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Synergistic PID and Output Feedback Control on Matrix Lie Groups ⋆

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Abstract: In this paper, we develop multiple synergistic hybrid feedback control laws for mechanical systems on matrix Lie groups with left-invariant metrics. With the goal of globally asymptotically tracking a desired reference trajectory, we propose a hybrid proportional-derivative (PD) type control law and an output feedback version which only utilizes configuration measurements. Moreover, to ensure global asymptotic tracking in the presence of a constant and unknown disturbance in the system dynamics, we introduce two novel proportional-integral-derivative (PID) type control laws with slightly different properties in terms of gain selection and integral action. The theoretical developments are validated through numerical simulation of an underwater vehicle.

Keywords: Hybrid feedback control, PID control, output feedback control, matrix Lie groups

1. INTRODUCTION

Synergistic control is a hybrid feedback control methodology that selects the state-feedback control action based on the value of multiple Lyapunov-like functions (Sanfelice, 2021; Mayhew, 2010; Mayhew et al., 2011). A hybrid synergistic control law solves the tracking problem robustly and globally on noncontractible manifolds, which is unattainable using conventional continuous or discontinuous control laws (Bhat and Bernstein, 2000; Mayhew et al., 2011).

The synergistic control paradigm is applied to orientation tracking control for rigid bodies utilizing quaternions in Mayhew et al. (2011), where a globally asymptotically stabilizing PD-controller is derived. The paper also introduces an output-feedback version of this control law. Synergistic potential functions are utilized to derive PD-controllers for global asymptotic tracking control for rigid body orientation on SO(3) in Mayhew and Teel (2013). A smoothing approach for the devised controller is presented, which ensures continuity of the control signal provided that desired acceleration is continuous. The work also presents a constructive procedure for a class of synergistic potential functions on SO(3). A class of global synergistic controllers with integral action for tracking control on SO(3) is presented in Lee (2015). The switching mechanism depends explicitly on the value of the integral state. Synergistic potential functions are generalized to synergistic Lyapunov functions and feedback pairs in Mayhew et al. (2011) and further generalized in Schmidt-Didlaukies et al. (2022), which allows the logic variable to change during flows. The synergistic framework has also been utilized in Marley et al. (2021), where synergistic control barrier functions are proposed.

An adaptive hybrid feedback control law ensuring global asymptotic tracking for marine vehicles is developed in Basso et al. (2022). This control law is derived from a set of potential functions and a hysteretic switching mechanism, which are required to satisfy a set of assumptions. These assumptions are less restrictive than the synergistic conditions in Mayhew (2010) and Mayhew and Teel (2013). However, the switching mechanism is not directly encoded through the potential functions as with synergistic potential functions.

A continuous intrinsic controller with integral action on compact Lie groups is presented in Maithripala and Berg (2015). The integral action stems from integration of the P-action in the controller. The controller ensures bounded tracking error in the presence of uncertainty, and almost global asymptotic stability in the absence of uncertainty. Several similar controllers with integral action are presented in Zhang et al. (2015). Here, the integral action stems from integrating the PD-action in the controller. This controller achieves almost global asymptotic stability in the presence of a constant disturbance.

The contribution of this paper is threefold. First, we propose a baseline synergistic PD control law ensuring global asymptotic tracking for mechanical systems on matrix Lie groups with a left-invariant Riemannian metric. The second contribution is a generalization of the synergistic output-feedback control law proposed for orientation control in Mayhew et al. (2011) to any system whose configuration space can be identified with a matrix Lie group. Finally, we present two novel synergistic PID type control laws, both of which ensure global asymptotic tracking in the presence of unknown constant disturbances.

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This paper is organized as follows. Section 2 introduces the notation used in the paper and a background on matrix Lie groups. Section 3 introduces the equations of motion and the assumptions on the desired trajectories. In Section 4, we derive the error system and give the problem statement, before Section 5 defines the concept of a synergistic function. Then, we present a synergistic hybrid PD control law in Section 6 and an output feedback version which only utilizes configuration measurements in Section 7. Section 8 introduces two novel synergistic control laws with integral action, both of which globally asymptotically track a given bounded reference trajectory in the presence of a constant and unknown disturbance. Finally, Section 9 presents a case study with simulation results of both PID control laws, before Section 10 concludes the paper.

2. PRELIMINARIES

The Euclidean inner product in $\mathbb{R}^n$ is written $\langle x, y\rangle$, and the Euclidean norm is denoted $|x| = \langle x, x\rangle^{1/2}$. The entry of a matrix $a \in \mathbb{R}^{n\times n}$ corresponding to the ith row and jth column is denoted a_{ij}. The closed ball of radius r in $\mathbb{R}^n$ is the set $r\mathbb{B} = \{x \in \mathbb{R}^n : |x| \leq r\}$. The range (or equivalently, the image) of a mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined as $\operatorname{rge} f = \{y \in \mathbb{R}^n \colon \exists x \in \mathbb{R}^m \text{ such that } y = f(x)\}$. A function $V : \mathcal{Y} \to \mathbb{R}$, where $\mathcal{Y} \subset \mathbb{R}^n$, is proper if the preimage of any compact set $K \subset \mathbb{R}$ under V is compact. For $f : \mathbb{R}^m \to \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times m}$, we say that f is A-monotone if $\langle f(x) - f(y), x - y\rangle \geq \langle A(x - y), x - y\rangle$ for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$. We say that f is monotone if it is 0-monotone. A matrix Lie group $\mathcal{G}$ is a closed subgroup of the general linear group $\mathrm{GL}(n) = \{g \in \mathbb{R}^{n\times n} : \det g \neq 0\}$. The Lie algebra of a matrix Lie group $\mathcal{G}$ is denoted $\mathfrak{g}$, and is defined as $\mathfrak{g} := \{a \in \mathbb{R}^{n\times n} : t \in \mathbb{R} \implies \exp(at) \in \mathcal{G}\}$, where $\exp : \mathbb{R}^{n\times n} \to \mathrm{GL}(n)$ is the matrix exponential. The Lie algebra $\mathfrak{g}$ is a real vector space with dimension equal to the dimension of $\mathcal{G}$ as a manifold. Therefore, there exists an isomorphism $(\cdot)^\wedge \colon \mathbb{R}^m \to \mathfrak{g}$ with inverse $(\cdot)^\vee \colon \mathfrak{g} \to \mathbb{R}^m$, where m denotes the dimension of $\mathcal{G}$. For $g \in \mathcal{G}, \xi \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^m$, we define the adjoint mappings $\mathrm{Ad} : \mathcal{G} \times \mathbb{R}^m \to \mathbb{R}^m$ and $\mathrm{ad} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ by $\mathrm{Ad}_g\, \xi := \big(g\xi^\wedge g^{-1}\big)^\vee$ and $\mathrm{ad}_\xi\, \zeta := \big(\xi^\wedge\zeta^\wedge - \zeta^\wedge\xi^\wedge\big)^\vee$, respectively. For each $\xi \in \mathbb{R}^k$, we define a left-invariant vector field $X_\xi(g) = g\xi^\wedge$ on $\mathcal{G}$ with $g \in \mathcal{G}$. The Lie derivative of a continuously differentiable function $V \colon \mathcal{G} \to \mathbb{R}$ along the vector field X_ξ can be written as $\langle\!\langle \mathrm{D}V(g), X_\xi(g)\rangle\!\rangle$, where $\langle\!\langle a, b\rangle\!\rangle := \operatorname{tr}(a^\mathsf{T} b)$ is the Frobenius inner product and

$$\mathrm{D}V(a) = \begin{pmatrix} \frac{\partial V}{\partial a_{11}} & \cdots & \frac{\partial V}{\partial a_{1j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial V}{\partial a_{i1}} & \cdots & \frac{\partial V}{\partial a_{ij}} \end{pmatrix}.$$

The Lie derivative can be rewritten using the Euclidean inner product by defining the mapping $\mathrm{d}V : \mathcal{G} \to \mathbb{R}^k$ by $\langle \mathrm{d}V(g), \xi\rangle := \langle\!\langle \mathrm{D}V(g), X_\xi(g)\rangle\!\rangle$. Finally, the bilinear map $\nabla^M \colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ induced by the inertia matrix M is defined by (Bullo and Lewis, 2005)

$$\nabla^M_\nu \eta := \tfrac{1}{2}\,\mathrm{ad}_\nu\, \eta - \tfrac{1}{2} M^{-1}[\mathrm{ad}_\nu^\mathsf{T} M\eta + \mathrm{ad}_\eta^\mathsf{T} M\nu]. \tag{1}$$

Observe that $M\nabla^M_\nu \nu = -\,\mathrm{ad}_\nu^\mathsf{T} M\nu$.

3. MODELING

Consider a fully actuated control-affine mechanical system whose configuration space can be identified with a matrix Lie group $\mathcal{G} \subset \mathbb{R}^{n\times n}$ with dimension m. Let $g \in \mathcal{G}$ denote the configuration and $\nu \in \mathbb{R}^m$ the body velocity. The equations of motion are given by

$$\dot{g} = g\nu^\wedge, \tag{2a}$$

$$M\dot{\nu} - \mathrm{ad}_\nu^\mathsf{T} M\nu = f(g, \nu) + \tau. \tag{2b}$$

where $M \in \mathbb{R}^{m\times m}$ is the inertia tensor and $\mathrm{ad}_\nu^\mathsf{T} M\nu$ describes inertial forces arising from curvature effects. Moreover, the continuous mapping $f : \mathcal{G} \times \mathbb{R}^m \to \mathbb{R}^m$ describes other external forces, and $\tau \in \mathbb{R}^m$ is an idealized control force.

Assumption 1. The desired configuration $t \mapsto g_d(t)$ and its derivatives up to the second order are bounded and continuous, and $t \mapsto \det g_d(t)$ is bounded away from zero.

For every desired configuration satisfying Assumption 1, there exist scalars $a \geq 0$ and $c \geq 0$ and a compact set $\Omega \subset \mathcal{G}$ such that the desired configuration and body velocity $t \mapsto (g_d(t), \nu_d(t))$ is a complete solution to the constrained differential inclusion

$$\left.\begin{aligned} \dot{g}_d &= g_d\nu_d^\wedge \\ \dot{\nu}_d &\in c\mathbb{B}^m \end{aligned}\right\} (g_d, \nu_d) \in \Omega \times a\mathbb{B} \tag{3}$$

4. ERROR SYSTEM AND PROBLEM STATEMENT

Define the left-invariant configuration and velocity errors by

$$g_e := g_d^{-1} g, \tag{4}$$

$$\nu_e := \nu - \nu_r = \nu - \mathrm{Ad}_{g_e}^{-1}\, \nu_d, \tag{5}$$

which results in the error system

$$\left.\begin{aligned} \dot{g}_e &= g_e\nu_e^\wedge \\ \dot{\nu}_e &= M^{-1}(\mathrm{ad}_\nu^\mathsf{T} M\nu + f(g,\nu) + \tau) - \dot{\nu}_r \\ \dot{g}_d &= g_d\nu_d^\wedge \\ \dot{\nu}_d &\in c\mathbb{B}^m \end{aligned}\right\} \quad x \in X \tag{6}$$

where $\dot{\nu}_r = \mathrm{Ad}_{g_e}^{-1}\, \dot{\nu}_d - \mathrm{ad}_{\nu_e}\, \nu_r$, $x := (g_e, \nu_e, g_d, \nu_d) \in X$ and $X := \mathcal{G} \times \mathbb{R}^m \times \Omega \times a\mathbb{B}$.

Problem statement: For a given compact set $\mathcal{A} \subset \mathcal{G}$, design a hybrid feedback control law with state $\xi \in \Xi$ and output $\tau \in \mathbb{R}^m$ such that the compact set

$$\mathcal{T} := \mathcal{A} \times \{0\} \times \Omega \times a\mathbb{B} \times \mathcal{S}, \tag{7}$$

is globally pre-asymptotically stable for the resulting hybrid closed-loop system, where $\mathcal{S} \subset \Xi$ is the controller state attractor.

5. SYNERGISTIC FUNCTIONS

The following definition of a synergistic function is similar to the ones found in Mayhew (2010) and Mayhew and Teel (2013).

Definition 2. Let $Q \subset \mathbb{R}$ be a finite set and $\mathcal{A} \subset \mathcal{G}$ be a compact set. A continuously differentiable function $V : \mathcal{G} \times Q \to \mathbb{R}$ is synergistic with respect to the set $\mathcal{A}$ if

- V is proper and positive definite with respect to a set $\mathcal{B} \subset \mathcal{A} \times Q$ defined such that for every $g \in \mathcal{A}$, there exists $q \in Q$ such that $(g, q) \in \mathcal{B}$.
- there exists $\delta > 0$ such that the synergy gap

  $$\mu_V(g, q) := V(g, q) - \min_{p\in Q} V(g, p), \tag{8}$$

  satisfies $\mu_V(g, q) > \delta$ for each $(g, q) \in (\mathcal{E} \cup (\mathcal{A}\times Q)) \setminus \mathcal{B}$, where

  $$\mathcal{E} := \{(g, q) \in \mathcal{G} \times Q : \mathrm{d}V(g, q) = 0\}. \tag{9}$$

The function V from Definition 2 can also be thought of as a family of potential functions indexed by Q, $\{V_q\}_{q\in Q}$, such that $V(g,q) = V_q(g)$ for all $(g,q) \in \mathcal{G} \times Q$. Furthermore, there exists a family of compact sets indexed by Q, $\{\mathcal{B}_q\}_{q\in Q}$, such that $\mathcal{A} = \cup_{q\in Q}\mathcal{B}_q$ and $\mathcal{B} = \cup_{q\in Q}(\mathcal{B}_q \times \{q\})$. For each $q \in Q$, V_q is a proper and positive definite function with respect to $\mathcal{B}_q$. It must be remarked that there may exist $q \in Q$ such that $\mathcal{B}_q = \emptyset$, in which case the corresponding function V_q is proper and everywhere positive.

Definition 2 also requires that if $g \in (\mathcal{E}_q \cup \mathcal{A}) \setminus \mathcal{B}_q$, where $\mathcal{E}_q$ denotes the set of critical points of V_q, then there exists $\delta > 0$ such that $V_q(g) - \min_{p\in Q} V_p(g) > \delta$. Hence, if g is a critical point of V_q that is not in $\mathcal{B}_q$, or g lies in $\mathcal{B}_p$ with $p \in Q\setminus\{q\}$, then the minimal value of $s \mapsto V_s(g)$ is at least δ lower than $V_q(g)$. A consequence of this fact, continuity of the potential functions, and positive definiteness of the potential functions is that, for $(p,q) \in Q \times Q$, either $\mathcal{B}_p \cap \mathcal{B}_q = \emptyset$ or $\mathcal{B}_p = \mathcal{B}_q$.

A synergistic function induces the following kinematic hybrid control law (Mayhew, 2010)

$$\begin{cases} \dot{q} = 0 & (g,q) \in C_V \\ q^+ \in G_V(g) & (g,q) \in D_V \\ y = -\,\mathrm{d}V(g,q) \end{cases} \tag{10}$$

with state $q \in Q$, input $g \in \mathcal{G}$ and output y. Moreover, the flow set $C_V \subset \mathcal{G} \times Q$, jump set $D_V \subset \mathcal{G} \times Q$ and jump map $G_V : \mathcal{G} \rightrightarrows Q$ are defined according to

$$C_V := \{(g,q) \in \mathcal{G} \times Q : \mu_V(g,q) \le \delta\}, \tag{11}$$
$$D_V := \{(g,q) \in \mathcal{G} \times Q : \mu_V(g,q) \ge \delta\}, \tag{12}$$
$$G_V(g) := \{q \in Q : \mu_V(g,q) = 0\}. \tag{13}$$

Note that the sets C_V and D_V are closed in $\mathcal{G} \times Q$ due to the continuity of μ_V. It follows that C_V and D_V are closed in $\mathrm{GL}(n) \times \mathbb{R}$, but not necessarily in $\mathbb{R}^{n\times n} \times \mathbb{R}$.

6. SYNERGISTIC PD CONTROL

In this section, we employ a synergistic function to design a hybrid PD controller with state $q \in Q$ which renders the closed-loop system globally pre-asymptotically stable. Moreover, we employ a novel feedforward control which is independent of the system velocities.

Defining the following velocity independent feedforward control

$$\kappa_{ff}(g_e, g_d, \nu_d, \dot{\nu}_d) := M\,\mathrm{Ad}_{g_e}^{-1}\dot{\nu}_d - \mathrm{ad}_{\nu_r}^{\mathsf{T}} M\nu_r - f(g,\nu_r), \tag{14}$$

we propose the following synergistic PD control law

$$\begin{cases} \dot{q} = 0 & (g_e,q) \in C_V \\ q^+ \in G_V(g_e) & (g_e,q) \in D_V \\ \tau = \kappa_{ff}(g_e,g_d,\nu_d,\dot{\nu}_d) - \mathrm{d}V(g_e,q) - K_d\nu_e \end{cases} \tag{15}$$

where V is synergistic with respect to a compact set $\mathcal{A}$ and $K_d \in \mathbb{R}^{m\times m}$. Observe that the feedback control law (15) comprises a proportional action $\mathrm{d}V$ and a derivative action $K_d\nu_e$.

Using Lemma A.1, the interconnection between the control law (15) and the error system (6) leads to the following closed-loop system

$$\mathcal{H}_1 : \begin{cases} \left.\begin{aligned} \dot{g}_e &= g_e\nu_e^{\wedge} \\ \dot{\nu}_e &= -\nabla^M_{\nu+\nu_r}\nu_e \\ &\quad + M^{-1}(f(g,\nu) - f(g,\nu_r)) \\ &\quad - M^{-1}(\mathrm{d}V(g_e,q) + K_d\nu_e) \\ \dot{g}_d &= g_d\nu_d^{\wedge} \\ \dot{\nu}_d &\in c\mathbb{B}^m \end{aligned}\right\} & x_1 \in C_1 \\ q^+ \in G_V(g_e) & x_1 \in D_1 \end{cases}$$

where $x_1 := (g_e, \nu_e, g_d, \nu_d, q) \in X_1$ and

$$\begin{aligned} X_1 &:= \mathcal{G} \times \mathbb{R}^m \times \Omega \times a\mathbb{B} \times Q, \\ C_1 &:= \{x_1 \in X_1 : (g_e,q) \in C_V\}, \\ D_1 &:= \{x_1 \in X_1 : (g_e,q) \in D_V\}. \end{aligned} \tag{16}$$

Theorem 3. If there exist $\varepsilon > 0$ and $K_d \in \mathbb{R}^{m\times m}$ such that $\nu \mapsto -f(g,\nu)$ is $(\varepsilon I - K_d)$-monotone for each $g \in \mathcal{G}$, then the hybrid control law (15) renders the compact set

$$\mathcal{T}_1 := \mathcal{A} \times \{0\} \times \Omega \times a\mathbb{B} \times Q, \tag{17}$$

globally pre-asymptotically stable for $\mathcal{H}_1$.

Proof. Consider the continuously differentiable function $W_1 : X_1 \to \mathbb{R}_{\ge 0}$ defined by

$$W_1(x_1) := V(g_e,q) + \frac{1}{2}\langle \nu_e, M\nu_e\rangle. \tag{18}$$

Evidently, W_1 is proper and positive definite with respect to the compact set $\mathcal{T}_1' := \{x_1 \in X_1 : (g_e,q) \in \mathcal{B}, \nu_e = 0\}$. Using Lemma A.2, the time derivative of W_1 satisfies

$$\dot{W}_1(x_1) \le -\varepsilon|\nu_e|^2 \quad \forall x_1 \in C_1.$$

Letting $G_1(x_1) := (g_e, \nu_e, g_d, \nu_d, G_V(g_e))$, the change of W_1 across jumps is $W_1(s_1) - W_1(x_1) \le -\delta$, for all $x_1 \in D_1$ and $s_1 \in G_1(x_1)$. Since W_1 is proper and non-increasing along flows and across jumps, it follows that $\mathcal{T}_1'$ is stable and every solution to $\mathcal{H}_1$ is bounded. In fact, every sublevel set of W_1 is compact and forward pre-invariant, that is, given $r \ge 0$, every solution starting in the set $W_1^{-1}([0,r])$ remains in it. Consider the hybrid system $\mathcal{H}_{1,r}$, defined such that it is equal to $\mathcal{H}_1$ with the flow and jump sets replaced by $C_{1,r} := C_1 \cap W_1^{-1}([0,r])$ and $D_{1,r} := D_1 \cap W_1^{-1}([0,r])$, respectively. Then, for each $r \ge 0$, the hybrid system $\mathcal{H}_{1,r}$ satisfies the hybrid basic conditions (Goebel et al., 2012, Assumption 6.5). Furthermore, every complete solution to $\mathcal{H}_1$ that starts in the set $W_1^{-1}([0,r])$ is a complete solution to $\mathcal{H}_{1,r}$. It then follows from Corollary 8.7 b) in Goebel et al. (2012) that, for each $r \ge 0$, complete solutions to $\mathcal{H}_{1,r}$ converge to the largest weakly invariant subset $\mathcal{W}_1$ contained in $W_1^{-1}(\gamma) \cap \{x_1 \in C_1 : \nu_e = 0\}$ for some $\gamma \in [0,r]$. The closed-loop system is such that $\nu_e \equiv 0$ implies $\mathrm{d}V(g_e,q) \equiv 0$. By construction, the only points in C_1 where $\mathrm{d}V(g_e,q) = 0$ are those for which $(g_e,q) \in \mathcal{B}$. It follows that $\mathcal{W}_1 \subset \mathcal{T}_1'$ and hence that every complete solution to $\mathcal{H}_{1,r}$ converges to $\mathcal{T}_1'$. Since every complete solution to $\mathcal{H}_1$ is a complete solution to $\mathcal{H}_{1,r}$ for some $r \ge 0$, every complete solution to $\mathcal{H}_1$ converges to $\mathcal{T}_1'$. Consequently, $\mathcal{T}_1'$ is globally pre-asymptotically stable for $\mathcal{H}_1$ since it is stable, all solutions are bounded and every complete solution converges to $\mathcal{T}_1'$. Since $\mathcal{T}_1' \subset \mathcal{T}_1$, it follows that $\mathcal{T}_1$ is globally pre-attractive. Moreover, $\mathcal{T}_1$ is forward pre-invariant because $\mathcal{T}_1'$ is forward pre-invariant and

$$\mathcal{T}_1 \setminus \mathcal{T}_1' = \{x_1 \in X_1 : (g_e,q) \in (\mathcal{A} \times Q) \setminus \mathcal{B}\} \subset D_1 \setminus C_1$$

is such that any maximal solution reaching $\mathcal{T}_1 \setminus \mathcal{T}_1'$ is immediately mapped to $\mathcal{T}_1'$ via a single jump. It then follows from Proposition 7.5 of Goebel et al. (2012) that

$\mathcal{T}_1$ is stable. Since $\mathcal{T}_1$ is stable and globally pre-attractive, it is globally pre-asymptotically stable. □

7. SYNERGISTIC OUTPUT FEEDBACK CONTROL

Due to the fact that the feedforward control in (14) is independent of the vehicle velocities, we can utilize it in the design of a Lyapunov-based output feedback tracking control law. To this end, let $U : \mathcal{G} \times H \to \mathbb{R}_{\geq 0}$ be synergistic with respect to $\mathcal{A}$, let $h \in H \subset \mathbb{R}$ be a logic variable, and consider the output feedback control law

$$\begin{cases} \dot{g}_f = g_f \left(\mathrm{Ad}_{g_o} K_f \,\mathrm{d}U(g_o, h)\right)^\wedge & (g_e, q, g_o, h) \in \widetilde{C}_2 \\ \left.\begin{array}{l} q^+ \in G_V(g_e) \\ h^+ \in G_U(g_o) \end{array}\right\} & (g_e, q, g_o, h) \in \widetilde{D}_2 \\ \tau = \kappa_{ff}(g_e, g_d, \nu_d, \dot{\nu}_d) - \mathrm{d}U(g_o, h) - \mathrm{d}V(g_e, q). \end{cases} \tag{19}$$

where $g_o := g_f^{-1} g_e$ is the filter error and

$$\widetilde{C}_2 := \{(g_e, q, g_o, h) : (g_e, q) \in C_V \text{ and } (g_o, h) \in C_U\}, \tag{20}$$

$$\widetilde{D}_2 := \{(g_e, q, g_o, h) : (g_e, q) \in D_V \text{ or } (g_o, h) \in D_U\}. \tag{21}$$

This results in the closed-loop system

$$\mathcal{H}_2 : \begin{cases} \left.\begin{array}{l} \dot{g}_e = g_e \nu_e^\wedge \\ \dot{\nu}_e = -\nabla^M_{\nu+\nu_r} \nu_e \\ \qquad + M^{-1}(f(g,\nu) - f(g,\nu_r)) \\ \qquad - M^{-1}(\mathrm{d}V(g_e,q) + \mathrm{d}U(g_o,h)) \\ \dot{g}_d = g_d \nu_d^\wedge \\ \dot{\nu}_d \in c\mathbb{B}^m \\ \dot{g}_o = g_o \left(\nu_e - K_f \,\mathrm{d}U(g_o,h)\right)^\wedge \end{array}\right\} & x_2 \in C_2 \\ \left.\begin{array}{l} q^+ \in G_V(g_e) \\ h^+ \in G_U(g_o) \end{array}\right\} & x_2 \in D_2 \end{cases}$$

where $x_2 = (g_e, \nu_e, g_d, \nu_d, q, g_o, h) \in X_2$ and

$$\begin{aligned} X_2 &:= \mathcal{G} \times \mathbb{R}^m \times \Omega \times a\mathbb{B} \times Q \times \mathcal{G} \times H, \\ C_2 &:= \{x_2 \in X_2 : (g_e, q) \in C_V \text{ and } (g_o, h) \in C_U\}, \\ D_2 &:= \{x_2 \in X_2 : (g_e, q) \in D_V \text{ or } (g_o, h) \in D_U\}. \end{aligned} \tag{22}$$

Theorem 4. If $\mathcal{A}$ is a finite set, K_f is positive definite and $\nu \mapsto -f(g,\nu)$ is monotone for every $g \in \mathcal{G}$, then the hybrid control law (19) renders the compact set

$$\mathcal{T}_2 := \mathcal{A} \times \{0\} \times \Omega \times a\mathbb{B} \times Q \times \mathcal{A} \times H, \tag{23}$$

globally pre-asymptotically stable for $\mathcal{H}_2$.

Proof. The continuously differentiable function $W_2 : X_2 \to \mathbb{R}_{\geq 0}$ defined by

$$W_2(x_2) := V(g_e, q) + U(g_o, h) + \frac{1}{2}\langle \nu_e, M\nu_e \rangle. \tag{24}$$

is proper and positive definite with respect to the compact set $\mathcal{T}_2' := \{x_2 \in X_2 : (g_e, q) \in \mathcal{B}, (g_o, h) \in \mathcal{B}, \nu_e = 0\}$. The derivative of W_2 along flows of the closed-loop system satisfies

$$\dot{W}_2(x_2) \leq -\langle K_f \,\mathrm{d}U(g_o,h), \mathrm{d}U(g_o,h)\rangle \quad \forall x_2 \in C_2$$

Letting $G_2(x_2) := (g_e, \nu_e, g_d, \nu_d, G_V(g_e), g_o, G_U(g_o))$, the change of W_2 across jumps is found to be $W_2(s_2) - W_2(x_2) \leq -\delta$, for all $x_2 \in D_2$ and $s_2 \in G_2(x_2)$. By a similar argument as in the proof of Theorem 3, it can be shown that $\mathcal{T}_2'$ is stable and that all solutions to $\mathcal{H}_2$ are bounded. Moreover, every complete solution to $\mathcal{H}_{2,r}$ converges to the largest weakly invariant subset $\mathcal{W}_2$ of $W_2^{-1}(\gamma) \cap \{x_2 \in C_2 : \mathrm{d}U(g_o,h) = 0\}$ for some $\gamma \in [0, r]$. In this set, it holds that $g_o \in \mathcal{A}$. Since $\mathcal{A}$ is finite, the closed-loop dynamics imply that $\dot{g}_o \equiv 0$ and hence that $\nu_e \equiv 0$. It then follows that $\mathrm{d}V(g_e, q) \equiv 0$, which implies that $(g_e, q) \in \mathcal{B}$. Therefore, $\mathcal{T}_2'$ is also globally pre-attractive, and this set is globally pre-asymptotically stable. The remainder of the proof uses an argument similar to the one utilized in the proof of Theorem 3. □

8. SYNERGISTIC CONTROL WITH INTEGRAL ACTION

We now assume that $f(g,\nu) = \bar{f}(g,\nu) + b$, where $\bar{f} : \mathcal{G} \times \mathbb{R}^m \to \mathbb{R}^m$ describes a known external force and $b \in \mathbb{R}^m$ is an unknown constant disturbance. Consider the following synergistic PID control law

$$\begin{cases} \dot{\varphi} = \mathrm{d}V(g_e, q) & (g_e, q, \varphi) \in \widetilde{C}_3 \\ q^+ \in G_V(g_e) & (g_e, q, \varphi) \in \widetilde{D}_3 \\ \tau = M\dot{\nu}_r - \bar{f}(g,\nu) \\ \qquad - (I + MK_d^{-1}K_i)\,\mathrm{d}V(g_e,q) - K_d\nu_e - K_i\varphi \end{cases} \tag{25}$$

where K_d and K_i are diagonal matrices with positive entries and

$$\widetilde{C}_3 := \{(g_e, q, \varphi) \in \mathcal{G} \times Q \times \mathbb{R}^m : (g_e, q) \in C_V\} \tag{26}$$

$$\widetilde{D}_3 := \{(g_e, q, \varphi) \in \mathcal{G} \times Q \times \mathbb{R}^m : (g_e, q) \in D_V\} \tag{27}$$

Observe that the feedback control law comprises a proportional term $(I + MK_d^{-1}K_i)\,\mathrm{d}V$, an integral term $K_i b$ and a derivative term $K_d\nu_e$.

Let $\varphi_e := \varphi - K_i^{-1} b$ denote the estimation error and define $x_3 := (g_e, \nu_e, g_d, \nu_d, q, \varphi_e) \in X_3$ with $X_3 := \mathcal{G} \times \mathbb{R}^m \times \Omega \times a\mathbb{B} \times Q \times \mathbb{R}^m$. The closed-loop system is then given by

$$\mathcal{H}_3 : \begin{cases} \left.\begin{array}{l} \dot{g}_e = g_e \nu_e^\wedge \\ \dot{\nu}_e = -M^{-1}(I + MK_d^{-1}K_i)\,\mathrm{d}V(g_e,q) \\ \qquad - M^{-1}K_d\nu_e - M^{-1}K_i\varphi_e) \\ \dot{g}_d = g_d\nu_d^\wedge \\ \dot{\nu}_d \in c\mathbb{B}^m \\ \dot{\varphi}_e = \mathrm{d}V(g_e,q) \end{array}\right\} & x_3 \in C_3 \\ q^+ \in G_V(g_e) & x_3 \in D_3 \end{cases}$$

where $C_3 := \{x_3 \in X_3 : (g_e, q) \in C_V\}$ and $D_3 := \{x_3 \in X_3 : (g_e, q) \in D_V\}$.

Theorem 5. If $\mathcal{A}$ is finite, K_d is diagonal and positive definite, and K_i is diagonal and positive definite, then the hybrid control law (25) renders the compact set

$$\mathcal{T}_3 := \mathcal{A} \times \{0\} \times \Omega \times a\mathbb{B} \times Q \times \{0\}, \tag{28}$$

globally pre-asymptotically stable for the closed-loop system $\mathcal{H}_3$.

Proof. Consider the continuously differentiable function $W_3 : X_3 \to \mathbb{R}$ defined by

$$W_3(x_3) := V(g_e, q) + \frac{1}{2}\langle M(\nu_e + K_d^{-1}K_i\varphi_e), \nu_e + K_d^{-1}K_i\varphi_e\rangle + \frac{1}{2}\langle \varphi_e, K_d^{-1}K_i\varphi_e\rangle.$$

W_3 is proper and positive definite relative to the compact set

$$\mathcal{T}_3' := \{x_3 \in X_3 : (g_e, q) \in \mathcal{B}, \nu_e = 0, \varphi_e = 0\} \tag{29}$$

Differentiating W_3 along the flows of the closed-loop system yields

$$\dot{W}_3(x_3) = -\langle K_d(\nu_e + K_d^{-1}K_i\varphi_e), \nu_e + K_d^{-1}K_i\varphi_e\rangle.$$

By a similar argument as in the proof of Theorem 3, it can be shown that $\mathcal{T}_3'$ is stable and that all solutions to $\mathcal{H}_3$ are bounded. Moreover, every complete solution to

$\mathcal{H}_{3,r}$ converges to the largest weakly invariant subset $\mathcal{W}_3$ of $W_3^{-1}(\gamma) \cap \{x_3 \in C_3 : \nu_e + K_d^{-1} K_i \varphi_e = 0\}$ for some $\gamma \in [0, r]$. Since $\nu_e + K_d^{-1} K_i \varphi_e \equiv 0$ implies that $\dot{\nu}_e + K_d^{-1} K_i \dot{\varphi}_e \equiv 0$, it follows from the closed-loop system $\mathcal{H}_3$ that $\mathrm{d}V(g_e, q) \equiv 0$. Therefore, from Definition 2 and the fact that $\mathcal{A}$ and hence $\mathcal{B}$ is finite, it follows that $\dot{g}_e \equiv 0$ which implies $\nu_e \equiv 0$. Thus, from $\dot{\nu}_e \equiv 0$ the closed-loop system implies that $\varphi_e \equiv 0$ must hold. Consequently, $\mathcal{T}_3'$ is also globally pre-attractive, and therefore globally pre-asymptotically stable for $\mathcal{H}_3$. The remainder of this proof is similar to the proof of Theorem 3. □

Although the cancellation term $MK_d^{-1}K_i \,\mathrm{d}V$ acts as a rescaling of the proportional action, it may often end up very small in practice because K_d is typically chosen considerably larger than K_i. The cancellation term can be removed and the assumptions on $\mathcal{A}$ and K_d can be relaxed by modifying the dynamics of the integral state and only allowing the integral gain to be scalar-valued. To this end, consider the following synergistic controller with integral action

$$\begin{cases} \dot{\varphi} = \mathrm{d}V(g_e, q) + K_d \nu_e & (g_e, q, \varphi) \in \widetilde{C}_3 \\ q^+ \in G_V(g_e) & (g_e, q, \varphi) \in \widetilde{D}_3 \\ \tau = M\dot{\nu}_r - \bar{f}(g, \nu) & \\ \qquad - \mathrm{d}V(g_e, q) - K_d \nu_e - k_i \varphi & \end{cases} \tag{30}$$

which leads to the closed-loop system

$$\mathcal{H}_4 : \begin{cases} \left.\begin{aligned} \dot{g}_e &= g_e \nu_e^\wedge \\ \dot{\nu}_e &= -M^{-1}(\mathrm{d}V(g_e, q) + K_d \nu_e + k_i \varphi_e) \\ \dot{g}_d &= g_d \nu_d^\wedge \\ \dot{\nu}_d &\in c\mathbb{B}^m \\ \dot{\varphi}_e &= \mathrm{d}V(g_e, q) + K_d \nu_e \end{aligned}\right\} & x_3 \in C_3 \\ q^+ \in G_V(g_e) & x_3 \in D_3 \end{cases}$$

Theorem 6. If $k_i > 0$ and $K_d - k_i M$ is positive definite, then the hybrid control law (30) renders the compact set

$$\mathcal{T}_4 := \mathcal{A} \times \{0\} \times \Omega \times a\mathbb{B} \times Q \times \{0\}, \tag{31}$$

globally pre-asymptotically stable for the closed-loop system $\mathcal{H}_4$.

Proof. Consider the continuously differentiable function $W_4 : X_3 \to \mathbb{R}$ defined by

$$\begin{aligned} W_4(x_3) := &\, V(g, q) + \tfrac{1}{2}\langle \nu_e, M\nu_e \rangle \\ &+ \tfrac{1}{2}\langle M(\nu_e + M^{-1}\varphi_e), \nu_e + M^{-1}\varphi_e \rangle \end{aligned} \tag{32}$$

W_4 is proper and positive definite with respect to the compact set $\mathcal{T}_3'$ defined in (29). Differentiating W_4 along the flows of the closed-loop system yields

$$\begin{aligned} \dot{W}_4(x_3) &= -\langle K_d \nu_e, \nu_e \rangle - 2k_i \langle \nu_e, \varphi_e \rangle - k_i \langle M^{-1}\varphi_e, \varphi_e \rangle \\ &= -\langle (K_d - k_i M)\nu_e, \nu_e \rangle \\ &\quad - k_i \langle M^{-1}(\varphi_e + M\nu_e), \varphi_e + M\nu_e \rangle \\ &\le -\varepsilon_1 |\nu_e|^2 - \varepsilon_2 |\varphi_e|^2 \end{aligned}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. The remainder of the proof is similar to the proof of Theorem 3. □

9. CASE STUDY

This section presents simulation results of a small fully actuated underwater vehicle. The configuration of an underwater vehicle can be identified with the matrix Lie group $\mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$. An element $g = (p, R) \in \mathrm{SE}(3)$ contains the position $p \in \mathbb{R}^3$ and orientation $R \in \mathrm{SO}(3)$ of a vehicle-fixed frame with respect to an inertial frame. The equations of motion for an underwater vehicle are given by (2) with the inertia tensor $M = M_{rb} + M_a$ comprising rigid body and hydrodynamic inertia. Moreover, $f(g, \nu) = d(\nu) + \gamma(g) + b$ where $d : \mathbb{R}^6 \to \mathbb{R}^6$ is the damping wrench comprising linear and nonlinear contributions $d(\nu) := -D_l \nu - D_n(\nu)\nu$, and $\gamma : \mathrm{SE}(3) \to \mathbb{R}^6$ denotes the net hydrostatic wrench including weight and buoyancy. More details are found in Fossen (2011). The center of gravity is $r_g = (0, 0, 0.02)$m, the center of buoyancy is $r_b = 0$ m, the dry mass is $m = 13.69$ kg, and the buoyancy is 1% larger than the weight. The hydrodynamic modeling is described by

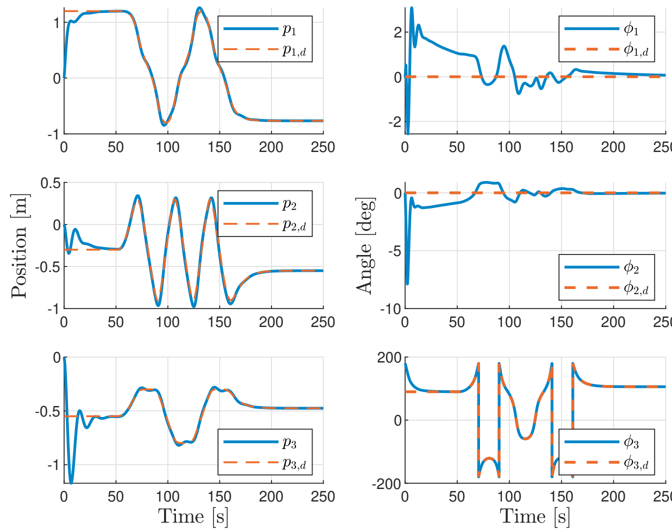

Fig. 1. The position p, desired position p_d, roll-pitch-yaw angles ϕ and desired roll-pitch-yaw angles ϕ_d.

$$\begin{aligned} M_a &:= \mathrm{diag}(5.5, 12.7, 14.6, 0.1, 0.1, 0.1) \\ D_l &:= \mathrm{diag}(4.0, 6.2, 5.2, 0.1, 0.1, 0.1) \\ D_n(\nu) &:= \mathrm{diag}(18.2|\nu_1|, 21.7|\nu_2|, 37.0|\nu_3|, 1.6|\nu_4|, 1.6|\nu_5|, 1.6|\nu_6|) \\ b &:= (2, 1, -1, -1, -1, 1) \end{aligned}$$

The control law (30) is developed by utilizing the universal covering group of SE(3), denoted $\widetilde{\mathrm{SE}}(3) := \mathbb{R}^3 \rtimes \mathrm{SU}(2)$, where SU(2) is the special unitary group of dimension two, which is isomorphic to the group of unit quaternions. See Basso et al. (2022) for more details. A unit quaternion is given by $z = (\eta, \epsilon) \in \mathbb{S}^3 := \{x \in \mathbb{R}^4 : |x| = 1\}$, where $\eta \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^3$ describe its real and imaginary components, respectively. Let $g_e = (p_e, z_e) := (R(z_d)(p - p_d), z_d^{-1} \otimes z)$, where $\otimes$ denotes the quaternion product and $R : \mathbb{S}^3 \to \mathrm{SO}(3)$ is defined by $R(z) := I + 2\eta\epsilon_\times + 2(\epsilon_\times)^2$, where $(\cdot)_\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by $\alpha_\times \beta := \alpha \times \beta$ with $\alpha, \beta \in \mathbb{R}^3$. It is easily verified that $V(g_e, q) = 2k(1 - q\eta_e) + \frac{1}{2}\langle K_p p_e, p_e \rangle$ satisfies Definition 2 with $k = 1$ and $K_p = 5I_3$. The other controller gains are set to $K_d = \mathrm{blkdiag}(10I_3, 2I_3)$ and $k_i = 0.1$. The controller has been implemented without the feedforward term that would cancel the acting hydrodynamic and hydrostatic wrenches to achieve a slightly more realistic picture of its performance. Simulation results are presented in Figures 1 to 3, from which we conclude that acceptable tracking performance is achieved despite this simplification of the control law.

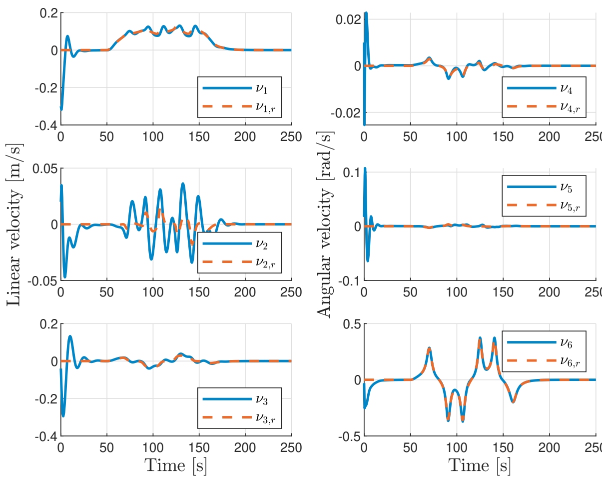

Fig. 2. The velocity ν and the desired velocity ν_r.

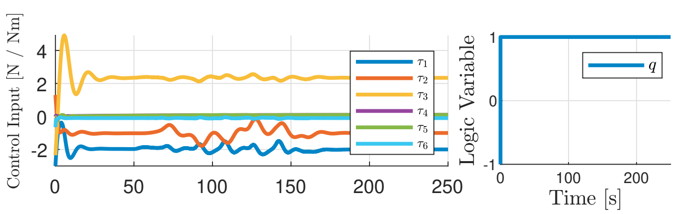

Fig. 3. The control inputs τ and logic variable q.

10. CONCLUSION

In this paper, we have introduced multiple synergistic control designs for mechanical systems described on matrix Lie groups. Specifically, we have proposed synergistic PD, output feedback, and PID type control laws ensuring global asymptotic tracking of a desired bounded reference trajectory. Additionally, the PID type control laws achieve global asymptotic tracking when the system dynamics are augmented with a constant and unknown disturbance.

REFERENCES

Basso, E.A., Schmidt-Didlaukies, H.M., Pettersen, K.Y., and Sørensen, A.J. (2022). Global asymptotic tracking for marine vehicles using adaptive hybrid feedback. *IEEE Trans. Autom. Control*.

Bhat, S.P. and Bernstein, D.S. (2000). A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, 39(1).

Bullo, F. and Lewis, A.D. (2005). *Geometric Control of Mechanical Systems*. Springer.

Fossen, T.I. (2011). *Handbook of Marine Craft Hydrodynamics and Motion Control*. Wiley.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid Dynamical Systems: Modeling Stability, and Robustness*. Princeton Univ. Press, Princeton, NJ.

Lee, T. (2015). Global exponential attitude tracking controls on SO(3). *IEEE Trans. Autom. Control*, 60(10).

Maithripala, D. and Berg, J.M. (2015). An intrinsic pid controller for mechanical systems on Lie groups. *Automatica*, 54, 189–200.

Marley, M., Skjetne, R., and Teel, A.R. (2021). Synergistic control barrier functions with application to obstacle avoidance for nonholonomic vehicles. In *Proc. 2021 American Control Conference*, 243–249.

Mayhew, C.G., Sanfelice, R.G., and Teel, A.R. (2011). Quaternion-based hybrid control for robust global attitude tracking. *IEEE Trans. Autom. Control*, 56(11).

Mayhew, C.G. (2010). *Hybrid control for topologically constrained systems*. University of California, Santa Barbara.

Mayhew, C.G., Sanfelice, R.G., and Teel, A.R. (2011). Synergistic Lyapunov functions and backstepping hybrid feedbacks. In *Proc. 2011 American Control Conference*, 3203–3208.

Mayhew, C.G. and Teel, A.R. (2013). Synergistic hybrid feedback for global rigid-body attitude tracking on SO(3). *IEEE Trans. Autom. Control*.

Rockafellar, R.T. and Wets, R.J.B. (2009). *Variational Analysis*. Springer.

Sanfelice, R.G. (2021). *Hybrid Feedback Control*. Princeton University Press, Princeton, NJ.

Schmidt-Didlaukies, H.M., Basso, E.A., and Pettersen, K.Y. (2022). A generalization of synergistic hybrid feedback control with application to maneuvering control of ships. In *Proc. 61st IEEE Conference on Decision and Control*. Cancún, Mexico.

Zhang, Z., Sarlette, A., and Ling, Z. (2015). Integral control on Lie groups. *Systems & Control Letters*.

Appendix A. LEMMAS

Lemma A.1. The feedforward control (14) results in the error dynamics

$$\dot{\nu}_e = -\nabla^M_{\nu+\nu_r}\nu_e + M^{-1}(f(g,\nu) - f(g,\nu_r)). \tag{A.1}$$

Proof. With $\tau = \kappa_{ff}(g_e, g_d, \nu_d, \dot{\nu}_d)$, (6) yields

$$M\dot{\nu}_e = M\,\mathrm{ad}_{\nu_e}\nu_r + \mathrm{ad}_{\nu}^{\top} M\nu - \mathrm{ad}_{\nu_r}^{\top} M\nu_r + f(g,\nu) - f(g,\nu_r).$$

Let $\tilde{\zeta}(\nu_e,\nu_r) = M\,\mathrm{ad}_{\nu_e}\nu_r + \mathrm{ad}_{\nu}^{\top} M\nu - \mathrm{ad}_{\nu_r}^{\top} M\nu_r$. Using the identities $\mathrm{ad}_{\zeta}\nu = \nabla^M_{\zeta}\nu - \nabla^M_{\nu}\zeta$, $\mathrm{ad}_{\zeta}\nu = -\mathrm{ad}_{\nu}\zeta$ and $\mathrm{ad}_{\nu}\nu = 0$, it holds that

$$\begin{aligned}
\tilde{\zeta}(\nu_e,\nu_r) &= \mathrm{ad}_{\nu}^{\top} M\nu - \mathrm{ad}_{\nu_r}^{\top} M\nu_r + M\,\mathrm{ad}_{\nu_e}\nu_r \\
&= M\nabla^M_{\nu_r}\nu_r - M\nabla^M_{\nu}\nu + M\,\mathrm{ad}_{\nu_e}\nu_r \\
&= M(\nabla^M_{\nu_r}\nu_r - \nabla^M_{\nu}\nu + \nabla^M_{\nu_e}\nu_r - \nabla^M_{\nu_r}\nu_e) \\
&= M(-\nabla^M_{\nu}\nu_e - \nabla^M_{\nu_r}\nu_e) \\
&= -M\nabla^M_{\nu+\nu_r}\nu_e. \qquad \square
\end{aligned}$$

Lemma A.2. For every $\xi \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^m$, it holds that

$$\langle \xi, M\nabla^M_{\zeta}\xi \rangle = 0, \tag{A.2}$$

Proof. From the identities $\langle \eta, \mathrm{ad}_{\xi}\zeta \rangle = \langle \mathrm{ad}_{\xi}^{\top}\eta, \zeta \rangle$ and $\mathrm{ad}_{\xi}\xi = 0$, it holds that

$$\begin{aligned}
2\langle \xi, M\nabla^M_{\zeta}\xi \rangle &= \langle \xi, M\,\mathrm{ad}_{\zeta}\xi - \mathrm{ad}_{\xi}^{\top} M\zeta - \mathrm{ad}_{\zeta}^{\top} M\xi \rangle \\
&= \langle \xi, M\,\mathrm{ad}_{\zeta}\xi - \mathrm{ad}_{\zeta}^{\top} M\xi \rangle. \qquad \text{(A.3)}
\end{aligned}$$

By rewriting the last term in (A.3) as $\langle \xi, -\mathrm{ad}_{\zeta}^{\top} M\xi \rangle = -\langle \mathrm{ad}_{\zeta}\xi, M\xi \rangle$, symmetry of M implies that $\langle \xi, M\,\mathrm{ad}_{\zeta}\xi \rangle = \langle M\xi, \mathrm{ad}_{\zeta}\xi \rangle$ and the result follows from linearity of the inner product. $\square$

Lemma A.3. Let $A \in \mathbb{R}^{m\times m}$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable. Then f is A-monotone if and only if the matrix $\nabla f(x) - A$ is positive semidefinite for all $x \in \mathbb{R}^m$.

Proof. Apply (Rockafellar and Wets, 2009, Prop. 12.3) to $f(x) - Ax$. $\square$