



Evolution driven by the infinity fractional Laplacian

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Abstract

We consider the evolution problem associated to the infinity fractional Laplacian introduced by Bjorland et al. (Adv Math 230(4–6):1859–1894, 2012) as the infinitesimal generator of a non-Brownian tug-of-war game. We first construct a class of viscosity solutions of the initial-value problem for bounded and uniformly continuous data. An important result is the equivalence of the nonlinear operator in higher dimensions with the one-dimensional fractional Laplacian when it is applied to radially symmetric and monotone functions. Thanks to this and a comparison theorem between classical and viscosity solutions, we are able to establish a global Harnack inequality that, in particular, explains the long-time behavior of the solutions.

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1 Introduction

In this paper we study a parabolic equation associated to the (normalized) infinity fractional Laplacian operator. We recall that the local version of the game had been introduced by Peres et al. in 2009 [33] where it is shown that the standard infinity Laplace equation is solved by the value function for a two-players random turn “tug-of-war” game. The game is as follows: a token is initially placed at a position $x_0 \in \Omega$ and every turn a fair coin is tossed to choose which of the players plays. This player moves the token to any point in the ball of radius

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$\varepsilon > 0$ around the current position. If, eventually, iterating this process, the token reaches a point $x_e \in \partial\Omega$, the players are awarded (or penalized) $f(x_e)$ (payoff function). For a PDE overview of the infinity Laplacian operator and its role as an absolute minimizer for the L^∞ norm of the gradient, see [26, 27].

In 2012 Bjorland et al. [6] introduced equations involving the so-called *infinity fractional Laplacian* as a model for a nonlocal version of the “tug-of-war” game. Following their explanation, instead of flipping a coin at every step, every player chooses a direction and it is an s -stable Lévy process that chooses both the active player and the distance to travel. The infinity fractional Laplacian, with symbol Δ_∞^s , is a nonlinear integro-differential operator, the original definition is given in Lemma 2.1 below. However, for the purpose of this paper, we also consider the alternative equivalent definition introduced in [6] (see also [16]) given by

$$\Delta_\infty^s \phi(x) := C_s \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_0^\infty (\phi(x + \eta y) + \phi(x - \eta \tilde{y}) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}} \quad \text{where } s \in (1/2, 1). \tag{1.1}$$

The constant is usually taken as $C_s = (4^s s \Gamma(\frac{1}{2} + s)) / (\pi^{\frac{1}{2}} \Gamma(1 - s))$ but the value is irrelevant for our results. In their paper [6] the authors study two stationary problems involving the infinity fractional Laplacian posed in bounded space domains, namely, a Dirichlet problem and a double-obstacle problem.

Here, we consider the evolution problem

$$\begin{cases} \partial_t u(x, t) = \Delta_\infty^s u(x, t), & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

with $s \in (1/2, 1)$ and $n \geq 2$. When $n = 1$ the operator $-\Delta_\infty^s$ is just the usual linear fractional Laplacian operator $(-\Delta)^s$ of order s , and Eq. (1.2) is just the well-known fractional heat equation [7, 23]. See also a detailed study of that equation using PDE techniques in [3, 8, 20, 38]. Note that for $n \geq 2$ the operator is nonlinear so a new theory is needed. A non-normalized version is introduced in [12] along with a well-posedness theory for the corresponding equations of the type (1.2)–(1.3). However, the two problems are not equivalent nor closely related.

The local counterpart of (1.2), given by $\partial_t u = \Delta_\infty u$, is studied in [21], where the authors present several applications such as image processing [10] or tug-of-war games over a finite time horizon [4]. In this case, the game finishes when the terminal time is reached, instead of when the boundary of the domain is reached. In the nonlocal case treated in the present paper, the application to image processing has not been described to our knowledge. However, a derivation similar to the one in [4], based on dynamic programming, seems to be reasonable.

Here, we develop an existence theory of suitable viscosity solutions for the parabolic problem (1.2)–(1.3), based on approximation with monotone schemes. We show that the obtained class of solutions enjoys a number of good properties. As in the elliptic case [6], we lack a uniqueness result in the context of viscosity solutions. However, we are able to prove an important comparison theorem relating two types of solutions, classical and viscosity ones, see Theorem 2.6. As a counterpart, we also obtain uniqueness and comparison of classical solutions. Moreover, we show that for smooth, radially symmetric functions which are nonincreasing along the radius in \mathbb{R}^n with $n \geq 2$, the operator $-\Delta_\infty^s$ reduces to the classical fractional Laplacian $(-\Delta)^s$ in dimension $n = 1$ (Theorem 6.1). A similar example regarding nondecreasing one-dimensional profiles can be found in Lemma 6.3. In this way we

may construct a large class of classical solutions that make the comparison theorem relevant (Theorems 2.10, 2.12). Note that no similar reduction to a lower dimensional problem applies in general, even in the radial case (see Sect. 6.2 for a counterexample).

Using the developed tools, we study the asymptotic behavior of the constructed solutions, and obtain a global Harnack type principle, see Theorem 2.13.

1.1 Related literature

It is interesting to compare the nonlocal model (1.2) with the local version of the infinity Laplacian that has been studied by many authors, both in the stationary and evolution cases, cf. [1, 2, 21, 26, 27, 33–35]. Asymptotic expansions for the game theoretical p -Laplacian in the local case and related approximation schemes in the elliptic case are studied in [19, 29, 30] and in the parabolic case in [28]. For the variational version of the p -Laplacian operator see [17].

There exist in the literature other nonlocal generalizations of the p -Laplacian and the infinity Laplacian. Let us mention (i) the normalized version [5, 6] with asymptotic expansions and game theoretical approach [9, 16, 25]; (ii) nonnormalized version [12] both elliptic and parabolic; (iii) Hölder infinity Laplacian [11]; and (iv) the (variational) fractional p -Laplacian [13, 14, 31, 32, 36, 37, 39].

2 Preliminaries and statement of main results

First let us fix some notation that we will use along the paper.

For given $\delta > 0$, standard mollifiers are denoted by ρ_δ . Following [6], we say that $\phi \in C^{1,1}(x)$ at some $x \in \mathbb{R}^n$ if there exists $p_x \in \mathbb{R}^n$ and $C_x, \eta_x > 0$ such that

$$|\phi(x + y) - \phi(x) - p_x \cdot y| \leq C_x |y|^2 \quad \text{for all } |y| < \eta_x. \tag{2.1}$$

Note that $C_b^2(B_R(x)) \subset C^{1,1}(x)$. Here $C_b^k(U)$ is the space of functions on the set U with bounded continuous derivatives of all orders in $[0, k]$. Let us also define:

$$\begin{aligned} B(\mathbb{R}^n) &:= \{\phi : \mathbb{R}^n \rightarrow \mathbb{R} \mid \phi \text{ is pointwisely bounded}\}, \\ UC(\mathbb{R}^n) &:= \{\phi : \mathbb{R}^n \rightarrow \mathbb{R} \mid \phi \text{ is uniformly continuous}\}, \\ BUC(\mathbb{R}^n) &:= B(\mathbb{R}^n) \cap UC(\mathbb{R}^n) \quad \text{with} \quad \|\phi\|_{C_b(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |\phi(x)|, \end{aligned}$$

and for $\beta \in (0, 1]$, we define $|\phi|_{C^{0,\beta}(\mathbb{R}^n)} = \sup_{x,y \in \mathbb{R}^n} |\phi(x) - \phi(y)|/|x - y|^\beta$ and

$$C^{0,\beta}(\mathbb{R}^n) := \{\phi \in C_b(\mathbb{R}^n) \mid \|\phi\|_{C^{0,\beta}} < \infty\} \quad \text{where} \quad \|\phi\|_{C^{0,\beta}} = \|\phi\|_{C_b} + |\phi|_{C^{0,\beta}}.$$

A modulus of continuity is a nondecreasing function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{r \rightarrow 0^+} \omega(r) = 0$. For a function $f \in BUC(\mathbb{R}^n)$, we define the corresponding modulus of continuity as follows:

$$\omega_f(r) = \sup_{|y| \leq r} \|f(\cdot + y) - f\|_{C_b(\mathbb{R}^n)}.$$

For a Hölder continuous function $f \in C^{0,\beta}(\mathbb{R}^n)$, $\omega_f(r) \leq |f|_{C^{0,\beta}} r^\beta$.

We will also need $e_i := (0, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, where 1 is at the i th component, and denote average integrals by

$$\int_{\varepsilon}^{\infty} g(\eta) \frac{d\eta}{\eta^{1+2s}} := \frac{1}{\int_{\varepsilon}^{\infty} \frac{d\eta}{\eta^{1+2s}}} \int_{\varepsilon}^{\infty} g(\eta) \frac{d\eta}{\eta^{1+2s}}.$$

2.1 Alternative characterization of the infinity fractional Laplacian

We have the following alternative characterization of operator Δ_{∞}^s that we will use throughout:

Lemma 2.1 (Alternative characterization) *Assume $\phi \in C^{1,1}(x) \cap B(\mathbb{R}^n)$. Then:*

- If $\nabla\phi(x) \neq 0$, then

$$\Delta_{\infty}^s \phi(x) = C_s \int_0^{\infty} (\phi(x + \eta\zeta) + \phi(x - \eta\zeta) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}} \quad \text{where } \zeta := \nabla\phi(x)/|\nabla\phi(x)|.$$

- If $\nabla\phi(x) = 0$, then

$$\begin{aligned} \Delta_{\infty}^s \phi(x) &= C_s \sup_{|y|=1} \int_0^{\infty} (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\ &\quad + C_s \inf_{|y|=1} \int_0^{\infty} (\phi(x - \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}}. \end{aligned}$$

The equivalence when $\nabla\phi(x) = 0$ follows from the fact that the integrals in this case are well-defined and can be combined to get (1.1). When $\nabla\phi(x) \neq 0$, it can be shown that the supremum and infimum of (1.1) is actually taken at ζ , see Proposition 2.2 in [16]. To sketch the proof, assume for simplicity that the supremum in (1.1) is taken at y , and let us argue that $y = \zeta$. Indeed, by splitting the integral and using the definitions of $C^{1,1}$ and the infimum,

$$\begin{aligned} \Delta_{\infty}^s \phi(x) &\leq C_s \int_0^{\infty} (\phi(x + \eta y) + \phi(x - \eta\zeta) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}} \\ &\leq C_s (\nabla\phi(x) \cdot (y - \zeta)) \int_0^{\eta_x} \eta \frac{d\eta}{\eta^{1+2s}} + C. \end{aligned}$$

Now, since $\Delta_{\infty}^s \phi(x)$ is well-defined and the integral diverges if $y \neq \zeta$, we must have $y = \zeta$. A similar argument holds for the infimum.

2.2 Existence of solutions and basic properties

We are able to construct a suitable class of viscosity solutions of (1.2)–(1.3). The two steps are as follows:

- (i) Approximating Δ_{∞}^s by removing the singularity, i.e., we introduce

$$\mathcal{L}_{\varepsilon}[\phi](x) := C_s \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_{\varepsilon}^{\infty} (\phi(x + \eta y) + \phi(x - \eta\tilde{y}) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}}.$$

- (ii) Discretizing in time by letting $\tau > 0$ and $t_j := j\tau$ for $j \in \mathbb{N}$, i.e., $t_j \in \tau\mathbb{N}$, and then considering the semidiscrete problem

$$\begin{cases} \frac{U^{j+1}(x) - U^j(x)}{\tau} = \mathcal{L}_{\varepsilon}[U^j](x), & x \in \mathbb{R}^n, j \in \mathbb{N}, \\ U^0(x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

$$\tag{2.3}$$

We study the properties of (2.2)–(2.3) in Sect. 3. Existence of viscosity solutions follows by taking the limit in this approximate scheme, as well as properties inherited from the approximations.

Theorem 2.2 (Existence and a priori results) *If $u_0 \in BUC(\mathbb{R}^n)$, then there is at least one viscosity solution $u \in C_b(\mathbb{R}^n \times [0, \infty))$ of (1.2)–(1.3). Moreover:*

- (a) (*C_b -bound*) For all $t > 0$, $\|u(\cdot, t)\|_{C_b(\mathbb{R}^n)} \leq \|u_0\|_{C_b(\mathbb{R}^n)}$.
- (b) (*Uniform continuity in space*) For all $y \in \mathbb{R}^n$ and all $t > 0$,
 $\|u(\cdot + y, t) - u(\cdot, t)\|_{C_b(\mathbb{R}^n)} \leq \omega_{u_0}(|y|)$.
- (c) (*Uniform continuity in time*) For all $t, \tilde{t} > 0$,
 $\|u(\cdot, t) - u(\cdot, \tilde{t})\|_{C_b(\mathbb{R}^n)} \leq \tilde{\omega}(|t - \tilde{t}|)$ where $\tilde{\omega}(r) := \inf_{\delta > 0} \left\{ \omega_{u_0}(\delta) + r \sup_{\varepsilon > 0} \|\mathcal{L}_\varepsilon[u_{0,\delta}]\|_{C_b(\mathbb{R}^n)} \right\}$
 is a modulus satisfying $\tilde{\omega}(r) \leq \omega_{u_0}(r^{1/3}) + C(r^{1/3} + r)$, $C := c_s \|u_0\|_{C_b(\mathbb{R}^n)} \|\nabla \rho\|_{L^1(\mathbb{R}^n)}^{2-2s}$
 $\|D^2 \rho\|_{L^1(\mathbb{R}^n)}^{2s-1}$, and ρ is a standard mollifier.

Remark 2.3 The definition of viscosity solutions is given Sect. 4 (Definition 4.3). We obtain viscosity solution as limits of monotone approximations of the problem in Sect. 3.

Note that if u_0 is Hölder continuous and $s \in (1/2, 1)$, then the above moduli will be (more) explicit.

Lemma 2.4 *If $u_0 \in C^{0,\beta}(\mathbb{R}^n)$ for $\beta \in (0, 1]$, then*

$$\omega_{u_0}(\delta) = |u_0|_{C^{0,\beta}} \delta^\beta \quad \text{and} \quad \|\mathcal{L}_\varepsilon[u_{0,\delta}]\|_{C_b(\mathbb{R}^n)} \leq c(s, \rho) |u_0|_{C^{0,\beta}} \delta^{\beta-2s}.$$

The above result will be proved at the end of Sect. 4.

It follows after a minimization in δ that $\tilde{\omega}(r) = c(s, \rho) |u_0|_{C^{0,\beta}} r^{\frac{\beta}{2s}}$, and the solution u will be Hölder continuous with the correct parabolic regularity.

Corollary 2.5 (Existence and a priori results) *If $u_0 \in C^{0,\beta}(\mathbb{R}^n)$ for $\beta \in (0, 1]$, then there is at least one viscosity solution $u \in C_b(\mathbb{R}^n \times [0, \infty))$ of (1.2)–(1.3). Moreover:*

- (a) (*C_b -bound*) For all $t > 0$ $\|u(\cdot, t)\|_{C_b(\mathbb{R}^n)} \leq \|u_0\|_{C_b(\mathbb{R}^n)}$.
- (b) (*Hölder in space*) For all $y \in \mathbb{R}^n$ and all $t > 0$,
 $\|u(\cdot + y, t) - u(\cdot, t)\|_{C_b(\mathbb{R}^n)} \leq |u_0|_{C^{0,\beta}} |y|^\beta$.
- (c) (*Hölder in time*) There is a constant $c(s, \rho)$ only depending on s and ρ such that for all $t, \tilde{t} > 0$,
 $\|u(\cdot, t) - u(\cdot, \tilde{t})\|_{C_b(\mathbb{R}^n)} \leq C |u_0|_{C^{0,\beta}} |t - \tilde{t}|^{\frac{\beta}{2s}}$.

2.3 Classical solutions, radial solutions, comparison, and uniqueness

There could be other ways of obtaining viscosity solutions, and unfortunately, we lack general comparison and uniqueness results. Nevertheless, we can obtain that classical solutions are unique and we can compare our constructed viscosity solutions with classical sub- and supersolutions of (1.2)–(1.3).¹

¹ We will work with classical solutions in C_b^2 . Actually, we can reduce to C_b^1 for the temporal variable, and to $C^{1,1} \cap B$ for the spatial variables.

Theorem 2.6 (Comparison between classical and viscosity solutions) *Assume $u_0 \in BUC(\mathbb{R}^n)$. Let $\underline{u}, \bar{u} \in C_b^2(\mathbb{R}^n \times [0, \infty))$ be respective classical sub- and supersolution of (1.2)–(1.3), and let $u \in BUC(\mathbb{R}^n \times [0, \infty)$ be a viscosity solution of (1.2)–(1.3) as constructed in Theorem 2.2. Then $\underline{u} \leq u \leq \bar{u}$ in $\mathbb{R}^n \times (0, \infty)$.*

The above result is proved in Sect. 7. We want to emphasize that it is done in a rather nonstandard way, since we inherit the comparison from the approximation scheme when the solution is classical. In general, this cannot be done in the context of viscosity solutions since the approximation scheme only converges up to a subsequence.

Remark 2.7 By Theorem 2.6, we can in addition get comparison of constructed viscosity solutions as long as the initial datas are separated by an initial data which produces a classical solution.

An immediate consequence of Theorem 2.6:

Corollary 2.8 (Comparison of classical sub- and supersolutions) *Let $u, v \in C_b^2(\mathbb{R}^n \times [0, \infty))$ be respective classical sub- and supersolutions of (1.2)–(1.3) with initial data u_0, v_0 . If $u_0 \leq v_0$, then $u \leq v$.*

Corollary 2.9 (Uniqueness of solutions) *Classical solutions of (1.2)–(1.3) in $C_b^2(\mathbb{R}^n \times [0, \infty))$ are unique.*

Theorem 2.6 might be an empty statement unless we provide a class of classical solutions of (1.2)–(1.3). The following result, proved in Sect. 6, solves this issue.

Theorem 2.10 (Existence of classical radial solutions) *Assume that $u_0 \in C_b^\infty(\mathbb{R}^n)$ is radial and radially nonincreasing. Then there exists a classical and radial solution $u \in C_b^\infty(\mathbb{R}^n \times [0, \infty))$ of (1.2)–(1.3). Moreover, if $U_0(r) := u_0(|x|)$ and $U_0(-r) := U_0(r)$ for $r = |x| \geq 0$, then*

$$u(x, t) = (P_s(\cdot, t) * U_0)(r) = \int_{-\infty}^{\infty} P_s(r - s, t)U_0(s) ds \quad \text{for all } |x| = r,$$

where P_s is the fundamental solution of the one-dimensional fractional heat equation (cf. 5.3).

Remark 2.11 (a) The idea in the above result is that, for radially nonincreasing radial functions, the operators $-\Delta_{\infty, \mathbb{R}^n}^s$ and $(-\Delta)_{\mathbb{R}^1}^s$ coincide (Proposition 6.1), and (1.2) then reduces to the one-dimensional fractional heat equation.

(b) In view of Theorem 2.6, this classical solution is also a viscosity solution in our sense.

Another class of classical solutions are:

Theorem 2.12 (Existence of classical solutions with one-dimensional profiles) *Assume that $U_0 \in C_b^\infty(\mathbb{R})$ is nondecreasing, and let $u_0 \in C_b^\infty(\mathbb{R}^n)$ be defined as*

$$u_0(x) := U_0(x_1).$$

Then there exists a classical solution $u \in C_b^\infty(\mathbb{R}^n \times [0, \infty))$ of (1.2)–(1.3). Moreover,

$$u(x, t) = (P_s(\cdot, t) * U_0)(x_1) = \int_{-\infty}^{\infty} P_s(x_1 - s, t)U_0(s) ds,$$

where P_s is the fundamental solution of the one-dimensional fractional heat equation (cf. (5.3)).

The proof is similar to the one of Theorem 2.10, and we will omit it. One just needs to note that $P_s(\cdot, t) * U_0$ is nondecreasing.

2.4 Asymptotic behavior and Harnack inequality

Having established Theorems 2.6 and 2.10, we can prove that solutions of (1.2)–(1.3) behave like solutions of the one-dimensional fractional heat equation, up to suitable constants. In Sect. 5, we recall some results on that equation and its fundamental solution denoted by P_s . In Sect. 8, we prove the following result.

Theorem 2.13 (Global Harnack principle) *Let $u \in BUC(\mathbb{R}^n \times [0, \infty))$ be a viscosity solution of (1.2)–(1.3), as constructed in Theorem 2.2, with initial data $u_0 \in BUC(\mathbb{R}^n)$ such that $u_0 \not\equiv 0$ and*

$$0 \leq u_0(x) \leq (1 + |x|^2)^{-\frac{1+2s}{2}} \quad \text{for all } |x| \geq R \geq 1.$$

Then, for all $\tau > 0$, there exist constants $C_1, C_2 > 0$ depending only on s, R , and u_0 , such that

$$C_1 P_s(|x|, t) \leq u(x, t) \leq C_2 P_s(|x|, t) \quad \text{for all } (x, t) \in \mathbb{R}^n \times [\tau, \infty).$$

Moreover, for all $\tau > 0$, there exist constants $\tilde{C}_1, \tilde{C}_2 > 0$ depending only on s, R , and u_0 , such that

$$\tilde{C}_1 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \leq u(x, t) \leq \tilde{C}_2 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \quad \text{for all } (x, t) \in \mathbb{R}^n \times [\tau, \infty).$$

In particular, $u > 0$ in $\mathbb{R}^n \times [\tau, \infty)$.

- Remark 2.14** (a) Note that u_0 is not necessarily in $L^1(\mathbb{R}^n)$ since the decay required for large x is the one corresponding to the one-dimensional fractional heat kernel P_s .
 (b) The above theorem provides a counterexample to conservation of mass for (1.2)–(1.3): For any smooth compactly supported $0 \leq u_0 \in L^1(\mathbb{R}^n)$, the corresponding solution u satisfies

$$\int_{\mathbb{R}^n} u(x, 1) \, dx \geq \tilde{C}_1 \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{\frac{1+2s}{2}}} \, dx.$$

The last integral is infinite if $1 + 2s \leq n$, and hence there is no conservation of mass for $n \geq 3$.

- (c) In Theorem 2.12, we construct other types of special solutions which could also be used to prove the global Harnack principle.

3 Properties of an approximation scheme

We will now start the detailed development of the theory. The basic idea we follow is to discretize explicitly in time and use the asymptotic expansion of Δ_∞^s found in [16] to provide a monotone zero-order approximation of the operator.

We recall that, for $s \in (1/2, 1)$,

$$\begin{aligned} \mathcal{L}_\varepsilon[\phi](x) &:= C_s \frac{1}{2s\varepsilon^{2s}} \left(\sup_{|y|=1} \int_\varepsilon^\infty \phi(x + \eta y) \frac{d\eta}{\eta^{1+2s}} + \inf_{|y|=1} \int_\varepsilon^\infty \phi(x + \eta y) \frac{d\eta}{\eta^{1+2s}} - 2\phi(x) \right) \\ &= C_s \left(\sup_{|y|=1} \int_\varepsilon^\infty \phi(x + \eta y) \frac{d\eta}{\eta^{1+2s}} + \inf_{|y|=1} \int_\varepsilon^\infty \phi(x + \eta y) \frac{d\eta}{\eta^{1+2s}} - \frac{1}{s\varepsilon^{2s}} \phi(x) \right) \\ &= C_s \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_\varepsilon^\infty (\phi(x + \eta y) + \phi(x - \eta \tilde{y}) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}}. \end{aligned} \tag{3.1}$$

Lemma 3.1 *The operator $\mathcal{L}_\varepsilon : C_b(\mathbb{R}^n) \rightarrow C_b(\mathbb{R}^n)$ is well-defined and bounded.*

Remark 3.2 Note that, in general, $\Delta_\infty^s : C_b^\infty(\mathbb{R}^n) \not\rightarrow C_b(\mathbb{R}^n)$. See Sect. 9.

Proof of Lemma 3.1 Let $\phi \in C_b(\mathbb{R}^n)$. Since $\int_\varepsilon^\infty \eta^{-(1+2s)} dt = \frac{1}{2s} \varepsilon^{-2s}$, we have $\|\mathcal{L}_\varepsilon[\phi]\|_{C_b(\mathbb{R}^n)} \leq \frac{4C_s}{2s} \|\phi\|_{C_b(\mathbb{R}^n)} \varepsilon^{-2s}$ for any $\varepsilon > 0$. It follows that $\mathcal{L}_\varepsilon[\phi]$ is bounded. If $\mathcal{L}_\varepsilon[\phi]$ is continuous, it also follows that \mathcal{L}_ε is a bounded operator on $C_b(\mathbb{R}^n)$. To show continuity at an arbitrary point $x_1 \in \mathbb{R}^n$, we fix $\epsilon > 0$. By the above bound there is (large) $R > 0$ such that

$$\|\mathcal{L}_R[\phi]\|_{C_b(\mathbb{R}^n)} < \frac{\epsilon}{4}.$$

For $x_2 \in B(x_1, 1)$, we find by the triangle inequality and $\sup \inf(\dots) - \sup \inf(\dots) \leq \sup \sup(\dots - \dots)$,

$$\begin{aligned} &|\mathcal{L}_\varepsilon[\phi](x_1) - \mathcal{L}_\varepsilon[\phi](x_2)| \\ &\leq C_s \sup_{|\tilde{y}|=1} \sup_{|y|=1} \int_\varepsilon^R |(\phi(x_1 + \eta y) + \phi(x_1 + \eta \tilde{y})) - (\phi(x_2 + \eta y) + \phi(x_2 + \eta \tilde{y}))| \frac{d\eta}{\eta^{1+2s}} \\ &\quad + 2C_s \int_\varepsilon^R \frac{d\eta}{\eta^{1+2s}} |\phi(x_1) - \phi(x_2)| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &\leq 2C_s \omega_{\phi,R}(x_1 - x_2) \int_\varepsilon^R \frac{d\eta}{\eta^{1+2s}} + \frac{\epsilon}{2}, \end{aligned}$$

where $\omega_{\phi,R}$ is the modulus of continuity of ϕ in the ball $B(0, |x_1| + 1 + R)$. Since the integral is finite, the last expression is less than ϵ when $|x_2 - x_1|$ is small enough and continuity of $\mathcal{L}_\varepsilon[\phi]$ follows. \square

To state the consistency, we introduce admissible test functions ϕ : There is $\eta_x > 0$, such that

$$(i) \ \phi \in C^2(\bar{B}_{\eta_x}) \quad \text{and} \quad (ii) \ \phi \in B(\mathbb{R}^n) \cap UC(\mathbb{R}^n \setminus \bar{B}_{\eta_x}).$$

Lemma 3.3 (Consistency, Theorem 1.1 in [16]) *Under the above assumptions on ϕ , for every $\varepsilon < \eta_x$,*

$$\left| \mathcal{L}_\varepsilon[\phi](x) - \Delta_\infty^s \phi(x) \right| = o_\varepsilon(1),$$

where the bound $o_\varepsilon(1)$ depends only on $|\nabla \phi(x)|^{-1}$, $\|D^2 \phi\|_{C_b(\bar{B}_{\eta_x})}$, and $\omega_{\phi, \bar{B}_{\eta_x}^c}$.

We also need ε independent bounds to send $\varepsilon \rightarrow 0$.

Lemma 3.4 (Uniform bound) *If $\phi \in C_b^{2s}(\mathbb{R}^n)$, then there is a constant $c(s)$ only depending on s such that*

$$|\mathcal{L}_\varepsilon[\phi](x)| \leq c(s) \|\nabla\phi\|_{C_b(\mathbb{R}^n)}^{2-2s} \|D^2\phi\|_{C_b(\mathbb{R}^n)}^{2s-1}.$$

Proof We add to \mathcal{L}_ε the gradient term

$$\sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_\varepsilon^1 t p_x \cdot (y - \tilde{y}) \frac{dt}{t^{1+2s}} = \int_\varepsilon^1 \eta \frac{d\eta}{\eta^{1+2s}} \sup_{|y|=1} \inf_{|\tilde{y}|=1} p_x \cdot (y - \tilde{y}) = 0,$$

Since the integrand is bounded, we then split the resulting integral in two—an integral with the inf and an integral with the sup. The result for the sup-part is:

$$\sup_{|\eta|=1} \int_\varepsilon^\infty (\phi(x + \eta y) - \phi(x) - t p_x \cdot y \eta^{1+2s}) \frac{d\eta}{\eta^{1+2s}}.$$

Splitting this integral in two, $\int_\varepsilon^r + \int_r^\infty$, and Taylor expanding, we find the following upper bound

$$\begin{aligned} \frac{1}{2} \|D^2\phi\|_{C_b} \int_0^r \eta^2 \frac{d\eta}{\eta^{1+2s}} + 2 \|\nabla\phi\|_{C_b} \int_r^\infty \eta \frac{d\eta}{\eta^{1+2s}} &\leq \frac{1}{2} \|D^2\phi\|_{C_b} \frac{1}{2-2s} r^{2-2s} \\ &+ 2 \|\nabla\phi\|_{C_b} \frac{1}{2s-1} r^{1-2s}. \end{aligned}$$

Minimizing with respect to r then proves the result for the sup-part. The inf-part is similar. □

Remark 3.5 Note that \mathcal{L}_ε is monotone in the following two ways:

- (i) $\mathcal{L}_\varepsilon[\phi] \leq 0$ at any global maximum of ϕ .
- (ii) In the sense of monotone approximations in viscosity solution theory:

$$\psi_1 \leq \psi_2 \quad \text{in } \mathbb{R}^n \quad \implies \quad L(\varepsilon, \psi_1, r) \leq L(\varepsilon, \psi_2, r) \quad \text{in } \mathbb{R}^n,$$

where $\mathcal{L}_\varepsilon[\psi](x) = L(\varepsilon, \psi, \psi(x))$ and $L : \mathbb{R}_+ \times BUC(\mathbb{R}^n) \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$L(\varepsilon, \psi, r) = C_s \left(\sup_{|y|=1} \int_\varepsilon^\infty \psi(x + \eta y) \frac{d\eta}{\eta^{1+2s}} + \inf_{|y|=1} \int_\varepsilon^\infty \psi(x + \eta y) \frac{d\eta}{\eta^{1+2s}} - \frac{1}{s\varepsilon^{2s}} r \right).$$

These properties are crucial in order to obtain approximation schemes that preserves the properties of the limit problem (1.2)–(1.3).

3.1 Semi-discrete scheme defined on $\mathbb{R}^n \times \{\mathcal{T}\mathbb{N} \cup \mathbf{0}\}$

We will now study the semi-discrete scheme (2.2)–(2.3).

Proposition 3.6 (Well-posedness and properties) *Assume $u_0 \in BUC(\mathbb{R}^n)$ and $\varepsilon, \tau > 0$. Then there exists a unique solution $U^j \in C_b(\mathbb{R}^n)$ of (2.2)–(2.3). Moreover, if*

$$\tau \leq \frac{s}{C_s} \varepsilon^{2s}, \tag{CFL}$$

then the following properties hold:

- (a) (C_b -stability) $\|U^j\|_{C_b(\mathbb{R}^n)} \leq \|u_0\|_{C_b(\mathbb{R}^n)}$.

- (b) (Comparison principle) Let U^j and V^j be sub- and supersolutions of (2.2)–(2.3) with respective initial data $u_0 \in BUC(\mathbb{R}^n)$ and $v_0 \in BUC(\mathbb{R}^n)$. If $u_0 \leq v_0$ in \mathbb{R}^n , then $U^j \leq V^j$ in \mathbb{R}^n for all $j \in \mathbb{N}$.
- (c) (C_b -contraction) Let U^j and V^j be solutions of (2.2)–(2.3) with respective initial data $u_0 \in BUC(\mathbb{R}^n)$ and $v_0 \in BUC(\mathbb{R}^n)$. Then

$$\|U^j - V^j\|_{C_b(\mathbb{R}^n)} \leq \|u_0 - v_0\|_{C_b(\mathbb{R}^n)} \quad \text{for all } j \in \mathbb{N}.$$

- (d) (Equicontinuity in space) For all $y \in \mathbb{R}^n$ and all $j \in \mathbb{N}$,

$$\|U^j(\cdot + y) - U^j\|_{C_b(\mathbb{R}^n)} \leq \omega_{u_0}(|y|).$$
- (e) (Equicontinuity in time) For all $j, k \in \mathbb{N}$ and all $0 < \varepsilon < 1$,

$$\|U^{j+k} - U^j\|_{C_b(\mathbb{R}^n)} \leq \tilde{\omega}(|t_{j+k} - t_j|), \quad \text{where } \tilde{\omega} \text{ is defined in Theorem 2.2 (c).}$$

Proof Since (2.2)–(2.3) is explicit and $\mathcal{L}_\varepsilon : C_b(\mathbb{R}^n) \rightarrow C_b(\mathbb{R}^n)$ is well-defined and bounded by Lemma 3.1, existence and uniqueness follows by construction.

Let us then show the different a priori estimates:

- (b) Since $u_0 \leq v_0$, we have $U^0 \leq V^0$. Then, by induction assume that $U^j \leq V^j$. By (2.2), we get

$$\begin{aligned} U^{j+1}(x) - V^{j+1}(x) &\leq U^j(x) - V^j(x) + \tau \left(\mathcal{L}_\varepsilon[U^j](x) - \mathcal{L}_\varepsilon[V^j](x) \right) \\ &= \left(U^j(x) - V^j(x) \right) \left(1 - \tau \frac{C_s}{s\varepsilon^{2s}} \right) \\ &\quad + \tau C_s \left(\sup_{|y|=1} \int_\varepsilon^\infty U^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} - \sup_{|y|=1} \int_\varepsilon^\infty V^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right) \\ &\quad + \tau C_s \left(\inf_{|y|=1} \int_\varepsilon^\infty U^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} - \inf_{|y|=1} \int_\varepsilon^\infty V^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right) \\ &\leq 0, \end{aligned}$$

where the last inequality follows from the induction hypothesis $U^j \leq V^j$ and CFL.

- (a) Note that

$$V^j := \inf_{x \in \mathbb{R}^n} \{u_0(x)\} \quad \text{and} \quad W^j := \sup_{x \in \mathbb{R}^n} \{u_0(x)\} \quad \text{for all } j \in \mathbb{N}$$

are solutions of (2.2)–(2.3). Since $\inf_{x \in \mathbb{R}^n} \{u_0(x)\} \leq u_0 \leq \sup_{x \in \mathbb{R}^n} \{u_0(x)\}$, we have by (b) that

$$\inf_{x \in \mathbb{R}^n} \{u_0(x)\} = V^j \leq U^j \leq W^j = \sup_{x \in \mathbb{R}^n} \{u_0(x)\}.$$

- (c) By the proof of (b) and the fundamental inequalities $|\sup(\dots) - \sup(\dots)| \leq \sup(|\dots - \dots|)$ and $|\inf(\dots) - \inf(\dots)| \leq \sup(|\dots - \dots|)$, we can also get that

$$\begin{aligned}
 &|U^{j+1}(x) - V^{j+1}(x)| \\
 &\leq \left| U^j(x) - V^j(x) \right| \left(1 - \tau \frac{C_s}{s\varepsilon^{2s}} \right) + \tau C_s \left| \sup_{|y|=1} \int_{\varepsilon}^{\infty} U^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right. \\
 &\quad \left. - \sup_{|y|=1} \int_{\varepsilon}^{\infty} V^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right| \\
 &\quad + \tau C_s \left| \inf_{|y|=1} \int_{\varepsilon}^{\infty} U^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} - \inf_{|y|=1} \int_{\varepsilon}^{\infty} V^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right| \\
 &\leq \left| U^j(x) - V^j(x) \right| \left(1 - \tau \frac{C_s}{s\varepsilon^{2s}} \right) + 2\tau C_s \sup_{|y|=1} \int_{\varepsilon}^{\infty} \left| U^j(x + \eta y) - V^j(x + \eta y) \right| \frac{d\eta}{\eta^{1+2s}} \\
 &\leq \|U^j - V^j\|_{C_b(\mathbb{R}^n)} \left(1 - \tau \frac{C_s}{s\varepsilon^{2s}} \right) + 2\tau C_s \|U^j - V^j\|_{C_b(\mathbb{R}^n)} \int_{\varepsilon}^{\infty} \frac{d\eta}{\eta^{1+2s}} \\
 &= \|U^j - V^j\|_{C_b(\mathbb{R}^n)}.
 \end{aligned}$$

In this way we have proved that

$$\|U^{j+1} - V^{j+1}\|_{C_b(\mathbb{R}^n)} \leq \|U^j - V^j\|_{C_b(\mathbb{R}^n)} \quad \text{for all } j \in \mathbb{N}.$$

An iteration then concludes the proof.

(d) This follows by using the translation invariant properties of (2.2)–(2.3) and part (c). More precisely, $W^j := U^j(\cdot + y)$ is the unique solution of (2.2)–(2.3) with initial data $w_0 := u_0(\cdot + y)$ for all $y \in \mathbb{R}^n$. Part (c) then yields

$$\|U^j(\cdot + y) - U^j\|_{C_b(\mathbb{R}^n)} = \|W^j - U^j\|_{C_b(\mathbb{R}^n)} \leq \|w_0 - u_0\|_{C_b(\mathbb{R}^n)} = \|u_0(\cdot + y) - u_0\|_{C_b(\mathbb{R}^n)}.$$

(e) Consider a mollification of the initial data $u_{0,\delta} := u_0 * \rho_{\delta}$, and denote the corresponding solution by U_{δ}^j . Choose $j = 1$ in (2.2)–(2.3) to get

$$\|U_{\delta}^1 - U_{\delta}^0\|_{C_b(\mathbb{R}^n)} \leq \tau \|\mathcal{L}_{\varepsilon}[U_{\delta}^0]\|_{C_b(\mathbb{R}^n)} = \tau \|\mathcal{L}_{\varepsilon}[u_{\delta}^0]\|_{C_b(\mathbb{R}^n)} := \tau K(u_{0,\delta}).$$

Now, define

$$V_{\delta}^j := U_{\delta}^{j+1} \quad \text{for all } j \in \mathbb{N}.$$

Then V_{δ}^j is the unique solution of (2.2)–(2.3) with initial data $V_{\delta}^0 = U_{\delta}^1$. By (c),

$$\begin{aligned}
 \|U_{\delta}^{j+1} - U_{\delta}^j\|_{C_b(\mathbb{R}^n)} &= \|V_{\delta}^j - U_{\delta}^j\|_{C_b(\mathbb{R}^n)} \leq \|V_{\delta}^0 - U_{\delta}^0\|_{C_b(\mathbb{R}^n)} \\
 &= \|U_{\delta}^1 - U_{\delta}^0\|_{C_b(\mathbb{R}^n)} \leq \tau K(u_{0,\delta}).
 \end{aligned} \tag{3.2}$$

Repeated use of the triangle inequality then yields

$$\|U_{\delta}^{j+k} - U_{\delta}^j\|_{C_b(\mathbb{R}^n)} \leq \sum_{i=0}^{k-1} \|U_{\delta}^{(j+i)+1} - U_{\delta}^{j+i}\|_{C_b(\mathbb{R}^n)} \leq k\tau K(u_{0,\delta}) = (t_{j+k} - t_j)K(u_{0,\delta}).$$

Then by (c),

$$\begin{aligned}
 \|U^{j+k} - U^j\|_{C_b(\mathbb{R}^n)} &\leq \|U^{j+k} - U_{\delta}^{j+k}\|_{C_b(\mathbb{R}^n)} + \|U_{\delta}^{j+k} - U_{\delta}^j\|_{C_b(\mathbb{R}^n)} + \|U_{\delta}^j - U^j\|_{C_b(\mathbb{R}^n)} \\
 &\leq 2\|u_0 - u_{0,\delta}\|_{C_b} + (t_{j+k} - t_j)K(u_{0,\delta}) \leq 2\omega_{u_0}(\delta) + (t_{j+k} - t_j)K(u_{0,\delta}),
 \end{aligned}$$

where we used that by properties of mollifiers, $\|u_0 - u_{0,\delta}\|_{C_b} \leq \sup_{|y| \leq \delta} \|u_0(\cdot + y) - u_0\|_{C_b(\mathbb{R}^n)} \leq \omega_{u_0}(\delta)$. Hence the result follows by the definition of $\tilde{\omega}$. \square

3.2 Semi-discrete scheme defined on $\mathbb{R}^n \times [0, \infty)$

In order to get uniform convergence of our approximation scheme, we need to define it on $\mathbb{R}^n \times [0, \infty)$. Let us therefore consider the solution of (2.2)–(2.3) $U_\varepsilon : \mathbb{R}^n \times \{\tau \mathbb{N} \cup 0\} \rightarrow \mathbb{R}$ and the function $u_\varepsilon : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ defined as:

$$\begin{cases} u_\varepsilon(x, 0) := U_\varepsilon^0(x) = u_0(x), \\ u_\varepsilon(x, t) := \frac{t_{j+1}-t}{\tau} U_\varepsilon^j(x) + \frac{t-t_j}{\tau} U_\varepsilon^{j+1}(x) \text{ if } t \in (t_j, t_{j+1}] \text{ with } j \in \mathbb{N}. \end{cases}$$

Corollary 3.7 (Well-posedness and properties) *Under the assumptions of Proposition 3.6, there exists a unique pointwise solution $u_\varepsilon \in BUC(\mathbb{R}^n \times [0, \infty))$ of (2.2)–(2.3) with initial data $u_0 \in BUC(\mathbb{R}^n)$. The solution, moreover, enjoys C_b -stability, comparison principle, C_b -contraction, continuity in space, and continuity in time in form of $\|u_\varepsilon(\cdot, t) - u_\varepsilon(\cdot, \tilde{t})\|_{C_b(\mathbb{R}^n)} \leq \tilde{\omega}(|t - \tilde{t}|)$ for all $t, \tilde{t} \in [0, \infty)$.*

Proof We easily inherit all properties from U_ε to u_ε , e.g.

$$\begin{aligned} \|u_\varepsilon(\cdot, t)\|_{C_b} &\leq \frac{(t_{j+1} - t)}{\tau} \|U_\varepsilon^j\|_{C_b} + \frac{(t - t_j)}{\tau} \|U_\varepsilon^{j+1}\|_{C_b} \\ &\leq \left(\frac{(t_{j+1} - t)}{\tau} + \frac{(t - t_j)}{\tau} \right) \|u_0\|_{C_b} = \|u_0\|_{C_b}. \end{aligned}$$

The other properties follows in a similar way, and we only explain the most difficult one, the continuity in time. Repeating the steps of the proof of Proposition 3.6(e), for $\tilde{t} \in (t_{j+k}, t_{j+k+1}]$ and $t \in (t_j, t_{j+1}]$,

$$\begin{aligned} &|(u_\varepsilon)_\delta(x, \tilde{t}) - (u_\varepsilon)_\delta(x, t)| \\ &\leq |(u_\varepsilon)_\delta(x, \tilde{t}) - (u_\varepsilon)_\delta(x, t_{j+k})| + |(u_\varepsilon)_\delta(x, t_{j+1}) - (u_\varepsilon)_\delta(x, t)| \\ &\quad + \sum_{l=j+1}^{j+k-1} |(u_\varepsilon)_\delta(x, t_{l+1}) - (u_\varepsilon)_\delta(x, t_l)| \\ &= |(u_\varepsilon)_\delta(x, \tilde{t}) - (U_\varepsilon^{j+k})_\delta(x)| + |(U_\varepsilon^{j+1})_\delta(x) - (u_\varepsilon)_\delta(x, t)| \\ &\quad + \sum_{l=j+1}^{j+k-1} |(U_\varepsilon^{l+1})_\delta(x) - (U_\varepsilon^l)_\delta(x)|. \end{aligned}$$

By the definition of linear interpolation

$$\begin{aligned} \|(u_\varepsilon)_\delta(\cdot, \tilde{t}) - (U_\varepsilon^{j+k})_\delta\|_{C_b(\mathbb{R}^n)} &\leq \frac{(\tilde{t} - t_{j+k})}{\tau} \|(U_\varepsilon^{j+k+1})_\delta - (U_\varepsilon^{j+k})_\delta\|_{C_b(\mathbb{R}^n)}, \\ \|(U_\varepsilon^{j+1})_\delta - (u_\varepsilon)_\delta(\cdot, t)\|_{C_b(\mathbb{R}^n)} &\leq \frac{(t_{j+1} - t)}{\tau} \|(U_\varepsilon^{j+1})_\delta - (U_\varepsilon^j)_\delta\|_{C_b(\mathbb{R}^n)}, \end{aligned}$$

and then by repeated use of (3.2),

$$\begin{aligned} \|(u_\varepsilon)_\delta(\cdot, \tilde{t}) - (u_\varepsilon)_\delta(\cdot, t)\|_{C_b(\mathbb{R}^n)} &\leq \left((\tilde{t} - t_{j+k}) + \sum_{l=j+1}^{j+k-1} \tau + (t_{j+1} - t) \right) K(u_{0,\delta}) \\ &= (\tilde{t} - t) K(u_{0,\delta}). \end{aligned}$$

We can then conclude the proof continuing as in the proof of Proposition 3.6(e). □

3.3 Compactness in $UC_{loc}(\mathbb{R}^n \times [0, \infty))$

Proposition 3.8 (Compactness) *Under the assumptions of Proposition 3.6, there exists a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ and a $u \in C_b(\mathbb{R}^n \times [0, \infty))$ such that*

$$u_{\varepsilon_k} \rightarrow u \quad \text{locally uniformly in } \mathbb{R}^n \times [0, \infty) \text{ as } k \rightarrow \infty.$$

Proof The sequence $\{u_\varepsilon\}_\varepsilon$ is equibounded and equicontinuous by Corollary 3.7 (see also Proposition 3.6). The result then follows from the Arzelà-Ascoli compactness theorem. \square

Taking pointwise limits in the a priori estimates of Corollary 3.7 (see also Proposition 3.6), the limit u immediately inherits these estimates.

Corollary 3.9 (A priori estimates) *Assume $u_0 \in BUC(\mathbb{R}^n)$. Then the limit u from Proposition 3.8 enjoys the following properties:*

- (a) (C_b -stability) *For all $t > 0$, $\|u(\cdot, t)\|_{C_b(\mathbb{R}^n)} \leq \|u_0\|_{C_b(\mathbb{R}^n)}$.*
- (b) (Uniform continuity in space) *For all $y \in \mathbb{R}^n$ and all $t > 0$,*

$$\|u(\cdot + y, t) - u(\cdot, t)\|_{C_b(\mathbb{R}^n)} \leq \omega_{u_0}(|y|).$$
- (c) (Uniform continuity in time) *For all $t, \tilde{t} > 0$,*

$$\|u(\cdot, t) - u(\cdot, \tilde{t})\|_{C_b(\mathbb{R}^n)} \leq \tilde{\omega}(t - \tilde{t}) \quad \text{where } \tilde{\omega} \text{ is defined in Theorem 2.2.}$$

4 Definitions, existence and properties of viscosity solutions

In this section we define the concept of viscosity solution. Before doing so we need to introduce two new operators that will be used when testing at zero gradient points.

Definition 4.1 For $\phi \in C^{1,1}(x) \cap B(\mathbb{R}^n)$,

$$\Delta_\infty^{s,+} \phi(x) := C_s \sup_{|y|=1} \int_0^\infty (\phi(x + \eta y) + \phi(x - \eta y) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}},$$

$$\Delta_\infty^{s,-} \phi(x) := C_s \inf_{|y|=1} \int_0^\infty (\phi(x + \eta y) + \phi(x - \eta y) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}}.$$

We immediately have:

Lemma 4.2 For $\phi \in C^{1,1}(x) \cap B(\mathbb{R}^n)$,

$$\Delta_\infty^{s,-} \phi(x) \leq \Delta_\infty^s \phi(x) \leq \Delta_\infty^{s,+} \phi(x).$$

Proof Recall Lemma 2.1. The result is trivial unless $\nabla\phi(x) = 0$. In that case,

$$\begin{aligned} \Delta_\infty^s \phi(x) &= C_s \sup_{|y|=1} \int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\ &\quad + C_s \inf_{|y|=1} \int_0^\infty (\phi(x - \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\ &= C_s \sup_{|y|=1} \int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\ &\quad - C_s \sup_{|y|=1} \left(- \int_0^\infty (\phi(x - \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \right) \\ &\leq C_s \sup_{|y|=1} \left(\left(\int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \right) \right. \\ &\quad \left. - \left(- \int_0^\infty (\phi(x - \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \right) \right) \\ &= \Delta_\infty^{s,+} \phi(x). \end{aligned}$$

The result $\Delta_\infty^{s,-} \phi(x) \leq \Delta_\infty^s \phi(x)$ follows in a similar way. □

We are now ready to define the concept of viscosity solution.

Definition 4.3 (*Viscosity solution*)

(a) A globally bounded upper semicontinuous function $u : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ is a *viscosity subsolution* of (1.2) if, for all $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, all $\phi \in C^2(B_R(x_0, t_0)) \cap BUC(\mathbb{R}^n \times (0, \infty) \setminus B_R(x_0, t_0))$ for some $R > 0$ and such that

- (i) $u(x_0, t_0) - \phi(x_0, t_0) = \sup_{(x,t) \in B_R(x_0,t_0)} (u(x, t) - \phi(x, t))$,
- (ii) $u(x_0, t_0) - \phi(x_0, t_0) > u(x, t) - \phi(x, t)$ for all $(x, t) \in B_R(x_0, t_0) \setminus (x_0, t_0)$,
- (iii) $u(x_0, t_0) - \phi(x_0, t_0) \geq u(x, t) - \phi(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty) \setminus B_R(x_0, t_0)$,

then

$$\begin{cases} \partial_t \phi(x_0, t_0) \leq \Delta_\infty^s \phi(x_0, t_0), & \text{if } \nabla \phi(x_0, t_0) \neq 0 \\ \partial_t \phi(x_0, t_0) \leq \Delta_\infty^{s,+} \phi(x_0, t_0), & \text{if } \nabla \phi(x_0, t_0) = 0. \end{cases} \quad (4.1)$$

(b) A globally bounded lower semicontinuous function $u : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ is a *viscosity supersolution* of (1.2) if, for all $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, all $\phi \in C^2(B_R(x_0, t_0)) \cap BUC(\mathbb{R}^n \times (0, \infty) \setminus B_R(x_0, t_0))$ for some $R > 0$ and such that

- (i) $u(x_0, t_0) - \phi(x_0, t_0) = \inf_{(x,t) \in B_R(x_0,t_0)} (u(x, t) - \phi(x, t))$,
- (ii) $u(x_0, t_0) - \phi(x_0, t_0) < u(x, t) - \phi(x, t)$ for all $(x, t) \in B_R(x_0, t_0) \setminus (x_0, t_0)$,
- (iii) $u(x_0, t_0) - \phi(x_0, t_0) \leq u(x, t) - \phi(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty) \setminus B_R(x_0, t_0)$,

then

$$\begin{cases} \partial_t \phi(x_0, t_0) \geq \Delta_\infty^s \phi(x_0, t_0), & \text{if } \nabla \phi(x_0, t_0) \neq 0 \\ \partial_t \phi(x_0, t_0) \geq \Delta_\infty^{s,-} \phi(x_0, t_0), & \text{if } \nabla \phi(x_0, t_0) = 0. \end{cases} \quad (4.2)$$

(c) A function $u \in C_b(\mathbb{R}^n \times [0, \infty))$ is a viscosity solution of (1.2) if it is both a viscosity subsolution and a viscosity supersolution.

(d) The viscosity solution u takes the initial data in a pointwise way: $u(x, 0) = u_0(x)$ for all $x \in \mathbb{R}^n$.

Remark 4.4 (a) In points where the gradient of u is zero, we only require $\partial_t u \in [\Delta_\infty^{s,-} u, \Delta_\infty^{s,+} u]$.

(b) In the local elliptic and homogeneous case [22], comparison follows without any condition at points where $\nabla\phi = 0$. In more general cases conditions are needed. We impose conditions (4.1) and (4.2) which are generalisations of the conditions introduced in the local parabolic case [1]. It is easy to show that comparison and uniqueness cannot hold without such conditions: E.g. $u(x, t) = 1$ and $v(x, t) = 2 \sin(t)$ would then both be viscosity solutions of (1.2) since $\nabla u = \nabla v = 0$ at every point. However comparison does not hold since $u(x, 0) = 1 \geq 0 = v(x, 0)$ while $u(x, \pi/2) = 1 \leq 2 = v(x, \pi/2)$. Let us check that v is not longer a viscosity solution when we impose (4.1). Let $K > 0$ and $0 \leq \psi \in C_b^2(\mathbb{R})$ be radial with $\psi(r) = r^2$ for $|r| < 1$ and $\psi(r) = 0$ for $|r| > 2$. We define

$$\phi(x, t) = v(t) + K\psi(|x|) + \psi(t - t_0) \quad \text{for some } t_0 \in (0, \infty).$$

It is then immediate that $v - \phi$ has a strict local max at $(0, t_0)$ and $\nabla\phi = \nabla v = 0$. Now let $t_0 = 2\pi$, then $\partial_t\phi(0, 2\pi) = \partial_t v(0, 2\pi) = 2 \cos(2\pi) = 2$, and by radial symmetry and followed by compact support of ϕ leads to

$$\begin{aligned} \Delta_\infty^{s,+}\phi(0, 2\pi) &= KC_s \int_0^\infty (\phi(\eta e_1, 2\pi) + \phi(-\eta e_1, 2\pi) - 2\phi(0, 2\pi)) \frac{d\eta}{\eta^{1+2s}} \\ &\leq KC_s \int_0^2 \|\psi''\|_{C_b} \eta^2 \frac{d\eta}{\eta^{1+2s}} = KC_s \|\psi''\|_{C_b} \frac{2^{2-2s}}{(2-2s)} \leq 1 \end{aligned}$$

if K is small enough. This contradicts (4.1) since

$$\partial_t\phi(0, 2\pi) = 2 \geq 1 \geq \Delta_\infty^{s,+}\phi(0, 2\pi).$$

We conclude that $v(x, t) = 2 \sin(t)$ is not a viscosity subsolution in the sense of Definition 4.3.

(c) Since $\partial_t, \Delta_\infty^s, \Delta_\infty^{s,+}$ and $\Delta_\infty^{s,-}$ are invariant under translations of ϕ by constants, without loss of generality, we can replace the conditions on the test function in Definition 4.3 by

- (i') $\phi(x_0, t_0) = u(x_0, t_0)$,
- (ii') $\phi > u$ (resp. $\phi < u$) in $B_R(x_0, t_0) \setminus (x_0, t_0)$,
- (iii') $\phi \geq u$ (resp. $\phi \leq u$) in $\mathbb{R}^n \times (0, \infty) \setminus B_R(x_0, t_0)$.

(d) We can also assume that the max is globally strict by adding a small in C_b^2 perturbation to ϕ supported in B_R^c , e.g. replacing ϕ by $\phi + \delta\psi$ where $\psi \in C_b^2$ is such that $0 \leq \psi \leq 1$, $\psi = 0$ in B_R and $\psi > 0$ in B_R^c . This new test function also satisfies (c), but with a strict inequality in part (iii'). Moreover, at the max point local derivatives up to order 2 coincide with those of ϕ , while nonlocal derivatives differ by an $O(\delta)$ term since $|\Delta_\infty^s\psi| + |\Delta_\infty^{s,+}\psi| + |\Delta_\infty^{s,-}\psi| \leq C\|\psi\|_{C_b^2}$. Before concluding the proof we then need to send $\delta \rightarrow 0$. Since this is never a problem, we will omit this modification in some proofs and simply assume globally strict max in the definition of viscosity solutions.

4.1 Existence and properties of solutions

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2 By Proposition 3.8 there is a subsequence u_ε such that

$$u_\varepsilon \rightarrow u \quad \text{locally uniformly in } \mathbb{R}^n \times [0, \infty) \text{ as } \varepsilon \rightarrow 0^+,$$

and by Corollary 3.9, $u \in BUC(\mathbb{R}^n \times [0, \infty))$ and satisfies the a priori estimates in Theorem 2.2. Let us prove that $\tilde{\omega}$ has the property claimed in part (c). To do this we use two basic facts about mollifiers: Since $u_0 \in BUC(\mathbb{R}^n)$, it follows that (differentiate ρ , use Young for convolutions)

$$\|D^k u_{0,\delta}\|_{C_b} \leq \|D^k \rho\|_{L^1} \|u_0\|_{C_b} \delta^{-k} \quad \text{for } k \in \mathbb{N}.$$

Then by Lemma 3.4, we have the following bound:

$$\|\mathcal{L}_\varepsilon[u_{0,\delta}]\|_{C_b} \leq c(s) \|\nabla \rho\|_{L^1}^{2-2s} \|D^2 \rho\|_{L^1}^{2s-1} \|u_0\|_{C_b} \delta^{-2s}.$$

Since $\delta^{-2s} \leq 1 + \delta^{-2}$, the estimate on $\tilde{\omega}$ follows after taking $\delta = r^{1/3}$.

It remains to check that u is a viscosity solution according to Definition 4.3. By Remark 4.4(d), consider $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ and $\phi \in C^2(B_R(x_0, t_0)) \cap BUC(\mathbb{R}^n \times (0, \infty) \setminus B_R(x_0, t_0))$ such that

- (i) $u(x_0, t_0) - \phi(x_0, t_0) = \sup_{(x,t) \in B_R(x_0,t_0)} (u(x, t) - \phi(x, t))$,
- (ii) $u(x_0, t_0) - \phi(x_0, t_0) > u(x, t) - \phi(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty) \setminus (x_0, t_0)$.

Local uniform convergence ensures that there exists a sequence $\{(x_\varepsilon, t_\varepsilon)\}_{\varepsilon>0}$ such that

- (i) $u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) = \sup_{(x,t) \in B_R(x_\varepsilon,t_\varepsilon)} (u_\varepsilon(x, t) - \phi(x, t)) := M_\varepsilon$,
- (ii) $u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon) > u_\varepsilon(x, t) - \phi(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty) \setminus (x_\varepsilon, t_\varepsilon)$,

and

$$(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, t_0) \quad \text{as } \varepsilon \rightarrow 0.$$

Recall that Corollary 3.7 ensures that u_ε solves the semidiscrete scheme. For simplicity, we use the notation in Remark 3.5. Let t_j be such that $t_\varepsilon \in (t_j, t_{j+1}]$. It is standard to check that

$$\frac{u_\varepsilon(x_\varepsilon, t_\varepsilon) - u_\varepsilon(x_\varepsilon, t_j)}{t_\varepsilon - t_j} = L(\varepsilon, u_\varepsilon, u_\varepsilon(x_\varepsilon, t_j)).$$

or equivalently

$$\frac{(u_\varepsilon(x_\varepsilon, t_\varepsilon) - M_\varepsilon) - (u_\varepsilon(x_\varepsilon, t_j) - M_\varepsilon)}{t_\varepsilon - t_j} = L(\varepsilon, u_\varepsilon - M_\varepsilon, u_\varepsilon(x_\varepsilon, t_j) - M_\varepsilon)$$

By defining $\tilde{u}_\varepsilon := u_\varepsilon - M_\varepsilon$, we have that $\tilde{u}_\varepsilon(x_\varepsilon, t_\varepsilon) = \phi(x_\varepsilon, t_\varepsilon)$ and $\phi > \tilde{u}_\varepsilon$ in $\mathbb{R}^n \times [0, \infty) \setminus (x_\varepsilon, t_\varepsilon)$. Let us then rewrite the scheme to obtain

$$\begin{aligned} \phi(x_\varepsilon, t_\varepsilon) &= \tilde{u}_\varepsilon(x_\varepsilon, t_\varepsilon) \\ &= \tilde{u}_\varepsilon(x_\varepsilon, t_j) \left(1 - (t_\varepsilon - t_j) \frac{C_s}{s\varepsilon^{2s}} \right) \\ &\quad + (t_\varepsilon - t_j) C_s \left(\sup_{|y|=1} \int_\varepsilon^\infty \tilde{u}_\varepsilon(x_\varepsilon + \eta y, t_j) \frac{d\eta}{\eta^{1+2s}} + \inf_{|y|=1} \int_\varepsilon^\infty \tilde{u}_\varepsilon(x_\varepsilon + \eta y, t_j) \frac{d\eta}{\eta^{1+2s}} \right) \\ &< \phi(x_\varepsilon, t_j) \left(1 - (t_\varepsilon - t_j) \frac{C_s}{s\varepsilon^{2s}} \right) \\ &\quad + (t_\varepsilon - t_j) C_s \left(\sup_{|y|=1} \int_\varepsilon^\infty \phi(x_\varepsilon + \eta y, t_j) \frac{d\eta}{\eta^{1+2s}} + \inf_{|y|=1} \int_\varepsilon^\infty \phi(x_\varepsilon + \eta y, t_j) \frac{d\eta}{\eta^{1+2s}} \right) \end{aligned}$$

that is,

$$\frac{\phi(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_j)}{t_\varepsilon - t_j} < L(\varepsilon, \phi, \phi(x_\varepsilon, t_j)). \tag{4.3}$$

Assume $\nabla\phi(x_0, t_0) > 0$ (the $\nabla\phi(x_0, t_0) < 0$ case is similar). Then for $\varepsilon_0 > 0$ small enough, $\nabla\phi(x_\varepsilon, t_j) > 0$ for $\varepsilon \leq \varepsilon_0$, and we use (4.3) og Lemma 3.3 to find that

$$\partial_t \phi(x_\varepsilon, t_\varepsilon) \leq \Delta_\infty^s \phi(x_\varepsilon, t_j) + o_\varepsilon(1) + O(\tau), \tag{4.4}$$

where o_ε depends on $\sup_{\varepsilon \leq \varepsilon_0} |\nabla\phi(x_\varepsilon, t_j)|^{-1}$ which is uniformly bounded by the above discussion. Since ϕ is smooth, for every $\eta \in [0, \infty)$,

$$\phi\left(x_\varepsilon \pm \frac{\nabla\phi(x_\varepsilon)}{|\nabla\phi(x_\varepsilon)|} \eta, t_j\right) \xrightarrow{\varepsilon \rightarrow 0^+} \phi\left(x_0 \pm \frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|} \eta, t_0\right).$$

The dominated convergence theorem then ensures that

$$\Delta_\infty^s \phi(x_\varepsilon, t_j) \rightarrow \Delta_\infty^s \phi(x_0, t_0) \quad \text{as } \varepsilon \rightarrow 0^+.$$

We thus pass to the limit in (4.4) and get the correct viscosity subsolution inequality.

When $\nabla\phi(x_0, t_0) = 0$ we have (see proof of Lemma 4.2)

$$\begin{aligned} L(\varepsilon, \phi, \phi(x_\varepsilon, t_j)) &\leq \sup_{|y|=1} \int_\varepsilon^\infty (\phi(x_\varepsilon + \eta y, t_j) + \phi(x_\varepsilon - \eta y, t_j) - 2\phi(x_\varepsilon, t_j)) \frac{d\eta}{\eta^{1+2s}} \\ &= \Delta_\infty^{s,+} \phi(x_\varepsilon, t_j) + o_\varepsilon(1), \end{aligned}$$

and it only remains to check $\Delta_\infty^{s,+} \phi(x_\varepsilon, t_j) \xrightarrow{\varepsilon \rightarrow 0^+} \Delta_\infty^{s,+} \phi(x_0, t_0)$. To do that, note that

$$\begin{aligned} &C_s^{-1} |\Delta_\infty^{s,+} \phi(x_\varepsilon, t_j) - \Delta_\infty^{s,+} \phi(x_0, t_0)| \\ &\leq \sup_{|y|=1} \left| \int_0^\infty \left((\phi(x_\varepsilon + \eta y, t_j) - \phi(x_0 + \eta y, t_0)) + (\phi(x_\varepsilon - \eta y, t_j) - \phi(x_0 - \eta y, t_0)) \right. \right. \\ &\quad \left. \left. - 2(\phi(x_\varepsilon, t_j) - \phi(x_0, t_0)) \right) \frac{d\eta}{\eta^{1+2s}} \right| \\ &\leq \sup_{|y|=1} \left| \int_0^{\frac{R}{4}} \left((\phi(x_\varepsilon + \eta y, t_j) - \phi(x_0 + \eta y, t_0)) + (\phi(x_\varepsilon - \eta y, t_j) - \phi(x_0 - \eta y, t_0)) \right. \right. \\ &\quad \left. \left. - 2(\phi(x_\varepsilon, t_j) - \phi(x_0, t_0)) \right) \frac{d\eta}{\eta^{1+2s}} \right| \\ &\quad + \sup_{|y|=1} \left| \int_{\frac{R}{4}}^\infty \left((\phi(x_\varepsilon + \eta y, t_j) - \phi(x_0 + \eta y, t_0)) + (\phi(x_\varepsilon - \eta y, t_j) - \phi(x_0 - \eta y, t_0)) \right. \right. \\ &\quad \left. \left. - 2(\phi(x_\varepsilon, t_j) - \phi(x_0, t_0)) \right) \frac{d\eta}{\eta^{1+2s}} \right| =: I_\varepsilon^1 + I_\varepsilon^2. \end{aligned}$$

Since $\phi \in BUC(\mathbb{R}^n \times (0, \infty))$,

$$I_\varepsilon^2 \leq 4\omega_\phi((x_\varepsilon - x_0, t_j - t_0)) \int_{\frac{R}{4}}^\infty \frac{d\eta}{\eta^{1+2s}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+.$$

Then note that $I_\varepsilon^1 \leq \int_0^{\frac{R}{4}} F_\varepsilon(\eta) \, d\eta$ where

$$F_\varepsilon(\eta) = \frac{1}{\eta^{1+2s}} \sup_{|y|=1} \left| \left(\phi(x_\varepsilon + \eta y, t_j) - \phi(x_0 + \eta y, t_0) \right) + \left(\phi(x_\varepsilon - \eta y, t_j) - \phi(x_0 - \eta y, t_0) \right) - 2 \left(\phi(x_\varepsilon, t_j) - \phi(x_0, t_0) \right) \right|.$$

By a second order Taylor expansion and continuity of all involved functions,

$$|F_\varepsilon(\eta)| \leq \|D^2\phi\|_{C_b(B_{R/2}(x_0, t_0))} \eta^{1-2s} \text{ for } \varepsilon \text{ small, and } F_\varepsilon(\eta) \rightarrow 0 \text{ pointwise as } \varepsilon \rightarrow 0.$$

By the dominated convergence theorem it follows that $I_\varepsilon^1 \rightarrow 0$. Finally, the initial condition trivially holds since $u(x, 0) = \lim_{\varepsilon \rightarrow 0^+} u_\varepsilon(x, 0) = u_0(x)$. □

It remains to proof Lemma 2.4 and then also Corollary 2.5 is proved.

Proof of Lemma 2.4 Since $u_0 \in C^{0,\beta}(\mathbb{R}^n)$, $\omega_{u_0}(\delta) \leq |u_0|_{C^{0,\beta}} \delta^\beta$, and basic facts about mollifiers yields

$$\|D^k u_{0,\delta}\|_{C_b} \leq c(\rho) |u_0|_{C^{0,\beta}} \delta^{-k+\beta} \text{ for } k \in \mathbb{N},$$

see e.g. [18, Appendix A] and [24]. Then, by Lemma 3.4, we have $\|\mathcal{L}_\varepsilon[u_{0,\delta}]\|_{C_b} \leq c(s, \rho) \delta^{-2s+\beta}$. □

5 Review of basic results on the fractional heat equation

Here we collect some well-known results on the fractional heat equation that we will need, see e.g. [3, 7, 8, 20]. The one-dimensional problem we consider is

$$\begin{cases} \partial_t v(x, t) + (-\partial_{xx}^2)^s v(x, t) = 0, & x \in \mathbb{R}, t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

$$\tag{5.2}$$

The fundamental solution of (5.1) is given by

$$P_s(x, t) = \mathcal{F}^{-1}(e^{-|\xi|^{2s}t})(x)$$

where \mathcal{F} denotes the Fourier transform and \mathcal{F}^{-1} its inverse. Since the Fourier symbol $e^{-|\xi|^{2s}t}$ is a tempered distribution, it follows that

$$P_s \in C^\infty(\mathbb{R} \times (0, \infty)).$$

Moreover, it is well-known that

$$P_s(x, t) = t^{-\frac{1}{2s}} F(|x|t^{-\frac{1}{2s}}), \tag{5.3}$$

with a profile $F(r)$ that is a smooth and strictly decreasing function of $r > 0$. We can also deduce that, for all $\tau > 0$, there exist constants $c_1, c_2 > 0$ depending only on s , such that

$$c_1 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \leq P_s(x, t) \leq c_2 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \text{ for all } (x, t) \in \mathbb{R} \times [\tau, \infty). \tag{5.4}$$

Once the basic properties of the fundamental solution are established, we also recall that given any $0 \leq v_0 \in L^1(\mathbb{R})$ (actually a bigger class can be considered), the unique (very weak) solution of (5.1)–(5.2) is given by convolution as

$$v(x, t) = (P_s(\cdot, t) * v_0)(x) = \int_{-\infty}^{\infty} P_s(x - y, t)v_0(y) \, dy. \tag{5.5}$$

Actually, since it is obtained by convolution with a C^∞ kernel, the solution with nonnegative L^1 -initial data will be C^∞ smooth in $\mathbb{R} \times (0, \infty)$. We will also need:

Lemma 5.1 (Classical solutions) *Let $v_0 \in C_b^\infty(\mathbb{R})$ be radially symmetric and radially nonincreasing. Then there exists a unique solution $v \in C_b^\infty(\mathbb{R} \times [0, \infty))$ of (5.1)–(5.2). Moreover, v is radial, radially nonincreasing, and given by (5.5).*

Let us also recall Theorem 8.1 in [8].

Lemma 5.2 (Global Harnack principle) *Let v be the very weak solution of (5.1)–(5.2) with initial data $v_0 \in L^1(\mathbb{R})$ such that $v_0 \not\equiv 0$ and*

$$0 \leq v_0(x) \leq (1 + |x|^2)^{-\frac{1+2s}{2}} \quad \text{for all } |x| \geq R \geq 1.$$

Then, for all $\tau > 0$, there exist constants $k_1, k_2 > 0$ depending only on s and R , such that

$$k_1 \|v_0\|_{L^1(\mathbb{R})} P_s(x, t) \leq v(x, t) \leq k_2 \|v_0\|_{L^1(\mathbb{R})} P_s(x, t) \quad \text{for all } (x, t) \in \mathbb{R} \times [\tau, \infty).$$

Moreover, by (5.4), for all $\tau > 0$, there exist constants $C_1, C_2 > 0$ depending only on s, R , and $\|v_0\|_{L^1(\mathbb{R})}$, such that

$$C_1 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \leq v(x, t) \leq C_2 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \quad \text{for all } (x, t) \in \mathbb{R} \times [\tau, \infty).$$

6 Smooth solutions and the 1d fractional heat equation

This section investigates different smooth solutions of (1.2)–(1.3).

6.1 Radial solutions

We will now focus on obtaining Theorem 2.10. To do so, we will demonstrate that for radially symmetric and radially nonincreasing functions, the operator Δ_∞^s reduces to the classical fractional Laplacian.

Proposition 6.1 *Assume that $\phi \in C^{1,1}(x) \cap B(\mathbb{R}^n)$ is radial and radially nonincreasing, i.e.,*

$$\phi(x) = \tilde{\Phi}(|x|) \quad \text{for all } x \in \mathbb{R}^n,$$

where $\tilde{\Phi} : [0, \infty) \rightarrow \mathbb{R}$ is nonincreasing. Then

$$\Delta_\infty^s \phi(x) = -(-\partial_{rr}^2)^s \Phi(|x|),$$

where Φ is the even extension of $\tilde{\Phi}$ to \mathbb{R} : $\Phi(r) = \tilde{\Phi}(r)$ and $\Phi(-r) = \tilde{\Phi}(r)$ for $r \in [0, \infty)$.

Remark 6.2 (a) When ϕ is radial and $\nabla\phi(x) \neq 0$, a similar observation has been done in the proof of Lemma 3.1 in [6].

(b) ‘‘Radially nonincreasing’’ is needed only when $\nabla\phi(x) = 0$, but it cannot be removed in general; see the Sect. 6.2 below.

Proof of Proposition 6.1 Assume $\nabla\phi(x) \neq 0$, and let $r = |x|$. We have that

$$\nabla\phi(x) = \frac{\Phi'(r)}{r}x, \quad |\nabla\phi(x)| = |\Phi'(r)|, \quad \frac{\nabla\phi(x)}{|\nabla\phi(x)|} = -\frac{x}{|x|},$$

Lemma 2.1 then yields

$$\begin{aligned} \Delta_\infty^s\phi(x) &= C_s \int_0^\infty \left(\phi\left(x\left(1 + \frac{\eta}{|x|}\right)\right) + \phi\left(x\left(1 - \frac{\eta}{|x|}\right)\right) - 2\phi(x) \right) \frac{d\eta}{\eta^{1+2s}} \\ &= C_s \int_0^\infty \left(\Phi\left(|x|\left(1 + \frac{\eta}{|x|}\right)\right) + \Phi\left(|x|\left(1 - \frac{\eta}{|x|}\right)\right) - 2\Phi(|x|) \right) \frac{d\eta}{\eta^{1+2s}} \quad (6.1) \\ &= C_s \int_0^\infty (\Phi(r + \eta) + \Phi(r - \eta) - 2\Phi(r)) \frac{d\eta}{\eta^{1+2s}} \\ &= -(-\partial_{rr}^2)^s\Phi(r). \end{aligned}$$

Assume $\nabla\phi(x) = 0$. Note that

$$\text{dist}(0, \partial B_\eta(x)) = \text{dist}\left(0, x - \eta\frac{x}{|x|}\right),$$

which implies that

$$\sup_{z \in \partial B_\eta(x)} \{\phi(z)\} = \phi\left(x - \eta\frac{x}{|x|}\right),$$

since $\phi = \Phi(|\cdot|)$ is radially nonincreasing. Then,

$$\begin{aligned} \int_0^\infty \left(\phi\left(x - \eta\frac{x}{|x|}\right) - \phi(x) \right) \frac{d\eta}{\eta^{1+2s}} &\leq \sup_{|y|=1} \int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\ &\leq \int_0^\infty \left(\sup_{z \in \partial B_\eta(x)} \{\phi(z)\} - \phi(x) \right) \frac{d\eta}{\eta^{1+2s}} \\ &= \int_0^\infty \left(\phi\left(x - \eta\frac{x}{|x|}\right) - \phi(x) \right) \frac{d\eta}{\eta^{1+2s}}, \end{aligned}$$

so that

$$\sup_{|y|=1} \int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} = \int_0^\infty \left(\phi\left(x - \eta\frac{x}{|x|}\right) - \phi(x) \right) \frac{d\eta}{\eta^{1+2s}}.$$

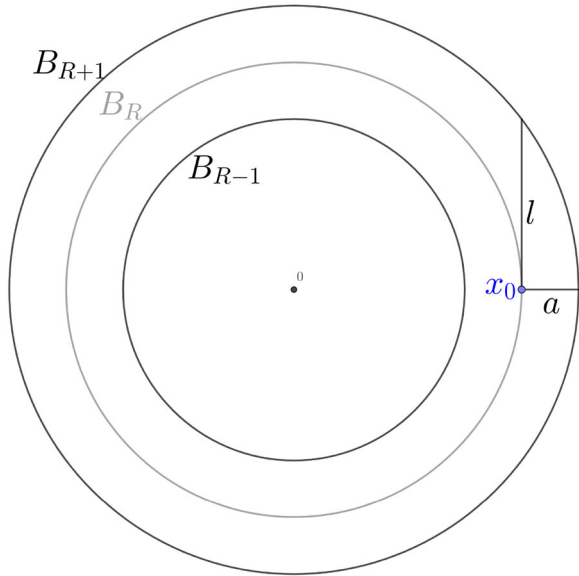
In the same way,

$$\inf_{|y|=1} \int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} = \int_0^\infty \left(\phi\left(x + \eta\frac{x}{|x|}\right) - \phi(x) \right) \frac{d\eta}{\eta^{1+2s}}.$$

Finally, Lemma 2.1 and the argument in (6.1) gives the result. □

Proof of Theorem 2.10 Define $v_0(r) := U_0(|x|)$ for $x \in \mathbb{R}^n$ and $r = |x|$, then $v_0 \in C_b^\infty(\mathbb{R})$ is radial and radially nonincreasing. Let v be the corresponding solution of (5.1)–(5.2). By Lemma 5.1, v is radial, radially nonincreasing, and C_b^∞ smooth. Then by Proposition 6.1, $u(x, t) := v(|x|, t)$ is a classical solution of (1.2)–(1.3). □

Fig. 1 Characteristic set for the radial counterexample



6.2 Counterexample for functions not being radially nonincreasing

We show now an example of a function ϕ not satisfying the radially nonincreasing assumption in the zero gradient case, and such that the operators Δ_∞^s and $-\partial_{rr}^2$ do not coincide. For $R > 0$, consider the radial function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\phi(x) = \mathbf{1}_{B_{R+1} \setminus B_{R-1}}(x)$, see Fig. 1.

We note that at $x_0 = (r, 0)$, we have that $\phi \in C^{1,1}(x_0) \cap B(\mathbb{R}^n)$ and $\nabla\phi(x_0) = 0$. Moreover, we have $a = 1$ and $R^2 + l^2 = (R + 1)^2$, so that $l = \sqrt{2R + 1}$. We denote by $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then

$$\begin{aligned} \sup_{|y|=1} \int_0^\infty (\phi(x_0 + \eta y) - \phi(x_0)) \frac{d\eta}{\eta^{1+2s}} &\geq \int_0^\infty (\phi(x_0 + \eta e_2) - \phi(x_0)) \frac{d\eta}{\eta^{1+2s}} \\ &= - \int_l^\infty \frac{d\eta}{\eta^{1+2s}} = -\frac{1}{2s} \frac{1}{(2R + 1)^s} \end{aligned}$$

and

$$\begin{aligned} \inf_{|y|=1} \int_0^\infty (\phi(x_0 + \eta y) - \phi(x_0)) \frac{d\eta}{\eta^{1+2s}} &= \int_0^\infty (\phi(x_0 + \eta e_1) - \phi(x_0)) \frac{d\eta}{\eta^{1+2s}} \\ &= - \int_1^\infty \frac{d\eta}{\eta^{1+2s}} = -\frac{1}{2s}. \end{aligned}$$

By Lemma 2.1,

$$\Delta_\infty^s \phi(x) \geq -\frac{C_s}{2s} \left(\frac{1}{(2R + 1)^s} + 1 \right).$$

On the other hand, we define $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\Phi(r) := \phi(|x|)$ when $r = |x|$ and $\Phi(-r) := \Phi(r)$, and let $r_0 = |x_0|$.

$$\begin{aligned}
 -(-\partial_{rr}^2)^s \Phi(r_0) &= C_s \int_0^\infty (\Phi(r_0 + \eta) + \Phi(r_0 - \eta) - 2\Phi(r_0)) \frac{d\eta}{\eta^{1+2s}} \\
 &= C_s \int_0^\infty \left(\phi\left(x_0 + \eta \frac{x_0}{|x_0|}\right) - \phi(x_0) \right) \frac{d\eta}{\eta^{1+2s}} \\
 &\quad + C_s \int_0^\infty \left(\phi\left(x_0 - \eta \frac{x_0}{|x_0|}\right) - \phi(x_0) \right) \frac{d\eta}{\eta^{1+2s}} \\
 &= -C_s \int_1^\infty \frac{d\eta}{\eta^{1+2s}} - C_s \int_1^{2R-1} \frac{d\eta}{\eta^{1+2s}} - C_s \int_{2R+1}^\infty \frac{d\eta}{\eta^{1+2s}} \\
 &= -\frac{C_s}{2s} \left(2 + \frac{1}{(2R+1)^{2s}} - \frac{1}{(2R-1)^{2s}} \right).
 \end{aligned}$$

Finally, taking R big enough, we get

$$-\Delta_\infty^s \phi(x_0) - (-\partial_{rr}^2)^s \Phi(r_0) \leq -\frac{C_s}{2s} \left(1 + \frac{1}{(2R+1)^{2s}} - \frac{1}{(2R-1)^{2s}} - \frac{1}{(2R+1)^s} \right) < 0.$$

Thus, the operators cannot coincide.

6.3 Another example of smooth solutions

We present here another example of functions for which Δ_∞^s reduces to a one-dimensional fractional Laplacian. As before, this allows to produce smooth solutions of (1.2)–(1.3). We will adopt the notation $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Lemma 6.3 *Assume that $\Phi \in C_b^2(\mathbb{R})$ is nondecreasing, and let $\phi \in C_b^2(\mathbb{R}^n)$ be defined as*

$$\phi(x) := \Phi(x_1).$$

Then

$$\Delta_\infty^s \phi(x) = -(-\partial_{x_1 x_1}^2)^s \Phi(x_1).$$

Remark 6.4 We could also take $\Phi \in C_b^2(\mathbb{R})$ and nonincreasing in the above result.

Proof of Lemma 6.3 Note that $\nabla\phi(x) = \Phi'(x_1)e_1$ and $\Phi'(x_1) \geq 0$. On one hand, if $\Phi'(x_1) = 0$, then it is clear that

$$\begin{aligned}
 \Delta_\infty^s \phi(x) &= C_s \sup_{|y|=1} \int_0^\infty (\phi(x + \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} + C_s \inf_{|y|=1} \int_0^\infty (\phi(x - \eta y) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\
 &= C_s \int_0^\infty (\phi(x + \eta e_1) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} + C_s \int_0^\infty (\phi(x - \eta e_1) - \phi(x)) \frac{d\eta}{\eta^{1+2s}} \\
 &= C_s \int_0^\infty (\Phi(x_1 + \eta) + \Phi(x_1 - \eta) - 2\Phi(x_1)) \frac{d\eta}{\eta^{1+2s}} \\
 &= -(-\partial_{x_1 x_1}^2)^s \Phi(x_1).
 \end{aligned}$$

On the other hand, if $\Phi'(x_1) > 0$, then $\zeta = \nabla\phi(x)/|\nabla\phi(x)| = e_1$ (cf. Lemma 2.1) and

$$\begin{aligned}
 \Delta_\infty^s \phi(x) &= C_s \int_0^\infty (\phi(x + \eta e_1) + \phi(x - \eta e_1) - 2\phi(x)) \frac{d\eta}{\eta^{1+2s}} \\
 &= -(-\partial_{x_1 x_1}^2)^s \Phi(x_1).
 \end{aligned}$$

□

7 Comparison and local truncation errors

We are not able to prove comparison (neither uniqueness) for the family of viscosity solutions constructed in Sect. 4. However, we are able to compare any constructed viscosity solution with any classical solution.

The argument is based on the fact that for classical solutions we can get full convergence of the scheme (2.2)–(2.3) (and not just compactness and convergence up to a subsequence). Then we can inherit the comparison result of the scheme to the limit solutions.

7.1 Comparison and convergence estimates under regularity assumptions

Proposition 7.1 *Assume CFL and $u_0 \in BUC(\mathbb{R}^n)$. Let $u, v \in C_b^2(\mathbb{R}^n \times [0, \infty))$ be respective classical sub- and supersolutions of (1.2)–(1.3). Then:*

(a) *Let $U_\varepsilon, V_\varepsilon$ respective super- and subsolutions of the scheme (2.2)–(2.3). Then, for all $T < \infty$,*

$$u + o_\varepsilon(1) + O(\tau) \leq U_\varepsilon \quad \text{and} \quad v + o_\varepsilon(1) + O(\tau) \geq V_\varepsilon, \quad \text{uniformly in } \mathbb{R}^n \times \{\tau \mathbb{N} \cup 0\}.$$

(b) *Let U_ε be a solution of the scheme (2.2)–(2.3). Then, for all $T < \infty$,*

$$u + o_\varepsilon(1) + O(\tau) \leq U_\varepsilon \leq v + o_\varepsilon(1) + O(\tau), \quad \text{uniformly in } \mathbb{R}^n \times \{\tau \mathbb{N} \cup 0\}.$$

We immediately get:

Corollary 7.2 *Assume CFL. Let $u \in C_b^2(\mathbb{R}^n \times [0, \infty))$ be a classical solution of (1.2)–(1.3), and U_ε be a solution of the scheme (2.2)–(2.3), both with initial data u_0 . Then, for all $T < \infty$,*

$$\max_{t_j \leq T} \|u(\cdot, t_j) - U_\varepsilon(\cdot, t_j)\|_{C_b(\mathbb{R}^n)} = o_\varepsilon(1) + O(\tau).$$

Proof of Proposition 7.1 (a) Define the local truncation error,

$$(R_\varepsilon)^j(x) := \frac{u(x, t_j + \tau) - u(x, t_j)}{\tau} - \mathcal{L}_\varepsilon[u(\cdot, t_j)](x). \tag{7.1}$$

Clearly, since u is a classical subsolution of (1.2) we have (from Lemma 3.3)

$$\begin{aligned} (R_\varepsilon)^j(x) &\leq \left(\frac{u(x, t_j + \tau) - u(x, t_j)}{\tau} - \partial_t u(x, t_j) \right) - (\mathcal{L}_\varepsilon[u(\cdot, t_j)](x) - \Delta_\infty^s[u(\cdot, t_j)](x)) \\ &\leq O(\tau) + o_\varepsilon(1) \end{aligned}$$

with uniform bounds in t_j and x .

Define now $e^j(x) = u(x, t_j) - (U_\varepsilon)^j(x) = u(x, t_j) - U_\varepsilon(x, t_j)$. By (2.2) and (7.1), we get

$$\begin{aligned}
 e^{j+1}(x) &= u(x, t_{j+1}) - (U_\varepsilon)^{j+1}(x) \\
 &\leq e^j(x) + \tau(\mathcal{L}_\varepsilon[u(\cdot, t_j)](x) - \mathcal{L}_\varepsilon[(U_\varepsilon)^j](x)) + \tau(R_\varepsilon)^j(x) \\
 &= e^j(x)(1 - \tau \frac{C_s}{s\varepsilon^{2s}}) + \tau C_s \left(\sup_{|\eta|=1} \int_\varepsilon^\infty u(x + \eta y, t_j) \frac{d\eta}{\eta^{1+2s}} - \sup_{|\eta|=1} \int_\varepsilon^\infty (U_\varepsilon)^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right) \\
 &\quad + \tau C_s \left(\inf_{|\eta|=1} \int_\varepsilon^\infty u(x + \eta y, t_j) \frac{d\eta}{\eta^{1+2s}} - \inf_{|\eta|=1} \int_\varepsilon^\infty (U_\varepsilon)^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} \right) + \tau(R_\varepsilon)^j(x) \\
 &\leq e^j(x)(1 - \tau \frac{C_s}{s\varepsilon^{2s}}) + 2\tau C_s \sup_{|\eta|=1} \int_\varepsilon^\infty e^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} + \tau(R_\varepsilon)^j(x) \\
 &\leq \sup_{x \in \mathbb{R}^n} e^j(x)(1 - \tau \frac{C_s}{s\varepsilon^{2s}}) + 2\tau C_s \sup_{|\eta|=1} \int_\varepsilon^\infty \sup_{x \in \mathbb{R}^n} e^j(x + \eta y) \frac{d\eta}{\eta^{1+2s}} + \tau \sup_{t_j \leq T} \sup_{x \in \mathbb{R}^n} (R_\varepsilon)^j(x) \\
 &= \sup_{x \in \mathbb{R}^n} e^j(x)(1 - \tau \frac{C_s}{s\varepsilon^{2s}}) + \tau \frac{C_s}{s\varepsilon^{2s}} \sup_{x \in \mathbb{R}^n} e^j(x) + \tau \sup_{t_j \leq T} \sup_{x \in \mathbb{R}^n} (R_\varepsilon)^j(x) \\
 &= \sup_{x \in \mathbb{R}^n} e^j(x) + \tau \sup_{t_j \leq T} \sup_{x \in \mathbb{R}^n} (R_\varepsilon)^j(x).
 \end{aligned}$$

I.e.,

$$\sup_{x \in \mathbb{R}^n} e^{j+1}(x) \leq \sup_{x \in \mathbb{R}^n} e^j(x) + \tau \sup_{t_j \leq T} \sup_{x \in \mathbb{R}^n} (R_\varepsilon)^j(x).$$

Iterating, we obtain

$$\begin{aligned}
 \sup_{x \in \mathbb{R}^n} e^j(x) &\leq \sup_{x \in \mathbb{R}^n} e^0(x) + j\tau \sup_{t_j \leq T} \sup_{x \in \mathbb{R}^n} (R_\varepsilon)^j(x) \\
 &\leq \sup_{x \in \mathbb{R}^n} (u(x, 0) - (U_\varepsilon)^0(x)) + T(O(\tau) + o_\varepsilon(1)) \\
 &\leq 0 + O(\tau) + o_\varepsilon(1).
 \end{aligned}$$

By changing the roles of u, U_ε with $-v, -V_\varepsilon$, we obtain the other inequality in a similar way.

(b) Follows directly from part (a). □

7.2 Comparison for classical sub- and supersolutions

In order to continue, we note that Proposition 7.1 and Corollary 7.2 hold exactly as before with the time interpolant u_ε replacing U_ε (cf. the proof of Corollary 3.7).

Proof of Theorem 2.6 The proof is similar for \bar{u}, \underline{u} , and we only provide it for \underline{u} . Since u is a constructed viscosity solution in the sense of Theorem 2.2, by Proposition 3.8 there is a sequence $u_{\varepsilon_j} \in BUC(\mathbb{R}^n \times [0, \infty))$ of time-interpolated solutions of (2.2)–(2.3) with initial condition u_0 such that

$$u_{\varepsilon_j} \rightarrow u \quad \text{locally uniformly in } \mathbb{R}^n \times [0, \infty) \text{ as } \varepsilon_j \rightarrow 0^+.$$

Then by taking the limit as $\varepsilon_j \rightarrow 0^+$ in Proposition 7.1(b), we get $\underline{u} \leq u$. □

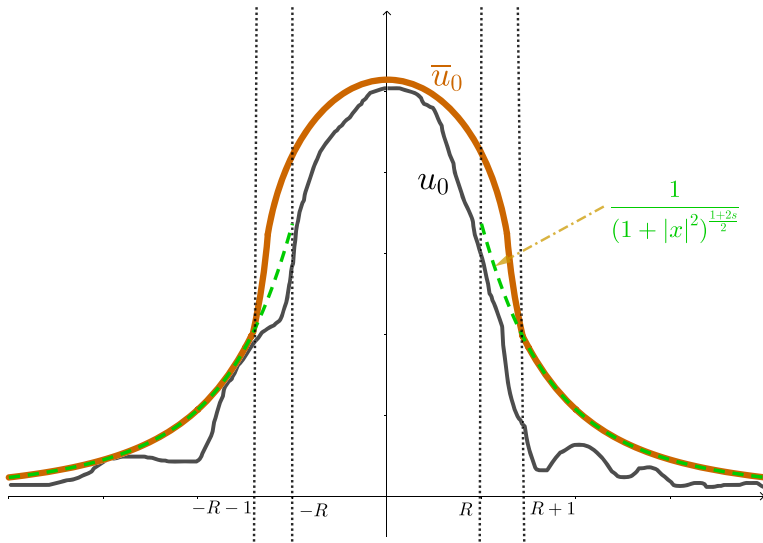


Fig. 2 Upper bound for u_0 in the proof of Theorem 2.13

8 Global Harnack principle

The proof of Theorem 2.13 is based on the relation between our problem and the smooth solutions of the fractional heat equation, the properties of smooth solutions in 1D for the fractional heat equation presented in the review Sect. 5, and the comparison principle of Theorem 2.6 for viscosity and classical solutions.

Proof of Theorem 2.13 A key point of the proof is the fact that if v is a smooth, radial, and radially nonincreasing solution of the fractional heat equation in one dimension, then $u(x, t) := v(|x|, t)$ is a solution of (1.2)–(1.3). See Theorem 2.10.

(1) *Upper bound.* Let $\bar{u}_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be such that (i) $\bar{u}_0(x) = (1 + |x|^2)^{-\frac{1+2s}{2}}$ if $|x| \geq R + 1$; (ii) \bar{u}_0 is radially symmetric and radially nonincreasing; (iii) $\bar{u}_0 \in C_b^\infty(\mathbb{R}^n)$; and (iv) $u_0 \leq \bar{u}_0$ in \mathbb{R}^n . Consult Fig. 2.

Moreover, let $\bar{v}_0 : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\bar{v}_0(r) := \bar{u}_0(|x|)$ with $r = |x|$ and $\bar{v}_0(-r) := \bar{v}_0(r)$. Clearly, $\bar{v}_0 \in C_b^\infty(\mathbb{R})$ is radially symmetric and radially nonincreasing. Let \bar{v} be the corresponding solution of the fractional heat equation (5.1)–(5.2) and define $\bar{u}(x, t) = \bar{v}(|x|, t)$. By Theorem 2.10, $\bar{u} \in C_b^\infty(\mathbb{R}^n \times [0, \infty))$ is a classical solution of (1.2)–(1.3). Moreover, \bar{u} is radial and radially nonincreasing. Since $\bar{u}_0(x) = (1 + |x|^2)^{-\frac{1+2s}{2}}$ if $|x| \geq R + 1$, then $\bar{v}_0(r) = (1 + |r|^2)^{-\frac{1+2s}{2}}$ if $|r| \geq R + 1$, so that, by Lemma 5.2, for all $t > \tau$ we have

$$\bar{u}(x, t) = \bar{v}(|x|, t) \leq k_2 \|\bar{v}_0\|_{L^1(\mathbb{R})} P_s(|x|, t) \leq C_1 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}}$$

Finally, since $\bar{u} \in C_b^\infty(\mathbb{R}^n \times [0, \infty))$ is a classical solution of (1.2)–(1.3) and $u_0 \leq \bar{u}_0$ we have, by Theorem 2.6, that $u(x, t) \leq \bar{u}(x, t)$.

(2) *Lower bound.* Without loss of generality, assume $u_0(0) = \sup_{x \in \mathbb{R}^n} u_0(x) > 0$. By continuity of u_0 , there exists $R_0 > 0$ such that $u_0(x) \geq u_0(0)/2$ for all $x \in B_{R_0}(0)$. Consider e.g. the

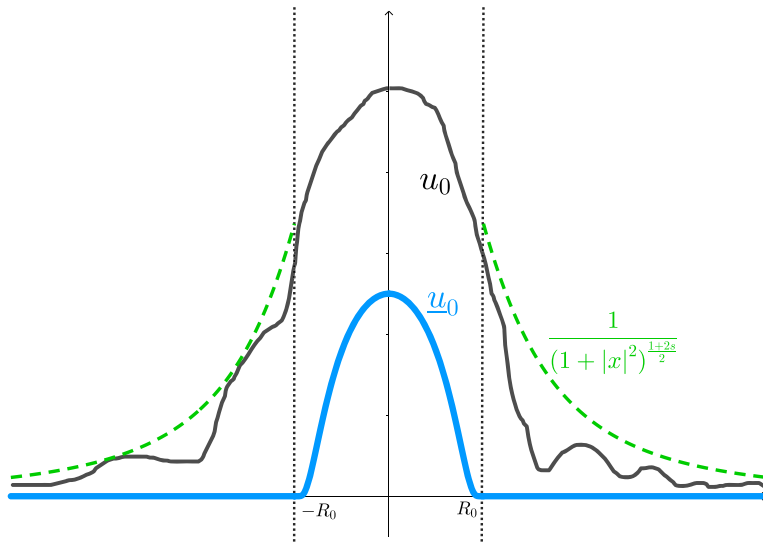


Fig. 3 Lower bound for u_0 in the proof of Theorem 2.13

scaled standard mollifier

$$\underline{u}_0(x) = \frac{u_0(0)}{2} e^{1 - \frac{R_0^2}{(R_0^2 - |x|^2)_+}}$$

Clearly, (i) $\underline{u}_0(x) = 0 \leq (1 + |x|^2)^{-\frac{1+2s}{2}}$ if $|x| \geq R_0$; (ii) \underline{u}_0 is radially symmetric and radially nonincreasing; (iii) $\underline{u}_0 \in C_b^\infty(\mathbb{R}^n)$; and (iv) $u_0 \geq \underline{u}_0$ in \mathbb{R}^n since

$$\underline{u}_0(x) \leq \underline{u}_0(0) = \frac{u_0(0)}{2} e^{1 - \frac{R_0^2}{(R_0^2 - 0)_+}} = \frac{u_0(0)}{2} \leq u_0(x) \quad \text{for all } x \in B_{R_0}(0)$$

and $\underline{u}_0(x) = 0 \leq u_0$ for $x \in \mathbb{R}^n \setminus B_{R_0}(0)$ (see Fig. 3). From here, the proof follows as in Step (1) by using the lower bound in Lemma 5.2. □

9 Extensions and open problems

- There is an important open problem concerning the uniqueness and general comparison principle of viscosity solutions, either defined in our way or another suitable way that includes existence. For the moment we know that the following two classes of *BUC* viscosity solutions are unique: (i) radial radially nonincreasing solutions and (ii) monotone solutions evolving in one dimension only. Uniqueness in these cases follows by comparison with classical solutions. The problem is also open for elliptic equations of the same type, cf. [6].
- A main question that we deal with here is: how different is the theory and its results from the linear case (fractional heat equation)? The answer seems to be that they are quite different if $n \geq 2$, since then the infinity fractional Laplacian is a heavily nonlinear operator.

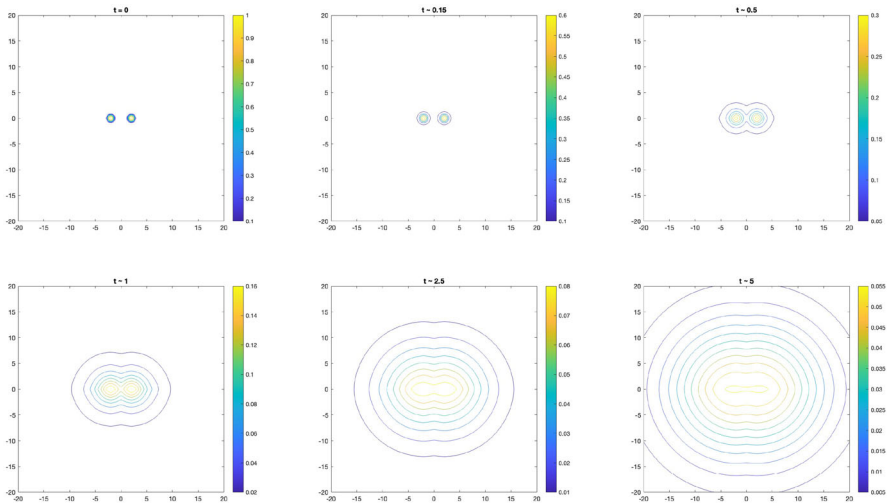


Fig. 4 Evolution of the level sets of the solution

- It is not clear whether for $n \geq 2$ the solutions evolve in time towards a radial profile (as in the local case, see below) or preserve a certain distortion. This is an interesting open problem to which we give a partial answer in our Sect. 8 with the global Harnack principle. In Fig. 4 (obtained with a rigorous finite difference scheme taken from the companion paper [15]) the distortion of the initial datum can still be observed for all the computed times.
- In particular, the evolution equation for the local version posed in the whole space has been studied by Portilheiro and the fourth author in [34, 35]. Then, there is a fine asymptotic behaviour as $t \rightarrow \infty$ that implies a sharp convergence rate to radially. The Aleksandrov Principle is a main ingredient in the proof. On the condition that the Aleksandrov Principle is true for some class of solutions of our Cauchy problem, we could also obtain a similar sharp asymptotic behaviour as $t \rightarrow \infty$ for such solutions. Such discussion is not included here.
- Large part of the concepts and results of this paper can be applied to the more general equation $\partial_t u = \Delta_\infty^s u + f(x, t)$. In particular, this could be applied to the stationary equation $\Delta_\infty^s u = f(x)$, thus relating the present results to the results of [6].
- We end the discussion by including an example demonstrating that the operator Δ_∞^s could indeed be pointwise discontinuous, even when applied to some smooth function ϕ . Consider $\Phi \in C_b^2(\mathbb{R})$ satisfying $\Phi(x_1) = \Phi(-x_1)$ and strictly decreasing for $x_1 \geq 0$. As in Lemma 6.3, we define $\phi(x) := \Phi(x_1)$ (see Fig. 5) where, for the sake of simplicity, $x = (x_1, x_2) \in \mathbb{R}^2$. On one hand, when $x_1 \neq 0$, we have $\zeta = \pm e_1$ (cf. Lemma 2.1) which yields $\Delta_\infty^s \phi(x_1, x_2) = -(-\partial_{x_1 x_1}^2)^s \Phi(x_1)$. On the other hand, when $x_1 = 0$, $\nabla \phi(0, x_2) = 0$ and by construction,

$$\inf_{|y|=1} \int_0^\infty (\phi(0, x_2) - \eta y) - \phi(0, x_2) \frac{d\eta}{\eta^{1+2s}} = \int_0^\infty (\Phi(x_1 - \eta) - \Phi(x_1)) \frac{d\eta}{\eta^{1+2s}}.$$

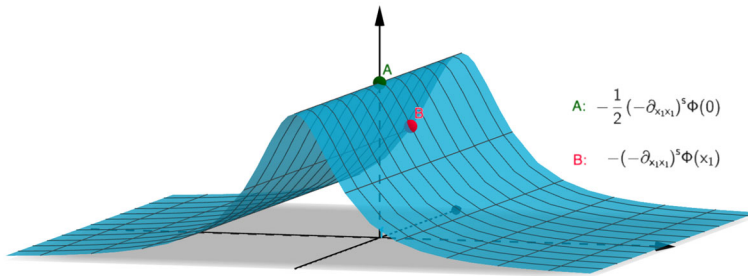


Fig. 5 Example of discontinuity of the operator Δ_∞^s

Since $x_1 = 0$ is a maximum point and $\phi(0, x_2) = \phi((0, x_2) + \eta e_2) = \Phi(x_1)$,

$$\begin{aligned} 0 &\geq \sup_{|y|=1} \int_0^\infty (\phi((0, x_2) + \eta y) - \phi(0, x_2)) \frac{d\eta}{\eta^{1+2s}} \\ &\geq \int_0^\infty (\phi((0, x_2) + \eta e_2) - \phi(0, x_2)) \frac{d\eta}{\eta^{1+2s}} = 0. \end{aligned}$$

We then conclude by Lemma 2.1 and symmetry of Φ that

$$\begin{aligned} \Delta_\infty^s \phi(0, x_2) &= C_s \int_0^\infty (\Phi(-\eta) - \Phi(0)) \frac{d\eta}{\eta^{1+2s}} \\ &= \frac{1}{2} C_s \int_0^\infty (\Phi(\eta) + \Phi(-\eta) - 2\Phi(0)) \frac{d\eta}{\eta^{1+2s}} \\ &= -\frac{1}{2} (-\partial_{x_1 x_1}^2)^s \Phi(0). \end{aligned}$$

Hence,

$$\Delta_\infty^s \phi(x_1, x_2) = \begin{cases} -(-\partial_{x_1 x_1}^2)^s \Phi(x_1), & \text{if } x_1 \neq 0, \\ -\frac{1}{2} (-\partial_{x_1 x_1}^2)^s \Phi(x_1), & \text{if } x_1 = 0. \end{cases}$$

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Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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References

1. Akagi, G., Juutinen, P., Kajikiya, R.: Asymptotic behavior of viscosity solutions for a degenerate parabolic equation associated with the infinity-Laplacian. *Math. Ann.* **343**(4), 921–953 (2009)
2. Aronsson, G., Crandall, M.G., Juutinen, P.: A tour of the theory of absolutely minimizing functions. *Bull. Am. Math. Soc. (N.S.)* **41**(4), 439–505 (2004)
3. Barrios, B., Peral, I., Soria, F., Valdinoci, E.: A Widder's type theorem for the heat equation with nonlocal diffusion. *Arch. Ration. Mech. Anal.* **213**(2), 629–650 (2014)
4. Barron, E.N., Evans, L.C., Jensen, R.: The infinity Laplacian, Aronsson's equation and their generalizations. *Trans. Am. Math. Soc.* **360**(1), 77–101 (2008)
5. Bjorland, C., Caffarelli, L., Figalli, A.: Non-local gradient dependent operators. *Adv. Math.* **230**(4–6), 1859–1894 (2012)
6. Bjorland, C., Caffarelli, L., Figalli, A.: Nonlocal tug-of-war and the infinity fractional Laplacian. *Commun. Pure Appl. Math.* **65**(3), 337–380 (2012)
7. Blumenthal, R.M., Gettoor, R.K.: Some theorems on stable processes. *Trans. Am. Math. Soc.* **95**, 263–273 (1960)
8. Bonforte, M., Sire, Y., Vázquez, J.L.: Optimal existence and uniqueness theory for the fractional heat equation. *Nonlinear Anal.* **153**, 142–168 (2017)
9. Bucur, C., Squassina, M.: An asymptotic expansion for the fractional p-Laplacian and for gradient-dependent nonlocal operators. *Commun. Contemp. Math.* **24**(4), 34 (2022)
10. Caselles, V., Morel, J.-M., Sbert, C.: An axiomatic approach to image interpolation. *IEEE Trans. Image Process.* **7**(3), 376–386 (1998)
11. Chambolle, A., Lindgren, E., Monneau, R.: A Hölder infinity Laplacian. *ESAIM Control Optim. Calc. Var.* **18**(3), 799–835 (2012)
12. Chasseigne, E., Jakobsen, E.R.: On nonlocal quasilinear equations and their local limits. *J. Differ. Equ.* **262**(6), 3759–3804 (2017)
13. Collier, T.A., Hauer, D.: A doubly nonlinear evolution problem involving the fractional p-Laplacian. [arXiv:2110.13401v1](https://arxiv.org/abs/2110.13401v1) [math.AP] (2021)
14. Coulhon, T., Hauer, D.: Functional inequalities and regularising properties of nonlinear semigroups—theory and application. To appear in *BCAM SpringerBriefs in Mathematics*. [arxiv:1604.08737](https://arxiv.org/abs/1604.08737) (2016)
15. del Teso, F., Endal, J., Jakobsen, E.R., Vázquez, J.L.: Finite-difference methods for evolution equations driven by the infinity fractional Laplacian. In preparation (2022)
16. del Teso, F., Endal, J., Lewicka, M.: On asymptotic expansions for the fractional infinity Laplacian. *Asymptot. Anal.* **127**(3), 201–216 (2022)
17. del Teso, F., Lindgren, E.: A mean value formula for the variational p-Laplacian. *NoDEA Nonlinear Differ. Equ. Appl.* **28**(3), 33 (2021)
18. del Teso, F., Lindgren, E.: Finite difference schemes for the parabolic p-Laplace equation. [arXiv:2205.13656v1](https://arxiv.org/abs/2205.13656v1) (2022)
19. del Teso, F., Manfredi, J.J., Parviainen, M.: Convergence of dynamic programming principles for the p-Laplacian. *Adv. Calc. Var.* **15**(2), 191–212 (2022)
20. Droniou, J., Gallouet, T., Vovelle, J.: Global solution and smoothing effect for a nonlocal regularization of a hyperbolic equation. *J. Evol. Equ.* **3**(3), 499–521 (2003)
21. Juutinen, P., Kawohl, B.: On the evolution governed by the infinity Laplacian. *Math. Ann.* **335**(4), 819–851 (2006)
22. Juutinen, P., Lindqvist, P., Manfredi, J.J.: On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.* **33**(3), 699–717 (2001)
23. Kolokoltsov, V.N.: *Markov Processes, Semigroups and Generators*. De Gruyter Studies in Mathematics, vol. 38, p. xviii+430. Walter de Gruyter & Co., Berlin (2011)

24. Krylov, N.V.: Lectures on Elliptic and Parabolic Equations in Hölder Spaces. Graduate Studies in Mathematics, vol. 12, p. xii+164. American Mathematical Society, Providence (1996)
25. Lewicka, M.: Non-local Tug-of-War with noise for the geometric fractional p -Laplacian. *Adv. Differ. Equ.* **27**(1–2), 31–76 (2022)
26. Lindqvist, P.: Notes on the p -Laplace equation. Vol. 102. Report. University of Jyväskylä Department of Mathematics and Statistics. University of Jyväskylä, Jyväskylä, pp. ii+80 (2006)
27. Lindqvist, P.: Notes on the Infinity Laplace Equation. SpringerBriefs in Mathematics. BCAM Basque Center for Applied Mathematics, p. ix+68. Springer, Bilbao (2016)
28. Manfredi, J.J., Parviainen, M., Rossi, J.D.: An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. *SIAM J. Math. Anal.* **42**(5), 2058–2081 (2010)
29. Manfredi, J.J., Parviainen, M., Rossi, J.D.: An asymptotic mean value characterization for p -harmonic functions. *Proc. Am. Math. Soc.* **138**(3), 881–889 (2010)
30. Manfredi, J.J., Parviainen, M., Rossi, J.D.: Dynamic programming principle for tug-of-war games with noise. *ESAIM Control Optim. Calc. Var.* **18**(1), 81–90 (2012)
31. Mazón, J.M., Rossi, J.D., Toledo, J.: Fractional p -Laplacian evolution equations. *J. Math. Pures Appl.* (9) **105**(6), 810–844 (2016)
32. Palatucci, G.: The Dirichlet problem for the p -fractional Laplace equation. *Nonlinear Anal.* **177**(part B), 699–732 (2018)
33. Peres, Y., Schramm, O., Sheffield, S., Wilson, D.B.: Tug-of-war and the infinity Laplacian. *J. Am. Math. Soc.* **22**(1), 167–210 (2009)
34. Portilheiro, M., Vázquez, J.L.: A porous medium equation involving the infinity-Laplacian. Viscosity solutions and asymptotic behavior. *Commun. Partial Differ. Equ.* **37**(5), 753–793 (2012)
35. Portilheiro, M., Vázquez, J.L.: Degenerate homogeneous parabolic equations associated with the infinity-Laplacian. *Calc. Var. Partial Differ. Equ.* **46**(3–4), 705–724 (2013)
36. Vázquez, J.L.: The evolution fractional p -Laplacian equation in \mathbb{R}^N . Fundamental solution and asymptotic behaviour. *Nonlinear Anal.* **199**, 112034 (2020)
37. Vázquez, J.L.: The Dirichlet problem for the fractional p -Laplacian evolution equation. *J. Differ. Equ.* **260**(7), 6038–6056 (2016)
38. Vázquez, J.L.: Asymptotic behaviour for the fractional heat equation in the Euclidean space. *Complex Var. Elliptic Equ.* **63**(7–8), 1216–1231 (2018)
39. Vázquez, J.L.: The fractional p -Laplacian evolution equation in \mathbb{R}^N in the sublinear case. *Calc. Var. Partial Differ. Equ.* **60**(4), 59 (2021)

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