Emilie Lyse-Olsen

On Eigenvectors and Eigenvalues

An Examination of University Students' Concept Images of Eigenvectors and Eigenvalues, and their Modes of Thinking

Master's thesis in Didactics of Mathematics
Supervisor: Yael Fleischmann
August 2023

Master's thesis

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Abstract

As one of the first courses students encounter at the university level, linear algebra introduces students to a world of concepts, objects and representations. However, these abstract notions and their representations can be difficult for beginning learners to navigate. Eigentheory, the domain of linear algebra encompassing eigenvectors, eigenvalues, and eigenspaces, emerges as a particularly valuable group of concepts. Despite its diverse applications, the inherent conceptual complexity of eigentheory can present difficulties for students at the onset of their learning journey.

As an assumed cornerstone of the learning process, homework assignments are widely used in universities across the world. In this study, we explore students' written and oral reasoning on a set of homework tasks, aiming to illuminate their understanding of eigenvectors and eigenvalues. However, the notion of understanding can be complex and even contentious. Thus, to capture and characterise their understanding, we draw upon Tall and Vinner's notion of the concept image, encompassing the cognitive structures that individuals associate with a concept. Additionally, we make us of Sierpinska's modes of thinking to capture the nuances of the students' reasoning.

Our study was conducted at the time when eigenvectors and eigenvalues were first introduced to the students, extending into the subsequent weeks. The written homework of 170 students were collected and their answers to two tasks, specifically designed to illuminate their comprehension of eigenvectors and eigenvalues, were analysed. To gain deeper insights, semi-structured interviews were conducted with five chosen participants after they had submitted their homework assignments.

Our findings reveal the diverse concept images held by students concerning eigenvectors and eigenvalues. These encompass a range of attributes, including their structural relationships with linear transformations, span and vector spaces, the arithmetic properties of their computation, as well as their geometric and visual characteristics, and spatial representations. Interestingly, a remarkable majority of the students displayed proficiency in their engagement with multiple modes of thinking. However, our observation that several of the students' answers fell in between the modes established by Sierpinska underscored a certain limitation in our application of this as an analytical framework. These experiences prompted us to expand upon the original modes by introducing our own, mixed categories. However, our study also highlighted potential challenges in students learning of eigenvectors and eigenvalues. These obstacles encompassed aspects such as a lacking awareness of the relations between a matrix (or linear transformation), its eigenvector(s) and corresponding eigenvalue(s), as well as confusion surrounding the number of eigenvectors associated with a given matrix or eigenvalue.

In closing, our study highlights aspects of students' comprehension that were left unexplored, which we consider compelling avenues for future research.

Keywords: linear algebra, concept image, modes of thinking, eigentheory, student understanding

Sammendrag

Som et av de første emnene studenter møter på universitetsnivå, introduserer linear algebra studentene for en verden av begreper, objekter og representasjoner. Imidlertid kan disse abstrakte begrepene og deres representasjoner være krevende å håndtere for nybegynnere. Egenteori, grenen av linear algebra som omhandler egenvektorer, egenverdier og egenrom, fremstår som en særlig nyttig gruppe begreper. Tross deres mangfoldige bruksområder, kan kompleksiteten disse begrepene innebærer være utfordrende for studenter i starten av læringsprosessen.

Som et antatt viktig ledd i læringsprosessen brukes øvinger (skriftlig hjemmearbeid) på universiteter over hele verden. I denne studien utforsker vi studenters skriftlige og muntlige resonnement i forbindelse med et sett oppgaver, hvor målet er å belyse deres forståelse av egenvektorer og egenverdier. Likevel kan begrepet *forståelse* fremstå som komplekst og muligens kontroversielt. Derfor tar vi i bruk Tall og Vinner's idé om begrepsbilde (engelsk: concept image), som omfatter alle de kognitive strukturene et individ assosierer med et begrep. Videre benytter vi Sierpinskas tenkemåter (engelsk: modes of thinking) for å belyse nyansene i studentenes resonnement.

Denne studien startet da studentene først ble introdusert for begrepene egenvektorer og egenverdier, og strakte seg utover de påfølgende ukene. Studentenes øvinger ble samlet inn og deres skriftlige svar på to oppgaver, spesielt utformet for å belyse deres forståelse av egenvektorer og egenverdier, ble analysert. For å få dypere innsikt ble semistrukturerte, individuelle intervjuer utført med fem utvalgte studenter etter at de hadde levert øvingene.

Våre funn avdekker de mangfoldige begrepsbildene studenter kan inneha av egenvektorer og egenverdier. Disse omfatter en rekke egenskaper, inkludert deres strukturelle forbindelser til lineære transformasjoner, spenn og vektorrom, aritmetiske egenskaper som prosedyrer for å bestemme dem, samt deres geometriske og visuelle tolkninger, og romlige representasjoner. Likevel observerte vi at flere av studentens svar falt mellom Sierpinskas etablerte tenkemåter, noe som understreker en viss begrensning i anvendelsen av dette som et analytisk rammeverk. Disse erfaringene fikk oss til å utvide de opprinnelige tenkemåtene ved å inkludere våre egne, blandede kategorier.

Studien vår avdekket også potensielle utfordringer i studentenes læring av egenvektorer og egenverdier. Disse inkluderte aspekter som manglende begrep om forholdene mellom en matrise (eller lineær transformasjon), dens egenvektor(er) og tilhørende egenverdi(er), samt en forvirring rundt antall egenvektorer som kan knyttes til en gitt matrise eller egenverdi.

Avslutningsvis belyser studien vår sider ved studenters forståelse som ikke lot seg utforske i denne masteroppgaven, men som vi anser som lovende områder for fremtidig forskning.

Nøkkelord: lineær algebra, egenvektor, egenverdi, begrepsbilde, forståelse

Preface

With this master's thesis in the didactics of mathematics, I conclude my six year journey of teacher education in science at NTNU. As I conclude this adventure, I eagerly anticipate embarking on the next one - pursuing a PhD in university didactics.

Throughout my journey as a student at NTNU, I have encountered a myriad of experiences - from late-night study sessions for exams, and enjoyable evenings with good friends and a few too many beers, with subsequent regrets the morning after. The memories of the mandatory field trip in the summer of 2018, reluctantly stirring dung in search of Aphondinae, continue to bring a smile to my face. Despite my initial annoyance, that trip turned out to be one of the most treasured experiences of my education, bringing me closer to my fellow students, who I firmly believe will become some of the coolest teachers in Norwegian schools.

I want to express my gratitude to all the kind people who have supported me throughout my time as a student. First of all, I want to thank my supervisor, Yael Fleischmann, for your dedication and guidance, which has been vital in realising the full potential of this project. I would also like to convey my deep appreciation to the course teachers for their cooperation and to all the students who participated in this study. Their contributions were vital in realising this research.

To my parents, Kristin and Ivar, thank you for your unwavering support, lending an ear during moments of doubt and worry, and for believing in me every step of the way through my 29 years. My friends William and Oline, your invaluable guidance has steered me away from the labyrinth of continuation exams and endless hallways of Sentralbygget. A special thank you goes to my best friend Benedikte, our weekly coffee breaks at NTNU have never failed to put a smile on my face. To all my friends in the NTNU Skydiving club, the unforgettable moments we shared both on the ground and in the sky have added a unique and cherished dimension to my student life. Last but not least, to my dear Mikael, thank you for being my rock and for constantly reminding me how to eat an elephant (one bite at a time).

Emilie Lyse-Olsen, Trondheim, August 2023

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Chapter 1

Introduction

Linear algebra is useful to science, technology, engineering and mathematics (also known as STEM fields). However, over the last decades, the problems in the teaching and learning of linear algebra have received increasing attention by researchers (Dorier & Sierpinska, 2001). In describing these challenges, Dorier (2002) states that "The teaching of linear algebra is universally recognised as difficult. Students usually feel that they land on another planet, they are overwhelmed by the number of new definitions and the lack of connection with previous knowledge." (p. 876).

In this study, we investigate students' understanding of eigentheory, the domain of linear algebra concerning eigenvectors, eigenvalues and eigenspaces, in an early stage of their university education. We focus on eigentheory because of its widespread use and conceptual complexity. First, eigentheory has important applications both in and outside mathematics. In physics, eigentheory can be used to solve differential equations or study Markov-chains. Eigentheory can also be applied to model predator-prey processes in statistics and biology. Second, eigentheory can be conceptually complex as students need to understand several key concepts in linear algebra, such as span, transformation and linear (in)dependence (Wawro et al., 2018, p. 275).

In our experience, a typical linear algebra course involves weekly lectures, homework and a final exam at the end of the semester. According to Gravesen et al. (2017), tasks are at the core of a learning situation. With the perspective that students develop their understanding when they are actively engaged in the content, we concentrate on the setting of homework tasks in eigentheory. Given the widespread use of homework in universities all over the world, it seems reasonable to assume that students' engagement with homework can be an important part of their learning process. Furthermore, it is assumed that students' homework may offer valuable insights into their current level of understanding.

1.1 Understanding

As described above, we are interested in the understanding of eigentheory that students exhibit. At this point, we may ask ourselves "What does it really mean to understand?". From our perspective, the notion of understanding can be challenging to define. There exists several characterisations of different kinds of understanding. For instance, Skemp (1978) distinguished between instrumental and relational understanding. The former is characterised as simply knowing how to apply rules to solve problems, in contrast to the latter, which also includes knowing *why* the rules apply. Similar descriptions can also be found in Hiebert's (1986) distinction between procedural and conceptual knowledge (or understanding) and Halmos' (1985) algorithmic and dialectic mathematics. However, such dichotomies have been criticised by researchers like Sfard (1991) and Sierpinska (2005) for being reductive or even false. We share the perspective that these dichotomies have some limitations, and taking into account that these distinctions are also not specific to linear algebra, they may not capture the many facets of understanding in this field. Therefore, we aim to develop our own concept of what it means to *understand*, which we will discuss in this thesis.

Harel (1997) lists several properties of what it means to understand in the context of linear algebra. These include the ability to think in general terms and to make connections between concepts, or in his own terms, "ideas" (p. 109). However, according to Dorier et al. (2000a, p. 94), definitions can be a source of difficulty for students in linear algebra. It is our perspective that this might be particularly true for eigentheory, where several key concepts, such as transformation, vector, matrix, etc., have intricate connections.

Linear algebra is characterised by the many representations of the mathematical objects under study (Hillel, 2000, p. 199). Thus, the ability to engage with diverse representations becomes an important facet of understanding linear algebra, in our view. Nonetheless, as we will later discuss, these representations can often pose challenges for students. It should be noted that the term *representation* holds distinct interpretations across various authors, a subject we will also delve into further down the line.

1.2 Aim and Research Questions

This thesis represents the initial and intermediate scientific outcome within an integrated PhD project. The research presented here lays the groundwork for further work, which will be carried out in a three-year PhD program with the intention of extending upon the findings. However, it is important to differentiate between the aims set for the master project and the broader goals pursued in the PhD project, despite their close interrelation and interconnection in terms of both objectives and planning. The primary objective of the master project is to investigate students' understanding of concepts from eigentheory. However, to provide a comprehensive overview, we briefly explain the aims of the PhD project as well.

1.2.1 Aim and Research Questions of the Master Project

Despite the growing amount of research on the teaching and learning of linear algebra, Wawro et al. (2019) state that research focused on eigentheory is "far from exhausted" (p. 275). In accordance with that, the master project aims to describe some aspects of students' understanding of two key concepts from eigentheory, namely eigenvectors and eigenvalues, as well as their challenges facing these concepts. The overarching question for the master is:

What characterises students' understanding of eigenvectors and eigenvalues?

To address this question, we have designed a set of tasks to be undertaken as part of students' compulsory homework in a first linear algebra course at the Norwegian University of Science and Technology (NTNU). The data material consists of the written works of 170 students, as well as sound recordings and transcripts of interviews with five selected individuals.

The current master study builds upon an additional preparatory work in form of a pilot study, which was conducted in the fall of 2022. There, the written the homework of 52 students were analysed to gain insights into their understanding of eigenvectors and eigenvalues. The results of the pilot study informed the design and conduction of the main study, whose results will be discussed in this thesis.

In the provided thesis, we aim to explore a comprehensive research inquiry by formulating theoretical foundations to guide our empirical investigation in search of answers to the overarching question. This will involve refining the primary question to precisely define the concept of *understanding* in the given context, resulting in the following, more specific research questions, which will guide our empirical analysis:

1. *What concept images can be described from the students' reasoning about eigenvectors and eigenvalues?*
2. *What modes of thinking can be identified in the students' reasoning about eigenvectors and eigenvalues?*

The process of arriving at these refined questions will be expounded in Chapter 3, within the context of the theoretical and methodological framework.

1.2.2 Aim and Overarching Research Question for the PhD Project

The broader project of the PhD builds on the master project and aims to develop a set of tasks to address students' challenges and enrich their understanding of linear algebra. The overarching research question for the PhD is:

What tasks can be designed to support university students' understanding of linear algebra?

It should be noted that the aforementioned research question is not intended as a final inquiry. Rather, it outlines our long-term intention to develop tasks to support students' learning process.

1.3 Structure

Chapter 2: Literature Review

In the next chapter, we will present a brief literature overview of relevant research on the teaching and learning of linear algebra and eigentheory in particular. This includes identifying some of the key concepts and theoretical insights that scientists have worked on in the context of teaching and learning linear algebra, and the respective challenges for students' understanding.

Chapter 3: Theory

In Chapter 3, we offer our perspective on how theories can be applied to address our research question(s). Afterwards, we delve into Tall and Vinner's notion of the concept image. Additionally, we explore the modes of thinking in linear algebra as described by Sierpinska, which will be one of the key concepts for the analysis of our data. Furthermore, we elaborate on the ontological and epistemological assumptions underlying our theoretical perspective and highlight the advantages of networking these theories.

Chapter 4: Methodology, Methods and Ethics

We begin Chapter 4 with a discussion on methodology, outlining the rationale of our chosen methods. We explain why we opted to design tasks for homework assignments and conduct subsequent interviews with students regarding their reasoning. Following this, we describe the methods employed in further detail, including an overview of the setting[1] and participants of our study, the process of data collection and the tasks. Then, we outline the steps involved in our thematic analysis of the data, accompanied by an elaboration on the characteristics defining the modes of thinking within the context of eigentheory. We end this chapter with our reflections regarding the ethical considerations involved in our study, including the measures taken to support trustworthiness and authenticity.

At this stage, we would like to inform the reader that Chapters 3 and 4 are inspired by two essays (Lyse-Olsen; 2022a; 2022b) on theoretical and methodological considerations for the master's project, that were written in conjunction with the courses MA-602 and MA-607 at the University of Agder in 2022. Therefore, parts of these chapters may reflect insights gained from writing the essays.

[1]By *setting* we mean the physical, social and cultural site in which the study is conducted.

Chapter 5: Results and Analysis

Chapter 5 outlines the findings of our study by showcasing examples of students' written and oral reasoning in response to the homework tasks we designed. To shed light on the students' understanding, we provide a thorough analysis of these responses using the notions of the concept image and the modes of thinking. In employing these theoretical lenses, we aim to illuminate the nuances of students' understanding and gain insights into how they interpret and conceptualise eigenvectors and eigenvalues.

Our analysis demonstrates that the students present concept images at various stages of their development. Furthermore, we discuss how students' answers align with Sierpinska's modes of thinking. It is noteworthy that several of the students' responses incorporate elements from multiple modes of thinking, indicating rich concept images.

Chapter 6: Discussion

In Chapter 6, we provide an overview of the main results of our study, comparing and contrasting these with the findings of previous research. We engage in a critical reflection of the theoretical lenses we employed, assessing their affordances and constraints, while also considering alternative theoretical lenses for our future research. Following this, we undertake a critical examination of our methods, carefully evaluating their benefits and limitations. We assess the extent to which our analysis of the data using our theoretical lenses allowed us to answer the research questions effectively. We conclude this chapter with an outlook towards future research, highlighting areas that warrant further exploration and development.

Chapter 7: Conclusion

We conclude this master's thesis with a concise answer to our research questions to demonstrate that we have effectively addressed the core objectives of the study. In addition, we offer some personal reflections that expand upon the insights gained from this study, as well as some thought-provoking questions for contemplation and reflection.

Chapter 2

Literature Review

In this chapter, we explore research on the teaching and learning of linear algebra at the tertiary education level, with a particular focus on eigentheory. Our analysis encompasses research spanning from the 1980s to the present, aiming to gain a comprehensive understanding of the existing knowledge and the challenges that students face in learning linear algebra and eigentheory in particular, as well as highlighting promising new research that has emerged in this area.

2.1 Dorier's Book

The 1980s marked the start of research on the learning of linear algebra, as researchers in France called upon the attention of the international community. They were concerned with students' persistent difficulties in understanding linear algebra concepts, even after completing one or two courses on the subject. Dorier's (2000) book, *On the Teaching of Linear Algebra*, is a significant collection of research on the teaching of linear algebra at the tertiary education level. It holds importance for laying the groundwork for future research and introduces several theoretical concepts which may be used to study students' understanding of linear algebra.

The first part of the book offers a historical survey and epistemological analysis of linear algebra, emphasising the abstract nature of linear algebra concepts and the unifying role of vector spaces (Dorier, 2000). The second part of the book concerns educational issues.The first four chapters of this part concern a research program starting in 1987 on the teaching and learning of linear algebra led by Dorier, Robert, Robinet and Rogalski. In Chapter 1, Dorier et al. (2000a) highlight some challenges students experience in learning linear algebra, specifically the *obstacle of formalism*. This refers to the challenges students face when dealing with symbols, notations and concepts in linear algebra, leading them to feel overwhelmed and hindering the development of a concrete understanding of the concepts. In Chapter 2, the notion of *level of conceptualisation* is introduced by Robert (2000) to analyse students' challenges in linear algebra. In Chapter 3, Rogalskí (2000) describes a long-term teaching project that combines initial diagnoses with new hypotheses to address students' challenges in learning linear algebra. Chapter 4 introduces the *meta level*, a teaching tool developed to address the obstacle of formalism, including three experimental illustrations and an evaluation of the challenges encountered in evaluating these experiments and remaining difficulties (Dorier et al., 2000b).

In Chapter 5, Harel (2000) stresses the importance of students feeling the necessity of abstract concepts and introduces three pedagogical principles for designing and implementing mathematics curricula. These are called the Concreteness Principle, the Necessity Principle and the Generalisability Principle. First, the *Concreteness Principle* states that abstract ideas should be introduced with concrete examples, visual aids or hands-on experiences to make them more tangible, thus bridging the gap between abstract concepts and students' pre-existing knowledge. Second, the *Necessity Principle* holds that instructional activities must be organised in such a way that the students perceive the knowledge at stake, be it a concept or a procedure, to be necessary to solve the problem at hand. Finally, the *Generalisability Principle* highlights the importance of enabling students to transfer their learning to different contexts by encouraging them to identify patterns, underlying principles and apply their knowledge to new situations, thus promoting flexible thinking.

In Chapter 6 of Dorier's book, Hillel (2000) addresses the problem of representations, exploring how geometric visualisation in lower dimensions can impact the learning of abstract concepts. He distinguishes between three modes of description in linear algebra: the abstract, algebraic and geometric mode. Within these modes, vectors and transformations are characterised by distinct notations, terminology and representations. The *abstract mode* employs formal language and concepts from the generalised n-space, like vector space, subspace and kernel. The *algebraic mode* is more specific, viewing vectors as n-tuples and transformations as matrices. Finally, the *geometric mode* uses the concepts from 2- and 3-space, such as points, lines and planes. Hillel's (2000) distinction is closely related to the modes of thinking introduced by Sierpinska in the subsequent chapter.

In Chapter 7, Sierpinska and her group explored the obstacle of formalism and concpetualised the modes of thinking (Sierpinska, 2000. These modes represent different ways of reasoning in linear algebra, namely the analytic-structural, analytic-arithmetic and synthetic-geometric mode of thinking. As we shall see later in this chapter, these have been used as an analytic framework for examining students' understanding in various areas of linear algebra, and will also play a fundamental role for the theoretical approach we have chosen for the study presented in this thesis.

Mental Conceptions

While Dorier's (2000) book provides an overview of the teaching and learning of linear algebra at the time of its publication, it also addressed the importance of mental conceptions in linear algebra education. Several studies, both within and outside the chapter's of Dorier's book, have studied students' mental conceptions of various topics in linear algebra. For example, Dubinsky (1997) emphasised the importance of visualising processes in linear algebra, and encouraging students to construct their own ideas and discuss them.

However, most studies have focused on specific concepts within linear algebra. For instance, Stewart and Thomas (2010) studied students' mental conceptions of key concepts like basis, span and linear independence, while Trigueros and Possani (2013) employed a modelling approach to explore students' mental constructions of linear combinations and linear independence. Furthermore, Salgado and Trigueros (2015) used a similar modelling approach to investigate students' mental constructions of eigenvectors and eigenvalues. Dogan (2018), on the other hand, used the framework of Sierpinska's modes of thinking to qualitatively analyse students' ideas of linear independence from interviews. Her findings suggest that relying solely on algebraic instructional tools, which is commonly practised in classrooms, may contribute to limit students' mental constructions to mere computational procedures.

We find it worth mentioning that the authors referenced here employ terms like *mental conceptions*, *mental structures* and *mental constructions* with slightly different meanings. However, our interpretation suggests that both terms largely concern the cognitive representations in the minds of individuals, which allow them to comprehend their experiences, organise knowledge and communicate their understanding.

2.2 Representations

In the context of linear algebra, and particularly concerning mental conceptions, the aspect of representations is a recurring theme. Jean-Francois Duval's (2006) research on representations in the teaching of mathematics generally emphasises the significance of implementing multiple representations in mathematics education. He stresses that engaging students in meaningful representational activities may promote their ability to make sense of mathematical ideas.

Returning to linear algebra, Sierpinska and her group (2000) notes that a common challenge for students is to perceive different representations as being independent mathematical objects themselves. This causes students to face the challenge of dealing with an ever increasing number of new mathematical objects. Duval (2006) raises the crucial question of how students can be expected to distinguish between the mathematical object and its representation, when their sole means of access to the objects are indeed through their respective representations.

According to Duval (2006), *changing* between the representations poses an additional challenge for students, some being more difficult than others. Hillel (2000) notes that instructors tend to perform these transfers without regard for the cognitive complexity involved for students. The importance of proficiently making transfers between representations is acknowledged by Alves Dias and Artigue (1995), who emphasise the necessity of training students to develop these competencies. They highlight that the responsibility of fostering these competencies cannot be left to the students alone. Rather it must be integrated in instructional practices.

2.3 Eigentheory

Concerning the challenges with representations, eigentheory emerges as a particularly interesting topic. According to Wawro et al. (2019), eigentheory encompasses a group of concepts with wide-ranging applications both in and outside linear algebra. In lower dimensions, like $\mathbb{R}^2$ and $\mathbb{R}^3$, eigenvectors and eigenvalues possess powerful geometric representations. However, Hillel (2000) notes that students may not be able to connect the geometric representations of eigenvectors and eigenvalues to the algebraic representations.

Delving deeper into students' understanding of eigentheory, Thomas and Stewart (2011) conducted a study, revealing both areas of proficiency and challenges in students' reasoning. While students demonstrated proficiency in performing the arithmetic procedures for computing eigenvectors and eigenvalues, they encountered difficulties when faced with their geometric representations. Despite using expressions like "being stretched" or "same direction" in describing eigenvectors, they struggled to apply these definitions in context. For instance, several students had troubles determining whether $\begin{bmatrix} 3 & -4 \end{bmatrix}^\mathsf{T}$ could be considered an eigenvector given that $\begin{bmatrix} -3 & 4 \end{bmatrix}^\mathsf{T}$ was in fact an eigenvector.

Their study also suggested that most students primarily perceived linear algebra as a mere "application of procedures" (p. 293), thought of eigenvectors and eigenvalues in a symbolic manner, and that few were aware of their embodied (or geometric) representations. A particular challenge the students encountered was performing the algebraic manipulations from the eigenequation, $A\vec{x} = \lambda\vec{x}$, to the homogeneous equation for computing eigenvectors, $(A - \lambda I)\vec{x} = \vec{0}$. According to Thomas and Stewart (2011), their struggles appeared to stem from their difficulties in conceptualising the matrix I in this context and the navigation of multiple mathematical objects, such as scalars, vectors and matrices. The solutions of $(A - \lambda I)\vec{x} = \vec{0}$ are a set of vectors, while the solution of $\det(A - \lambda I) = 0$ yields numbers.

In light of these algebraic representations, Salgado and Trigueros (2015) emphasised the importance of understanding the equivalence between the equations $A\vec{x} = \lambda\vec{x}$ and $(A - \lambda I)\vec{x} = \vec{0}$. Moreover, Wawro et al. (2018) issued a cautionary note against an excessive focus on the computation of bases for eigenspaces before students have developed a robust understanding of eigenvectors and eigenvalues. Potentially, such an overemphasis can lead to misconceptions, such as believing that there is only one eigenvector corresponding to an eigenvalue (when there are in fact infinitely many).

While the studies discussed so far have revealed that undergraduate students often encounter challenges facing the concepts of eigentheory, research also suggests several promising approaches for alleviating these challenges. Incorporating dynamic geometric software has the potential to enrich students' geometric understanding of eigenvectors and eigenvalues. For instance, Gol Tabaghi and Sinclair (2013) found that students' engagement with the dynamic geometry software *Geometer's Sketchpad* could support a synthetic-geometric mode of thinking about eigentheory, as conceptualised in Sierpinska's (2000) framework. As their students' conceptions of eigenvectors were heavily motion-based, Gol Tabaghi and Sinclair characterised their modes of thinking as dynamic-synthetic-geometric, thus further contributing to the framework of Sierpinska.

2.4 The IOLA Project

The study conducted by Gol Tabaghi and Sinclair (2013) represents one of several initiatives over the past decades attempting to support the teaching of linear algebra in tertiary education. Another important initiative is the Inquiry-Oriented Linear Algebra (IOLA) project, which aims to enhance the teaching of linear algebra through an inquiry-based approach (Wawro et al., 2013). Inquiry, in this sense, involves students being actively engaged in challenging problems in an authentic setting. For instructors, inquiry-based teaching entails actively listening to the ideas of the students, responding and building upon their ideas to enhance the mathematical competencies in the classroom. Thus, in aligning the principles of inquiry-oriented instruction and inquiry-based learning, the IOLA project aims to promote active exploration of mathematical ideas for both students and instructors.

For the study presented here, and the PhD project it is a part of, the work of the IOLA project has been an inspiration. We share the wish to develop learning materials that can support students in their learning of linear algebra, based on a scientific foundation, and to accompany these attempts with didactic analyses of the content knowledge of linear algebra.

Chapter 3

Theory

We begin this chapter by presenting our view on what theory is, why we need theory and how we apply theory in the context of this study. Next, we introduce the theories employed in this research, namely radical constructivism, Tall and Vinner's concept image (as described e.g. in Tall and Vinner, 1981), and Sierpinska's modes of thinking (as introduced in Sierpinska, 2000). Subsequently, we offer the rationale for our choice of theories, emphasising their affordances and limitations, as well as comparing their ontological and epistemological assumptions to ensure compatibility. We conclude this chapter by restating the aim and research questions of our study in terms of the theoretical perspective we have adopted.

What is theory?

In mathematics education research, the notion of theory can be defined in various ways and serve multiple functions. Niss (2007, pp. 1308-1309) describes a theory as an organised system of concepts and principles that can be used to describe, interpret, predict or explain phenomena. Assude et al. (2008) claims theory can be seen as a tool for interpreting aspects of reality and generating new knowledge. According to Simon (2009), a theory may also be used as a *lens* that helps researchers to focus on specific aspects of a phenomenon. While this focus may obscure other aspects, he argues it allows for a deeper and more nuanced analysis of the research data. The importance of theory (or theories) in research comes from their ability to give meaning to the data. Lester (2010) asserts that the data do not speak for themselves and stresses that theory is a necessary condition for comprehending and interpreting data, allowing us to "transcend common sense" (p. 70).

Levels of theories

Theories may be classified into different levels; local, middle-range and grand theories (Bikner-Ahsbahs & Prediger, 2006; Assude et al., 2008). *Local theories* are developed within specific contexts, focusing on problems or tasks. *Middle-range theories*, on the other hand, cut across contexts and encompass classrooms or institutions. Finally, *grand theories* are developed outside mathematics education, and imply ontological and epistemological assumptions.

Theory as a lens

In order to come to know about and to describe students' understanding of eigenvectors and eigenvalues, we need both data and theory. For the purpose of this study, we employ theory as a lens to illuminate specific facets of their understanding, at the cost of disregarding other aspects. However, *understanding* is a complex phenomenon, inherently challenging to define and directly observe. In the following, we explain our decision to adopt radical constructivism as the theoretical foundation informing our ontological and epistemological assumptions on what understanding entails. Then, we introduce Tall and Vinner's notion of the concept image, which serves as a theoretical lens for describing students' understanding of mathematical concepts. Lastly, we delve into Sierpinska's modes of thinking, which provides a specialised lens, tailored to capture nuances in students' reasoning within the domain of linear algebra. Thus, it should be noted that while radical constructivism underpins the study, the primary theoretical lenses guiding the analysis are the concept image and the modes of thinking.

3.1 Radical Constructivism

For this study, radical constructivism, as introduced by Ernst von Glasersfeld in the 1970s (Walshe, 2020), forms the theoretical background perspective on learning for this study. It is our perspective that radical constructivism can be considered a grand theory of learning, providing us with valuable insights into the nature of learning and how it occurs.

Taking a constructivist perspective on learning, students are not passive recipients of knowledge, but active constructors of their own personal interpretation of the knowledge at stake (Lerman, 1996). Hence, in the context of our study, this implies that students develop their understanding of eigenvectors and eigenvalues by actively engaging with the subject matter. That is, in their interaction with definitions, examples and representations, students strive to reinterpret and make sense of the information at hand.

Radical constructivism distinguishes itself from conventional constructivism by adopting a radical stance about the existence of an objective reality or objective knowledge, such as an inherently "true" understanding of the concept of eigenvectors. While radical constructivism does not outright deny the existence of such an objective knowledge, it claims we cannot directly access it (von Glasersfeld, 1990a; 1990b). Instead, we may only interpret it through our subjective experiences. Thus, in describing students' conceptions of eigenvectors and eigenvalues, we may only do so in comparison to our own construction of these concepts.

3.2 Concept Image and Concept Definition

In this subchapter, we introduce Tall and Vinner's notions of the concept image and concept definition, and their relations to the notion of understanding. We explore the development of the concept image, factors influencing its formation, and conflicts that may arise within different parts of the concept image, as well as between the concept image and concept definition.

3.2.1 What is the Concept Image?

According to Vinner (2002), definitions can be a source of students' difficulties because the axiomatic structure of mathematics may not align with the process of learning mathematics. In 1981, Tall and Vinner coined the terms concept image and concept definition to explain how students' associations with a concept may be inconsistent with the formal definition.

The *concept image*, as defined by Tall and Vinner (1981), refers to the mental representation or internalisation of a mathematical concept that individuals develop through their experiences and interactions with mathematical ideas. Hence, the concept image can be described as an individual's subjective understanding of a concept, shaped by their personal experiences, interpretations, and prior knowledge.

For example, in the case of eigenvectors, we propose that a student's concept image may consist of visual examples illustrating how a linear transformation scales the eigenvector. Additionally, the concept image may include the eigenequation ($A\vec{x} = \lambda\vec{x}$), the homogeneous equation ($(A - \lambda I)\vec{x} = \vec{0}$) and the computational steps for deriving this latter equation.

3.2.2 What is the Concept Definition?

The *concept definition* is a verbal definition that explains the concept in a precise and non-circular way (Vinner, 1983). Tall and Vinner (1981) makes a further distinction between a *personal* and a *formal* concept definition. The latter is a definition accepted by the mathematical community and is often presented to students in lectures and textbooks. The personal concept definition may be the result of rote learning of a formal concept definition or it could be the individual's reconstructed version of it. The concept definition generates its own concept image, which forms a part of the total concept image possessed by an individual. This may be more or less consistent with other parts of the concept image.

3.2.3 The Development of the Concept Image

While a formal concept definition can be considered a static entity, Tall and Vinner (1981) emphasise that the concept image is considered an evolving, dynamic construct. The concept image is continuously shaped and refined through the individual's engagement with mathematical tasks, their encounters with different instructional approaches, and participation in social interactions with others. In the context of eigenvectors and eigenvalues, we propose that a student's concept images of eigenvectors and eigenvalues may be influenced by the formal concept definition, as well as examples and visual representations of them. In the analysis, we shall see examples of different contributions to students' concept images.

3.2.4 The Concept Image and Concept Definition in Relation to Understanding

In his work, Vinner (2002) addresses the notion of understanding and cautions that memorising a concept definition does not guarantee understanding of it. Instead, he proposes that understanding a concept involves the possession of a concept image, stating that "To understand, so we believe, means to have a concept image" (p. 69). However, while acknowledging the significance of the concept image, we are of the opinion that it may not encompass every facet of the complex phenomenon of understanding. Therefore, we refine this perspective by asserting that possessing a concept image is a necessary condition for understanding. We illustrate our perspective on the relationship between understanding, and the concept image in Figure 3.1. It is important to note that the proportions depicted in the diagram are not intended to represent their respective magnitudes.

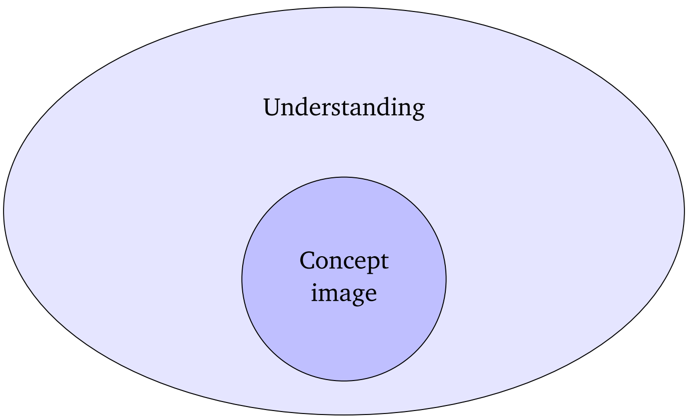


Figure 3.1: Visualisation of the concept image as an integral part of the notion of understanding.

3.2.5 Conflict Factors

As previously described, the concept image is considered to be multifaceted and dynamic. Tall and Vinner (1981) propose that when a particular stimulus is presented, a subset of the concept image is activated, referred to as the *evoked concept image*. Hence, not all aspects of an individual's concept image are necessarily activated simultaneously, as different stimuli evoke different subsets of the concept image.

Tall and Vinner (1981) further claim that an individual's concept image comprises various parts, which may differ in their level of consistency and even exhibit conflicts. They employ the term *potential conflict factor* to describe the elements within the concept image (or concept definition) which may be in conflict with other elements. These conflict factors may not always be triggered in circumstances that cause the individual to experience an actual cognitive conflict. However, when they *are* activated simultaneously, they are referred to as *cognitive conflict factors*.

Thus, if conflicting parts of the concept image are evoked simultaneously, it can trigger a cognitive conflict, which we believe presents as a valuable learning opportunity. The conflict arises when individuals are confronted with the inconsistencies or contradictions in their understanding, which can prompt them to critically examine and resolve these conflicts. We argue that engaging with cognitive conflict can facilitate the development of the concept image towards a more robust and accurate understanding of the concept. In the analysis of the data collected for iyr study, we will see how the students' interactions with both the written tasks and the interviewer lead to situations where conflicting parts of their concept images were evoked.

3.2.6 Summary and Limitations

So far, we have provided an overview of the concept image, elucidating its components, and the process of its development. Additionally, we have presented the concept definition and discussed the potential conflicts which can arise within an individual's concept image, or between their concept image and the concept definition.

From our perspective, understanding the concept image is paramount for educators as it provides insights into students' cognitive processes and their abilities of reasoning. By recognising and addressing gaps or misconceptions in students' concept images, teachers can facilitate more effective instructional approaches to support students' development of a rich and accurate understanding of mathematical concepts.

However, to adequately describe students' concept images in the domain of linear algebra, we require a theoretical approach that allows us to describe phenomena specific to the field. Therefore, we turn to Sierpinska's modes of thinking (as introduced in Sierpinska 2000), which offer a framework for analysing students' reasoning regarding linear algebra concepts. In the following, we will provide an overview of the modes of thinking, their visual representations and some challenges students may have with these modes.

3.3 Modes of Thinking

Recall that within the realm of linear algebra, students often encounter challenges in navigating the many abstract concepts and the representations associated with these mathematical objects. To shed light on these difficulties and better understand students' reasoning in linear algebra, Sierpinska (2000) introduced the *modes of thinking*. Though she does not provide a rigid definition of this term, we interpret it as distinct ways of reasoning that individuals employ when engaging with linear algebra. As we shall see, the modes can encompass different ways of understanding, representing, and interpreting mathematical ideas.

In our perspective, the modes of thinking can be considered to describe a part of students' understanding. In particular, we consider the modes of thinking to partly overlap with the notion of the concept image, as illustrated in Figure 3.2. It should be noted that the diagram depicts a partial overlap between the sets, but the proportions are not to scale.

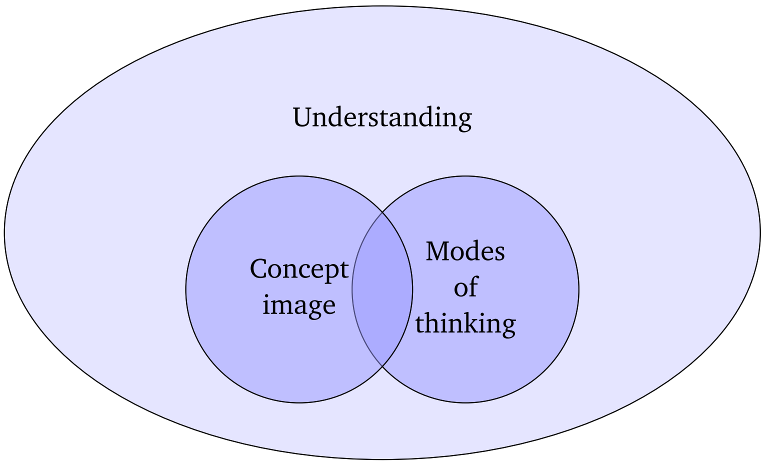


Figure 3.2: Venn diagram illustrating the concept image and modes of thinking as partially overlapping aspects of *understanding*.

3.3.1 Background

Sierpinska (2000) distinguishes between three modes of thinking in linear algebra. She calls them the synthetic-geometric, analytic-arithmetic and analytic-structural modes of thinking. The names of these categories have two levels; In the first level, she distinguishes between *synthetic* and *analytic* modes of thinking.

In the second level, a further distinction is made between the *geometric*, *arithmetic* and *structural* modes. This hierarchic structure is illustrated in Figure 3.3.

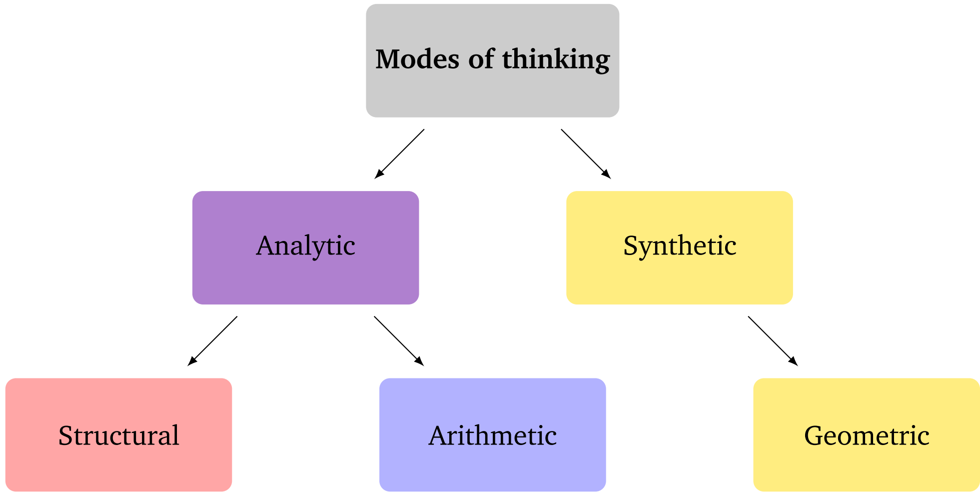


Figure 3.3: The modes of thinking can be divided into two subcategories; analytic and synthetic. In a second level, Sierpinska distinguishes between structural, arithmetic and geometric modes of thinking. The arithmetic and structural modes are analytic, and the geometric mode is synthetic.

3.3.2 Analytic vs Synthetic Modes of Thinking

In describing the difference between the synthetic and analytic modes of thinking, Sierpinska (2000) states that:

> [I]n the synthetic mode the objects are, in a sense, given directly to the mind which then tries to describe them, while, in the analytic mode they are given indirectly: In fact, they are only constructed by the definition of the properties of their elements. (p. 233)

Sierpinska (2000) illustrates the distinction with the example of a straight line which, in the synthetic mode, is understood as a "pre-given object of a certain shape lying somewhere in space" (p. 233), and in the analytic mode "the straight line is defined as a certain specific relationship between the coordinates of points or vectors in a space of a given dimension" (p. 233). In contrast to the analytic mode, the synthetic mode can only describe the line, not define it.

Yet, it is our perspective that the distinction between the analytic and synthetic modes allows for interpretation in various contexts, necessitating further elaboration for our application. In our view, the synthetic mode of thinking involves describing a mathematical object through a *representation*. We use the term representation here in the sense of Duval (2006), where semiotic representation refers to using signs, symbols or notations to depict mathematical objects or operations. Thus, a synthetic mode of thinking could be expressed as describing a function through its graph.

However, representations inherently cannot capture all facets of the mathematical object. In contrast, the analytic mode of thinking aims to provide a well-defined and precise description of the object. At this stage, we draw the reader's attention to what we perceive as a similarity between Sierpinska's differentiation of synthetic and analytic modes, and Duval's (2006) observation that students sometimes struggle to distinguish the mathematical object from its representation.

3.3.3 The Three Modes of Thinking

In the analytic-arithmetic mode, an object is defined by the formula that enables its computation, while the analytic-structural mode is concerned with the characteristic properties defining the object. Sierpinska (2000) provides the example of inverses to illustrate the distinction from the analytic-structural mode of thinking. While the analytic-arithmetic mode pertains to the process of *calculating* the inverse of a given element, the analytic-structural mode focuses on the property of this element of *having* an inverse.

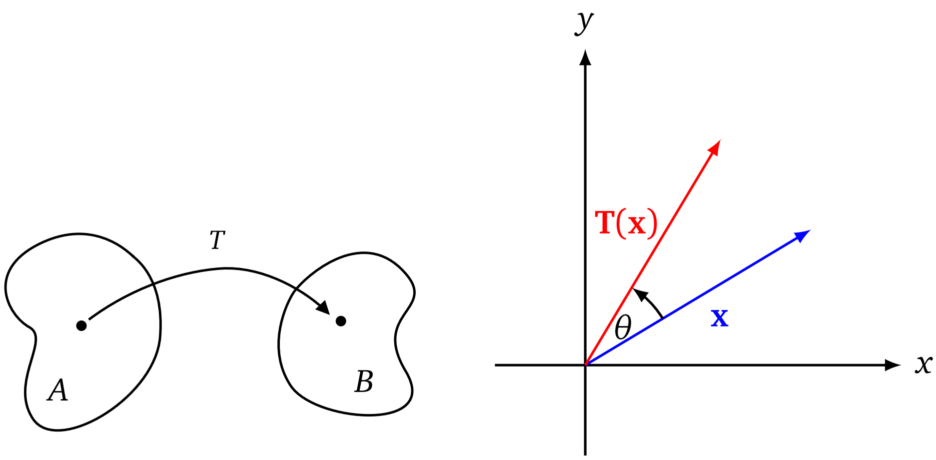
According to Sierpinska (2000, p. 234), the analytic mode is concerned with making computations correctly and efficiently, while the structural mode aims to extend our knowledge about the concepts and their connections. She elaborates this distinction by discussing the notion of linear transformation. Traditionally, a linear transformation was defined as the *substitution of variables*, where variables y_i are linear combinations of the variables x_i for $i \in \{1, 2, ..n\}$, with x and y representing numbers (Sierpinska, 2000, p. 234; Daintith & Nelson, 1989, pp. 201-202). However, in modern undergraduate texts, a linear transformation is defined as a *mapping* from one vector space to another. Here, the definition does not provide a formula for computing the resulting image, and the elements could be numbers, vectors or matrices. Thus, this latter definition corresponds to an analytic-structural mode of thinking (Sierpinska, 2000).

Finally, the *synthetic-geometric* mode employs the vocabulary of geometric figures, such as points, lines and planes. This mode is concerned with the geometric characteristics of the objects and their visual representations. For example, a vector could be conceptualised as an arrow lying somewhere in space in the synthetic-geometric mode, or as an n-tuple in the analytic-arithmetic mode. From our understanding, the synthetic-geometric mode emphasises visual and spatial aspects of the objects, while the analytic-arithmetic mode focuses on their algebraic representations. In the following, we will further discuss the modes of thinking in terms of their corresponding representations.

3.3.4 Visual Representations

According to Sierpinska (2000), each of the three modes of thinking in linear algebra corresponds to "a specific system of representations" (p. 234). While she does not define the notion of "representation" or "system of representations", we interpret them in the sense of Duval (2006). The analytic-structural and synthetic-geometric modes of thinking have at least three common features. Firstly, they can be independent of any coordinate system. Secondly, they are based on their properties rather than calculations. Thirdly, the synthetic-geometric and the analytic-structural mode include visual interpretations, yet they exhibit distinct characteristics from one another.

Regarding the distinction between synthetic-geometric and analytic-structural visualisations, Sierpinska (2000) states that "the latter is more metaphoric and/or diagrammatic than the other" (p. 236). She elaborates this using the example of a linear transformation. In a structural mode of thinking, a linear transformation could be illustrated as two shapes (representing sets) linked by an arrow (representing the transformation). In contrast, a visual representation of a transformation corresponding to a synthetic-geometric mode of thinking could be visualised as a line or a vector in the coordinate system which is transformed in a specific way such as being rotated by an angle θ. From our interpretation, a representation corresponding to an analytic-structural mode of thinking is more abstract, in this case illustrating a general map, while a synthetic-geometric representation is more specific, in this case, illustrating properties of a particular transformation. Based on Sierpinska's (2000, p. 236) description, we have attempted to illustrate this in Figure 3.4a and Figure 3.4b.



(a) A transformation, T, maps an element from the domain, A, to its corresponding element in the codomain, B.

(b) A transformation, T, rotates a vector $\vec{x}$ by an angle θ in the two-dimensional, real plane.

Figure 3.4: Visual representations of transformations associated with different modes of thinking.

3.3.5 Students' Challenges with the Modes of Thinking

According to Sierpinska (2000), each of the three modes of thinking in linear algebra can pose challenges for students. She notes that while students may have access to all three modes, their arguments tend to lie somewhere between the three modes presented here. Sierpinska (2000) explains that it would be unreasonable to claim that students prefer one mode over the others and suggests that they tend to employ whichever mode (or combination of modes) they perceive to be more convenient for a given situation (p. 236).

Sierpinska (2000) further notes that students often encounter difficulties in the attempt to transition between the modes of thinking. Interestingly, in our view, these difficulties bear resemblance to the challenge students face in converting between different representational systems, as described by Duval (2006). However, it is our hope that this study can contribute to further develop the modes of thinking as an analytical framework[1] for exploring students' reasoning in linear algebra, with a specific focus on eigentheory.

3.4 Epistemology, Ontology, and Coherence Across Theories

In this chapter, we shall briefly discuss our understanding of the notion of research paradigm and explain why we align our study with an interpretative paradigm. We will present our ontological assumptions of knowledge as constructed by the learning individual, as well as our epistemological assumptions. That is, our perspective of how we can come to know about students' understanding through an analysis of their reasoning about eigenvectors and eigenvalues. Afterwards, we discuss the affordances and constraints provided by our chosen theories. Furthermore, we compare the fundamental assumptions underlying these theories, aiming to establish their compatibility and potential for networking.

3.4.1 Research Paradigm

There are at least 21 different meanings to the notion of research paradigm (Masterman, 1970). According to Zakariah (2021), the meanings appear to stem from different fields of study. A widely accepted definition of the term is that of Thomas Kuhn. In his book *The structure of scientific revolutions* (1970), Kuhn described a research paradigm as "the entire constellation of beliefs, values, techniques, and so on shared by the members of a given community." (p. 175). Similarly, Morgan (2007) refers to paradigms as "shared beliefs among members of a specialty area". Hence, a research paradigm can be understood as a shared set of beliefs, values and methods shared amongst a group of researchers. In other words, which

[1]By *analytical framework*, we mean a structured approach to analyse the data, facilitating the interpretation of relationships, identification of patterns and exploration of underlying meanings.

paradigm a study corresponds to depends on the underpinning epistemology, ontology, methodology and methods. According to Robson and McCartan (2016), explicating one's research paradigm is important as it can highlight the significance of one's research questions and account for what constitutes sound answers to the questions. In the following, we will describe the ontological and epistemological assumptions that were applied in this work, that is, the assumptions on what knowledge is and how one can come to know about the objects under study, and how they have come to influence the methodological considerations, which will be described in the next chapter.

3.4.2 Ontology

Ontology is related to philosophy and it can be described as the study of being, becoming, existence and reality. Hence, ontology concerns the nature of reality and social beings, that is, what is real and what it means to exist (Bolstad, 2020, p. 21). From an ontological perspective, a key question is whether the object(s) of study should be considered as existing objectively and independently of social actors, or as subjective and constructed by our minds. For the project described in this thesis, we aim to describe students' concept images of eigenvectors and eigenvalues, and their modes of thinking. First, we may ask whether these concepts can exist independently of social actors? In other words, is there an objective, "true" definition of eigenvector (and eigenvalue), existing independently of any social being? It is our perspective that there is no foundation for neither denying nor confirming the existence of such an objective knowledge, aligning with a radical constructivist perspective.

As described in Section 3.1, adopting a radical constructivist perspective implies that students are not passive receivers of knowledge, but active constructors of their own learning. Furthermore, radical constructivism assumes that knowledge resides in the mind of individuals. However, our modified perspective holds that knowledge at least *partly* resides in the mind. We would like to acknowledge the existence of alternative forms of knowledge and thinking, such as the close relationship between thinking and communication (Sfard, 2008) and the interconnectedness between mathematical thinking and movement (Gandell, 2022; Ingold, 2013). However, these aspects are not the primary focus of the current study and will therefore not be discussed in greater detail. Instead, we accept that we cannot gain direct access to what is (and what is not) in the students' heads. How we can come to know about students' concept images is an epistemological issue that will be discussed in the following.

3.4.3 Epistemology

According to Bryman (2016, p. 690), *epistemology* deals with what is (or should be) regarded as acceptable knowledge in the discipline. Hence, epistemological issues concern questions like 'How can we come to know about the objects we study?', 'Can we rely on subjective meanings or observable phenomena?' and 'What are the limitations of this knowledge?'. In this section, we shall discuss the epistemological issues related to our study, that is, how we can come to know about students' concept images and modes of thinking.

Epistemological Implications of Radical Constructivism

Keep in mind that the ontological assumption that knowledge (at least partially) resides in the minds of individuals, implies that students' understanding of eigentheory cannot be directly accessed. However, this should not be misconstrued as suggesting that their understanding is beyond study. On the contrary, we believe that this highlights the importance of formulating appropriate research questions concerning observable indicators of understanding and establishing suitable methodologies to answer the questions. Let us recall that understanding, according to Vinner (2002), implies the presence of a concept image. It is our perspective that the notion of concept image as proposed by Tall and Vinner (1981), provides several indicators for describing students' comprehension of mathematical concepts. Combining with Sierpinska's (2000) modes of thinking, we achieve a specialised analytical tool for examining students' reasoning within the realm of linear algebra. We still acknowledge that direct observation of thinking is beyond reach in this study. Nevertheless, it is our perspective that Sierpinska's descriptive account of the characteristics of modes of thinking in both written and oral reasoning allows us to utilise the categorisation to gain insights into students' reasoning about eigenvectors and eigenvalues.

Assuming an epistemological perspective similar to that of radical constructivism has at least three implications for our study. First, we must acknowledge that students' communication is essentially a representation of their understanding (here, the term *representation* is used in the sense of an observable depiction), and that this representation is inherently fallible. Our thoughts, which may be non-verbal and "volatile", may differ from the structure of written, and to some extent, oral sentences. Hence, in the process of translating ideas into verbal reasoning, a process of structurisation occurs, where thoughts are organised into a coherent, linear sequence of words. Thus, culminating in a (more or less) meaningful sentence. However, this translation is not necessarily one-to-one, as some aspects may be lost, added or transformed in the process. Thus, akin to how a representation of a mathematical object may not perfectly capture all its facets, the students' explicit reasoning may not reflect all parts of their understanding. Our challenge then becomes to determine observable indicators to examine students' concept images and modes of thinking.

Second, an analysis of these indicators may only result in our own, constructed version of students' knowledge. Necessarily, our construction too is fallible. Third, as we cannot directly access the mathematical objects (whose existence can be interpreted from different ontological perspectives), we may only compare our construction of students' knowledge to our own concept images of eigenvectors and eigenvectors (which may contain imperfections). Therefore, any results presented in our study will inevitably involve an element of subjectivity.

Interpretative Research Paradigm

Amidst the aforementioned challenges, we align our study with an interpretative research paradigm, where we seek to understand students' reasoning through the lens of subjective meaning-making (Bryman, 2016, p. 26; p. 692). Thus, we acknowledge the significance of subjectivity in our interpretations and recognise their role in shaping our upcoming analysis and the findings of our study. We contend that by providing rich descriptions of students' reasoning, actively reflecting upon our own biases and presumptions, we may achieve trustworthy and authentic research that captures the nuances of students' understanding. Later, we shall explore measures taken to support trustworthiness and authenticity in this study. Nevertheless, in line with our interpretative stance, we recognise that other interpretations, shaped by different experiences, may coexist and be equally valid to our interpretation.

In the next subchapter, we will explore the ontological and epistemological underpinnings of our three theories to compare and argue for their compatibility. Furthermore, we shall justify their inclusion in our study by highlighting their affordances and constraints.

3.4.4 Networking of Theories

According to Prediger and Bikaner-Ahsbahs (2014), theories can be networked to explore a phenomenon, facilitate understanding and interpret empirical findings. Networking, in this context, involves establishing connections between components of theoretical perspectives, while respecting their distinctiveness and integrity (Prediger et al., 2008). There are several strategies of networking which can be categorised on a scale based on the level of integration. In the following, we will discuss the affordances and constraints of the theoretical perspectives adopted for this study in order to justify our choice and demonstrate how they can be effectively networked in our study.

Rationale for the Chosen Theories

Our study on students' learning and understanding of eigenvectors and eigenvalues is underpinned by the theoretical background of radical constructivism, shaping our ontological and epistemological perspectives on the nature of understanding and the process of coming to understand. It is our perspective that radical constructivism aligns with our interest in exploring authentic products of students' work, such as their homework. We consider these to be more accurate representations of their constructed knowledge than materials or situations that are more directly influenced by a present teacher.

However, given that radical constructivism in our view constitutes a grand theory of learning, we recognise that it may not fully address the specific nuances of mathematics education and students' reasoning in this domain. Therefore, to accurately describe and analyse students' comprehension of eigenvectors and eigenvalues, we acknowledge the need for a specialised theory tailored to address their understanding of these mathematical concepts. In this regard, we rely on Tall and Vinner's (1981) notion of the concept image, alongside Sierpinska's (2000) modes of thinking, as the analytical framework guiding our analysis. As we shall argue, we believe that these specialised theories allow us to delve deeper into students' reasoning in linear algebra.

For our purpose of exploring students' understanding of eigenvectors and eigenvalues, we find that Tall and Vinner's notion of the concept image presents as a suitable theoretical lens. It offers a range of indicators, such as the use of examples, formulas and representations, that may effectively capture specific aspects of students' conceptions of mathematical objects. However, it is important to note that these notions were developed for mathematics in general, and not for linear algebra in particular. As a result, we recognise the need to supplement this theory with additional perspectives addressing the intricacies of the field. Thus, we turn to Sierpinska's modes of thinking.

Within linear algebra, students encounter various "languages" and representations, which might be confusing for them (Hillel, 2000). To better understand students' reasoning in this subject, numerous theories and concepts have been proposed, including Tall's three worlds (Tall, 2004) and Hillel's modes of description (2000). However, our pilot study conducted in 2022 (as described in Section 1.2.1) revealed challenges in distinguishing between Hillel's (2000) abstract and algebraic modes of description. In this context, we find that Sierpinska's (2000) modes of thinking, a theory specifically developed for analysing students' reasoning in linear algebra., offer a more precise and well-defined categorisation to capture the nuances of their reasoning. Nevertheless, it is noteworthy that Sierpinska (2000) herself states that students' reasoning tends to fall somewhere between the modes (p. 240). It is our hope and anticipation that our study may contribute to further develop the modes of thinking as an analytical framework and enhance our knowledge of students' understanding in linear algebra.

Compatibility

In our perspective, neither of the presented theories alone is sufficient for effectively describing students' understanding of eigenvectors and eigenvalues. As previously described, each of them contributes to our aim in different ways. Consequently, it becomes necessary to examine the compatibility of their fundamental assumptions regarding ontology and epistemology before attempting to combine them.

As previously explained, the adoption of the radical constructivist theoretical lens implies the assumption that knowledge is constructed. Furthermore, in assuming that knowledge resides in the mind of the individual, radical constructivism is a theory focused on the individual's learning processes, as opposed to collective learning processes. However, it is our perspective that the underlying perspective on learning of Tall and Vinner's (1981) concept image and Sierpinska's (2000) modes of thinking is not explicitly stated. Consequently, we need to interpret and analyse these theories to uncover their underlying ontological and epistemological assumptions. In doing so, we aim to evaluate their compatibility with radical constructivism, which we will do now.

Previously, the concept image was described as a result of the individual accumulating experiences with the concept through their engagement with tasks, instructional approaches and discussions. Additionally, conflict factors were identified as a potential learning opportunity when evoked simultaneously in the mind of the individual. It is our perspective that these descriptions align the concept image with a constructivist perspective on learning. Furthermore, our examination of students' individually submitted homework aligns with the inclusion of the concept image as a theoretical lens.

The formal concept definition, on the other hand, appears to represent a form of objective knowledge. Several studies have compared students' concept images to the concept definition in order to identify deficiencies in their understanding. However, this approach does not align with our aim. Instead, we adopt the radical constructivist perspective, refuting the idea of direct access to an objective knowledge of eigenvectors and eigenvalues. Instead, we compare our interpretation of students' concept images to our own knowledge of these concepts, assuming that ours is somewhat further developed than that of the students. Thus, we argue that we may ensure compatibility of these theories by applying them in this particular manner.

Concerning Sierpinska's modes of thinking, we observe that both the sociocultural and the constructivist approach can be applied in this context. While Sierpinska refers to Vygotsky's sociocultural theory (e.g. Vygotsky, 1987), we do not perceive the modes of thinking to be fundamentally based on this grand theory of learning. Instead, we consider the modes of thinking to describe the cognitive strategies and ideas used by individuals to solve mathematical problems. This aligns with the radical constructivist stance of thinking as a process occurring within individuals' minds. Nevertheless, we acknowledge that the sociocultural learning process may influence the development of the concept image and the modes of thinking, although this is not the focus of our analysis.

Finally, it is our perspective that in combining these theories, we achieve triangulation in theory which is beneficial for several reasons. Each of the chosen theories can provide unique insights to different aspects of students' understanding. Thus, in combining these perspectives, we can gain a more comprehensive understanding of students' reasoning about eigenvectors and eigenvalues. As we have discussed, none of these theories alone are sufficient to describe the complexity of students' understanding. Thus, we argue that combining theories to understand this phenomenon is akin to shining a light in a dark area. While a single light source may only provide a limited understanding, multiple light sources can reveal different aspects of the phenomenon that would otherwise remain concealed in the shadows. Furthermore, triangulation in theories can enhance the validity of the study in reducing bias in the results obtained. However, the matter of validity will be further discussed in a later chapter.

3.5 Aim and Research Questions

In this section, we outline the objectives, research questions, and unit of analysis for the current master study. Afterwards, we provide an overview of the overarching aim of the PhD project, of which the master study constitutes an integral component.

3.5.1 Specific Research Questions and Unit of Analysis in the Master Study

Recall that the aim of this thesis is to describe aspects of students' understanding of eigenvectors and eigenvalues. As described in the introduction, understanding in the context of linear algebra entails the ability to connect concepts, to work with different representations of objects, and to effectively make transfers between these representations. In the theory chapter, we have seen that *understanding* entails having a concept image and that the modes of thinking can characterise different ways of reasoning about a concept. Thus, it is our perspective that both the concept image and the modes of thinking can offer valuable insights in our quest to characterise students' understanding of eigenvectors and eigenvalues.

Specific Research Questions

Thus, in light of our theoretical perspective, we rephrase our overarching research question in terms of the more specific research questions:

1. *What concept images can be described from the students' reasoning about eigenvectors and eigenvalues?*
2. *What modes of thinking can be identified in the students' reasoning about eigenvectors and eigenvalues?*

Unit of Analysis

In our perspective, specifying the unit of analysis can help to clearly identify and delimit the object of study, as well as select appropriate research methods. While there are many definitions of the unit of analysis, we define it as the specific entity that researchers gather data about and analyse in order to understand the phenomena under study. In this particular study, we are interested in describing students' concept images and modes of thinking in the days and weeks following their initial introduction to the concepts of eigenvectors and eigenvalues. Hence, we consider both students' concept images and their modes of thinking, as expressed in their reasoning on our tasks during this time interval, to constitute the unit(s) of analysis in this study.

3.5.2 Overall Aim of the PhD Study

We remind the reader that the insights gained from investigating these questions are intended to contribute to the PhD project. The aim of which is not only to identify students' challenges in understanding algebra but also to develop a set of (homework) tasks addressing these challenges. Thus, the master study is considered to be a very first step in the broader aim of providing practical supporting solutions to support a more robust understanding.

Chapter 4

Methodology, Research Design and Ethics

In this chapter, we explore the methodological implications provided by our ontological and epistemological perspectives. This includes critical reflections of our chosen methods, their affordances and constraints. Afterwards, we provide a detailed account of our research design, including the context of the linear algebra course in which our study is situated, together with a brief description of the participants and their prior knowledge. Subsequently, we will present the tasks that were designed to explore students' concept images and modes of thinking, together with a deeper analysis of potential solution strategies in light of our conceptual framework. We end this chapter by stating the ethical considerations guiding our study and the measures taken to support trustworthiness and authenticity in our research.

4.1 Methodology

Kothari (2004) posits that methodology is the study of methods and their underlying rationale. Additionally, it entails a critical reflection on the limitations of the chosen approach, as well as the rejection of alternative methods. From our perspective, methodology is where theory meets methods.

In this section, we will explicate our approach to answering the research questions, namely by analysing our data using the theoretical lens described in the previous chapter. We shall argue for our chosen methods of collecting written homework and interview data to shed light on students' concept images of eigenvectors and eigenvalues, as well as their modes of thinking.

Rationale for Collecting Written Homework

In our experience, mathematics in university involves several written materials, such as textbooks, assignments, lecture notes and written exams. Several studies (e.g. Wawro et al., 2018, 2019; Donevska-Todorova 2016; Bouhjar et al., 2018) have made use of students' written work (often in combination with other data sources) to examine students' understanding of linear algebra. In our perspective, the widespread use of homework (both at our university and worldwide), together with the fact that homework is often mandatory in order to access the exam, indicates the significant role homework holds in students' learning process.

To come to know about students' concept images of eigenvectors and eigenvalues, it is desirable to get an impression of their written, as well as their oral reasoning. Therefore, it was planned to analyse both written works and conduct interviews with students.

The inclusion of written homework is motivated by several factors. First, homework is a regular component of the students' coursework, making it an authentic source of their work (a further discussion of measures taken to support authenticity will be elaborated in Section 4.3.3). Second, a large number of students complete homework assignments, which allows us to collect a large number of responses. This, in turn, gives a good basis for the selection of students to interview, who are desired to represent as many different concept images as possible. Given our objective to describe students' concept images and modes of thinking, we decided to design specific tasks to address these aspects of students' understanding.

In addition to the immediate research objectives, the more long-term goal of the PhD project is to contribute to the improvement of Linear Algebra teaching practices through the design and scientific evaluation of homework tasks. Homework tasks play a crucial role in students' learning efforts and academic achievements, as evidenced by several studies (e.g. Zhu & Leung, 2012; Fernández-Alonso et al., 2016). The evaluation of students' homework assignments within this master project aligns with and serves as a first step towards the overarching objective of the PhD project.

Rationale for Conducting Interviews

It is our perspective that solely relying on students' written answers presents limitations in capturing the entirety of their understanding. Our position stems from our assumption that students may prioritise providing correct answers over articulating their full thought process in writing. With this in mind, we opted to conduct interviews as they may offer valuable insights into students' concept images, complementing the insights gained from their written responses.

We consider both homework assignments and interviews to be potential learning situations. However, while the oral data obtained through the interviews can illuminate aspects of students' thinking *process*, their written homework can only provide a glimpse of the *outcome* of their reasoning. Hence, the decision to collect homework and conduct interviews may allow us to understand both aspects of students' reasoning, thus contributing to data triangulation.

Our decision to conduct semi-structured individual interviews and audio-record them was based on several methodological considerations. While students have the option to work either individually or collaboratively, they are required to submit their work individually. Given the objective of describing students' concept images (which are inherently individual) we opted for *individual* interviews to capture their unique perspectives and modes of thinking.

Our choice of *semi-structured* interviews was influenced by Sierpinska's (2000) note that students' reasoning tend to fall somewhere between the modes of thinking described in her framework. The interviews provide an opportunity to delve deeper into their modes of thinking and explore which modes are dominant in their arguments. With a predefined list of topics to cover, the semi-structured interviews offer flexibility including spontaneous follow-up questions to uncover further nuances in students' modes of thinking (Corbin & Strauss, 2015, p. 39).

The rationale for exclusively collecting *audio-recordings* of the interviews, rather than video-recordings, stems from a careful consideration of the objectives of our study, and advantages and disadvantages associated with the methods. In a pilot study conducted in the 2022, it was observed that students' often referred to specific parts of their written work using terms like "these" and "that" during the interviews. This observation led to the exploration of video-recordings as a potential remedy for this issue.

Although video-recordings may capture gestures and non-verbal cues, which may be valuable in some research contexts, our research questions did not necessitate such an analysis. Moreover, video-recordings come with certain drawbacks. For instance, Moschkovich (2019) cautions that individuals might be less willing to participate when being filmed. Additionally, video-recordrings may cause students to be more self-conscious and less at ease to engage in discussions. Conversely, it was hoped that audio-recordings could contribute to a more relaxed and candid conversation, enhancing the quality of the data collected.

4.2 Research Design

In this section, we present an overview of the setting and participants of our study, describing the learning activities and our assumptions regarding students' prior knowledge. Next, we describe our means of data collection and participant recruitment. Subsequently, we present the tasks together with an initial analysis of anticipated responses and their corresponding modes of thinking. We conclude this section detailing how the students' written responses were analysed using a thematic coding approach and how the interviews supported this analysis.

4.2.1 Setting and Participants

This study was situated in a first linear algebra course during the fall of 2022 at a Norwegian university. About 700 students from various engineering study programs were enrolled in the course, most of which were in their second year of study and aged in their early 20s.

Learning Activities

The instructional approach of the course employed a flipped classroom style, where students were expected to watch a series of short lecture videos prior to attending weekly interactive lectures with a duration of 2x45 minutes.In the lecture videos, key concepts, theorems and examples were demonstrated as an introduction to the topic, providing students with a foundation before the interactive sessions on campus. The students were also provided lecture notes and links to YouTube-videos, primarily from the channel *3blue1brown*, where key concepts and procedures were explained and visualised.

During the interactive lectures, the interactive learning platform *Mentimeter* was employed to actively engage students in the learning process. Throughout the sessions, students were given a few minutes to independently or collaboratively attempt tasks relevant to the topic. Subsequently, the instructor demonstrated effective solution strategies and highlighted common errors on the blackboard.

Additionally, there were weekly exercise lectures where an instructor (myself) demonstrated potential solution strategies for basic tasks and exam problems on the blackboard. I personally selected the tasks for these sessions, but students were given the opportunity to request specific tasks in advance. Moreover, students were encouraged to prepare for the lectures by watching the lecture videos and/or reading the lecture notes beforehand. While the students were given the opportunity to review the tasks prior to the lecture, they were not obligated to do so.

Assessment

The students' final grade in the course was determined by a written exam with a time limit of four hours administered at the end of the semester. To be eligible for the exam, students were required to complete and submit a minimum of four out of six written homework assignments. Apart from this requirement, the students' performance on the homework did not affect their grade in the course.

For assistance or collaboration, students had the opportunity to attend exercise classes, called *Mattelab*, multiple times per week, where they could engage in discussions with their peers and/or students regarding the homework (or other academic questions). In addition, students were provided a digital platform where they could engage in similar discussions.

Prior Knowledge

This study was conducted in the weeks following the introduction of the concepts of eigenvectors and eigenvalues to the students. The students were required to hand in their homework the same week as they were first introduced to the concepts of eigenvectors and eigenvalues, and the interviews were conducted in the weeks following this.

Through the interviews and communication with the students, it was evident that not all students engaged with every learning material available to them. Instead, many chose to engage with parts of it, although we lack a comprehensive overview of which resources each student utilised. Nevertheless, it was anticipated that by the time of this study, most students had been exposed to the definition of eigenvectors and eigenvalues (See Definition 4.2.1 by NTNU (2021), our translation) and the computational procedure for determining eigenvalues and eigenvectors, as well as some visual interpretations of them in $\mathbb{R}^2$, as these were presented in both the video lectures, lecture notes and YouTube-videos.

Definition 4.2.1 (Eigenvector and Eigenvalue) *Let $T : V \to V$ be a linear transformation. A scalar λ is called an **eigenvalue** of T if there exists a vector $\vec{v} \neq \vec{0}$ in V such that*

$$T(\vec{v}) = \lambda\vec{v}$$

*The vector v is called an **eigenvector** of T corresponding to the eigenvalue λ. When T is given as an $n \times n$ matrix A, λ is called an eigenvalue of A and v is an eigenvector of A corresponding to the eigenvalue λ.*

4.2.2 Data Collection

To come to know about the students' concept images of eigenvectors and eigenvalues, and their modes of thinking, four tasks were designed and implemented as part of the students' written homework. In addition, interviews with five students were conducted by the author of this thesis, further exploring their reasoning in completing the tasks and their overall experiences with the course, approximately 3-5 weeks after submitting their homework.

Recruitment

Out of the 700 students enrolled in the course, consent from 170 students to collect their written homework could be obtained. To recruit participants, the author of this thesis attended the interactive lectures, informing the students about the purpose and scope of the study. Their rights as participants were emphasised, as well as the option to decline participation. The information provided to students was given in writing (refer to Appendix A) and a short summary of its content was presented orally to the class. Then, students were asked to fill out a consent form (See Appendix B), indicating their willingness to participate. While all the students submitted their homework through the digital platform *Ovsys2*, only the assignments of the students who provided their consent were collected for analysis. A more comprehensive description of the ethical considerations pertaining to recruitment, data collection, analysis and other aspects of this study will be presented in Section 4.3.

Interviews

Based on a preliminary analysis of their written homework, six students were identified for interviews. To motivate students to take part in the interviews, students were approached individually via email, communicating why their participation is both interesting and important. The selection aimed to include students exhibiting different modes of thinking and concept images at various stages of their development. Specifically, students who gave brief answers offering limited information about their concept images were selected, as well as students who presented comprehensive descriptions of eigenvectors, indicating several modes of description and rich concept images.

Subsequently, the chosen students were contacted via email, providing the students with information regarding the purpose and duration of the interviews (refer to Appendix C). Additionally, they were informed in regards to the handling of their personal data to ensure transparency. Out of the six students who were contacted, five willingly agreed to participate in the interviews. In preparation for the interviews, semi-structured interview guides were created for each participant (see Appendix D for an example). These guides were designed to guide the exploration of students' perspectives.

The interviews were conducted individually, with each session lasting up to 45 minutes, in designated study rooms on campus to ensure a suitable environment for discussions. To ensure accuracy in capturing students' responses and uncover nuances in their modes of thinking, the interviews were audio-recorded. During the interviews, students were provided with a written copy of the tasks and their answers. Additionally, they were equipped with pen and paper, allowing the students to express their reasoning in written explanations and/or visual representations, in case they found it helpful.

The structure of the interviews adhered to the approach suggested by Robson and McCartan (2016, p. 290). It commenced with an introduction of the interviewer (myself) and a clarification of the purpose of the interviews. To establish a trustworthy and confidential environment, students were reminded of their anonymity and right to withdraw their consent at any moment. Following the introduction, a warm-up phase was initiated, allowing participants to talk about their interests, study programs and learning strategies. This phase aimed to establish a comfortable atmosphere, allowing both parties to ease into the situation. Subsequently, the students' were asked to explain their reasoning concerning the four tasks that were designed in the preparation of the study, aiming to better understand their concept images and modes of thinking. Towards the end of the interview, a cool-down phase with easier questions was implemented to relieve any potential tension or stress which may have accumulated during the discussions. The interviews were concluded by expressing gratitude to the participants for their cooperation and for making the time to participate in the study. The participants were compensated for their efforts with a small, edible gift. Moreover, an additional 30 minutes was allotted for each participant, offering an opportunity to discuss any academic matters, ask questions or receive assistance with their homework. This additional time was not audio-recorded and is not considered in this study.

4.2.3 The Tasks

To gain insights into students' concept images and modes of thinking, four tasks (numbered 9, 10, 11 and 12) were designed and implemented as part of the students' written homework. For this master's study, we restrict our analysis to Tasks 9 and 10. These tasks were selected because they align with the aim and research questions of our study.

4.2.3.1 Task 9

The first task, referred to as Task 9, required students to provide their own explanation of the concepts of eigenvectors and eigenvalues. The exact phrasing of Task 9 can be found in Figure 4.1, which presents our translated version from Norwegian to English.

Task 9: Explain in your own words:

a) What is an eigenvector?
b) What is an eigenvalue?

You may use sketches to illustrate.

Figure 4.1: In Task 9, students were asked to explain eigenvectors and eigenvalues in their own terms, and encouraged to provide a sketch.

The open-ended structure of this task allows students the freedom to prioritise specific aspects of eigenvectors and eigenvalues, directly addressing our research questions concerning students' concept images and modes of thinking. Furthermore, students could determine the extent of their answers, allowing us to explore the breadth of their concept images. As we will later elaborate, a more comprehensive response may indicate a more advanced concept image, yet a briefer answer does not necessarily imply a limited concept image.

Our Interpretation of Sierpinska's Modes of Thinking

In order to effectively characterise students' modes of thinking, it is essential to first understand what the analytic-structural, analytic-arithmetic and synthetic-geometric modes entail in the context of eigenvectors and eigenvalues. However, it is important to note that since Sierpinska does not provide a comprehensive explanation of this, we had to develop our own interpretation, which we elucidate in the following.

Now, recall that in the *analytic-structural mode*, an object is described by the properties defining it. Hence, an analytic-structural description of eigenvectors captures aspects that hold true for all eigenvectors, not just particular examples. Since eigenvectors and eigenvalues are defined by the corresponding matrix (or linear transformation), an analytic-structural description of an eigenvector can include the relationships that exist between the eigenvector, eigenvalue and matrix/linear transformation. Expanding upon Sierpinska's (2000) description, we recognise an additional intrinsic characteristic common to all eigenvectors, their ability to maintain their original span when imaged by a linear transformation. A sketch illustrating how a general linear transformation would affect an eigenvector could also be considered to align with this mode of thinking.

Furthermore, in the *analytic-arithmetic mode*, an object is defined by the formula allowing its computation. Applied to eigenvectors and eigenvalues, this implies that eigenvectors can be characterised as the solutions of the homogeneous equation $(A-\lambda I)\vec{x} = \vec{0}$, where A represents the matrix, λ represents the eigenvalue, I represents the identity matrix and x represents the eigenvector. Similarly, eigenvalues can be described as the roots of the characteristic polynomial, that is, the solutions λ of $\det(A-\lambda I) = 0$.

Finally, in the *synthetic-geometric mode*, objects were described based on their geometric or visual properties[1]. In the case of eigenvectors, this may involve visual descriptions of their representations or geometric characterisations of their behaviour under a matrix multiplication or linear transformation. For example, eigenvectors may be characterised as remaining on the same line or preserving their direction. For eigenvalues, a synthetic-geometric description could involve characterising the eigenvalue as the factor by which an eigenvector is stretched or compressed. Furthermore, sketches illustrating how a specific eigenvector is affected by the linear transformation it corresponds to align with a synthetic-geometric mode of thinking. From our understanding, descriptions in the synthetic-geometric mode describe certain characteristic properties of some eigenvectors and eigenvalues. These descriptions do not not define them because they are not true for all eigenvectors and eigenvalues.

[1]We understand visual properties as referring to observable characteristics that can be graphically represented, like length and direction. Geometric properties, on the other hand, encompass mathematical aspects like orthogonality, parallelism, and angles. While visual and geometric properties do exhibit a degree of overlap, it is important to note that our study does not centre on delineating this distinction.

In Table 4.1, we present a short summary of the characteristics of the modes of thinking in the context of eigenvectors and eigenvalues.

Table 4.1: An overview of our interpretation of how Sierpinska's (2000) modes of thinking could present in Task 9.

Mode of Thinking	**Description**
Analytic-structural	Description based on the defining properties of eigenvectors and eigenvalues
Analytic-arithmetic	Description based on the equations for computing eigenvectors ($(A-\lambda I)\vec{x} = \vec{0}$) and eigenvalues ($\det(A-\lambda I) = 0$)
Synthetic-geometric	Description based on geometric or visual properties of eigenvectors and eigenvalues, as well as sketches

Students were also encouraged to produce a sketch accompanying their written answers. In encouraging students to provide sketches alongside their written explanations, we hoped to capture any visual interpretations they may possess of eigenvectors and eigenvalues. As we shall see, only a portion of the students provided sketches. However, there were interesting variations within the sketches.

4.2.3.2 Task 10

In Task 10, students were given the vector $\begin{bmatrix}-1 & 2\end{bmatrix}^\intercal$, and were asked to justify whether or not it was an eigenvector to the given matrix $A = \begin{bmatrix}1 & 1\\ -2 & -2\end{bmatrix}$. The exact phrasing of this task is given in Figure 4.2 (our translation from Norwegian to English).

Task 10: Justify why $\vec{x} = \begin{bmatrix}-1\\ 2\end{bmatrix}$ is or is not an eigenvector of the matrix $A = \begin{bmatrix}1 & 1\\ -2 & -2\end{bmatrix}$.

Figure 4.2: In Task 10, students were asked to justify whether a specific vector $\vec{x}$ was or was not an eigenvector of a given matrix A.

The purpose of this task was to identify the modes of thinking employed by students to support their decision regarding the vector's status as an eigenvector of A. This task was selected due to its capacity to encompass multiple solution strategies corresponding to our interpretation of Sierpinska's (2000) modes of thinking in the context of eigenvectors and eigenvalues. In the previous section, we discussed our interpretation of these modes in the context of Task 9. Now, we extend our interpretation to Task 10.

In the context of Task 10, an *analytic-structural* argument would make use of the defining properties of eigenvectors to verify that $\vec{x}$ is in fact an eigenvector of A. As described in the previous section, one defining property of eigenvectors is fulfilling the eigenequation. Hence, an analytic-structural mode of thinking in Task 10 could present as a verification that the eigenequation is fulfilled for the given vector $\vec{x}$ and matrix A.

Furthermore, an *analytic-arithmetic* argument would make use of the formulas for computing eigenvectors and eigenvalues to determine $\vec{x}$'s status as an eigenvector of A. Thus, employing the procedures to compute the eigenvalues of A and the corresponding eigenvectors, and verifying that the given vector $\vec{x}$ is one of them represents an analytic-arithmetic mode of thinking in Task 10.

Finally, a *synthetic-geometric* mode of thinking here would involve an argument using the geometric or visual properties of eigenvectors. Accordingly, this could present as computing the matrix product $A\vec{x}$ and observing that the resulting vector is a scalar multiple or lying on the same line as the given vector $\vec{x}$. Additionally, a sketch illustrating the visual or geometric properties of $\vec{x}$ as an eigenvector of A would also align with a our interpretation of a synthetic-geometric mode of thinking. This could entail employing a coordinate system to portray the vectors $\vec{x}$ and as arrows that lie on the same line or vectors that are scalar multiples of each other. The reader is referred to Figure 4.3 for an example of a potential solution that includes such a sketch.

For the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$ and the vector $\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Compute $A\vec{x} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

As the vectors $\vec{x}$ and $A\vec{x}$ lie on the same line (see illustration below), and $A\vec{x}$=-$\vec{x}$. Thus, $\vec{x}$ is an eigenvector of A with corresponding eigenvalue $\lambda = -1$.

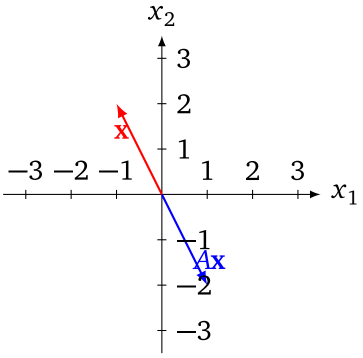


Figure 4.3: Our proposed solution to Task 10, exemplifying a synthetic-geometric mode of thinking. This approach features the computation of the matrix-vector product $A\vec{x}$ and an accompanying sketch that demonstrates the collinearity of vectors $\vec{x}$ and $A\vec{x}$.

In Table 4.2, we present a short overview of the arguments identified in our preliminary analysis of Task 10, alongside their corresponding modes of thinking.

Table 4.2: An overview of our interpretation of how Sierpinska's (2000) modes of thinking could present in Task 10.

Mode of Thinking	Argument
Analytic-structural	Checking if A and $\vec{x}$ fulfil the eigenequation for some scalar λ
Analytic-arithmetic	Computing eigenvalues and eigenvectors and checking that $\vec{x}$ is indeed one of them
Synthetic-geometric	Geometric or visual description of $\vec{x}$ and $A\vec{x}$ lying on the same line (or exhibiting a similar relationship) or a drawing illustrating this

4.2.3.3 Tasks 11 and 12

The third task, Task 11, asked students to reason about a set of statements concerning eigenvectors and eigenvalues. Finally, in Task 12, students were tasked with analysing several visual representations of vectors and the resulting vector after a matrix multiplication in $\mathbb{R}^2$. Based on these representations, students were asked to determine whether these were eigenvectors of the matrix, and if so, provide an estimate for the corresponding eigenvalues. For the precise phrasing of all our tasks, the reader is referred to Appendix E. The students' answers to these tasks will be addressed in a future study.

4.2.4 Data Analysis

To allow an in-depth analysis of students' concept images and modes of thinking, we engaged in thematic coding, as outlined by Braun and Clarke (2006), of the students' written answers to Tasks 9 and 10. The coding was conducted separately for each task, with the interviews providing further insights and context to support the written answers. In our analysis, we did not make a distinction between correct and incorrect answers. The interview recordings and transcripts, while not subjected to the same thematic coding, played a supplementary role, providing additional perspectives and enriching the insights gained from the written data.

4.2.4.1 Analysis of Task 9

In Task 9, our coding approach was inspired by Wawro et al. (2019) and involved two levels of coding. The codes developed in this first level were descriptive codes as they were based on words or short phrases used by the students, aimed at capturing the essence of their ideas.

First Level

In the first level, we employed a cyclical process to label all students' answers, creating a coding handbook containing all codes, their descriptions and examples of their use on the way. These first level codes were in vivo codes, based on the words or short phrases used by the students (Miles et al., 2014, p. 74).

Given the considerable number of 16 first level codes employed in our analysis of Task 9, it is not feasible to explain each one in detail here. Instead we present two examples of such codes. For instance, the code "9-transformation" was assigned to answers to Task 9 relating eigenvectors to the notion of transformation. Similarly, responses describing eigenvectors as preserving their direction were coded "9-direction". The reader is referred to Appendix F for the full coding handbook. It should be noted that the majority of the students gave their answers in Norwegian. During our analysis, we made the decision to translate their responses to English and generate codes based on this translation.

Second Level

In the second level of coding, we created codes corresponding to the modes of thinking and categorised the first level codes accordingly.

- Thus, first level codes corresponding to defining properties of eigenvectors, such as the preservation of the span, were assigned to the *analytic-structural* mode of thinking. Similarly, answers making use of key concepts closely related to these properties, such as "image" and "vector space", were also coded and assigned to this mode.
- First level codes assigned to answers making use of the formulas for computing eigenvalues and eigenvectors were classified as *analytic-arithmetic*.
- Finally, codes describing geometric or visual properties of eigenvectors and eigenvalues were grouped in the *synthetic-geometric mode of thinking*. These answers described eigenvectors in terms of geometric or visual properties like "being scaled" or "not being rotated". As we shall see later, while several students gave such descriptions, they were actually not able to capture the geometric properties of all eigenvectors.

The previously described coding handbook in Appendix F includes these second level codes as well.

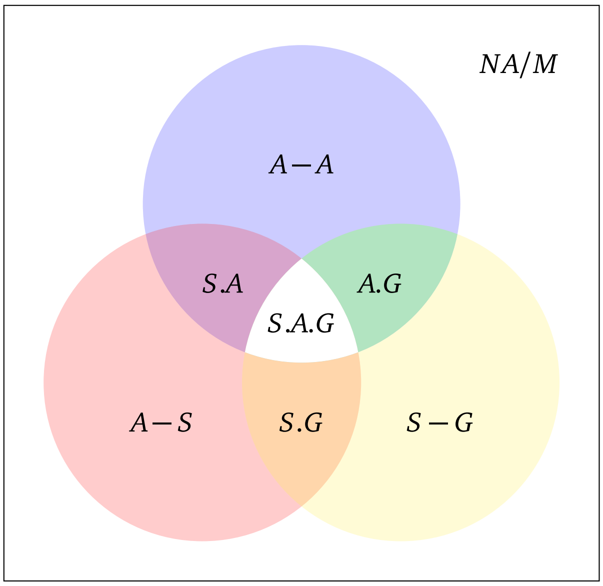
Multiple Modes of Thinking

We quickly came to realise that many of the students' answers aligned with multiple modes of thinking. To accommodate this complexity, we created four additional categories representing these combined or in-between modes. To simplify categorisation, we established the following codes:

- Answers aligning with both the analytic-structural and analytic-arithmetic modes were categorised as *structural-arithmetic*. This mode was characterised by describing eigenvectors and eigenvalues as fulfilling the eigenequation, $A\vec{x} = \lambda\vec{x}$. It is our perspective that this equation represents a defining property of eigenvectors, thus aligning with an analytic-structural mode of thinking. However, the equations for computing eigenvectors ($(A-\lambda I)\vec{x} = \vec{0}$) and eigenvalues ($\det(A-\lambda I) = 0$) can fairly easily be derived from the eigenequation through arithmetic procedures. Thus, we argue that describing eigenvectors in terms of the eigenequation can also align with the analytic-arithmetic mode of thinking. Consequently, answers employing the eigenequation would fall between these two modes and we handled this complexity by categorising them as structural-arithmetic.
- Answers combining analytic-structural and synthetic-geometric modes were categorised as *structural-geometric*. This combined mode was characterised by describing eigenvectors and eigenvalues from their defining properties, as well as geometric or visual properties. Recall that defining properties are general, while the geometric or visual properties are necessarily more specific (e.g., not all eigenvectors are stretched by the matrix/linear transformation it corresponds to).
- Answers combining analytic-arithmetic and synthetic-geometric modes were categorised as *arithmetic-geometric*. This combined mode was characterised by describing eigenvectors and eigenvalues in relation to the formulas for computing them ($(A-\lambda I)\vec{x} = \vec{0}$ and $\det(A-\lambda I) = 0$), and their geometric or visual properties.
- Finally, answers incorporating elements from all three modes of thinking were categorised as structural-arithmetic-geometric.

Figure 4.4 provides an overview of the modes of thinking identified in Task 9, illustrating the "pure" modes, as well as their combinations. The "pure" modes, analytic-structural, analytic-arithmetic and synthetic-geometric are represented in red, blue and yellow, respectively. The overlapping regions, displayed in purple, orange, green, and white, depict the merged modes of thinking. The purple area represents the structural-arithmetic mode, where elements of both analytic-structural and analytic-arithmetic thinking converge. The orange area represents the structural-geometric mode, combining elements of the analytic-structural and synthetic-geometric mode of thinking. The green area represents the arithmetic-geometric mode, encompassing features of the analytic-arithmetic and synthetic-geometric modes. Finally, the white area in the middle represents the structural-arithmetic-geometric mode, which involves the integration of all three pure modes.

In our naming of the combined modes, we omitted the explicit distinction between analytic and synthetic modes of thinking. This is because the distinction is inherent in the differentiation between structural and arithmetic modes on the one hand, and the geometric mode on the other hand. The reader is referred to Figure 3.3 for a reminder of the levels in Sierpinska's (2000) modes of thinking.



Abbreviation	Explanation
$A-S$	Analytic-structural
$A-A$	Analytic-arithmetic
$S-G$	Synthetic-geometric
$S.A$	Structural-arithmetic
$A.G$	Structural-geometric
$A.G$	Arithmetic-geometric
$S.A.G$	Structural-arithmetic-geometric
NA/M	Not answered/ Miscellaneous

Figure 4.4: Overview with the modes of thinking identified in Task 9, illustrating both Sierpinska's (2000) three modes, and their combinations.

4.2.4.2 Analysis of Task 10

Due to the inherent differences in the nature of Tasks 9 and 10, a slightly different approach, consisting of just one level, was employed to code the students' answers to Task 10. As outlined in Section 4.2.3.2, our initial analysis of Task 10 identified three main solution strategies aligning with Sierpinska's (2000) modes of thinking:

- Answers that verified the fulfilment of the eigenequation ($A\vec{x} = \lambda\vec{x}$) were coded as indicative of an *analytic-structural* mode of thinking.
- Conversely, answers that computed the eigenvalues and corresponding eigenvectors were coded as indicative of an *analytic-arithmetic* mode of thinking.
- Finally, answers that drew upon geometric or visual descriptions of eigenvectors to evaluate the vector $\vec{x}$'s status as an eigenvector of A were coded as representing a synthetic-geometric mode of thinking. As previously mentioned, a sketch illustrating that the vectors $\vec{x}$ and $A\vec{x}$ are scalar multiples of each other would also align with a *synthetic-geometric* mode.

We would like to highlight a distinction in the interpretation of modes of thinking between answers utilising the eigenequation in Task 9 and Task 10, arising from the inherent differences in the nature of these tasks. In Task 9, answers making use of the eigenequation ($A\vec{x} = \lambda\vec{x}$ or $T(\vec{x}) = \lambda\vec{x}$) were interpreted as aligning with a structural-arithmetic mode of thinking, as fulfilling this equation can be seen as both a defining property and a formula for computing eigenvectors and eigenvalues. However, in Task 10, answers employing the eigenequation to check the vector $\vec{x}$'s status as an eigenvector were considered to employ an analytic-structural mode of thinking.

It is our perspective that Task 10 was more "closed", implicitly offering students three distinct paths to follow: verifying the fulfilment of the eigenequation, computing the eigenvectors, or reasoning about the geometric or visual properties of eigenvectors. However, as we shall see later, it is likely that this task primarily emphasised the first two strategies and their corresponding modes of thinking. In contrast, Task 9 was open-ended and did not obviously gear students towards any particular modes of thinking in the same manner. Thus, in the context of Task 9 it was important to acknowledge that describing eigenvectors as fulfilling the eigenequation could align with both the analytic-structural and analytic-arithmetic modes of thinking. However, to accurately capture the nuances in students' arguments in Task 10, it was necessary to classify arguments employing the eigenequation as belonging to the analytic-structural mode. This distinction allowed us to capture the difference between *verifying* the eigenequation and *constructing* the eigenvectors by applying the procedure for computing them. While we acknowledge that these reflections might be more suitably discussed later, we felt it was important to clarify any potential confusion at this early stage. However, we will come back to this matter in Chapter 6.

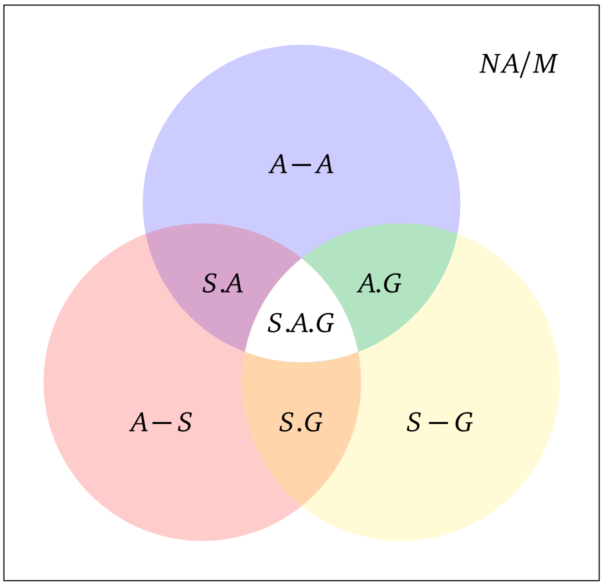
Multiple Modes of Thinking

During the analysis of students' responses to Task 10, instances were observed where combinations of our pre-defined solution strategies were employed, indicating the utilisation of multiple modes of thinking. To account for these observations, similar to the approach used in the second level of coding in Task 9, the following codes were introduced: structural-arithmetic, structural-geometric, arithmetic-geometric, and structural-arithmetic-geometric.

- The code *structural-arithmetic* was assigned to responses that utilised the arithmetic procedure to compute eigenvalues (by solving the equation $\det(A - \lambda I) = 0$) and verified that the eigenequation was fulfilled for one of the computed eigenvalues, the given vector $\vec{x}$, and the matrix A.
- The code *structural-geometric* was given to responses that verified the fulfilment of the eigenequation for the given vector $\vec{x}$, the matrix A, and some scalar λ, and incorporated arguments referring to geometric properties of eigenvectors.
- The code *arithmetic-geometric* was applied to answers employing the arithmetic procedure for computing eigenvalues and/or eigenvectors, while also referring to their geometric properties.
- Finally, the code *structural-arithmetic-geometric* was utilised for responses

that combined the eigenequation, the computational procedure and the geometric properties of eigenvectors and eigenvalues.

Similar to Task 9, the pure and combined modes in Task 10 are represented using a Venn diagram, employing the same color system for consistency and visual clarity (see Figure 4.5).



Abbreviation	Explanation
$A-S$	Analytic-structural
$A-A$	Analytic-arithmetic
$S-G$	Synthetic-geometric
$S.A$	Structural-arithmetic
$A.G$	Structural-geometric
$A.G$	Arithmetic-geometric
$S.A.G$	Structural-arithmetic-geometric
NA/M	Not answered/ Miscellaneous

Figure 4.5: Overview of the modes of thinking identified in Task 10, illustrating both Sierpinska's (2000) three modes, and their combinations.

4.2.4.3 The Objects and their Relations: An Aspect of the Concept Image

During the analysis, various aspects of the concept image were encountered that were not captured by the modes of thinking. To address one of these aspects (which was recurrent in the data material), an inductive approach was adopted, and the notion of "the objects and their relations" was developed. In the following, a comprehensive outline of this aspect of the concept image is provided.

The definition of eigenvectors encompasses at least three essential mathematical objects[2], namely the eigenvector, eigenvalue and matrix or linear transformation, depending on the definition employed. Recall that the definition in the students' curriculum (refer to Definition 4.2.1), defines eigenvectors and eigenvalues in relation to both the linear transformation and the eigenvalue.

[2]By mathematical object, we mean an abstract entity or concept that is defined within mathematics. Examples of such mathematical objects are numbers, functions, vector spaces etc.

These mathematical objects are all interconnected:

- First, the *matrix* and the *linear transformation* share a close **relationship**. A matrix is a collection of elements (numbers, vectors, functions or any other mathematical objects that satisfy the properties of matrix operations), and can be represented as a rectangular array. In the context of eigentheory, the linear transformation is a mathematical operation fulfilling specific properties and mapping elements from one vector space to the same vector space. Multiplying an element with the matrix yields the same result as applying the linear transformation, making the matrix and the linear transformation related yet distinct mathematical objects.
- Second, an *eigenvalue* is a scalar value **associated** with a square matrix (or a linear transformation) which represents a specific property of said matrix or linear transformation.
- Finally, an *eigenvector* is a non-zero vector **corresponding** to an eigenvalue (and consequently also to the matrix/linear transformation).

Thus, the matrix (or linear transformation) defines an operation or a transformation, while the eigenvalues and eigenvectors serve as properties of the matrix (or linear transformation), describing specific characteristics of the linear transformation. These objects and their relationships are illustrated in Figure 4.6.

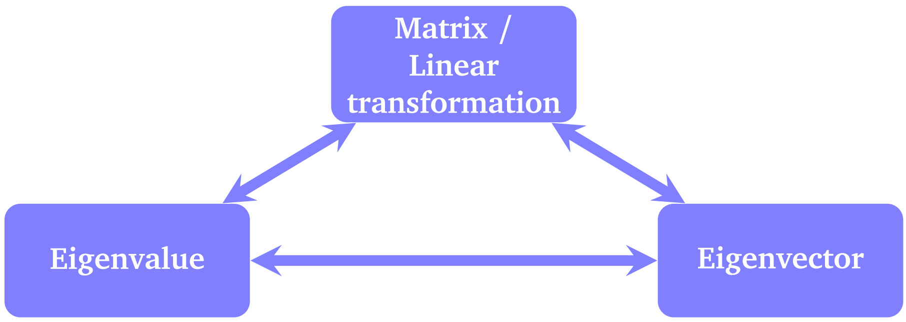


Figure 4.6: Illustration of the relations that exist between the matrix or linear transformation, eigenvalue and eigenvector.

In the upcoming chapter on results and analysis, we explore examples of students' responses where objects were omitted. Additionally, we examine instances where the roles of these objects appear to be mixed up. By examining these examples, we aim to gain deeper insights into the challenges students face when dealing with these objects and the relations that exist between them.

4.2.4.4 Analysis of Interviews

Our analysis of the interviews began by listening through the complete recordings. Next, we transcribed the relevant portions pertaining to Tasks 9 and 10 using the built-in transcription feature in Word Web App, from Microsoft. Following this, we compared the recordings and corrected inaccuracies. The data obtained from these interviews provided valuable support for our analysis of the students' written answers, enabling us to uncover nuances in their modes of thinking and identify additional elements of their concept images. It is important to emphasise that although the interviews were conducted in Norwegian, the mother tongue of both the participants and interviewer, the excerpts included in the analysis have been translated to English (our translation).

4.3 Ethical Considerations

In this subchapter, we reflect upon what ethics in research entails, expand upon our ethical considerations and outline the accompanying measures implemented in this study to address them. Following this, we delve into the significance of trustworthiness and authenticity as alternative criteria for assessing the quality of qualitative research, distinguishing them from the commonly associated criterions of reliability and validity in quantitative research. Additionally, we elucidate specific measures undertaken to foster trustworthiness and authenticity in the context of our study.

4.3.1 Ethics

As explained by Denzin and Lincoln (2000), ethics in research encompasses a careful and comprehensive consideration of the research objectives, the treatment of participants, as well as the handling and presentation of data. According to The National Committee for Research Ethics in the Social Sciences and Humanities, also known as NESH, all researchers are responsible for upholding current ethical standards in their studies (NESH, 2021, p. 8). These ethical norms encompass various aspects, such as ensuring that the methods are transparent and verifiable and presenting findings in a fair and truthful manner, and that the participants otherwise are treated fairly and respectfully.

NSD

In compliance with NTNU's ethical guidelines, this study has been duly reported to NSD (the Norwegian Centre for Research Data), the authority responsible for managing research data involving people and society. Prior to commencing recruitment and data collection, the study underwent an evaluation by NSD to ensure its adherence to their ethical principles. These include obtaining informed consent, maintaining confidentiality and our duty of secrecy, and in any preventing harm to participants. In the following, we outline the specific measures implemented to promote compliance with the current guidelines as evaluated and approved by NSD.

Informed Consent

To attend to the principle of *informed consent*, all students received comprehensive information about the study in both written and oral form. This information included details about the purpose and scope of the study, their anonymity and their rights regarding access, correction or deletion of personal data. Students who were willing to participate were asked to fill out a consent form, thereby signifying their consent to take part in our study.

Confidentiality

Confidentiality in research entails that information about the participants is accessible only to authorised individuals. The purpose of confidentiality is to prevent misuse of personal data (Fossheim & Ingierd, 2013). In line with our notification form for NSD, we have implemented measures to ensure confidentiality throughout the study. To safeguard data, we have employed password-protected cloud storage enforced with two factor authentication and approved third-party data processors. Access to sensitive material, such as students' complete homework, the names and contact information of participants, and interview recordings and transcripts, has been limited to the author of this thesis and the responsible supervisor.

Participants have been duly informed, both orally and in writing, about their rights to access, correct or delete any personal data about themselves. Additionally, they have been informed that anonymised information, such as excerpts from their homework and interview transcripts, may be used in this master study and upcoming papers. In compliance with the notification form submitted to NSD, we are committed to delete any personal data, that is, details which may identify individuals, by the conclusion of the PhD project (expected autumn 2026).

Duty of Secrecy

Another important aspect of ethics in research which is closely related to confidentiality, is the duty of secrecy. As outlined by Fossheim and Ingierd (2013), the *duty of secrecy* imposes a responsibility upon the researcher(s) to preserve secrecy regarding certain information provided by participants. Aligning with the notification form submitted to NSD, we have implemented measures to uphold this duty.

During the interviews with students, we came across information extending beyond the scope of our research questions, such as students' opinions regarding organisational matters related to the course. Regardless of the nature of these opinions, we have honoured our duty of secrecy by reporting solely on information relevant to our objectives.

In the presentation of our findings, we have taken diligent steps to promote the anonymity of our participants. This includes restraining from divulging information we deem to potentially being personally identifiable, such as names, gender or study programs of individual participants. Thus, the names used in the analysis are gender-neutral aliases and not the real names of the participants.

Harm to Participants

The duty of secrecy further involves the researcher's responsibility to *avoid harm to participants* (Fossheim & Ingierd, 2013). To fulfil this obligation, our duty has been meticulously coordinated with course lecturers, ensuring that our tasks indeed align with the curriculum and do not disrupt organisation matters within the course. Furthermore, measures have been taken to ensure that the participants of our study do not gain unfair benefits over those who chose not to participate. For example, it is important to note that all students were required to complete the homework, regardless of their participation in our study. Therefore, those who agreed to have their homework collected for the purposes of our study did not gain any unfair advantage (or disadvantage). While the interviews may be viewed as a potential learning situation, we argue that similar opportunities were available to students through activities, such as *Mattelab*. Moreover, it is our perspective that any small edible gifts offered to students after participating in the interviews are insignificant within the broader context of this study.

Data Minimisation

In research ethics, the principle of data minimisation plays an important aspect of ethical research practices. As stated by the European Commission (2021, p. 11), this principle emphasises that all data collected has to be relevant and limited to what is strictly necessary to answer the research questions. To accommodate this, a careful analysis was conducted prior to data collection to determine which data was necessary to meet the aims of our study. The decision to collect written homework and supplement with interviews is outlined in the methodology section.

4.3.2 Trustworthiness

According to Lincoln and Guba (1985), trustworthiness in research concerns how the researcher can demonstrate that the findings of the study are worthy of the attention of the audience. To assess the trustworthiness of qualitative research, Lincoln (1995) poses four criteria: credibility, transferability, dependability and confirmability. In the following, we shall delve deeper into what these criteria entail and the steps taken to support their fulfilment in our research.

4.3.2.1 Credibility

Credibility, which aligns with internal validity in quantitative research, concerns the internal consistency of the research. That is, how we can ensure and convince others that our research is conducted with rigour (Gasson, 2004, p. 95). An important approach we have adopted is triangulation in data, wherein we have collected both written homework and conducted interviews with students, thereby enforcing credibility of our research (Mok & Clarke, 2015; Lincoln & Guba, 1985).

4.3.2.2 Transferability

Transferability parallels external validity in quantitative research and concerns the extent to which the findings of the study can be applied in other contexts (Morrow, 2005, p. 252). In the context of qualitative research, Geertz (1973) argues that transferability can be supported by a so-called "thick description", that is, rich and detailed information about the context of the study and how it was conducted. To enhance transferability of our research, we have provided a comprehensive account of the setting and participants of our study in Section 4.2.1. This includes detailed information on how the course was organised, students' prior knowledge and learning materials. In providing this information, we aim to facilitate a better understanding of the context, enabling readers to assess the applicability of our findings in their own settings.

4.3.2.3 Dependability

Dependability corresponds to reliability in quantitative research, and concerns the consistency of methods, data analysis and findings, across different times, researchers and techniques of analysis (Gasson, 2004). While our study adopts an interpretative perspective, acknowledging that our findings are inherently shaped by our values, expectations and interpretations of the data, we argue that dependability remains an important aspect.

However, according to Robson and McCartan (2016, p. 470), thematic analysis has often been subject to criticism for lacking transparency in terms of how the analysis was conducted. To meet these concerns and contribute to an audit trail, we developed a detailed coding handbook with descriptions and illustrative examples of thor use. This documentation serves to promote transparency and in our analysis and enhance transferability (Gasson, 2004).

4.3.2.4 Confirmability

Confirmability, as proposed by Lincoln and Guba (1985), serves as an alternative criterion to objectivity in quantitative research. According to Gasson (2004, p. 93), confirmability in research entails that the findings should accurately represent the situation under research, rather than the researcher's beliefs and biases. By framing our research within an interpretivist paradigm, we acknowledge the inherent influence of our presumptions and values upon our findings. However, Morrow (2005, p. 252) claims that the integrity of the study can be maintained by connecting the data, the analysis and the findings, while being transparent about what is descriptions and what is interpretations.

Morrow (2005) further explains that confirmability can be supported through many of the same measures as dependability. To support a fair and accurate analysis of the data, in the initial level of coding, we utilised data-driven codes that closely align with students' own words and phrases. To foster intra-rater reliability, which refers to consistent measurements or coding by the same observer, a coding handbook was developed and continuously refined throughout the coding process (Bryman, 2016, p. 294). The second level of coding involves interpreting the data in relation to the modes of thinking or their combinations. In our upcoming analysis, we present multiple examples of students' written work and quotes from the interviews, accompanied by our arguments for their corresponding modes of thinking. Based on our interpretation of Morrow (2005), we argue that these steps can promote transparency of the analysis, clarifying the distinction between *descriptions* of data and *interpretations* of findings. Consequently, both dependability and confirmability may be supported.

4.3.3 Authenticity

Authenticity in research, as described by James (2008) pertains to the genuine and credible conduct and evaluation of the study. In the context of our study, we contend that the homework setting is authentic as students would need to complete their homework assignment regardless of our study. Consequently, it is our expectation that students will be equally motivated and committed to engage with the tasks as they would with their "regular" homework.

To assess the authenticity of the tasks, we turn to Vos (2011; 2020), who suggests that tasks can be deemed authentic if they resemble questions actors within the given context would pose. More specifically, our tasks may be considered authentic if they resemble the typical tasks of students' regular coursework.

Task 9 was designed to present students with a conceptually challenging task, differing from the standard calculations or re-inventions of theorems one might associate with "traditional" homework assignments. Thus, it was expected that this task might be perceived as less authentic and that the students' answers could offer valuable insights into their concept images of eigenvectors and eigenvalues. For a more typical and authentic task, we included Task 10. By incorporating both these tasks, we sought to enhance the overall authenticity and relevance of our study, aligning with our research questions.

In contrast, we recognise that the interview setting may lack the same degree of authenticity as the homework setting, since one-to-one interviews typically do not occur in the students' daily learning activities. Nevertheless, the oral communication of mathematics is an important part of students' university education in mathematical subjects, for example in the communication with fellow students or teaching assistants. In these respects, an oral conversation on the course contents can be considered an authentic challenge for the students. Furthermore, the interviews offer an opportunity to explore students' experiences with the tasks, thus checking whether they were in fact perceived as “different” and/or challenging.

However, the interview setting may bear some resemblance to oral examinations, which our students may have experienced. While oral examinations are authentic situations, this can be problematic, as the resemblance can cause students to feel more stressed and concerned about giving incorrect answers. To create a more relaxed atmosphere, we commenced the interviews by restating the purpose of the interviews and reassuring students that the interview would not affect their grade in the course in any way.

4.3.3.1 Relevance

James (2008) further explains that authenticity in research encompasses the relevance and value of the research to the members of the community being researched. In line with this perspective, we propose that research in the didactics of mathematics should contribute to the advancement of mathematics education by supporting teaching and learning processes.

As the primary objective of the master study is to gain deeper insights into students’ comprehension of eigenvectors and eigenvalues, we contend that our research aligns with what Lester (2010) characterises as use-inspired basic research. By exploring students’ understanding of these concepts, we aim to generate knowledge that can contribute to practical implications for the teaching and learning of linear algebra in tertiary education. Furthermore, we hope that our analysis can further develop the modes of thinking described by Sierpinska (2000) as an analytical framework for understanding students’ reasoning in linear algebra.

Additionally, we anticipate that the knowledge gained from this master study will function as a valuable foundation for our future research, such as the PhD project, which aims to improve students’ understanding of linear algebra through task-design and possibly also teaching sequences.

Chapter 5

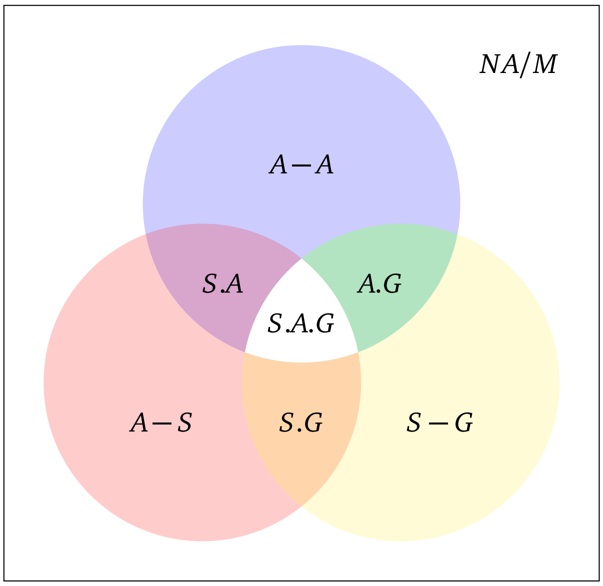
Results and Analysis

In this chapter, we analyse students' written responses to Tasks 9 and 10, aiming to characterise their concept images and modes of thinking, as conceptualised by Tall and Vinner (1981) and Sierpinska (2000). As we shall see, the analysis reveals a range of concept images. Several students presented rich concept images encompassing multiple modes of thinking, while others gave more general descriptions, suggesting concept images at the early stages of their development.

Structure

We begin with an examination of students' answers to Task 9, then followed by their responses to Task 10. The analysis of each task comprises two main parts. In the first part, we present examples to illustrate how students' answers align with Sierpinska's modes of thinking. The modes, namely the analytic-structural, analytic-arithmetic and synthetic-geometric, are represented respectively as red, blue and yellow in Figure 4.4. Through these examples, we aim to build a robust understanding of these modes in the context of eigentheory. It is important to note that while the upcoming examples are selected to illustrate characteristics of these three "pure" modes of thinking, they may also contain elements that correspond to other modes or that fall between Sierpinska's modes. However, this is the focus of the next part of the analysis.

In the second part, our attention shifts to conceptualising the combined modes of thinking, which we have named the structural-arithmetic, structural-geometric, arithmetic-geometric and the structural-arithmetic-geometric modes. These modes pertain, respectively, to the purple, orange, green and white (in the middle) areas of the Venn diagram. Figure 4.4, introduced in the previous chapter, is repeated here in Figure 5.1 to illustrate this. To conceptualise the combined modes, we will revisit some examples presented in the first part of our analysis, and introduce new examples incorporating elements from multiple modes of thinking. As we shall see, the majority of the students' answers encompassed multiple modes of thinking, while only a minority strictly adhered to a single mode.



Abbreviation	Explanation
$A-S$	Analytic-structural
$A-A$	Analytic-arithmetic
$S-G$	Synthetic-geometric
$S.A$	Structural-arithmetic
$A.G$	Structural-geometric
$A.G$	Arithmetic-geometric
$S.A.G$	Structural-arithmetic-geometric
NA/M	Not answered/ Miscellaneous

Figure 5.1: Overview with the modes of thinking identified in Task 9, illustrating both Sierpinska's (2000) three modes, and their combinations.

To address the inherent limitations of relying solely on students' written answers, as discussed in the previous chapter, we supplement our analysis with excerpts from the interviews whenever available. It is our perspective that these excerpts can aid in establishing a more comprehensive and nuanced understanding of students' concept images and their modes of thinking.

We conclude this chapter with an examination of the parts of students' concept images that were not captured by their modes of thinking. This includes a discussion of the role of the matrix, linear transformation, the eigenvector and the eigenvalue, as well as the relations that exist between these objects. Furthermore, we delve into the topic of the number of eigenvectors associated with an eigenvalue or a matrix. Finally, we consider students' experiences in working with our tasks.

5.1 Task 9

Our analysis commences with Task 9, where students were asked to provide their own explanations of the concepts of eigenvectors and eigenvalues. Recall that this task was intentionally designed with an open phrasing to allow students the freedom to express their ideas, and thus, allowing us to capture the nuances of their concept images and modes of thinking. By encouraging students to provide a sketch, we hoped to tap into the students' visual interpretations of eigenvectors and eigenvalues.

In Sections 5.1.1-5.1.4, we will examine examples of students' answers incorporating elements that correspond to Sierpinska's (2000) modes of thinking, namely analytic-structural, analytic-arithmetic, and synthetic-geometric modes. It is important to acknowledge that we are unable to provide examples for all first level codes associated with these modes. Nevertheless, we contend that the chosen examples included here allow us to characterise modes of thinking evident in our students' responses. In Section 5.1.5, we explore our additional modes of thinking, namely the structural-arithmetic, structural-geometric, arithmetic-geometric and structural-arithmetic-geometric modes. We conclude our analysis of students' reasoning in Task 9 with a summary and overview in Section 5.1.6.

5.1.1 Analytic-Structural Mode of Thinking

In line with our characterisation in Section 4.2.4.1, an analytic-structural description of eigenvectors highlights their defining properties, such as their correspondence to a matrix or linear transformation and their ability to preserve their span. In this section, we showcase examples of students' answers incorporating elements of this particular mode of thinking.

5.1.1.1 The Notion of Span

Recall that the property of preserving the original span after undergoing a linear transformation was considered to be a characteristic property of the eigenvectors of a matrix or linear transformation. An example of an answer making use of the notion of span to explain the concept of eigenvector was provided by a student we shall call Student A. The answer is depicted in Figure 5.2.

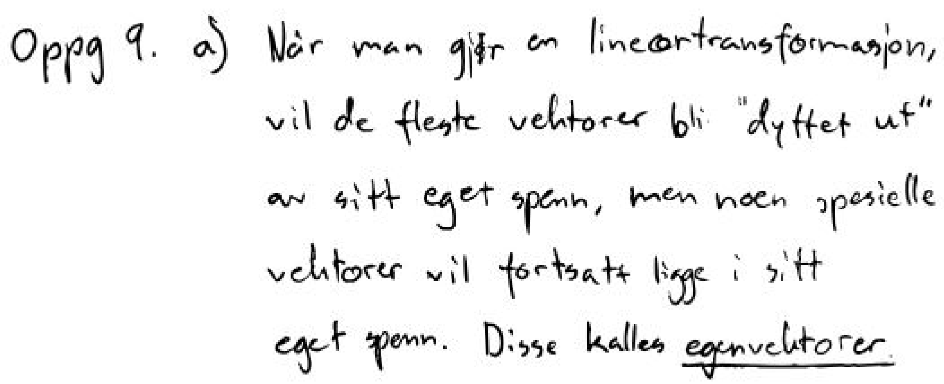


Figure 5.2: Student A's written answer to Task 9a), describing eigenvectors in relation to the notion of span.

This student, henceforth called Student A, stated that: "When you do a linear transformation, most vectors will be 'pushed out' of their own span, but some special vectors will still lie in their own span. These are called eigenvectors.”. Here, Student A observed that a linear transformation could affect a vector by changing their span. In characterising eigenvectors as “special vectors” which preserve their span, the student exhibited an analytic-structural mode of thinking.

However, it is worth noting that the nullvector also preserves its original span under a linear transformation. Yet, the definition of eigenvectors in the students’ curriculum (refer to Definition 4.2.1), does not allow the nullvector to be an eigenvector. However, this possibility was not excluded in Student A’s response to Task 9 a), and consequently, the answer does not fully qualify as a definition of an eigenvector. Rather, it describes a defining property inherent to eigenvectors. Among the 170 homework assignments we collected, 15 students described eigenvectors in relation to the notion of span.

Later in this chapter, we shall delve deeper into the examination of such “lacks” in students’ descriptions of eigenvectors and eigenvalues. We emphasise that the term “lacks” is not used in a normative sense to imply any deficiency or inadequacy in students’ answers. As the students’ were not asked to provide formal definitions, it would be inappropriate to evaluate their answers based on that criterion. However, prior to this examination, we discuss students’ visual interpretations of eigenvectors and eigenvalues that align with an analytic-structural mode of thinking.

5.1.1.2 Analytic-Structural Sketches

Recall now that Tall and Vinner’s (1981) notion of concept image encompasses all the cognitive structures an individual associates with a concept, including visual interpretations. Furthermore, Sierpinska (2000) highlights a distinction between the concept’s representation associated with synthetic-geometric and structural modes of thinking. A representation related to a structural mode of thinking is not reliant on a coordinate, but rather based on the inherent properties of the concept represented. In this study, two students gave sketches illustrating the notions of eigenvectors and eigenvalues, which were identified as examples of such structural representations. One of these students, Student B, gave the sketch in Figure 5.3.

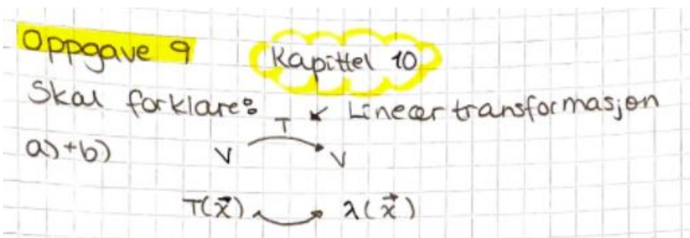


Figure 5.3: The sketch by Student B illustrates two Vs linked by an arrow labelled T, symbolising a linear transformation. Below, the linear transformation T acts upon a vector $\vec{x}$, mapping it to $\lambda(\vec{x})$.

The sketch shows two Vs connected by an arrow labelled T, which we interpreted as symbolising a linear transformation within a vector space denoted V. Below, the linear transformation T is depicted as operating on a vector $\vec{x}$ and mapping it to $\lambda(\vec{x})$. We understand this sketch as depicting how a linear transformation T acts upon a corresponding eigenvector $\vec{x}$ by scaling it with a factor of , the eigenvalue. However, we would typically interpret the notation $(\vec{x})$ as "$\vec{x}$ as a function of λ", rather than "the scalar product $\lambda \cdot \vec{x}$". Nevertheless, Student B's intention can not be determined based on their written answer alone.

In subsequent sections of this chapter, we will discuss other sketches demonstrating the concepts of eigenvectors and eigenvalues, namely those attributed to a synthetic-geometric mode of thinking. However, before we delve into those examples, we consider descriptions interpreted as having elements of an analytic-arithmetic mode of thinking.

5.1.2 Analytic-Arithmetic Mode of Thinking

In the sense of Sierpinska (2000), the analytic-arithmetic mode of thinking was characterised by describing mathematical objects based upon the procedures for computing them. In the case of eigenvectors and eigenvalues, our interpretation suggests that an analytic-arithmetic mode of thinking may include a description of eigenvectors as the solution of the homogeneous equation, $(A-\lambda I)\vec{x} = \vec{0}$, or eigenvalues as the roots of the characteristic polynomial, $\det(A-\lambda I) = 0$. Responses that made use of these equations from the computational procedure were coded as "9-procedure".

5.1.3 Prodecure

In this study, a small number of nine students provided descriptions of eigenvectors and/or eigenvalues by referring to the procedures for computing them, either by stating the equations or verbal rephrasing of them. For example, a student, henceforth referred to as Student C, stated that: "An eigenvector corresponds to $\vec{x}$ in $(A-\lambda I)\vec{x} = \vec{x}$ [sic], $\vec{x} \neq \vec{0}$. The eigencvector is a self-willed vector which can only be scaled by A." (See Figure 5.4).

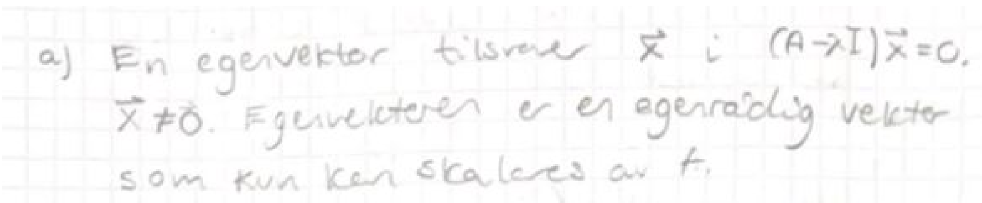


Figure 5.4: Student C's answer to Task 9 a), describing an eigenvector as the vector x in the equation$(A-\lambda I)\vec{x} = \vec{x}, \vec{x} \neq \vec{0}$, and as a "self-willed" vector.

As student C characterised eigenvectors as fulfilling the (homogeneous) equation $(A - \lambda I)\vec{x} = \vec{x}$, their answer was coded as "9-procedure" and interpreted as corresponding to an analytic-arithmetic mode of thinking. Later, we shall see that this answer not only encompasses elements of the analytic-arithmetic mode, but also contains elements of a synthetic-geometric mode, thus making it an answer corresponding to multiple modes of thinking.

In the latter part of the response, Student C noted that an eigenvector possesses a quality of being "self-willed". We interpret the description of an eigenvector as "self-willed" as a result of Student C's own reconstructed version of the concept of eigenvector. It may serve as a mnemonic or heuristic for understanding the nature of eigenvectors rather than a formal concept definition. We further argue that such mnemonics may be an important step for students in the process of constructing and internalising the knowledge of eigenvectors, that is, in developing their concept image.

Other students described the concept of eigenvalue according to the procedure for computing them, namely by computing the determinant of the matrix $(A - \lambda I)$ and the roots of the resulting polynomial. For instance, a student, which we shall call Student D, stated that: "Eigenvalue or characteristic value is a solution of the characteristic equation $\det(A - \lambda I) = 0$." (See Figure 5.5). In describing an eigenvalue as the solution of an equation, the answer was deemed as aligning with an analytic-arithmetic mode of thinking.

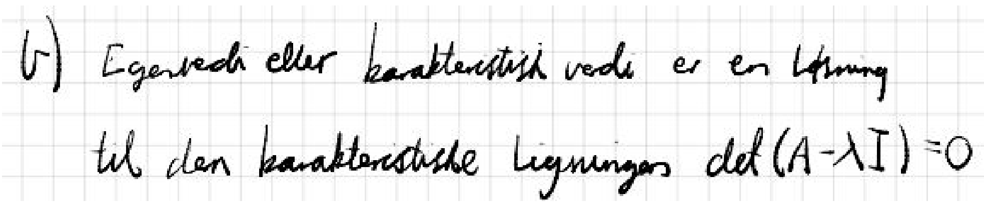

b) Egenverdi eller karakteristisk verdi er en løsning
til den karakteristiske ligningen det(A-λI)=0

Figure 5.5: Student D's answer to Task 9 b), where an eigenvalue (or characteristic value) is characterised as a solution of the characteristic equation, $\det(A - \lambda I) = 0$.

Next, we shall consider answers aligning with an alternative mode of thinking, namely the synthetic-geometric mode.

5.1.4 Synthetic-Geometric Mode of Thinking

While only a limited number of students gave explanations of eigenvectors and eigenvalues aligning with an analytic-arithmetic mode of thinking, several offered descriptions highlighting their geometric and/or visual properties. Recall that according to our interpretation of Sierpinska's (2000) modes of thinking, descriptions of how eigenvectors are affected by the linear transformation (or matrix multiplication) in terms of stretching, shrinking etc. are consistent with a synthetic-geometric mode of thinking.

In this study, we came across a range of visual characterisations of eigenvectors and eigenvalues. Among the collected homework assignments, 36 students described eigenvectors as maintaining their direction, 36 described eigenvectors as being scaled, and 33 explained how eigenvectors may be stretched, shrunk, flipped or left unchanged. Furthermore, five students characterised the original vector $\vec{x}$ as parallel to the resulting vector ($T(\vec{x})$ or $A\vec{x}$), while four students stated that $\vec{x}$ is not rotated under the linear transformation or matrix multiplication. Finally, one student noted that the resulting vector would lie on the same line as the original.

Among the 170 homework assignments collected for this study, a total of 69 written answers included one or more of these descriptions of eigenvectors and eigenvalues. In the following, we shall discuss examples of answers using geometric or visual descriptions of how a linear transformation (or matrix multiplication) may affect a corresponding eigenvector. Subsequently, we revisit the topic of students' sketches, this time focusing on those aligning with a synthetic-geometric mode of thinking.

5.1.4.1 Geometric or Visual Descriptions

Among the 33 students who explained ways in which a linear transformation or matrix multiplication may affect the length of its eigenvectors, using expressions like "stretching", "shrinking", "changing length" or "preserving length". For example, one of these students, which we shall call Student E, stated that "[The] Eigenvalue is how much the vector is stretched." (see Figure 5.6).

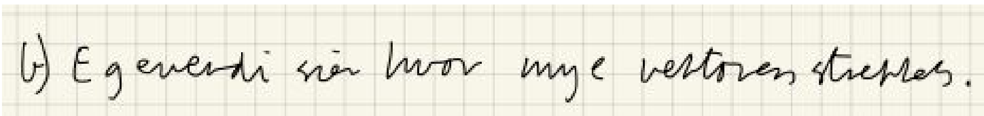


Figure 5.6: Student E's answer to Task 9 b), characterising eigenvalues as describing how much an eigenvector is stretched.

In explaining that the eigenvalue represents how much the eigenvector is stretched, Student E gave a visual description of a particular type of eigenvalues, namely real eigenvalues with an absolute value greater than 1 $|\lambda| \leq 1, \lambda \in \mathbb{R}$). Because the description is visual and limited to only some eigenvalues, the answer was evaluated to correspond to a synthetic-geometric mode of thinking. In fact, 13 out of these 33 students gave similar descriptions of eigenvalues being factors of stretching or eigenvectors being stretched, thus excluding the possibilities of shrinking or preserving the length. However, the definition presented to students (See Definition 4.2.1) does not imply such restrictions.

In this study, none of the 170 participating students gave a comprehensive description of *all* the ways an eigenvector can be affected by the linear transformation (or matrix multiplication). These effects encompass various possibilities, including stretching, shrinking, rotating by 180 degrees, preserving its length or direction, as well as combinations of the aforementioned.

It is possible that the 13 students who described eigenvectors as being stretched or attributing the stretching to eigenvalues, may stem from being exposed to graphical representations highlighting this aspect of stretching. Such illustrations might serve as *prototypical examples* of eigenvectors and eigenvalues, representing a common or typical way of visually interpreting these concepts. While these examples may serve as useful starting points for making sense of complex mathematical ideas, they may not capture all their facets. As Dorier and Sierpinska (2001, p. 264) noted, relying solely on prototypical examples might restrict students' understanding of the concepts. It is plausible that our students have not encountered other examples or have yet to contemplate the definition (see 4.2.1) sufficiently to grasp the opportunities beyond stretching. This observation led us to believe that these 13 students may possess concept images at early stages of their development, yet it should be noted that we do not imply this in a normative sense.

However, it is important to recognise that absence of explicit mentions of these possibilities does not necessarily imply that students lack awareness of them. For instance, it is plausible that students possess an understanding that eigenvectors may also be compressed or preserve their length, despite not having articulated it in their written answers. In other words, students' concept images may encompass additional dimensions that are not reflected in their written answers.

5.1.4.2 Direction

In the previous example, we observed how students described how the length of an eigenvector is influenced by the linear transformation or matrix multiplication. Now, our attention shifts towards answers discussing addressing the impact on the *direction* of eigenvectors. For instance, a student which we shall call Robin, gave the following characterisation of eigenvectors "[A] vector which does not change direction" (refer to Figure 5.7).

The claim that eigenvectors do not change direction, can be considered a visual interpretation pertaining to the synthetic-geometric mode of thinking. The fact that "not changing direction" is only true for some eigenvectors (not eigenvectors in general) further supports this classification. Characterising eigenvectors in general as preserving their direction is, in our opinion, somewhat inaccurate. In the case of real eigenvectors, their direction may be reversed by the linear transformation or matrix if the eigenvalue is negative. Furthermore, eigenvectors associated with complex eigenvalues can undergo both scaling and rotation. Thus, their direction[1] is not maintained in the same sense as for real eigenvectors. Consequently, Robin's written answer was deemed to represent a concept image restricted to real eigenvectors, and possibly also positive eigenvalues.

[1]The notion of *direction* for complex vectors differs from that of real vectors. Thus, in the context of eigentheory, we find it more appropriate to discuss their magnitude and phase.

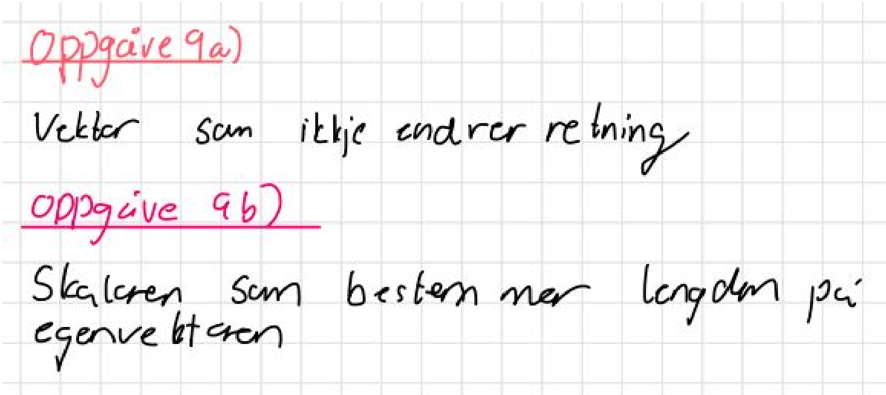


Figure 5.7: Robin's answer to Tasks 9 a) and 9 b), characterising eigenvectors as not changing direction and eigenvalues as scalars which determine the length of the eigenvector.

Robin was selected for an interview in order to further explore their concept images of eigenvectors and eigenvalues. When asked to explain the concept of eigenvectors in the interview, the student gave a verbal rephrasing of the eigenequation, identifying the role of the eigenvector and eigenvalue in the equation:

> Interviewer: May I ask you to explain what an eigenvector and an eigenvalue is?
>
> Robin: [long pause] Yes, that, if you have a... uhm. A matrix, then you can... And you multiply it with the eigenvalue, then you will have the same as if you multiply the eigenvector... an eigenvector with the eigenvalue. Is actually the only thing I know about that... Uh... So, basically, an eigenvalue is a value which you can multiply by the matrix and a vector and obtain the same result.

Thus, it appeared the written task evoked a part of the student's concept image associated with a synthetic-geometric mode of thinking, while the interview evoked another part of their concept image. Later, we shall see how answers describing the eigenequation can be associated with both analytic-structural and analytic-arithmetic modes of thinking. However, to see if the student could link the written and oral response, the student was reminded of their written description:

> Interviewer: It seems to me that you know another thing, because you wrote that... uhm. In [Task 9] a) you wrote: "A vector which does not change direction"?
>
> Robin: Oh... Yes... Uhm... [long pause]. I don't really know what I meant by that. If I... Maybe I meant, if I... That I multiplied by a number and then... No... Now I'm not quite sure what that means.

The purpose of reminding the student of their written response was to assist them in establishing a connection between the two descriptions. However, instead of achieving the desired clarity, the student appeared confused and uncertain about the accuracy of their written response. Nevertheless, when prompted to recall their written explanation of eigenvalues affecting the length of the eigenvector, the student appeared to align their oral and written answer:

> Interviewer: In [Task 9] b) you wrote: "The scalar which determines the length of the eigenvector".
>
> Robin: That makes a bit more sense. Maybe if you take a vector and you multiply it by a number, it would change length.

Based on the excerpt above, it appears that the student is capable of visually interpreting scalar multiplication as a means of altering the length of a vector (in this case, an eigenvector). Considering both the students' written and oral answers, we perceive them as indicating an evolving concept image at the beginning of its development. Later in this chapter, we shall revisit this example to examine other aspects of their concept image, not captured by the modes of thinking.

5.1.4.3 Synthetic-Geometric Sketches

Despite explicit encouragement for students to provide sketches illustrating the concepts of eigenvectors and eigenvalues, the number of participants who did so amounted to a mere 29, that is, less than 20% of the total 170. In Section 5.1.1, we discussed how two of these sketches aligned with an analytic-structural mode of thinking. Moving forward, our focus shifts to examining sketches exemplifying a synthetic-geometric mode of thinking. These sketches are characterised by depicting particular examples of eigenvectors in a coordinate system, either as arrows or dots.

A noteworthy example of these sketches, provided by Student F, shows a two-dimensional coordinate system with dots labelled $\vec{u}$ and $A\vec{u}$ connected by a curved arrow, as well as $\vec{v}$ and the equation $A\vec{v} = \dot{k}\vec{v}$ linked by another arrow. It is further noted that k is a real scalar and that A is an $n \times n$ matrix (see Figure 5.8). This sketch bears resemblance to the figures presented in the course's lecture notes (The reader is referred to NTNU, (2021)). Upon closer examination of the sketch, it seems that the dot labelled v has coordinates $(1, -1)$, while the dot labelled $A\vec{v} = \dot{k}\vec{v}$ is located at the coordinate $(3, 4)$. Consequently, while the equation $A\vec{v} = \dot{k}\vec{v}$ implies that v is scaled by a real factor k when multiplied with A, the figure suggests that $A\vec{v}$ is subjected to both rotation and stretching. Hence, the scaling in the equation $A\vec{v} = \dot{k}\vec{v}$ appears contradicted in the sketch.

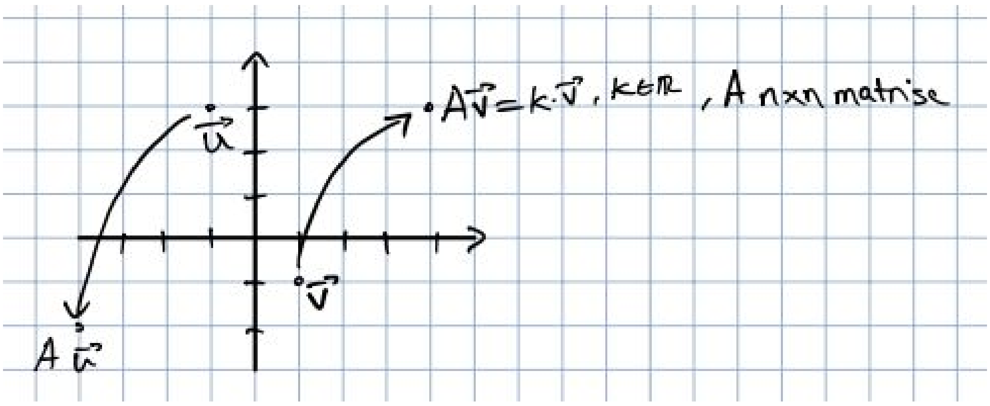


Figure 5.8: Student F's sketch shows a coordinate system and dots labelled $\vec{u}$ and $A\vec{u}$ connected by a curved arrow, and another arrow connecting v and the equation $A\vec{v} = \dot{k}\vec{v}$.

Accompanying this sketch, Student F gave a written description of eigenvectors as "a vector that together with a matrix only gives the same vector multiplied by a factor" and eigenvalue as "the factor in front of the eigenvector" (refer to Figure 5.9). Considering this written explanation alongside the sketch and the discrepancy therein, we wonder whether this student fully understands the visual interpretation of eigenvectors and scalar multiplication in general. Our impression is rather that Student F perceives eigenvectors as fulfilling the eigenequation, $A\vec{v} = \dot{k}\vec{v}$, for a matrix A, a vector $\vec{v}$ and a scalar k. In the following, we shall discuss such answers in greater detail.

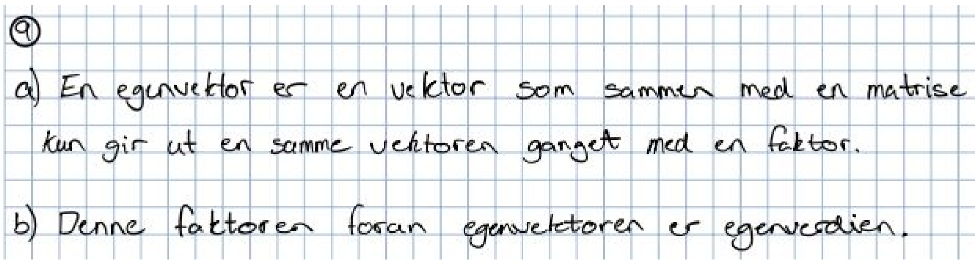


Figure 5.9: Student F's written answers to Tasks 9 a) and b), giving a verbal description of the eigenequation.

5.1.5 Multiple Modes of Thinking

As previously stated in this chapter, several students demonstrated engagement with multiple modes of thinking in their written answers to Task 9. Out of the total 170 participants in our study, 98 students exhibited elements of two modes of thinking, while 54 students incorporated aspects of all three modes described by Sierpinska (2000). In the upcoming sections, we discuss examples of such answers corresponding to the structural-arithmetic, structural-geometric and arithmetic-geometric modes of thinking. Furthermore, we present examples where students integrated all three modes, which we have categorised as the structural-arithmetic-geometric mode of thinking.

5.1.5.1 Structural-Arithmetic Mode of Thinking

In this study, we identified an overwhelming majority of 123 answers employing the symbolic eigenequation of the linear transformation ($T(\vec{x}) = \lambda(\vec{x})$ or matrix ($A\vec{x} = \lambda\vec{x}$)), or a verbal rephrasing of it. For instance, a student, referred to as Student G, gave the following explanation of eigenvectors and eigenvalues: "If one has a matrix A and a vector $\vec{x}$, the product will give a number λ multiplied by $\vec{x}$. Then will be an eigenvalue and $\vec{x}$ an eigenvector: $A\vec{x} = \vec{x}$." (see Figure 5.10).

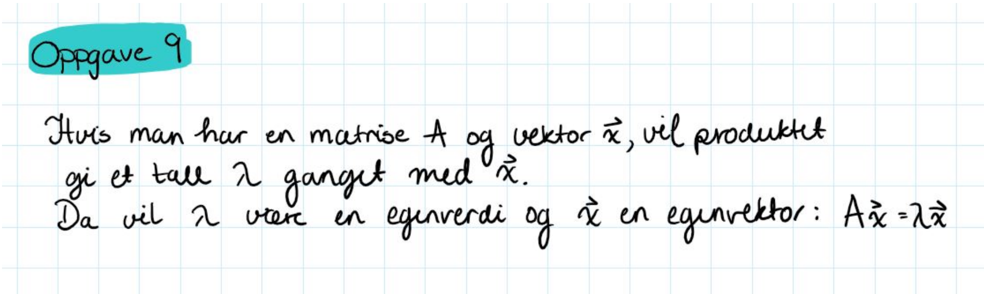


Figure 5.10: Student G's written response to Task 9 a) and b), describing the eigenequation. The answer has been reproduced for increased legibility.

In characterising eigenvectors as fulfilling the eigenequation, the answer demonstrates elements of an analytic-structural mode of thinking. However, the answer essentially describes an *equation* (which in turn can be manipulated to compute eigenvectors and eigenvalues) and matrix multiplication, thereby exhibiting elements of an analytic-arithmetic mode of thinking. Consequently, Student G's answers can be seen as falling between the realms of both the analytic-structural and analytic-arithmetic mode, leading us to categorise it as a manifestation of structural-arithmetic thinking.

Nevertheless, we note that the answer is brief and provides little other information regarding the Student G's concept images of eigenvectors and eigenvalues. There is no explicit mention of the correspondence between the eigenvector and eigenvalue or other key concepts closely related to eigentheory such as linear transformation or span. Moreover, there is no sketch or description of geometric and/or visual properties included either. Thus, the answer illustrates the limitations of relying solely on students' written answers to gain insigths into their concept images. Later in this chapter, we shall revisit this example to further discuss the importance of understanding the relations that exist between the eigenvector, eigenvalue and matrix (or linear transformation).

Another student, Iben, who described eigenvectors and eigenvalues by rephrasing the eigenequation wrote the following: "An eigenvector is a vector that you can multiply with an eigenvalue (number) such that it becomes the same as multiplying a square matrix with this vector." (see Figure 5.11).

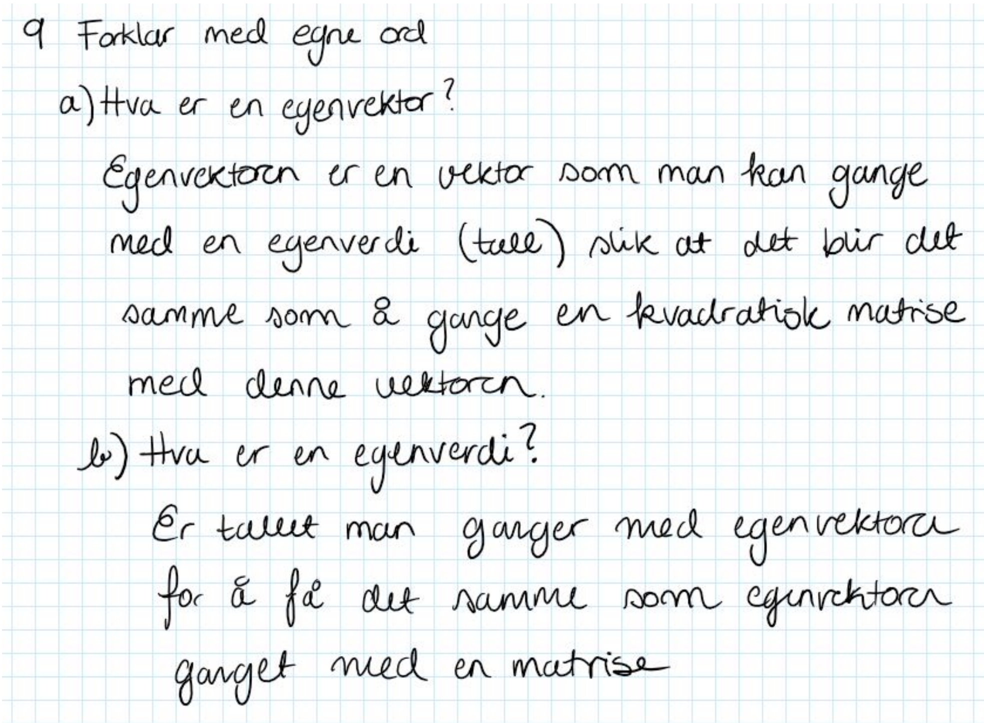

9 Forklar med egne ord
a) Hva er en egenvektor?
Egenvektorn er en vektor som man kan gange med en egenverdi (tall) slik at det blir det samme som å gange en kvadratisk matrise med denne vektorn.
b) Hva er en egenverdi?
Er tallet man ganger med egenvektorn for å få det samme som egenvektorn ganget med en matrise

Figure 5.11: Iben's answers to Task 9 a) and b), giving a verbal rephrasing of the eigenequation. The answer has been reproduced to enhance readability.

In the interview, when the student was asked to explain the concepts of eigenvectors and eigenvalues, they admitted that they found such tasks to be somewhat challenging:

> Iben: I find explanatory tasks to be somewhat difficult. I do not know if people notice, but I tend to respond with somewhat vague answers to them. [Short pause] But at least I wrote that an eigenvector is a vector that can be multiplied by an eigenvalue, so it is equivalent to multiplying a square matrix by this vector. And, that is what it is. . .

They proceed to write the eigenequation , $A\vec{x} = \lambda\vec{x}$, with the provided pen and paper, and added:

> Iben: But uh. . . I get it, and I can apply it, but maybe I struggle to understand what it is on a deeper level than that? Uhm. . . Yeah.

Hence, although Iben is correct that fulfilling the eigenequation is a necessary condition for being an eigenvector (we note that it is not sufficient in itself), they are of the opinion that their answer is a bit vague. This reflection, combined with their remark of understanding on a 'deeper level', suggests to us that perhaps Iben is unfamiliar with such tasks, and unsure what kind of answer is expected from them. Later, we shall revisit the aspect of students' experiences with the tasks. First, we explore another combined mode of thinking, namely the structural-geometric mode.

5.1.5.2 Structural-Geometric Mode of Thinking

In our study, we observed several answers exhibiting characteristics aligning with both analytic-structural and synthetic-geometric modes of thinking. Specifically, these answers described eigenvectors by their defining properties and visual interpretations.

One of these students, Student H, articulated their understanding of the notion of eigenvector as follows: "A vector that remains in the same span when it is transformed, and that the only effect the linear transformation has on the vector is to shorten/lengthen it.". They further noted that the eigenvector is the factor by which the eigenvalue is shortened/lengthened and gave two sketches (see Figure 5.12).

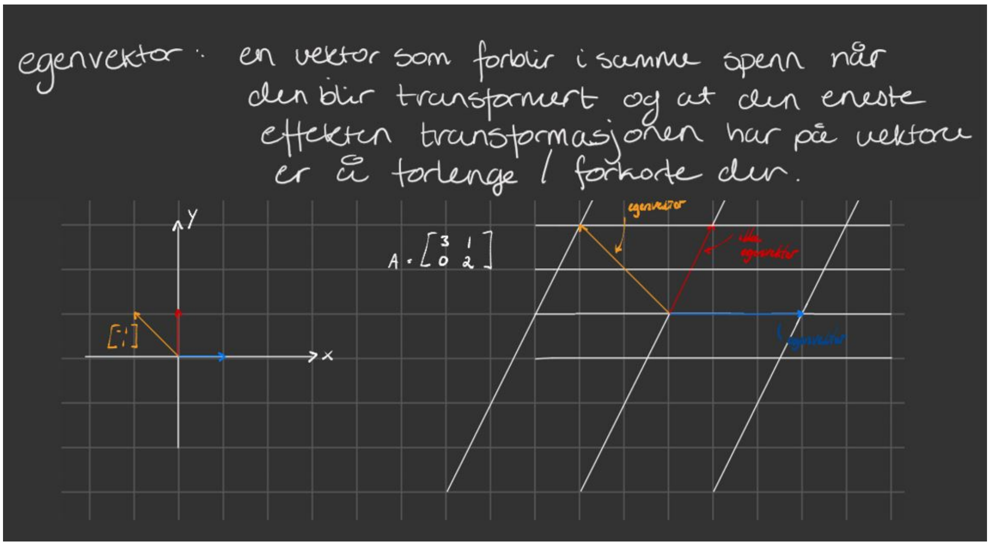


Figure 5.12: Student H's answer to Task 9 a) and b), including sketches. The text is reproduced for legibility.

According to our analysis, this answer exhibited a combination of both analytic-structural and synthetic-geometric modes of thinking. By relating eigenvectors to the notions of span and (linear) transformation, Student H demonstrated an analytic-structural mode of thinking. Additionally, by describing the effect of the transformation, specifically as shortening or lengthening the eigenvectors, the answer also incorporated a synthetic-geometric mode of thinking.

Furthermore, Student H supplemented their written response with a sketch depicting a two-dimensional coordinate system with axes labelled x and y. The sketch included three vectors: Orange, red and blue. Adjacent to it, another sketch displayed a grid that appeared rotated and skewed compared to the first. Between the two sketches, Student H included the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, possibly to illustrate the impact of the matrix on the vector space. In the latter sketch, the orange and blue vectors were labelled as eigenvectors and appear to be stretched with respect to the initial coordinate system. However, the red vector appears both stretched and rotated, and is labelled “not eigenvector”. Consequently, we infer that Student H’s concept image encompasses visual examples as well as non-examples of eigenvectors.

The two sketches presented by Student H bear notable resemblance to one of the supplementary course materials accessible through the course website, namely a video by 3blue1brown (2016) called *Eigenvectors and eigenvalues | Chapter 14, Essence of linear algebra*. This video introduces the concepts of eigenvectors and eigenvalues, highlighting their geometric properties and the dynamics associated with linear transformations, matrices, eigenvectors and eigenvalues. Hence, the Student H’s inclusion of these sketches suggest their engagement with these external materials.

In total, 23 students gave such answers that were interpreted as representing a structural-geometric mode of thinking.

5.1.5.3 Arithmetic-Geometric Mode of Thinking

A less common combination within our dataset were answers incorporating both analytic-arithmetic and synthetic-geometric modes of thinking in their written answers. Notably, this particular combination was only identified in answers provided by two students. To better understand this combination, we revisit the example of Student C’s answer which was discussed in Section 5.1.3, as illustrated again in Figure 5.4.

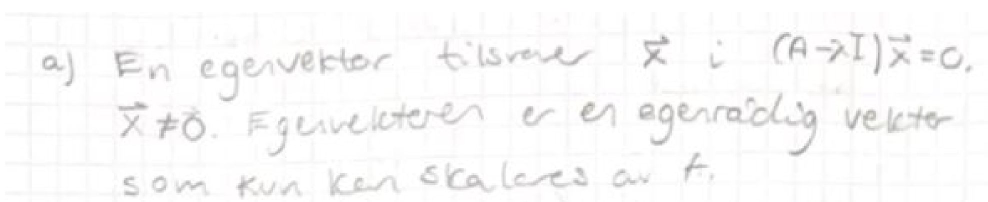


Figure 5.13: Student C’s answer to Task 9 a), describing an eigenvector as the vector x in the equation$(A - \lambda I)\vec{x} = \vec{x}, \vec{x} \neq \vec{0}$, and as a "self-willed" vector.

First, recall that the answer was previously identified as exemplifying an analytic-arithmetic mode of thinking. This classification was based upon the characterisation of the eigenvector as the vector $\vec{x}$ in the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$, which may be solved to compute the eigenvector. On the other hand, the student also demonstrated a synthetic-geometric mode of thinking by stating that an eigenvector is a vector which may only be *scaled* by A (the matrix). Thus, the answer exemplifies both these modes of thinking.

Next, we discuss answers incorporating all three modes of thinking.

5.1.5.4 Structural-Arithmetic-Geometric Mode of Thinking

As many as 54 of our 170 students (amounting to more than 30% of participants) gave answers with elements corresponding to all three modes of thinking described by Sierpinska (2000). A particularly interesting example, provided by Student I, is illustrated in Figure 5.14.

Student I gave a sketch of a coordinate system with axes x_1 and x_2, the vectors $\vec{x}$ and $A\vec{x}$, and the angle θ between them. Adjacent to the sketch, the student provided additional information, noting that A is an $n \times n$ matrix. Furthermore, Student I included the equation $T(\vec{x}) = A\vec{x}$ describing the linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$. According to the student's description, T scales the vector $\vec{x}$ and rotates it by an angle θ.

Below the sketch, Student I wrote that "An eigenvector of a matrix (example: *A*) is a vector which is only scaled, and not rotated. Cannot be nullvector.". Our interpretation is that Student I first explains and illustrates how a general linear transformation may act upon any vector $\vec{x}$, and in Task 9 a) the student specifies that the vectors which are only scaled are eigenvectors, so long as they are not the nullvector.

Upon a thorough examination of Student I's extensive response, which spans a full page, we identified several elements corresponding to all three of Sierpinska's (2000) modes of thinking. We list some of them here:

- By relating eigenvectors to the notion of linear transformation, the answer can be characterised as incorporating an analytic-structural mode of thinking.
- By further stating that eigenvectors are scaled, not rotated, Student I provides a visual description of typical examples of eigenvectors, aligning with a synthetic-geometric mode of thinking.
- Furthermore, by stating the equation $(A - \lambda I)\vec{x} = \vec{x}$, the answer could indicate an analytic-arithmetic mode of thinking as well.

Consequently, our analysis suggests this student possesses a concept image encompassing multiple characteristics of eigenvectors, indicating a developed understanding of them.

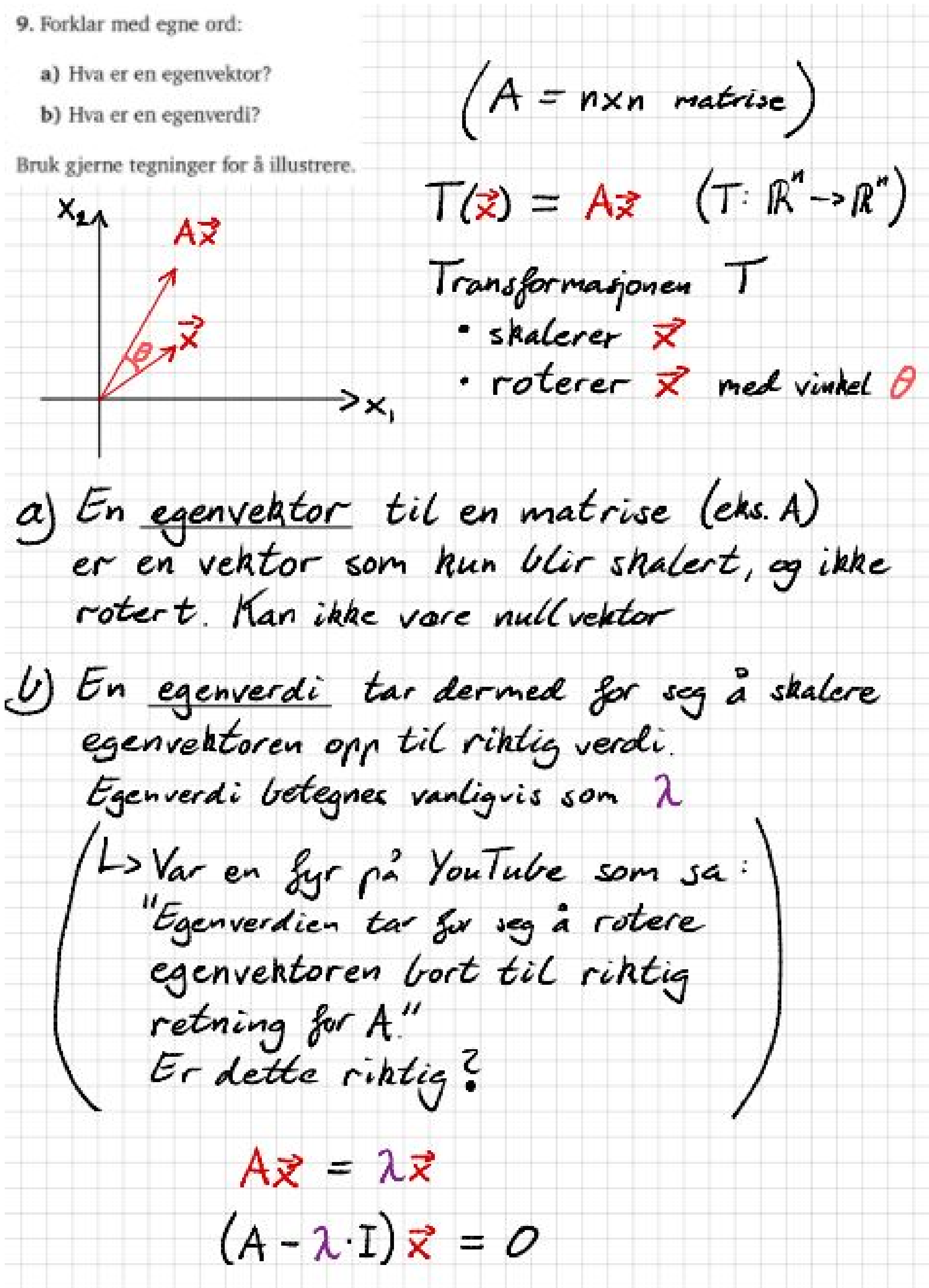


Figure 5.14: Student I's written answer and accompanying sketch to Tasks 9 a) and b), demonstrating all three modes of thinking.

In the Student I's response to Task 9 b), they explain the concept of eigenvalue, stating that: "An eigenvalue deals with scaling the vector up to the correct value. Eigenvalue is usually denoted λ.". Additionally, Student I included a quote from a YouTube-video: "The eigenvalue deals with rotating the eigenvector to the correct direction of *A*" and asked for confirmation regarding its accuracy.

Upon careful consideration, we find this quote to be inaccurate, at best, and possibly incorrect. From our perspective, it is evident that Student I possesses a sufficiently developed concept image to question the statement. However, their understanding may not be robust enough to definitively determine its inaccuracy. Our interpretation suggests that Student I is currently in a process of connecting various elements of their concept image and establishing links between the modes of thinking.

5.1.6 Summary and Overview

So far, we have seen various examples of students' written responses that align with each of the three modes of thinking, as well as combinations thereof. Figure 5.15 presents an overview of the distribution of modes of thinking exhibited by students.

Initially, our expectation was that most students would adhere to one mode of thinking in their responses. However, it became evident that the majority of the students incorporated more than one mode of thinking in their responses to Task 9. Among those presenting a singular mode of thinking, three were characterised as analytic-structural (indicated by red area), while none exclusively adopted an analytic-arithmetic mode (indicated by blue area). Additionally, six students exclusively exhibited synthetic-geometric modes (indicated by yellow area).

Instead, combining two modes of thinking was much more prevalent, with 73 incorporating a structural-arithmetic mode (indicated by purple area) and 23 demonstrating a structural-geometric mode (orange area). As we have previously discussed, the arithmetic-geometric mode was rare, with only two students exhibiting this particular combination. Interestingly, 54 students gave answers which encompassed all three modes of thinking, and were thus characterised as employing a structural-arithmetic-geometric mode (indicated by white area in the middle).

In Figure 5.15, the number 9 outside the Venn diagram represent the written answers that could not be classified into either mode of thinking, either because they left the task unanswered (4 students) or gave a description that did not align with any of the modes (5 students). There are several possible reasons for unanswered tasks. Unanswered tasks can be attributed to various factors, such as a lack of understanding, limited time or simply overlooking the task. Whatever the reason, unanswered tasks provide no information on students' concept images.

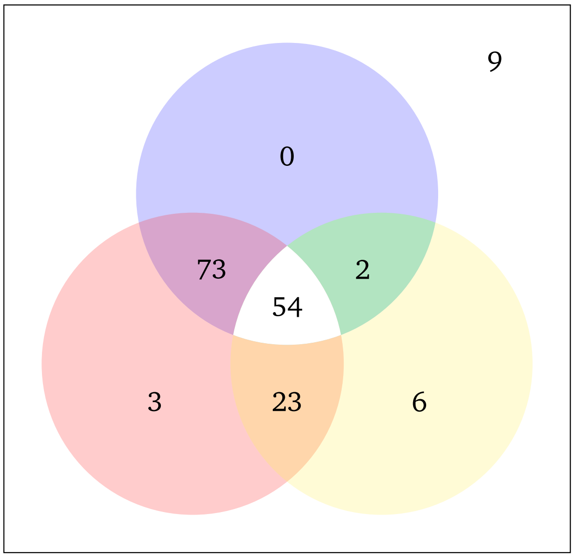


Figure 5.15: Overview of the distribution of modes of thinking exhibited in the students' answers to Task 9.

Uncategorised answers

Among the answers that did not align with the modes of thinking, one student, which we shall refer to as Kim, described eigenvectors in relation to the notion of diagonalisation (see Figure 5.16).

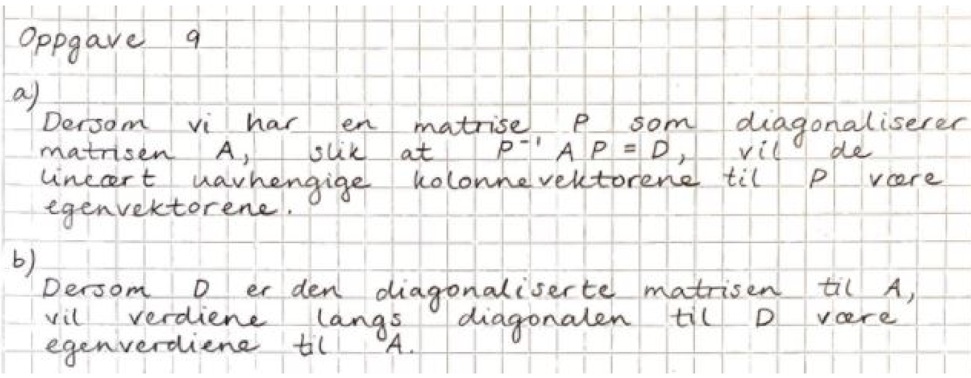

Oppgave 9
a)
Dersom vi har en matrise P som diagonaliserer matrisen A, slik at $P^{-1} A P = D$, vil de lineært uavhengige kolonnevektorene til P være egenvektorene.
b)
Dersom D er den diagonaliserte matrisen til A, vil verdiene langs diagonalen til D være egenverdiene til A.

Figure 5.16: Kim's written answer to Tasks 9 a) and b), explaining eigenvectors and eigenvalues in relation to diagonalisation.

In Task 9 a), Kim explained: "If we have a matrix *P* which diagonalises the matrix *A*, such that $P^{-1}AP = D$, then the linearly independent columns of *P* will be the eigenvectors." In Task 9 b) they stated that: "If *D* is the diagonalised matrix of *A*, then the values along the diagonal of *D* will be the eigenvalues of *A*."

In the interview, Kim first expressed some difficulty in explaining the concepts orally:

> Interviewer: How... What is your understanding of this now? If you were to explain to me what an eigenvector and eigenvalue is?
>
> Kim: Well, uhm, I think it is really easy... Or, you know... When we talk about matrices, it is like... Uhm... Oh, it is a little difficult to explain right now... I mean, I kind of know what it is.

After being encouraged to consider their written response, Kim appeared to remember their reasoning when working on the task:

> Kim: Mm... Yes, right. So if we are talking about a matrix, then it is that, uhm, the eigenvectors are, uh, the columns of the diagonalised matrix.
>
> Interviewer: Can you show me which is the diagonalised matrix? Just to make sure I understand.
>
> Kim: Uhm, yeah. It is that one [Points to the equation] matrix D in the equation PDP^{-1}. D is the diagonalised matrix.

Thus, Kim's oral answer supported their written description of what an eigenvector is and which matrix represents the so-called "diagonalised matrix". It is our perspective that since this answer does not describe defining properties of eigenvectors and eigenvalues, nor the formulas for computing them, nor their geometric properties, the answer cannot be identified as clearly belonging to any modes of thinking.

5.2 Task 10

In Task 9, we have seen aspects of students' concept images as expressed in their answers. This task was intended to explore and describe their concept image, rather than testing an application of their knowledge. Moving on to Task 10 now, we aim to observe how students make use of their concept image in a situation where they have to solve a problem related to eigentheory. Our interest lies in whether the modes of thinking that can be found here align with the distribution we observed in our analysis of their answers to Task 9.

In Task 10, students were asked to determine whether a given vector $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ is or is not an eigenvector to the matrix $A = \begin{bmatrix}1 & 1\\ -2 & -2\end{bmatrix}$, and to provide a justification. The purpose of this task was to examine which modes of thinking, in the sense of Sierpinska (2000), students employed to argue for their answer. In Section 4.2.3, we identified the following arguments corresponding to Sierpinska's (2000) modes of thinking:

- An argument based on the vector fulfilling the eigenequation was characterised as aligning with an analytic-structural mode of thinking.
- Performing computations to determine the eigenvectors and verify that the given vector is among them was regarded as demonstrating an analytic-arithmetic mode of thinking.
- An argument based on visual or geometric properties of eigenvectors, such as the vector $\vec{x}$ lying on the same line as the vector $A\vec{x}$ etc., or illustrating this with a sketch was categorised as synthetic-geometric.

Additionally, combinations of the arguments above and other solution strategies were also found. As we shall see next, an analytic-structural argument was the most prevalent of those described above.

5.2.1 Analytic-Structural Mode of Thinking

Among the 170 students who took part in the study, a remarkable majority of 140 students exhibited an analytic-structural mode of thinking. This was evident in their approach of checking examining whether the given vector x and the matrix A fulfilled the eigenequation, $A\vec{x} = \lambda\vec{x}$, where λ is the corresponding eigenvalue. Specifically, the students computed the matrix product $A\vec{x}$ and sought to identify a scalar λ such that $\lambda\vec{x}$ would equal the vector $A\vec{x}$. If such a scalar could be determined, the vector x was deemed to be an eigenvector of A. It should be noted that in our classification, we did not differentiate between correct or incorrect answers due to computational errors.

An example of such an argument making use of the eigenequation was observed in Robin's answer which is depicted in Figure 5.17.

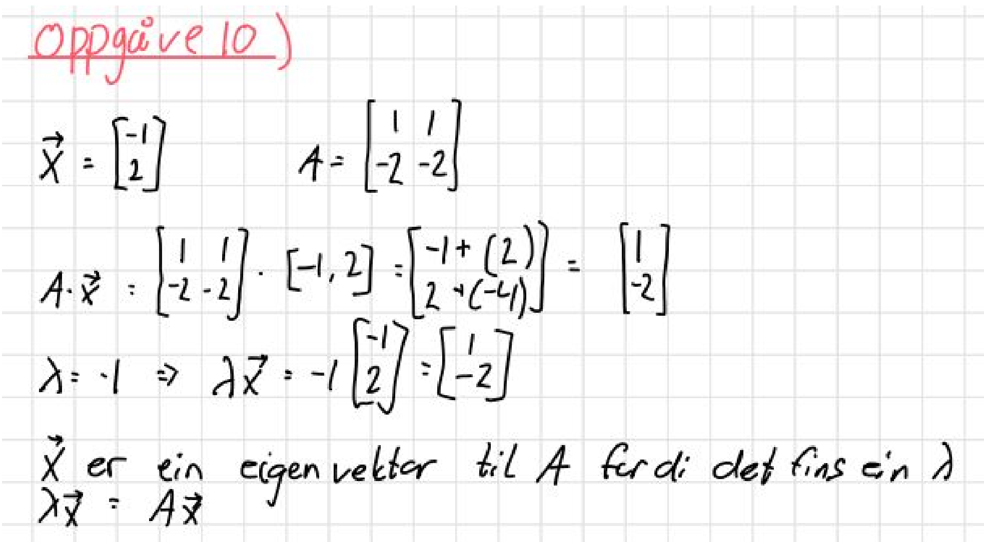


Figure 5.17: Robin's written answer to Task 10, interpreted as aligning with an analytic-structural mode of thinking.

Here, Robin first computed the matrix product $A\vec{x}$ to be $\begin{bmatrix}1 & -2\end{bmatrix}$. Afterwards, Robin recognised that an eigenvalue of $\lambda = -1$ would yield the same vector when multiplied by $\vec{x}$. Based on this, Robin correctly concluded that $\vec{x}$ was in fact an eigenvector of A with (corresponding) eigenvalue $\lambda = -1$. Because Robin based their argument on the defining property of $\vec{x}$ fulfilling the eigenequation ($A\vec{x} = \lambda\vec{x}$), the answer was characterised as corresponding to an analytic-structural mode of thinking.

At the beginning of the interview, Robin had expressed that examples were an important part of their learning process and that they enjoyed Task 10:

> Robin: I think that... Like, Task 10. That one I felt was kind of, it is a very kind of... task that I like, a kind of basic task which I really like. Kind of just testing a bit basic... basic things.

However, when asked to explain their reasoning in Task 10, Robin struggled a bit to remember or restate their argument:

> Interviewer: I would like to ask you about Task 10 as well. Can you explain to me how you solved this task? You said that you liked this task.
>
> Robin: I just [slight cough] tested and kind of that... uh... uh... I think, I just multiplied them together, kind of, that one and that one, and then...
>
> Interviewer: The vector and the matrix?
>
> Robin: I multiplied the vector and the matrix and then I saw that I, well, and then I got, uh... uh, a new vector and then I saw that it was possible to find a new number then, which I could multiply by uh... the vector to obtain... No, I mean, I just saw that if I found a number that, if I multiply it [the vector] by that number, I will get... uhm, the vector. No, wait a minute... No, I saw that I found it, a number such that if I multiply it by negative 1, then I will get the same result as when I multiply the matrix by the vector.
>
> Interviewer: Mm...
>
> Robin: And...Well, yeah, it was, uhm, it was a formula that I had written down that... uh... yeah, that the vector multiplied by the eigenvalue is equal to the matrix multiplied by the vector.

In the final part of the excerpt, the student acknowledged that the formula, the eigenequation, was just something they had written down. However, it remains unclear whether the student fully appreciates this equation as a defining characteristic of eigenvectors or merely views it as a useful "test" which can be applied.

Upon considering both the written and oral answers together, it becomes apparent that they may reflect different modes of thinking. As we have argued, the written argument is based on the defining property of satisfying the eigenequation, which aligns with an analytic-structural mode of thinking. In contrast, the oral answer suggests that the student might be unaware of this property being defining at all. Instead, it seems the student perceived Task 10 as a matter of "solving an equation," which leans more towards an analytic-structural mode of thinking.

Nevertheless, the statement "It was a formula that I had written down" evokes the concept of instrumental understanding, as discussed by Skemp (1978), wherein formulas are applied without comprehension of their underlying rationale. However, identifying instrumental understanding falls beyond the scope of our research question.

Another student, which we shall refer to as Student J, produced an answer with the same underlying argument, yet with a different sequence of steps (see Figure 5.18). Similar to the first example, Student J began by computing the matrix product $A\vec{x}$, obtaining the vector $\begin{bmatrix} 1 & -2 \end{bmatrix}^\intercal$. Then, they constructed two equations. The first, $a \cdot (-1) = 1$, likely pertains to a scalar multiplied by the first coordinate of the vector $\vec{x}$, equal to the first coordinate of the vector $A\vec{x}$. Similarly, the second equation, $b \cdot 2 = -2$, likely pertains to a scalar b multiplied by the second coordinate of the vector $\vec{x}$ to the second coordinate of the vector $A\vec{x}$. For the purpose of clarity, we restate a more general version of this in Figure 5.19.

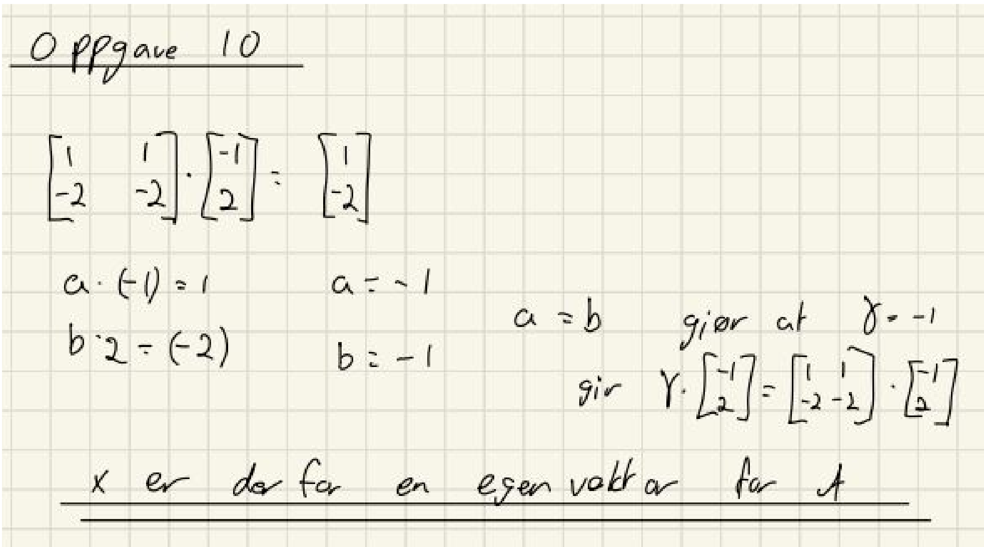


Figure 5.18: Student J's answer to Task 10, based on solving a system of two equations and two unknowns.

Subsequently, Student J solved these equations, determining that a=b=-1. Consequently, they inferred that the eigenvalue called γ would be equal to -1. Based on this computation, the student concluded that since gamma multiplied by $\vec{x}$ equals $A\vec{x}$, then $\vec{x}$ is indeed an eigenvector of A.

Let A be a 2×2 matrix and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a vector.
Then $A\mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.
To compute the scalar value (the eigenvalue), construct a system of two equations.
Let

$$\begin{cases} x_1 \cdot a = y_1, \\ x_2 \cdot b = y_2. \end{cases}$$

If there exists a solution such that $a = b$, then a is an eigenvalue of A.

Figure 5.19: An illustration of student J's solution strategy in Task 10 (our version).

However, we consider this argument to be somewhat disorganised and unclear, as Student J did not explicitly state the origin of the two equations. Consequently, we had to make our own interpretation to reconstruct the argumentation. From our perspective, this example suggests an important aspect of an analytic-*arithmetic* mode, to perform computations with efficiency and clarity. In our perspective, Student J's response highlights a challenge they faced in effectively communicating the computations involved in their reasoning. Nevertheless, the fundamental argument appears to be that the eigenequation is fulfilled, hence $\vec{x}$ is an eigenvector, and the answer was characterised as representing an analytic-*structural* mode of thinking. Next, our attention shifts to answers that predominantly align with an analytic-arithmetic mode of thinking.

5.2.2 Analytic-Arithmetic Mode of Thinking

While verifying that the eigenequation is fulfilled can be considered an effective solution strategy, there was an alternative approach which required more extensive written work. This involved first computing the eigenvalues of the matrix A, followed by determining the corresponding eigenvectors. The additional written work stems from the calculations required to compute eigenvalues and eigenvectors. As this approach relied on the procedure for computing eigenvectors, it was categorised as aligning with an analytic-arithmetic mode of thinking. Despite our initial anticipation, only a limited number of students adopted this particular solution strategy. In Figure 5.20, we showcase an example of such an answer, provided by Student K.

Student K first computed the eigenvalues of A by computing the roots of the characteristic polynomial in solving the equation $\det(A - \lambda I) = 0$. Then, they computed the corresponding eigenvectors for each eigenvalue, $\lambda = 0$ and $\lambda = -1$, by applying the Gaussian algorithm to solve the homogeneous equation,
$\det(A - \lambda I)\vec{x} = \vec{0}$. For the eigenvalue $\lambda = 0$, they found a non-trivial solution

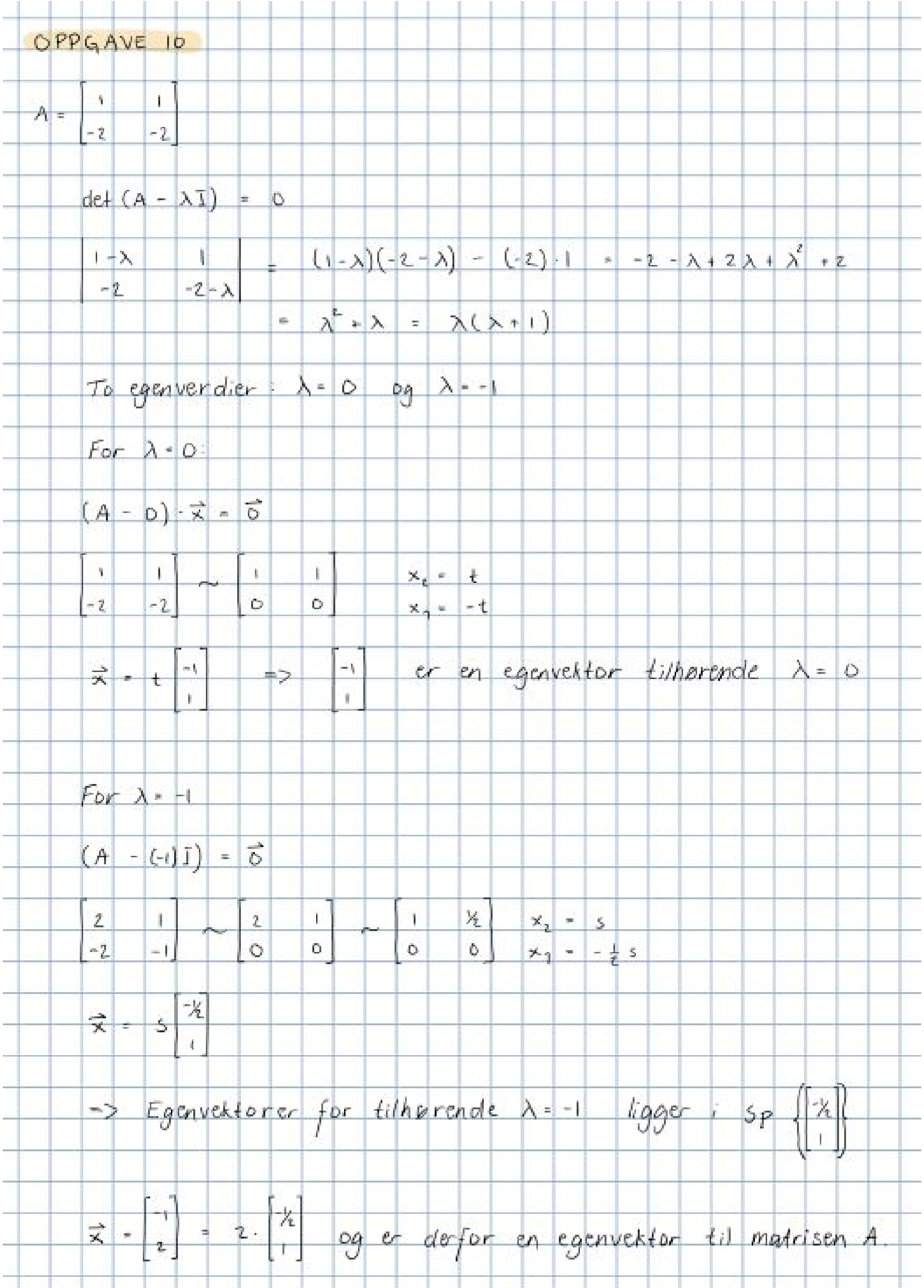


Figure 5.20: Student K's written answer to Task 10, where they employed the procedure for computing eigenvectors to argue for why $\vec{x}$ is an eigenvector of A.

(or rather infinitely many non-trivial solutions), $\vec{x} = s \begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathsf{T}}$. From there, it seems they selected $s = 1$ and concluded that $\begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathsf{T}}$ served as an eigenvector corresponding to the eigenvalue $\lambda = 0$.

For the eigenvalue $\lambda = -1$, Student K discovered non-trivial solutions of the form $\vec{x} = s \begin{bmatrix} \frac{-1}{2} & 1 \end{bmatrix}^{\mathsf{T}}$ and stated that the eigenvectors corresponding to $\lambda = -1$ lie within the span of the vector $\begin{bmatrix} \frac{-1}{2} & 1 \end{bmatrix}^{\mathsf{T}}$. Following this, they concluded that since the given vector $\vec{x} = \begin{bmatrix} -1 & 2 \end{bmatrix}^{\mathsf{T}}$ could be expressed as a scalar multiple of the previous vector (specifically for the scalar 2), then the given vector $\vec{x}$ was in fact an eigenvector for the matrix A. Considering that their answer relied on the procedure for computing eigenvectors, it was classified as corresponding to an analytic-arithmetic mode of thinking, along with eight other answers with the same argument.

It is our perspective that the provided answer is comprehensive and indicative of an understanding that there exist infinitely many eigenvectors for a given eigenvalue. As we shall see later, there were students who appeared unaware of this aspect, thinking there was a limited number of eigenvectors to an eigenvalue or matrix/linear transformation. Nevertheless, we remark that the student's answer could have been enhanced in terms of clarity by using indices for the eigenvalues and eigenvectors, to avoid confusion with the given vector x=$\vec{x} = \begin{bmatrix} -1 & 2 \end{bmatrix}^{\mathsf{T}}$. However, this is an aspect of formal accuracy, which is not at the focus of our study. Moving forward, we delve into the discussion of the synthetic-geometric mode of thinking in the context of Task 10.

5.2.3 Synthetic-Geometric Mode of Thinking

As mentioned earlier, a synthetic-geometric mode of thinking in Task 10 could involve an argument that the vectors $\vec{x}$ and $A\vec{x}$ lie on the same line or that $A\vec{x}$ can be expressed as a scaled version of $\vec{x}$. Additionally, a sketch illustrating this relationship would also be interpreted as an argument aligning with a synthetic-geometric mode of thinking.

However, within this study, there were only three answers incorporating such elements of a synthetic-geometric mode of thinking in Task 10, and these were always in conjunction with other modes. For an example of a solution pertaining exclusively to a synthetic-geometric mode of thinking, the reader is referred to Figure 4.3 in Section 4.2.3.2. Shortly, we explore examples of how the synthetic-geometric mode presented in combination with other modes of thinking.

5.2.4 Multiple modes of Thinking

In contrast to Task 9, where a notable number of students exhibited multiple modes of thinking in their answers, only eleven answers in Task 10 incorporated such combinations. In the following, we present examples of students' answers to Task 10 that align with the combinations of modes of thinking. That is, the structural-arithmetic, structural-geometric and arithmetic-geometric mode. Before we close this section, we also provide an example of a student's answer that included all three modes of thinking.

5.2.4.1 Structural-Arithmetic Mode of Thinking

Among the ten students who incorporated two modes of thinking in their answers, eight utilised the structural-arithmetic combination. These arguments involved both the procedure for computing eigenvalues or eigenvectors (aligning with an analytic-arithmetic mode), as well as the eigenequation to verify that the given vector $\vec{x}$ was in fact an eigenvector (aligning with an analytic-structural mode). In Figure 5.21, we present an example of an answer classified as structural-arithmetic, provided by Student L.

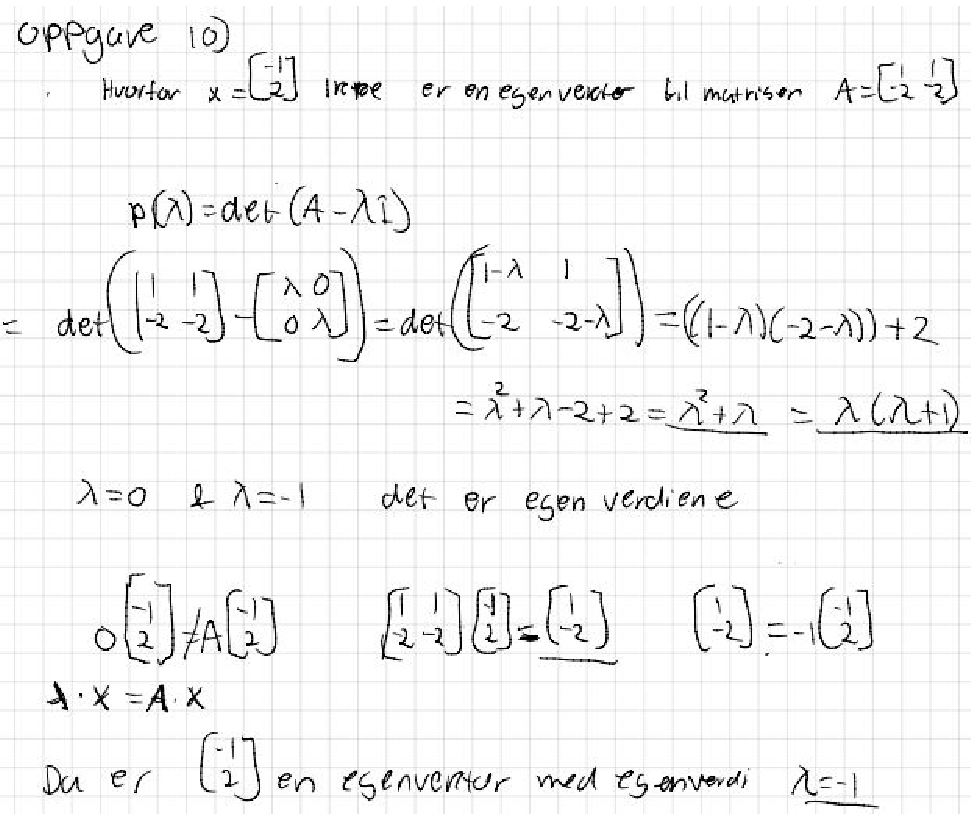


Figure 5.21: Student L's answer to Task 10, utilising the procedure for computing eigenvalues as well as the eigenequation to verify that $\vec{x}$ is an eigenvector.

Student L began by introducing the characteristic polynomial, $p(\lambda)$ as the determinant of the matrix $(A - \lambda I)$. They proceeded to compute the determinant, obtaining the second-order polynomial $\lambda(\lambda + 1)$. We presume that Student L obtained the eigenvalues of the matrix as $\lambda = 0$ and $\lambda = -1$ by computing the roots of this polynomial.

Subsequently, Student L seemed to set up the eigenequation ($\lambda\vec{x} = A\vec{x}$), substituting the eigenvalue $\lambda = 0$, the given vector $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ and the matrix A. They concluded that the left side of the equation was not equal to the right, implying that $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ was not an eigenvectors corresponding to $\lambda = 0$. Next, Student L computed the vector Ax to be $\begin{bmatrix}1 & -2\end{bmatrix}^{\mathsf{T}}$ and recognised that this was equal to the given vector $\vec{x}$ scaled by a factor of -1, i.e. $A\vec{x} = -1\vec{x}$. They concluded by writing "Then $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ is an eigenvector with eigenvalue $\lambda = -1$."

In summary, the answer demonstrated a structural-arithmetic mode of thinking by computing the eigenvalues from the characteristic polynomial and using the eigenequation to verify that the given vector $\vec{x}$ was an eigenvector corresponding to the eigenvalue $\lambda = -1$. In our interpretations, answers of this nature allow students to demonstrate their computational skills within matrices and their recognition of a characteristic property of eigenvectors, namely fulfilling the eigenequation.

In the following section, we discuss an example of another answer that had elements of both analytic-structural and synthetic-geometric modes of thinking.

5.2.4.2 Structural-Geometric Mode of Thinking

In our study, we encountered only one answer to Task 10 suggesting a combination of an analytic-structural and synthetic-geometric mode of thinking (see Figure 5.22).

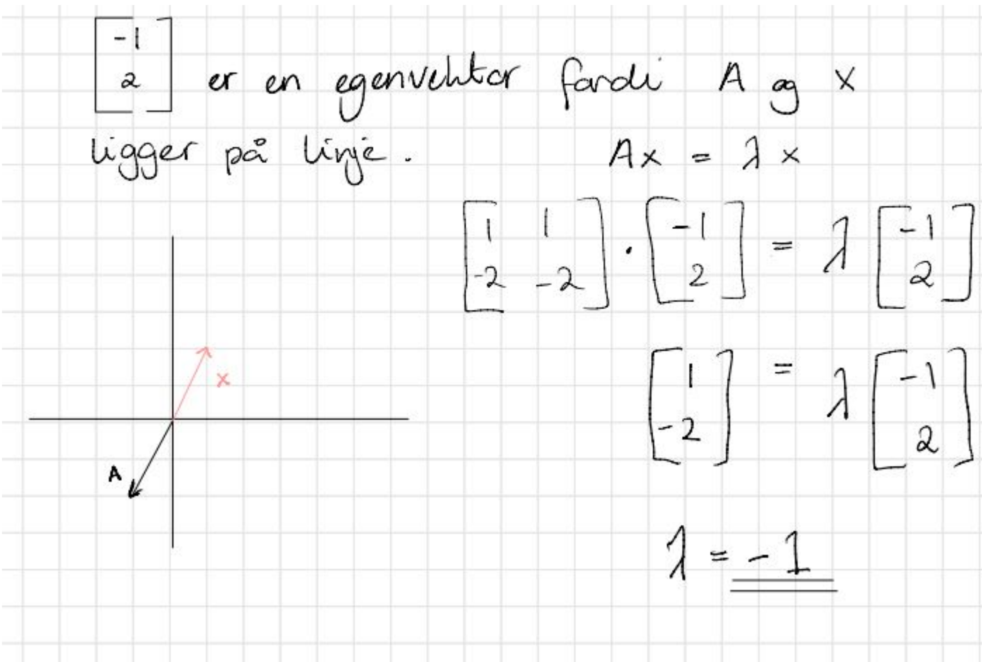


Figure 5.22: Student M's answer to Task 10, where the eigenequation is solved to derive the eigenvalue $\lambda = -1$ together with a sketch.

This student, henceforth called Student M, initiated their answer by stating that "$\begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ is an eigenvector because A and $\vec{x}$ lie on the same line." Below, they included a sketch of a two-dimensional coordinate system, with a red vector labelled $\vec{x}$ and a black vector labelled A, positioned in the opposite direction to $\vec{x}$. Their statement together with the sketch was identified to align with a synthetic-geometric mode of thinking, as the argument was based on geometric properties of particular eigenvectors. Based on this, we understood that the Student L's concept image included visual interpretations of eigenvectors.

However, it is important to note that A, as given in Task 10, is a 2×2 matrix, not a vector. While vectors, such as $\begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ are commonly represented as arrows in a coordinate system, matrices do not have direct visual representation in the same manner. In the context of a linear transformation, a matrix can be seen as a rule mapping input vectors to output vectors. These may be rotated, scaled or displaced relative to the original input vector. Thus, it is possible to visually interpret the effect of applying the matrix to objects in the vector space. Consequently, stating that "A [the matrix] and $\vec{x}$ [the vector] line on the same line" is inaccurate. Instead, it would be more appropriate to argue that $\vec{x}$ and $A\vec{x}$ are on the same line, and thus, $\vec{x}$ is an eigenvector of A.

Adjacent to the sketch, Student M wrote the eigenequation, $A\vec{x} = \lambda\vec{x}$. Subsequently, they substituted the values of A and $\vec{x}$ into the equation and computed the matrix product $A\vec{x}$ to be $\begin{bmatrix}1 & -2\end{bmatrix}^{\mathsf{T}}$. Below, the student wrote that $\lambda = -1$, indicating their recognition that this value satisfies the eigenequation. By verifying the fulfilment of the eigenequation for the given matrix A, the given vector $\vec{x}$ and the corresponding eigenvalue $\lambda = -1$, Student M exhibited an analytic-structural mode of thinking about eigenvectors as well.

Upon revisiting the sketch, it becomes evident that the depicted arrows do not accurately represent the vectors $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ and $A\vec{x} = \begin{bmatrix}1 & -2\end{bmatrix}^{\mathsf{T}}$ (as illustrated in our proposed solution in Figure 4.3, Section 4.2.3.2). Instead, the student drew arrows corresponding to the vectors $\begin{bmatrix}1 & 2\end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix}-1 & -2\end{bmatrix}^{\mathsf{T}}$. However, we consider this to be a likely result of oversight, rather than a significant misconception regarding the visual interpretation of vectors.

Because of the particularly interesting and exceptional character of Student M's answer in Task 10, we will compare it with their answer in Task 9, even if such comparisons between the two tasks' answers are not in the main focus of our study. In Task 9, Student M explained eigenvectors as fulfilling the eigenequation, stating that: "The eigenvector is the vector that, based on the eigenvalue, satisfies the requirements of $A\vec{x} = \lambda\vec{x}$". In doing so, their answer represented a structural-arithmetic mode of thinking. However, in part b), Student M described the concept of eigenvalue as "a concept which together with the eigenvector fulfils the equation $(A - \lambda I)\vec{x} = 0$ [sic]", suggesting an analytic-arithmetic mode of thinking (see Figure 5.23).

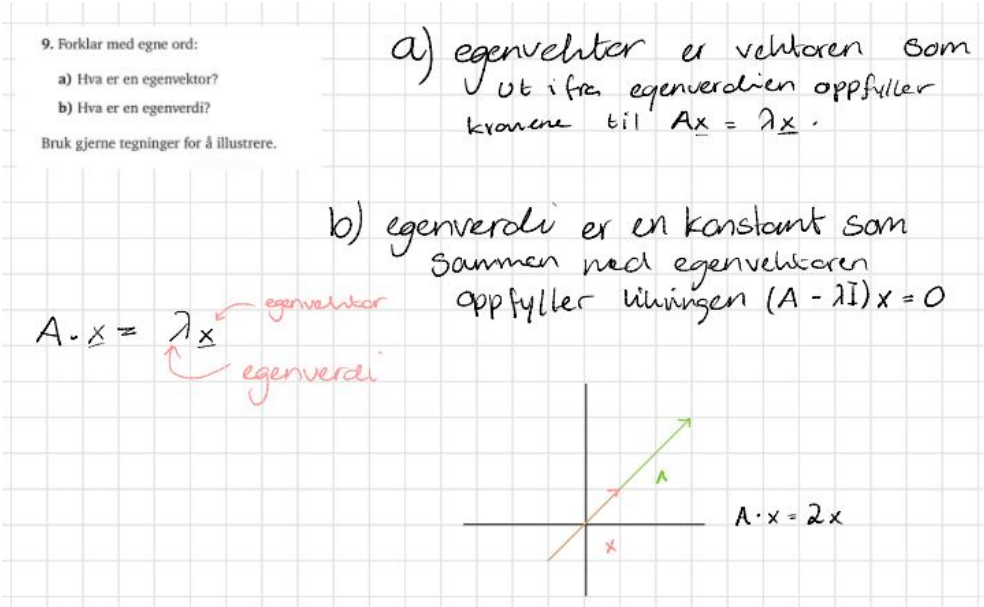


Figure 5.23: Student M's answer to Task 9 exhibited a structural-arithmetic-geometric mode of thinking.

Additionally, they gave a sketch of a coordinate system and a red vector labelled $\vec{x}$ and a green vector labelled A, aligning with a synthetic-geometric mode of thinking. Adjacent, they stated the equation "$A\vec{x} = 2\vec{x}$", possibly indicating that the eigenvalue corresponding to $\vec{x}$ was 2. Thus, the student's answer to Task 9 exhibited a structural-arithmetic-geometric mode, as opposed to a structural-geometric mode in Task 10.

The example illustrates that students occasionally exhibited different across tasks. However, it should be noted that a comparison between what a student wrote in Task 9 and what they wrote in Task 10 was not the main focus of our research question, and therefore was not done for all students.

5.2.4.3 Arithmetic-Geometric Mode of Thinking

Another infrequent combination of modes of thinking in Task 10, was the arithmetic-geometric mode of thinking. This mode was distinguished by the inclusion of both the computational procedure for determining eigenvalues and/or eigenvectors, as well as an argument based on the geometric properties of the eigenvector or a corresponding sketch. It is noteworthy that in our study, only one student, called Student N, provided an answer exhibiting this particular combination of modes in Task 10, as depicted in Figure 5.24.

Student N initiated their response by writing the matrix A and computing the determinant of the matrix $(A - \lambda I)$ to obtain the characteristic polynomial. By factoring the resulting second-degree polynomial in first degree polynomials, they

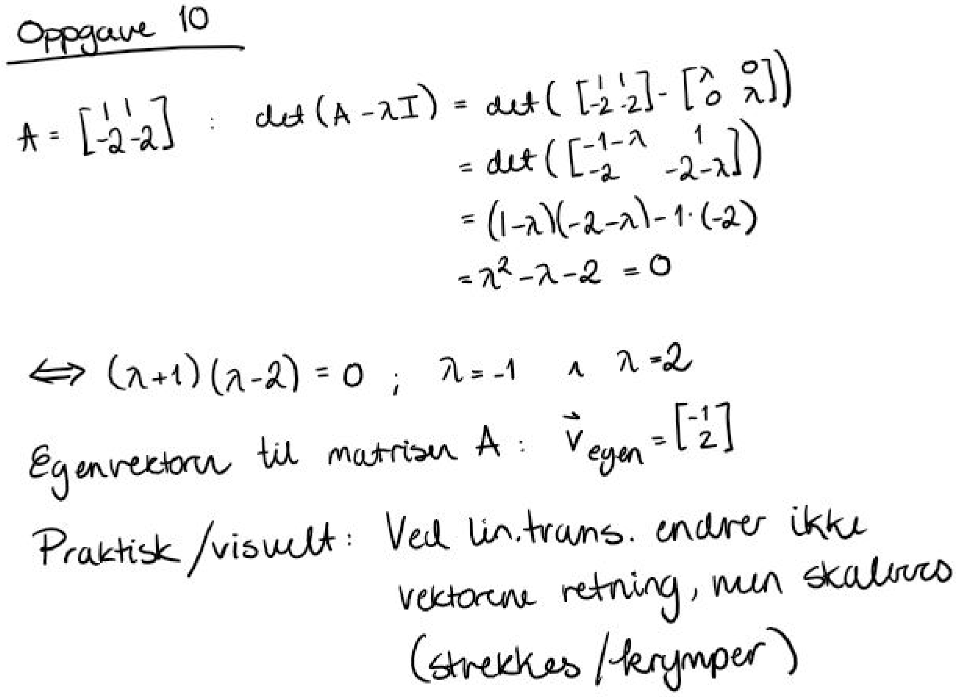


Figure 5.24: Student N's handwritten answer to Task 10, recreated for improved readability.

were able to determine the two eigenvalues, $\lambda = -1$ and $\lambda = 2$. Given that the answer involved the procedure for computing eigenvalues, it was considered to align with an analytic-arithmetic mode of thinking.

In the second part of the answer, Student N concluded that the vector $\begin{bmatrix}-1 & 2\end{bmatrix}^\mathsf{T}$, which they referred to as "$\vec{v}_{\text{egen}}$", is an eigenvector of A. However, despite the request for a justification, the student did not provide an explicit justification as to *why* the given vector $\begin{bmatrix}-1 & 2\end{bmatrix}^\mathsf{T}$ is indeed an eigenvector of A. The reason for this omission is uncertain. It could be attributed to overlooking the requirement for a justification, perceiving the claim as self-evident or a lack of understanding for what constitutes a proper justification. However, without further information, it is difficult to determine the exact reason for this omission.

Adjacent, Student N presented an additional interpretation for their answer by stating: "Practical/visually: The vector will not change direction under a lin.trans. [short for linear transformation], but it will be scaled (stretching/shrinking)." However, it is unclear whether Student N refers to how the *particular* eigenvector $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^\mathsf{T}$ is affected by the specific linear transformation expressed by the matrix A (indicating a synthetic mode of thinking), or if they refer to how eigenvectors *in general* are affected by the linear transformation they correspond to (indicating an analytic mode of thinking). Nevertheless, as Student N characterised the eigenvector as being stretched or shrunk, the dominant mode of thinking in the statement was considered to align with a synthetic-geometric mode of thinking.

Before we delve into the discussion of answers combining all three modes of thinking, it is necessary to acknowledge a computational error that likely occurred during the computation of the characteristic polynomial. This resulted in the polynomial $\lambda^2 - \lambda - 2$ instead of the correct polynomial $\lambda^2 + \lambda$. Upon solving the equation $\lambda^2 - \lambda - 2 = 0$, the previously mentioned computational error resulted in the identification of one correct eigenvalue $\lambda = -1$ and an incorrect eigenvalue $\lambda = 2$. Fortunately, the eigenvalue $\lambda = -1$ was indeed the eigenvalue corresponding to the given vector $\vec{x}$. It is worth noting that there were other instances in which computational errors resulted in incorrect conclusions regarding the status of $\vec{x}$ as an eigenvector. However, our analysis has focused on characterising the underlying argument in terms of the modes of thinking employed, regardless of correct or incorrect conclusions stemming from computational errors.

5.2.4.4 Structural-Arithmetic-Geometric Mode of Thinking

In Task 9, a remarkable number of 55 students gave answers incorporating elements from all three modes of thinking described by Sierpinska (2000), aligning with what we have named a structural-arithmetic-geometric mode of thinking. In contrast, in Task 10, this combined mode of thinking was notably less prevalent, as only one student, henceforth referred to as Student O, exemplified this mode, as illustrated in Figure 5.25.

Student O began by restating the task: "Justify why $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^\intercal$ is or is not an eigenvector to the matrix $A = \begin{bmatrix}1 & 1\\ -2 & -2\end{bmatrix}$". Below, they wrote: "To find the eigenvectors of A we may first find the eigenvalues by using the relation $A\vec{x} = \lambda\vec{x}$". They proceeded to perform the computations to derive the equation $\det(A - \lambda I) = 0$ from the eigenequation. Next, they computed the eigenvalues to be $\lambda_1 = 0$ and $\lambda_2 = -1$ by solving the equation.

Then, Student O wrote: "Now checking the eigenvectors corresponding to the two eigenvalues! → Find the nullspace:" For each eigenvalue, they set up a matrix, likely corresponding to the homogenous equations $(A - \lambda_1 I)\vec{x} = \vec{0}$ and $(A - \lambda_2 I)\vec{x} = \vec{0}$, and computed the corresponding eigenvectors. In employing this procedure, Student O's answer was considered to incorporate an analytic-arithmetic mode of thinking.

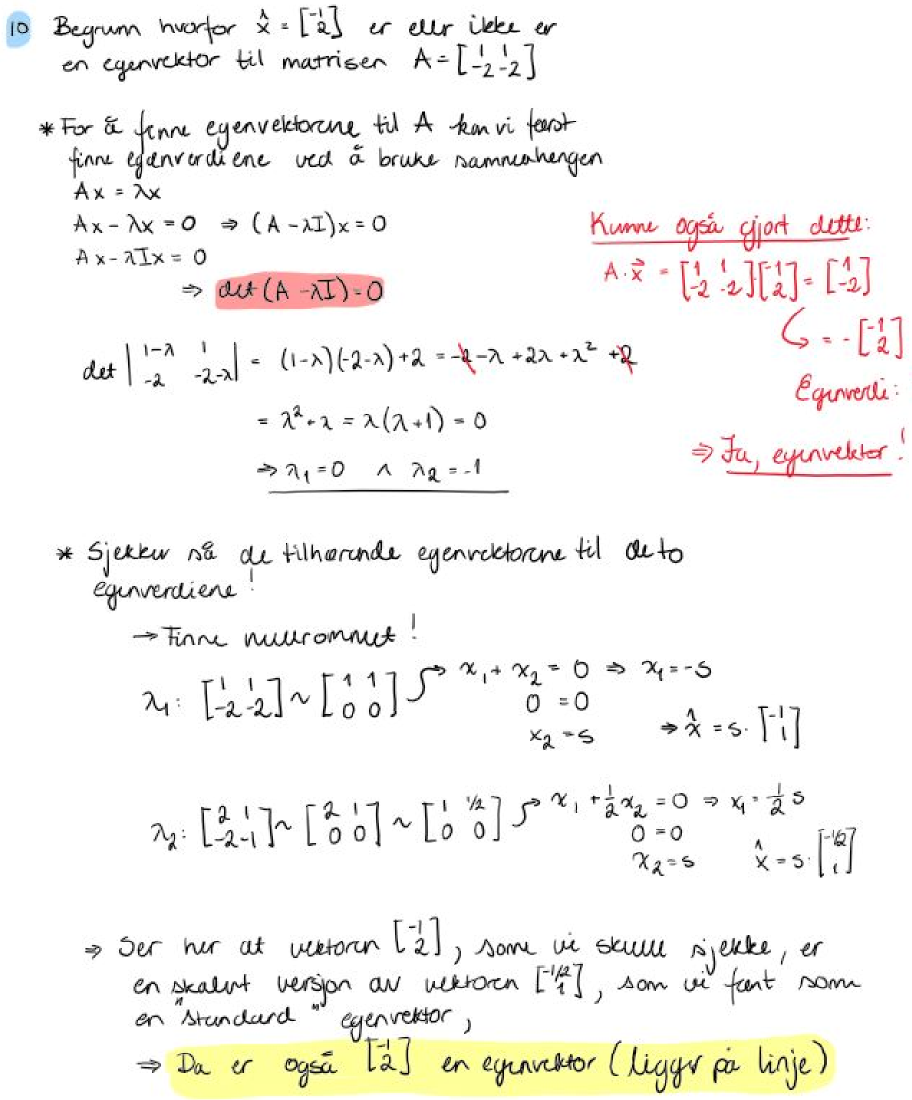


Figure 5.25: Student O's answer to Task 10, exhibiting elements aligning with a structural-arithmetic-geometric mode of thinking. The answer has been reproduced to enhance readability.

Below, Student O observed that the given vector $\begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ was a *scaled version* of the computed eigenvector $\begin{bmatrix}\frac{-1}{2} & 1\end{bmatrix}^{\mathsf{T}}$, and argued that this implied that $\begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$ was also an eigenvector. They further noted "(lie on the same line)", likely referring to the vectors $\begin{bmatrix}\frac{-1}{2} & 1\end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix}-1 & 2\end{bmatrix}^{\mathsf{T}}$. By recognising this, Student O's answer was interpreted as encompassing a synthetic-geometric mode of thinking in their answer.

On the right hand side, Student O presented an alternative solution strategy in red, corresponding to checking the eigenequation, thus verifying that the given vector [$\vec{x} = \begin{bmatrix} -1 & 2 \end{bmatrix}^\intercal$] was in fact an eigenvector (of A). In doing so, they demonstrated an analytic-structural mode of thinking, thus incorporating all three modes in their answer to Task 10. This indicates a concept image that is rich and flexible enough to generate two, equally-valid solution strategies and interpret the results visually.

In the subsequent section, we summarise our findings and present an overview of the modes and their distribution among the responses to Task 10.

5.2.5 Summary and Overview

So far, we have examined several examples of students' written responses to Task 10, showcasing the alignment with the modes of thinking and combinations of them. Figure 5.26 presents an overview of the distribution of the modes of thinking exhibited by the students' written answers.

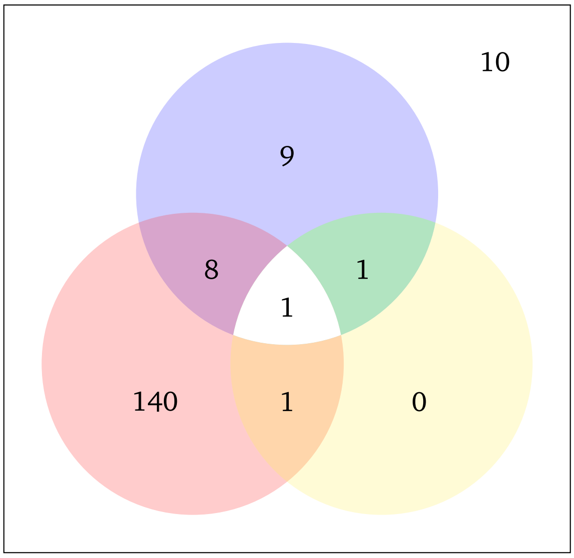


Figure 5.26: Overview of the distribution of modes of thinking exhibited in the students' answers to Task 10.

Remarkably, an overwhelming majority of 80% of the students' presented an analytic-structural mode of thinking (indicated in red) in checking the eigenequation to verify that $\vec{x}$ was an eigenvector. In contrast, fewer presented a purely analytic-arithmetic mode (indicated in blue) in computing the eigenvectors and verifying that $\vec{x}$ was indeed one of them, with only eight students doing so. Notably, no students gave a standalone synthetic-geometric argument (indicated in yellow area). We find this intriguing, as so many students gave such descriptions of eigenvectors in Task 9.

In contrast to Task 9, where multiple modes of thinking were commonly observed, our analysis of the students' answers to Task 10 revealed that only a small number of the students incorporated a synthetic-geometric mode alongside other modes of thinking. Specifically, one student presented a structural-geometric mode (indicated in green), while another gave an argument aligning with an arithmetic-geometric mode (indicated in orange). Finally, only one student incorporated elements from all three modes and was classified as structural-arithmetic-geometric (indicated in white area in the middle).

Similar to Task 9, there were responses in Task 10 that could not be categorised within any of the identified modes of thinking. This was either due to some students leaving the task unanswered (3 students) or providing arguments that did not align with any of the modes (7 students). These unclassified responses are represented by the number 10 in Figure 5.26.

Uncategorised answers

An example of an answer that could not be categorised within the identified modes of thinking was provided by Student P, as illustrated in Figure 5.27.

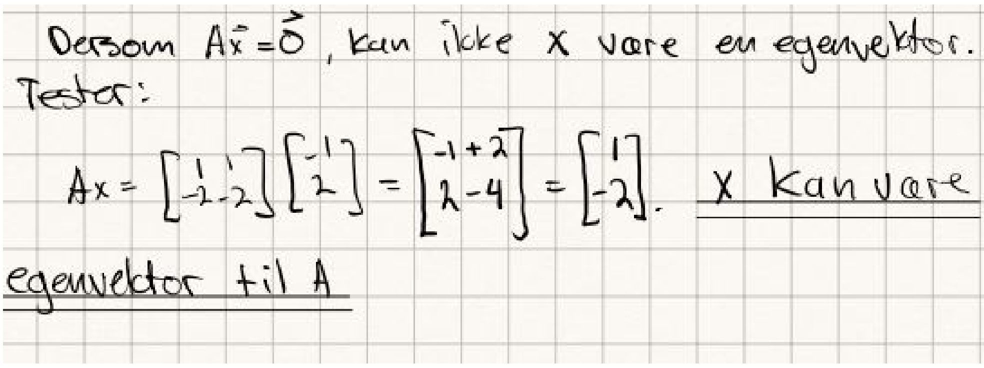


Figure 5.27: Student P's answer to Task 10, which did not appear to align with any of the modes of thinking.

Here, Student P argued that as long as the matrix product $A\vec{x}$ does not result in the nullvector, the vector $\vec{x}$ can be an eigenvector. Testing this, they computed the matrix product for the given matrix A and vector $\vec{x}$. Upon calculating the resulting vector as $\begin{bmatrix}1 & -2\end{bmatrix}^\mathsf{T}$, Student P concluded that $\vec{x}$ could potentially be an eigenvector of A. It should be noted that this argumentation does not definitely conclude that $\vec{x}$ *is* an eigenvector. Rather, it signifies that Student P thinks $\vec{x}$ *could* be an eigenvector of A. Thus, it appears that Student P implicitly conveys the idea that there is insufficient information available to definitively draw a conclusion about $\vec{x}$'s status as an eigenvector.

The assumption that $A\vec{x} = \vec{0}$ is impossible for eigenvectors is incorrect. In general, for eigenvectors with an eigenvalue of zero, the matrix product $A\vec{x}$ will yield the nullvector. This can also occur when A is the nullmatrix. Therefore, the Student

P's argument suggests a potential misconception about the possibility of $A\vec{x} = \vec{0}$ for eigenvectors. It is possible that this confusion stems from conflating the exclusion of the null vector as an eigenvector, as stated in Definition 4.2.1. However, further information is necessary to draw any conclusive insights into this matter.

In the next subchapter, our attention shifts to other facets of students' concept images that extend beyond their modes of thinking. Thus, we delve into additional elements that contribute to students' understanding of eigenvectors and eigenvalues.

5.3 Concept Images

Thus far, we have made brief observations on elements of students' concept images of eigenvectors and eigenvalues which are not directly expressed in their modes of thinking. In some cases, we have also assessed the level of development of these concept images based on their written, and when available, oral responses. We have observed answers suggesting that students possess rich concept images, displaying several ideas about eigenvectors and eigenvalues, although some were not yet at the stage of being able to unify these ideas into a coherent whole. Conversely, other students have presented more brief answers and confusion surrounding these concepts.

Recalling that the modes of thinking were considered to partially overlap with the notion of concept image, we now delve deeper into additional aspects of students' concept images, not reflected in their modes of thinking. These encompass the interrelationships that exist among matrices and linear transformations, eigenvectors and eigenvalues, as well as the nature of these relations. Furthermore, we briefly touch upon the topic of the number of eigenvectors associated with an eigenvalue of a matrix. We argue that this area is particularly interesting due to its connections with other key concepts in linear algebra, such as eigenspaces and bases.

Concluding this subchapter, we turn our attention to examining students' experiences in engaging with these tasks. We argue that such insights can provide valuable information regarding their concept images, as well as their strengths, challenges, motivation, learning styles, which in turn are important for our future research.

5.3.1 The Objects and their Relations

To illustrate students' descriptions of the objects and their relations, we revisit a previous example of a students' answer to Task 9 (see Figure 5.7). Here, Robin described the notion of eigenvector as a "Vector which does not change direction" and eigenvalue as "The scalar that determines the magnitude of the vector". However, the matrix or linear transformation was omitted in both these descriptions.

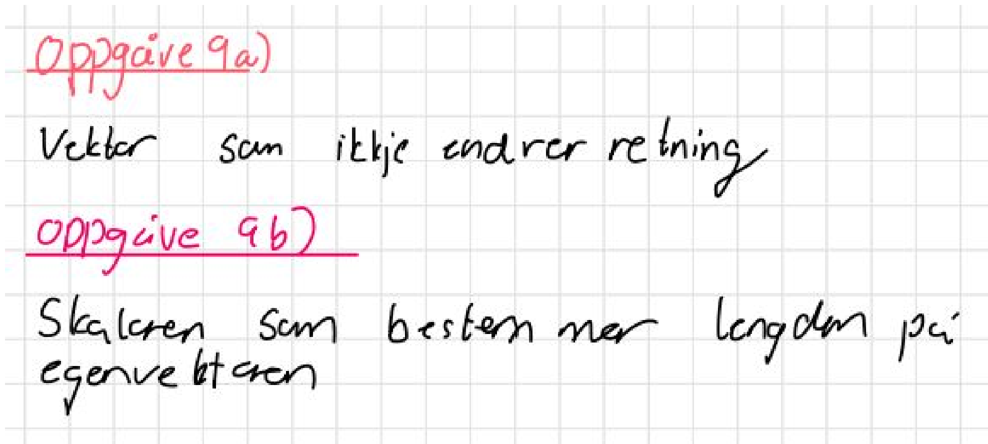


Figure 5.28: Robin's answer to Tasks 9 a) and 9 b), characterising eigenvectors as not changing direction and eigenvalues as scalars which determine the length of the eigenvector.

However, as we saw earlier, another part of Robin's concept image was evoked in the interview, as they explained eigenvectors and eigenvectors in relation to the matrix, and as fulfilling the eigenequation.

> Robin: ... Uh... So, basically, an eigenvalue is a value which you can multiply by the matrix and a vector and obtain the same result.

Thus, Robin's written answer indicated a synthetic-geometric mode of thinking, whereas the oral discussion of the eigenequation aligned with a structural-arithmetic mode. This example demonstrates how the interviews provided students with an opportunity to further develop their understanding of eigenvectors and eigenvalues, and shed light on aspects of their concept images that may not have been evident in their written responses.

However, an interesting discrepancy between Robin's oral and written answers arose in regards to the (in)dependence of the matrix. The written answer omitted the matrix completely, while the oral answer depicted eigenvalues as dependent on the matrix. This discrepancy suggested potential conflicting parts of Robin's concept image, or in Tall and Vinner's (1981) terms, these aspects of the student concept image represent *potential conflict factors*. To further explore this, Robin was asked if eigenvectors and eigenvalues could exist without matrices, aiming to provoke a cognitive conflict and potential alignment or modification of their descriptions. An excerpt from this dialogue is illustrated below:

> Interviewer: And I wonder, can we have eigenvectors, and eigenvalues for that matter, without having a matrix?
>
> [long pause]
>
> Robin: According to what I wrote here [points to Task 9], you can

> indeed have that if a vector is just something... a... or if... If an eigenvector is simply something that... a vector that does not change direction. So... it should be possible to have an eigenvector and an eigenvalue without kind of having a matrix.
>
> Interviewer: Mm...
>
> Robin: So I guess 'Yes'.

In the excerpt above, Robin begins by reflecting on their written answer, admitting that if what they wrote was correct, then eigenvectors and eigenvalues could indeed exist independently of matrices. In terms of our previous visual representations of the relations between the objects, we can illustrate the students' answer as in Figure 5.29:

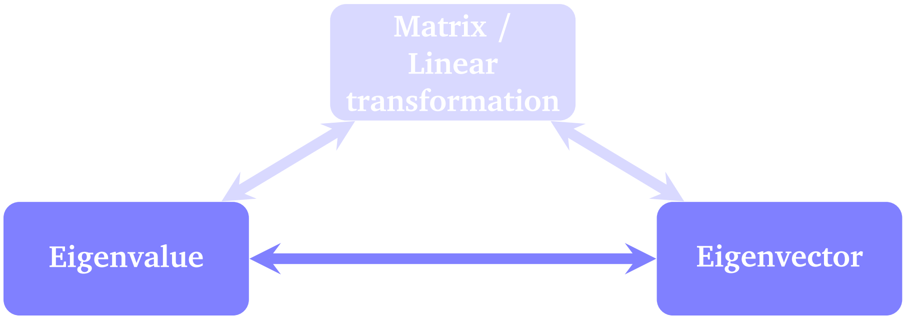


Figure 5.29: A visual interpretation of the students' description of the relation between eigenvectors and eigenvalues, illustrating the possibility of their existence independent of a matrix.

Hence, our prompt did not seem to provoke an actual cognitive conflict and or lead to a revision of Robin's concept image. Instead, Robin appeared to presume the correctness of their written answer, and infer that eigenvectors and eigenvalues could exist independently of matrices.

However, it is our perspective that awareness of the fact that eigenvectors and eigenvalues can be understood as *properties* of matrices is crucial as it highlights the fundamental *relations* between these objects and the linear transformation. Recognising these relations between the objects can serve as important motivation for various applications of eigenvectors and eigenvalues, such as the solution of differential equations or stability and equilibriu analysis. Thus consciously omitting any of these objects is, in our opinion, a clear indication that their concept images are at an early stage of development. However, it is important to note that Robin is clear that they are guessing, and in doing so they admit a degree of uncertainty. Thus, we do not find it appropriate to categorise this as a misconception per se. Instead, we see this as a potential angle to further guide the student to reflect upon and develop their concept image.

Next, we consider a related aspect of students' concept images of students' concept images of eigenvectors and eigenvalues. That is, the apparent confusion in students' answers regarding the relationships that exist between the matrix or linear transformation, the eigenvalue and the eigenvector.

5.3.2 Confusion of Roles

In the data, we also observed examples where students confused the roles of the matrix, eigenvector and eigenvalue, particularly in regards to which objects act upon which, and the resulting effects. For instance, one student, which we will call Student Q, stated that: "An eigenvector is a vector which does not change the direction of a matrix in a transformation, but could change size." (see Figure 5.30).

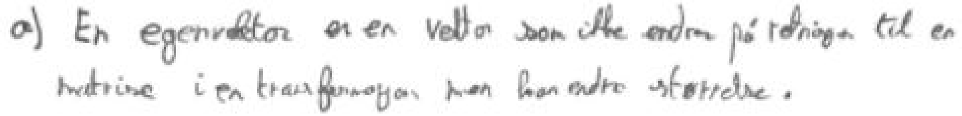

a) En egenvektor er en vektor som ikke endrer på retningen til en matrise i en transformasjon men kan endre størrelse.

Figure 5.30: Student Q's answer to Task 9 a), giving a description of how an eigenvector acts upon a matrix by changing its magnitude.

As we have observed earlier, a matrix acts upon its corresponding eigenvector by scaling it, allowing the magnitude to change or remain unchanged. Furthermore, in the real case, the direction of the vector is maintained or reversed. Thus, it is inaccurate to suggest that the eigenvector acts upon the matrix, as described by Student Q. A more appropriate description would imply that it is the matrix that acts upon the eigenvector. Additionally, as we discussed in an earlier example, it is not meaningful to discuss the direction of a matrix. The concept of direction is typically associated with vectors. Thus, it appears Student Q is aware that there is a relationship between the matrix, linear transformation and eigenvector, and that the relationship has visual interpretations. However, there appears to be some confusion concerning the nature of these relations.

However, so far we have only discussed Student Q's answer to Task 9 a), where they were asked to explain the concept of eigenvectors. As eigenvectors and eigenvalues are closely interconnected, we may examine their description of eigenvalues in b) to gain further insights into their concept images (See Figure 5.31). Here, Student Q described the relationship between the eigenvalue and the eigenvector as follows: "The eigenvalue indicates the size of the eigenvector and which direction. So, 1 means the same size, $2 = 2x$ the length, $-1 =$ opposite direction.".

Considering this, we are of the impression that Student Q tried to describe how multiplication by the eigenvalue affects the *eigenvector* (not the matrix) by preserving its magnitude ($\lambda = 1$), doubling its magnitude ($\lambda = 2$) or reversing its direction and preserving magnitude ($\lambda = -1$). Thus, our analysis of parts a) and b) reveal potential inconsistencies in their concept image. Additionally, the description in Task 9 a) can illustrate inconsistencies between the students' concept

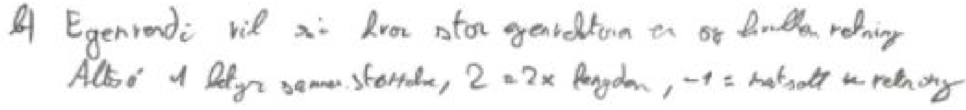


Figure 5.31: Student Q's answer to Task 9 b), describing the eigenvalues' effect on the corresponding eigenvectors, along with specific examples..

image of eigenvectors and the formal concept definition, which Tall and Vinner (1981) characterised as a more problematic form of inconsistency. However, an alternative interpretation is that our analysis can exemplify how the translation from ideas to written responses may result in the loss or distortion of information. This example further underscores the importance of considering these closely interrelated concepts together to gain a comprehensive understanding of the students' concept images.

Next, we consider another important aspect of eigentheory, namely the number of eigenvectors associated with an eigenvalue, and by extension, a matrix.

5.3.3 The Number of Eigenvectors

The number of eigenvalues associated with an $n \times n$ matrix is n. However, some eigenvalues may be repeated (have a multiplicity greater than 1), resulting in at most n distinct eigenvalues for an $n \times n$ matrix. Conversely, as indicated by Student K (see Figure 5.20) and Student O (see Figure 5.25), there are infinitely many eigenvectors associated with an eigenvalue, as any scalar multiple of an eigenvector is also an eigenvector with the same eigenvalue. Hence, if $\vec{x}$ is an eigenvector corresponding to the eigenvalue λ, then $k\vec{x}$ is also an eigenvector corresponding to λ for all scalars k. In examining students' written answers, we came across answers suggesting students may have concept images conflicting with these aspects. An excerpt of a such an answer, provided by Student R, is depicted in Figure 5.32.

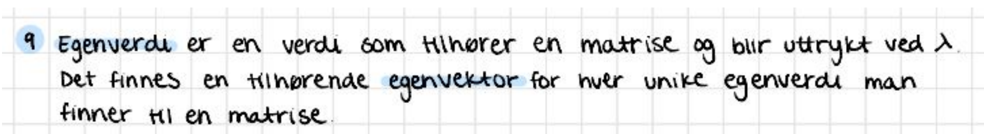


Figure 5.32: An excerpt of Student R's answer to Task 9 addressing the issue of the number of eigenvectors associated with each, unique eigenvalue.

In describing eigenvectors and eigenvalues, Student R stated that: "An eigenvalue is a value that belongs to a matrix and is denoted by λ. For every unique eigenvalue found for a matrix, there exists a corresponding eigenvector.". While it is true that each eigenvalue has a corresponding eigenvector, we were curious if students were aware *how many* eigenvectors were associated with an eigenvalue. Thus, we decided to include this aspect in the subsequent interviews. However, it is important to note that Student R did not take part in the interviews. Thus, the following excerpts are collected from interviews with other students.

In discussing Task 10, it was established that $\vec{x} = \begin{bmatrix}-1 & 2\end{bmatrix}^\mathsf{T}$ was an eigenvector to the matrix *A*. Then, the student was asked to explain if there were more possible eigenvectors of *A*, upon which the student explained that a 2×2 matrix has two possible eigenvectors. Consider the following excerpt:

> Interviewer: Are there other eigenvectors here?
>
> Robin: Mm.. Probably... Maybe? Mm...
>
> [long pause]
>
> Robin: [clears throat] There is something like, since it is... a matrix is... or I feel like I remember something about if it is... It is a matrix in $\mathbb{R}^2$, then there are two eigenvectors, or something like that, so then I guess there are two. Or there is one more possible eigenvector.
>
> Interviewer: Mm...
>
> Robin: Um... Yeah.

This answer could suggest that the student believes that a real 2×2 matrix has only two eigenvectors, rather than recognising that there are infinitely many possible eigenvectors. It is our perspective that the number of eigenvectors associated with an eigenvalue or a matrix is an interesting topic because it is closely connected to other key concepts in linear algebra that the students are familiar with. These include the notion of basis, span, eigenspace and linear (in)dependence.

However, an alternative interpretation is that Robin meant that the matrix *A* can have two *linearly independent* eigenvectors. The concept of linear independence was introduced to the students approximately five weeks prior to the homework on eigenvalues and eigenvectors. In another interview, Kim was asked to state the number of eigenvectors of *A*, and responded that it has two linearly independent eigenvectors:

> Interviewer: So if we consider the matrix A in Task 10, how many eigenvectors does this matrix have?
>
> Kim: Well, I would think it has two. Or that it could have two. But it... But it is... Because it becomes zero if you... Could it matter? Since it is... It is not symmetric, but it could cancel itself. But, uhm. If... If *A*... is diagonalisable, well, then it must have two linearly independent eigenvectors. So if you find out if *A* is diagonalisable, then you know if there are two eigenvectors or one.

Similar to the previous example, Kim initially stated that the matrix *A* from Task 10 has two eigenvectors. However, they then corrected themselves, in stating that *A could* have two eigenvectors. Additionally, they made connections to the notion of linear independence diagonalisation in stating that if *A* was diagonalisable, it would have two linearly independent eigenvectors. Making connections between concepts is considered an important aspect of understanding in linear algebra, as highlighted by Harel (1997). In our opinion, Kim's ability to make these connections indicates a developed concept image at an early stage in their learning process.

Next, we explore the aspect of students' experiences in working with the tasks we designed and implemented as part of their coursework.

5.3.4 The Students' Experiences with the Tasks

The interviews presented an opportunity to inquire about students' experiences with the tasks. We contend that by exploring their experiences, we can better understand their concept images of eigenvectors and eigenvalues, as well as their modes of thinking. Additionally, these insights into their experiences provide valuable information about their competencies and the challenges they encounter in learning about eigenvectors and eigenvalues.

Furthermore, it is important to note that our tasks were intentionally designed to elicit several aspects of students' concept images, possibly going beyond the scope of "traditional" tasks focused on developing computational proficiency. Consequently, we anticipated that at least some of our tasks would be perceived as different or inauthentic. As we shall see, the interviews provided an opportunity to test this hypothesis.

On a side note, understanding students' experiences can generate valuable insights for the future PhD study, where we aim to design instructive tasks and improve teaching practices at the university level.

Upon analysing the interview recordings and transcripts, we observed that students had various experiences in working with the tasks, particularly in relation to the perceived level of difficulty. When prompted to describe the level of difficulty, Kim expressed that they had trouble meeting the deadline, which consequently limited the amount of time and effort they could allocate to our tasks.

> Interviewer: Okay, so this homework assignment. Now it is... I will mainly ask you about these tasks [pointing to Tasks 9-12], their level of difficulty.
>
> Kim: Uhm... Well, uhm, I remember that when I worked on the assignment, I had limited time, so those tasks were a bit deprioritised."

Kim went on to explain that the tasks turned out to be more challenging than they had anticipated:

> Kim: Yeah, I guess I heard that they were kind of... Since we had not covered that topic much yet, there would be some easy tasks on eigenvectors and stuff.

Interviewer: Mm...

Kim: Uhm, but at that time, I did not think they were easy.

As we shall see later, Kim was not the only one who found the tasks to be challenging. However, another student, Sam, expressed the opinion that our tasks were easier compared to the other tasks in the homework assignment:

Sam: Uhm... So it was this last chapter that you had? Then it was a lot easier.

The students who found the tasks challenging gave several reflections on what made the tasks difficult. In Kim's own words:

Kim: And I felt it was more about understanding than actually calculating. Uhm...

Interviewer: Mm...

Kim: So then, I did not have a complete understanding of what... what eigenvectors and eigenvalues were.

In the excerpt above, Kim explained that they perceived the tasks to be more oriented towards what they refer to as "understanding" rather than calculation. This suggests that our tasks were perceived as somewhat different from the tasks they were accustomed to. In other words, they could be seen as inauthentic. However, it is important to acknowledge that the students' perception of "understanding", a complex and potentially contentious term, may differ from our own.

Recall now that in Section 5.1.5.1, Iben expressed their discomfort with explaining concepts and their confidence in performing computations. They expressed feeling more comfortable with "applying" concepts, rather than explaining them. It is our perspective that this highlights the student's confidence, and possibly also their preference for hands-on procedures, rather than abstract conceptualisation. This sentiment was not unique to Iben, as Robin also reflected upon their experience working with Task 9:

Interviewer: What about Task 9, the first task?

[Long pause]

Robin: I found that one a bit difficult, like... uhm. To articulate, because I kind of understood how to find it and how to test it and stuff like that, and then we were supposed to explain what it actually was. So... That was a bit trickier.

The excerpt above suggests the students perceive themselves as confident and competent in performing calculation-based procedures, such as computing eigenvectors and eigenvalues. However, they expressed a perceived lack of competence in explaining concepts. We found this both surprising and interesting, considering the majority of the students in our study were able to produce relatively accurate descriptions of eigenvectors and eigenvalues. Together, the students presented various concept images and modes of thinking. While some responses were elaborate and accurate, reflecting several aspects of their concept image, others were less comprehensive. Nevertheless, all students were able to communicate some aspects of their concept images, which according to our interpretation of Vinner (2002), is a necessary condition for understanding.

The attentive reader may have noticed that our analysis includes transcripts of utterances from four out of the five interviewed students. Quotes from the fifth student were omitted due to their responses closely aligning with those of Sam.

Chapter 6

Discussion

This master study aimed to gain more knowledge on students' understanding of eigenvectors and eigenvalues. In particular, we aimed to describe students' concept images and their modes of thinking as expressed in their written answers to our tasks and subsequent interviews. We addressed the followings research questions:

1. *What concept images can be described from the students' reasoning about eigenvectors and eigenvalues?*
2. *What modes of thinking can be identified in the students' reasoning about eigenvectors and eigenvalues?*

In the following, we offer an overview of our main findings to demonstrate that our analysis effectively addressed our research questions, which were approached and answered together. We engage in a brief comparative analysis of our results with studies from the literature review and theory, to enhance our understanding of students' comprehension of eigentheory and showcase the contributions of this master study to the existing body of knowledge.

Subsequently, we engage in a critical reflection on our chosen theoretical perspective, evaluating the extent to which it enabled us to address our research questions. Furthermore, we briefly introduce an alternative theoretical perspective that may inform our future research.

We then turn our attention to the limitations inherent in our chosen methods and their impact on the extent to which our research questions were addressed. We include some reflections on how these limitations can be mitigated in future studies involving students' written work and interviews. We conclude our discussion by offering an outlook on potential future research areas that have emerged from this study, to support students in their learning of linear algebra at the tertiary education level.

6.1 Main Findings and Empirical Comparison

Our analysis has highlighted the presence of diverse concept images and modes of thinking in students' written and oral answers. Not only have we observed the manifestation of Sierpinska's (2000) analytic-structural, analytic-arithmetic and synthetic-geometric mode, we have also identified answers encompassing elements from multiple modes of thinking. This discovery led us to develop additional categories for the combinations of modes, namely the structural-arithmetic, structural-geometric, arithmetic-geometric and structural-arithmetic-geometric mode.

Through our analysis of Task 9, we discovered that a considerable number of students adeptly integrated elements of an analytic-structural mode of thinking. They skillfully connected eigenvectors and eigenvalues with other crucial concepts of linear algebra, such as span, linear transformations, and vector spaces — an ability emphasized by Harel (1997) as vital for understanding in linear algebra. Additionally, when describing eigenvectors and eigenvalues using relevant formulas like the homogeneous system of equations ($(A-\lambda I)\vec{x} = \vec{0}$) and the roots of the characteristic polynomial ($\det(A-\lambda I) = 0$), the students showcased their mathematical prowess.

Our analysis of students' answers to Task 9 revealed that many students incorporated elements of an analytic-structural mode of thinking. They connected eigenvectors and eigenvalues to several key concepts, such as span, linear transformations and vector spaces - an ability highlighted by Harel (1997) to be an important aspect of understanding linear algebra. In describing eigenvectors and eigenvalues using the formulas for computing them, some students were able to demonstrate their engagement with an arithmetic mode of thinking as well.

Remarkably, as many as half of the students participating in this study could aptly describe the geometric or visual properties of eigenvectors and eigenvalues, such as scaling, flipping, or remaining on the same line, showcasing a synthetic-geometric mode of thinking. Although we explicitly encouraged the inclusion of sketches, only some students took this approach. However, the diversity in their sketches, with some reflecting analytic-structural modes of thinking and others suggesting synthetic-geometric modes of thinking, demonstrated an ability to visualise both examples and non-examples of eigenvectors and eigenvalues. This ability to handle representations, according to Duval (2006), may promote a deeper understanding of the mathematical concepts.

Interestingly, our analysis revealed that the majority of students engaged with multiple modes of thinking in their characterisations of eigenvectors and eigenvalues. It is our perspective that in doing so, the students demonstrated a cognitive flexibility in thinking about these concepts, and concept images encompassing various elements.

Sierpinska (2000) found no empirical foundation to assert that students generally prefer one mode of thinking over another. Upon reflecting on our own analysis, we concur with this observation. Instead, we have developed an impression that specific types of tasks might elicit particular modes of thinking in students. In Task 9, where students were asked to explain eigenvectors and eigenvalues, we observed that most students presented answers incorporating elements from multiple modes of thinking. In contrast, an overwhelming majority of students gave answers corresponding to a single mode of thinking in Task 10, namely the analytic-structural mode.

Diverging from the findings of Thomas and Stewart (2011), our study revealed that many students demonstrated an ability to describe eigenvectors and eigenvalues in terms of their geometric properties in Task 9. We found it particularly interesting that while numerous students incorporated such elements of the synthetic-geometric mode in describing eigenvectors and eigenvalues, only a few students applied these characteristics in Task 10 to determine whether $\vec{x}$ was an eigenvector of the matrix A. Based on these findings, we hypothesise that our tasks may have lead students to particular modes of thinking. This observation highlights a potential limitation of our current task design.

The Students' Experiences with the Tasks

To obtain a comprehensive and nuanced understanding of students' concept images and modes of thinking, we conducted individual, semi-structured interviews with a targeted sample of five students. By incorporating our analysis of these interviews alongside their written answers, we aimed to bridge the gap between their expressed knowledge and the knowledge they may possess. This approach deepened our insights into students' concept images of eigenvectors and eigenvalues and their modes of thinking about these objects. Furthermore, the interviews allowed us to uncover another interesting aspect of students' learning process, namely their experiences in working with the tasks.

In the previous chapter, we saw that the majority of these students experienced difficulties with some of our tasks, with Task 9 proving particularly challenging. Interestingly, some of these students expressed general struggles with tasks involving explanation, while expressing confidence in performing computation, such as in Task 10. These findings align with an earlier study by Thomas and Stewart (2011), which highlighted students' proficiency in performing computations.

Our findings suggest that students may not have received adequate training in tasks involving explanations. Nonetheless, we contend that the ability to provide clear and coherent explanations is not only an essential characteristic of understanding a concept, but also an important skill in personal and professional life. Consequently, if our students perceive themselves as ill-equipped to handle such tasks, we posit that it is necessary to provide them with targeted-training to foster this skill. In doing so, we may effectively close the gap between their current proficiency and the desired level of competence and confidence in solving tasks involving explanations.

6.2 Reflections on the Theoretical Framework

In this master's study, we made use of Tall and Vinner's concept image and Sierpinska's modes of thinking as theoretical lenses to illuminate aspects of students' understanding of eigentheory. Additionally, we adopted Von Glasersfeld's radical constructivism as a background theory to clarify our perspective on the nature of learning and understanding, and how they occur. However, it should be noted that we do not consider radical constructivism as a main part of our analytical framework. Taking a reflective stance, we now engage in a critical evaluation of the strengths and limitations of these theories in providing insights and addressing our objectives.

6.2.1 Radical Constructivism as the Theoretical Foundation

Radical constructivism was adopted as the underlying perspective on learning, which posits that knowledge is constructed and resides in minds of individual's, rendering direct observation of students' understanding of eigenvectors and eigenvalues inaccessible. Instead, we relied on students' written and oral answers to gain insights into their thinking, assuming that their communication reflects their understanding adequately. However, our experience revealed the limitations of radical constructivism in providing explicit guidance on which aspects of communication to focus on when examining students' thinking.

This has led us to consider Anna Sfard's commognition, where communication and cognition are viewed as closely connected processes, while still respecting their differences (the reader is referred to Sfard, 2015; 2020 for further insights on this notion). The commognitive perspective encompasses both individual cognitive construction as well as the co-construction of meaning through social interaction and communication. Furthermore, commognition offers a detailed analytic approach to examining students' thinking by considering the dynamic interactions in communication and thinking in the learning process. Thus, it is our opinion that the commognititve perspective presents as a promising lens for investigating the nuances of students' understanding beyond the individual aspect emphasised by radical constructivism, and that it could be applied in our future research.

6.2.2 The Concept Image as a Theoretical Lens

In our quest to gain insights into students' understanding of eigenvectors and eigenvalues, we made use of Tall and Vinner's notion of the concept image, wherein the possessing of a concept image was deemed a necessary condition for understanding the concept. Here, the concept image was desribed as consisting of all the cognitive structures associated with a concept, including all its visual representations, formulas, examples and non-examples. This holistic perspective provided us with a range of observable indicators to effectively describe students' understanding of the concepts of eigenvectors and eigenvalues.

Nevertheless, the complexity and richness inherent in the concept image rendered it too vast to describe in its entirety. To address this challenge, we made the decision to narrow our focus to Sierpinska's modes of thinking, which we identified as partially overlapping with the concept image. In adopting this deductive approach, we aimed to shed light on specific aspects of the concept image, while acknowledging that this decision would leave some aspects of the concept image unexplored.

During our analysis of students' responses, we came across intriguing aspects of students' concept images that were not adequately accounted for within the modes of thinking framework. In response, we developed an inductive approach to describe these aspects, which we referred to as the objects and their relations. By incorporating this lens into our analysis, we sought to describe a broader portion of students' concept images. However, it is important to acknowledge that despite our efforts, there may still be dimensions of students' concept images pertaining to eigenvectors and eigenvalues that remain unexplored in our study.

6.2.3 The Modes of Thinking as a Theoretical Lens

The incorporation of Sierpinska's (2000) modes of thinking in our selection of theories allowed us to effectively describe specific facets of students' concept images. However, it is important to acknowledge that this theory was originally developed in the context of linear algebra in general, not for eigentheory in particular. As a consequence, we had to interpret and adapt her framework to suit our specific focus on the concepts of eigenvectors and eigenvalues.

During our analysis of students' written responses, we encountered a challenge with our interpretation of the three modes, as many of the students' answers exhibited characteristics that fell between or combined multiple modes. As a result, we found it necessary to expand upon Sierpinska's framework by introducing additional categories, namely the structural-arithmetic, structural-geometric, arithmetic-geometric and structural-arithmetic-geometric mode, to better capture the complexities and nuances observed in the data. Thus, the Venn diagram (see Figure 5.15 and Figure 5.26) reflect this process of modifying and expanding upon Sierpinska's (Sierpinska, 2000) original modes of thinking.

It is worth noting that our study is not alone in building upon this framework. In a previous study conducted by Gol Tabaghi and Sinclair (2013), they also employed the modes of thinking to analyse students' answers. In their work, they identified a mode of thinking which they named the dynamic-synthetic-geometric mode, which emerged from students' emphasis on the dynamic aspects of eigenvectors.

Despite our efforts to develop the modes of thinking as an analytical framework, we encountered difficulties in distinguishing between them at times. A notable example arises from how the inclusion of the eigenequation was interpreted as aligning with different modes of thinking in Tasks 9 and 10. As we have seen, in the context of Task 9, *describing* eigenvectors and eigenvalues as fulfilling the eigenequation could suggest both analytic-structural and analytic-arithmetic modes of thinking. As fulfilling the eigenequation can be considered a defining property of eigenvectors and eigenvalues, the inclusion of the eigenequation could suggest an analytic-structural way of thinking about these concepts. However, the eigenequation can be utilised to compute eigenvectors and eigenvalues through a series of arithmetic steps, suggesting the potential presence of an analytic-arithmetic mode of thinking as well. In Task 10 on the other hand, the use of the eigenequation to *verify* the vector $\vec{x}$'s status as an eigenvector was associated with an analytic-structural mode of thinking exclusively. These complexities highlight the limitations and potential subjectivity in interpreting and employing these modes to characterise students' thinking.

In aligning our study with an interpretative research paradigm, we recognise that our interpretation of the modes might not be flawless, as it could be influenced by our subjective meanings and experience. Moreover, we acknowledge that others might perceive the modes differently than we do. Thus, we contend that our future studies might profit from considering alternative theoretical lenses, which may better capture the intricacies and variations in students' understanding of linear algebra and eigentheory in particular. However, we caution that different theoretical lenses will inevitably highlight certain aspects of students' reasoning, while leaving others unseen.

Thus far, we have discussed some affordances and constraints of employing the chosen theoretical lenses to investigate the research questions and suggested some alternative approaches for future studies. In the following section, we engage in a critical reflection on the benefits and limitations associated with the selected methods, aiming to evaluate their ability to effectively address the research questions.

6.3 Reflections on the Methods

In this study, we collected students' written homework and conducted semi-structured individual interviews to explore students' concept images of eigenvectors and eigenvalues, as well as their modes of thinking.

6.3.1 Reflections on the Tasks

In light of our objective to describe students' understanding of eigenvectors and eigenvalues, we designed four tasks as part of students' written homework. However, for the purpose of our analysis, we chose to focus specifically on students' answers to Tasks 9 and 10, as they aligned more closely with our research questions.

Through our analysis of students' written answers to these tasks, we were able to identify elements of students' concept images. However, we must acknowledge that due to the sheer vastness of the concept image and our choice of theoretical lenses, we were unable to capture students' concept images in their entirety. During our analysis, we encountered an interesting example of a student exhibiting one mode of thinking in Task 10 and additional modes in Task 9. However, we did not perform such comparisons across tasks for all individual students. Consequently, it is likely that additional insights could have been gained had we compared each student's response to Task 9 with their response to Task 10. We see such comparisons and the inclusion of Tasks 11 and 12 as potential for our future research.

6.3.2 Reflections on the Interviews

In considering students' written answers as a fallible representation of their knowledge, we acknowledged the inherent limitations of solely relying on students' written answers to address the research objectives. To address these limitations, we decided to conduct interviews to gain a more comprehensive and nuanced understanding of students' concept images and modes of thinking.

In the previous chapter, we witnessed an interesting example of a student who demonstrated different modes of thinking in their written answer compared to their oral response during the interviews. This observation supports our belief that triangulation of data can yield additional insights regarding students' reasoning about eigenvectors and eigenvalues. We posit that students may not express their complete line of thinking in writing, as they may strive to produce a polished version with exclusively correct answers for their homework assignments. Consequently, we conjecture that their written responses primarily reflect the outcomes of their learning process. However, through the interviews, we observed that students' oral answers offered a glimpse into the stages of their thinking, allowing us to witness their learning process in action. Thus, we hypothesise that students' oral reasoning may align more closely with their actual thinking compared to their written answers.

However, it is important to acknowledge the inherent limitations associated with interviews as well. For this master study, we were only able to interview a limited selection of five students out of a total of 170. Consequently, it is likely that our findings may not capture the full range of concept images among these 170 participants. Furthermore, the interviews took place 2-4 weeks after the students submitted their homework, potentially affecting their ability to recall and reason about these concepts. Thus, it is possible that the passage of time influenced their explanations and insights provided during the interviews.

During the interviews, we observed a potential influence of social desirability bias, where interviewees (in our case, the students) tend to provide answers they perceive to be socially acceptable or desirable (Bordens & Abbott, 2011, p. 147; p. 273). As mentioned earlier, the interview setting might resemble an oral examination for students, where an expert (in this case, the interviewer) tests and evaluates the students' understanding of a topic (in our case, eigenvectors and eigenvalues). This could have led students to tailor their answers to align with their perception of the interviewer's expectations, such as providing correct answers about eigenvectors and eigenvalues or adjusting their experiences with the tasks to match what they believed the interviewer desired. As a result, we contemplate whether the students' reasoning about eigenvectors and eigenvalues would have differed if they had engaged in conversations with peers instead. While we were unable to test or establish the presence of such biases at the time, it is crucial to acknowledge the potential influence of bias on our research findings.

6.3.3 Reflections on the Combined Methods

Through our analysis of both students' written and oral answers, we observed concept images exhibiting various stages of development. Some answers revealed an ability to engage in multiple modes of thinking, indicating the presence of rich and comprehensive concept images. Other responses were brief, providing limited insight into students' concept images. Moreover, we came across answers that suggested flawed concept images, such as believing there exists a finite number of eigenvectors corresponding to a matrix. These findings align with an observation made by Wawro et al. (2018), who cautioned against an overemphasis on computational procedures as this may lead students to erroneously deduce that there is only one eigenvector to an eigenvalue (when there are in fact infinitely many).

Our approach not only allowed us to identify areas of students' proficiency, such as their ability to compute eigenvectors and eigenvalues, and to characterise them through multiple modes of thinking. It also enabled us to recognise potential flaws and misconceptions in their concept images of these objects. It is our perspective that these insights hold great promise for the development of future tasks that effectively address these challenges and support students in building a more robust understanding of these concepts. For instance, tasks could be designed to prompt students to reflect upon the number of eigenvectors associated with a matrix or eigenvalue. Furthermore, prompting students to connect this reflection to the visual representations of eigenvectors and their corresponding eigenspaces, could contribute to deepen their understanding of these mathematical concepts.

6.4 Summary and Outlook

In our quest to describe students' understanding of eigenvectors and eigenvalues, it is important to acknowledge that as students express their ideas in sentences, (whether written or oral) and illustrations, there is a potential for meaning to be lost, transformed or even added to their utterances. Therefore, we must acknowledge that our methods do not enable us to identify all concept images and modes of thinking employed by the students. However, by analysing the collected data through our theoretical lenses, we have been able to provide descriptions of certain aspects of students' concept images of eigenvectors and eigenvalues, and highlight some modes of thinking that emerged from the data.

Thus, our theoretical lenses and methodological approaches enabled us to effectively address the research questions, while leaving several potential pathways to be explored in future research. For instance, we believe that further insights can be gained from analysing the students' written responses and oral reasoning concerning Tasks 11 and 12, and comparing their answers across all four tasks.

The findings of this master's study are expected to inform the future PhD study, which seeks to develop tasks that effectively address students' challenges and foster a deeper understanding of key concepts in linear algebra. While this master study has focused on individual students' concept images and modes of thinking, it could be interesting to explore the collective meaning-making as students collaborate in future studies. We contend that the ability to collaborate effectively to solve conceptually challenging problems is an important skill both in academic and professional contexts.

Chapter 7

Conclusion

In this study, we aimed to explore students' concept images of eigenvectors and eigenvalues, and the modes of thinking, as presented their written answers and oral reasoning concerning a set of specifically designed tasks. Through the analysis of these answers, we have encountered a range of concept images pertaining to eigenvectors and eigenvalues among the participating students. Furthermore, we have observed answers that align with our interpretation of Sierpinska's modes of thinking, as well as our added modes, which we called the structural-arithmetic, structural-geometric, arithmetic-geometric and structural-geometric-arithmetic. Our analysis has shed light on concept images at various stages of their development. While some students gave comprehensive answers suggesting rich concept images, others presented more brief answers suggesting concept images at the early stages of formation. It is worth noting that our examination also revealed instances of confusion and occasional conflicts within students' concept images, underscoring the dynamic nature of the concept image and its role in students' learning process.

Through the interviews, like the one conducted with Robin in Section 5.3.1, we made a noteworthy observation that students appeared to develop their concept images and incorporate additional modes of thinking in articulating their ideas orally. We posit that a similar learning process is likely to unfold as students engage in the process of writing their homework. However, if we solely analyse students' written work, we gain insights solely into the learning outcome, devoid of understanding for the learning process that occurs in the shadows. This raises an intriguing question of how we can encourage students to integrate their thinking process into their written responses, presenting an avenue to be explored in future research.

From this study, I personally have seen the valuable insights that can be gained from allowing students to express their knowledge in multiple ways. Simultaneously, it has revealed the limitations inherent in relying solely on students' written work to evaluate their understanding of mathematics. Engaging in conversations with students', both in and outside the interview situation, has shed light on another interesting aspect, as students report spending a significant amount of time to complete their homework. Yet, their final grade in the course is determined solely by a four-hour written exam at the end of the term. This creates a stark distinction between the continuous learning activities during the semester and the isolated assessment that awaits them. It is my perspective that as educators and researchers, we should carefully consider whether the current approach to learning and assessment can truly and accurately reflect students' comprehension. Perhaps embracing a more holistic approach with formative assessment, where assessment and learning walk hand in hand, could foster more meaningful and reflective learning experiences

Chapter 8

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Appendix A

Information Part 1

Appendix A contains the information provided to students about the purpose of collecting their homework, handling of the data and their rights as (potential) participants.

DEL 1

Emilie Lyse-Olsen
Mellomila 88
7018 Trondheim
Telefon: 94186805
E-post: emilie.l.olsen@ntnu.no

Trondheim, 28.06.2022

Til studenter i emnet TMA4110 høsten 2022 ved NTNU

Anmodning om tillatelse til innsamling av øvingsbesvarelser

Formål
Jeg er student på lektorprogrammet i realfag og følger integrert PhD-utdanning ved NTNU. Høsten 2022 fortsetter jeg arbeidet med min masteroppgave i matematikkdidaktikk, og skal i den forbindelse gjennomføre en undersøkelse med en gruppe studenter. Målet med undersøkelsen er å få kunnskap om studenters forståelse og utfordringer knyttet til et tema innen lineær algebra. På bakgrunn av denne undersøkelsen, vil jeg utvikle ressurser som kan bidra til å øke læringsutbyttet og berike studenters forståelse av lineær algebra gjennom arbeidet med både masteroppgaven og doktorgradsavhandlingen.

Datamateriale
Denne undersøkelsen består av studentenes skriftlige besvarelser på øvingsopplegget i emnet TMA4110 Matematikk 3. De skriftlige besvarelsene vil bli samlet inn gjennom den digitale læringsplattformen (Blackboard eller ovsys) hvor du normalt leverer øvingene dine. Ved å gi ditt samtykke kan du, på bakgrunn av din besvarelse, senere kan bli kontaktet med forespørsel om å delta i et oppfølgingsintervju, men du er ikke forpliktet til å delta på dette.

Behandling
Din besvarelse vil bli sett av meg, Emilie Lyse Olsen, og muligens også min veileder Yael Fleischmann, førsteamanuensis ved institutt for matematiske fag. I masteroppgaven og doktorgraden vil alle personer bli anonymisert for å unngå at enkeltpersoner kan identifiseres.

Varighet
Besvarelsen vil bli analysert og benyttet til å skrive min masteroppgave (planlagt levert våren 2023) og doktorgrad (planlagt levert høsten 2026). Det er ønskelig å oppbevare og eventuelt benytte datamaterialet fram til avsluttet doktorgrad. For å ta høyde for eventuelle forsinkelser settes varigheten til 31.12.2028.

Rett til innsyn i, retting eller sletting av personopplysninger
Forutsetningen for tillatelsen er at alt innsamlet materiale blir behandlet med respekt og blir anonymisert, og at prosjektet ellers følger gjeldende retningslinjer for etikk og personvern. Det er frivillig å delta i denne undersøkelsen og du kan til enhver tid trekke seg fra deltakelse uten å måtte oppgi noen grunn til det. Så lenge du kan identifiseres i datamaterialet har du rett til innsyn i hvilke personopplysninger som er registrert om deg, rett til å få utlevert en kopi av personopplysningene som er registrert om deg, rett til å få rettet personopplysninger om deg, rett til å få slettet personopplysninger om deg, og rett til å sende klage til Datatilsynet om

behandlingen av dine personopplysninger. Ved ønske om innsyn, retting eller sletting, ta kontakt med meg, Emilie Lyse-Olsen (se kontaktinformasjon øverst på første side).

Dersom du vil vite mer om dette prosjektet, eller hva det innsamlede materialet skal brukes til, kan du kontakte meg per e-post eller telefon (se øverst på første side for kontaktinformasjon).

Kontaktinfo for øvrige involverte
Veileder er Yael Fleischmann: tlf.: 96732597; e-post: yael.fleischmann@ntnu.no.

NTNUs personvernombud er Thomas Helgesen: tlf. 93079038; e-post: thomas.helgesen@ntnu.no.

Dersom du har spørsmål knyttet til NSD sin vurdering av prosjektet, ta kontakt med:
- NSD – Norsk senter for forskningsdata AS på epost (personverntjenester@nsd.no) eller på telefon: 55 58 21 17.

Jeg håper du synes denne forskningen er av verdi, og at du er villig til å være med på den.

På forhånd takk!

Vennlig hilsen

Emilie Lyse Olsen

Appendix B

Consent Form

Appendix B contains a copy of the digital consent form in Nettskjema.

Nettskjema

Samtykkeskjema TMA4110

Hva heter du (fornavn og etternavn)?

Hva er din e-postadresse?

Hva er ditt mobilnummer?

Hva er din fødselsdato?

Hvilket studieprogram følger du?

Som del av arbeidet med min masteroppgave og doktorgradsavhandling, ber jeg om tillatelse til å anvende og gjengi deler av dine besvarelser på øvingsopplegget i TMA4110. Forutsetningen for tillatelsen er at arbeidet blir anonymisert og behandlet med respekt, samt at prosjektet følger retningslinjer for etikk og personvern i tråd med vurderingen fra Norsk senter for forskningsfata (NSD). Formålet med prosjektet er å utvikle læringsressurser og undervisningssekvenser som støtter studenters læring. Du kan lese mer om prosjektet, samt finne min kontaktinformasjon dersom du har spørsmål eller ønsker å trekke tilbake ditt samtykke her:
https://drive.google.com/file/d/1CqcX4JNOYYvogSofbp_oB22HobxK6JC_/view?usp=sharing
Huk av nedenfor dersom du vil gi tillatelse.

Jeg gir tillatelse til å anvende og gjengi deler av min besvarelse på øvingsopplegget i TMA4110.

Appendix C

Information Part 2

Appendix C contains the information provided to students about the purpose and duration of the interviews, handling of the data and their rights as (potential) participants.

DEL 2

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Trondheim, 28.06.2022

Til studenter i emnet TMA4110 høsten 2022 ved NTNU

Anmodning om tillatelse til video-/lydopptak av intervju

Formål
Jeg er student på lektorprogrammet i realfag og følger integrert PhD-utdanning ved NTNU. Høsten 2022 fortsetter jeg arbeidet med min masteroppgave i matematikkdidaktikk, og skal i den forbindelse gjennomføre en undersøkelse med en gruppe studenter. Målet med undersøkelsen er å få kunnskap om studenters forståelse og utfordringer knyttet til et tema innen lineær algebra. På bakgrunn av denne undersøkelsen, vil jeg utvikle ressurser som kan bidra til å øke læringsutbyttet og berike studenters forståelse av lineær algebra gjennom arbeidet med både masteroppgaven og doktorgradsavhandlingen.

Datamateriale
Denne undersøkelsen består av studentenes skriftlige besvarelser på øvingsopplegget i emnet TMA4110 Matematikk 3, samt video-/lydopptak av intervjuer med enkelte studenter. På bakgrunn av din besvarelse ber jeg deg om å delta på et kort intervju (maksimalt 45 minutter) hvor du kan bli bedt om å utdype, forklare og begrunne svarene på din øving. For å få så nøyaktige data som mulig, er det ønskelig å ta video-/lydopptak av intervjuet. Intervjuet vil gjennomføres fysisk eller digitalt vha. videosamtaletjenesten Zoom, avhengig av hva som passer best for deg. Det understrekes at intervjuet ikke vil benyttes i vurderingen av ditt arbeid i emnet TMA4110.

Behandling
Opptaket og eventuelle transkripsjoner av dette vil kun bli sett av meg, Emilie Lyse Olsen, og muligens også min veileder Yael Fleischmann, førsteamanuensis ved institutt for matematiske fag. I masteroppgaven og doktorgraden vil alle personer bli anonymisert for å unngå at enkeltpersoner kan identifiseres.

Varighet
Opptaket og eventuelle transkripsjoner av dette vil bli analysert og benyttet til å skrive min masteroppgave (planlagt levert våren 2023) og doktorgrad (planlagt levert høsten 2026). Det er ønskelig å oppbevare og eventuelt benytte datamaterialet fram til avsluttet doktorgrad. For å ta høyde for eventuelle forsinkelser settes varigheten til 31.12.2028.

Rett til innsyn i, retting eller sletting av personopplysninger
Forutsetningen for tillatelsen er at alt innsamlet materiale blir behandlet med respekt og blir anonymisert, og at prosjektet ellers følger gjeldende retningslinjer for etikk og personvern. Det er frivillig å delta i denne undersøkelsen og du kan til enhver tid trekke seg fra deltakelse

uten å måtte oppgi noen grunn til det. Så lenge du kan identifiseres i datamaterialet har du rett til innsyn i hvilke personopplysninger som er registrert om deg, rett til å få utlevert en kopi av personopplysningene som er registrert om deg, rett til å få rettet personopplysninger om deg, rett til å få slettet personopplysninger om deg, og rett til å sende klage til Datatilsynet om behandlingen av dine personopplysninger. Ved ønske om innsyn, retting eller sletting, ta kontakt med meg, Emilie Lyse-Olsen (se kontaktinformasjon øverst på første side).

Dersom du vil vite mer om dette prosjektet, eller hva det innsamlede materialet skal brukes til, kan du kontakte meg per e-post eller telefon (se øverst på første side for kontaktinformasjon).

Kontaktinfo for øvrige involverte
Veileder er Yael Fleischmann: tlf.: 96732597; e-post: yael.fleischmann@ntnu.no.

NTNUs personvernombud er Thomas Helgesen: tlf. 93079038; e-post: thomas.helgesen@ntnu.no.

Dersom du har spørsmål knyttet til NSD sin vurdering av prosjektet, ta kontakt med:

- NSD – Norsk senter for forskningsdata AS på epost (personverntjenester@nsd.no) eller på telefon: 55 58 21 17.

Jeg håper du synes denne forskningen er av verdi, og at du er villig til å være med på den.

På forhånd takk!

Vennlig hilsen

Emilie Lyse Olsen

Appendix D

Interview Guide (Example)

Appendix D gives an example of an interview guide. It should be noted that while all the interviews followed the same basic structure, the semi-structured approach provided the flexibility to omit certain questions while incorporating others.

Intervjuguide

Innledning

- **Takk:** Takke respondenten for oppmøte
- **Formålet med intervjuet**: Få dypere innsikt i studentens forståelse av egenvektorer og resonnement i oppgavene.
- **Anonymitet**: Informere om databehandling og rett til å trekke seg fra intervjuet.
- **Innhold**:
  - Bakgrunn (studieprogram, arbeidsvaner osv)
  - Spørsmål om oppgavene 9.-12.
  - Spørsmål om din opplevelse av innleveringen og intervjuet
  - Siste 10 min styrer respondenten (ekstra tid)
- **Tid**: Ca 45 minutter.
- **Hjelpemidler**
  - Du får skrivesaker, ark, oppgaveteksten og en kopi av egen besvarelse

Spørsmål

Bakgrunn
- Hva heter du?
- Hvilket studieprogram følger du?
- Hvor langt har du kommet i progresjonen på ditt studieprogram?

Arbeidsvaner
- Hvordan jobber du med emnet (videoforelesninger, interaktive, mattelab, plenumsregning)?
- Hvordan jobber du med innleveringene (alene/sammen med andre, på campus/hjemme)?
- Hvilke ressurser bruker du for å løse oppgavene (youtube, forelesningsnotater, mattelab, geogebra, kalkulatorer på nett etc.)?
- Hvor mye tid brukte du på innleveringen?
- Hvordan opplever du vanskelighetsgraden/arbeidsmengden i emnet?
- Hvordan lærer du best (løse eksamensoppgaver, forelesning, lese etc.)?

Spørsmål om temaet og oppgavene

<u>Oppgave 9</u>
- Hva er en egenvektor og hva er en egenverdi?
  - Er det sånn at hvis en vektor er en egenvektor for en matrise A, så må den også være det for en annen matrise B?
  - Har alle kvadratiske matriser egenvektorer/egenverdier?

Oppgave 10

- Hvordan løste du oppgave 10?
    - Kunne du løst den på en annen måte?
    - Hvor mange egenvektorer har denne matrisen?

Oppgave 11

- Hvordan resonnerte du i oppgave b)?
    - c)?
    - d)?
- Du har skrevet at Ax ikke kan være 0-vektoren. Hvordan resonnerte du her?
    - Hva med nullmatrisen?
- Hvorfor kan ikke lambda være 0?
- Hvorfor kan ikke x være 0-vektoren?

Oppgave 12 - Byttet om egenverdiene

- Hvordan tenkte du for å løse denne oppgaven?
- Så her har du skrevet at e) ikke er en egenvektor fordi Ax = 0
- Hvordan ser du at egenverdien i a) er ½?
- Hvordan ser du at egenverdien i f) er 3)?

Tanker rundt oppgavene og emnet

- Hvordan opplever du arbeidet med å løse disse oppgavene?
- Var det noen oppgaver du opplevde som spesielt krevende? Evt hvilke og hvorfor?
- Hvordan vil du beskrive din egen forståelse av temaet egenvektorer, egenverdier, egenrom?

Avslutningsspørsmål

- Er det noe mer du vil legge til eller noen spørsmål?
- Kan jeg kontakte deg igjen dersom det blir aktuelt?
- Takk for at du stilte opp!

Appendix E

Tasks (Norwegian Version)

Appendix E shows the Norwegian version of Tasks 9-12, as it was presented to the students of this study,

Oppgaver til kapittel 10

9. Forklar med egne ord:

a) Hva er en egenvektor?

b) Hva er en egenverdi?

Bruk gjerne tegninger for å illustrere.

10. Begrunn hvorfor $\boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ er eller ikke er en egenvektor til matrisen $A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$.

11. La A være en $n \times n$-matrise og $\lambda \in \mathbb{R}$ være en egenverdi av A med tilhørende egenvektor $\boldsymbol{x} \in \mathbb{R}^n$. Avgjør i hver av deloppgavene nedenfor om påstanden er sann eller usann og forklar hvorfor:

a) $A\boldsymbol{x} = \lambda\boldsymbol{x}$

b) $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

c) $\det(A - \lambda I) = 0$

d) Multiplikasjon med $\boldsymbol{x}$ skalarer A med en faktor λ

e) Multiplkasjon med A skalerer $\boldsymbol{x}$ med en faktor λ

f) $A\boldsymbol{x}$ kan ikke være $\boldsymbol{0}$

g) λ kan ikke være 0

h) $\boldsymbol{x}$ kan ikke være $\boldsymbol{0}$

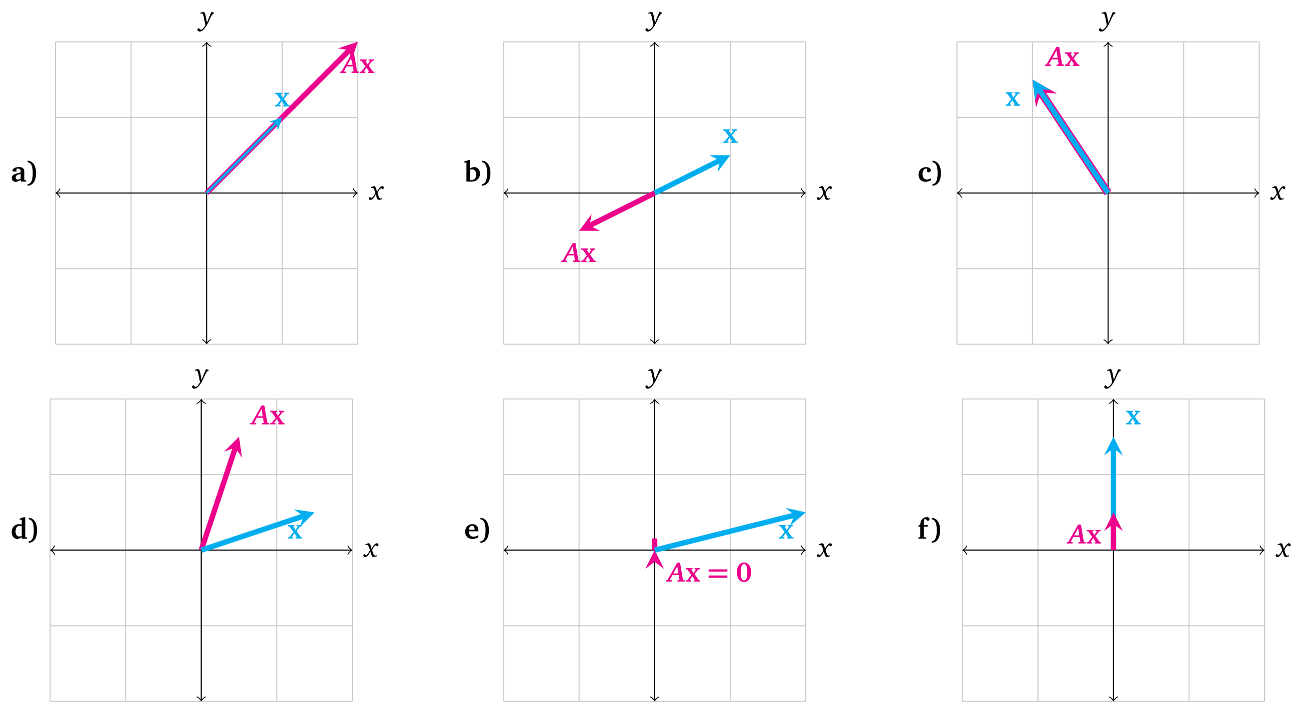
12. La A være en 2×2-matrise og $\boldsymbol{x}$ være en todimensjonal vektor. Svar på spørsmålene

1) Er $\boldsymbol{x}$ en egenvektor for A?

2) Hvorfor/hvorfor ikke?

3) Hvis $\boldsymbol{x}$ *er* en egenvektor, hva tror du egenverdien $\lambda \in \mathbb{R}$ er lik?

for hvert av tilfellene **a)–f)** nedenfor:



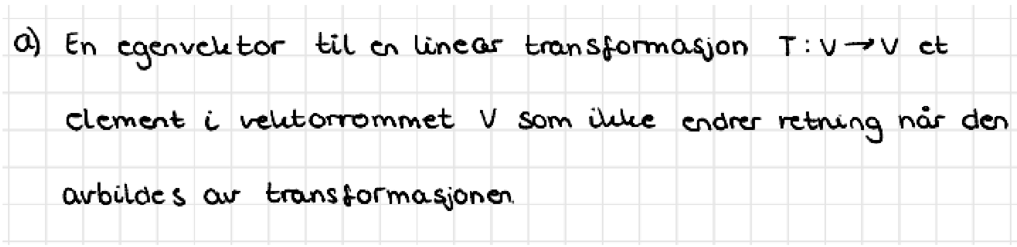
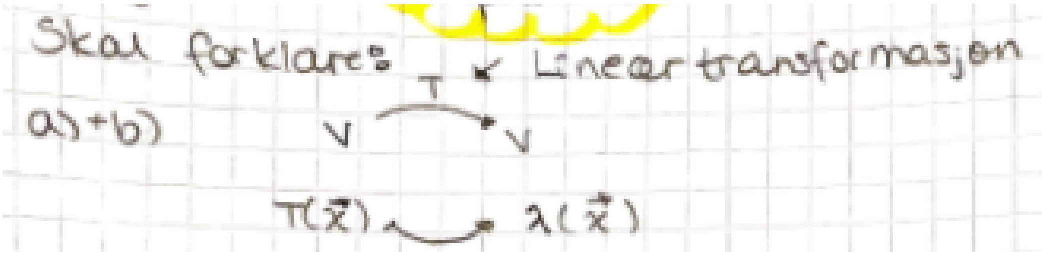
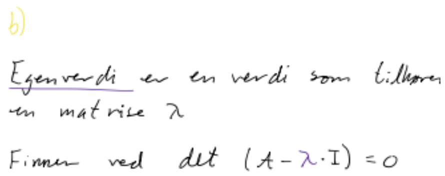
Appendix F

Coding handbook (Task 9)

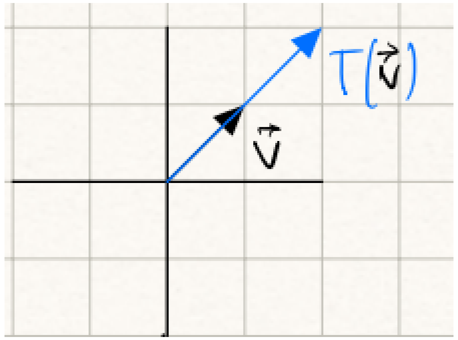
Appendix F shows the coding handbook that was created to code the students' answers to Task 9. The handbook contains the codes and a description of their use, some chosen examples and the corresponding modes of thinking.

Task 9

Code	Description	Example	Theme
9-transformation	**Transformation** Describing eigenvectors as vectors that are transformed by a linear transformation.	"En egenvektor til en lineær transformasjon T:V->V et element i vektorrommet V som ikke endrer retning når det avbildes av transformasjonen"	Analytic-structural
9-span	**Span** Describes eigenvectors as preserving their span (during a transformation/matrix multiplication).	"An eigenvector is a vector remaining in its own span after a transformation." "An eigenvector is a vector of a lin-trans which spans the same space before and after undergoing a transformation."	Analytic-structural
9-vector space	**Vector space** Tying eigenvectors and eigenvalues to vector spaces. Either describing eigenvectors as elements of vector spaces (fulfilling certain properties) or as vectors of transformations that span the same space before and after the transformation.	a) En egenvektor til en linear transformasjon $T: V \to V$ et element i vektorrommet V som ikke endrer retning når den avbildes av transformasjonen a) En egenvektor er en vektor til en lin.trans. som utspenner samme rommet før og etter å ha gjennomgått transformasjonen	Analytic-structural

9-image	**Image** Describing eigenvectors as imaged by linear transformations.	 a) En egenvektor til en linear transformasjon T : V → V et element i vektorrommet V som ikke endrer retning når den avbildes av transformasjonen	Analytic-structural
9-real.eig.val	**Figure_structural** A general sketch illustrating the notions of eigenvectors and eigenvalues, not restricted to particular examples of eigenvectors or eigenvalues.		Analytic-structural
9-eigen equation	**Eigen equation** Describing eigenvectors as fulfilling the eigenequation (Either $Ax = \lambda x$, $T(x) = \lambda x$ or a verbal rephrasing of either of these equations).	"$A\vec{x} = \vec{x}$. From the equation above is $\vec{x}$ - the vector, eigenvector, if $x \neq 0$ and A is an $n \times n$ matrix. [...].."	Structural-arithmetic
9-procedure	**Procedure** Describing eigenvectors and eigenvalues as stemming from the process for computing them.	 b) Egenverdi er en verdi som tilhører en matrise λ Finnes ved det (A − λ·I) = 0	Analytic-arithmetic

9-scaling	**Scaling** Describing eigenvectors s vectors that are scaled (by a transformation/matrix multiplication).	Egenvektor: En vektor som blir det samme som & gangeopp (skalert) når man tar A på $\vec{v}$ Egenverdi : tallet man skalerert $\vec{v}$ med når $\vec{v}$ er en egenvektor.	Synthetic-geometric
9-direction	**Direction** Describing eigenvectors as preserving their direction (under a transformation/matrix multiplication).	"An eigenvector of a linear transformation T: V->V [is] an element in the vector space V which *does not change direction* [emphasis added] when imaged by the transformation."	Synthetic-geometric
9-line	**Line** Describing eigenvectors as remaining on the same line (after undergoing a transformation/matrix multiplication).	En enhetsvekter er en vekter som holder seg på samme linje under en lineær trans-formasjon. λ vil være en skalar av egen-vektor som da kalles en egenverdi.	Synthetic-geometric
9-rotation/angle	**Rotation-angle** Describing eigenvectors as not being rotated or moved by an angle.	"En egenvektor er en vektor som ikke endrer **vinkel** i forhold til origo når matrisen påvirker rommet den eksisterer i, den blir bare skalert med en faktor."	Synthetic-geometric
9-parallel	**Parallel** Describing the eigenvector x as parallel with the transformed (multiplied) vector, Ax.	"En vektor som er **parallel** med seg selv etter den er blitt transformert av matrisen…"	Synthetic-geometric

9-visual description	**Visual description** A visual description of eigenvectors, including “changing length”, "stretching", "shrinking", "flipping" etc.	b) Egenverdien er et uttrykk for hvor mye egenvektoren strekkes av den lineære transformasjonen.	Synthetic-geometric
9-figure	**Figure_geometric** Student provided a sketch of eigenvectors and/or eigenvalues, illustrating a particular example of them.		Synthetic-geometric
9-misc	**Miscellaneous** The answer does not fit into any other codes and differs significantly from other answers.	9 a) egenvektor er en vektor som kan trekkes ut etter en matematisk operasjon er utført. b) egenverdi er det som er igjen som en konstant foran en egenvektor.	--
9-NA	**Not answered** The student did not provide an answer.	--	--

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