, 1])^2$ (cf. (4.19)), which coincides with $\{0\} \cup \sigma(A_{t,N}^M)$ by Lemma 3.7(ii) and Remark 3.11, and $\sigma(D_\Gamma)$ denotes the spectrum of D_Γ on $L^2(\Gamma)$, which coincides with the essential spectrum by Corollary 4.2. The proof of this result follows that of Theorem 4.13, noting that the argument of Lemma 4.12 applies to each of the operator families $\tilde{K}_{t,N}^{\pm,M}$ and $\tilde{L}_{t,N}^{\pm,M}$ so that [where $\tilde{K}_{t,N}^M$ is defined by (4.64)] $\{\tilde{K}_{t,N}^M : t \in [-\pi, \pi], M, N \in \mathbb{N}\}$ is collectively compact.

Theorem 4.19 *Choose sequences* $(m_N)_{N \in \mathbb{N}}$, $(M_N)_{N \in \mathbb{N}} \subset \mathbb{N}$ *such that* $m_N, M_N \to \infty$ *as* $N \to \infty$, *and for each* N, *let* $T_N := \{\pm(k - 1/2)\pi/m_N : k = 1, \ldots, m_N\}$. *Then, as* $N \to \infty$,

$$\sigma^N(D_\Gamma) := \bigcup_{t \in T_N} \sigma(\tilde{K}_{t,N}^{M_N}) = \{0\} \cup \bigcup_{t \in T_N} \sigma(A_{t,N}^{M_N}) \overset{\text{H}}{\to} \sigma(D_\Gamma) = \sigma_{\text{ess}}(D_\Gamma). \tag{4.71}$$

Remark 4.20 Using Remark 3.11, we see that Remark 4.14 applies also to $\sigma^N(D_\Gamma)$ given by (4.71).

Our second main result of this subsection, obtained by applying Theorem 3.8, noting Remark 3.11, is proved in the same way as the analogous result, Theorem 4.15, in the one-sided case. In this theorem $A_{t,N}^M$, B_N^M, $C_1(M)$, C_3, and C_4, are defined by (4.63), (4.68), (4.66), (4.69), and (4.70), respectively.

Theorem 4.21 *Under our standing assumption* (4.44), *define* $g_\pm$ *by* (4.45) *so that, by Lemma 4.7,* $g_\pm$ *have analytic continuations to* Σ_c, *for some* $c > 0$, *such that* $g_\pm$, $g'_\pm$, *and* $g''_\pm$ *are bounded on* Σ_c, *and, without loss of generality, assume that* (4.54) *holds. Suppose that* $\rho_0 > 0$, $m, M, N \in \mathbb{N}$, *with* $M \geq 2$. *For* $k = 1, \ldots, m$, *set* $t_k := (k - 1/2)\pi/m$, *suppose that* $\rho(A_{t_k,N}^M) < \rho_0$ *for* $k = 1, \ldots, m$, *and recursively define* $\mu_{k,\ell}$, *for* $\ell = 1, 2, \ldots$, *by* $\mu_{k,1} := \rho_0$, *and by* (4.42). *Further, for* $k = 1, \ldots, m$, *let* n_k *denote the smallest integer such that* $\sum_{\ell=1}^{n_k} \nu_{k,\ell} \geq 4\pi\rho_0$, *and let*

$$R_{m,M,N} := \min_{\substack{k=1,\ldots,m \\ \ell=1,\ldots,n_k}} \left(\frac{\nu_{k,\ell}}{4} + \frac{\nu_{k,\ell+1}}{2}\right).$$

If

$$\mathcal{L}_c(f, N) := \frac{C_4}{e^{2\pi Nc} - 1} < \mathcal{R}_c(f, m, M, N)$$

$$:= r_0^2 \left(1 + C_3 \left(R_{m,M,N} - C_1(M) - \frac{\pi}{2m} \|B_N^M\|_\infty \right)^{-1} \right)^{-1}, \quad (4.72)$$

then $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$. Conversely, if $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$, then, provided m and M are sufficiently large, there exists $N_0 \in \mathbb{N}$ such that (4.72) holds and $\rho(A_{t,N}^M) < \rho_0$ for $t \in [0, \pi]$ and all $N \geq N_0$.

Remark 4.22 The condition (4.72) can be written as

$$S_c(\Gamma, m, M, N) < 0,$$

where $S_c(\Gamma, m, M, N) := \mathcal{L}_c(f, N) - \mathcal{R}_c(f, m, M, N)$. For any $c > 0$ such that the conditions of the first sentence of the above theorem are satisfied (and these conditions are necessarily satisfied for all sufficiently small $c > 0$), we note that:

(i) The proof of the above theorem, which follows that of Theorem 4.15, shows that, provided $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$, $R_{m,M,N}$ is positive and bounded away from zero for all sufficiently large $m, M, N \in \mathbb{N}$, so that the same holds for $\mathcal{R}_c(f, m, M, N)$. Since also $\mathcal{L}_c(f, N) \to 0$ as $N \to \infty$, $S_c(\Gamma, m, M, N) < 0$ for all sufficiently large m, M, and N, if $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$.

(ii) For fixed $m, M, N \in \mathbb{N}$, evaluation of the functionals $\mathcal{L}(f, N)$ and $\mathcal{R}_c(f, m, M, N)$ requires inputs relating to the functions $f_\pm$ defined by (4.45), namely: the constant $\alpha \in (0, 1)$ and the bounds $\|g_\pm\|_0$, $\|g'_\pm\|_0$, and $\|g''_\pm\|_0$ on g defined by (4.45) (to evaluate $C_1(M)$ and C_3); the bounds $\|g_\pm\|_c$, $\|g'_\pm\|_c$, and $\|g''_\pm\|_c$ (to evaluate C_4); the values $f_\pm(\alpha^{x_{p,N}+j})$, $f'_\pm(\alpha^{x_{p,N}})$, and $f''_\pm(\alpha^{x_{p,N}})$, for $p = 1, \ldots, N$ and $j = -M, \ldots, M$ (to compute the entries of the matrices $A_{t_k,N}^M$, for $k = 1, \ldots, m$).

(iii) If exact values of $\|g_\pm\|_0$, $\|g'_\pm\|_0$, $\|g''_\pm\|_0$, $\|g_\pm\|_c$, $\|g'_\pm\|_c$, and $\|g''_\pm\|_c$ are not available, it is enough to replace them with upper bounds which leads to upper bounds $\widehat{C}_1(M)$, $\widehat{C}_3$ and $\widehat{C}_4$ for $C_1(M)$, C_3 and C_4, respectively (with $\widehat{C}_1(M) \to 0$ as $M \to \infty$ at the same rate as $C_1(M)$). Let $\widehat{\mathcal{L}}_c(f, N)$ and $\widehat{\mathcal{R}}_c(f, m, M, N)$ be defined by (4.43) with $C_1(M)$, C_3, C_4, replaced by their upper bounds, so that $\mathcal{L}_c(f, N) \leq \widehat{\mathcal{L}}_c(f, N)$ and $\mathcal{R}_c(f, m, M, N) \leq \widehat{\mathcal{R}}_c(f, m, M, N)$. Then the conclusions of Theorem 4.21 hold with $\mathcal{L}_c(f, N)$ and $\mathcal{R}_c(f, m, M, N)$ replaced by $\widehat{\mathcal{L}}_c(f, N)$ and $\widehat{\mathcal{R}}_c(f, m, M, N)$, respectively; in particular $S_c(\Gamma, m, M, N) < 0$ if $\widehat{\mathcal{L}}_c(f, N) < \widehat{\mathcal{R}}_c(f, m, M, N)$ and, provided $\rho(D_\Gamma; L^2(\Gamma)) < \rho_0$, $\widehat{\mathcal{L}}_c(f, N) < \widehat{\mathcal{R}}_c(f, m, M, N)$ for all sufficiently large m, M, and N.

4.4 The 2D case: lower bounds for the numerical range

This paper is motivated by the question at the end of Sect. 1.1 which notes that there exist (e.g., [11, Fig. 3]) 2D examples of $\Gamma \in \mathcal{D}$ with $w_{\text{ess}}(D_\Gamma) \gg \frac{1}{2}$ and asks: is $\rho_{L^2(\Gamma),\text{ess}} \geq \frac{1}{2}$ for any of these examples? We will explore this in Sect. 6 where we will see that $\rho_{L^2(\Gamma),\text{ess}} < \frac{1}{2}$ for each example we treat, supporting Conjecture 1.2. We will also see that $w_{\text{ess}}(D_\Gamma) > \frac{1}{2}$ for at least some of these examples, indeed that $W_{\text{ess}}(D_\Gamma) \supset B_R := \{z \in \mathbb{C} : |z| < R\}$ for R substantially larger than $\frac{1}{2}$.

As the route to obtain these estimates for $W_{\text{ess}}(D_\Gamma)$, in this subsection we obtain lower bounds[19] for $W(D_\Gamma) = W_{\text{ess}}(D_\Gamma)$ in the case when Γ is a dilation invariant graph, precisely in the 2D case when either (4.8) or (4.44) applies. It is enough to restrict attention to the one-sided case (4.8) as one easily sees (e.g., [11, Sect. 2.3]) that if Γ satisfies (4.44) and $\widetilde{\Gamma} := \{(x, f(x)) : x > 0\}$, then $\widetilde{\Gamma}$ satisfies (4.8) and

$$W(D_{\widetilde{\Gamma}}) \subset W(D_\Gamma). \tag{4.73}$$

We obtain lower bounds for $W(D_\Gamma)$ in the case that (4.8) applies via the Nyström method that we used in Sect. 4.2 to approximate $\sigma(D_\Gamma)$. Firstly, by Corollary 4.4, and since, for $t \in [-\pi, \pi]$, $\tilde{L}_t$, defined in the proof of Proposition 4.10, is unitarily equivalent to $\tilde{K}_t$ defined by (4.13), which is unitarily equivalent to K_t, we have that $W(\tilde{L}_t) \subset W(D_\Gamma)$. For $p \in \mathbb{N}_0$, let $\mathcal{T}_p \subset L^2(0, 1)$ denote the set of trigonometric polynomials of degree at most p. An orthonormal basis of $\mathcal{T}_p$ is given by $\{e_{-p}, \ldots, e_p\}$, where $e_j(x) := e^{2\pi i j x}$. Consider the restricted numerical range

$$W_p(\tilde{L}_t) := \left\{ \langle \tilde{L}_t \phi, \phi \rangle : \phi \in \mathcal{T}_p, \|\phi\|_{L^2(0,1)} = 1 \right\}.$$

It is clear that $W_p(\tilde{L}_t) \subset W(\tilde{L}_t)$ for $t \in [-\pi, \pi]$, and that (e.g., [11, Sect. 2.3]) $W_p(\tilde{L}_t) = W(T^{p,t})$, where $T^{p,t} = (T^{p,t}_{jk})^p_{j,k=-p}$ is the $(2p+1) \times (2p+1)$ matrix with $T^{p,t}_{jk} := \langle \tilde{L}_t e_k, e_j \rangle$, $j, k = -p, \ldots, p$. Recalling from the proof of Proposition 4.10 that $\tilde{L}_t$ has kernel $\tilde{L}_t(x, y) = e^{i(x-y)t} \tilde{K}_t(x, y)$, we will approximate $T^{p,t}$ by $T^{p,t,N}$, where

$$T^{p,t,N}_{jk} := \frac{1}{N^2} \sum_{m,n=1}^N e^{i(x_{m,N}-x_{n,N})t} \tilde{K}_t(x_{m,N}, x_{n,N}) e_k(x_{n,N}) \overline{e_j(x_{m,N})}, \quad j, k = -p, \ldots, p, \tag{4.74}$$

recalling that $x_{m,N} := \frac{1}{N}\left(m - \frac{1}{2}\right)$, $m = 1, \ldots, N$, and then approximate $T^{p,t,N}$ further by $T^{p,t,N,M}$, defined by (4.74) but with $\tilde{K}_t$, given by the infinite sum (4.15), replaced by $\tilde{K}_t^M$, given by the finite sum (4.32). The following result, which uses the notation (4.34), will enable us to estimate the difference between $W_p(\tilde{L}_t) = W(T^{p,t})$ and $W(T^{p,t,N,M})$.

---

[19] By a *lower bound* for $W(D_\Gamma)$ we mean, simply, a set $S \subset \mathbb{C}$ such that $S \subset W(D_\Gamma)$.

Proposition 4.23 *Let f and $c > 0$ be as in Proposition 4.8, in particular satisfying (4.21), so that $\tilde{L}_t(x, \cdot)$ and $\tilde{L}_t(\cdot, y)$ are analytic and bounded in Σ_c for all $x, y, t \in \mathbb{R}$, let $\tilde{L}_t^M(x, y) := e^{i(x-y)}\tilde{K}^M(x, y)$, for $x, y \in \mathbb{R}$ and $M \in \mathbb{N}$, and suppose that g is a bounded, analytic, 1-periodic function on Σ_c. Then, for $M, N \in \mathbb{N}$ and $t \in [-\pi, \pi]$,*

$$\left| \langle \tilde{L}_t g, g \rangle - \frac{1}{N^2} \sum_{m,n=1}^{N} \tilde{L}_t^M(x_{m,N}, x_{n,N}) g(x_{n,N}) \overline{g(x_{m,N})} \right|$$

$$\leq 2 \|g\|_0 \|g\|_c \, e^{\pi c} \frac{\|\tilde{K}_t\|_{0,c} + \|\tilde{K}_t\|_{c,0}}{e^{2\pi Nc} - 1} + C_1(M).$$

Proof

$$\left| \langle \tilde{L}_t g, g \rangle - \frac{1}{N^2} \sum_{m,n=1}^{N} \tilde{L}_t(x_{m,N}, x_{n,N}) g(x_{n,N}) \overline{g(x_{m,N})} \right|$$

$$\leq \int_0^1 |g(x)| \left| \int_0^1 \tilde{L}_t(x, y) g(y) \, \mathrm{d}y - \frac{1}{N} \sum_{n=1}^{N} \tilde{L}_t(x, x_{n,N}) g(x_{n,N}) \right| \mathrm{d}x$$

$$+ \frac{1}{N} \sum_{n=1}^{N} \left| \int_0^1 \overline{g(x)} \tilde{L}_t(x, x_{n,N}) \, \mathrm{d}x - \frac{1}{N} \sum_{m=1}^{N} \overline{g(x_{m,N})} \tilde{L}_t(x_{m,N}, x_{n,N}) \right| |g(x_{n,N})|$$

$$\leq \|g\|_0 \frac{2\|\tilde{L}_t\|_{0,c} \|g\|_c}{e^{2\pi Nc} - 1} + \frac{2 \|g\|_c \|\tilde{L}_t\|_{c,0}}{e^{2\pi Nc} - 1} \|g\|_0,$$

by Theorem 4.11, noting that g and, from the proof of Proposition 4.10, $\tilde{L}_t(x, \cdot)$, for $x \in \mathbb{R}$, and $\tilde{L}_t(\cdot, y)$, for $y \in \mathbb{R}$, are all 1-periodic. The claimed result now follows immediately from (4.33) and by noting that, from the proof of Proposition 4.10, $\|\tilde{L}_t\|_{0,c} \leq e^{\pi c}\|\tilde{K}_t\|_{0,c}$ and $\|\tilde{L}_t\|_{c,0} \leq e^{\pi c}\|\tilde{K}_t\|_{c,0}$. □

Clearly, $g \in \mathcal{T}^p$ with $\|g\|_{L^2(0,1)} = 1$ if and only if

$$g = \sum_{j=-p}^{p} c_j e_j \quad \text{with} \quad \sum_{j=-p}^{p} |c_j|^2 = 1, \tag{4.75}$$

in which case

$$\|g\|_c \leq \sum_{j=-p}^{p} |c_j| \, \|e_j\|_c \leq \sum_{j=-p}^{p} |c_j| \, \|e_j\|_c \leq \sum_{j=-p}^{p} |c_j| e^{2\pi pc} \leq \sqrt{2p+1} \, e^{2\pi pc}.$$

The following corollary follows immediately from this observation and Proposition 4.23, noting that, if g and $c_{-p}, \ldots, c_p$ are related by (4.75), then the summation in Proposition 4.23 coincides with $\sum_{j,k=-p}^{p} T_{j,k}^{p,t,N,M} c_k \overline{c_j} \in W(T^{p,t,N,M})$.

Corollary 4.24 *For $p \in \mathbb{N}_0$, $M, N \in \mathbb{N}$, $t \in [-\pi, \pi]$, where $c > 0$ is as in Proposition [4.23](),*

$$d_H(W_p(\tilde{L}_t), W(T^{p,t,N,M})) \leq C_7(p, N, M)$$

$$:= 2(2p + 1)e^{\pi c(2p+1)} \frac{\|\tilde{K}_t\|_{0,c} + \|\tilde{K}_t\|_{c,0}}{e^{2\pi Nc} - 1} + C_1(M).$$

We can approximate $W(T^{p,t,N,M})$ by a standard method that dates back to Johnson [31]. Choose $n \in \mathbb{N}$ and, for $\ell = 0, \ldots, n$, let $\theta_\ell := 2\pi\ell/n$, let λ_ℓ denote the largest eigenvalue of $\mathrm{Re}(e^{-i\theta_\ell} T^{p,t,N,M})$ and x_ℓ an associated unit eigenvector, and let $z_\ell := \langle T^{p,t,N,M} x_\ell, x_\ell \rangle$, for $\ell = 0, 1, \ldots, n - 1$, and $z_n := z_0$. Then $W_n :=$ conv($\{z_0, \ldots, z_n\}$) $\subset W(T^{p,t,N,M})$ and (Johnson [31]) $W_n \overset{H}{\to} W(T^{p,t,N,M})$ as $n \to \infty$. The following simple corollary of the above results, in which we use the notations just introduced, will enable us to show that $B_R \subset W(\tilde{L}_t) \subset W(D_\Gamma)$ for concrete values of $R > 0$ in the examples we treat in Sect. [6]().

Corollary 4.25 *Suppose that $p \in \mathbb{N}_0$, $M, N, n \in \mathbb{N}$, $t \in [-\pi, \pi]$, that $c > 0$ is as in Proposition [4.23](), and that 0 is an interior point of W_n, in which case $z_\ell = \exp(i\gamma_\ell)|z_\ell|$, $\ell = 0, \ldots, n$, with $\gamma_0 \leq \gamma_1 \leq \ldots \leq \gamma_n = \gamma_0 + 2\pi$. Then, where $\theta_{\max} := \max_{\ell=1,\ldots,n}(\gamma_\ell - \gamma_{\ell-1}) \in (0, 2\pi]$ and $R_{\min} := \min_{\ell=0,\ldots,n} |z_\ell| > 0$, if $R^* := R_{\min}\cos(\theta_{\max}/2) - C_7(p, t, N, M) > 0$, then $\overline{B_{R^*}} \subset W(D_\Gamma)$.*

Proof Under the above assumptions, $\overline{B_R} \subset W_n$, where $R := R_{\min}\cos(\theta_{\max}/2)$. Since $W_n \subset W(T^{p,t,N,M})$ and $W_p(\tilde{L}_t) \subset W(\tilde{L}_t) \subset W(D_\Gamma)$, it follows that $\overline{B_{R^*}} \subset W(D_\Gamma)$ by Corollary [4.24](). $\qquad\square$

5 Localization and deformation

Let $\Omega_- \subset \mathbb{R}^d$, $d \geq 2$, be a Lipschitz domain with boundary $\Gamma = \partial\Omega_-$ and outward-pointing unit normal vector $n_y = n_y^\Gamma$ at almost every $y \in \Gamma$. If $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism, then $\psi(\Omega_-)$ is a Lipschitz domain with boundary $\psi(\Gamma)$ by Hofmann et al. [29, Theorem 4.1]. Note that this is no longer true if ψ is only assumed to be a bi-Lipschitz map (see [21, Lemma 1.2.1.4], and the discussion in [29]). As before, D_Γ will be the double-layer operator on Γ. To simplify notation we will use $\sim$ to denote equality up to compact operators, and abbreviate $D_\Gamma - \lambda I$ as $D_\Gamma - \lambda$. For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}$.

5.1 Localization without deformation

To start, we will consider domains with locally-dilation-invariant boundaries $\Gamma \in \mathscr{D}$; recall Definition [1.3](). Note that being locally dilation invariant at x is equivalent to the existence of an isometry $\psi_x : \mathbb{R}^d \to \mathbb{R}^d$ with $\psi_x(x) = 0$ and $\psi_x(B(x, \delta(x)) \cap \Gamma) = B(0, \delta(x)) \cap \Gamma_x$, for some $\delta(x) > 0$, where Γ_x is the graph of a Lipschitz continuous function with $\alpha\Gamma_x = \Gamma_x$. We also assume that ψ_x preserves the orientation of the outer normal vector field, that is, the outward-pointing normal vector field on $B(x, \delta(x)) \cap \Gamma$

is mapped to the upwards-pointing normal vector field on $B(0, \delta(x)) \cap \Gamma_x$. Similarly, Γ is locally C^1 at x if there is a $\delta(x) > 0$ and an Euclidean map $\psi_x : \mathbb{R}^d \to \mathbb{R}^d$ with $\psi_x(x) = 0$ and $\psi_x(B(x, \delta(x)) \cap \Gamma) = B(0, \delta(x)) \cap \Gamma_x$, where Γ_x is the graph of a C^1-function with compact support. Again, we assume that ψ_x preserves the orientation of the outer normal vector field. We will use extensively below the notation $\delta(x)$, for $x \in \Gamma$, which will have one of the above meanings, depending on whether Γ is locally dilation invariant at x or is C^1 at x.

Note that D_Γ is compact for every C^1 domain [19], as we recalled in Sect. 1.2; this implies that also $D_{\widetilde{\Gamma}}$ is compact if $\widetilde{\Gamma}$ is the graph of a C^1-function with compact support. Consequently, if Γ consists of different parts that are separated by C^1 areas, we are able to localize using sharp cut-off functions. In the following we equip Γ with the topology induced from $\mathbb{R}^d$ and the standard surface measure. In particular, χ_E denotes the characteristic function of a subset $E \subset \Gamma$ that is measurable with respect to the surface measure on Γ.

Lemma 5.1 *Let $E \subset \Gamma$ be a measurable subset and assume that Γ is locally C^1 at every $x \in \partial E$. Then D_Γ essentially commutes with χ_E, i.e., the commutator $[D_\Gamma, \chi_E] := D_\Gamma \chi_E - \chi_E D_\Gamma$ is compact.*

Proof Since ∂E is a compact subset of Γ, there is an $\varepsilon > 0$ such that

$$(\partial E + B(0, \varepsilon)) \cap \Gamma \subset \bigcup_{x \in \partial E} (B(x, \delta(x)) \cap \Gamma).$$

Set $E_o := (\partial E + B(0, \frac{\varepsilon}{2})) \cap \Gamma$. We know that $\chi_A D_\Gamma \chi_B$ is compact, even Hilbert–Schmidt, whenever A and B have positive distance from each other. We have

$$E = (E \cap E_o) \cup (E \setminus E_o) \quad \text{and} \quad E^c = (E^c \cap E_o) \cup (E^c \setminus E_o),$$

where $E^c := \Gamma \setminus E$. By inspection, we see that the four sets on the right of these equalities, if non-empty, have pairwise positive distance from each other except for $E \cap E_o$ and $E^c \cap E_o$. We thus have

$$\chi_E D_\Gamma \chi_{E^c} \sim \chi_{E \cap E_o} D_\Gamma \chi_{E^c \cap E_o} = \chi_E \chi_{E_o} D_\Gamma \chi_{E_o} \chi_{E^c},$$
$$\chi_{E^c} D_\Gamma \chi_E \sim \chi_{E^c \cap E_o} D_\Gamma \chi_{E \cap E_o} = \chi_{E^c} \chi_{E_o} D_\Gamma \chi_{E_o} \chi_E.$$

The operator $\chi_{E_o} D_\Gamma \chi_{E_o}$ is compact because $\overline{E}_o$ is C^1 everywhere by construction. We conclude that $[D_\Gamma, \chi_E] = \chi_{E^c} D_\Gamma \chi_E - \chi_E D_\Gamma \chi_{E^c}$ is compact as well. $\qquad \square$

In particular, this shows that if Γ is locally C^1 at some point $x \in \Gamma$, then D_Γ is not Fredholm, since if $E = B(x, \frac{\delta(x)}{2})$, then $\chi_E D_\Gamma \chi_E$ is compact, and Lemma 5.1 implies that $D_\Gamma \sim \chi_E D_\Gamma \chi_E + \chi_{E^c} D_\Gamma \chi_{E^c}$. Hence

$$0 \in \sigma_{\text{ess}}(\chi_E D_\Gamma \chi_E) \subset \sigma_{\text{ess}}(D_\Gamma). \tag{5.1}$$

More generally, if $E_1, \ldots, E_N$ are measurable subsets of Γ such that $\Gamma = \cup_{j=1}^N E_j$, $E_j \cap E_k = \emptyset$ for $j \neq k$ and Γ is locally C^1 at every $x \in \cup_{j=1}^N \partial E_j$, then $D_\Gamma \sim$

$\sum_{j=1}^{N} \chi_{E_j} D_{\Gamma} \chi_{E_j}$ and thus

$$\sigma_{\text{ess}}(D_{\Gamma}) = \bigcup_{j=1}^{N} \sigma_{\text{ess}}(\chi_{E_j} D_{\Gamma} \chi_{E_j}). \tag{5.2}$$

Equation (5.2) is a localisation result. We have also the following, more substantial localisation result (cf. [11, 15, 16, 48]) for the case where $\Gamma \in \mathcal{D}$.

Theorem 5.2 *Let $\Omega_- \subset \mathbb{R}^d$ be a bounded Lipschitz domain such that $\Gamma = \partial\Omega_- \in \mathcal{D}$, and pick $x_1, \ldots, x_N \in \Gamma$ for which $\Gamma \subset \bigcup_{j=1}^{N} B(x_j, \delta(x_j))$. Then*

$$\sigma_{\text{ess}}(D_{\Gamma}) = \bigcup_{x \in \Gamma} \sigma_{\text{ess}}(D_{\Gamma_x}) = \bigcup_{j=1}^{N} \sigma_{\text{ess}}(D_{\Gamma_{x_j}}).$$

Proof Let $x \in \Gamma$. If Γ is locally C^1 at x, then, by (5.1) and since D_{Γ_x} is compact, $\sigma_{\text{ess}}(D_{\Gamma_x}) = \{0\} \subset \sigma_{\text{ess}}(D_{\Gamma})$. So assume that Γ is locally dilation invariant at x. Then there is an $\alpha \in (0, 1)$ such that $\alpha\Gamma_x = \Gamma_x$. Let $\eta \colon \Gamma \to [0, 1]$ be a Lipschitz continuous function with support in $B(x, \delta(x)) \cap \Gamma$ that is equal to 1 in a neighbourhood of x. Let $\psi_x \colon \mathbb{R}^d \to \mathbb{R}^d$ be an isometry with $\psi_x(x) = 0$ and $\psi_x(B(x, \delta(x)) \cap \Gamma) = B(0, \delta(x)) \cap \Gamma_x$. Define $\Psi_x \colon L^2(\Gamma_x) \to L^2(\Gamma)$ by

$$\Psi_x \phi(y) = \begin{cases} \phi(\psi_x(y)) & \text{if } y \in B(x, \delta(x)) \cap \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

which has adjoint $\Psi'_x \colon L^2(\Gamma) \to L^2(\Gamma_x)$ given by

$$\Psi'_x \phi(y) = \begin{cases} \phi(\psi_x^{-1}(y)) & \text{if } y \in B(0, \delta(x)) \cap \Gamma_x, \\ 0 & \text{otherwise.} \end{cases}$$

Ψ_x and Ψ'_x are partial isometries with initial spaces $L^2(B(0, \delta(x)) \cap \Gamma_x)$ and $L^2(B(x, \delta(x)) \cap \Gamma)$, respectively. For $m \in \mathbb{N}$ the composition $\Psi'_x \eta^m \Psi_x \colon L^2(\Gamma_x) \to L^2(\Gamma_x)$ is the operation of multiplication by $(\eta')^m$, where $\eta'(y) := \eta(\psi_x^{-1}(y))$, for $y \in B(x, \delta(x)) \cap \Gamma_x$, $\eta'(y) := 0$, $y \in \Gamma_x \backslash B(x, \delta(x))$.

Let V_α be the unitary dilation as defined in (4.2) but with Γ replaced by Γ_x. Then

$$V_\alpha^n \eta' V_\alpha^{-n} \phi(y) = \begin{cases} \eta(\psi_x^{-1}(\alpha^n y))\phi(y) & \text{if } \alpha^n y \in B(0, \delta(x)) \cap \Gamma_x, \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbb{N}$, $y \in \Gamma_x$ and $\phi \in L^2(\Gamma_x)$. By dominated convergence, it follows that $V_\alpha^n \eta' V_\alpha^{-n} \phi \to \phi$ as $n \to \infty$ for all $\phi \in L^2(\Gamma_x)$. (We can see, similarly, that, for every $m \in \mathbb{N}$, $V_\alpha^n (\eta')^m V_\alpha^{-n} \phi \to \phi$ as $n \to \infty$ for all $\phi \in L^2(\Gamma_x)$.) Using that ψ_x is

an isometry, we further observe that $\Psi'_x \eta D_\Gamma \eta \Psi_x = \eta' D_{\Gamma_x} \eta'$. This implies, since also $D_{\Gamma_x} V_\alpha = V_\alpha D_{\Gamma_x}$, that

$$V_\alpha^n \Psi'_x \eta D_\Gamma \eta \Psi_x V_\alpha^{-n} = V_\alpha^n \eta' D_{\Gamma_x} \eta' V_\alpha^{-n} = (V_\alpha^n \eta' V_\alpha^{-n}) D_{\Gamma_x} (V_\alpha^n \eta' V_\alpha^{-n}) \to D_{\Gamma_x} \tag{5.3}$$

in the strong operator topology.

Now assume that $D_\Gamma - \lambda$ is Fredholm and let $A \colon L^2(\Gamma) \to L^2(\Gamma)$ be a Fredholm regularizer of $D_\Gamma - \lambda$, so that the products $A(D_\Gamma - \lambda)$ and $(D_\Gamma - \lambda)A$ are compact perturbations of the identity. Then, for any $\phi \in L^2(\Gamma_x)$,

$$\|\phi\| \le \left\| V_\alpha^n \Psi'_x \eta A \eta (D_\Gamma - \lambda) \eta \Psi_x V_\alpha^{-n} \phi \right\| + \left\| V_\alpha^n (I - \Psi'_x \eta A \eta (D_\Gamma - \lambda) \eta \Psi_x) V_\alpha^{-n} \phi \right\|. \tag{5.4}$$

The first term can be estimated as

$$
\begin{aligned}
\left\| V_\alpha^n \Psi'_x \eta A \eta (D_\Gamma - \lambda) \eta \Psi_x V_\alpha^{-n} \phi \right\| &\le \|A\| \left\| \eta (D_\Gamma - \lambda) \eta \Psi_x V_\alpha^{-n} \phi \right\| \\
&= \|A\| \left\| V_\alpha^n \Psi'_x \eta (D_\Gamma - \lambda) \eta \Psi_x V_\alpha^{-n} \phi \right\|,
\end{aligned}
$$

where we used that V_α is an isometry, Ψ'_x is an isometry on the range of the multiplication operator η, and $\|\eta\| \le 1$. By (5.3),

$$\left\| V_\alpha^n \Psi'_x \eta (D_\Gamma - \lambda) \eta \Psi_x V_\alpha^{-n} \phi \right\| \to \left\| (D_{\Gamma_x} - \lambda) \phi \right\|,$$

as $n \to \infty$. The second term in (5.4) can be estimated as

$$
\begin{aligned}
&\left\| V_\alpha^n (I - \Psi'_x \eta A \eta (D_\Gamma - \lambda) \eta \Psi_x) V_\alpha^{-n} \phi \right\| \\
&\le \left\| V_\alpha^n (I - \Psi'_x \eta A (D_\Gamma - \lambda) \eta^2 \Psi_x) V_\alpha^{-n} \phi \right\| + \left\| V_\alpha^n \Psi'_x \eta A[\eta, D_\Gamma] \eta \Psi_x V_\alpha^{-n} \phi \right\| \\
&\le \left\| V_\alpha^n (I - \eta'^3) V_\alpha^{-n} \phi \right\| + \left\| V_\alpha^n \Psi'_x \eta (A(D_\Gamma - \lambda) - I) \eta^2 \Psi_x V_\alpha^{-n} \phi \right\| \\
&\quad + \left\| V_\alpha^n \Psi'_x \eta A[\eta, D_\Gamma] \eta \Psi_x V_\alpha^{-n} \phi \right\|.
\end{aligned}
$$

Note that both $A(D_\Gamma - \lambda) - I$ and $[\eta, D_\Gamma]$ are compact. We also observe that for any compact operator $K \colon L^2(\Gamma_x) \to L^2(\Gamma_x)$ we have $V_\alpha^n K V_\alpha^{-n} \to 0$ in the strong operator topology, since $V_\alpha^{-n} \to 0$ in the weak operator topology. Therefore, and using that $V_\alpha^n \eta'^3 V_\alpha^{-n} \to I$, we obtain that the second term in (5.4) tends to zero as $n \to \infty$. We conclude that $\|\phi\| \le \|A\| \left\| (D_{\Gamma_x} - \lambda) \phi \right\|$ for every $\phi \in L^2(\Gamma_x)$. The same argument also shows that $\|\phi\| \le \|A\| \left\| (D_{\Gamma_x} - \lambda)' \phi \right\|$ for every $\phi \in L^2(\Gamma_x)$. It follows that $D_{\Gamma_x} - \lambda$ is invertible and thus Fredholm.

Conversely, choose $x_1, \ldots, x_N \in \Gamma$ such that $\Gamma \subset \cup_{i=1}^N B(x_i, \delta(x_i))$, and suppose that $D_{\Gamma_{x_j}} - \lambda$ is Fredholm for $j = 1, \ldots, N$, and let A_j be a Fredholm regularizer of $D_{\Gamma_{x_j}} - \lambda$. For every j let $\psi_{x_j} \colon \mathbb{R}^d \to \mathbb{R}^d$ be an isometry with $\psi_{x_j}(x_j) = 0$ and $\psi_{x_j}(B(x_j, \delta(x_j)) \cap \Gamma) = B(0, \delta(x_j)) \cap \Gamma_{x_j}$. Choose Lipschitz continuous functions $\eta_1, \ldots, \eta_N$ such that $\left\{ \eta_j^2 : j = 1, \ldots, N \right\}$ is a partition of unity of Γ subordinate to the sets $B(x_j, \delta(x_j))$. Now note that, since $D_{\Gamma_{x_j}}$ essentially commutes with η'_j, A_j

essentially commutes with η'_j as well. This implies that

$$\sum_{j=1}^{N} \Psi_{x_j} \eta'_j A_j \Psi'_{x_j} \eta_j (D_\Gamma - \lambda)$$

$$\sim \sum_{j=1}^{N} \Psi_{x_j} A_j \eta'_j \Psi'_{x_j} (D_\Gamma - \lambda) \eta_j = \sum_{j=1}^{N} \Psi_{x_j} A_j \Psi'_{x_j} \eta_j (D_\Gamma - \lambda) \eta_j$$

$$= \sum_{j=1}^{N} \Psi_{x_j} A_j \eta'_j (D_{\Gamma_{x_j}} - \lambda) \eta'_j \Psi'_{x_j} \sim \sum_{j=1}^{N} \Psi_{x_j} \eta'_j A_j (D_{\Gamma_{x_j}} - \lambda) \eta'_j \Psi'_{x_j}$$

$$\sim \sum_{j=1}^{N} \Psi_{x_j} \eta'^2_j \Psi'_{x_j} = \sum_{j=1}^{N} \eta^2_j = I.$$

Similarly, $(D_\Gamma - \lambda) \sum_{j=1}^{N} \eta_j \Psi_{x_j} A_j \eta'_j \Psi'_{x_j} \sim I$. Thus $D_\Gamma - \lambda$ is Fredholm. $\square$

5.2 Deformations

Let $\beta \in (0, 1)$. We will now consider the situation where a Lipschitz domain Ω_- with boundary Γ is deformed by a $C^{1,\beta}$-diffeomorphism $\psi \colon \mathbb{R}^d \to \mathbb{R}^d$ that is conformal or anti-conformal at a point $x \in \Gamma$, meaning that the Jacobian $J\psi(x)$ of ψ at x lies in $\mathbb{R}O(d)$, where $\mathbb{R}O(d) := \{\lambda A \in \mathbb{R}^{d \times d} : \lambda \in \mathbb{R}, A \text{ orthogonal}\}$. Due to the invariance of the double-layer potential under transformations from $\mathbb{R}O(d)$, we will for our purposes be able to assume that $J\psi(x) = I$. Let $D_\Gamma(\cdot, \cdot)$ denote the kernel of D_Γ. Likewise, V_j will be the integral operator with kernel $V_j(\cdot, \cdot)$, $|D_\Gamma|$ will be the integral operator with kernel $|D_\Gamma(\cdot, \cdot)|$, and so on. We write

$$D_{\psi(\Gamma)}(\psi(z), \psi(y)) = D_\Gamma(z, y)$$

$$+ \underbrace{D_{\psi(\Gamma)}(\psi(z), \psi(y)) - \frac{(z-y) \cdot n^\Gamma(y)}{c_d |\psi(z) - \psi(y)|^d} n^{\psi(\Gamma)}(\psi(y)) \cdot J\psi(y) n^\Gamma(y)}_{V_1(z,y)}$$

$$+ \underbrace{D_\Gamma(z, y)(n^{\psi(\Gamma)}(\psi(y)) \cdot J\psi(y) n^\Gamma(y) - 1)}_{V_2(z,y)}$$

$$+ \underbrace{D_\Gamma(z, y) n^{\psi(\Gamma)}(\psi(y)) \cdot J\psi(y) n^\Gamma(y) \left[\frac{|z-y|^d}{|\psi(z) - \psi(y)|^d} - 1 \right]}_{V_3(z,y)}. \quad (5.5)$$

The term V_1 is weakly singular, satisfying an estimate of the form $|V_1(z, y)| \leq C_\beta |z - y|^{1+\beta-d}$ for $\beta \in (0, 1)$ and almost every $z, y \in \Gamma$, see [44, Lemma 6]. (In order to apply Medková's results from Medková [44], note that the reduced boundary $\partial_* \Omega_- \subset \Gamma$

has full surface measure for a Lipschitz domain Ω_-, cf. [17, p. 209].) Therefore the operator $V_1 \colon L^2(\Gamma) \to L^2(\Gamma)$ is compact.

We view V_2 as the composition of D_Γ with a multiplication operator, noting that by Medková [44, Lemma 3] we have

$$\lim_{\rho \to 0^+} \operatorname{ess\,sup}_{y \in \Gamma \cap B(x,\rho)} |n^{\psi(\Gamma)}(\psi(y)) \cdot J\psi(y)n^\Gamma(y) - 1| = 0.$$

In particular, for every $\varepsilon > 0$, there is $\rho > 0$ such that the operator norm of

$$V_2 \colon L^2(\Gamma \cap B(x,\rho)) \to L^2(\Gamma)$$

is smaller than ε.

For a bounded Lipschitz domain with boundary Γ, let $\mathscr{A}_2 = \mathscr{A}_2(\Gamma)$ denote the algebra of integral operators K such that $|K| \colon L^2(\Gamma) \to L^2(\Gamma)$ is bounded. We will work within the class of domains such that $D_\Gamma \in \mathscr{A}_2$ in this subsection. The preceding considerations show that $D_{\psi(\Gamma)} \in \mathscr{A}_2(\psi(\Gamma))$ if $D_\Gamma \in \mathscr{A}_2(\Gamma)$. Since the relationship between Γ and $\psi(\Gamma)$ is symmetric, we see that

$$D_{\psi(\Gamma)} \in \mathscr{A}_2(\psi(\Gamma)) \text{ if and only if } D_\Gamma \in \mathscr{A}_2(\Gamma). \tag{5.6}$$

Note that $D_\Gamma \in \mathscr{A}_2$ for any $C^{1,\beta}$-boundary Γ, since D_Γ has a weakly singular kernel in this case. It is also known that $|D_{\widetilde{\Gamma}}| \colon L^2(\widetilde{\Gamma}) \to L^2(\widetilde{\Gamma})$ is bounded for any of the following *graphs* $\widetilde{\Gamma}$:

  (i) a 2D wedge (modelling a polygonal corner) [49];
 (ii) a 3D wedge (modelling a polyhedral edge) [52];
(iii) a 3D polyhedral cone [16];
 (iv) a 3D smooth cone (such as a circular cone) [53].

Furthermore, this statement may be extended to dilation-invariant domains which are built from such graphs.

Lemma 5.3 *Suppose that $\widetilde{\Gamma}$ is a dilation-invariant graph, i.e., $\widetilde{\Gamma} = \alpha\widetilde{\Gamma}$, for some $\alpha \in (0,1)$, and suppose that, at every $x \in \widetilde{\Gamma} \setminus \{0\}$, $\widetilde{\Gamma}$ is either locally $C^{1,\beta}$ for some $\beta \in (0,1)$, or coincides locally with a graph of type (i)–(iii), or (iv). Then $|D_{\widetilde{\Gamma}}|$ is bounded on $L^2(\widetilde{\Gamma})$.*

Proof By hypothesis, we know that $|D_j| \colon L^2(\widetilde{\Gamma}_0) \to L^2(\widetilde{\Gamma}_0)$ is bounded for $j = -1, 0, 1$, where D_j and $\widetilde{\Gamma}_0$ are as in Sect. 4.1. The lemma therefore follows from Proposition 4.3 and (4.4). □

A simple localization argument now demonstrates that the condition $D_\Gamma \in \mathscr{A}_2$ is quite general.

Theorem 5.4 *Suppose that at every $x \in \Gamma$, Γ is locally $C^{1,\beta}$, for some $\beta \in (0,1)$, or coincides locally with a graph satisfying the hypothesis of Lemma 5.3. Then $D_\Gamma \in \mathscr{A}_2(\Gamma)$.*

Returning to the decomposition of $D_{\psi(\Gamma)}(\psi(z), \psi(y))$, we note, for the third term V_3, the following consequence of the fact that $J\psi(x) = I$.

Lemma 5.5 *Suppose that* $D_\Gamma \in \mathscr{A}_2$. *Then* $|V_3|(z, y) = |D_\Gamma|(z, y)Q(z, y)$, *where* $Q \in L^\infty(\Gamma \times \Gamma)$ *is non-negative and* $\lim_{\rho \to 0+} \text{ess sup}_{y,z \in \Gamma \cap B(x,\rho)} Q(z, y) = 0$, *where the essential supremum is taken with respect to the product measure on* $\Gamma \times \Gamma$.

Let $\eta_\rho \colon \Gamma \to [0, 1]$ be a Lipschitz continuous function with supp $\eta_\rho \subset B(x, \rho)$ such that η_ρ is equal to 1 in a neighbourhood of x. Furthermore, denote the operator of composition with ψ by

$$C_\psi \colon L^2(\psi(\Gamma)) \to L^2(\Gamma), \quad C_\psi \phi(z) = \phi(\psi(z)),$$

and let $\eta'_\rho = \eta_\rho \circ \psi^{-1}$. Note also that we have the change of variables formula

$$\int_{\psi(\Gamma)} \phi(y) \, ds_{\psi(\Gamma)}(y) = \int_\Gamma (1 + T(y))\phi(\psi(y)) \, ds_\Gamma(y), \tag{5.7}$$

where $T \in L^\infty(\Gamma)$ and $\lim_{y \to x} T(y) = 0$. The operator

$$\eta'_\rho D_{\psi(\Gamma)} \eta'_\rho \colon L^2(\psi(\Gamma)) \to L^2(\psi(\Gamma))$$

is then similar to

$$C_\psi \eta'_\rho D_{\psi(\Gamma)} \eta'_\rho C_{\psi^{-1}} \colon L^2(\Gamma) \to L^2(\Gamma),$$

which, by (5.7), is an integral operator with kernel $(1 + T(y))\eta_\rho(z)D_{\psi(\Gamma)}(\psi(z), \psi(y))\eta_\rho(y)$. By the previous calculations, if $D_\Gamma \in \mathscr{A}_2$, we therefore find that

$$C_\psi \eta'_\rho D_{\psi(\Gamma)} \eta'_\rho C_{\psi^{-1}} \sim \eta_\rho D_\Gamma \eta_\rho + \eta_\rho \tilde{V} \eta_\rho, \tag{5.8}$$

where $\tilde{V}$ is an integral operator induced by a kernel of the form $\tilde{V}(z, y) = D_\Gamma(z, y)\tilde{Q}(z, y)$ with $\tilde{Q} \in L^\infty(\Gamma \times \Gamma)$ such that

$$\lim_{\rho \to 0+} \text{ess sup}_{y,z \in \Gamma \cap B(x,\rho)} |\tilde{Q}(z, y)| = 0. \tag{5.9}$$

For $\phi_1, \phi_2 \in L^2(\Gamma)$ with support in $B(x, \rho)$ we have

$$\left| \langle \tilde{V}\phi_1, \phi_2 \rangle \right| \leq \int_\Gamma \int_\Gamma \left| \phi_2(z)\tilde{V}(z, y)\phi_1(y) \right| \, ds(y) \, ds(z)$$

$$\leq \text{ess sup}_{y,z \in \Gamma \cap B(x,\rho)} |\tilde{Q}(z, y)| \, \||D_\Gamma\|| \, \|\phi_1\| \, \|\phi_2\|.$$

Since $D_\Gamma \in \mathscr{A}_2$, (5.9) implies that

$$\lim_{\rho \to 0+} \|\eta_\rho \tilde{V} \eta_\rho\| = 0. \tag{5.10}$$

Taking powers, we conclude that

$$\lim_{\rho \to 0^+} \|(\eta_\rho D_\Gamma \eta_\rho)^n - C_\psi (\eta_\rho' D_{\psi(\Gamma)} \eta_\rho')^n C_{\psi^{-1}}\|_{\mathrm{ess}} = 0 \tag{5.11}$$

for every $n \geq 1$.

We now prove the main theorem of this section. When $R = 1/2$, it says the following: if, for every point x, Γ is locally obtained by deforming a domain that satisfies the spectral radius conjecture, then Γ satisfies the spectral radius conjecture as well.

Theorem 5.6 *Let $R > 0$ and $\beta > 0$. Suppose that for every $x \in \Gamma$ there exists a $C^{1,\beta}$-diffeomorphism $\psi_x \colon \mathbb{R}^d \to \mathbb{R}^d$, conformal or anti-conformal at x, and a Lipschitz cut-off function $\eta_x \colon \mathbb{R}^d \to [0,1]$ such that η_x is equal to 1 in a neighbourhood of x, $D_{\psi_x(\Gamma)} \in \mathscr{A}_2$, and*

$$\sigma_{\mathrm{ess}}(\eta_x' D_{\psi_x(\Gamma)} \eta_x') \subset B(0, R),$$

where $\eta_x' = \eta_x \circ \psi_x^{-1}$. Then $D_\Gamma \in \mathscr{A}_2$ and $\sigma_{\mathrm{ess}}(D_\Gamma) \subset B(0, R)$.

Proof As noted previously, the condition on Γ, even for just one x, implies that $D_\Gamma \in \mathscr{A}_2$.

Fix $x \in \Gamma$. Since the commutator $[D_{\psi_x(\Gamma)}, \eta_x']$ is compact, we see that $(\eta_x' D_{\psi_x(\Gamma)} \eta_x')^n - \eta_x''^n D_{\psi_x(\Gamma)}^n \eta_x''^n$ is compact for every $n \geq 1$. By the hypothesis, choose $n = n(x)$ sufficiently large so that

$$\|(\eta_x' D_{\psi_x(\Gamma)} \eta_x')^n\|_{\mathrm{ess}}^{\frac{1}{n}} = \|\eta_x''^n D_{\psi_x(\Gamma)}^n \eta_x''^n\|_{\mathrm{ess}}^{\frac{1}{n}} < R \|C_{\psi_x}\|^{-\frac{1}{n}} \|C_{\psi_x^{-1}}\|^{-\frac{1}{n}}.$$

If we choose a new cut-off function $\tilde{\eta}_x \colon \mathbb{R}^d \to [0,1]$ such that $\mathrm{supp}\,\tilde{\eta}_x \subset \{y \in \mathbb{R}^d : \eta_x(y) = 1\}$, note that, since $(\eta_x' D_{\psi_x(\Gamma)} \eta_x')^n - \eta_x''^n D_{\psi_x(\Gamma)}^n \eta_x''^n$ is compact, we have

$$\|(\tilde{\eta}_x' D_{\psi_x(\Gamma)} \tilde{\eta}_x')^n\|_{\mathrm{ess}}^{\frac{1}{n}} = \|\tilde{\eta}_x''^n D_{\psi_x(\Gamma)}^n \tilde{\eta}_x''^n\|_{\mathrm{ess}}^{\frac{1}{n}}$$

$$\leq \|\eta_x''^n D_{\psi_x(\Gamma)}^n \eta_x''^n\|_{\mathrm{ess}}^{\frac{1}{n}} < R \|C_{\psi_x}\|^{-\frac{1}{n}} \|C_{\psi_x^{-1}}\|^{-\frac{1}{n}},$$

so that

$$\|C_{\psi_x}(\tilde{\eta}_x' D_{\psi_x(\Gamma)} \tilde{\eta}_x')^n C_{\psi_x^{-1}}\|_{\mathrm{ess}}^{\frac{1}{n}} \leq \|\eta_x''^n D_{\psi_x(\Gamma)}^n \eta_x''^n\|_{\mathrm{ess}}^{\frac{1}{n}} \|C_{\psi_x}\|^{\frac{1}{n}} \|C_{\psi_x^{-1}}\|^{\frac{1}{n}} < R.$$

Hence, by making the support of η_x sufficiently small and applying (5.11), we obtain that

$$\|(\eta_x D_\Gamma \eta_x)^n\|_{\mathrm{ess}}^{\frac{1}{n}} = \|\eta_x^n D_\Gamma^n \eta_x^n\|_{\mathrm{ess}}^{\frac{1}{n}} < R.$$

Now choose $\delta(x)$ sufficiently small so that $\eta_x \equiv 1$ in a neighbourhood of $B(x, \delta(x)) \cap \Gamma$. By compactness, we may pick points $x_1, \ldots, x_N$ so that $\Gamma \subset \cup_{j=1}^N B(x_j, \delta(x_j))$.

Let

$$r := \max_{j=1,\dots,N} \|(\eta_{x_j} D_\Gamma \eta_{x_j})^{n(x_j)}\|_{\mathrm{ess}}^{\frac{1}{n(x_j)}} < R$$

and let $\{\tau_j : j = 1, \dots, N\}$ be a Lipschitz partition of unity of Γ subordinate to the sets $B(x_j, \delta(x_j))$. Choose m to be a large integer such that $n(x_j)$ divides m for all j. Then

$$D_\Gamma^m = \sum_{j=1}^N \tau_j D_\Gamma^m = \sum_{j=1}^N \tau_j \eta_{x_j}^{2m} D_\Gamma^m$$

and therefore, recalling that $[D_\Gamma, \eta_{x_j}]$ is compact,

$$\begin{aligned}
\|D_\Gamma^m\|_{\mathrm{ess}} &= \left\| \sum_{j=1}^N \tau_j (\eta_{x_j} D_\Gamma \eta_{x_j})^m \right\|_{\mathrm{ess}} \le \sum_{j=1}^N \left\| (\eta_{x_j} D_\Gamma \eta_{x_j})^m \right\|_{\mathrm{ess}} \\
&\le \sum_{j=1}^N \left\| (\eta_{x_j} D_\Gamma \eta_{x_j})^{n(x_j)} \right\|_{\mathrm{ess}}^{\frac{m}{n(x_j)}} \\
&\le N r^m < N R^m.
\end{aligned}$$

Therefore $\sigma_{\mathrm{ess}}(D_\Gamma) \subset B(0, R)$. □

Referring back to Lemma 5.3, all graphs $\widetilde{\Gamma}$ of type (i)–(iii) satisfy (the graph version of) the spectral radius conjecture, results which can also be found in the corresponding references. Graphs of type (iv) that are convex also satisfy the spectral radius conjecture, by Fabes et al. [18]. If $\eta : \mathbb{R}^3 \to [0, 1]$ is a compactly supported Lipschitz function, we thus have that

$$\|(\eta D_{\widetilde{\Gamma}} \eta)^n\|_{\mathrm{ess}} = \|\eta^n D_{\widetilde{\Gamma}}^n \eta^n\|_{\mathrm{ess}} \le \|D_{\widetilde{\Gamma}}^n\|_{\mathrm{ess}} < 2^{-n}$$

for n sufficiently large. Applying Theorem 5.6 we obtain the following.

Corollary 5.7 *Let $d = 3$ and let Γ be the boundary of a Lipschitz domain. Assume that for every $x \in \Gamma$ there exists a $C^{1,\beta}$-diffeomorphism $\psi_x : \mathbb{R}^3 \to \mathbb{R}^3$, conformal or anti-conformal at x, such that $\psi_x(\Gamma)$ is locally a subset of the boundary of a polyhedral cone or a convex smooth cone. Then D_Γ satisfies the spectral radius conjecture.*

The class of domains considered in Corollary 5.7 encompasses all domains in 3D that may reasonably be referred to as a Lipschitz curvilinear polyhedra. From Theorem 5.6 we of course also obtain the analogous corollary for 2D domains, a well known result; the corresponding class of curves precisely describes the $C^{1,\beta}$-curvilinear polygons in 2D.

5.3 Possible extensions and further questions

Theorem 5.6 in particular shows that the spectral radius conjecture only depends on the local behaviour of Γ, under the extraneous assumption that $D_\Gamma \in \mathscr{A}_2$. To achieve the same result for a general Lipschitz domain, we need a different tool to treat the term V_3 which does not rely on the introduction of absolute values. Since $V_3(z, y) = \widetilde{Q}(z, y) D_\Gamma(z, y)$, for a specific kernel $\widetilde{Q}(z, y)$, we would like an answer to the following question.

Question 5.8 Describe suitable classes of kernels B such that the kernel $B(z,y) D_\Gamma(z,y)$ defines a bounded operator on $L^2(\Gamma)$, and estimate the norm of the corresponding operator.

When B is of the form $B(z, y) = a(z) - a(y)$, for some function a on Γ, the resulting operator is the commutator $[a, D_\Gamma]$. Commutators of singular integral operators are very well studied, but we have been unable to identify or correctly apply existing results to the kernel multiplier of interest to us, namely,

$$B(z, y) = \eta_\rho(z) \left(\frac{|z - y|^d}{|\psi(z) - \psi(y)|^d} - 1 \right) \eta_\rho(y).$$

Other versions of Question 5.8 also seem interesting. For example, one could ask for the stronger property that $B(z, y)$ be a kernel multiplier of each of the Riesz transforms of Γ. We also note the similarity between the term V_3 and the kernel of the Clifford–Cauchy integral operator, as presented in [4, Consequence 3.6].

Of course, one would like to know that not only the spectral radius conjecture is local, but that the entire essential spectrum is as well.

Question 5.9 For a domain Ω_- with boundary Γ, is the essential spectrum local and invariant under locally conformal deformations?

We can give a positive answer to this question if we, in addition to the hypotheses of Theorem 5.6, assume that we do not have too many singular points, by which we mean points where the boundary is not C^1. In the statement, we let $\chi_{x,\rho} := \chi_{B(x,\rho)}$ and $\chi'_{x,\rho} := \chi_{x,\rho} \circ \psi_x^{-1}$ for $x \in \Gamma$ and $\rho > 0$.

Theorem 5.10 *Let $\beta > 0$ and assume that Γ has at most countably many singular points. Further suppose that for every $x \in \Gamma$ there exists a $C^{1,\beta}$-diffeomorphism $\psi_x : \mathbb{R}^d \to \mathbb{R}^d$, conformal or anti-conformal at x, such that $D_{\psi_x(\Gamma)} \in \mathscr{A}_2$. Then $D_\Gamma \in \mathscr{A}_2$ and*

$$\sigma_{\mathrm{ess}}(D_\Gamma) \subset \bigcup_{x \in \Gamma} \sigma_{\mathrm{ess}}(\chi'_{x,\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\rho_x})$$

for arbitrary $\rho_x > 0$. Moreover, there exist $\rho_x > 0$ such that

$$\sigma_{\mathrm{ess}}(D_\Gamma) = \bigcup_{x \in \Gamma} \sigma_{\mathrm{ess}}(\chi'_{x,\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\rho_x}).$$

Proof By (5.6), $D_{\psi_x(\Gamma)} \in \mathscr{A}_2$ for just one $x \in \Gamma$ already implies $D_\Gamma \in \mathscr{A}_2$.

Let $x \in \Gamma$, $\rho_x > 0$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $0 \in \sigma_{\text{ess}}(\chi'_{x,\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\rho_x})$ either by (5.1) (if $\chi'_{x,\rho_x} \equiv 1$) or by construction (otherwise). So assume that $\lambda \neq 0$ and suppose that $\chi'_{x,\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\rho_x} - \lambda$ is Fredholm for every $x \in \Gamma$.

Fix x for a moment. As Γ has at most countably many singular points and ψ_x is a diffeomorphism, we know that $\chi'_{x,\tilde\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\tilde\rho_x} - \lambda$ is Fredholm for all but at most countably many $\tilde\rho_x \in (0, \rho_x]$. In this case the essential norm of the Fredholm regularizer $A_{x,\tilde\rho_x}$ of $\chi'_{x,\tilde\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\tilde\rho_x} - \lambda$ is bounded by

$$\left\| A_{x,\tilde\rho_x} \right\|_{\text{ess}} \leq \max \left\{ \left\| A_{x,\rho_x} \right\|_{\text{ess}}, |\lambda|^{-1} \right\}.$$

Therefore, by (5.8) and (5.10), for every $x \in \Gamma$ we can choose $\tilde\rho_x$ sufficiently small such that $\chi_{x,\tilde\rho_x} D_\Gamma \chi_{x,\tilde\rho_x} - \lambda$ is also Fredholm, and such that $\partial B(x, \tilde\rho_x)$ contains none of the singularities of Γ.

By compactness, we may choose finitely many points x_j and corresponding radii $\tilde\rho_{x_j}$ such that the balls $B(x_j, \tilde\rho_{x_j})$ cover Γ. Define recursively

$$E_1 := B(x_1, \tilde\rho_{x_1}) \cap \Gamma, \quad E_{j+1} = \left(B(x_{j+1}, \tilde\rho_{x_{j+1}}) \cap \Gamma \right) \setminus \bigcup_{k=1}^{j} E_k.$$

Now we can apply (5.2) to obtain that $D_\Gamma - \lambda$ is Fredholm.

Conversely, assume that $D_\Gamma - \lambda$ is Fredholm and fix $x \in \Gamma$. By (5.1), we must have $\lambda \neq 0$. As we only have countably many singularities, $\chi_{x,\rho} D_\Gamma \chi_{x,\rho} - \lambda$ is also Fredholm for all but at most countably many $\rho > 0$ by (5.2). Now note that the situation between D_Γ and $D_{\psi_x(\Gamma)}$ is symmetric. Hence, the same argument as above shows that $\chi'_{x,\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\rho_x} - \lambda$ is also Fredholm for a sufficiently small $\rho_x > 0$. □

In particular, we may extend Theorem 5.2 to domains that are only *approximately* locally dilation invariant in the following sense.

Corollary 5.11 *Let $\beta > 0$ and assume that Γ has at most countably many singular points. Denote the set of singular points by $\mathcal{S}$. Suppose that for every $x \in \mathcal{S}$ there exists a $C^{1,\beta}$-diffeomorphism $\psi_x \colon \mathbb{R}^d \to \mathbb{R}^d$ such that*

- (i) *ψ_x is conformal or anti-conformal at x,*
- (ii) *$\psi_x(\Gamma)$ is locally $C^{1,\beta}$ or locally dilation invariant at $\psi_x(x)$,*
- (iii) *$D_{\psi_x(\Gamma)} \in \mathscr{A}_2$ (e.g., a domain with local behaviour as in Theorem 5.4).*

Then there exist $\rho_x > 0$ such that $\sigma_{\text{ess}}(D_\Gamma) = \{0\} \cup \bigcup_{x \in \mathcal{S}} \sigma_{\text{ess}}(\chi'_{x,\rho_x} D_{\psi_x(\Gamma)} \chi'_{x,\rho_x})$.

6 Synthesis and numerical examples

In this final section we bring earlier results together to study the case where $\Gamma \in \mathscr{D}_\mathcal{A}$ (recall Definition 1.4), meaning that Γ is the boundary of a bounded Lipschitz domain $\Omega_- \subset \mathbb{R}^2$ that is locally dilation invariant and is piecewise analytic. We then illustrate

the various results of the paper by several examples. Regarding our first aim, the following result is immediate from Theorems 5.2, 4.19, and 4.21.

Theorem 6.1 *Suppose that $\Gamma \in \mathscr{D}_{\mathcal{A}}$ and let $F \subset \Gamma$ be the finite set of points at which Γ is not locally analytic but is locally dilation invariant, so that Γ coincides locally near $x \in F$ with Γ_x, a dilation invariant graph (in some rotated coordinate system centred at x). Then, where σ^N is as defined in Theorem 4.19, $\Sigma^N(D_\Gamma) := \cup_{x \in F} \sigma^N(D_{\Gamma_x}) \xrightarrow{\text{H}} \sigma_{\text{ess}}(D_\Gamma; L^2(\Gamma))$ as $N \to \infty$. Further, suppose that $c > 0$ is small enough such that the conditions of the first sentence of Theorem 4.21 are satisfied for $\Gamma = \Gamma_x$, $x \in F$. Then, for every $\rho_0 > 0$, where $R_N(D_\Gamma)$ is defined by (1.13), $\rho_{\text{ess}}(D_\Gamma; L^2(\Gamma)) < \rho_0$ if: i) $R_N(D_\Gamma) < \rho_0$; and ii) for some $m, M, N \in \mathbb{N}$, with $M \geq 2$,*

$$\mathscr{S}_c(\Gamma, m, M, N) := \max_{x \in F} S_c(\Gamma_x, m, M, N) < 0, \tag{6.1}$$

where $S_c(\cdot, \cdot, \cdot, \cdot)$ is as defined in Remark 4.22. Conversely, if $\rho_{\text{ess}}(D_\Gamma; L^2(\Gamma)) < \rho_0$, then, for all sufficiently large N, $R_N(D_\Gamma) < \rho_0$, and, if $m, M \in \mathbb{N}$ are also sufficiently large, then $\mathscr{S}_c(\Gamma, m, M, N) < 0$.

Proof Since $\Gamma \in \mathscr{D}_{\mathcal{A}} \subset \mathscr{D}$, Γ is locally C^1 at x if $x \in \Gamma \backslash F$, while, if $x \in F$, Γ_x satisfies (4.44) (in some local coordinate system centred at x). Thus, and by Theorem 5.2, $\sigma_{\text{ess}}(D_\Gamma) = \cup_{x \in \Gamma} \sigma_{\text{ess}}(D_{\Gamma_x}) = \cup_{x \in F} \sigma_{\text{ess}}(D_{\Gamma_x})$, since $\sigma_{\text{ess}}(D_{\Gamma_x}) = \{0\}$ if $x \in \Gamma \backslash F$, and $0 \in \sigma_{\text{ess}}(D_{\Gamma_x})$ if $x \in F$, by Theorem 4.4 and Corollary 4.6. The result thus follows from Theorems 4.19 and 4.21. $\qquad \square$

The following examples illustrate the above result and the results of Sects. 4 and 5. In each example, whether Γ is a dilation invariant graph or $\Gamma \in \mathscr{D}_{\mathcal{A}}$, we demonstrate that $\rho_{\text{ess}}(D_\Gamma; L^2(\Gamma)) < \frac{1}{2}$, providing new evidence in support of the spectral radius conjecture.

Example 1 We first consider an example where Γ satisfies (4.8), with $f : \mathbb{R}_+ \to \mathbb{R}$ given by

$$f(x) := x \sin^2(\pi \log_\alpha(x)), \quad x > 0, \tag{6.2}$$

for some $\alpha \in (0, 1)$, so that [recall (4.10)] $g(x) = \sin^2(\pi x) = (1 - \cos(2\pi x))/2$, for $x \in \mathbb{R}$. As α increases in $(0, 1)$ the graph of f (see Fig. 2) becomes increasingly oscillatory and its Lipschitz character increases; elementary calculations give that the maximum and minimum of f' are $f'_{\max} = \cos^2(\theta_\alpha/2) + \pi \sin(\theta_\alpha)/|\log \alpha| > 1$ and $f'_{\min} = 1 - f'_{\max} < 0$, where $\theta_\alpha := \arctan(2\pi/|\log \alpha|)$; note that $f'_{\max} = \pi/|\log \alpha| + 1/2 + O(1 - \alpha)$ as $\alpha \to 1^-$. To apply Theorem 4.15 we need to compute $C_1(M)$, C_3, and C_4; see (4.34), (4.39), and the definition below (4.40). This requires computation of quantities that are defined in Proposition 4.8 in terms of[20]

$$\| \operatorname{Im} g^{(j)} \|_c = \frac{(2\pi)^j}{2} \sinh(2\pi c), \quad j = 0, 1, \quad c \geq 0,$$

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[20] In this example, and the other examples below, we are able to compute these norms exactly. We can, instead, just compute upper bounds; the theory and algorithm apply essentially unchanged—see Remark 4.22(iii).
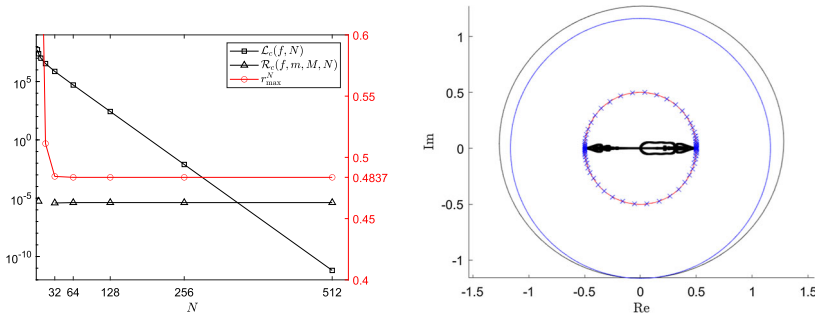
(a) $\mathcal{L}_c(f, N)$, $\mathcal{R}_c(f, m, M, N)$, for $\rho_0 = \frac{1}{2}$, and the maximum of the spectral radii of $A^M_{t_k, N}$, $k = 1, \ldots, m$, plotted against N for $m = 16,000$ and $M = 100$. The right-hand scale (in red) is to be used for $r^N_{\max}$ (plotted in red). The $\mathcal{R}_c(f, m, M, N)$ value is not shown for $N = 16$ because it is negative.

(b) $\sigma^N(D_\Gamma) = \{0\} \cup \bigcup_{t \in t_N} \sigma(A^{M_N}_{t,N})$, for $N = 512$, $m_N = 16,000$, $M_N = 100$, an approximation to $\sigma(D_\Gamma) = \sigma_{\text{ess}}(D_\Gamma)$ by Theorem 4.13. Also shown are the boundary of W_{100} (for $p = 10$, $N = 512$, $M = 100$ and $t = \pi/18$) in black, $R^* \mathbb{T} \subset W_{100} \subset W(D_\Gamma)$ in blue (see Corollary 4.25 for definitions), $\frac{1}{2}\mathbb{T}$ in red. The crosses on $\frac{1}{2}\mathbb{T}$ are $\mu_{k,\ell}$ for $k = 16,000$ and $\ell = 1, \ldots, n_k = 67$ (see Theorem 4.15)

Fig. 4 Numerical results for Example 1; f given by (6.2), $\alpha = \frac{3}{4}$

$$\|g\|_c = \cosh^2(\pi c), \quad \|g^{(j)}\|_c = \frac{(2\pi)^j}{2} \cosh(2\pi c), \quad j = 1, 2, \quad c \geq 0.$$

Theorem 4.15 applies for all $c > 0$ such that (4.21) applies, i.e. provided

$$c \leq \frac{\arccos(\alpha)}{|\log \alpha|} \quad \text{and} \quad c < \frac{1}{2\pi} \operatorname{arsinh}\left(\frac{2|\log \alpha|}{2\pi + |\log \alpha|}\right). \tag{6.3}$$

In Fig. 4 we plot results for the case $\alpha = \frac{3}{4}$ (see Fig. 2), when $f'_{\max} \approx 11.43$ and the above conditions reduce to $c \leq \arccos(3/4)/\log(4/3) \approx 2.51$ and $c < \operatorname{arsinh}\left(2\log(4/3)/(2\pi + \log(4/3))\right)/(2\pi) \approx 0.0139$. Choosing $c = 0.013$ we plot in Fig. 4a, for the case $\rho_0 = \frac{1}{2}$, $\mathcal{L}_c(f, N)$ and $\mathcal{R}_c(f, m, M, N)$, given by (4.43), against N, for $N = 2^j$, $j = 3, 4, \ldots, 9$, choosing $m = 16,000$ and $M = 100$. We see that with these choices $\mathcal{R}_c(f, m, M, N)$ is positive and bounded away from zero for sufficiently large N, while, as is clear from its definition, $\mathcal{L}_c(f, N)$ decreases exponentially with N; note that (4.43) is satisfied for $N = 512$. To apply Theorem 4.15 to conclude that $\rho(D_\Gamma) = \rho_{\text{ess}}(D_\Gamma) < \frac{1}{2}$ we also need to check that $\rho(A^M_{t,N}) < \frac{1}{2}$, for $M = 100$, N = 512, and $t = t_k = (k - 1/2)/m$, $k = 1, \ldots, m$, with $m = 16,000$; equivalently, that $r^N_{\max} := \max_{z \in \sigma^N(D_\Gamma)} |z| < \frac{1}{2}$, where $\sigma^N(D_\Gamma)$ is as defined by (4.41) with $N = 512$, $m_N = 16,000$, $M_N = 100$. For these parameter values $\sigma^N(D_\Gamma)$, an approximation to $\sigma(D_\Gamma) = \sigma_{\text{ess}}(D_\Gamma)$ by Theorem 4.13, is plotted in Fig. 4b, and $r^N_{\max} \approx 0.4837 < \frac{1}{2}$. Our calculations are in standard double-precision floating-point arithmetic rather than exact arithmetic,[21] but this is convincing evidence, by application of Theorem 4.15, that $\rho(D_\Gamma) < \frac{1}{2}$.

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[21] In all these examples the spectral radii of the matrices $A^M_{t,N}$ as well as the norms of the $(A^M_{t,N} - \lambda_l I)^{-1}$ are computed using standard Matlab routines; our codes are available at https://github.com/Raffael-Hagger/DLP.

In Fig. 4b we also plot, for the parameter values $p = 10$, $N = 512$, $M = 100$, $n = 100$, and $t = \pi/18$, the bounded domain W_n, which[22] is contained in and is an approximation to $W(T^{p,t,N,M})$ (these notations defined in Sect. 4.4). Where R^* is as defined in Corollary 4.25, we also plot the circle $R^*\mathbb{T}$ of radius $R^* \approx 1.163$; by Corollary 4.25 it is guaranteed that $\overline{B_{R^*}} \subset W(D_\Gamma) = W_{\text{ess}}(D_\Gamma)$, so that $w(D_\Gamma) = w_{\text{ess}}(D_\Gamma) \geq R^*$ is significantly larger than $\frac{1}{2}$ for this example. We note that, for these parameter values, $C_7(p, N, M) \leq 3.964 \times 10^{-5}$ (see Corollaries 4.24 and 4.25).

In Fig. 4b we additionally plot, to illustrate the application of Theorem 4.15 and the adaptive definition of the parameters defined by (4.42), the points $\mu_{k,\ell}$ at which the resolvent of $A^M_{t_k,N}$ is calculated when $\rho_0 = \frac{1}{2}$, $N = 512$, $M = 100$, $m = 16,000$, for the case $k = 16,000$ (so that $t_k \approx \pi$) and $\ell = 1, \ldots, n_k = 67$. This value of k is fairly typical; n_k varies in the range [67, 110] as a function of k. It is clear that the adaptive algorithm of Theorem 4.15 (and see Corollary 3.5) is significantly more efficient than the uniform grid on $\rho_0\mathbb{T}$ of Corollary 3.3 (recall the discussion above Lemma 3.4). For $k = 16,000$ we have $\min_\ell v_{k,\ell} \approx 0.0101$ and $\max_\ell v_{k,\ell} \approx 0.2216$ [see (4.42) for this notation]. The ratio of these maximum and minimum values, which is approximately the ratio of the maximum to the minimum spacing of the points $\mu_{k,\ell}$, is about 20.2.

Example 2 We now turn to examples where we can apply the theory of Sect. 4.3. First we consider the case of a cone, that is $\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$ where

$$f(x) := \mu |x|, \quad x \in \mathbb{R}, \tag{6.4}$$

for some $\mu \in \mathbb{R}$. This clearly satisfies (4.44), for any $\alpha \in (0, 1)$, and, where $f_\pm$ and $g_\pm$ are defined by (4.45), $f_\pm(x) = \mu x$, $g_\pm(x) = \mu$, for $x > 0$. For this example the spectrum and spectral radius are known, viz.

$$\sigma_{\text{ess}}(\Gamma) = \sigma(D_\Gamma) = \{0\} \cup \left\{ \pm \frac{\sin(\arctan(|\mu|)(1 - iy))}{2\sin(\pi(1 - iy)/2)} : y \in \mathbb{R} \right\}$$

$$\text{so that } \rho(D_\Gamma) = \frac{|\mu|}{2\sqrt{1 + \mu^2}} \tag{6.5}$$

(see, e.g., [49]) and, since D_Γ is diagonalised by the Mellin transform when Γ is a 2D cone, D_Γ is normal so that $W_{\text{ess}}(D_\Gamma) = \overline{W(D_\Gamma)} = \text{conv}(\sigma(D_\Gamma))$ and $\|D_\Gamma\| = w(D_\Gamma) = \rho(D_\Gamma)$.

To make comparison of these known results with the methods of Sect. 4.3 we can choose, in principle, any $\alpha \in (0, 1)$, but the choice of α affects the choice of c via (4.54), and thus the rate of decrease with N of $\mathcal{L}_c(f, N)$, defined by (4.72). The operators $\tilde{L}^\pm_t$, given by (4.48), and thus the matrices $A^M_{t_k,N}$, given by (4.63), also depend on α, while the operators $\tilde{K}^\pm_t$, given by (4.47), vanish in this case. It is worth noting that the blocks of $A^M_{t_k,N}$ corresponding to $\tilde{L}^\pm_t$ are Toeplitz matrices in this example.

---

[22] In our computations of W_n we neglect the factor $e^{i(x_{m,N} - x_{n,N})t}$ when using (4.74) with $\tilde{K}_t$ replaced by $\tilde{K}^M_t$. The resulting matrix is unitarily equivalent to $T^{p,t,N,M}$ as defined in Sect. 4.4, so has the same numerical range.
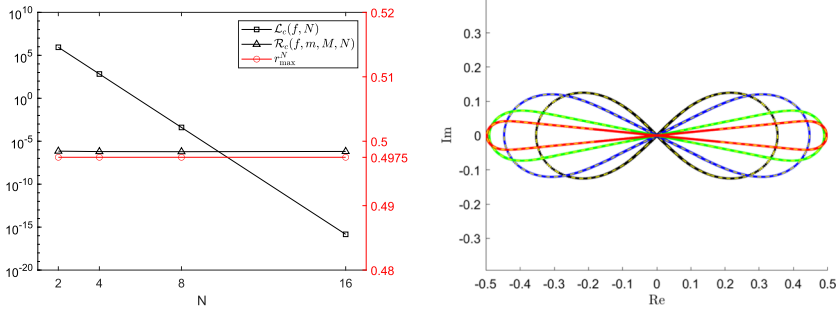
(a) $\mathcal{L}_c(f, N)$, $\mathcal{R}_c(f, m, M, N)$, for $\rho_0 = \frac{1}{2}$, and the maximum of the spectral radii of $A_{t_k,N}^M$, $k = 1, \ldots, m$, plotted against N for $m = 2,000$ and $M = 200$ in the case $\mu = 10$.

(b) $\sigma^N(D_\Gamma) = \{0\} \cup \bigcup_{t \in T_N} \sigma(A_{t,N}^M)$ for $N = 16$, $m_N = 2,000$, $M_N = 200$, and $\mu = 1$ (black), $\mu = 2$ (blue), $\mu = 5$ (green), $\mu = 10$ (red). The yellow dashed lines indicate the lemniscates (6.5), i.e. the (exact) spectrum of D_Γ in each case.

Fig. 5 Numerical results for Example 3; f given by (6.4), $\alpha = \frac{7}{8}$

The conditions (4.54) reduce to $c \leq \arccos(\alpha)/|\log \alpha|$ and $|\mu|c \leq \alpha^2/|\log \alpha|$. As the right hand sides of these inequalities are increasing on $(0, 1)$, it is beneficial to choose α closer to 1 in order to be able to choose a larger c. We select $\alpha = \frac{7}{8}$, so that $\arccos(\alpha)/|\log \alpha| \approx 3.78$ and $\alpha^2/|\log \alpha| \approx 5.73$, and then take $c = 0.57$, so that (4.54) is satisfied for $|\mu| \leq 10$. For $\mu = 10$ we see from Fig. 5a that, with $M = 200$, $m = 2000$ and $N = 16$, the inequality (4.72) is satisfied and $\max_{k=1,\ldots,m} \rho(A_{t_k,N}^M) \approx 0.4975 < \frac{1}{2}$, so that $\rho(D_\Gamma) < \frac{1}{2}$ by Theorem 4.21, in agreement with (6.5) which gives $\rho(D_\Gamma) = 5/\sqrt{101} \approx 0.4975$. We see in Fig. 5b that the approximations $\sigma^N(D_\Gamma)$ to $\sigma(D_\Gamma)$, given by Theorem 4.19, agree closely, for the given parameter values, with the expected lemniscates given by (6.5) for different values of μ.

Example 3 In this example we take

$$f(x) := |x| \sin^2(\pi \log_\alpha |x|), \quad x \in \mathbb{R}, \tag{6.6}$$

and Γ satisfies (4.44), but now just for one $\alpha \in (0, 1)$. Note that $f_\pm$ and $g_\pm$, given by (4.45), are the same as f and g in Example 1, so that $g_\pm$ satisfy the same bounds as g in Example 1, and Γ has Lipschitz character $f'_{\max}$ as defined in that example. Thus the conditions (4.54) reduce in this example to (6.3) plus the condition that $\sinh(2\pi c)/2 + c|\log \alpha| \cosh^2(\pi c) < \alpha^2$. We choose $\alpha = \frac{2}{3}$ (see Fig. 3), for which $c = 0.019$ satisfies this condition and (4.54), and $f'_{\max} \approx 8.26$. We see from Fig. 6a that, with $M = 60$, $m = 10,000$ and $N = 256$, the inequality (4.72) is satisfied and $\max_{k=1,\ldots,m} \rho(A_{t_k,N}^M) < \frac{1}{2}$, so that $\rho(D_\Gamma) < \frac{1}{2}$ by Theorem 4.21. Figure 6b plots an approximation $\sigma^N(D_\Gamma)$ to $\sigma(D_\Gamma)$ given by Theorem 4.19, which is contained in the circle of radius $\frac{1}{2}$ (in red). By contrast, by (4.73), and arguing as in Example 1, the numerical range $W(D_\Gamma)$ contains at least the closed disc of radius $R^* \approx 0.8179$ (shown in blue in Fig. 6b), where R^* is as given in Corollary 4.25.

Figure 6 is also, by Theorem 6.1, relevant to the bounded Lipschitz domain Ω_- shown in Fig. 1b. This has boundary $\Gamma \in \mathscr{D}_\mathcal{A}$ which is C^1 except at the single point
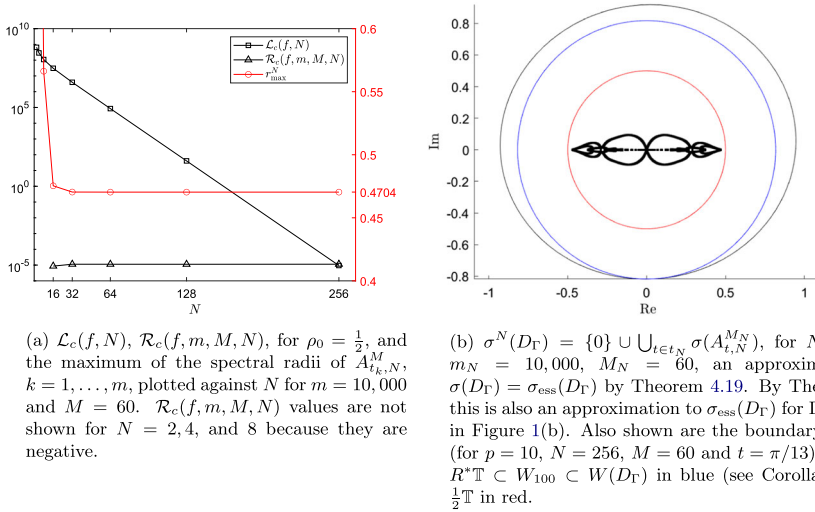
(a) $\mathcal{L}_c(f,N)$, $\mathcal{R}_c(f,m,M,N)$, for $\rho_0 = \frac{1}{2}$, and the maximum of the spectral radii of $A_{t_k,N}^M$, $k = 1, \ldots, m$, plotted against N for $m = 10,000$ and $M = 60$. $\mathcal{R}_c(f,m,M,N)$ values are not shown for $N = 2, 4$, and 8 because they are negative.

(b) $\sigma^N(D_\Gamma) = \{0\} \cup \bigcup_{t \in t_N} \sigma(A_{t,N}^{M_N})$, for $N = 256$, $m_N = 10,000$, $M_N = 60$, an approximation to $\sigma(D_\Gamma) = \sigma_{ess}(D_\Gamma)$ by Theorem 4.19. By Theorem 6.1 this is also an approximation to $\sigma_{ess}(D_\Gamma)$ for $\Gamma \in \mathscr{D}_\mathcal{A}$ as in Figure 1(b). Also shown are the boundary of W_{100} (for $p = 10$, $N = 256$, $M = 60$ and $t = \pi/13$) in black, $R^*\mathbb{T} \subset W_{100} \subset W(D_\Gamma)$ in blue (see Corollary 4.25), $\frac{1}{2}\mathbb{T}$ in red.

Fig. 6 Numerical results for Example 3; f given by (6.6), $\alpha = \frac{2}{3}$

$x = 0$ where it coincides locally with the graph of f given by (6.6). It follows by the above calculations and Theorem 6.1 that $\rho_{ess}(D_\Gamma) < \frac{1}{2}$ for $\Gamma = \partial\Omega_-$ and that Fig. 6b is also a plot of the approximation $\Sigma^N(D_\Gamma)$, defined in Theorem 6.1, to the essential spectrum of D_Γ for $\Gamma = \partial\Omega_-$. While $\rho_{ess}(D_\Gamma) < \frac{1}{2}$, $W_{ess}(D_\Gamma)$, for $\Gamma = \partial\Omega_-$, contains the closed disc of radius $R^* \approx 0.8179$ shown in Fig. 6b, by the above result for the graph Γ given by (6.6), and a localisation result [11, Theorem 3.2] (and see [11, Theorem 3.17]) for the essential numerical range, analogous to Theorem 5.2. Thus also, by (1.8), $\|D_\Gamma\|_{ess} \geq w_{ess}(D_\Gamma) \geq R^*$.

Example 4 In this example we define f by (6.2) for $x > 0$ and set $f(x) := 0$ for $x \leq 0$, so that Γ and f satisfy (4.44). Recalling (4.45), we see that f_+ and g_+ are the same as f and g in Example 1, while $f_- = 0$ and $g_- = 0$. The graph Γ has Lipschitz character $L = (f'_{max} - f'_{min})/2 = f'_{max} - 1/2$, where f'_{max} and f'_{min} are as defined in Example 1. The conditions (4.54) reduce in this example to the same conditions as in Example 3, and again we choose $\alpha = \frac{2}{3}$ and $c = 0.019$, so that Γ has Lipschitz character $L = f'_{max} - 1/2 \approx 7.76$. Similarly to the previous example, we see from Fig. 7a that, with $M = 50$, $m = 5,000$, and $N = 256$, the inequality (4.72) is satisfied and $\max_{k=1,\ldots,m} \rho(A_{t_k,N}^M) < \frac{1}{2}$, so that $\rho(D_\Gamma) < \frac{1}{2}$ by Theorem 4.21. The approximation $\sigma^N(D_\Gamma)$ to $\sigma(D_\Gamma)$, given by Theorem 4.19 and plotted in Fig. 7b, is contained in the circle of radius $\frac{1}{2}$, while the numerical range $W(D_\Gamma)$ contains at least the closed disc of radius $R^* \approx 0.8179$ shown in Fig. 7b.

Figure 7 is also, by Theorem 6.1, relevant to the bounded Lipschitz domain Ω_- shown in Fig. 1a. This has boundary $\Gamma \in \mathscr{D}_\mathcal{A}$ which is C^1 except at the single point $x = 0$ where it coincides locally with the graph of the function f described above. It follows by the above calculations and Theorem 6.1 that $\rho_{ess}(D_\Gamma) < \frac{1}{2}$ for $\Gamma = \partial\Omega_-$ and that Fig. 6b is also a plot of the approximation $\Sigma^N(D_\Gamma)$, defined in Theorem 6.1, to the essential spectrum of D_Γ for $\Gamma = \partial\Omega_-$. Arguing as in the previous example,
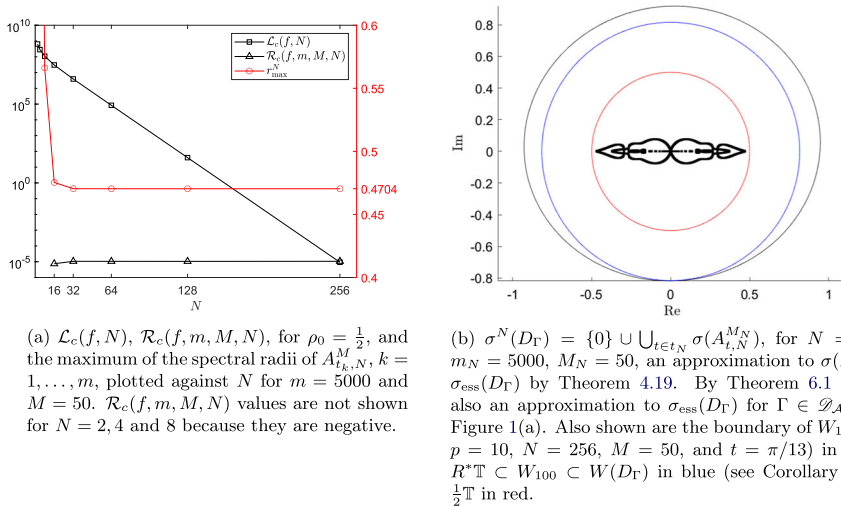
(a) $\mathcal{L}_c(f, N)$, $\mathcal{R}_c(f, m, M, N)$, for $\rho_0 = \frac{1}{2}$, and the maximum of the spectral radii of $A_{t_k, N}^M$, $k = 1, \ldots, m$, plotted against N for $m = 5000$ and $M = 50$. $\mathcal{R}_c(f, m, M, N)$ values are not shown for $N = 2, 4$ and 8 because they are negative.

(b) $\sigma^N(D_\Gamma) = \{0\} \cup \bigcup_{t \in t_N} \sigma(A_{t,N}^{M_N})$, for $N = 256$, $m_N = 5000$, $M_N = 50$, an approximation to $\sigma(D_\Gamma) = \sigma_{\mathrm{ess}}(D_\Gamma)$ by Theorem 4.19. By Theorem 6.1 this is also an approximation to $\sigma_{\mathrm{ess}}(D_\Gamma)$ for $\Gamma \in \mathscr{D}_\mathcal{A}$ as in Figure 1(a). Also shown are the boundary of W_{100} (for $p = 10$, $N = 256$, $M = 50$, and $t = \pi/13$) in black, $R^* \mathbb{T} \subset W_{100} \subset W(D_\Gamma)$ in blue (see Corollary 4.25), $\frac{1}{2} \mathbb{T}$ in red.

Fig. 7 Numerical results for Example 4; $f_+ = f$ given by (6.2), $f_- = 0$, $\alpha = \frac{2}{3}$

while $\rho_{\mathrm{ess}}(D_\Gamma) < \frac{1}{2}$, $W_{\mathrm{ess}}(D_\Gamma)$, for $\Gamma = \partial\Omega_-$, contains the closed disc of radius $R^* \approx 0.8179$ plotted in Fig. 7b, and $\|D_\Gamma\|_{\mathrm{ess}} \geq w_{\mathrm{ess}}(D_\Gamma) \geq R^*$.

Remark 6.2 (*Symmetry of the spectrum and essential spectrum*) For Example 2 it is immediate from (6.5) that $\sigma(D_\Gamma)$ is symmetric with respect to the origin; if $z \in \sigma(D_\Gamma)$ then $-z \in \sigma(D_\Gamma)$. We conjecture, based on the numerical results for Examples 3 and 4 and similar calculations, that this same symmetry holds in 2D whenever Γ is a dilation invariant graph satisfying (4.44). If this conjecture is true, then, for all $\Gamma \in \mathscr{D}_\mathcal{A}$, $\sigma_{\mathrm{ess}}(D_\Gamma)$ is symmetric with respect to the origin by Theorem 6.1. We note that symmetry results in the 2D case of this sort are proved for $D_\Gamma : L^2(\Gamma) \to L^2(\Gamma)$ when Γ is the boundary of a bounded C^2 domain Ω_- in [35, Proposition 6], and for D_Γ' as an operator on the natural energy space for general Lipschitz Ω_- [25, Theorem 2.1] (in both these cases the relevant spectrum lies in $[-1, 1]$). In 3D, even when Γ is a polyhedron, the spectrum and essential spectrum of D_Γ need not be symmetric with respect to the origin (see, e.g., [15]).

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