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NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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On the equality case of the Alexandrov-Fenchel inequality

Master's thesis in Mathematical Sciences (MSMNFMA)
Supervisor: Xu Wang
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Abstract

In this thesis, we look at the Alexandrov-Fenchel inequality and its equality case. We lay out some relevant results in Hodge theory, T-Hodge theory, and the theory of Delzant polytopes and their associated varieties and line bundles. We conclude by proving our main result stating that two Delzant polytopes, with the same Delzant variety, attain Alexandrov-Fenchel equality if and only if they are homothetic.

Sammendrag

Denne oppgaven omhandler Alexandrov-Fenchel ulikheten og dens likhets tilfelle. Vi viser til relevante resultater om Hodge-teori, T-Hodge-teori og rundt Delzant polytoper, varieteter og linjebunter. Vi avslutter oppgaven med vårt hovedresultat. Det sier at to Delzant polytoper med samme Delzant varietet, gir Alexandrov-Fenchel likhet hvis og bare hvis de er homotetiske.

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Contents

Introduction

The problem of finding which closed curves maximize the area they enclose, while also minimizing their arclength, is an ancient one, and is called the isoperimetric problem. The answer to the problem has long been "known" to be that such a curve must be a circle, though a rigorous proof was first given towards the tail-end of the 1800s. The problem can be restated as: *"For which S do we attain equality in the, similarly named, isoperimetric inequality*

$$|\partial S| \geq n|S|^{\frac{n-1}{n}}|\mathbb{B}|^{\frac{1}{n}}, \quad S \subset \mathbb{R}^n \text{ closed and compact."}$$

An important generalization of the isoperimetric inequality, also dating back to the end of the 1800s, is the Brunn-Minkowski inequality. It gives a lower bound for the measure of Minkowski-sums of convex bodies,

$$(\mu(A+B))^{\frac{1}{n}} \geq (\mu(A))^{\frac{1}{n}} + (\mu(B))^{\frac{1}{n}}, \quad A, B \subset \mathbb{R}^n,$$

where μ denotes the Lebesgue measure. A functional version of this inequality was given in 1973 by Prékopa [1]. Brascamp and Lieb gave a proof of this Prékopa theorem which reduces to an L^2-Hörmander estimate [2].

An even further generalization of the generalization of the Brunn-Minkowski, is the Alexandrov-Fenchel inequality. Fully solving for equality in this inequality is a longstanding open problem. In [3], X. Wang gives a new proof of the Alexandrov-Fenchel inequality, using methods similar to the Brascamp-Lieb proof.

We will in this thesis, like in [3], consider an alternate description of the Alexandrov-Fenchel inequality where it is given by log convexity of a certain integral of Kähler forms. This is the so-called Khovanskii-Teissier inequality, a Kähler-geometry analog of the Alexandrov-Fenchel inequality. We consider the equality case for a type of convex bodies called Delzant polytopes. Restricting ourselves to the Delzant case will simplify certain computations of these integrals, making the classification of the equality case simpler. Specifically, it will establish a direct link between the Khovanskii-Teissier and Alexandrov-Fenchel inequalities.

In Chapter 1 we give some preliminaries on Hodge theory, stating some necessary and useful results. We also give some generalized results in the T-Hodge theory setting. Chapter 2 concerns the Alexandrov-Fenchel inequality and the related Khovanskii-Teissier inequality, as well as some basic concept and results related to these. It will contain most of the setup for proving our main result. Lastly, in the third and final chapter, we lay out some theory regarding Delzant polytopes and their related varieties, culminating in the statement and proof of our main result.

Chapter 1

Hodge Theory

1.1 Hodge Theory

The study of Kähler manifolds incorporates elements from symplectic, complex and Riemannian geometry. Hodge theory allows us to study Kähler manifolds and their cohomology through the lense of differential equations. In this section we will present relevant and useful results for this thesis. Proofs of these results may be found in [4], [5], and [6].

Let h be a Hermitian metric on a complex manifold X. That is to say it is a hermitian inner product on each complex tangent plane. Let dz_j and $d\overline{z}_k$ denote local dual coordinates for $T_p^{1,0}(X)$ and $T_p^{0,1}(X)$ respectively. Then we have

$$h = \sum h_{j,k} dz_j \otimes d\overline{z}_k,$$

where $h_{j,k}$ is such that for a

$$\mu = \sum \mu_j \frac{\partial}{\partial z_j},$$

we have

$$|\mu|^2 = \sum h_{j,k} \mu_j \overline{\mu_k}.$$

We call the form

$$\omega = \sum h_{jk} dz_j \wedge d\overline{z}_k,$$

the $(1,1)$-form associated to h. For clarity, we might write it as ω_h. The metric is said to be *Kähler* if $d\omega = 0$, where d is the exterior derivative. A *Kähler form* is then a real closed positive definite $(1,1)$-form. A real smooth function f on X is called strictly plurisubharmonic if the form $\omega = i\partial\overline{\partial} f$ is Kähler. Then f is called a *Kähler potential* for ω.

Definition 1.1.1. A *Kähler manifold* (X, ω) is a complex manifold X along with a Kähler form ω.

Equivalently one might define it as either a Riemmanian manifold or symplectic manifold with extra structure. That is to say that a Kähler manifold is an even-dimensional Riemmanian manifold (X, g) together with a complex structure J on each tangent space, that preserves the metric and is itself preserved by parallel transport. Or, one might define it as a symplectic manifold (X, ω) along with an integrable almost complex structure J that agrees with ω.

Definition 1.1.2. Let $\partial, \overline{\partial}$ be the Dolbeault operators on (p, q)-forms on X. Then the exterior derivative

$$d : \Omega^k(X) \to \Omega^{k+1}$$

is given by $d = \partial + \overline{\partial}$. We then define $d^c = -\frac{i}{2}(\partial - \overline{\partial})$.

Note that we have $dd^c = i\partial\overline{\partial}$.

Since a Kähler manifold is endowed with a Riemannian metric we get a L^2-inner product, $\langle \cdot, \cdot \rangle_{L^2}$, on forms given by

$$\langle \mu, \nu \rangle_{L^2} = \int \langle \mu, \nu \rangle \frac{\omega^n}{n!},$$

where the inner product in the integral is just the bilinear form associated to the Riemannian metric. With respect to this inner product, we can define formal adjoints of our differential operators. For example, the adjoint of ∂ is the operator ∂^* such that $\langle \partial\mu, \nu \rangle_{L^2} = \langle \mu, \partial^*\nu \rangle_{L^2}$ for any smooth form μ , and any smooth form ν with compact support. Note that if we have a differential operator T of degree k, then the adjoint T^* must necessarily be of degree $-k$. With these adjoints, we can define the following Laplacians.

Definition 1.1.3. Each of the differential operators $\partial, \overline{\partial}, d$ have related Laplacian operators. They are defined as follows,

- $\Delta_\partial := \partial\partial^* + \partial^*\partial$
- $\Delta_{\overline{\partial}} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$
- $\Delta_d := dd^* + d^*d.$

For each of these, we may define the corresponding space of harmonic (p, q)-forms as the kernel of the corresponding Laplacian,

$$\mathcal{H}_d^{(p,q)}(X) := ker(\Delta_d), \quad \mathcal{H}_\partial^{(p,q)}(X) := ker(\Delta_\partial), \quad \mathcal{H}_{\overline{\partial}}^{(p,q)}(X) := ker(\Delta_{\overline{\partial}}).$$

Definition 1.1.4. Let (X, ω) be a Kähler manifold. We then define the *Lefschetz operator*

$$L : \Omega^k \to \Omega^{k+2}$$

by $Lu := \omega \wedge u$. We denote its adjoint by $\Lambda := L^*$.

A useful tool for describing the adjoints of our relevant operators will be the star operator.

Definition 1.1.5. Given our Kähler form ω we get a bilinear form ω^{-1}, defined by

$$\omega^{-1}(dz_j, d\overline{z_k}) = -\omega^{-1}(dz_k, d\overline{z_j}) = \delta_{jk}, \quad \omega^{-1}(dz_j, dz_k) = \omega^{-1}(d\overline{z_j}, d\overline{z_k}) = 0.$$

We may also define it on k-forms by $\omega^{-1}(\mu, \nu) = det(\omega^{-1}(\mu_1, \nu_1))$ where

$$\mu_1 \wedge \cdots \wedge \mu_k,\, \nu = \nu_1 \wedge \cdots \wedge \nu_k.$$

We then define the *Hodge star operator* $*$ by the property

$$\mu \wedge *\overline{\nu} = (\omega^{-1}(\mu, J\overline{\nu}))\frac{\omega^n}{n!}.$$

Remark 1.1.6. The star operator also has a symplectic version $*_s$ which is defined by

$$\mu \wedge (*_s \nu) = \omega^{-1}(\mu, \nu)\frac{\omega^n}{n!}.$$

Then we have

$$* = *_s \circ J = J \circ *_s.$$

The symplectic star operator has the property $*_s^2 = 1$ and the Hodge star operator has the property $*^2 = (-1)^{k(n-k)}$

The star operator works as an "adjointer" for differential operators. We have the following adjoints:

- $d^* = - * \circ d \circ *$
- $\partial^* = - * \circ \partial *$
- $\overline{\partial}^* = - * \circ \overline{\partial} \circ *$
- $d^{c*} = - * \circ d^c \circ *$
- $L^* = \Lambda = *^{-1} \circ L \circ *$

The next theorem, the so-called Kähler identities, describe interactions between some of the operators we have given so far.

Theorem 1.1.7. *(Kähler identities) Let* $[\cdot, \cdot]$ *denote the commutator. Then we have the following identities:*

- $[\overline{\partial}, L] = [\partial, L] = [\partial^*, \Lambda] = [\overline{\partial^*}, \Lambda] = 0.$
- $[\overline{\partial^*}, L] = -[\partial^*, L] = i\partial$
- $[\Lambda, \overline{\partial}] = i\partial^*$, $[\Lambda, \partial] = i\overline{\partial^*}$
- $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$
- $[\Delta, T] = 0$ *for* $T = d, \partial, \overline{\partial}, L, d^*, \partial^*, \overline{\partial}^*, \Lambda, *.$

Instead of looking at differential operators on the usual space of (p, q)-forms, one might consider forms with values in some holomorphic vector bundle $E \to X$. So at a point p, a form μ of total degree k is a smooth multilinear alternating map

$$\mu_p : \overbrace{T_p^{\mathbb{C}} \circ \cdots T_p^{\mathbb{C}}}^{k \text{ times}} \to E_p.$$

Let h be a metric on E. We denote the differential operators defined on these forms by the subscript $-_E$, e.g. d_E. There is then a unique connection that is compatible with h, namely the Chern connection $D_E = \partial_E + \overline{\partial}_E$. We write its associated curvature form as $\Theta(E, h)$. The Kähler identities not involving the Laplacians also hold in this case, and we have the following identities for the Laplacians.

Theorem 1.1.8. *(Nakano identities) Let* (E, X) *be a holomorphic vector bundle with Hermitian metric* h*. Then we have*

- $\Delta_{\partial_E} + \Delta_{\overline{\partial}_E} = \Delta_{D_E}$
- $\Delta_{\overline{\partial}_E} - \Delta_{\partial_E} = [i\Theta(E, h) \wedge -, \Lambda]$. *(Bochner-Kodaira-Nakano identity)*

Theorem 1.1.9. *(Hodge decomposition) Let (X,ω) be a compact Käher manifold. Then we have the following orthogonal decompositions on (p,q)-forms.*

- $\Omega^{(p,q)}(X) = \mathcal{H}^{p,q}_{\partial} \oplus im(\partial) \oplus im(\partial^*)$
- $\Omega^{(p,q)}(X) = \mathcal{H}^{p,q}_{\overline{\partial}} \oplus im(\overline{\partial}) \oplus im(\overline{\partial}^*)$
- $\Omega^{(p,q)}(X) = \mathcal{H}^{p,q}_{d} \oplus im(d) \oplus im(d^*)$

Note that by the identity $\Delta_{\partial} = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$ we have

$$\mathcal{H}^{p,q}_d(X) \simeq \mathcal{H}^{p,q}_{\partial}(X) \simeq \mathcal{H}^{p,q}_{\overline{\partial}}(X),$$

and we may therefore choose to write any of these simply as $\mathcal{H}^{p,q}(X)$. As a corollary of the above decomposition, we get the $\partial\overline{\partial}$-lemma, a useful tool for working on compact Kähler manifolds.

Theorem 1.1.10. *($\partial\overline{\partial}$-lemma) Let $\alpha \in \Omega^{p,q}(X)$ be d-closed, i.e. $d\alpha = 0$, on a compact Kähler manifold (X,ω). Then the following are equivalent:*

- *α is d-exact, that is to say there is some $(p-1,q-1)$-form β with $\alpha = d\beta$*
- *α is ∂-exact. ($\alpha = \partial\beta$)*
- *α is $\overline{\partial}$-exact. ($\alpha = \overline{\partial}\beta$)*
- *α is $\partial\overline{\partial}$-exact. ($\alpha = i\partial\overline{\partial}\beta$)*
- *α is orthogonal to the subspace of harmonic (p,q)-forms $\mathcal{H}^{p,q}(X)$.*

Remark 1.1.11. Though we have restricted ourselves to the compact case, there is a local version of the $\partial\overline{\partial}$-lemma that also holds in the non-compact case. This can be thought of as a complex/Kähler version of the Poincaré lemma. This assures that every Kähler form has a local Kähler potential.

As well as the harmonic Hodge decomposition, we have the Lefschetz decomposition.

Theorem 1.1.12. *Let (X,ω) be a compact Kähler manifold. Then the map*

$$\begin{aligned} L^k := \overbrace{L \circ \cdots \circ L}^{k} : \Omega^k(X) &\longrightarrow \Omega^{2m-k}(X) \\ u &\longmapsto \omega^{m-k} \wedge u \end{aligned}$$

is an isomorphism for each $0 \leq k \leq n$. We call a form primitive *if it is in the kernel of L^k for some k. We denote these kernels by*

$$P^{2n-k}(X) := ker(L^{k+1}).$$

We then have the following decomposition on k-forms:

$$\Omega^k(X) = \bigoplus_{i \geq 0} L^i P^{k-2i}(X).$$

1.2 T-Hodge Theory

A generalization of Hodge theory that will be useful to us is that of the mixed T-Hodge theory studied by Timorin in [5]. It is akin to adding a weight or shift to the space of forms. Most of the above results have a T-Hodge theory counterpart. In this section, we deliberate upon these results. Proofs may be found in [5] and [4].

Let $m \leq n$ and $\alpha_{m+1}, \dots, \alpha_n$ be smooth positive $(1,1)$-forms on a Kähler manifold (X, ω) of complex dimension n. We define

$$T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n.$$

Timorins mixed T-Hodge theory concerns the image of multiplication with T and of what classical results from Hodge theory still apply to this image.

Definition 1.2.1. Let T and (X, ω) be as above, and let (E, h_E) be a holomorphic vector bundle and compatible metric. We denote the multiplication-by-T operator by

$$f_T(u) := T \wedge u,$$

and denote the images of f_T by

$$\Omega_T^{p,q}(X) := f_T(\Omega^{p,q}(X)), \quad \Omega_T^k(X) := \bigoplus_{p+q=k} \Omega_T^{p,q}(X), \quad \Omega_T(X) := \bigoplus_{0 \leq p,q \leq n} \Omega_T^{p,q}(X),$$

where $\Omega^{p,q}(X)$ is the space of E-valued (p,q)-forms.

The operators $d, \partial, \overline{\partial}$ and their adjoints are defined on $\Omega_T^k(X)$ in the obvious way. Our first theorem in T-Hodge theory is a generalization of the Hard Lefschetz theorem.

Theorem 1.2.2. *With T and (X, ω) as above, then*

$$f_T : \Omega^m(X) \to \Omega^{2n-m}(X)$$

defines is an isomorphism. Further, just as the Hard Lefschetz theorem holds in the case where $T = 1$, i.e. $m = n$, we have that for $0 \leq k \leq m$

$$L^k := \overbrace{L \circ \cdots \circ L}^{k} : \Omega_T^k(X) \longrightarrow \Omega_T^{2n-k}(X)$$
$$u \longmapsto \omega^{n-k} \wedge u$$

is an isomorphism. Similarly to the standard case, we call a form $u \in \Omega_T^k(X)$ primitive if it is in the kernel of some L^k. Then every $u \in \Omega_T^k$ has a primitive decomposition

$$u = \sum L_r u_r, \quad L_r := \frac{L^r}{r!},$$

with $u_r \in \Omega_T^{k-2r}$ primitve.

Mixed versions of some of the Kähler/Nakano identities still hold in this setting.

Theorem 1.2.3. *Let T be d-closed on a Kähler manifold (X, ω) with holomorphic vector bundle (E, h_E). Then on $\Omega_T(X)$ we have the $D_E := \overline{\partial}_E + \partial_E$ related identities*

- $D_E^* = [\Lambda, D_E^c]$
- $(D_E^c)^* = [D_E, \Lambda]$,
- $\Delta_{D_E} = \Delta_{D_E^c} = \Delta_{\overline{\partial}_E} + \Delta_{\partial_E}$

and the Bochner-Kodaira-Nakano identity

$$\Delta_{\overline{\partial}_E} - \Delta_{\partial_E} = [i\Theta(, h_E) \wedge, \Lambda].$$

As in the standard setting, we can use the Lefschetz decomposition to define a symplectic and a Hodge star operator, still denoted $*_s$ and $*$ respectively. On a primitive form, say $u \in V_T^k$, we define

$$*_s(L_r u) := (-1)^{\frac{k(k+1)}{2}} L_{m-r-k} u,$$

and extend it to an operator on all of Ω_T. Then the Hodge star operator on Ω_T is given as $* := *_s \circ J$, where $J = i^{p-q}$. There is also an analog of the Hodge-Riemann bilinear relations given by

$$(-1)^{\frac{k(k+1)}{2}} T_{k+1} \wedge u \wedge \overline{Ju} > 0.$$

We use these two facts to define an L^2 norm and inner product on Ω_T defined by

$$\|u\|_T := \int u \wedge \overline{*T \wedge u}.$$

Chapter 2

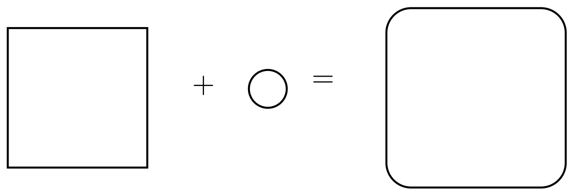
The Alexandrov-Fenchel inequality

Definition 2.0.1. Let A, B be non-empty sets in $\mathbb{R}^n$. Define their *Minkowski sum* as

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

One may think of this as placing a copy of A at every point of B or vice versa.

Example 2.0.2. Let $S = \{(x, y) \mid \max x, y \leq 1\}$ the unit square and $\mathbb{B}_\epsilon$ a small ball. Then their sum $S + \mathbb{B}_\epsilon$ would be a square with rounded corners.



A classical result of Minkowski states that given some *convex bodies*, i.e. compact convex sets with non-empty interiors, $A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$, the function

$$F(t_1, t_2, \ldots, t_n) : [0, 1]^k \to \mathbb{R}$$

given by $(t_1, \ldots, t_k) \mapsto Vol(t_1A_1 + t_2A_2 + \cdots + t_kA_k)$ is a homogeneous polynomial. We will denote the coefficient of the $t_1t_2 \cdots t_k$ term by $V(A_1, \ldots, A_k)$. So

$$V(A_1, \ldots, A_k) := \frac{\partial^k F(A_1, \ldots, A_k)}{\partial t_1 \cdots \partial t_k}.$$

Then the Alexandrov-Fenchel inequality says the following.

Theorem 2.0.3. (Alexandrov-Fenchel inequality)

Given a k-tuple of convex bodies $(A_1, \dots, A_k) \subset \mathbb{R}^n$ we have the following inequality

$$V(A_1, A_2, \dots, A_k)^2 \geq V(A_1, A_1, A_3, \dots, A_k)V(A_2, A_2, A_3, \dots, A_k).$$

As discussed in the introduction, an immediate consequence is the isoperimetric inequality. And even generalizations of the isoperimetric inequality are contained as special cases of the Alexandrov-Fenchel inequality.

Example 2.0.4. Let $\mathbb{B}$ denote the unit ball in $\mathbb{R}^n$, then for some convex body C, we define the j-th *quermassintegral* of C as

$$V(\underbrace{C, \dots, C,}_{n-j \text{ times}} \overbrace{\mathbb{B}, \dots, \mathbb{B}}^{j \text{ times}}).$$

It gives the average volume of the projection of C onto a random j-dimensional subspace. Then by 2.0.3 we have an isoperimetric type inequality

$$V(\mathbb{B}, C, C)^2 \geq V(C, C, C)V(\mathbb{B}, \mathbb{B}, C).$$

That is to say, the area of ∂C is bounded from below by the product of the volume of C and its average width.

2.1 The Legendre transform

To show the above-mentioned result of Minkowski we will construct a diffeomorphism from our convex body A to all of $\mathbb{R}^n$. We do this using the Legendre transform.

Definition 2.1.1. Let $A \subset \mathbb{R}^n$ be a non-empty, compact, convex set and let $\phi : int(A) \to \mathbb{R}$ be a convex function on the interior of A. Then we define the Legendre transform of ϕ by

$$\phi^*(y) := \sup_{x \in int(A)} x \cdot y - \phi(x),$$

where $x \cdot y$ denotes the standard Euclidean inner-product.

Example 2.1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^2$. Then

$$f^*(y) = \sup_{x \in \mathbb{R}} x \cdot y - f(x) = \sup_{x \in \mathbb{R}} xy - x^2.$$

Now fix a some y_0. Then $xy - x^2$ attains its max at $x = \frac{y}{2}$

$$\implies f^*(y) = \frac{y^2}{4}$$

Theorem 2.1.3. *Let $A \subset \mathbb{R}^n$ be a non-empty convex body, and let $\phi : A \to \mathbb{R}$ be a smooth, strictly convex function. If ϕ tends to infinity at the boundary of A then*

1. *the Legendre transform ϕ^* is also smooth and strictly convex*
2. *the gradient map*
$$\nabla\phi : int(A) \to \mathbb{R}^n$$
is a diffeomorphism with $\nabla\phi^*$ as its inverse.

Proof. We start by showing (2). By strict convexity of ϕ, at every interior point of A, the Hessian matrix $(\frac{\partial^2\phi}{\partial x_i \partial x_j})$ is positive definite. So it is invertible, and by the inverse function theorem $\nabla\phi$ is a local diffeomorphism. Thus, to show that $\nabla\phi$ is a diffeomorphism we only need to show that it is bijective.

Consider the smooth, strictly convex function $\phi^{y_0}(x) = \phi(x) - y_0 \cdot x$ for some fixed $y_0 \in \mathbb{R}^n$. Let $\nabla\phi(x_1) = \nabla\phi(x_2) = y_0$. Then

$$\nabla\phi^{y_0}(x) = \nabla\phi(x) - y_0.$$

So $\nabla\phi^{y_0}(x_1) = \nabla\phi^{y_0}(x_2) = 0$. Since $\nabla\phi^{y_0}$ is a strictly convex function it must also be strictly convex on any line segment. Specifically, it is strictly convex on the line segment connecting x_1 and x_2. But now a strictly convex function has at most one critical point so

$$\nabla\phi^{y_0}(x_1) = \nabla\phi^{y_0}(x_2) = 0 \implies x_1 = x_2.$$

So $\nabla\phi$ is injective.

As stated above the function ϕ^{y_0} is strictly convex for any fixed $y \in \mathbb{R}^n$. Further, since ϕ tends to infinity at the boundary, so will ϕ^y for any $y \in \mathbb{R}^n$. These two facts give that ϕ^y has a unique minimal point $x \in int(A)$

$$\implies \nabla\phi^y(x) = \nabla\phi(x) - y = 0 \implies \nabla\phi(x) = y.$$

So $\nabla\phi$ is bijective.

Now since $\nabla\phi$ is smooth and bijective we can see that

$$\phi^*(\nabla\phi(x)) = \sup_{y\in\mathbb{R}^n} \nabla(x) \cdot y - \phi(y).$$

So we have $\phi^*(\nabla\phi) = \nabla\phi(x) \cdot x - \phi(x)$ is smooth, and since $\nabla\phi$ is surjective so is ϕ^*. □

The reason we do this construction is that it gives us another description of the volume of our convex body A. Letting $\psi : int(A) \to \mathbb{R}$ be a strictly convex function that blows up at the boundary, as in the theorem, and setting $\phi := \psi^*$ we get that

$$|A| = \int_{\mathbb{R}^n} Hess(\phi) dx.$$

Note that for a convex polytope, i.e. the convex hull of a finite set $\{p_j\} \subset \mathbb{R}^n$, one can choose

$$\phi(x) = log \left(\sum_j e^{\langle p_j, x \rangle} \right).$$

Moreover, we have the following.

Theorem 2.1.4. *Let $\phi_i, i = 1, 2, \ldots, n$ each such that $\nabla \phi_i$ is a diffeomorphism from $\mathbb{R}^n$ to $int(A)$ and let $0 < t_i$ then we have*

$$|t_1 A_1 + \cdots + t_n A_n| = \int Hess(t_1 \phi_1 + \cdots + t_n \phi_n) dx.$$

A consequence of the above theorem, as mentioned in the introduction, is that $|t_1 A_1 + \cdots + t_N A_n|$ is actually given by a homogeneous polynomial in the variables $t_1, \ldots, t_n$.

2.2 Alexandrov-Fenchel inequality as log convexity

For the proof of our main theorem, we will interpret the Alexandrov-Fenchel inequality as a special case of its complex geometric analog, the Khovanskii-Teissier inequality. Proofs of which may be found in [7]. See [3] for how to Khovanskii-Teissier implies Alexandrov-Fenchel in the general (i.e. not only Delzant) case.

Theorem 2.2.1. *(Khovanskii-Teissier inequality) Let X be a n-dimensional compact Kähler manifold, and let $\omega_1, \ldots, \omega_n$ be Kähler forms. Then, putting $T := \omega_3 \wedge \cdots \wedge \omega_n$, we have*

$$\left(\int_X \omega_1 \wedge \omega_2 \wedge T \right)^2 \geq \left(\int_X \omega_1^2 \wedge T \right) \left(\int_X \omega_2^2 \wedge T \right).$$

In addition, the following lemma will give us equivalent descriptions of both inequalities.

Lemma 2.2.2. *Let C be a cone over $\mathbb{R}$. We say that a function $f : C \to \mathbb{R}$ is 1-homogeneous if for every $t \in \mathbb{R}^+$ we have*

$$f(tx) = tf(x), \quad \forall x \in C.$$

If f is 1-homogeneous then the following are equivalent:

1. $f(x + y) \geq f(x) + f(y)$
2. $-log(f)$ *is convex*
3. $t \mapsto -log(f(tx + (1 - t)(y)))$ *is convex on for $t \in (0, 1)$ for any $x, y \in C$.*

Proof. See [3] □

With this lemma, we can see that the Alexandrov-Fenchel inequality described above is equivalent to the following.

Theorem 2.2.3. *Let* $A_1, \ldots, A_k$ *be convex bodies and define* $A_t := tA_1 + (1-t)A_2$*. Then the function*

$$t \mapsto -log(V(A_t, A_t, A_3, \ldots, A_k))$$

is convex.

And the Khovanskii-Teissier inequality is equivalent to:

Theorem 2.2.4. *Let* $(X, \hat{\omega})$ *be a* n*-dimensional compact Kähler manifold, and let* $\omega_1, \ldots, \omega_n$ *be Kähler forms. Put* $T := \omega_3 \wedge \cdots \wedge \omega_n$*, and let* $\omega := t\omega_1 + (1-t)\omega_2$*. Then the function*

$$t \mapsto -log \int_X \frac{\omega^2}{2} \wedge T \tag{2.1}$$

is convex on $(1, 0)$*.*

Note that this looks quite similar to the Prekopa (see for example [8, Chapter 1.2]). And the proof of it in [3] is also quite similar to the proof of Prekopa by Brascamp and Lieb. We will use the methods in the proof of Wang to solve for equality in our main result. Below we discuss how one would go about proving it and what results are needed, but refer the reader to [3] for in-depth proofs of these.

Remark 2.2.5. The way we have stated the Khovanskii-Teissier inequality above is just a special case of the one given in [3]. The most significant difference is that the more general result of Wang concerns complete, not only compact, Kähler manifolds of finite volume.

First we define G to be the function such that $\frac{d}{dt}(\frac{\omega^2}{2} \wedge T) = -G(\frac{\omega^2}{2} \wedge T)$, and call the function in 2.1 f. Then a computation gives that

$$f_t = \frac{d}{dt}\left(-log \int_X \frac{\omega^2}{2} \wedge T\right) = \frac{\int G\frac{\omega^2}{2} \wedge T}{\int \frac{\omega^2}{2} \wedge T}.$$

In order to make this expression and subsequent discussion less messy we will use the following notation:

$$d\mu := \frac{\frac{\omega^2}{2} \wedge T}{\int \frac{\omega^2}{2} \wedge T}, \quad E_\mu(G) := \int_X G d\mu.$$

With this notation we get that $f_t = E_\mu(G)$ and one can compute aswell that

$$f_{t,t} = \int_X (G_t - (G - E_\mu(G))^2) d\mu.$$

This is the same kind of expression as in the proof of the Prekopa theorem. So to show convexity of f we need to show

$$\int_X G_t d\mu \geq \int_X (G - E_\mu(G)^2) d\mu.$$

This is solved by the following lemma (3.3 and 3.4 of [3]).

Lemma 2.2.6. *Let $\|\cdot\|_{T,\omega}$ denote the T-Hodge norm. And let $\theta := \frac{d}{dt}\omega = \omega_1 - \omega_2$. Then, using the same notation as above we have*

$$\int_X G_t d\mu = e^f \|\theta\|_{T,\omega},$$

and

$$\int_X (G - E_\mu(G))^2 d\mu = e^f \|G - E_\mu(G)\|^2_{T,\omega}.$$

Further, we have that

$$T \wedge G = -\Lambda(T \wedge \theta), \quad \textit{(in T-Hodge theory)},$$

and $T \wedge (E_\mu(G) - G)$ is an L^2-minimal solution to the equation

$$d(-) = (d^c)^*(T \wedge \theta),$$

with $\|G - E_\mu(G)\|_{T,\omega} \leq \|\theta\|_{T,\omega}$

We end this chapter by looking at some lemmas concerning the equality cases of 2.2.2. Define the function $V(t) := (A_t, A_t, A_3, \ldots, A_n)^{\frac{1}{2}}$ for $t \in (0,1)$.

Lemma 2.2.7. *$(A_1, \ldots, A_n)$ attains Alexandrov-Fenchel equality $\iff V(t) = tV(0) + (1-t)V(1)$ for some $t \in (0,1)$.*

Proof. We will consider the $n = 2$ case, but the arguments that we make will hold for any n. We will also, for the time being, assume that $t = \frac{1}{2}$. First note that

$$V(\frac{1}{2})^2 = V(\frac{1}{2}(A_1 + A_2), \frac{1}{2}(A_1 + A_2)) = \frac{1}{4}V(A_1, A_1) + \frac{1}{4}V(A_2, A_2) + \frac{1}{2}V(A_1, A_2),$$

where the last inequality comes from the multilinearity of the mixed volume. Secondly, we note that

$$(\frac{1}{2}V(0) + \frac{1}{2}V(1))^2 = \frac{1}{4}V(A_1, A_1) + \frac{1}{4}V(A_2, A_2) + \frac{1}{2}V(A_1, A_1)^{\frac{1}{2}}V(A_2, A_2)^{\frac{1}{2}}.$$

By our assumption these two are the same and by canceling like terms we get exactly Alexandrov-Fenchel equality as desired. Note that by linearity we could have chosen any t, so the choice of $t = \frac{1}{2}$ is simply to make the calculations easier. □

By scaling either of A_1 or A_2 we can get $V(0) = V(1)$, in which case the Alexandrov-Fenchel equality is simply equivalent to $V(t) \equiv C$.

Lemma 2.2.8. *For any function f with $f(0) = f(1)$ we have*

$$-log(f)\text{is affine} \iff -log(f(t)) \equiv C.$$

Proof. $f(0) = f(1) \implies -log(f(0)) = -log(f(1)) \overset{affine}{\implies} -log(f) \equiv C$ □

Chapter 3

Delzant Polytopes and Varieties

3.1 Delzant polytopes

We start this chapter by introducing the objects of interest in our main theorem, namely Delzant polytopes. As we will discover, these polytopes are of interest to us due to their nice combinatorial properties. These properties will allow us to make a nice connection between the Alexendrov-Fenchel inequality for the polytopes and the Khovanskii-Teissier inequality for their corresponding varieties.

Definition 3.1.1. A convex *polytope* $P \subset \mathbb{R}^n$ is a compact finite intersection of half-planes with non-empty interior. That is to say, there exists some normal vectors $\alpha_j \in \mathbb{R}^n$ together with scalars $\beta_j \in \mathbb{R}, \quad 1 \leq j \leq N$, such that

$$P = \{x \in \mathbb{R}^n \mid \langle x, \alpha_j \rangle \leq \beta_j, \forall j\}.$$

We say that a polytope P is Delzant if in addition:

1. $\alpha_j \in \mathbb{Z}^n$ and $\beta_j \in \mathbb{Z}$
2. At every vertex ν exactly n half-planes meet. That is to say, if we define the following index set
$$I_\nu := \{j \mid \langle \alpha_j, \nu \rangle = \beta_j\},$$
then $|I_\nu| = n$.
3. The set $\{\alpha_j \mid j \in I_\nu\}$ generates the lattice $\mathbb{Z}^n$ for all ν and
$$\bigcup_{\nu \text{ vertex}} I_\nu = \{1, 2, \dots, N\}.$$

Let's consider some examples and non-examples.

Example 3.1.2. The very simplest example of a Delzant polytope would be to consider the one-dimensional case. Then every Delzant polytope is just an interval of the form $[a, b]$ for some integers a, b.

Example 3.1.3. Another simple example is that of a triangle. One way would be to construct it as follows. Let $\{\alpha_j\} = \{(1,0), (0,1), (-1,-1)\}$ and set $\{\beta_j\} = \{1,1,1\}$.

Example 3.1.4. $\{\alpha_j\} = \{\pm e_i\} \subset \mathbb{R}^2$ together with $b_j = 1, \forall j$ gives a Delzant square with a side lenght of 2 and center in the origin.

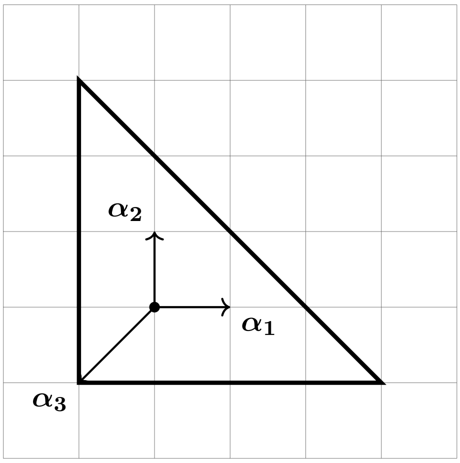


Figure 3.1: A Delzant triangle

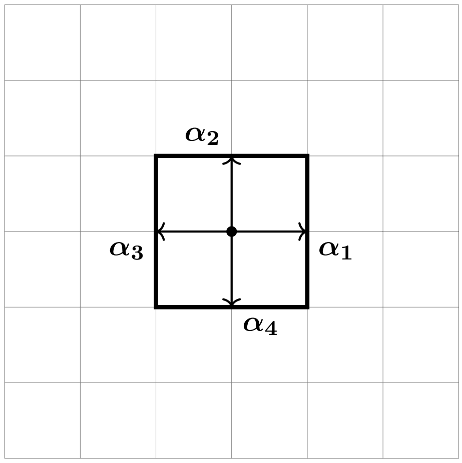


Figure 3.2: A Delzant square

The following are examples of polytopes that aren't Delzant.

Example 3.1.5. A pyramid in $\mathbb{R}^3$, say the convex hull of a square and a point, can not be Delzant as it does not satisfy condition 2) in the definition 3.1.1. That is more than three half-planes meet at the top vertex.

Example 3.1.6. The same construction of the square as described above, but replacing the normal vectors with the vectors $\{(2,0), (0,2), (-2,0), (0,-2)\}$, would not be Delzant. This is because at each vertex, even though the associated normal vectors are integral lattice points and linearly independent, they would not generate $\mathbb{Z}^n$. They would generate $(2\mathbb{Z})^n$.

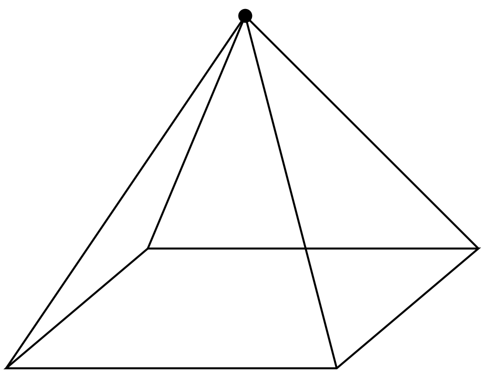


Figure 3.3: A pyramid in $\mathbb{R}^3$ can not be Delzant

Given a Delzant polytope we can easily construct a complete fan.

Definition 3.1.7. Given a vertex ν of a Delzant polytope P we define its associated cone as

$$\sigma_\nu := cone(\{\alpha_i\}_{i\in I_\nu}) = \left\{ \sum_{i\in I_\nu} \mid t_i\alpha_i,\ t_i \geq 0 \right\}.$$

We also define the fan of the polytope to be

$$\Sigma_P := \bigcup_\nu \sigma_\nu$$

Recall that a complex manifold consists of a collection of open subsets of $\mathbb{C}^n$ together with holomorphic charts between them. With this in mind, we proceed with the construction of a complex manifold from a Delzant polytope. Given a vertex ν let $\sigma_\nu^\vee$ denote the dual cone at that vertex. Since σ_ν is rational polyhedral we know that $\sigma_\nu^\vee$ is as well. Denote the dual basis by $\{\alpha_j^\vee\}_{j\in I_\nu} \subset \mathbb{Z}^n$. This basis is also a generating set for $\mathbb{Z}^n$. Then for each ν we get an embedding

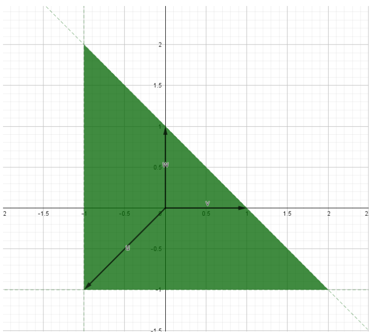
$$\Phi_\nu : (\mathbb{C}^*)^n \to \mathbb{C}^n,$$

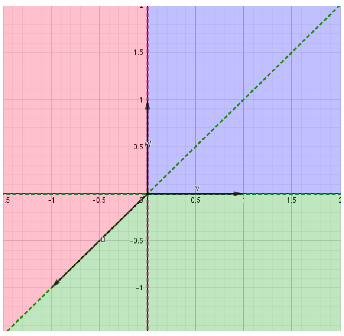
given by

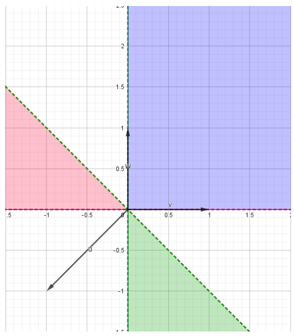
$$\Phi_\nu(z_1, \dots, z_n) = (z_1^{-\alpha_{j_1}^\vee}, \dots, z_n^{-\alpha_{j_n}^\vee}).$$

These embeddings give us a complex manifold X_P covered by N copies of $\mathbb{C}^n$, defined by gluing these embeddings together via $\Phi_{\nu_i} \circ \Phi_{\nu_j}$.

One way of visualizing the dual cone $\sigma_\nu^\vee$ geometrically is that it is "generated by the corner" at the vertex ν. Below is an illustration of a Delzant polytope, its fan consisting of the cones of each vertex and its dual fan consisting of the dual cones of each vertex. In that order.







So given a Delzant polytope P, we can recover all information on its vertices and normal vectors up to scaling. And we know that two homothetic Delzant polytopes give the same fans and dual fans, and therefore also the same varieties.

Example 3.1.8. Consider $P = [0, n]$ for some integer n and let the normal vector to 0 be -1 and the normal vector to n be 1. So since P has two veritces, we know that X_p is covered by two copies of $\mathbb{C}$. And we have two isomorphisms of $\mathbb{C}^*$, namely

$$\Phi_0 = id, \quad \Phi_n(w) = w^{-1}.$$

And since Φ_n is its own inverse we get that

$$X_P \simeq \mathbb{C} \coprod \mathbb{C}/(z = \frac{1}{W}).$$

The right-hand side can be seen to be the Riemann sphere, so we get that $X_P \simeq \mathbb{P}^1$.

Example 3.1.9. If we were to consider the P as the Delzant triangle above, then $X_P \simeq \mathbb{P}^2$. In fact, if we let $P \subset \mathbb{R}^n$ be bounded by the planes $\{x_i = -1\}$ and the plane $\{\sum_j^n x_j = 1\}$, i.e. P is something like a n-simplex, then $X_P \simeq \mathbb{P}^n$

Example 3.1.10. Let P be the Delzant square as above. Then $X_P \simeq \mathbb{P}^1 \times \mathbb{P}^1$

As with any convex body, we can construct a function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that its gradient is a diffeomorphism onto the interior of P. And since P is a polytope, i.e. the hull of a finite set of points, we can choose

$$\phi(x) := log\left(\sum_{u \in P \cap \mathbb{Z}^n} e^{\langle u, x \rangle}\right).$$

In the same vein, we can define a function on X_P that is surjective onto P, continuous and proper. Using the above function ϕ we get a function $\varphi : X_P \to P$ on each copy of $(\mathbb{C}^*)^n$ by setting

$$\varphi(z) := \nabla\phi(log|z_1|^2, \dots, log|z_n|^2).$$

This map has a unique extension to all of X_P with the desired properties.

3.1.1 Delzant line bundles

We may also add more structure to the variety X_P in the form of a line bundle. This construction will again use, and encode, the combinatorial Delzant information of P. Recall that a line bundle may be constructed by gluing together open sets $U_i \times \mathbb{C} \subset X \times \mathbb{C}$, identifying $(z, \xi) \sim (z, g_{i,j}(\xi))$ using transition maps $g_{i,j}$ on $U_i \cap U_j$.

Given a Delzant polytope P, we construct a line bundle on X_P in the following way. On each toric embedding, i.e. for each vertex, we give the following embedding into $\mathbb{C}^n \times \mathbb{C}$,

$$\Psi_\nu(z, \xi) = (\phi_\nu(z), z^{-\nu}\xi).$$

Then we glue together these N copies of $\mathbb{C}^n \times \mathbb{C}$ along the maximal extensions of $\Psi_\nu \circ \Psi_\nu^{-1}$. We will call this line bundle L_P. The following theorem gives one reason for using this specific line bundle construction.

Theorem 3.1.11. *Let $u \in P \cap \mathbb{Z}^n$, then u gives a global section $s_u \in H^0(X_P, L_P)$ of L_P. In fact we have that $H^0(L_P, X_P) \simeq span_{\mathbb{C}}\{s_u\}_{u \in P \cap \mathbb{Z}^n}$*

Proof. We wish to construct a section of $(\mathbb{C}^*) \times \mathbb{C}$ that also has a holomorphic extension on each embedding Ψ_ν. The section we want to extend is defined by $(z) \mapsto (z, z^u)$. So we need $\Psi_\nu(z, z^u) = (\Phi_\nu, z^{u-\nu})$ to be holomorphic on the coordinates $\{z^{-\alpha_k^\vee}\}_{k \in I_\nu}$. But that is simply to say that $u - \nu$ is in the dual cone $\sigma_\nu^\vee$, which is guaranteed by $P - \nu \in \sigma_\nu^\vee$, which we again get from our geometric description of $\sigma_\nu^\vee$. And in fact, since this must hold simultaneously for all the vertices, for this construction to work for an arbitrary $u' \in \mathbb{Z}^n$ we must have $u \in P$. So we have proved the first part of the statement.

For the second part, consider the $(\mathbb{C}^*)^n$ action on L_P defined by the action

$$\chi_t(z, \xi) = (t_1 z_1, \ldots, t_n z_n, \xi), \quad t \in (\mathbb{C}^*)^n$$

on each embedding of $(\mathbb{C}^*)^n \times \mathbb{C}$. Then by the below lemma, we can write the Laurent series expansion of a global section

$$s = \sum_{u \in \mathbb{Z}^n} c_u z^u.$$

Then we must have that $z^u \in H^0(X_P, L_P)$ as long as $c_u \neq 0$. But as we have already shown, this can only happen when $u \in P$. $\square$

Lemma 3.1.12. *Let λ be a group action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^m$. Then there is a basis $\{e_i\}$ of $\mathbb{C}^m$ and a $k \in \mathbb{Z}^n$ such that*

$$\lambda(t)e_i = t^k e_i.$$

Proof. Setting

$$W_k := \{z \in \mathbb{C}^m \mid \lambda(t)z = t^k z\},$$

we have a decomposition of $\mathbb{C}^m$ given

$$\mathbb{C}^m = \bigoplus_{k \in \mathbb{Z}^n} W_k.$$

This is a standard result in representation theory. See for example [9, 3.2.3]. $\square$

This description of the space of global sections will be useful for connecting the Khovanskii-Teissier inequality to the Alexandrov-Fenchel inequality in the Delzant case.

We can on each toric embedding define a Hermitian h_P metric on the L_P by

$$h_P(z,\xi) = \frac{|z^{-\nu}\xi|^2}{\sum_{u\in P\cap\mathbb{Z}^n}|z^{u-\nu}|^2}.$$

This extends to all of X_P. and we get that L_P is positively curved by the Chern curvature

$$i\Theta(L_P,h_P) = i\partial\overline{\partial}log(\sum_{u\in P\cap\mathbb{Z}^n}|z^{u-\nu}|^2).$$

Positivity will allow us to make use of the Hörmander L^2 estimate and its consequences.

Theorem 3.1.13. *Let P be a Delzant polytope, then $vol(L_P) = |P|$*

Proof. Noting that

$$(X_{mP}, L_{mP}) = (X_P, mL_P),$$

for any positive integer m, the Bergmann kernel asymptotic formula [10] gives

$$vol(L_P) := \lim_{m\to\infty}\frac{H^0(X_P,mL_P)}{m^n} = \int\frac{(\frac{i\Theta(L_P,h_P)}{2\pi})^n}{n!} =: \frac{L_P}{n!}.$$

By our description of the space of global sections, we get

$$\lim_{m\to\infty}\frac{H^0(X_P,mL_P)}{m^n} = \lim_{m\to\infty}\frac{\{mP\cap\mathbb{Z}^n\}}{m^n}.$$

The right-hand side, however, is just the Lebesgue measure of P. So we get $|P| = vol(L_P)$. □

We used in this proof the statement that $(X_{mP}, L_{mP}) = (X_P, mL_P)$. One might then wonder, can we show something similar for an arbitrary Minksowski sum of Delzant polytopes? It turns out that we can!

We start by considering a general question concerning the Minkowski sum of polytopes. Recall that a polytope is given by a set of normal vectors $\{\alpha_j\}$ and corresponding scalars $\{\beta_j\}$. We look at two, not necessarily Delzant, polytopes P_r, P_s with the same set of normal vectors α_j but different scalars $\{s_j = s\}$ and $\{r_j\} = r$ respectively. We then ask whether or not we have the homomorphism-like property

$$P_s + P_r = P_{r+s},$$

where

$$P_{r+s} := \{x\in\mathbb{R}^n \mid \langle\alpha,x\rangle \le r_j + s_j\}.$$

It is not hard to see that the inclusion $P_r + P_s \subset P_{r+s}$ holds, but the other direction is not true in general.

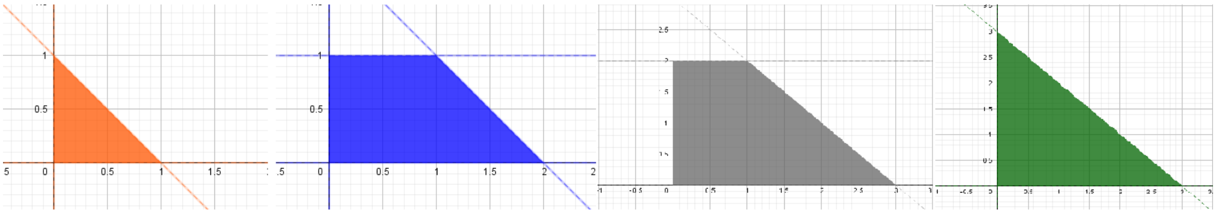


Figure 3.4: From left to right, P_r, P_s, $P_r + P_s$, P_{r+s}

Example 3.1.14. Let $\{\alpha_j\} = \{(-1,0),(0,-1),(1,1),(0,1)\}$, and let $\{r_j\} = \{0,0,1,2\}$ and $\{s_j\} = \{0,0,2,1\}$. Then $P_r + P_s \neq P_{r+s}$. (See the figure, 3.4)

In the Delzant case, however, we always have equality. We will not give a detailed proof, but do give an outline of how one would prove it. The idea is to represent line bundles of X_P by divisors, which may always be done in the Delzant case. One can show that if P_s and P_r define the same toric variety, then

$$L_{P_s+P_r} = L_{P_s} + L_{P_r} = L_{P_{r+s}}.$$

Further, one shows that if two Delzant P_r, P_s define the same X_P then P_{r+s} also defines X_P, and that if P_r and P_s are Delzant then they always define the same X_P. In more detailed terms we have the following result.

Theorem 3.1.15. *Let P_r, P_s be defined using the notation used above, and let both be Delzant. Then they give the same variety X_P, and P_{r+s} also gives the same variety X_P. In addition, we have both that*

$$L_{P_s} + L_{P_r} = L_{P_{s+r}} = L_{P_s} + L_{P_r},$$

and

$$P_r + P_s = P_{r+s}.$$

Using this we can establish a connection between the Khovanskii-Teissier inequality for X_P and Alexandrov-Fenchel inequality for P_i.

Theorem 3.1.16. *Let P_{s_i}, $1 \leq i \leq n$, be Delzant polytopes with the same normal vectors $\{\alpha_j\}_{j=1,\ldots,N}$.*

Then we have the following Alexandrov-Fenchel type inequality:

$$(V(P_{s_1},\ldots,P_{s_n}))^2 \geq V(P_{s_1},P_{s_1},P_{s_3},\ldots,P_{s_n})V(P_{s_2},P_{s_2},P_{s_3},\ldots,P_{s_n})$$

Proof. For a set of Delzant polytopes P_i, we define

$$L_{P_1}\cdots L_{P_n}.$$

Then the result follows by the proof of the previous theorem and the Khovanskii-Teissier inequality

$$(L_{P_1}\cdots L_{P_n})^2 \geq (L_{P_1}L_{P_1}L_{P_3}\cdots L_{P_n})(L_{P_2}L_{P_2}L_{P_3 P_n}).$$

□

3.2 Main result and proof

Theorem 3.2.1. *Let P, Q be Delzant polytopes defining the same Delzant variety X. Then we attain, for any tuple of Delzant polytopes $(A_1, \dots, A_{n-2})$, Alexandrov-Fenchel equality*

$$V(P, Q, A_1, \dots, A_{n-2})^2 = V(P, P, A_1, \dots, A_{n-2})V(Q, Q, A_1, \dots, A_{n-2})$$

if and only if P and Q are homothetic.

Proof. We handle first the $n = 2$. We will assume that $V(0) = V(1)$ as in the discussion of 2.2.7, 2.2.8. Using 2.2.7, 2.2.8, 2.2.6 and the fact that the Alexandrov-Fenchel inequality is equivalent to the Khovanskii-Teissier inequality in the Delzant case, we know that P and Q attaining Alexandrov-Fenchel equality is equivalent to

$$\left(t \mapsto -log\left(\int_X \omega^2\right)\right)'' = 0, \quad \omega = ti\Theta(L_P, h_P) + (1-t)i\Theta(L_Q, h_Q).$$

This again by lemma 3.4 of [3] is equivalent to saying, using their notation,

$$\|E_\mu(G) - G\|^2 = \|\theta\|^2, \quad \theta := i\Theta(L_P, h_P) - i\Theta(L_Q h_Q). \tag{3.1}$$

Note that since we are in the 2-dimensional case, $n = 2$, we have that $T = 1$ so the T-Hodge theory norm is just the standard norm. So we know that $u := E_\mu(G) - G$ is the L^2-minimal solution of

$$d(\cdot) = (d^c)^*(\theta).$$

Using the Hodge-decomposition we can split θ into its harmonic and minimal parts, $\theta = \theta_{min} + \theta_{harm}$, and since it is an orthogonal decomposition we get

$$\|\theta\| = \|\theta_{min}\| + \|\theta_{harm}\|.$$

Then we have

$$\|\theta\| = \|u\| = \|u_{min}\| = \|\theta_{min}\|.$$

So we get that 3.1 is equivalent to $\|\theta_{harm}\| = 0 \iff \theta_{harm} = 0$. By the $\partial\overline{\partial}$-lemma, this implies that $\theta = dd^c(G)$ for some smooth globally defined G. Note that G must be bounded since X_P is compact.

On each toric embedding, we have $dd^c(G) = \theta = dd^c(\phi_P - \phi_Q)$, where $\nabla\phi_P(\mathbb{R}^n) = P$, $\nabla\phi_Q(\mathbb{R}^n) = Q$. In other words,

$$dd^c(\phi_P - \phi_Q - G) = 0.$$

So $dd^c(\phi_P - \phi_Q - G)$ is simultaneously plurisubharmonic and plurisuperharmonic. On each toric embedding

$$(\mathbb{C}^*)^n \simeq \mathbb{R}^n \times \mathbb{T}^n$$

with coordinates $(x+iy)$, we have that $\phi_P - \phi_Q - G$ is only dependent on x. So plurisubharmonicity and plurisupeharmonicity are equivalent to convexity and concavity respectively. Then what we have shown so far is that, up to scaling of each of $dd^c(\phi_P)$ and $dd^c(\phi_Q)$, the function

$$(\phi_P - \phi_Q - G)(x)$$

is affine and therefore equal to $\langle \alpha, x\rangle + \beta$ for some fixed α, β. Then we get

$$\nabla(\phi_P - \phi_Q - G) = \nabla(\langle \alpha, x\rangle + \beta) = \alpha.$$

Since ϕ_P and ϕ_Q differ by only a bounded function we get that $\nabla\phi_P(\mathbb{R}^n) = \nabla\phi_Q(\mathbb{R}^n) + \alpha$. Taking into consideration the scaling assumption we made at the start, we have shown what we needed. In other words

$$Q = kP + \alpha,$$

for some scalar $k \in \mathbb{R}$.

Increasing n and again assuming $V(0) = V(1)$ we still only need to solve for

$$\left(t \mapsto -log\left(\int_X \omega^2 \wedge T\right)\right)'' = 0,$$

where $T := (i\Theta(L_{A_1}, h_{A_1}) \wedge \cdots \wedge i\Theta(L_{A_{n-2}}, h_{A_{n-2}})$. Now note that by 2.2.6, this is equivalent to solving for

$$\|T \wedge u\| = \|T \wedge \theta\|.$$

And as in the $n = 2$ case we get

$$\|T \wedge u\|_{min} = \|T \wedge u\| = \|T \wedge \theta\|_{min} \implies \|T \wedge \theta\|_{harm} = 0.$$

And again by the $\partial\overline{\partial}$-lemma we have that $\theta \wedge T$ is d-exact. So we get $\theta \wedge T = d(u \wedge T)$, and by the Leibniz rule and $dT = 0$ we get

$$\theta \wedge T - d(u \wedge T) = (\theta - du) \wedge T = 0.$$

Finally, by the Hard Lefschetz theorem for T-Hodge theory 1.2.2, we get $\theta - du = 0$. So we have showed that θ is d-exact, and therefore also dd^c-exact. Then the result follows from the proof of the $n = 2$ case. □

3.3 Closing remarks

Our main result is by no means novel. In fact, a relatively recent paper by van Handel and Shenfeld [11] classifies the equality case in a much more general setting. With the notation in our main result, they solved for equality when P, Q are any two convex bodies, and $A_1, \ldots, A_{n-2}$ are convex polytopes (not necessarily Delzant). The methods in this thesis however differ significantly from theirs. One possible avenue for furthering the methods in this thesis is to consider not only Delzant polytopes. Instead, we would look at $V(P_1, \dots, P_n)$ where there are convex bodies $A_1, \dots, A_N$ such that each P_j can be given as

$$P_j = \sum_{i=1}^{N} t_{ij} A_i, \quad t_{ij} > 0.$$

We conjecture that in this case, a similar result to our main result holds. That is to say

$$V(P_1, \dots, P_n)^2 = V(P_1, P_1, P_3, \dots, P_n) V(P_2, P_2, P_3, \dots, P_n) \iff P_1, P_2 \textit{ are homothetic}.$$

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