Amund Skretting Bergset

Towards a theory for fractional Mean Field Games in Hölder spaces

Master's thesis in Industrial Mathematics
Supervisor: Espen R. Jakobsen
Co-supervisor: Artur Rutkowski
June 2023

Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

Amund Skretting Bergset

Towards a theory for fractional Mean Field Games in Hölder spaces

Master's thesis in Industrial Mathematics
Supervisor: Espen R. Jakobsen
Co-supervisor: Artur Rutkowski
June 2023

Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of Science and Technology

Abstract

In this master's thesis, we study a Mean Field Game system in the whole space driven by a fractional Laplacian $-(-\Delta)^{\alpha/2}$ of order $\alpha \in (1,2)$. We prove existence and uniqueness of classical solutions to the Hamilton-Jacobi-Bellman and Fokker-Planck equations, and discuss how our results contribute to the study of the coupled Mean Field Game system. Unlike previous work, we assume Hölder continuous initial and source terms, and provide improved spatial regularity estimates for our solutions. The proofs use a combination of fixed point arguments on a Duhamel map, fractional heat kernel estimates, interpolation in Hölder spaces and comparison principles.

Sammendrag

I denne masteroppgaven studerer vi et Mean Field Game-system i hele rommet drevet av en fraksjonell Laplace-operator $-(-\Delta)^{\alpha/2}$ med orden $\alpha \in (1,2)$. Vi beviser eksistens og entydighet av klassiske løsninger til Hamilton-Jacobi-Bellman- og Fokker-Planck-ligningene, og diskuterer hvordan resultatene våre bidrar til å studere det koblede Mean Field Game-systemet. Til forskjell fra tidligere arbeid antar vi Hölderkontinuerlig initial- og randdata, og presenterer forbedrede romlige regularitetsestimater for løsningene våre. Bevisene benytter en kombinasjon av fikspunktargumenter på Duhamelavbildninger, estimater for den fraksjonelle varmekjernen, interpolasjon i Hölderrom og sammenligningsprinsipper.

Preface

This thesis is the result of my work in *TMA4900 - Industrial Mathematics, Master's Thesis*, and concludes my Master's degree in Applied Physics and Mathematics at the Norwegian University of Science and Technology (NTNU). I would like to thank my supervisors Espen R. Jakobsen and Artur Rutkowski for their invaluable guidance and support. This thesis would not have been the same without either of them. I would also like to thank my fellow students for making the last five years truly memorable.

Amund Skretting Bergset

June 2023

Trondheim

Table of Contents

1 Introduction

Game theory is a branch of mathematics that aims to model strategic interaction between rational agents. The field was first introduced by John von Neumann in 1928 [23], and has since seen numerous advancements both mathematically and application-wise. A particularly noteworthy contribution was made by John Nash who explored the concept of equilibria in non-cooperative games in 1951 [19]. Several applications have since emerged within fields such as social sciences and economics.

In classical game theory, each agent's action depends on the behavior of all other agents. This causes the number of interactions to grow rapidly as we increase the population, and computational approaches become unviable. A natural idea that arises is to look at the agents from a statistical point of view instead. Since each agent has negligible impact upon a large population system, we choose to view the agents as a statistical distribution. This is the main idea behind Mean Field Games (MFGs) which was introduced by Jean-Michel Lasry and Pierre-Louis in 2006 [17], and almost simultaneously by Huang, Malhamé and Caines [11].

The MFG system consists of a Hamilton-Jacobi-Bellman (HJB) equation, essentially an optimal control problem, coupled with a Fokker-Planck (FP) equation representing the distribution of agents.

$$\begin{cases} -\partial_t v - \varepsilon \Delta v + H\left(x, m, Dv\right) = F\left(m\left(t\right), x\right) & \text{in } \left[0, T\right) \times \mathbb{R}^d, \\ v\left(T, \cdot\right) = G\left(m\left(T\right), \cdot\right) & \text{in } \mathbb{R}^d, \\ \partial_t m - \varepsilon \Delta m + \nabla \cdot \left(D_p H\left(x, m, Dv\right) m\right) = 0 & \text{in } \left(0, T\right] \times \mathbb{R}^d, \\ m\left(0, \cdot\right) = m_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{1}$$

Much of the early work studied variations of (1), with particular emphasis on existence and uniqueness of classical solutions [1, 5, 17]. There has since been a variety of generalizations. Of particular interest to us are the Mean Field Games with nonlocal fractional operators in the work by Olav Ersland and Espen R. Jakobsen [8]. Instead of only working with the Laplace operator Δ, they showed existence and uniqueness of classical solutions to a large class of fractional MFGs. Operators considered were of order $\alpha \in (1, 2)$, and initial and source terms were imposed with C_b^k and $W^{k,\infty}$-type assumptions in space.

The core of this thesis is to study MFGs with less regular initial and source terms. We consider a special case of the nonlocal coupling system in [8] of the form

$$\begin{cases} -\partial_t v + \left(-\Delta\right)^{\frac{\alpha}{2}} v + H\left(Dv\right) = F\left(m\left(t\right), x\right) & \text{in } \left[0, T\right) \times \mathbb{R}^d, \\ v\left(T, \cdot\right) = G\left(m\left(T\right), \cdot\right) & \text{in } \mathbb{R}^d, \\ \partial_t m + \left(-\Delta\right)^{\frac{\alpha}{2}} m + \nabla \cdot \left(D_p H\left(Dv\right) m\right) = 0 & \text{in } \left(0, T\right] \times \mathbb{R}^d, \\ m\left(0, \cdot\right) = m_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2}$$

where $\alpha \in (1, 2)$. The spatial regularity of the solution (v, m) will be closely related to the operator and assumptions imposed on the source term. For the HJB equation, we expect v to be an order of α more regular in space than F. Similarly, m is expected to be an order of $(\alpha - 1)$ more regular than the $D_p H(Dv)$-term. Since we are working with a fractional operator, however, we will not be able to express the entire regularity through C_b^k-spaces. This is nicely highlighted in [8], where we go from $F(m, \cdot) \in C_b^2(\mathbb{R}^d)$ to $v(t, \cdot) \in C_b^3(\mathbb{R}^d)$, hence only gaining an order of 1 in terms of spatial regularity in the HJB equation. In order to utilize the operator to its fullest, we therefore need a new way of looking at spatial regularity. This motivates our main deviation from earlier work, namely the introduction of Hölder spaces, which is a generalization of the C_b^k-spaces (see sections 2.1 and 2.8).

The main part of the text is dedicated to proving existence and uniqueness of classical solutions to the HJB and FP equations. We view the equations seperately, and finish the thesis by discussing how these results contribute to the study of the coupled MFG system in (2). Main results include

(i) *Existence and uniqueness of classical solutions to the HJB equation, given spatial β-Hölder continuity of initial and source terms where $\beta \in (0, 1)$. The resulting solution is $(\alpha + \beta - \varepsilon)$-Hölder continuous in space for any $\varepsilon > 0$. We refer to Chapter 3.*

(ii) *Existence and uniqueness of very weak solutions to the FP equation, given ν-Hölder continuous initial data where $\nu \in (0,1)$, and spatial μ-Hölder continuity of the drift term where $\mu \in (0,\alpha)$. The resulting solution is $(\alpha+\mu-\varepsilon-1)$-Hölder continuous in space for any $\varepsilon > 0$, and classical whenever $\mu > 1$. We refer to Chapter 4.*

Existence results are mainly tackled through fixed point arguments, using a combination of fractional heat kernel estimates and interpolation in Hölder spaces. Uniqueness for the HJB equation is shown through the comparison principle, whereas positivity and mass preservation properties are used in the Fokker-Planck case. The proofs in the thesis will closely follow [8], but differ in some key aspects which we will briefly review.

As we only assume Hölder continuity on the terminal data $G(m(T),\cdot)$ in (2), Dv will not exist at time $t = T$. This requires us to impose a global Lipschitz condition on $H(Dv)$, since we lose control over its function argument close to the terminal time. When tackling uniqueness results for the FP equation, we work with so-called very weak solutions, drawing inspiration from [13]. This is due to insufficiency of the corresponding proof in [8] when considering less regular initial and source terms. Lastly, we comment on the choice of operator. Although we are only working with the fractional Laplacian $-(-\Delta)^{\alpha/2}$, the majority of our assumptions hold for all operators considered in [8] (see sections 2.5 and 2.6). Results in the thesis may therefore hold for these operators as well, but this requires further exploration.

The HJB equation was originally treated in the present author's project thesis [2] for the case where $\alpha+\beta \geqslant 2$. In order to consider $\alpha+\beta < 2$ as well, we have heavily revised our approach and present mostly new proofs in Chapter 3. Much of the preliminary material is also from the project thesis, and we will specify where this is the case later in the text.

The structure of the thesis is as follows:

- Chapter 2 presents the preliminary material. This includes technical results in Hölder spaces, fractional heat kernel estimates and well-known results from analysis and measure theory.
- Chapter 3 contains existence and uniqueness results for classical solutions to the Hamilton-Jacobi-Bellman equation. We also present spatial Hölder regularity estimates for this solution.
- Chapter 4 treats similar results for very weak solutions to the Fokker-Planck equation. We also prove positivity and L^1-regularity of the solution, and study when it is classical.
- Chapter 5 discusses further work and how results in the thesis contribute to the study of the coupled Mean Field Game system.

2 Preliminary Material

Before we begin with our main analysis, we give a review of preliminary material. This chapter is mainly from the present author's project thesis [2], and we will therefore omit most of the proofs. Particular focus is placed upon new results, including the generalized Grönwall inequality (Lemma 2.11) and some of the technical results for Hölder seminorms in Section 2.4.

2.1 Spaces and norms

We begin by defining spaces and norms that will be referred to frequently throughout the text.

Definition 2.1 (The spaces C^s and C^s_b). *For $s \in \mathbb{N}_0$, we denote by $C^s\left(\mathbb{R}^d\right)$ the space of continuous functions on $\mathbb{R}^d$ with s continuous derivatives. Furthermore, we define*

$$C^s_b\left(\mathbb{R}^d\right) := \left\{g \in C^s\left(\mathbb{R}^d\right) : \|g\|_{C^s(\mathbb{R}^d)} < \infty\right\}, \tag{3}$$

where the $\|\cdot\|_{C^s(\mathbb{R}^d)}$-norm is defined as

$$\|g\|_{C^s(\mathbb{R}^d)} := \|g\|_{L^\infty(\mathbb{R}^d)} + \sum_{j=1}^{s} [g]_{C^j(\mathbb{R}^d)} \quad \text{where} \quad [g]_{C^j(\mathbb{R}^d)} := \max_{|k|=j} \|D^k g\|_{L^\infty(\mathbb{R}^d)}, \tag{4}$$

and k is a multi-index, meaning that

$$D^k g := \frac{\partial^{|k|} g}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_d^{k_d}} \quad \text{where} \quad |k| = k_1 + k_2 + ... + k_d. \tag{5}$$

The spaces above provide us with important information about the regularity of a function and its derivatives. We are able to derive even finer regularity results by introducing the notion of Hölder continuity, which is stronger than uniform continuity but weaker than Lipschitz.

Definition 2.2 (Hölder space and seminorm). *For $s \in \mathbb{N}_0$ and $\beta \in (0,1)$, we define the Hölder space*

$$C^{s,\beta}\left(\mathbb{R}^d\right) := \left\{g \in C^s_b\left(\mathbb{R}^d\right) : \|g\|_{C^s(\mathbb{R}^d)} + [g]_{C^{s,\beta}(\mathbb{R}^d)} < \infty\right\}, \tag{6}$$

where the Hölder seminorm is defined as

$$[g]_{C^{s,\beta}}\left(\mathbb{R}^d\right) = \max_{|k|=s} \left[D^k g\right]_{C^{0,\beta}(\mathbb{R}^d)} \quad \text{where} \quad [g]_{C^{0,\beta}(\mathbb{R}^d)} = \sup_{\substack{x,h\in\mathbb{R}^d \\ h\neq 0}} \frac{|g(x+h) - g(x)|}{|h|^\beta}. \tag{7}$$

We say that a function g is β-Hölder continuous if $[g]_{C^{0,\beta}(\mathbb{R}^d)}$ is finite. All Hölder spaces are Banach spaces, as stated in Section 3.1 in [15]. Intuitively, the Hölder spaces work as a continuation of the C^s_b-spaces, which is nicely highlighted through the following embedding theorem.

Theorem 2.3 (Embedding theorem for Hölder spaces). *Let $s \in \mathbb{N}_0$ and $\beta, \mu \in (0,1)$ such that $\beta < \mu$. Then,*

$$C^{s,\beta}\left(\mathbb{R}^d\right) \subset C^{s,\mu}\left(\mathbb{R}^d\right) \quad \text{and} \quad C^{s,\beta}\left(\mathbb{R}^d\right) \subset C^{s+1}_b\left(\mathbb{R}^d\right).$$

Proof. The result is proven in Theorem 2.3 in [2]. □

The embedding results above give us a key understanding of how Hölder spaces of different orders relate to each other. We will investigate these relations further in Section 2.8 where we look at interpolation inequalities for the Hölder seminorms. In order to simplify our calculations later in the text, we introduce some remarks regarding notation.

Remark 2.4 (Notation). *For simplicity, we will refer to spaces C^γ for a noninteger $\gamma > 0$. By this, we mean the space $C^{s_\gamma,\beta_\gamma}$ where $\gamma = s_\gamma + \beta_\gamma$ such that $s_\gamma \in \mathbb{N}_0$ and $\beta_\gamma \in (0,1)$. Similarly, we let $[\cdot]_{C^\gamma(\mathbb{R}^d)} := [\cdot]_{C^{s_\gamma,\beta_\gamma}(\mathbb{R}^d)}$. When working with Hölder seminorms over $\mathbb{R}^d$, we simplify our notation by letting $[\cdot]_{C^{s,\beta}} := [\cdot]_{C^{s,\beta}(\mathbb{R}^d)}$. Similarly, we let $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^d)}$. Furthermore, for any $s \in \mathbb{N}$, we use the conventions $\|D^s \cdot\|_\infty := [\cdot]_{C^s}$ and $[D^s \cdot]_{C^{0,\beta}} := [\cdot]_{C^{s,\beta}}$.*

2.2 The Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman equation (HJB) is a nonlinear PDE arising from optimal control problems, and serves as the first of two equations in the Mean Field Game system (2). We will throughout the text consider a specific version of this equation driven by a fractional Laplacian $-(-\Delta)^{\alpha/2}$. In addition, we assume that the Hamiltonian H only depends on the derivative of the solution.

Definition 2.5 (The Hamilton-Jacobi-Bellman equation (HJB)). *For some terminal time $T > 0$ and $\alpha \in (1, 2)$, we consider the equation*

$$\frac{\partial}{\partial t} v(t,x) + (-\Delta)^{\frac{\alpha}{2}} v(t,x) + H(Dv(t,x)) = f(t,x) \qquad \text{in } (0,T] \times \mathbb{R}^d, \tag{8a}$$

$$v(0,x) = v_0(x) \qquad \text{in } \mathbb{R}^d. \tag{8b}$$

We refer to $f(t,x)$ as the *source term*. Working with this PDE is difficult for multiple reasons. Since the Hamiltonian is general and only subject to regularity assumptions, its dependence on Dv makes us unable to solve the equation explicitly. Furthermore, the fractional Laplacian $-(-\Delta)^{\alpha/2}$ is a nonlocal operator and has no simple explicit definition in the real space. We discuss the fractional Laplacian in detail in Section 2.5.

2.3 The Fokker-Planck equation

The second equation in the Mean Field Game system is the Fokker-Planck equation (FP), which describes the time evolution of a probabilistic distribution.

Definition 2.6 (The Fokker-Planck equation (FP)). *For some terminal time $T > 0$ and $\alpha \in (1, 2)$, we consider the equation*

$$\frac{\partial}{\partial t} m(t,x) + (-\Delta)^{\frac{\alpha}{2}} m(t,x) + \nabla \cdot (b(t,x)\, m(t,x)) = 0 \qquad \text{in } (0,T] \times \mathbb{R}^d, \tag{9a}$$

$$m(0,x) = m_0(x) \qquad \text{in } \mathbb{R}^d. \tag{9b}$$

We refer to $b(t,x)$ as the *drift* term. This PDE is easier to deal with than the HJB equation, due to the abscence of the Hamiltonian term. There are, however, some additional considerations to be taken. Since $m(t,\cdot)$ is generally viewed as a probabilistic distribution, we need to ensure positivity and L^1-regularity of possible solutions.

2.4 Some central results

In this section, we recall some well-known results from analysis and measure theory. A core ingredient in our analysis is Banach's fixed point theorem, which will be used when deriving existence results for the HJB and FP equations in sections 3.1 and 4.1.

Theorem 2.7 (Banach's fixed point theorem (Theorem 5.1 in [9])). *Let $(X, \|\cdot\|_X)$ denote a real non-empty Banach space, and let $\phi : X \to X$ be a contraction mapping, that is, there exists $L < 1$ such that for any $u, w \in X$,*

$$\|\phi(u) - \phi(w)\|_X \leqslant L\|u - w\|_X.$$

Then, ϕ has a unique fixed point, meaning that $\exists! v \in X$ such that $\phi(v) = v$.

We proceed by introducing two well-known results for interchanging limits and integrals, namely the Fubini-Tonelli theorem and Lebesgue's dominated convergence theorem. Both results can be found in [3], and will prove useful when showing existence of classical solutions to the HJB and FP equations in sections 3.3 and 4.4.

Theorem 2.8 (The Fubini-Tonelli theorem). *Let* $X \subseteq \mathbb{R}^{d_1}$ *and* $Y \subseteq \mathbb{R}^{d_2}$ *for some* $d_1, d_2 \in \mathbb{N}$. *If* f *is a measurable function, it follows that*

$$\int_X \left(\int_Y |f(x,y)|\, dy \right) dx = \int_Y \left(\int_X |f(x,y)|\, dx \right) dy = \int_{X \times Y} |f(x,y)|\, d(x,y).$$

Furthermore, if any of these integrals are finite, then

$$\int_X \left(\int_Y f(x,y)\, dy \right) dx = \int_Y \left(\int_X f(x,y)\, dx \right) dy = \int_{X \times Y} f(x,y)\, d(x,y).$$

Theorem 2.9 (Lebesgue's dominated convergence theorem). *Let* $X \subseteq \mathbb{R}^{d_1} \to \mathbb{R}$ *for some* $d_1 \in \mathbb{N}$ *and let* $f : X \to \mathbb{R}$. *Suppose that* (f_n) *is a sequence of measurable functions on* X *that converges pointwise to* f *almost everywhere, and that there exists* $g : X \to \mathbb{R}$ *such that* $|f_n(x)| \leqslant g(x)$ *a.e. for all* n *in the index set, and*

$$\int_X |g(x)|\, dx < \infty.$$

Then, it follows that

$$\lim_{n \to \infty} \int_X f_n(x)\, dx = \int_X f(x)\, dx.$$

The next result provides us with a useful relation between Lipschitz continuity and differentiability.

Theorem 2.10 (Rademacher's theorem (Theorem 2.2.4 in [22])). *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a Lipschitz continuous function. Then,* f *is totally differentiable at* x_0 *for almost every* $x_0 \in \mathbb{R}^d$.

Lastly, we introduce a generalized version of the Grönwall inequality. The result is inspired by Corollary 1 in [24], and will be used when deriving regularity results for the HJB equation in Section 3.2. We present it slightly different from [24], and will therefore give a proof.

Lemma 2.11 (Generalized Grönwall inequality). *Let* $a_0, a_{T_0}, c \geqslant 0$, $\gamma, \zeta < 1$ *and* $T_0 > 0$. *Suppose that* $u(t)$ *is nonnegative and locally integrable on* $[0, T_0]$, *and that for any* $t \in [0, T_0]$,

$$u(t) \leqslant a_0 t^{-\gamma} + a_{T_0} + c \int_0^t (t-s)^{-\zeta} u(s)\, ds.$$

Then, there exist constants $b_0, b_{T_0} \geqslant 0$ *independent of* t *such that* $u(t) \leqslant b_0 t^{-\gamma} + b_{T_0}$ *for any* $t \in [0, T_0]$.

Proof. Notice that $a_0 t^{-\gamma} + a_{T_0}$ is nonnegative and locally integrable on $[0, T_0]$. By Corollary 1 in [24], we then have that

$$u(t) \leqslant a_0 t^{-\gamma} + a_{T_0} + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(c\Gamma(1-\zeta))^n}{\Gamma(n(1-\zeta))} (t-s)^{n(1-\zeta)-1} \left(a_0 s^{-\gamma} + a_{T_0} \right) \right) ds, \quad 0 \leqslant t < T_0. \tag{10}$$

It suffices to show that the integral is finite and uniformly bounded in time for $t \in [0, T_0]$. Notice that

$$(t-s)^{n(1-\zeta)-1} = \left((t-s)^{1-\zeta} \right)^{n-1} (t-s)^{-\zeta} \leqslant \left(T_0^{1-\zeta} \right)^{n-1} (t-s)^{-\zeta}, \quad \forall n \geqslant 1,$$

where the inequality holds since $(1-\zeta)(n-1) \geqslant 0$. The gamma function $\Gamma(z)$ is positive whenever $0 < z \in \mathbb{R}$. This follows quite directly from its definition (see [14]). By $1 - \zeta > 0$ and $n(1-\zeta) > 0$, we then get that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(c\Gamma(1-\zeta))^n}{\Gamma(n(1-\zeta))} (t-s)^{n(1-\zeta)-1} \leqslant c\Gamma(1-\zeta)(t-s)^{-\zeta} \sum_{n=1}^{\infty} \frac{\left(c\Gamma(1-\zeta) T_0^{1-\zeta} \right)^{n-1}}{\Gamma(n(1-\zeta))} \\ &= c\Gamma(1-\zeta)(t-s)^{-\zeta} E_{1-\zeta, 1-\zeta}\left(c\Gamma(1-\zeta) T_0^{1-\zeta} \right) \leqslant c\Gamma(1-\zeta)(t-s)^{-\zeta} C_{E, T_0}. \end{aligned} \tag{11}$$

Here, $E_{1-\zeta,1-\zeta}$ is the two-parametric Mittag-Leffler function, which is an entire function in $\mathbb{C}$ of order $1/(1-\zeta)$ (see [10], Section 4). It is therefore finite at any $z \in \mathbb{R}$, and we can bound it by some $C_{E,T_0} \geqslant 0$ at the particular point where it is evaluated. Inserting (11) into (10) yields

$$\begin{aligned} u(t) &\leqslant a_0 t^{-\gamma} + a_{T_0} + C_{E,T_0} c\Gamma(1-\zeta) \int_0^t (t-s)^{-\zeta} \left(a_0 s^{-\gamma} + a_{T_0}\right) ds \\ &= a_0 t^{-\gamma} + a_{T_0} + C_{E,T_0} c\Gamma(1-\zeta) \left(a_0 t^{1-\zeta-\gamma} \int_0^1 (1-r)^{-\zeta} r^{-\gamma} ds + a_{T_0} t^{1-\zeta} \int_0^1 (1-r)^{-\zeta} ds \right), \end{aligned}$$

where we have used the substitution $r = s/t$. Since all exponents on r and $1-r$ are greater than -1, we have integrability. Hence, there exist constants $C_1, C_2 \geqslant 0$ independent of t such that

$$u(t) \leqslant a_0 t^{-\gamma} + a_{T_0} + C_1 t^{1-\zeta-\gamma} + C_2 t^{1-\zeta} \leqslant a_0 t^{-\gamma} + a_{T_0} + C_1 T_0^{1-\zeta} t^{-\gamma} + C_2 T_0^{1-\zeta},$$

where we have let $t \to T_0$ since $1-\zeta \geqslant 0$. By letting $b_0 := a_0 + C_1 T_0^{1-\zeta}$ and $b_{T_0} := a_{T_0} + C_2 T_0^{1-\zeta}$, the proof is complete. □

2.5 The fractional Laplacian

We proceed by familiarizing ourselves with the fractional Laplacian, written $-(-\Delta)^{\alpha/2}$ where $\alpha \in (1,2)$. Intuitively, this operator can be thought of as a generalization of the ordinary Laplacian operator Δ, in the sense that it extends the notion of spatial derivatives to fractal powers. Unlike the ordinary Laplacian, $-(-\Delta)^{\alpha/2}$ is a nonlocal operator. This means that for any function f and $x \in \mathbb{R}^d$, we cannot determine $-(-\Delta)^{\alpha/2} f(x)$ solely by looking at some neighborhood of x. Instead, we need to consider function values $f(y)$ for all $y \in \mathbb{R}^d$, which makes it hard to define the operator explicitly. A summary of some of the existing definitions can be found in [16], where the complexity of defining the operator is nicely highlighted. We will define the fractional Laplacian in two seperate ways, as a Fourier multiplier and as a singular integral. Both definitions are stated and shown equivalent to each other in [16].

Definition 2.12 (The fractional Laplacian (Fourier definition))**.** *For $\alpha \in (1,2)$ and a sufficiently regular function f, we define the fractional Laplacian $-(-\Delta)^{\alpha/2}$ as the operator satisfying the following relation in Fourier space:*

$$\mathcal{F}\left\{-(-\Delta)^{\frac{\alpha}{2}} f\right\}(\xi) = -|\xi|^{\alpha} \mathcal{F}\{f\}(\xi). \tag{12}$$

The consistency with the Fourier transform of the ordinary Laplacian becomes evident by letting $\alpha = 2$ above. We will use this definition when working with the fractional heat kernel in the next section.

Definition 2.13 (The fractional Laplacian (Singular integral definition))**.** *For $\alpha \in (1,2)$, we define the fractional Laplacian $-(-\Delta)^{\alpha/2}$ applied to some function f on $\mathbb{R}^d$ by the singular integral*

$$-(-\Delta)^{\frac{\alpha}{2}} f(x) = \lim_{r \to 0^+} \int_{\mathbb{R}^d \setminus B_r(0)} (f(x+z) - f(x)) \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz, \tag{13}$$

where $B_r(0) := \left\{x \in \mathbb{R}^d : |x| < r\right\}$, and $c_{d,\alpha} > 0$ is a constant only depending on d and α.

We can also write the singular integral in terms of second-order differences. This follows quite easily by considering the radial symmetry of the domain $\mathbb{R}^d \setminus B_r(0)$, and is inspired by Definition 2.7 in [16]. The second-order difference representation will only be used in combination with Definition 2.13 for specific subdomains of $\mathbb{R}^d \setminus B_r(0)$. We will therefore only formulate the equivalence in domains on the form $B_{r,R}(0) = B_R(0) \setminus B_r(0)$. It can, however, be extended to hold in the entire $\mathbb{R}^d \setminus B_r(0)$.

Proposition 2.14. *Let f be a function on $\mathbb{R}^d$ and let $0 < r < R < \infty$. Then,*

$$\int_{B_{r,R}(0)} (f(x+z) - f(x)) \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz = \frac{1}{2} \int_{B_{r,R}(0)} (f(x+z) - 2f(x) + f(x-z)) \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz, \tag{14}$$

where $B_{r,R}(0) = B_R(0) \setminus B_r(0)$.

Proof. The result is proven in Proposition 2.8 in [2]. □

As for the ordinary Laplacian and derivatives in general, the finiteness of $-(-\Delta)^{\alpha/2} f(x)$ is highly connected to the regularity of f. In order for $-(-\Delta)^{\alpha/2}$ to be consistent with the ordinary Laplacian, we expect $-(-\Delta)^{\alpha/2} f$ to be finite whenever $f \in C^{\alpha}(\mathbb{R}^d)$. A slightly weaker result is shown in [2], requiring that f is $(\alpha+\delta)$-Hölder continuous for some small $\delta > 0$. We present a somewhat rewritten result, where we include an explicit bound for the L^∞-norm of $-(-\Delta)^{\alpha/2} f$.

Proposition 2.15. *Let $f \in C^{\alpha+\delta}(\mathbb{R}^d)$ for $\alpha \in (1,2)$ and $\delta > 0$ where $\alpha+\delta < 2$. Then, $-(-\Delta)^{\frac{\alpha}{2}} f \in L^\infty(\mathbb{R}^d)$. Furthermore, there exist constants $C_1, C_2 \geqslant 0$ such that*

$$\left\| -\left(-\Delta^{\frac{\alpha}{2}}\right) f \right\|_\infty \leqslant C_1 \|f\|_\infty + C_2 [f]_{C^{\alpha+\delta}}. \tag{15}$$

Proof. The proof follows quite directly from the singular integral definition in (13). Note that

$$\left| -\left(-\Delta^{\frac{\alpha}{2}}\right) f(x) \right| \leqslant \lim_{r\to 0^+} \int_{\mathbb{R}^d \setminus B_r(0)} |f(x+z) - f(x)| \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz.$$

By the definition of Hölder spaces, $f \in C^{\alpha+\delta}(\mathbb{R}^d)$ implies $f \in C_b(\mathbb{R}^d)$. Then, $\|f\|_\infty$ and $[f]_{C^{\alpha+\delta}}$ are finite. By splitting the integral above into integrals over $\mathbb{R}^d \setminus B_1(0)$ and $B_{r,1}(0)$, and using Proposition 2.14 on the integral over the latter domain, we get that

$$\begin{aligned} \left| -\left(-\Delta^{\frac{\alpha}{2}}\right) f(x) \right| \leqslant & \int_{\mathbb{R}^d \setminus B_1(0)} |f(x+z) - f(x)| \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz \\ & + \lim_{r\to 0^+} \frac{1}{2} \int_{B_{r,1}(0)} |f(x+z) - 2f(x) + f(x-z)| \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz. \end{aligned} \tag{16}$$

Since f is continuous, and by the mean value theorem, there exists some $\xi_{x,z} \in (x-z, x)$ such that

$$f(x+z) - 2f(x) + f(x-z) = |z| \left(\nabla_z f(\xi_{x,z} + z) - \nabla_z f(\xi_{x,z})\right),$$

where $\nabla_z f$ denotes the directional derivative of f along the z-direction. By $\alpha+\delta < 2$, we get that

$$\frac{|f(x+z) - 2f(x) + f(x-z)|}{|z|^{d+\alpha}} \leqslant \frac{|\nabla_z f(\xi_{x,z}+z) - \nabla_z f(\xi_{x,z})|}{|z|^{d+\alpha-1}} \leqslant d \frac{[Df]_{C^{0,\alpha+\delta-1}}}{|z|^{d-\delta}} = d \frac{[f]_{C^{\alpha+\delta}}}{|z|^{d-\delta}}.$$

We can now estimate (16) by

$$\left| -\left(-\Delta^{\frac{\alpha}{2}}\right) f(x) \right| \leqslant 2\|f\|_\infty \int_{\mathbb{R}^d \setminus B_1(0)} \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz + \lim_{r\to 0^+} \frac{d}{2} [f]_{C^{\alpha+\delta}} \int_{B_1(0)\setminus B_r(0)} \frac{c_{d,\alpha}}{|z|^{d-\delta}} dz. \tag{17}$$

Since we integrate in d dimensions, the second integral will not attain a singularity at $z = 0$, and will be finite in the limit $r \to 0^+$. Furthermore, since $\alpha > 0$, the first integral is also finite. This follows by

$$\int_{\mathbb{R}^d \setminus B_1(0)} \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz = A_d \int_1^\infty \frac{c_{d,\alpha}}{\rho^{d+\alpha}} \rho^{d-1} d\rho = \frac{A_d c_{d,\alpha}}{\alpha},$$

where A_d is the surface area of the d-dimensional unit sphere. We can then take the supremum over $x \in \mathbb{R}^d$ in (17) to deduce that there exist constants $C_1, C_2 \geqslant 0$ such that (15) holds. It follows directly that $-(-\Delta)^{\alpha/2} f \in L^\infty(\mathbb{R}^d)$. □

We finish the section with a result on how the fractional Laplacian behaves in global maxima of a function. The result will be used in Section 3.4, where we use comparison principles to show uniqueness of solutions to the HJB equation.

Proposition 2.16 (The fractional Laplacian in a global maximum)**.** *Let $\alpha \in (1,2)$ and let $f \in C^{\alpha+\delta}(\mathbb{R}^d)$ for some small $\delta > 0$. Suppose that $x_0 \in \mathbb{R}^d$ is a global maximum of f. Then, $-(-\Delta)^{\frac{\alpha}{2}} f(x_0) \leqslant 0$.*

Proof. The result is proven in Proposition 2.10 in [2]. □

2.6 The fractional heat kernel

Since much of our analysis involves the fractional Laplacian, it seems useful to introduce its fundamental solution. This is called the fractional heat kernel, and we define it implicitly through an initial value problem.

Definition 2.17 (The fractional heat kernel)**.** *The fractional heat kernel for the operator* $-(-\Delta)^{\frac{\alpha}{2}}$*, where* $\alpha \in (1,2)$*, is the function* $K(t,x)$ *solving the initial value problem*

$$\partial_t K(t,x) = -(-\Delta)^{\frac{\alpha}{2}} K(t,x), \tag{18a}$$

$$K(0,x) = \delta(0), \tag{18b}$$

where $\delta(x)$ *is the Dirac measure.*

Since the initial value problem above includes a fractional Laplacian, it is not possible to define $K(t,x)$ explicitly in real space. This is because we lack an explicit definition of $-(-\Delta)^{\alpha/2}$, as discussed in the last section. We can, however, define the fractional heat kernel explicitly on the Fourier side, by using the Fourier definition of the fractional Laplacian.

Lemma 2.18 (The Fourier transform of the fractional heat kernel)**.** *The Fourier transform* $\widehat{K}(t,\xi)$ *of the fractional heat kernel* $K(t,x)$ *in* (18) *with respect to* x *is*

$$\hat{K}(t,\xi) = e^{-t|\xi|^\alpha}. \tag{19}$$

Proof. By the definition of the Fourier transform, (18b) yields $\hat{K}(0,\xi) = 1$. Furthermore, by the Fourier definition of the fractional Laplacian in (12), we have

$$\mathcal{F}\left\{-(-\Delta)^{\frac{\alpha}{2}} K(t,x)\right\}(\xi) = -|\xi|^\alpha \hat{K}(t,\xi).$$

Since our Fourier transform is applied with respect to x, differentiation with respect to t remains unchanged. Thus, we get the initial value problem

$$\partial_t \hat{K}(t,\xi) = -|\xi|^\alpha \hat{K}(t,\xi), \tag{20a}$$

$$\hat{K}(0,\xi) = 1, \tag{20b}$$

which has the solution $\hat{K}(t,\xi) = e^{-t|\xi|^\alpha}$. □

Although we lack an explicit definition of the fractional heat kernel, properties and estimates of this mathematical object have been extensively studied. A summary of some of the fundamental results for the one-dimensional case can be found in Proposition 1 in [7]. Since we will be working in multiple dimensions, however, we need to know whether these hold in the d-dimensional case as well. Two such results are proven here.

Proposition 2.19. *Let* $K(t,x)$ *be the fractional heat kernel. Then, for any* $t > 0$ *and* $x \in \mathbb{R}^d$*,*

$$\int_{\mathbb{R}^d} K(t,x)\, dx = 1.$$

Proof. By the inverse Fourier transform,

$$\int_{\mathbb{R}^d} K(t,x)\, e^{ix\cdot\xi} dx = e^{-t|\xi|^\alpha}.$$

Letting $\xi = 0$, the Proposition is proven. □

Proposition 2.20. *For any* $t, \tau > 0$ *and* $x \in \mathbb{R}^d$*, we have that*

$$K(t+\tau, x) = K(\tau,\cdot) * K(t,\cdot)(x).$$

Proof. By the Fourier transform of the fractional heat kernel in Lemma 2.18, and since multiplication in Fourier space translates to convolution in real space,

$$K(t+\tau,x) = \mathcal{F}^{-1}\left\{e^{-(t+\tau)|\xi|^{\alpha}}\right\} = \mathcal{F}^{-1}\left\{e^{-\tau|\xi|^{\alpha}}e^{-t|\xi|^{\alpha}}\right\} = K(\tau,\cdot) * K(t,\cdot)(x).$$

□

We proceed by stating two well-known results for the fractional heat kernel, namely a pointwise bound (see [6]) and an L^1-estimate (see [8]).

Proposition 2.21. *Let K be the fractional heat kernel. There exists a constant $c_K > 0$ such that for all $t > 0$ and $x \in \mathbb{R}^d$,*

$$K(t,x) \leqslant c_K \min\left\{t^{-\frac{d}{\alpha}}, \frac{t}{|x|^{d+\alpha}}\right\}.$$

Proposition 2.22. *Let K be the fractional heat kernel. For any $t > 0$,*

$$\|K(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1.$$

The next result is quite technical but provides us with a useful relation between space and time regularity of convolutions with the heat kernel. Specifically, we will see that we can estimate $(K(\tau,\cdot) * g)(x) - g(x)$ only by the time τ and the spatial regularity of g. Note also that by letting $\tau \to 0^+$ in the Lemma that follows, we immediately get that $(K(\tau,\cdot) * g)(x) \to g(x)$. This proves useful when deriving time continuity of solutions to the HJB and FP equations in sections 3.1 and 4.1.

Lemma 2.23 (Corrected from Lemma 2.17 in [2])**.** *Let K be the fractional heat kernel and let $g \in C^{\gamma}(\mathbb{R}^d)$ for some $0 < \gamma \leqslant 1$. Let $L_g = [g]_{C^{0,\gamma}}$ if $\gamma < 1$ and $L_g = \|Dg\|_{\infty}$ if $\gamma = 1$. For any given time $\tau > 0$, we have that*

$$|(K(\tau,\cdot) * g)(x) - g(x)| \leqslant A_d c_K \left(\frac{2\|g\|_{\infty}}{\alpha}\tau^{\frac{\gamma}{2(d+\gamma)}} + \frac{L_g}{d+\gamma}\tau^{\frac{\gamma}{2\alpha}}\right), \tag{21}$$

where c_K is the constant from Proposition 2.21 and A_d is the surface area of the d-dimensional unit sphere.

Proof. The result was incorrectly proven in Lemma 2.17 in [2], and we will provide a revised proof. By Proposition 2.19, we have that

$$g(x) = \int_{\mathbb{R}^d} K(\tau, x-y)\, g(x)\, dy,$$

for any $\tau > 0$. Combining this with the definition of the convolution, the left hand side of (21) can be rewritten as

$$|(K(\tau,\cdot) * g)(x) - g(x)| = \int_{\mathbb{R}^d} |K(\tau, x-y)(g(y) - g(x))|\, dy. \tag{22}$$

In order to derive an estimate for this expression, recall the pointwise inequality for the fractional heat kernel in Proposition 2.21. For some $\rho > 0$ yet to be determined, we divide our integral into integrals over $\mathbb{R}^d \backslash B_{\tau^{\rho}}(x)$ and $B_{\tau^{\rho}}(x)$. It follows that

$$|(K(\tau,\cdot) * g)(x) - g(x)| \leqslant c_K \int_{\mathbb{R}^d \backslash B_{\tau^{\rho}}(x)} \frac{\tau\,|g(y) - g(x)|}{|x-y|^{d+\alpha}} dy + c_K \int_{B_{\tau^{\rho}}(x)} |g(y) - g(x)|\, \tau^{-\frac{d}{\alpha}} dy. \tag{23}$$

Since $g \in C^{\gamma}(\mathbb{R}^d)$, $\|g\|_{\infty}$ is finite and

$$|g(x) - g(y)| \leqslant L_g\, |x-y|^{\gamma}.$$

By (23), we then get that

$$|(K(\tau,\cdot)*g)(x)-g(x)| \leqslant 2c_K\|g\|_\infty \int_{\mathbb{R}^d\backslash B_{\tau^\rho}(x)} \frac{\tau}{|x-y|^{d+\alpha}}dy + c_K L_g \int_{B_{\tau^\rho}(x)} |x-y|^\gamma \tau^{-\frac{d}{\alpha}}dy. \quad (24)$$

Substituting $r=|x-y|$, and letting A_d be the surface area of the d-dimensional unit sphere, we get that

$$\begin{aligned} |(K(\tau,\cdot)*g)(x)-g(x)| &\leqslant 2A_d c_K\|g\|_\infty \int_{\tau^\rho}^\infty \frac{\tau}{r^{\alpha+1}}dr + A_d c_K L_g \int_0^{\tau^\rho} \tau^{-\frac{d}{\alpha}} r^{\gamma+d-1}dr \\ &= 2A_d c_K\|g\|_\infty \frac{\tau}{\alpha}\frac{1}{\tau^{\rho\alpha}} + \frac{A_d c_K L_g}{d+\gamma}\tau^{-\frac{d}{\alpha}}\tau^{\rho(d+\gamma)}. \end{aligned}$$

By inserting $\rho=(2d+\gamma)/(2\alpha(d+\gamma))$, it follows that

$$|(K(\tau,\cdot)*g)(x)-g(x)| \leqslant A_d c_K\left(\frac{2\|g\|_\infty}{\alpha}\tau^{\frac{\gamma}{2(d+\gamma)}} + \frac{L_g}{d+\gamma}\tau^{\frac{\gamma}{2\alpha}}\right),$$

and the proof is complete. □

The remainder of this section is dedicated to deriving L^1-estimates for derivatives of the heat kernel. This will be a key ingredient when showing existence of solutions to the HJB and FP equations. We begin by presenting a self-similarity result.

Lemma 2.24 (Self-similarity for the fractional heat kernel)**.** *Let K be the fractional heat kernel. For any $x\in\mathbb{R}^d$, we have that*

$$K(t,x) = t^{-\frac{d}{\alpha}}K\left(1, xt^{-\frac{1}{\alpha}}\right). \quad (25)$$

Furthermore, for any multi-index k,

$$D^kK(t,x) = t^{-\frac{d+|k|}{\alpha}}D^kK\left(1, xt^{-\frac{1}{\alpha}}\right). \quad (26)$$

Proof. The result is proven in Lemma 3.2 in [2]. □

By utilizing this result, we are able to derive pointwise and L^1-estimates for the derivatives of the heat kernel. The following results were incorrectly proven in [2], and revised proofs are attached in Appendix A. The general approach is inspired by Proposition 1 in [7] where similar results are shown for the one-dimensional case.

Lemma 2.25 (Pointwise estimate for $D^kK(1,u)$)**.** *Let K be the fractional heat kernel and let k be any multi-index such that $|k|\geqslant 1$. Then, there exists a constant $C>0$ such that for any $u\in\mathbb{R}^d\backslash\{0\}$,*

$$\left|D^kK(1,u)\right| \leqslant \frac{C}{|u|^{|k|}}.$$

Theorem 2.26 (L^1-estimate for $D^kK(t,x)$)**.** *Let K be the fractional heat kernel. There exists a constant $\lambda>0$ such that for any $t>0$ and any multi-index k,*

$$\|D^kK(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leqslant \lambda t^{-\frac{|k|}{\alpha}}.$$

2.7 The Duhamel principle

One of the most common methods of obtaining solutions to inhomogenous PDEs is Duhamel's principle. In general, this approach consists of finding a solution to the homogenous version of a problem, and afterwards including the inhomogenities with an integral over time. In simpler PDEs, the Duhamel formula may provide us with an explicit formula for solutions. Due to the complexity of the HJB and FP equations, however, this seems challenging in our case. We therefore need a more general way of looking at existence, motivating the introduction of Duhamel maps.

Definition 2.27 (Duhamel map for the HJB equation)**.** *Given functions v_0, f and H as in Definition 2.5, we define the Duhamel map for the HJB equation by*

$$\phi\left(v\right)\left(t,x\right)=K\left(t,\cdot\right)*v_0\left(\cdot\right)\left(x\right)-\int_0^t K\left(t-s,\cdot\right)*\left(H\left(Dv\left(s,\cdot\right)\right)-f\left(s,\cdot\right)\right)\left(x\right)ds. \tag{27}$$

Definition 2.28 (Duhamel map for the FP equation)**.** *Given functions m_0 and b as in Definition 2.6, we define the Duhamel map for the FP equation by*

$$\psi\left(m\right)\left(t,x\right)=K\left(t,\cdot\right)*m_0\left(\cdot\right)\left(x\right)-\sum_{i=1}^{d}\int_0^t \partial_{x_i}K\left(t-s,\cdot\right)*\left(b_i\left(s,\cdot\right)m\left(s,\cdot\right)\right)\left(x\right)ds. \tag{28}$$

These maps coincide with ordinary Duhamel formulas whenever $\phi\left(v\right)=v$ and $\psi\left(m\right)=m$. This gives rise to the definition of mild solutions.

Definition 2.29 (Mild solution)**.** *We say that v is a mild solution to the HJB equation (8) if it is a fixed point of ϕ. Similarly, m is a mild solution to the FP equation (9) if it is a fixed point of ψ.*

We will see in sections 3.3 and 4.4 that mild solutions correspond to classical solutions when the source and drift terms are sufficiently regular.

A key observation is that Duhamel maps work independently of the initial time, in the sense that we for any $t_0>0$ achieve similar maps by substituting v_0 and m_0 with $\phi\left(v\right)\left(t_0,\cdot\right)$ and $\psi\left(m\right)\left(t_0,\cdot\right)$ respectively. The following result is heavily inspired by Lemma B.1 in [13] and is shown for $\phi\left(v\right)$ in Lemma 2.19 in [2]. Here, we provide a result that holds for both $\phi\left(v\right)$ and $\psi\left(m\right)$ by considering a Duhamel map on a more general form.

Lemma 2.30. *Given a terminal time $T>0$, let $u_0\in L^\infty\left(\mathbb{R}^d\right)$ and denote by $\omega\left(t,x\right)$ the function*

$$\omega\left(t,x\right)=K\left(t,\cdot\right)*u_0\left(\cdot\right)\left(x\right)-\int_0^t K\left(t-s,\cdot\right)*g\left(s,\cdot\right)\left(x\right)ds,\quad\forall\left(t,x\right)\in\left(0,T\right]\times\mathbb{R}^d, \tag{29}$$

where g is a function defined on $\left(0,T\right]\times\mathbb{R}^d$. Let $t_0,\tau>0$ where $t_0+\tau\leqslant T$ and assume that there exists a constant $C>0$ such that

$$\int_0^{t_0}\left|K\left(t_0-s,\cdot\right)*g\left(s,\cdot\right)\left(x\right)\right|ds\leqslant C. \tag{30}$$

Then, the following holds:

$$\omega\left(t_0+\tau,x\right)=K\left(\tau,\cdot\right)*\omega\left(t_0,\cdot\right)\left(x\right)-\int_{t_0}^{t_0+\tau}K\left(t_0+\tau-s,\cdot\right)*g\left(s,\cdot\right)\left(x\right)ds. \tag{31}$$

Proof. By dividing the integral in (31) into integrals over $\left(0,t_0\right)$ and $\left(t_0,t_0+\tau\right)$, as well as using Proposition 2.20, we have

$$\begin{aligned}\omega\left(t_0+\tau,x\right)=&\left(K\left(\tau,\cdot\right)*K\left(t_0,\cdot\right)*u_0\left(\cdot\right)\right)\left(x\right)\\&-\int_0^{t_0}\left(K\left(\tau,\cdot\right)*K\left(t_0-s,\cdot\right)*g\left(s,\cdot\right)\right)\left(x\right)ds\\&-\int_{t_0}^{t_0+\tau}K\left(t_0+\tau-s,\cdot\right)*g\left(s,\cdot\right)\left(x\right)ds.\end{aligned} \tag{32}$$

In order to complete the proof, we need to take $K\left(\tau,\cdot\right)$ outside the second integral. We begin by writing out the convolution.

$$\begin{aligned}&\int_0^{t_0}\left(K\left(\tau,\cdot\right)*K\left(t_0-s,\cdot\right)*g\left(s,\cdot\right)\right)\left(x\right)ds\\&=\int_0^{t_0}\int_{\mathbb{R}^d}K\left(\tau,x-y\right)\left(K\left(t_0-s,\cdot\right)*g\left(s,\cdot\right)\left(y\right)\right)dyds.\end{aligned} \tag{33}$$

The inner and outer integrals can be interchanged by the Fubini-Tonelli theorem (Theorem 2.8) if the integrand is absolutely integrable over $(0, t_0) \times \mathbb{R}^d$. We have that

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_0^{t_0} \left| K(\tau, x-y) \left(K(t_0 - s, \cdot) * g(s, \cdot)(y) \right) \right| ds dy \\ &\leqslant \int_{\mathbb{R}^d} |K(\tau, x-y)| \int_0^{t_0} |K(t_0 - s, \cdot) * g(s, \cdot)(y)| \, ds dy.\end{aligned}$$

The inner integral is finite by the assumption in (30), and it follows that

$$\begin{aligned}&\int_{\mathbb{R}^d} |K(\tau, x-y)| \int_0^{t_0} \left| \left(K(t_0 - s, \cdot) * g_u(s, \cdot) \right)(y) \right| ds dy \\ &\leqslant \int_{\mathbb{R}^d} C \, |K(\tau, x-y)| \, dy = C \| K(\tau, \cdot) \|_{L^1(\mathbb{R}^d)} = C,\end{aligned}$$

where the last equality holds by Proposition 2.22. Hence, the integrand in (33) is absolutely integrable, and we can use the Fubini-Tonelli theorem to deduce that

$$\begin{aligned}&\int_0^{t_0} \left(K(\tau, \cdot) * K(t_0 - s, \cdot) * g(s, \cdot) \right)(x) \, ds \\ &= \int_{\mathbb{R}^d} K(\tau, x-y) \int_0^{t_0} K(t_0 - s, \cdot) * g(s, \cdot)(y) \, ds dy \\ &= K(\tau, \cdot) * \left(\int_0^{t_0} K(t_0 - s, \cdot) * g(s, \cdot) \, ds \right)(x).\end{aligned}$$

Combining this with (32), we complete the proof by

$$\begin{aligned}&\omega(t_0 + \tau, x) = K(\tau, \cdot) * \left(K(t_0, \cdot) * u_0(\cdot) - \int_0^{t_0} K(t_0 - s, \cdot) * g(s, \cdot) \, ds \right)(x) \\ &- \int_{t_0}^{t_0 + \tau} K(t_0 + \tau - s, \cdot) * g(s, \cdot)(x) \, ds \\ &= K(\tau, \cdot) * \omega(t_0, \cdot)(x) - \int_{t_0}^{t_0 + \tau} K(t_0 + \tau - s, \cdot) * g(s, \cdot)(x) \, ds.\end{aligned}$$

□

By letting $\omega = \phi(v)$ and $u_0 = v_0$, (29) is identical to the Duhamel map for the HJB equation (27). A similar result holds for the FP equation by introducing $\psi(m)$ and m_0 in the lemma. Note that by subtracting $\omega(t_0, x)$ from both sides in (31) and using Lemma 2.23, we are able to bound $\omega(t_0 + \tau, x) - \omega(t_0, x)$ only by the time difference τ and the spatial reguarity of $\omega(t_0, \cdot)$. This relation between space and time regularity will be used when deriving time continuity of solutions to the HJB and FP equations in sections 3.1 and 4.1.

2.8 Technical results in Hölder spaces

Much of our analysis involves estimating Hölder seminorms. In particular, we need estimates for convolutions, interpolations and generalized versions of the product and chain rules in differentiation. This section presents a brief review, and will frequently refer to [2] where most of the results are proven.

A particularly important result for estimating convolutions is Young's convolution inequality, which can be found in Theorem 3.9.4 in [3].

Theorem 2.31 (Young's convolution inequality). *Let $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, and let $1 \leqslant p, q, r \leqslant \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

It follows that $f * g \in L^r\left(\mathbb{R}^d\right)$ *and that*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

This inequality will be used extensively for the case where $p = 1$ and $r, q = \infty$ in sections 3.1 and 4.1. We also need a similar result for Hölder seminorms, which follows quite directly.

Proposition 2.32 (Convolution inequality for Hölder seminorms)**.** *Let* $s \in \mathbb{N}_0$ *and* $\beta \in (0,1)$. *For functions* $f \in W^{s,1}\left(\mathbb{R}^d\right)$ *and* $g \in C^{0,\beta}\left(\mathbb{R}^d\right)$, *we have that*

$$[f * g]_{C^{s,\beta}} \leqslant \|D^s f\|_{L^1(\mathbb{R}^d)} [g]_{C^{0,\beta}}.$$

Proof. The result is proven in Proposition 2.22 in [2]. ☐

Relations between Hölder seminorms and ordinary derivatives of functions will be important in the upcoming analysis. By drawing inspiration from [15], specifically chapters 3.2 and 3.3, we can derive interpolation inequalities that provide us with such relations.

Theorem 2.33 (Hölder interpolation between 0 and μ)**.** *Given a function* $g \in C^{0,\mu}\left(\mathbb{R}^d\right)$ *where* $\beta, \mu \in (0,1)$ *such that* $\beta < \mu$, *there exists a constant* $C_{\beta,\mu} > 0$ *only dependent on* β *and* μ *such that the following inequality holds:*

$$[g]_{C^{0,\beta}} \leqslant C_{\beta,\mu} \|g\|_\infty^{\frac{\mu-\beta}{\mu}} [g]_{C^{0,\mu}}^{\frac{\beta}{\mu}}.$$

Proof. The result is proven in Theorem 4.1 in [2]. ☐

Theorem 2.34 (Hölder interpolation between 0 and 1)**.** *Given a function* $g \in C_b^1\left(\mathbb{R}^d\right)$ *where* $\beta \in (0,1)$, *there exists a constant* $C_{\beta,1} > 0$ *only dependent on* β *such that the following inequality holds:*

$$[g]_{C^{0,\beta}} \leqslant C_{\beta,1} \|g\|_\infty^{1-\beta} \|Dg\|_\infty^{\beta}.$$

Proof. The result is proven in Theorem 4.2 in [2]. ☐

We will frequently need to estimate terms on the form $[K * f]_{C^{0,\mu}}$ where $f \in C^{0,\beta}\left(\mathbb{R}^d\right)$ and K is the fractional heat kernel. Since we only have estimates for the derivatives of K (see Theorem 2.26), we need to put an integer order of regularity on the heat kernel. This motivates our last interpolation inequality, where we interpolate between β and $1+\beta$. The resulting terms can then be estimated as $[K * f]_{C^{0,\beta}} \leqslant \|K\|_1 [f]_{C^{0,\beta}}$ and $[DK * f]_{C^{0,\beta}} \leqslant \|DK\|_1 [f]_{C^{0,\beta}}$ by Young's inequality, hence utilizing the entire regularity of f.

Theorem 2.35 (Hölder interpolation between β and $1+\beta$)**.** *Given* $\beta, \mu \in (0,1)$ *and a function* $g \in C^{1+\beta}\left(\mathbb{R}^d\right)$, *there exists a constant* $C_I > 0$ *only depending on* β *and* μ *such that the following statements hold:*

$$\begin{aligned}
&(a) \qquad [g]_{C^{0,\mu}} \leqslant C_I [g]_{C^{0,\beta}}^{1+\beta-\mu} [Dg]_{C^{0,\beta}}^{\mu-\beta}, \qquad \text{if } \mu > \beta,\\
&(b) \qquad [Dg]_{C^{0,\mu}} \leqslant C_I [g]_{C^{0,\beta}}^{\beta-\mu} [Dg]_{C^{0,\beta}}^{1+\mu-\beta}, \qquad \text{if } \mu < \beta,\\
&(c) \qquad \|Dg\|_{L^\infty} \leqslant C_I [g]_{C^{0,\beta}}^{\beta} [Dg]_{C^{0,\beta}}^{1-\beta}.
\end{aligned}$$

Proof. The result follows directly from Exercise 3.3.7 in [15]. ☐

We proceed by presenting generalized versions of the chain and product rules in differentiation. Particular focus is placed upon the latter, which is not proven in [2].

Lemma 2.36 (Generalized chain rule for Hölder seminorms)**.** *Given a function $f \in C^{0,\beta}\left(\mathbb{R}^d\right)$ such that $f : \mathbb{R}^d \to \mathbb{R}^d$, and a globally Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$, consider the composite function $g\left(f\left(\cdot\right)\right)$. We have that*

$$[g\left(f\left(\cdot\right)\right)]_{C^{0,\beta}(\mathbb{R}^d)} \leqslant L_g\left[f\right]_{C^{0,\beta}(\mathbb{R}^d)},$$

where $L_g > 0$ is the Lipschitz constant, meaning that

$$\left|g\left(y_2\right) - g\left(y_1\right)\right| \leqslant L_g\left|y_2 - y_1\right|, \quad \forall y_1, y_2 \in \mathbb{R}^d.$$

Proof. The result is proven in Lemma 4.4 in [2]. □

Lemma 2.37 (Generalized product rule for Hölder seminorms)**.** *Let $\beta \in (0,1)$ and let $f, g \in C^{0,\beta}\left(\mathbb{R}^d\right)$. Then, the following inequality holds:*

$$[fg]_{C^{0,\beta}(\mathbb{R}^d)} \leqslant \|f\|_{L^\infty(\mathbb{R}^d)}\left[g\right]_{C^{0,\beta}(\mathbb{R}^d)} + \left[f\right]_{C^{0,\beta}(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)}. \tag{34}$$

Proof. By writing out the definition of the Hölder seminorm,

$$\begin{aligned}
[fg]_{C^{0,\beta}(\mathbb{R}^d)} &= \sup_{\substack{x,h\in\mathbb{R}^d\\ h\neq 0}} \frac{\left|f\left(x+h\right)g\left(x+h\right) - f\left(x\right)g\left(x\right)\right|}{|h|^\beta} \\
&\leqslant \sup_{\substack{x,h\in\mathbb{R}^d\\ h\neq 0}} \frac{\left|f\left(x+h\right)\left(g\left(x+h\right) - g\left(x\right)\right)\right| + \left|\left(f\left(x+h\right) - f\left(x\right)\right)g\left(x\right)\right|}{|h|^\beta} \\
&\leqslant \sup_{\substack{x,h\in\mathbb{R}^d\\ h\neq 0}} \left|f\left(x+h\right)\right| \frac{\left|g\left(x+h\right) - g\left(x\right)\right|}{|h|^\beta} + \left|g\left(x\right)\right| \frac{\left|f\left(x+h\right) - f\left(x\right)\right|}{|h|^\beta} \\
&\leqslant \|f\|_{L^\infty(\mathbb{R}^d)}\left[g\right]_{C^{0,\beta}(\mathbb{R}^d)} + \left[f\right]_{C^{0,\beta}(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)}. \qquad (35)
\end{aligned}$$

□

A similar estimate can be derived for when the product of functions is convolved with a third function. This is a quite specific result, but proves useful when deriving Hölder regularity in Theorem 3.3 (c).

Proposition 2.38. *Let $\beta \in (0,1)$ and let $f, g \in C^{0,\beta}\left(\mathbb{R}^d\right)$. Furthermore, let $p \in L^1\left(\mathbb{R}^d\right)$. The following inequality holds:*

$$\left[p * (fg)\right]_{C^{0,\beta}(\mathbb{R}^d)} \leqslant \|f\|_{L^\infty(\mathbb{R}^d)}\left[p * g\right]_{C^{0,\beta}(\mathbb{R}^d)} + \|g\|_{L^\infty(\mathbb{R}^d)}\left[p * f\right]_{C^{0,\beta}(\mathbb{R}^d)}.$$

Proof. By writing out the convolution, we get that

$$\left[p * (fg)\right]_{C^{0,\beta}(\mathbb{R}^d)} \leqslant \sup_{\substack{x,h\in\mathbb{R}^d\\ h\neq 0}} \left|\int_{\mathbb{R}^d} \frac{p\left(y\right)\left(f\left(x+h-y\right)g\left(x+h-y\right) - f\left(x-y\right)g\left(x-y\right)\right)}{|h|^\beta} dy\right|.$$

A calculation similar to (35) yields

$$\begin{aligned}
\left[p * (fg)\right]_{C^{0,\beta}(\mathbb{R}^d)} \leqslant{}& \|f\|_{L^\infty(\mathbb{R}^d)} \sup_{\substack{x,h\in\mathbb{R}^d\\ h\neq 0}} \left|\int_{\mathbb{R}^d} \frac{p\left(y\right)\left(g\left(x+h-y\right) - g\left(x-y\right)\right)}{|h|^\beta} dy\right| \\
&+ \|g\|_{L^\infty(\mathbb{R}^d)} \sup_{\substack{x,h\in\mathbb{R}^d\\ h\neq 0}} \left|\int_{\mathbb{R}^d} \frac{p\left(y\right)\left(f\left(x+h-y\right) - f\left(x-y\right)\right)}{|h|^\beta} dy\right| \\
={}& \|f\|_{L^\infty(\mathbb{R}^d)}\left[p * g\right]_{C^{0,\beta}(\mathbb{R}^d)} + \|g\|_{L^\infty(\mathbb{R}^d)}\left[p * f\right]_{C^{0,\beta}(\mathbb{R}^d)}.
\end{aligned}$$

□

We finish the section with Hölder's inequality, which can be found in Theorem 2.11.1 in [3].

Lemma 2.39 (Hölder's inequality)**.** *Let $1 \leqslant p, q \leqslant \infty$ where $1/p + 1/q = 1$. Assume that $f \in L^p\left(\mathbb{R}^d\right)$ and $g \in L^q\left(\mathbb{R}^d\right)$. Then, $fg \in L^1\left(\mathbb{R}^d\right)$ and*

$$\|fg\|_{L^1(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

2.9 The comparison principle

In order to investigate uniqueness results for the HJB equation, we need to show that any two solutions are equal at each point in the given domain. By drawing inspiration from the maximum principle, we can reduce the problem to a finite number of points, which seems desirable.

Maximum principles and their applications have been studied extensively, and previous work can be found in for instance [9] and [18]. To motivate its application for uniqueness, we introduce the principle in its most simple form.

Lemma 2.40 (The maximum principle). *Given a terminal time $T > 0$, let $\Omega = [0, T] \times \mathbb{R}^d$ and let $u : \Omega \to \mathbb{R}^d$ be a sufficiently regular function. Let $\mathcal{L}$ be any linear spatial operator such that $\mathcal{L}u\,(t_0, x_0) \leqslant 0$ whenever (t_0, x_0) is a global maximum of u.*

If $\partial u/\partial t\,(t, x) - \mathcal{L}u\,(t, x) \leqslant 0$ in $\Omega \backslash \left(\{0\} \times \mathbb{R}^d\right)$ and $u < 0$ on $\{0\} \times \mathbb{R}^d$, it follows that $u < 0$ in the entire Ω.

This lemma is inspired by Lemma 2.1 in [18] where a proof using a contradiction by considering a global maximum can be found.

Maximum principles allow us to bound a function in a domain only by restrictions on its initial data and an inequality involving the terms in the PDE. Its application to uniqueness becomes clear when we let u be the difference between two possible solutions of a PDE. Since the initial data is the same for both solutions, we should be able to use the maximum principle to bound the difference between the two solutions in the whole domain, which can be optimized to show uniqueness. This generalization of the maximum principle is called the comparison principle. In order to introduce this, we need to define the notion of sub- and supersolutions.

Definition 2.41 (Sub- and supersolution). *Let Ω and T be defined as in Lemma 2.40 and consider a general PDE on the form*

$$\begin{cases} \partial_t u\,(t, x) - \mathcal{L}u\,(t, x) = F\,(t, x)\,, & \text{in } \Omega, \\ u\,(0, x) = g\,(x)\,, & \text{in } \{0\} \times \mathbb{R}^d. \end{cases} \tag{36}$$

The function u^- is called a subsolution to (36) if it is sufficiently regular and satisfies

$$\begin{cases} \partial_t u^-\,(t, x) - \mathcal{L}u\,(t, x) \leqslant F\,(t, x)\,, & \text{in } \Omega, \\ u^-\,(0, x) \leqslant g\,(x)\,, & \text{in } \{0\} \times \mathbb{R}^d. \end{cases}$$

Similarly, u^+ is a supersolution to (36) if it is sufficiently regular and satisfies

$$\begin{cases} \partial_t u^+\,(t, x) - \mathcal{L}u\,(t, x) \geqslant F\,(t, x)\,, & \text{in } \Omega, \\ u^+\,(0, x) \geqslant g\,(x)\,, & \text{in } \{0\} \times \mathbb{R}^d. \end{cases}$$

By using sub- and supersolutions and a similar argument as in the proof for Lemma 2.40, we can show that $u^-\,(t, x) \leqslant u^+\,(t, x)$ holds in the entire Ω. Proving this is quite technical, and will be different for each PDE. We therefore omit the proof for now, and prove it specifically for the HJB equation in Section 3.4. The relation between the inequality $u^-\,(t, x) \leqslant u^+\,(t, x)$ and uniqueness is then evident by the following result.

Lemma 2.42 (Uniqueness by the comparison principle). *Let Ω and T be defined as in Lemma 2.40 and assume that (36) has a solution $u : \Omega \to \mathbb{R}$. If $u^-\,(t, x) \leqslant u^+\,(t, x)$ holds in the entire Ω for any pair of sub- and supersolutions (u^-, u^+), the solution u to (36) is unique.*

Proof. The result is proven in Lemma 2.25 in [2]. □

2.10 A penalization method

Since the comparison principle requires the existence of a global maximum, we run into complications when looking at unbounded domains, even for bounded functions. Let $u \in L^\infty\left([0,T] \times \mathbb{R}^d\right)$ be a function bounded by some constant $C > 0$, and let $u(x) \to C$ asymptotically as $|x| \to \infty$. Since u has no maxima, the comparison principle cannot be used directly. By utilizing the boundedness of the function, however, we can make a modified function that attains a maximum by penalizing the function outside some compact set in $\mathbb{R}^d$.

Lemma 2.43 (Existence of a smooth penalty function)**.** *Let $R, K > 0$. There exists a function $\Phi \in C^\infty\left(\mathbb{R}^d\right)$ such that*

$$\Phi(x) = \begin{cases} 0, & |x| \leqslant R, \\ K, & |x| \geqslant 2R, \end{cases}$$

where Φ is monotone increasing with respect to $|x|$ in $R < |x| < 2R$.

Proof. The result follows from Corollary 2.3 in [20] and is proven in Lemma 2.26 in [2]. □

Lemma 2.44 (Global maximum by penalization)**.** *Consider $\Omega := [0,T] \times \mathbb{R}^d$. Let $u \in L^\infty(\Omega)$ be bounded by some constant $C > 0$. Furthermore, assume that u has no global maxima. Let $\varphi_R \in C^\infty\left(\mathbb{R}^d\right)$ be the function from Lemma 2.43 where we let $K = 2C + \delta$ such that*

$$\varphi_R(x) = \begin{cases} 0, & |x| \leqslant R, \\ 2C + \delta, & |x| \geqslant 2R, \end{cases}$$

for some $\delta > 0$. Then, $\tilde{u}(t,x) := u(t,x) - \varphi_R(x)$ has a global maximum in $[0,T] \times \{x : |x| \leqslant 2R\}$ for any $R > 0$.

Proof. The result is proven in Lemma 2.27 in [2]. □

3 The Hamilton-Jacobi-Bellman Equation

The first equation in the Mean Field Game system (2) is the Hamilton-Jacobi-Bellman equation (HJB), which we briefly introduced in Section 2.2. In this chapter, we show existence and uniqueness of classical solutions to this equation, as well as providing spatial Hölder regularity estimates. The chapter is based on [2], where similar results were shown for the case where $\alpha + \beta \geqslant 2$. In order to also consider $\alpha + \beta < 2$, the text is heavily revised and we will comment on the differences.

We consider the HJB equation of the form

$$\begin{cases} \partial_t v(t,x) + (-\Delta)^{\frac{\alpha}{2}} v(t,x) + H(Dv(t,x)) = f(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ v(0,x) = v_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{37}$$

where $\alpha \in (1,2)$. The initial data v_0 is assumed to be β-Hölder continuous where $\beta \in (0,1)$. We impose the same spatial regularity on the source term f, as well as continuity in time, hence $f \in C_b\left([0,T]; C^{0,\beta}\left(\mathbb{R}^d\right)\right)$. For the Hamiltonian, we assume $H \in C^1\left(\mathbb{R}^d\right)$ and that it is globally Lipschitz.

Our results are summarized in Theorem 3.8, where we conclude with the existence of a unique classical solution $v \in C_b\left([0,T] \times \mathbb{R}^d\right)$, where $v(t,\cdot)$ is $(\alpha + \beta - \varepsilon)$-Hölder continuous in space for any $t \in (0,T]$ and $\varepsilon > 0$. This is slightly less than the optimal regularity, as it should be possible to show that $v(t,\cdot)$ is $(\alpha + \beta)$-Hölder continuous. The additional regularity we gain when going from source term to solution is highly connected to the order of our operator. Hence, since $f(t,\cdot)$ is β-Hölder continuous, we expect $v(t,\cdot) \in C^{\alpha+\beta}\left(\mathbb{R}^d\right)$ for any $t \in (0,T]$. We will later see that letting $\varepsilon = 0$ causes singularity issues in our regularity estimates. Showing $(\alpha + \beta)$-Hölder continuity is therefore outside the scope of this thesis, and will likely involve scaling properties and more delicate bounds for the fractional heat kernel.

Another key observation in the upcoming results is the blowup on derivatives and Hölder seminorms of v as $t \to 0$. As an example, in order to bound the first order derivative uniformly in time, we will need to multiply it by $t^{\frac{1-\beta}{\alpha}}$. This is because our initial data is only β-Hölder continuous, meaning that Dv_0 does not exist.

Due to the complexity of the HJB equation, it seems challenging to find explicit solutions. This is especially caused by the Hamiltonian term $H(Dv)$ and its dependence on v. Since H is considered a general function, only subject to regularity assumptions, it may introduce nonlinearities to the system. These are often hard to tackle using ordinary solution methods, and we need a new approach for studying existence.

Recall the Duhamel map $\phi(v)$ for the HJB equation in (27). As stated in Definition 2.29, any fixed point of ϕ is a mild solution to (37). This motivates our existence approach, where we use Banach's fixed point theorem (Theorem 2.7) to show that there exists $v(t,x)$ such that $\phi(v) = v$. Although this is a promising approach, it has its limitations. In order to use Banach's fixed point theorem, we need to ensure that ϕ is a contraction mapping in some Banach space X. We will later see that this restricts our existence proof to some short time interval $[0,T_0]$. This is discussed in detail in Section 3.1, where we show short time existence of mild solutions to the HJB equation. The section will also entail continuity and boundedness results for the first spatial derivative Dv. We follow the general approach in [2], but in order to consider cases where $\alpha + \beta < 2$, the fixed point argument is revised and does not address Hölder regularity.

Spatial regularity estimates for mild solutions are instead studied in the subsequent section. These results play a vital role when showing that our solution is classical in Section 3.3, since this requires $-(-\Delta)^{\alpha/2} v$ to be well-defined. By Proposition 2.15, we then need $(\alpha + \delta)$-Hölder continuity in space for some $\delta > 0$. We will specifically show that $v(t,\cdot)$ is $(\alpha + \beta - \varepsilon)$-Hölder continuous in space for any $t \in (0,T_0]$ and $\varepsilon > 0$, as well as commenting on why our approach is insufficient for the case where $\varepsilon = 0$. In addition, we show that the β-Hölder seminorm of $v(t,\cdot)$ is uniformly bounded in time, and prove existence of $D^2 v$ whenever $\alpha + \beta \geqslant 2$.

Section 3.3 addresses whether our mild solution is classical and is mostly based on [2]. This also applies to Section 3.4, where we show that classical solutions to (37) are unique. Section 3.5 proves the existence of a unique classical solution in the entire $[0,T]$. Here, we use a patching argument by combining unique short time solutions on overlapping time intervals. Finally, we finish the chapter with a uniform continuity result for v and its derivatives in Section 3.6.

3.1 Short time existence results

We begin with proving short time existence of a mild solution in Theorem 3.1. By combining Banach's fixed point theorem (Theorem 2.7) with the Duhamel map $\phi(v)$ in (27), we show that there exists a fixed point $\phi(v) = v$ where $v \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ and $t^{\frac{1-\beta}{\alpha}} Dv \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$ for some $T_0 \in (0,T)$. By Definition 2.29, this is a mild solution to the HJB equation (37). The time blowup on Dv as $t \to 0$ appears since v_0 is only β-Hölder continuous, implying that Dv_0 does not exist.

Unlike [2], we will not include the second derivative $D^2 v$ or Hölder regularity estimates in our fixed point argument. This is done in order to avoid differentiating the Hamiltonian, which would complicate the contraction argument for ϕ drastically.

The proof uses a combination of Young's inequality (Theorem 2.31), L^1-estimates for the fractional heat kernel (Theorem 2.26) and interpolation results from Section 2.8. Time continuity of the Duhamel map is shown by using its relation to spatial regularity through the fractional heat kernel (see Lemma 2.23).

Theorem 3.1 (Short time existence of mild solutions)**.** *Let $\alpha \in (1,2), \beta \in (0,1)$ and suppose that $v_0 \in C^{0,\beta}\left(\mathbb{R}^d\right)$. Let λ be the constant defined in Theorem 2.26, and assume that $f \in C_b\left([0,T]; C^{0,\beta}\left(\mathbb{R}^d\right)\right)$. Furthermore, suppose $H \in C^1\left(\mathbb{R}^d\right)$ is globally Lipschitz with Lipschitz constant $L_H = \|\partial_p H\|_\infty \geqslant 0$. Given a terminal time $T > 0$, there exists $T_0 \in (0,T)$ only depending on α, β, λ and L_H such that there exists a unique mild solution $v \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ to the HJB equation (37) where $t^{\frac{1-\beta}{\alpha}} Dv \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$.*

Proof. Let X be the Banach space

$$X = \left\{ v : v, t^{\frac{1-\beta}{\alpha}} Dv \in C_b\left((0,T_0] \times \mathbb{R}^d\right) \right\}, \tag{38}$$

and define the corresponding norm by $\|v\|_X := \sup_{t \in (0,T_0]} \|v(t,\cdot)\|_X$ where

$$\|v(t,\cdot)\|_X = \|v(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + \|t^{\frac{1-\beta}{\alpha}} Dv(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}. \tag{39}$$

We note that the norm $\|\cdot\|_X$ depends on t, but we skip it in the notation for the sake of readability. Recall the definition of the Duhamel map $\phi(v)$ in (27).

$$\phi(v)(t,x) = (K(t,\cdot) * v_0(\cdot))(x) - \int_0^t (K(t-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot)))(x)\, ds. \tag{40}$$

We want to use Banach's fixed point theorem to show the existence of a fixed point $\phi(v) = v$. This requires $\phi : X \to X$, meaning that $\phi(v), t^{\frac{1-\beta}{\alpha}} D\phi(v) \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$ for any $v \in X$. In order to show these regularity requirements for ϕ, we need L^∞-bounds on H and f. By $f \in C_b\left([0,T]; C^{0,\beta}\left(\mathbb{R}^d\right)\right)$, there exists $C_f > 0$ such that

$$\sup_{t \in [0,T]} \|f(t,\cdot)\|_\infty \leqslant C_f. \tag{41}$$

Since H is not necessarily bounded, we need to use its global Lipschitz condition to arrive at an L^∞-bound. Denote the Lipschitz constant by $L_H \geqslant 0$ such that

$$|H(p_2) - H(p_1)| \leqslant L_H |p_2 - p_1|, \quad \forall p_1, p_2 \in \mathbb{R}^d.$$

By letting $H(0) := H_0$, it follows that

$$\begin{aligned} \|H(Dv(t,\cdot))\|_\infty &\leqslant \|H(Dv(t,\cdot)) - H(0)\|_\infty + H_0 \\ &\leqslant L_H \|Dv(t,\cdot)\|_\infty + H_0 = L_H t^{-\frac{1-\beta}{\alpha}} \|t^{\frac{1-\beta}{\alpha}} Dv(t,\cdot)\|_\infty + H_0. \end{aligned} \tag{42}$$

Notice that $\|t^{\frac{1-\beta}{\alpha}} Dv(t,\cdot)\|_\infty$ is bounded uniformly in $(0,T_0]$ by the definition of X.

We proceed by estimating $\|\phi(v)(t,\cdot)\|_X$, looking at the two parts of the norm in (39) individually. Convolution and interpolation inequalities from Section 2.8 will be used quite extensively, as well as the L^1-estimates for the heat kernel in Theorem 2.26. The constants λ and C_I come from Theorem 2.26 and 2.35. Note that since both parts of $\|\cdot\|_X$ include a supremum, we can move the integral outside the norms and seminorms in the inequalities below.

We begin by estimating $\|\phi(v)(t,\cdot)\|_\infty$. Recall that $\|K(t,\cdot)\|_\infty = 1$ from Proposition 2.22. By the triangle inequality, Young's convolution inequality in Theorem 2.31, (41) and (42),

$$\begin{aligned}
\|\phi(v)(t,\cdot)\|_\infty &\leqslant \|K(t,\cdot) * v_0\|_\infty + \int_0^t \|K(t-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot))\|_\infty ds \\
&\leqslant \|K(t,\cdot)\|_1 \|v_0\|_\infty + \int_0^t \|K(t-s,\cdot)\|_1 (\|H(Dv(s,\cdot))\|_\infty + \|f(s,\cdot)\|_\infty) ds \\
&\leqslant \|v_0\|_\infty + \int_0^t \left(L_H s^{-\frac{1-\beta}{\alpha}} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty + H_0 + C_f \right) ds \\
&\leqslant \|v_0\|_\infty + L_H \frac{\alpha}{\alpha+\beta-1} t^{\frac{\alpha+\beta-1}{\alpha}} \sup_{s\in(0,t)} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty + t(H_0 + C_f) \\
&\leqslant \|v_0\|_\infty + L_H \frac{\alpha}{\alpha+\beta-1} T_0^{\frac{\alpha+\beta-1}{\alpha}} \sup_{s\in(0,T_0]} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty + T_0(H_0 + C_f),
\end{aligned} \tag{43}$$

where the last inequality holds by the positivity of L_H, H_0, C_f and the exponent $(\alpha+\beta-1)/\alpha$. Since the upper bound in (43) is independent of t, it follows that $\|\phi(v)(t,\cdot)\|_\infty$ is uniformly bounded in $(0,T_0]$ by the definition of X and since $v_0 \in C^{0,\beta}(\mathbb{R}^d)$.

We proceed by estimating $\|t^{\frac{1-\beta}{\alpha}} D\phi(v)(t,\cdot)\|_\infty$. Notice first that

$$\|t^{\frac{1-\beta}{\alpha}} D\phi(v)(t,\cdot)\|_\infty \leqslant t^{\frac{1-\beta}{\alpha}} \|DK(t,\cdot) * v_0\|_\infty + t^{\frac{1-\beta}{\alpha}} \int_0^t \|DK(t-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot))\|_\infty ds. \tag{44}$$

The first term in (44) is estimated by using the interpolation inequality from Theorem 2.35 (c) on $K(t,\cdot) * v_0$, heat kernel estimates from Theorem 2.26 and the convolution inequality in Proposition 2.32. We get that

$$\begin{aligned}
&t^{\frac{1-\beta}{\alpha}} \|DK(t,\cdot) * v_0\|_\infty \leqslant C_I t^{\frac{1-\beta}{\alpha}} [K(t,\cdot) * v_0]_{C^{0,\beta}}^{\beta} [DK(t,\cdot) * v_0]_{C^{0,\beta}}^{1-\beta} \\
&\leqslant C_I t^{\frac{1-\beta}{\alpha}} [v_0]_{C^{0,\beta}} \|K(t,\cdot)\|_1^{\beta} \|DK(t,\cdot)\|_1^{1-\beta} \leqslant C_I \lambda^{1-\beta} t^{\frac{1-\beta}{\alpha}} t^{-\frac{1-\beta}{\alpha}} [v_0]_{C^{0,\beta}} = C_I \lambda^{1-\beta} [v_0]_{C^{0,\beta}},
\end{aligned} \tag{45}$$

which is bounded since $v_0 \in C^{0,\beta}(\mathbb{R}^d)$. For the second term in (44), we differentiate the heat kernel in the integral, and use bounds for f and H from (41) and (42).

$$\begin{aligned}
&t^{\frac{1-\beta}{\alpha}} \int_0^t \|DK(t-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot))\|_\infty ds \\
&\leqslant t^{\frac{1-\beta}{\alpha}} \int_0^t \|DK(t-s,\cdot)\|_1 (\|H(Dv(s,\cdot))\|_\infty + \|f(s,\cdot)\|_\infty) ds \\
&\leqslant t^{\frac{1-\beta}{\alpha}} \int_0^t \lambda (t-s)^{-\frac{1}{\alpha}} \left(L_H s^{-\frac{1-\beta}{\alpha}} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty + H_0 + C_f \right) ds \\
&\leqslant \lambda L_H \sup_{s\in(0,T_0]} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty T_0^{\frac{\alpha-1}{\alpha}} \int_0^1 (1-r)^{-\frac{1}{\alpha}} r^{-\frac{1-\beta}{\alpha}} dr + \lambda(H_0 + C_f) T_0^{\frac{\alpha-\beta}{\alpha}} \int_0^1 (1-r)^{-\frac{1}{\alpha}} dr.
\end{aligned} \tag{46}$$

We can let $t \to T_0$ in the last inequality above by $L_H, H_0, C_f \geqslant 0$ and since the exponents $(\alpha-1)/\alpha$ and $(\alpha-\beta)/\alpha$ are positive. Furthermore, integrability is ensured since all exponents on r and $1-r$ are strictly greater than -1. We conclude with boundedness in the last line by the definition of X. Having shown boundedness of both terms in (44), with upper bounds independent of t, it follows that $\|t^{\frac{1-\beta}{\alpha}} D\phi(v)(t,\cdot)\|_\infty$ is uniformly bounded in $(0,T_0]$.

We proceed with showing continuity in space, namely that $\phi(v)(t,\cdot), t^{\frac{1-\beta}{\alpha}} D\phi(v)(t,\cdot) \in C_b(\mathbb{R}^d)$ for each $t \in (0, T_0]$. Notice that since t is fixed, we do not need to consider blowup in time. For every $t_0 \in (0, T_0]$, we then have that $\|D\phi(v)(t_0,\cdot)\|_\infty$ is bounded, and it follows that $\phi(v)(t_0,\cdot)$ is Lipschitz continuous in space.

For $D\phi(v)(t_0,\cdot)$, we use an interpolation as in Theorem 2.34 combined with heat kernel estimates from Theorem 2.26. Then, for any $t_0 \in (0, T_0]$,

$$\begin{aligned}
[D\phi(v)(t_0,\cdot)]_{C^{0,\beta}} &\leqslant \lambda t_0^{-\frac{1}{\alpha}} [v_0]_{C^{0,\beta}} + \lambda C_{\beta,1} \int_0^{t_0} (t_0 - s)^{-\frac{1+\beta}{\alpha}} \left(\|H(Dv(s,\cdot))\|_\infty + \|f(s,\cdot)\|_\infty\right) ds \\
&\leqslant \lambda t_0^{-\frac{1}{\alpha}} [v_0]_{C^{0,\beta}} + \lambda C_{\beta,1} \int_0^{t_0} (t_0 - s)^{-\frac{1+\beta}{\alpha}} \left(L_H s^{-\frac{1-\beta}{\alpha}} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty + H_0 + C_f\right) ds, \qquad (47)
\end{aligned}$$

where we used (41) and (42) to bound the L^∞-norms of f and H. Integrability above is ensured by the uniform boundedness of $\|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty$ in time, and since the exponents on $t_0 - s$ and s are greater than -1. It follows that $D\phi(v)(t_0,\cdot)$ is β-Hölder continuous in space, implying $t^{\frac{1-\beta}{\alpha}} D\phi(v)(t,\cdot) \in C_b(\mathbb{R}^d)$ for any $t \in (0, T_0]$.

Next, we show that $\phi(v)$ and $t^{\frac{1-\beta}{\alpha}} D\phi(v)$ are continuous in time. Whenever $t < T_0$, it suffices to show that $\phi(v)(t_0+\tau, x) - \phi(v)(t_0, x)$ and $D\phi(v)(t_0+\tau, x) - D\phi(v)(t_0, x)$ go to zero as $\tau \to 0^+$ for any $(t_0, x) \in (0, T_0) \times \mathbb{R}^d$.

Letting $\omega := \phi(v)$, $g := H(Dv) - f$ and $u_0 := v_0$ in Lemma 2.30, and subtracting $\phi(v)(t_0, x)$ from both sides in (31) yields

$$\begin{aligned}
\phi(v)(t_0+\tau, x) - \phi(v)(t_0, x) &= (K(\tau,\cdot) * \phi(v)(t_0,\cdot))(x) - \phi(v)(t_0, x) \\
&\quad - \int_{t_0}^{t_0+\tau} (K(t_0+\tau-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot)))(x)\, ds. \qquad (48)
\end{aligned}$$

The latter term is estimated as in (43), such that

$$\begin{aligned}
&\left|\int_{t_0}^{t_0+\tau} (K(t_0+\tau-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot)))(x)\, ds\right| \\
&\leqslant L_H \frac{\alpha}{\alpha+\beta-1} \sup_{s\in(0,T_0]} \|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty \left|(t_0+\tau)^{\frac{\alpha+\beta-1}{\alpha}} - t_0^{\frac{\alpha+\beta-1}{\alpha}}\right| + \tau(H_0 + C_f). \qquad (49)
\end{aligned}$$

Notice that $0 < \frac{\alpha+\beta-1}{\alpha} \leqslant 1$. Then, $0 < (t_0+\tau)^{\frac{\alpha+\beta-1}{\alpha}} - t_0^{\frac{\alpha+\beta-1}{\alpha}} \leqslant \tau^{\frac{\alpha+\beta-1}{\alpha}}$, and by the uniform boundedness of $\|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty$ in time, the right hand side of (49) goes to zero as $\tau \to 0^+$.

It remains to derive an estimate for the first term in (48). Letting $g := \phi(v)(t_0,\cdot)$ and $\gamma = 1$ in Lemma 2.23, we get that

$$\begin{aligned}
&|(K(\tau,\cdot) * \phi(v)(t_0,\cdot))(x) - \phi(v)(t_0, x)| \\
&\qquad \leqslant A_d c_K \left(\frac{2\|\phi(v)(t_0,\cdot)\|_\infty}{\alpha} \tau^{\frac{1}{2(d+1)}} + \frac{\|D\phi(v)(t_0,\cdot)\|_\infty}{d+1} \tau^{\frac{1}{2\alpha}}\right). \qquad (50)
\end{aligned}$$

Since both $\|\phi(v)(t_0,\cdot)\|_\infty$ and $\|D\phi(v)(t_0,\cdot)\|_\infty$ are finite for $t_0 \in (0, T_0]$, the right hand side goes to zero as $\tau \to 0^+$. By combining (49) and (50), we get from (48) that

$$\begin{aligned}
&\lim_{\tau\to 0^+} |\phi(v)(t_0+\tau, x) - \phi(v)(t_0, x)| \leqslant \lim_{\tau\to 0^+} |(K(\tau,\cdot) * \phi(v)(t_0,\cdot))(x) - \phi(v)(t_0, x)| \\
&+ \lim_{\tau\to 0^+} \left|\int_{t_0}^{t_0+\tau} (K(t_0+\tau-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot)))(x)\, ds\right| = 0.
\end{aligned}$$

It follows that $\phi(v)(t_0,\cdot)$ is continuous in time for any $t_0 \in (0, T_0)$. In order to show continuity in the entire $(0, T_0]$, we need to let $t_0 = T_0$ above. This would imply $t_0 + \tau > T_0$, which is outside our scope in terms of regularity assumptions. By instead repeating the argument with $\phi(v)(T_0, x) - \phi(v)(T_0 - \tau, x)$ in (48), we circumvent this problem. The proof is very similar, and we conclude that $\phi(v) \in C_b((0, T_0] \times \mathbb{R}^d)$.

The approach for deriving time continuity of $D\phi(v)$ is similar. By differentiating (31), subtracting $D\phi(v)(t_0,x)$ from both sides and taking the absolute value, we get that

$$\begin{aligned}
&|D\phi(v)(t_0+\tau,x) - D\phi(v)(t_0,x)| \leqslant |(K(\tau,\cdot) * D\phi(v)(t_0,\cdot))(x) - D\phi(v)(t_0,x)| \\
&+ \left| \int_{t_0}^{t_0+\tau} (DK(t_0+\tau-s,\cdot) * (H(Dv(s,\cdot)) - f(s,\cdot)))(x)\, ds \right| \\
&\leqslant |(K(\tau,\cdot) * D\phi(v)(t_0,\cdot))(x) - D\phi(v)(t_0,x)| \\
&+ C_1 \sup_{s\in(0,T_0]} \| s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty \left| (t_0+\tau)^{\frac{\alpha+\beta-2}{\alpha}} - t_0^{\frac{\alpha+\beta-2}{\alpha}} \right| + C_2 \left| (t_0+\tau)^{\frac{\alpha-1}{\alpha}} - t_0^{\frac{\alpha-1}{\alpha}} \right|, \qquad (51)
\end{aligned}$$

for some $C_1, C_2 \geqslant 0$ independent of t_0 and τ. In the last line, we estimated similarly to (46). The third term goes to zero as $\tau \to 0^+$ by $0 < \frac{\alpha-1}{\alpha} < 1$ and an argument like in (49). If $\alpha+\beta \geqslant 2$, this argument holds also for the second term in (51). In the case where $1 < \alpha+\beta < 2$, let $k = -\frac{\alpha+\beta-2}{\alpha}$. Then, $k \in (0,1)$ and we get that

$$\left| (t_0+\tau)^{\frac{\alpha+\beta-2}{\alpha}} - t_0^{\frac{\alpha+\beta-2}{\alpha}} \right| = \frac{1}{t_0^k} - \frac{1}{(t_0+\tau)^k} \leqslant \frac{(t_0+\tau)^k - t_0^k}{t_0^k (t_0+\tau)^k}. \qquad (52)$$

The numerator goes to zero as $\tau \to 0^+$ by $k \in (0,1)$ and an argument like in (49). It follows that the second term in (51) goes to zero.

For the first term, we use that $D\phi(v)$ is β-Hölder continuous in space. By letting $g := D\phi(v)(t_0,\cdot)$ and $\gamma = \beta$ in Lemma 2.23, we get that

$$\begin{aligned}
&|K(\tau,\cdot) * D\phi(v)(t_0,\cdot)(x) - D\phi(v)(t_0,x)| \\
&\qquad \leqslant A_d c_K \left(\frac{2\|D\phi(v)(t_0,\cdot)\|_\infty}{\alpha} \tau^{\frac{\beta}{2(d+\beta)}} + \frac{[D\phi(v)(t_0,\cdot)]_{C^{0,\beta}}}{d+\beta} \tau^{\frac{\beta}{2\alpha}} \right), \qquad (53)
\end{aligned}$$

which goes to zero as $\tau \to 0^+$. Since all terms in (51) go to zero as $\tau \to 0^+$, we conclude that

$$\lim_{\tau\to 0^+} |D\phi(v)(t_0+\tau,x) - D\phi(v)(t_0,x)| = 0.$$

This holds for all $t_0 \in (0,T_0)$. For $t_0 = T_0$, the argument is repeated with $D\phi(v)(T_0,x) - D\phi(v)(T_0-\tau,x)$ in (51). We then get time continuity of $D\phi(v)$ in the entire $(0,T_0]$, and since $t^{\frac{1-\beta}{\alpha}}$ is continuous in time, we have $t^{\frac{1-\beta}{\alpha}} D\phi(v) \in C_b\left((0,T_0]\times\mathbb{R}^d\right)$. Combining this with $\phi(v) \in C_b\left((0,T_0]\times\mathbb{R}^d\right)$, we get that $\phi: X \to X$.

In order to use Banach's fixed point theorem (Theorem 2.7), it remains to show that ϕ is a contraction mapping. This means that there must exist $L > 0$ such that $\|\phi(u) - \phi(w)\|_X \leqslant L\|u-w\|_X$ for any $u, w \in X$. Notice that

$$\|\phi(u)(t,\cdot) - \phi(w)(t,\cdot)\|_X \leqslant \int_0^t \|K(t-s,\cdot) * (H(Du(s,\cdot)) - H(Dw(s,\cdot)))\|_X ds. \qquad (54)$$

We proceed with looking at the two parts of $\|\cdot\|_X$ seperately, similar to what we did earlier in the proof. Recall that by the definition of $\|\cdot\|_X$, $\|Du(s,\cdot) - Dw(s,\cdot)\|_\infty \leqslant s^{-\frac{1-\beta}{\alpha}} \|u-w\|_X$. Using that H is globally Lipschitz, there then exists a constant $c_0 > 0$ only depending on α, β, λ and L_H such that

$$\begin{aligned}
\|\phi(u)(t,\cdot) - \phi(w)(t,\cdot)\|_\infty &\leqslant \int_0^t \|K(t-s,\cdot)\|_1 L_H \|Du(s,\cdot) - Dw(s,\cdot)\|_\infty ds \\
&\leqslant \int_0^t \lambda s^{-\frac{1-\beta}{\alpha}} L_H \|u-w\|_X ds \leqslant \lambda L_H t^{\frac{\alpha+\beta-1}{\alpha}} \|u-w\|_X. \qquad (55)
\end{aligned}$$

Similarly, by differentiating K in the convolution, there exists $c_1 > 0$ depending on the same constants such that

$$\begin{aligned}
&\|t^{\frac{1-\beta}{\alpha}} (D\phi(u)(t,\cdot) - D\phi(w)(t,\cdot))\|_\infty \leqslant t^{\frac{1-\beta}{\alpha}} \int_0^t \|DK(t-s,\cdot)\|_1 L_H \|Du(s,\cdot) - Dw(s,\cdot)\|_\infty ds \\
&\leqslant t^{\frac{1-\beta}{\alpha}} \int_0^t \lambda (t-s)^{-\frac{1}{\alpha}} s^{-\frac{1-\beta}{\alpha}} L_H \|u-w\|_X ds \leqslant c_1 t^{\frac{\alpha-1}{\alpha}} \|u-w\|_X. \qquad (56)
\end{aligned}$$

By combining these estimates, and noticing that both $(\alpha+\beta-1)/\alpha$ and $(\alpha-1)/\alpha$ are positive exponents, we can take the supremum over $(0, T_0]$ and deduce that

$$\|\phi(u)-\phi(w)\|_X \leqslant \left(c_0 T_0^{\frac{\alpha+\beta-1}{\alpha}} + c_1 T_0^{\frac{\alpha-1}{\alpha}}\right) \|u-w\|_X.$$

By choosing T_0 sufficiently small, there exists $L<1$ such that $\|\phi(u)-\phi(w)\|_X \leqslant L\|u-w\|_X$. Then, ϕ is a contraction mapping. Furthermore, since c_0 and c_1 only depend on α, β, λ and L_H, it follows that T_0 depends solely on these constants as well.

Since $\phi : X \to X$ and ϕ is a contraction mapping, it follows from Banach's fixed point theorem (Theorem 2.7) that there exists a unique $v \in X$ such that $\phi(v) = v$. This is a mild solution to the HJB equation in (37) by Definition 2.29. Finally, since $v(0,\cdot) = v_0 \in C_b(\mathbb{R}^d)$, we get that $v \in C_b([0,T_0]\times\mathbb{R}^d)$ and the proof is complete. □

3.2 Regularity estimates

In the last section, we showed that given certain regularity assumptions on v_0, f and H, there exists a short time mild solution $v \in C_b([0,T_0]\times\mathbb{R}^d)$ to the HJB equation where $t^{\frac{1-\beta}{\alpha}} Dv \in C_b((0,T_0]\times\mathbb{R}^d)$. We will now derive additional spatial regularity results for this solution. By using the generalized Grönwall inequality (Lemma 2.11), we show that $v(t,\cdot)$ is $(\alpha+\beta-\varepsilon)$-Hölder continuous in space for all $t \in (0,T_0]$ and $\varepsilon > 0$. Since v_0 is only β-Hölder continuous, however, this estimate will blow up as $t \to 0$. We will therefore consider the function $t^{\frac{\alpha-\varepsilon}{\alpha}} v(t,\cdot)$ instead, such that its Hölder seminorm can be bounded uniformly in time. Spatial Hölder regularity without time blowup is also considered, as we show uniform boundedness in time for $[v(t,\cdot)]_{C^{0,\beta}}$ as well. These results are presented in Theorem 3.3.

Our approach will be insufficient for proving $(\alpha+\beta)$-Hölder continuity in space. This is due to singularity issues in the upcoming analysis. Notice specifically that the first integral in (69) blows up if $\varepsilon = 0$.

Since $\alpha \in (1,2)$ and $\beta \in (0,1)$, we will encounter cases where $\alpha+\beta > 2$. This implies existence of $D^2 v$, which we will investigate in Lemma 3.2. Continuity of the second derivative is proven together with the Hölder regularity results in Theorem 3.3. We begin with proving existence of $D^2 v$ whenever $\alpha+\beta>2$, using a combination of Rademacher's theorem (Theorem 2.10) and the generalized Grönwall inequality (Lemma 2.11).

Lemma 3.2. *Let $\alpha \in (1,2)$, $\beta \in (0,1)$ and assume that $\alpha+\beta > 2$. Let assumptions on f and H be as in Theorem 3.1, and let v be the corresponding mild solution obtained in the theorem. Then, $D^2 v(t,x)$ exists everywhere in $(0,T_0]\times\mathbb{R}^d$. In addition, $t^{\frac{2-\beta}{\alpha}} D^2 v \in L^\infty((0,T_0]\times\mathbb{R}^d)$.*

Proof. We begin with showing that $D^2 v(t,\cdot)$ exists almost everywhere in space for any fixed $t \in (0,T_0]$. By Rademacher's theorem (Theorem 2.10), it suffices to show that $Dv(t,\cdot)$ is Lipschitz continuous in $\mathbb{R}^d$, which we will prove using the generalized Grönwall inequality in Lemma 2.11.

Note that $v(t,x)$ is a fixed point of the Duhamel map in (27) since it is a mild solution of the HJB equation. By translating in space, subtracting and differentiating, we deduce that for $(t,x) \in (0,T_0]\times\mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$,

$$\begin{aligned} Dv(t,x+h) - Dv(t,x) &= (DK(t,\cdot) * v_0)(x+h) - (DK(t,\cdot) * v_0)(x) \\ &\quad - D\int_0^t (K(t-s,\cdot) * f(s,\cdot))(x+h) - (K(t-s,\cdot) * f(s,\cdot))(x)\, ds \\ &\quad - D\int_0^t K(t-s,\cdot) * (H(Dv(s,\cdot+h)) - H(Dv(s,\cdot)))(x)\, ds. \end{aligned} \tag{57}$$

Recall that for any function $g(x)$ with a bounded derivative $g'(x)$, we have that

$$|g(x+h) - g(x)| = \left|\int_x^{x+h} g'(y)\, dy\right| \leqslant |h|\, \|g'\|_\infty. \tag{58}$$

This lets us bound the spatial differences in (57). For the first term, we use the fact that $\|D(DK(t,\cdot)*v_0)\|_\infty \leqslant \lambda C_I t^{-\frac{2-\beta}{\alpha}}[v_0]_{C^{0,\beta}}$ by Young's inequality and an interpolation like in Theorem 2.35 (c). It follows that $D(DK(t,\cdot)*v_0)(x)$ is bounded in $\mathbb{R}^d$ for any fixed $t \in (0,T_0]$. Boundedness of $D(DK(t-s,\cdot)*f(s,\cdot))(x)$ can be derived similarly, resulting in the bound $\|D(DK(t-s,\cdot)*f(s,\cdot))\|_\infty \leqslant \lambda C_I(t-s)^{-\frac{2-\beta}{\alpha}}\tilde{C}_f$, where we let $\tilde{C}_f := \sup_{s\in(0,T_0]}[f(s,\cdot)]_{C^{0,\beta}}$. This holds for any fixed $t \in (0,T_0]$ and $s \in (0,t)$.

Dividing (57) by h, taking the L^∞-norm over $\mathbb{R}^d$ and using the relation in (58) yields

$$\begin{aligned}\left\|\frac{Dv(t,x+h)-Dv(t,x)}{|h|}\right\|_\infty &\leqslant \|D(DK(t,\cdot)*v_0)\|_\infty + \int_0^t \|D(DK(t-s,\cdot)*f(s,\cdot))\|_\infty ds \\ &\quad + \int_0^t \|DK(t-s)\|_1 \left\|\frac{H(Dv(s,\cdot+h))-H(Dv(s,\cdot))}{|h|}\right\|_\infty ds \\ \leqslant \lambda C_I\Big(t^{-\frac{2-\beta}{\alpha}}[v_0]_{C^{0,\beta}} + \tilde{C}_f\int_0^t (t-s)^{-\frac{2-\beta}{\alpha}}ds\Big) &+ \lambda L_H\int_0^t (t-s)^{-\frac{1}{\alpha}}\left\|\frac{Dv(s,\cdot+h)-Dv(s,\cdot)}{|h|}\right\|_\infty ds,\end{aligned} \tag{59}$$

where we used interpolation and heat kernel estimates, the $\|\cdot\|_\infty$-bounds calculated above and the fact that H is globally Lipschitz. The first integral in (59) is bounded and will reach its supremum at $t = T_0$. This follows by integrating and noticing that $-\frac{2-\beta}{\alpha} > -1$. Then, there exist constants $a_0, a_{T_0} \geqslant 0$ independent of t and h such that

$$\left\|\frac{Dv(t,\cdot+h)-Dv(t,\cdot)}{|h|}\right\|_\infty \leqslant a_0 t^{-\frac{2-\beta}{\alpha}} + a_{T_0} + \lambda L_H\int_0^t (t-s)^{-\frac{1}{\alpha}}\left\|\frac{Dv(s,\cdot+h)-Dv(s,\cdot)}{|h|}\right\|_\infty ds. \tag{60}$$

We will now use the generalized Grönwall inequality in Lemma 2.11 to show boundedness of the left hand side in (60) for any $t \in (0,T_0]$. The resulting upper bound will be independent of h, thus immediately implying that $Dv(t,\cdot)$ is Lipschitz continuous. Here are the details.

Let $\gamma = (2-\beta)/\alpha$ and $\zeta = 1/\alpha$. Notice that $\gamma, \zeta < 1$. Furthermore, let $c = \lambda L_H \geqslant 0$ and define $u_h(t)$ as the left hand side in (60). By the triangle inequality, we get that

$$0 \leqslant u_h(t) := \left\|\frac{Dv(t,\cdot+h)-Dv(t,\cdot)}{|h|}\right\|_\infty \leqslant \frac{2}{|h|}t^{-\frac{1-\beta}{\alpha}}\|t^{\frac{1-\beta}{\alpha}}Dv(t,\cdot)\|_\infty. \tag{61}$$

Recall from Theorem 3.1 that $t^{\frac{1-\beta}{\alpha}}Dv$ is bounded over the entire $(0,T_0]\times\mathbb{R}^d$. By fixing $h \in \mathbb{R}^d\setminus\{0\}$, it follows that $u_h(t)$ is integrable over $(0,T_0]$. We can then use the generalized Grönwall inequality to deduce that for any $t \in (0,T_0]$, there exist constants $b_0, b_{T_0} \geqslant 0$ independent of t and h such that

$$\left\|\frac{Dv(t,\cdot+h)-Dv(t,\cdot)}{|h|}\right\|_\infty \leqslant b_0 t^{-\frac{2-\beta}{\alpha}} + b_{T_0}. \tag{62}$$

By noticing that this holds for any fixed $h \in \mathbb{R}^d\setminus\{0\}$, we conclude that $Dv(t,\cdot)$ is Lipschitz continuous in space for any fixed $t \in (0,T_0]$. By Rademacher's theorem (Theorem 2.10), this implies that $D^2v(t,\cdot)$ exists almost everywhere in $\mathbb{R}^d$.

In order to show existence everywhere, we use the a.e. existence together with the Duhamel formula. We begin by expressing D^2v as a limit. Dividing (57) by h and letting $h \to 0$ yields

$$\begin{aligned}\lim_{h\to0}\frac{Dv(t,x+h)-Dv(t,x)}{h} &= \lim_{h\to0}\frac{(DK(t,\cdot)*v_0)(x+h)-(DK(t,\cdot)*v_0)(x)}{h} \\ -\lim_{h\to0}\int_0^t &\frac{DK(t-s,\cdot)*(H(Dv(s,\cdot+h))-H(Dv(s,\cdot)))(x)}{h}ds \\ -\lim_{h\to0}\int_0^t &\frac{(DK(t-s,\cdot)*f(s,\cdot))(x+h)-(DK(t-s,\cdot)*f(s,\cdot))(x)}{h}ds.\end{aligned} \tag{63}$$

By the definition of the derivative, D^2v exists everywhere if all limits on the right hand side of (63) exist for any $(t,x) \in (0,T_0]\times\mathbb{R}^d$. The first limit exists by $v_0 \in C_b(\mathbb{R}^d)$ and the L^1-estimate for D^2K in Theorem 2.26. For the second term, we fix $(t,x) \in (0,T_0] \times \mathbb{R}^d$ and define

$$\Lambda_h(s,y) = DK(t-s,x-y)\,\frac{H(Dv(s,y+h)) - H(Dv(s,y))}{h}, \tag{64}$$

such that

$$\lim_{h\to 0}\int_0^t \frac{DK(t-s,\cdot) * (H(Dv(s,\cdot+h)) - H(Dv(s,\cdot)))(x)}{h}ds = \lim_{h\to 0}\int_0^t\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dy. \tag{65}$$

Recall that almost everywhere behavior is sufficient for using dominated convergence (see Theorem 2.9). Hence, by utilizing that D^2v exists almost everywhere, we can iteratively move the limit inside the integrals. We begin by showing that $\lim_{h\to 0}\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dy$ exists for any fixed $s \in (0,t)$ and is equal to $\int_{\mathbb{R}^d}\lim_{h\to 0}\Lambda_h(s,y)\,dy$.

Since $H \in C^1(\mathbb{R}^d)$, we have that for every $y \in \mathbb{R}^d$ such that $D^2(s,y)$ exists,

$$\lim_{h\to 0}\frac{H(Dv(s,y+h)) - H(Dv(s,y))}{h} = \partial_y H(Dv(s,y)) = D^2v(s,y)\,\partial_p H(Dv(s,y)). \tag{66}$$

It follows that $\lim_{h\to 0}\Lambda_h(s,y)$ exists almost everywhere in space for every $s \in (0,t)$. In order to use dominated convergence, it remains to find a dominating function which is integrable over $\mathbb{R}^d$. Notice that for any $y \in \mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$,

$$|\Lambda_h(s,y)| \leqslant L_H\,|DK(t-s,x-y)|\sup_{h\neq 0}\left\|\frac{Dv(s,y+h) - Dv(s,y)}{|h|}\right\|_\infty. \tag{67}$$

The right hand side dominates Λ_h and is integrable over $\mathbb{R}^d$ by (62) and $DK(t-s,\cdot) \in L^1(\mathbb{R}^d)$. It follows by dominated convergence that $\lim_{h\to 0}\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dy = \int_{\mathbb{R}^d}\lim_{h\to 0}\Lambda_h(s,y)\,dy$ exists for any $s \in (0,t)$.

It remains to show that the limit in (65) can be moved inside the time integral. Since we already know that $\lim_{h\to 0}\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dy$ exists, we only need to find a function which is integrable over $(0,t)$ and dominates $\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dy$. By (67) together with (62) and heat kernel estimates, we get that

$$\int_{\mathbb{R}^d}|\Lambda_h(s,y)|\,dy \leqslant L_H\lambda(t-s)^{-\frac{1}{\alpha}}\left(a_0 s^{-\frac{2-\beta}{\alpha}} + a_{T_0} + b_{T_0}\right).$$

The right hand side dominates $\int_{\mathbb{R}^d}|\Lambda_h(s,y)|\,dy$ and is integrable over $(0,t)$ since the exponents on $(t-s)$ and s are greater than -1. It follows by dominated convergence that

$$\lim_{h\to 0}\int_0^t\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dyds = \int_0^t\lim_{h\to 0}\int_{\mathbb{R}^d}\Lambda_h(s,y)\,dyds = \int_0^t\int_{\mathbb{R}^d}\lim_{h\to 0}\Lambda_h(s,y)\,dyds,$$

implying that the second limit in (63) exists.

We use a similar approach for the third limit. In order to move the limit inside the integral, recall from (59) that $\|D(DK(t-s,\cdot) * f(s,\cdot))\|_\infty \leqslant \lambda C_I(t-s)^{-\frac{2-\beta}{\alpha}}\tilde{C}_f$, which is integrable over $(0,t)$. By using (58), this is a dominating function to the last integrand in (63). In addition, the limit of this integrand as $h \to 0$ exists since $D(DK(t-s,\cdot) * f(s,\cdot))$ exists for all $s \in (0,t)$. By dominated convergence, it then follows that the third limit in (63) exists. Finally, since all limits on the right hand side of (63) exist, $D^2v(t,x)$ exists everywhere in $(0,T_0] \times \mathbb{R}^d$.

It only remains to show that $t^{\frac{2-\beta}{\alpha}}D^2v \in L^\infty((0,T_0] \times \mathbb{R}^d)$. By (62), we have that

$$t^{\frac{2-\beta}{\alpha}}\|D^2v(t,\cdot)\|_\infty \leqslant t^{\frac{2-\beta}{\alpha}}\left(b_0 t^{-\frac{2-\beta}{\alpha}} + b_{T_0}\right) \leqslant b_0 + T_0^{\frac{2-\beta}{\alpha}}.$$

Since the right hand side is finite and independent of t, we deduce that $t^{\frac{2-\beta}{\alpha}}D^2v$ is bounded uniformly in $(0,T_0] \times \mathbb{R}^d$, and the proof is complete. □

We proceed with showing spatial Hölder regularity for $v(t,x)$. As explained earlier in the section, we need to consider cases where $\alpha + \beta \leqslant 2$ and $\alpha + \beta > 2$ seperately due to the appearance of the second derivative. We will prove $(\alpha+\beta-\varepsilon)$-Hölder continuity of $v(t,\cdot)$ with time blowup, as well as β-Hölder continuity without time blowup. When $\alpha + \beta > 2$, we will in addition show that $t^{\frac{2-\beta}{\alpha}}D^2v \in C_b((0,T_0] \times \mathbb{R}^d)$, as we did for v and $t^{\frac{1-\beta}{\alpha}}Dv$ in Theorem 3.1.

Theorem 3.3 (Spatial regularity estimates for mild solutions)**.** *Let $\alpha \in (1,2)$, $\beta \in (0,1)$ and $\varepsilon > 0$, and let assumptions on v_0, f and H be as in Theorem 3.1. Furthermore, let $v \in C_b\left([0,T_0]\times\mathbb{R}^d\right)$ be the solution obtained in Theorem 3.1. Then, there exist constants $C_1, C_2 > 0$ such that*

$$
\begin{aligned}
&(a) && [v(t,\cdot)]_{C^{0,\beta}} \leqslant C_1, && \forall t \in [0,T_0],\\
&(b) && \text{If } \alpha+\beta \leqslant 2, \text{ then } \left[t^{\frac{\alpha-\varepsilon}{\alpha}} Dv\right]_{C^{0,\alpha+\beta-\varepsilon-1}} \leqslant C_2, && \forall t \in (0,T_0],\\
&(c) && \text{If } \alpha+\beta > 2, \text{ then } \left[t^{\frac{\alpha-\varepsilon}{\alpha}} D^2 v\right]_{C^{0,\alpha+\beta-\varepsilon-2}} \leqslant C_2, && \forall t \in (0,T_0],\\
&(d) && \text{If } \alpha+\beta > 2, \text{ then } t^{\frac{2-\beta}{\alpha}} D^2 v \in C_b\left((0,T_0]\times\mathbb{R}^d\right).
\end{aligned}
$$

Proof of Theorem 3.3 (a). By Theorem 3.1, we know that v is a mild solution to the HJB equation, and that it is a fixed point of the Duhamel map for $t \in (0,T_0]$. By a similar calculation as in (47), working with one less derivative, we get that

$$
[v(t,\cdot)]_{C^{0,\beta}} \leqslant [v_0]_{C^{0,\beta}} + \lambda^\beta C_{\beta,1}\int_0^t (t-s)^{-\frac{\beta}{\alpha}}\left(L_H s^{-\frac{1-\beta}{\alpha}}\|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty + H_0 + C_f\right) ds. \quad (68)
$$

We know from Theorem 3.1 that $\|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty$ is uniformly bounded in $(0,T_0]$. By an integration similar to (46), there exist constants $C, \tilde{C} \geqslant 0$ such that

$$
\begin{aligned}
[v(t,\cdot)]_{C^{0,\beta}} &\leqslant [v_0]_{C^{0,\beta}} + \sup_{s\in(0,T_0]}\|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty C t^{\frac{\alpha-1}{\alpha}} + \tilde{C} t^{\frac{\alpha-\beta}{\alpha}}\\
&\leqslant [v_0]_{C^{0,\beta}} + \sup_{s\in(0,T_0]}\|s^{\frac{1-\beta}{\alpha}} Dv(s,\cdot)\|_\infty C T_0^{\frac{\alpha-1}{\alpha}} + \tilde{C} T_0^{\frac{\alpha-\beta}{\alpha}} =: C_1.
\end{aligned}
$$

The last inequality holds since the exponents on t are positive. It follows that there exists $C_1 > 0$ independent of t such that $[v(t,\cdot)]_{C^{0,\beta}} \leqslant C_1$ for any $t \in (0,T_0]$. Finally, by noting that $[v(0,\cdot)]_{C^{0,\beta}} = [v_0]_{C^{0,\beta}} \leqslant C_1$, this holds in the entire $[0,T_0]$. □

Proof of Theorem 3.3 (b). Let $\eta := \alpha+\beta-\varepsilon-1$. We want to use the generalized Grönwall inequality (Lemma 2.11) to show that there exists $C_2 \geqslant 0$ such that $[t^{\frac{\alpha-\varepsilon}{\alpha}} Dv(t,\cdot)]_{C^{0,\eta}} \leqslant C_2$ for every $t \in (0,T_0]$. The most intuitive approach would be to let $u(t) := [Dv(t,\cdot)]_{C^{0,\eta}}$ in Lemma 2.11 and use the Duhamel formula to conclude. This requires $[Dv(s,\cdot)]_{C^{0,\eta}}$ to be integrable in $(0,T_0]$, however, which is not yet known. In order to circumvent this issue, we will instead consider Hölder quotients with fixed $h \in \mathbb{R}^d\setminus\{0\}$. This is similar to our approach in Lemma 3.2. For any fixed $h \in \mathbb{R}^d\setminus\{0\}$, we have that

$$
\begin{aligned}
\left\|\frac{Dv(t,\cdot+h)-Dv(t,\cdot)}{|h|^\eta}\right\|_\infty &\leqslant [DK(t,\cdot)*v_0]_{C^{0,\eta}} + \int_0^t [DK(t-s,\cdot)*f(s,\cdot)]_{C^{0,\eta}}\, ds\\
&\quad + \int_0^t \|DK(t-s)\|_1 \left\|\frac{H(Dv(s,\cdot+h))-H(Dv(s,\cdot))}{|h|^\eta}\right\|_\infty ds,
\end{aligned}
$$

where we have taken the supremum over $h \in \mathbb{R}^d\setminus\{0\}$ in the first two terms on the right hand side. In addition, we used Young's convolution inequality in the last integral. By using the interpolation inequality in Theorem 2.35 (a) on the first two terms, heat kernel estimates from Theorem 2.26 and the fact that H is globally Lipschitz, we get that

$$
\begin{aligned}
\left\|\frac{Dv(t,\cdot+h)-Dv(t,\cdot)}{|h|^\eta}\right\|_\infty &\leqslant \lambda C_I t^{-\frac{\alpha-\varepsilon}{\alpha}}[v_0]_{C^{0,\beta}} + \lambda C_I\int_0^t (t-s)^{-\frac{\alpha-\varepsilon}{\alpha}}[f(s,\cdot)]_{C^{0,\beta}}\, ds\\
&\quad + \lambda L_H\int_0^t (t-s)^{-\frac{1}{\alpha}}\left\|\frac{Dv(s,\cdot+h)-Dv(s,\cdot)}{|h|^\eta}\right\|_\infty ds. \quad (69)
\end{aligned}
$$

The second term on the right hand side is integrable and reaches its supremum when $t = T_0$. Then, there exist constants $a_0, a_{T_0} \geqslant 0$ such that

$$
\left\|\frac{Dv(t,\cdot+h)-Dv(t,\cdot)}{|h|^\eta}\right\|_\infty \leqslant a_0 t^{-\frac{\alpha-\varepsilon}{\alpha}} + a_{T_0} + \lambda L_H\int_0^t (t-s)^{-\frac{1}{\alpha}}\left\|\frac{Dv(s,\cdot+h)-Dv(s,\cdot)}{|h|^\eta}\right\|_\infty ds. \quad (70)
$$

We can now use the generalized Grönwall inequality in Lemma 2.11. Let $\gamma = (\alpha - \varepsilon)/\alpha$ and $\zeta = 1/\alpha$ and notice that $\gamma, \zeta < 1$. Furthermore, let $c = \lambda L_H \geqslant 0$ and define $u_h(t)$ as the left hand side in (70). We estimate u_h similarly to (61) and get that

$$0 \leqslant u_h(t) := \left\| \frac{Dv(t, \cdot + h) - Dv(t, \cdot)}{|h|^\eta} \right\|_\infty \leqslant \frac{2}{|h|^\eta} t^{-\frac{1-\beta}{\alpha}} \| t^{\frac{1-\beta}{\alpha}} Dv(t, \cdot) \|_\infty. \tag{71}$$

Since this upper bound is integrable in $(0, T_0]$ for any fixed $h \in \mathbb{R}^d \setminus \{0\}$, it follows that u_h is integrable in $(0, T_0]$ as well. Hence, by the generalized Grönwall inequality, there exists $b_0, b_{T_0} \geqslant 0$ independent of t an h such that

$$\left\| \frac{Dv(t, \cdot + h) - Dv(t, \cdot)}{|h|^\eta} \right\|_\infty \leqslant b_0 t^{-\frac{\alpha-\varepsilon}{\alpha}} + b_{T_0}.$$

Finally, by taking the supremum over $h \in \mathbb{R}^d \setminus \{0\}$ and multiplying by $t^{\frac{\alpha-\varepsilon}{\alpha}}$, we get that

$$t^{\frac{\alpha-\varepsilon}{\alpha}} [Dv(t, \cdot)]_{C^{0,\eta}} \leqslant b_0 + b_{T_0} t^{\frac{\alpha-\varepsilon}{\alpha}} \leqslant b_0 + b_{T_0} T_0^{\frac{\alpha-\varepsilon}{\alpha}} =: C_2.$$

Hence, there exists $C_2 \geqslant 0$ independent of $t \in (0, T_0]$ such that $[t^{\frac{\alpha-\varepsilon}{\alpha}} Dv(t, \cdot)]_{C^{0,\eta}} \leqslant C_2$, and the proof is complete. □

Proof of Theorem 3.3 (c). Let $\eta := \alpha + \beta - \varepsilon - 2$. We want to show that $[t^{\frac{\alpha-\varepsilon}{\alpha}} D^2 v]_{C^{0,\eta}} \leqslant C_2$. By Theorem 3.1, we know that $v \in C_b([0, T_0] \times \mathbb{R}^d)$ and $t^{\frac{1-\beta}{\alpha}} Dv \in C_b((0, T_0] \times \mathbb{R}^d)$. In addition, $D^2 v$ exists and $t^{\frac{2-\beta}{\alpha}} D^2 v \in L^\infty((0, T_0] \times \mathbb{R}^d)$ by Lemma 3.2.

We want to fix $h \in \mathbb{R}^d \setminus \{0\}$ and use the generalized Grönwall inequality (Lemma 2.11). By calculations similar to the previous proof, there exist constants $a_0, a_{T_0} \geqslant 0$ such that

$$\begin{aligned} \left\| \frac{D^2 v(t, \cdot + h) - D^2 v(t, \cdot)}{|h|^\eta} \right\|_\infty &\leqslant a_0 t^{-\frac{\alpha-\varepsilon}{\alpha}} + a_{T_0} \\ &+ \int_0^t \left\| \frac{DK(t-s, \cdot) * (\partial_x H(Dv(s, \cdot + h)) - \partial_x H(Dv(s, \cdot)))}{|h|^\eta} \right\|_\infty ds. \end{aligned} \tag{72}$$

The main differences in (72) from the previous proof are that we use Theorem 2.35 (b) instead of (a) for the interpolation, and that we put one derivative on H in the convolution. Define u_h as the left hand side in (72), and notice that

$$0 \leqslant u_h(t) := \left\| \frac{D^2 v(t, \cdot + h) - D^2 v(t, \cdot)}{|h|^\eta} \right\|_\infty \leqslant \frac{2}{|h|^\eta} t^{-\frac{2-\beta}{\alpha}} \| t^{\frac{2-\beta}{\alpha}} D^2 v(t, \cdot) \|_\infty. \tag{73}$$

Since $t^{\frac{2-\beta}{\alpha}} D^2 v \in L^\infty((0, T_0] \times \mathbb{R}^d)$ by Lemma 3.2, it follows that u_h is integrable in $(0, T_0]$. In order to use the Grönwall inequality, we need to isolate $u_h(s)$ in the integrand in (72), and arrive at an expression similar to (70). In the previous proof, we simply used the global Lipschitz condition of H. This is not possible in (70), however, since we now are dealing with the derivative of H. This complicates our argument, as we need to use the chain rule to estimate this term. Omitting the function arguments, we have that $DK * \partial_x H(Dv) = DK * D^2 v \partial_p H(Dv)$. Hence, the integrand in (72) essentially becomes a Hölder seminorm of a function convolved with a product of functions. We can estimate this term by using an analogoue of Proposition 2.38 for Hölder quotients (i.e. without the supremum over h). One can easily see that the Proposition holds for fixed $h \in \mathbb{R}^d \setminus \{0\}$ by removing the suprema over h in the proof. It follows that

$$\begin{aligned} &\left\| \frac{DK(t-s, \cdot) * (\partial_x H(Dv(s, \cdot + h)) - \partial_x H(Dv(s, \cdot)))}{|h|^\eta} \right\|_\infty \\ &\leqslant \| D^2 v(s, \cdot) \|_\infty [DK(t-s, \cdot) * \partial_p H(Dv(s, \cdot))]_{C^{0,\eta}} \\ &+ \| \partial_p H(Dv(s, \cdot)) \|_\infty \left\| \frac{DK(t-s, \cdot) * (D^2 v(s, \cdot + h) - D^2 v(s, \cdot))}{|h|^\eta} \right\|_\infty. \end{aligned} \tag{74}$$

In the first term, we want to put as much regularity on DK as possible. Since H is globally Lipschitz, we can bound $\partial_p H(Dv)$ by the Lipschitz constant L_H. Furthermore, by our interpolation result in Theorem 2.34, Young's inequality and heat kernel estimates from Theorem 2.26, we get that

$$\begin{aligned}[DK(t-s,\cdot)*\partial_p H(Dv(s,\cdot))]_{C^{0,\eta}} &\leqslant C_{\eta,1}\|DK(t-s,\cdot)\|_1^{1-\eta}\|D^2K(t-s,\cdot)\|_1^{\eta}\|\partial_p H(Dv(s,\cdot))\|_\infty \\ &\leqslant C_{\eta,1}\lambda(t-s)^{-\frac{1+\eta}{\alpha}}L_H = C_{\eta,1}\lambda(t-s)^{-\frac{\alpha+\beta-\varepsilon-1}{\alpha}}L_H.\end{aligned}$$

By recalling that $t^{\frac{2-\beta}{\alpha}}D^2v(t,\cdot)\in L^\infty\left((0,T_0]\times\mathbb{R}^d\right)$, we have $\|D^2v(t,\cdot)\|_\infty\leqslant C't^{-\frac{2-\beta}{\alpha}}$ in $(0,T_0]$ for some $C'\geqslant 0$. It follows that

$$\|D^2v(s,\cdot)\|_\infty[DK(t-s,\cdot)*\partial_p H(Dv(s,\cdot))]_{C^{0,\eta}}\leqslant C_{\eta,1}L_HC'\lambda(t-s)^{-\frac{\alpha+\beta-\varepsilon-1}{\alpha}}s^{-\frac{2-\beta}{\alpha}}, \tag{75}$$

which is integrable over $(0,t)$ since $\beta<1$ and $\alpha+\beta>2$. In order to use Grönwall, we need $u_h(s)$ to appear in the last term in (74). We get that

$$\begin{aligned}&\|\partial_p H(Dv(s,\cdot))\|_\infty\left\|\frac{DK(t-s,\cdot)*\left(D^2v(s,\cdot+h)-D^2v(s,\cdot)\right)}{|h|^\eta}\right\|_\infty \\ &\leqslant L_H\|DK(t-s,\cdot)\|_1\left\|\frac{D^2v(s,\cdot+h)-D^2v(s,\cdot)}{|h|^\eta}\right\|_\infty\leqslant\lambda L_H(t-s)^{-\frac{1}{\alpha}}u_h(s).\end{aligned} \tag{76}$$

By inserting (75) and (76) into (74) and recalling (72), it follows that

$$\begin{aligned}u_h(t)&\leqslant a_0t^{-\frac{\alpha-\varepsilon}{\alpha}}+a_{T_0}+\int_0^t C_{\eta,1}L_HC'\lambda(t-s)^{-\frac{\alpha+\beta-\varepsilon-1}{\alpha}}s^{-\frac{2-\beta}{\alpha}}ds+\lambda L_H\int_0^t(t-s)^{-\frac{1}{\alpha}}u_h(s)\,ds \\ &\leqslant a_0t^{-\frac{\alpha-\varepsilon}{\alpha}}+a_{T_0}+a_1t^{-\frac{1-\varepsilon}{\alpha}}+\lambda L_H\int_0^t(t-s)^{-\frac{1}{\alpha}}u_h(s)\,ds \\ &\leqslant\left(a_0+a_1T_0^{\frac{\alpha-1}{\alpha}}\right)t^{-\frac{\alpha-\varepsilon}{\alpha}}+a_{T_0}+\lambda L_H\int_0^t(t-s)^{-\frac{1}{\alpha}}u_h(s)\,ds,\end{aligned}$$

for some $a_1\geqslant 0$. In the last inequality, we used that $t^{-\frac{1-\varepsilon}{\alpha}}\leqslant T_0^{\frac{\alpha-1}{\alpha}}t^{-\frac{\alpha-\varepsilon}{\alpha}}$ since $\frac{\alpha-1}{\alpha}>0$.

We can now use the generalized Grönwall inequality in Lemma 2.11. Let $\gamma=(\alpha-\varepsilon)/\alpha$ and $\zeta=1/\alpha$ and notice that $\gamma,\zeta<1$. We already know that u_h is integrable from (71). It follows by the Grönwall inequality that there exist constants $b_0,b_{T_0}\geqslant 0$ independent of t and h such that

$$u_h(t)\leqslant b_0t^{-\frac{\alpha-\varepsilon}{\alpha}}+b_{T_0}.$$

Finally, by letting $h\to 0$ and multiplying with $t^{\frac{\alpha-\varepsilon}{\alpha}}$, we get that

$$\left[t^{\frac{\alpha-\varepsilon}{\alpha}}D^2v(t,\cdot)\right]_{C^{0,\eta}}\leqslant b_0+b_{T_0}t^{\frac{\alpha-\varepsilon}{\alpha}}\leqslant b_0+b_{T_0}T_0^{\frac{\alpha-\varepsilon}{\alpha}}=:C_2.$$

Hence, there exists $C_2\geqslant 0$ independent of $t\in(0,T_0]$ such that $[t^{\frac{\alpha-\varepsilon}{\alpha}}D^2v(t,\cdot)]_{C^{0,\eta}}\leqslant C_2$, and the proof is complete. □

Proof of Theorem 3.3 (d). This is very similar to the proof of $t^{\frac{1-\beta}{\alpha}}D\phi(v)\in C_b\left((0,T_0]\times\mathbb{R}^d\right)$ in Theorem 3.1, and we will be brief. From Theorem 3.3 (c), we know that $t^{\frac{\alpha-\varepsilon}{\alpha}}D^2v(t,\cdot)$ is η-Hölder continuous in space, where $\eta=\alpha+\beta-\varepsilon-2$. By fixing $t\in(0,T_0]$, we can ignore the time blowup and get that $t^{\frac{2-\beta}{\alpha}}D^2v(t,\cdot)\in C_b\left(\mathbb{R}^d\right)$. Recall that $v(t,x)$ is a fixed point of the Duhamel map in (27), and assume first that $t_0<T_0$. By differentiating (31) twice, subtracting $D^2v(t_0,x)$ and taking the absolute value, we get that

$$\begin{aligned}&\left|D^2v(t_0+\tau,x)-D^2v(t_0,x)\right|\leqslant\left|\left(K(\tau,\cdot)*D^2v(t_0,\cdot)\right)(x)-D^2v(t_0,x)\right| \\ &+\left|\int_{t_0}^{t_0+\tau}\left(D^2K(t_0+\tau-s,\cdot)*(H(Dv(s,\cdot))-f(s,\cdot))\right)(x)\,ds\right| \\ &\leqslant\left|\left(K(\tau,\cdot)*D^2v(t_0,\cdot)\right)(x)-D^2v(t_0,x)\right| \\ &+C_1\sup_{s\in(0,T_0]}\|s^{\frac{2-\beta}{\alpha}}D^2v(s,\cdot)\|_\infty\left|(t_0+\tau)^{\frac{\alpha+\beta-3}{\alpha}}-t_0^{\frac{\alpha+\beta-3}{\alpha}}\right|+C_2\left|(t_0+\tau)^{\frac{\alpha+\beta-2}{\alpha}}-t_0^{\frac{\alpha+\beta-2}{\alpha}}\right|,\end{aligned} \tag{77}$$

for constants $C_1, C_2 \geqslant 0$. In the last inequality, we used a calculation similar to (59) and that $t^{\frac{2-\beta}{\alpha}} D^2 v \in L^\infty\left((0, T_0] \times \mathbb{R}^d\right)$ by Lemma 3.2. By noticing that $0 < \frac{\alpha+\beta-2}{\alpha} < 1$, we get that $0 < (t_0+\tau)^{\frac{\alpha+\beta-2}{\alpha}} - t_0^{\frac{\alpha+\beta-2}{\alpha}} \leqslant \tau^{\frac{\alpha+\beta-2}{\alpha}}$. The last term in (77) then goes to zero as $\tau \to 0^+$. For the second term, we argue as in (52) since $-1 < \frac{\alpha+\beta-3}{\alpha} < 0$. Finally, we use Lemma 2.23 with $g := D^2 v(t_0, x)$ and $\gamma = \eta$ to deduce that

$$\begin{aligned} &\left|K(\tau,\cdot) * D^2 v(t_0,\cdot)(x) - D^2 v(t_0, x)\right| \\ &\qquad \leqslant A_d c_K \left(\frac{2\|D^2 v(t_0,\cdot)\|_\infty}{\alpha} \tau^{\frac{\eta}{2(d+\eta)}} + \frac{\left[D^2 v(t_0,\cdot)\right]_{C^{0,\eta}}}{d+\eta} \tau^{\frac{\eta}{2\alpha}} \right). \end{aligned} \tag{78}$$

All terms on the right hand side of (77) then go to zero as $\tau \to 0^+$, and it follows that $D^2 v(t_0, x)$ is continuous in time for any $t_0 \in (0, T_0)$. For $t_0 = T_0$, we repeat the argument with $D^2 v(T_0, x) - D^2 v(T_0 - \tau, x)$ in (77) and argue as in Theorem 3.1. Finally, since $t^{\frac{2-\beta}{\alpha}}$ is continuous, it follows that $t^{\frac{2-\beta}{\alpha}} D^2 v \in C_b\left((0, T_0] \times \mathbb{R}^d\right)$, and the proof is complete. □

3.3 Existence of a classical short time solution

In this section, we show that the mild solution $v(t, x)$ obtained in Theorem 3.1 is classical. The proof is fairly technical and uses a combination of Fubini-Tonelli and dominated convergence arguments (see Theorem 2.8 and 2.9). We will use that $v(t,\cdot) \in C^{\alpha+\beta-\varepsilon}\left(\mathbb{R}^d\right)$ for any fixed $t \in (0, T_0]$ as we need at least $(\alpha+\delta)$-Hölder continuity in space for some $\delta > 0$ to ensure that $-(-\Delta)^{\alpha/2} v(t, x)$ is well-defined (see Proposition 2.15). This follows from our regularity estimates in Theorem 3.3.

Theorem 3.4 (Existence of a classical solution)**.** *Let assumptions on v_0, f and H be as in Theorem 3.1, and let v be the mild solution obtained in the theorem. Then, v is a classical solution to the HJB equation (37) in $(0, T_0] \times \mathbb{R}^d$.*

The proof is similar to Lemma 5 in [12] where equivalence between the time derivative and the fractional Laplacian of the integral in the Duhamel map is shown. Notice, however, that this Lemma assumes spatial C^2-regularity on the source term. Since we only have $f(t,\cdot) \in C^{0,\beta}\left(\mathbb{R}^d\right)$, we cannot use the result directly. We will therefore provide a proof in the case of a β-Hölder continuous source term below. The proof is mostly self contained, and will only assume the following result, which is easily derived from Proposition 1 in [12].

Proposition 3.5 (Proposition 1 in [12])**.** *Let $K(t, x)$ be the fractional heat kernel and let $w \in C_b\left(\mathbb{R}^d\right)$. Then, for any $t > 0$ and any $x \in \mathbb{R}^d$, we have that*

$$\frac{\partial}{\partial t}\left((K(t,\cdot) * w)(x)\right) = -(-\Delta)^{\frac{\alpha}{2}}\left((K(t,\cdot) * w)(x)\right).$$

Proof of Theorem 3.4 (Corrected from [2]). The result was incorrectly proven in [2], and we give a revised proof. By Theorem 3.1, v is a fixed point of the Duhamel map in (27), meaning that

$$v(t, x) = (K(t,\cdot) * v_0)(x) - \int_0^t \left(K(t-s,\cdot) * \left(H(Dv(s,\cdot)) - f(s,\cdot)\right)\right)(x)\, ds. \tag{79}$$

Since $v_0 \in C^{0,\beta}\left(\mathbb{R}^d\right)$ and $K(t,\cdot) \in C_b^\infty\left(\mathbb{R}^d\right)$, it follows that $K(t,\cdot) * v_0 \in C_b^\infty\left(\mathbb{R}^d\right)$, and by Proposition 2.15 that $-(-\Delta)^{\frac{\alpha}{2}}(K(t,\cdot) * v_0)$ is bounded. Furthermore, since $v_0 \in C^{0,\beta}\left(\mathbb{R}^d\right)$ implies $v_0 \in C_b\left(\mathbb{R}^d\right)$, we can use Proposition 3.5 to derive that

$$-(-\Delta)^{\frac{\alpha}{2}}(K(t,\cdot) * v_0)(x) = \frac{\partial}{\partial t}(K(t,\cdot) * v_0)(x). \tag{80}$$

For simplicity, let $g(s, x) := H(Dv(s, x)) - f(s, x)$. We can show that

$$\|g(s,\cdot)\|_\infty \leqslant C_{g_0} + \tilde{C}_{g_0} s^{-\frac{1-\beta}{\alpha}} \quad \text{and} \quad [g(s,\cdot)]_{C^{0,\beta}} \leqslant C_{g_\beta} + \tilde{C}_{g_\beta} s^{-\frac{1}{\alpha}}, \tag{81}$$

for constants $C_{g_0}, \tilde{C}_{g_0}, C_{g_\beta}, \tilde{C}_{g_\beta} \geqslant 0$. The L^∞-estimate follows directly from (42) and uniform boundedness of f. For the Hölder estimate, we use that $f(s,\cdot) \in C^{0,\beta}(\mathbb{R}^d)$, H is globally Lipschitz, and interpolate $[Dv(s,\cdot)]_{C^{0,\beta}}$ between $\|Dv(s,\cdot)\|_\infty$ and $[v(s,\cdot)]_{C^{\alpha+\beta-\varepsilon}}$ using Theorem 2.33. The $s^{-1/\alpha}$-factor in (81) appears from the time blowup on these terms in Theorem 3.1 and 3.3.

By (79) and (80), it only remains to show that

$$-(-\Delta)^{\frac{\alpha}{2}}\left(\int_0^t (K(t-s,\cdot) * g(s,\cdot))(x)\,ds\right) = \frac{\partial}{\partial t}\left(\int_0^t (K(t-s,\cdot) * g(s,\cdot))(x)\,ds\right) - g(t,x), \quad (82)$$

for v to be a classical solution to the HJB equation. Let $\sigma(t,s,x) := (K(t-s,\cdot) * g(s,\cdot))(x)$. By Young's inequality and $\|K(t-s,\cdot)\|_1 = 1$, we have that $\|\sigma(t,s,\cdot)\|_\infty \leqslant \|g(s,\cdot)\|_\infty$ which is integrable on $0 < s < t$ by (81). Furthermore, using heat kernel estimates from Theorem 2.26 and an interpolation as in Theorem 2.35, we get that

$$[\sigma(t,s,\cdot)]_{C^{\alpha+\beta-\varepsilon}} \leqslant C_I \lambda (t-s)^{-\frac{\alpha-\varepsilon}{\alpha}} [g(s,\cdot)]_{C^{0,\beta}}, \quad (83)$$

which is integrable on $0 < s < t$ by (81). It follows that

$$\int_0^t \|\sigma(t,s,\cdot)\|_\infty ds < \infty \quad \text{and} \quad \int_0^t [\sigma(t,s,\cdot)]_{C^{\alpha+\beta-\varepsilon}}\,ds < \infty. \quad (84)$$

By the singular integral definition of K in (13) and the linearity of the integral, we have that

$$-(-\Delta)^{\frac{\alpha}{2}}\left(\int_0^t \sigma(t,s,x)\,ds\right) = \lim_{r\to 0^+}\int_{\mathbb{R}^d\setminus B_r(0)}\int_0^t \frac{c_{d,\alpha}}{|z|^{d+\alpha}}(\sigma(t,s,x+z)-\sigma(t,s,x))\,dsdz. \quad (85)$$

The limit exists if the left hand side is finite, which follows from Proposition 2.15 since (84) implies that $\int_0^t \sigma(t,s,\cdot)\,ds \in C^{\alpha+\beta-\varepsilon}(\mathbb{R}^d)$.

In order to show (82), we need to move the fractional Laplacian inside the integral. This corresponds to interchanging the integrals and limit in (85), which requires us to use the Fubini-Tonelli theorem, as well as Lebesgue's dominated convergence theorem. These are stated in Theorem 2.8 and Theorem 2.9 respectively. We can interchange the integrals in (85) by Fubini-Tonelli if the integrand is absolutely integrable over $(\mathbb{R}^d\setminus B_r(0)) \times (0,t)$ for any fixed $r > 0$. Notice that

$$\left|\frac{c_{d,\alpha}}{|z|^{d+\alpha}}(\sigma(t,s,x+z)-\sigma(t,s,x))\right| \leqslant \frac{2c_{d,\alpha}}{|z|^{d+\alpha}}\|\sigma(t,s,\cdot)\|_\infty. \quad (86)$$

The factor $|z|^{-(d+\alpha)}$ ensures integrability in $\mathbb{R}^d\setminus B_r(0)$ since we integrate in d dimensions and $\alpha > 1$. Furthermore, the resulting function is integrable over $(0,t)$ by (84). We can now use the Fubini-Tonelli theorem to deduce that

$$-(-\Delta)^{\frac{\alpha}{2}}\left(\int_0^t \sigma(t,s,x)\,ds\right) = \lim_{r\to 0^+}\int_0^t\int_{\mathbb{R}^d\setminus B_r}\frac{c_{d,\alpha}}{|z|^{d+\alpha}}(\sigma(t,s,x+z)-\sigma(t,s,x))\,dzds. \quad (87)$$

In order to interchange the limit and the outer integral in (87), we need to use dominated convergence. Let $G_r(t,s,x)$ be the inner integral above such that

$$-(-\Delta)^{\frac{\alpha}{2}}\left(\int_0^t \sigma(t,s,x)\,ds\right) = \lim_{r\to 0^+}\int_0^t G_r(t,s,x)\,ds,$$

and let $G^*(t,s,x) = -(-\Delta)^{\frac{\alpha}{2}}\sigma(t,s,x)$. Notice that $G_r(t,s,x)$ converges pointwise to $G^*(t,s,x)$ as $r \to 0^+$ for any $s \in (0,t)$ by the singular integral definition of $-(-\Delta)^{\alpha/2}$. In order to use dominated convergence, we need to find a function which is absolutely integrable over $(0,t)$, and that dominates $G_r(t,s,x)$ for any $r > 0$. Notice that

$$\begin{aligned}
|G_r(t,s,x)| &\leqslant \int_{\mathbb{R}^d\setminus B_r(0)}\frac{c_{d,\alpha}}{|z|^{d+\alpha}}|\sigma(t,s,x+z)-\sigma(t,s,x)|\,dz\\
&\leqslant [\sigma(t,s,\cdot)]_{C^{\alpha+\beta-\varepsilon}}\int_{B_1(0)\setminus B_r(0)}\frac{c_{d,\alpha}}{|z|^{d-\beta+\varepsilon}}dz + \|\sigma(t,s,\cdot)\|_\infty\int_{\mathbb{R}^d\setminus B_1(0)}\frac{c_{d,\alpha}}{|z|^{d+\alpha}}dz\\
&\leqslant [\sigma(t,s,\cdot)]_{C^{\alpha+\beta-\varepsilon}}\int_{B_1(0)}\frac{c_{d,\alpha}}{|z|^{d-\beta+\varepsilon}}dz + C_2\|\sigma(t,s,\cdot)\|_\infty\\
&\leqslant C_1[\sigma(t,s,\cdot)]_{C^{\alpha+\beta-\varepsilon}} + C_2\|\sigma(t,s,\cdot)\|_\infty, \quad (88)
\end{aligned}$$

for constants $C_1, C_2 > 0$, since the integrands above are integrable in their respective domains. The resulting bound is a dominating function and is absolutely integrable over $(0,t)$ by (84). Then, by Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned} -(-\Delta)^{\frac{\alpha}{2}}\left(\int_0^t \sigma(t,s,x)\,ds\right) &= \lim_{r\to 0^+}\int_0^t G_r(t,s,x)\,ds \\ &= \int_0^t G^*(t,s,x)\,ds = \int_0^t -(-\Delta)^{\frac{\alpha}{2}}\sigma(t,s,x)(x)\,ds. \end{aligned}$$

Having moved the fractional Laplacian inside the integral, we can once more use Proposition 3.5 to show that

$$-(-\Delta)^{\frac{\alpha}{2}}\left(\int_0^t \sigma(t,s,x)\,ds\right) = \int_0^t -(-\Delta)^{\frac{\alpha}{2}}\sigma(t,s,x)\,ds = \int_0^t \frac{\partial}{\partial t}\sigma(t,s,x)\,ds. \tag{89}$$

The last equality holds by Proposition 3.5 since $\sigma(t,s,x) := (K(t-s,\cdot) * g(s,\cdot))(x)$, and $g(s,\cdot) \in C_b(\mathbb{R}^d)$ for any fixed $s \in (0,t)$.

By (82) and (89), it only remains to show that

$$\int_0^t \frac{\partial}{\partial t}\sigma(t,s,x)\,ds = \frac{\partial}{\partial t}\left(\int_0^t \sigma(t,s,x)\,ds\right) - g(t,x). \tag{90}$$

We will once more use dominated convergence to complete the proof. By writing the derivative in the left hand side as a difference, we have that

$$\int_0^t \frac{\partial}{\partial t}\sigma(t,s,x)\,ds = \int_0^t \lim_{\tau\to 0^+}\frac{\sigma(t+\tau,s,x)-\sigma(t,s,x)}{\tau}ds. \tag{91}$$

Define $\Lambda_\tau(t,s,x) = (\sigma(t+\tau,s,x)-\sigma(t,s,x))/\tau$ and let $\Lambda^*(t,s,x)$ be its limit as $\tau \to 0$, which coincides with $\partial_t\sigma(t,s,x)$. Then, Λ_τ converges pointwise to Λ^* for any $s \in (0,t)$. By $g(s,\cdot) \in C_b(\mathbb{R}^d)$, the definition of σ and since K is continuous and differentiable in time, it follows that $\sigma(t,s,x)$ is continuous and differentiable with respect to t. We can then use the mean value theorem to deduce that for any fixed $t \in (0,T_0)$ and $s \in (0,t)$, there exists $\eta_{s,t}(\tau) \in (0,\tau)$ such that

$$|\Lambda_\tau(t,s,x)| = \frac{1}{\tau}|\sigma(t+\tau,s,x)-\sigma(t,s,x)| = \left|\frac{\partial}{\partial t}\sigma(t+\eta_{s,t}(\tau),s,x)\right|.$$

By $\sigma(t+\eta_{s,t}(\tau),s,x) = K(t+\eta_{s,t}(\tau)-s,\cdot) * g(s,\cdot)(x)$ and $g(s,\cdot) \in C_b(\mathbb{R}^d)$, it follows from Proposition 3.5 that

$$|\Lambda_\tau(t,s,x)| = \left|\frac{\partial}{\partial t}\sigma(t+\eta_{s,t}(\tau),s,x)\right| = \left|-(-\Delta)^{\frac{\alpha}{2}}\sigma(t+\eta_{s,t}(\tau),s,x)\right|. \tag{92}$$

Furthermore, by a calculation similar to (88), we have

$$|\Lambda_\tau(t,s,x)| \leqslant C_1[\sigma(t+\eta_{s,t}(\tau),s,\cdot)]_{C^{\alpha+\beta-\varepsilon}} + C_2\|\sigma(t+\eta_{s,t}(\tau),s,\cdot)\|_\infty, \tag{93}$$

for constants $C_1, C_2 > 0$. We estimate as in (83), and get that

$$\begin{aligned} |\Lambda_\tau(t,s,x)| &\leqslant C_1C_I\lambda(t+\eta_{s,t}(\tau)-s)^{-\frac{\alpha-\varepsilon}{\alpha}}[g(s,\cdot)]_{C^{0,\beta}} + C_2\|g(s,\cdot)\|_\infty \\ &\leqslant C_1C_I\lambda(t-s)^{-\frac{\alpha-\varepsilon}{\alpha}}[g(s,\cdot)]_{C^{0,\beta}} + C_2\|g(s,\cdot)\|_\infty. \end{aligned} \tag{94}$$

In the last inequality, we used that $\eta_{s,t}(\tau) \geqslant 0$ and that $-\frac{\alpha-\varepsilon}{\alpha}$ is negative. Observe that the right hand side in (94) dominates $\Lambda_\tau(t,s,x)$ for any $\tau \geqslant 0$, and is absolutely integrable on $0 < s < t$. We can then use dominated convergence in (91) to deduce that

$$\begin{aligned} &\int_0^t \frac{\partial}{\partial t}\sigma(t,s,x)\,ds = \lim_{\tau\to 0^+}\int_0^t \frac{\sigma(t+\tau,s,x)-\sigma(t,s,x)}{\tau}ds \\ &= \lim_{\tau\to 0^+}\frac{1}{\tau}\left(\int_0^{t+\tau}\sigma(t+\tau,s,x)\,ds - \int_0^t \sigma(t,s,x)\,ds\right) - \lim_{\tau\to 0^+}\frac{1}{\tau}\int_t^{t+\tau}\sigma(t+\tau,s,x)\,ds \\ &= \frac{\partial}{\partial t}\left(\int_0^t \sigma(t,s,x)\,ds\right) - \int_0^1 \lim_{\tau\to 0^+}\sigma(t+\tau,t+r\tau,x)\,dr, \end{aligned} \tag{95}$$

where we have used the substitution $s = t + r\tau$. The limit is moved inside the integral in the last equality by dominated convergence. The corresponding dominating function was derived by letting $\tau < \tau_0$ for some $\tau_0 < T_0 - t$ such that for any $r \in (0, 1)$,

$$\begin{aligned} |\sigma(t+\tau, t+r\tau, x)| &\leqslant \|K(t+\tau-(t+r\tau), \cdot)\|_{L^1(\mathbb{R}^d)} \|g(t+r\tau, \cdot)\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant \|g\|_{L^\infty((t,t+\tau)\times\mathbb{R}^d)} \leqslant \|g\|_{L^\infty((t,t+\tau_0)\times\mathbb{R}^d)}. \end{aligned} \tag{96}$$

Integrability of the dominating function was ensured by using the L^∞-estimate for g in (81), which holds in the entire $(t, t+\tau_0) \times \mathbb{R}^d$ since $t + \tau_0 < T_0$.

By writing out the definition of σ in the last integral in (95), we get that

$$\begin{aligned} \int_0^t \frac{\partial}{\partial t}\sigma(t,s,x)\,ds &= \frac{\partial}{\partial t}\left(\int_0^t \sigma(t,s,x)\,ds\right) - \int_0^1 \lim_{\tau\to 0^+} K((1-r)\tau,\cdot) * g(t+r\tau,\cdot)(x)dr \\ &= \frac{\partial}{\partial t}\left(\int_0^t \sigma(t,s,x)\,ds\right) - \int_0^1 \lim_{\tau\to 0^+} g(t+r\tau,x)\,dr \\ &\quad - \int_0^1 \lim_{\tau\to 0^+} K((1-r)\tau,\cdot) * g(t+r\tau,\cdot)(x) - g(t+r\tau,\cdot)(x)dr. \end{aligned}$$

The integrand in the last integral goes to zero as $\tau \to 0^+$ by Lemma 2.23 since $g(t+r\tau,\cdot) \in C^{0,\beta}(\mathbb{R}^d)$ for any $r \in (0,1)$ and $\tau < \tau_0$. Finally, we deduce that

$$\begin{aligned} \int_0^t \frac{\partial}{\partial t}\sigma(t,s,x)\,ds &= \frac{\partial}{\partial t}\left(\int_0^t \sigma(t,s,x)\,ds\right) - \int_0^1 \lim_{\tau\to 0^+} g(t+r\tau,x)\,dr \\ &= \frac{\partial}{\partial t}\left(\int_0^t \sigma(t,s,x)\,ds\right) - g(t,x), \end{aligned} \tag{97}$$

where the last equality holds since $g(\cdot,x)$ is continuous in $(t, t+\tau)$ for any $x \in \mathbb{R}^d$ and $\tau < \tau_0$. This time continuity follows from the fact that H is globally Lipschitz, and since f and Dv are continuous in time. By combining (80), (89) and (97), it follows that $v(t,x)$ is a classical solution to the HJB equation in $(0, T_0) \times \mathbb{R}^d$.

For $t = T_0$, we write the difference in (91) as $(\sigma(T_0,s,x) - \sigma(T_0-\tau,s,x))/\tau$ and repeat the argument. The main difference in the proof is in (94), where we get $(t - \eta_{s,t}(\tau) - s)^{-\frac{\alpha-\varepsilon}{\alpha}}$ instead. By letting $\tau < \tau_0$, we can bound it by $(t - \tau_0 - s)^{-\frac{\alpha-\varepsilon}{\alpha}}$ which is independent of τ, thus reaching a dominating function. The other calculations are similar, and we conclude that our solution is classical at $t = T_0$. It follows that v is classical solution to the HJB equation (37) in the entire $(0, T_0] \times \mathbb{R}^d$. □

3.4 Uniqueness results

In Section 3.1, we proved the existence of a unique mild solution to the HJB equation (37). Although this solution was shown to be classical in Theorem 3.4, it does not necessarily mean that it is unique among the classical solutions. We will therefore need to investigate uniqueness in a seperate argument by using the comparison principle. This was briefly introduced in Section 2.9.

Inspired by the theory on sub- and supersolutions in Section 2.9, we say that v^-, v^+ are sub- and supersolutions to the HJB equation respectively if they satisfy the relations

$$\begin{cases} \partial_t v^-(t,x) + (-\Delta)^{\frac{\alpha}{2}} v^-(t,x) + H(Dv^-(t,x)) \leqslant f(t,x) \\ \partial_t v^+(t,x) + (-\Delta)^{\frac{\alpha}{2}} v^+(t,x) + H(Dv^+(t,x)) \geqslant f(t,x) \end{cases} \quad \text{in } (0,T_0] \times \mathbb{R}^d, \tag{98}$$

and if $v^-(0,x) \leqslant v_0(x) \leqslant v^+(0,x)$ in $\mathbb{R}^d$.

The classical solution v from Theorem 3.4 is both a subsolution and a supersolution since the inequalities above are non-strict. The existence of functions v^- and v^+ is therefore evident.

It suffices to show that the inequalities in (98) imply $v^- \leqslant v^+$ in the whole $[0, T_0] \times \mathbb{R}^d$, as uniqueness then follows by an argument similar to Lemma 2.42. Notice, however, that since our domain is unbounded in the spatial dimensions, existence of a global maximum is not guaranteed. As the comparison principle depends on the existence of global maxima, we will need to modify our functions through penalization to prove uniqueness. Such a method was introduced in Section 2.10.

Lemma 3.6 (Comparison principle for the HJB equation)**.** *Suppose v^- and v^+ are sub- and supersolutions to the HJB equation defined in (98), and suppose that $v^-, v^+ \in C_b\left([0, T_0] \times \mathbb{R}^d\right)$. Given functions v_0, f and H as in Theorem 3.1, it follows that $v^- \leqslant v^+$ in $[0, T_0] \times \mathbb{R}^d$.*

Proof. Define the function $v_d :\equiv v^- - v^+$ as the difference between the sub- and supersolution. It suffices to show that $v_d \leqslant 0$ in $(0, T_0] \times \mathbb{R}^d$ to prove the Lemma. By (98), we have that

$$\frac{\partial}{\partial t} v_d(t,x) + (-\Delta)^{\frac{\alpha}{2}} v_d(t,x) + \left(H\left(Dv^-(t,x)\right) - H\left(Dv^+(t,x)\right)\right) \leqslant 0, \quad \text{in } (0, T_0] \times \mathbb{R}^d, \tag{99a}$$

$$v_d(0,x) \leqslant 0, \quad \text{in } \mathbb{R}^d. \tag{99b}$$

Hence, $v_d(t,x) \leqslant 0$ is already satisfied whenever $t = 0$. To prove that this inequality holds in the entire domain, it suffices to show that any supremum of v_d is attained at $t = 0$. We begin by assuming that v_d has a global maximum in $[0, T_0] \times \mathbb{R}^d$. As this may not be true, we will need to inspect the case where only a supremum is attained as well.

Let (t_0, x_0) be a global maximum of v_d such that $t_0 \in [0, T_0]$ and $x_0 \in \mathbb{R}^d$. We want to show that $t_0 = 0$ must hold. Assume by contradiction that there exists some global maximum (t_0, x_0) where $t_0 \in (0, T_0]$. The spatial first-order-derivatives are clearly zero at this point. The Hamiltonian term in (99a) will then cancel due to the Lipschitz continuity of H since

$$\left|H\left(Dv^-(t_0,x_0)\right) - H\left(Dv^+(t_0,x_0)\right)\right| \leqslant L_H \left|Dv^-(t_0,x_0) - Dv^+(t_0,x_0)\right| = L_H \left|Dv_d(t_0,x_0)\right| = 0,$$

where $L_H > 0$ is the Lipschitz constant. Furthermore, $\partial_t v_d(t_0, x_0) \geqslant 0$ must hold, since $t_0 \in (0, T_0]$ implies a zero time derivative and $t_0 = T_0$ implies a positive time derivative in order for it to be a global maximum. Combining these observations with (99a), it follows that $(-\Delta)^{\alpha/2} v_d(t_0, x_0) \leqslant 0$ must hold. However, by Proposition 2.16, we know that $(-\Delta)^{\alpha/2} v_d(t_0, x_0) \geqslant 0$ whenever (t_0, x_0) is a global maximum. This nearly leads to a contradiction, but since both inequalities are non-strict, it is not entirely enough. In order to arrive at a contradiction, we will instead consider a modified difference function with time penalization.

Let $\tilde{v_d}(t,x) = v^-(t,x) - v^+(t,x) - qt$ for some $q > 0$. Since we assume that v_d has a global maximum in $(0, T_0] \times \mathbb{R}^d$, it follows that $\tilde{v_d}$ has a global maximum in $(0, T_0] \times \mathbb{R}^d$ as well. Consider now (t_0, x_0) to be a maximum point of $\tilde{v_d}$. This implies $(-\Delta)^{\frac{\alpha}{2}} v_d(t_0, x_0) \leqslant -q < 0$ by (99a), which contradicts $(-\Delta)^{\frac{\alpha}{2}} v_d(t_0, x_0) = (-\Delta)^{\frac{\alpha}{2}} \tilde{v_d}(t_0, x_0) \geqslant 0$. It follows that there cannot exist any global maxima (t_0, x_0) of $\tilde{v_d}$ for $t_0 > 0$ in $(0, T_0] \times \mathbb{R}^d$. Since $\tilde{v_d}$ has a global maximum by assumption, the only possibility left is that the maximum is attained at some point $(0, x_0)$ along $t = 0$. It follows that

$$v_d(t,x) = \tilde{v_d}(t,x) + qt \leqslant \tilde{v_d}(0,x_0) + qt = v_d(0,x_0) + qt \leqslant qt. \tag{100}$$

Since this holds for all $t \in [0, T_0]$ and $x \in \mathbb{R}^d$, we can choose q arbitrarily small and deduce that $v_d(t,x) \leqslant 0$.

Suppose now that v_d has no global maxima. Since $v^-, v^+ \in C_b\left([0, T_0] \times \mathbb{R}^d\right)$ by assumption, our sub- and supersolutions are uniformly bounded. This allows us to draw inspiration from the penalization method introduced in Lemma 2.44 to arrive at a function with a global maximum.

Let $C = \|v^-\|_{L^\infty([0,T_0]\times\mathbb{R}^d)} + \|v^+\|_{L^\infty([0,T_0]\times\mathbb{R}^d)}$. Define the penalization function $\varphi_1(x) \in C_b^\infty\left(\mathbb{R}^d\right)$ by

$$\varphi_1(x) = \begin{cases} 0, & |x| \leqslant 1, \\ 2C + \delta, & |x| \geqslant 2, \end{cases} \tag{101}$$

for some $\delta > 0$, and let the function be monotone increasing in $1 < |x| < 2$. The existence of such a function follows from Lemma 2.43. Since $\phi_1 \in C_b^\infty\left(\mathbb{R}^d\right)$, its derivatives are bounded. Let $\|D\varphi_1\|_\infty \leqslant K_1$ and $\|D^2\varphi_1\|_\infty \leqslant K_2$ for constants $K_1, K_2 > 0$, and let $\varphi_R(x) := \varphi_1(x/R)$.

We can now define our penalized difference function. Let $\tilde{v}_d(t,x) = v^-(t,x) - v^+(t,x) - qt - \varphi_R(x)$ for some small $q > 0$ and $R > 0$. By the boundedness of $v^-(t,x) - v^+(t,x) - qt$ and the definition of $\varphi_R(x)$, it follows by Lemma 2.44 that $\tilde{v}_d$ has a global maximum in $[0,T_0] \times \{x : |x| \leqslant 2R\}$. Let (t_0, x_0) be a global maximum of $\tilde{v}_d$, and assume by contradiction that $t_0 > 0$. By (99a) and observations of the time and first-order spatial derivatives of $\tilde{v}_d$ in the global maximum point, we get that

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}} v_d(t_0,x_0) &\leqslant \left|H\left(Dv^-(t_0,x_0)\right) - H\left(Dv^+(t_0,x_0)\right)\right| - q \\ &< L_H\left|D\tilde{v}_d(t_0,x_0) - D\varphi_R(x_0)\right| - q = L_H\left|D\varphi_R(x_0)\right| - q,\end{aligned} \tag{102}$$

where we once more used that H is globally Lipschitz. Since (t_0,x_0) is a global maximum, we also have by Proposition 2.16 that

$$(-\Delta)^{\frac{\alpha}{2}}\tilde{v}_d(t_0,x_0) \geqslant 0 \implies (-\Delta)^{\frac{\alpha}{2}} v_d(t_0,x_0) \geqslant (-\Delta)^{\frac{\alpha}{2}}\varphi_R(x_0). \tag{103}$$

Arriving at a contradiction by (102) and (103) seems harder than in the case where a global maximum was assumed. In order to conclude the proof with a similar argument as before, we somehow need to let $|D\varphi_R(x_0)|$ and $\left|(-\Delta)^{\frac{\alpha}{2}}\varphi_R(x_0)\right|$ go towards zero.

Notice that since $\|D\varphi_1\|_\infty \leqslant K_1$ implies $\|D\varphi_R\|_\infty \leqslant K_1/R$, it follows that $\|D\varphi_R\|_\infty \to 0$ as $R \to \infty$, implying $|D\varphi_R(x_0)| \to 0$. For the fractional Lapacian, recall its singular integral definition in Definition 2.13 and Proposition 2.14. We get that

$$\begin{aligned}-(-\Delta)^{\frac{\alpha}{2}}\varphi_R(x_0) &= c_{d,\alpha}\int_{\mathbb{R}^d\setminus B_1(0)}\frac{\varphi_R(x_0+z) - \varphi_R(x_0)}{|z|^{d+\alpha}}dz \\ &+ \frac{c_{d,\alpha}}{2}\lim_{r\to 0^+}\int_{B_1(0)\setminus B_r(0)}\frac{\varphi_R(x_0+z) - 2\varphi_R(x_0) + \varphi_R(x_0-z)}{|z|^{d+\alpha}}dz.\end{aligned} \tag{104}$$

When evaluating the integrals above as $R \to \infty$, it is easier to work with the derivative of $\varphi_R(x)$ than the function itself, since the derivative will converge to zero as shown above. Notice that similarly to the first derivative, we have that $\|D^2\varphi_R\|_\infty \leqslant K_2/R^2$ such that $\left|D^2\varphi_R(x_0)\right| \to 0$ as $R \to \infty$. By a similar argument as in Proposition 2.15, we deduce that

$$\begin{aligned}\left|(-\Delta)^{\frac{\alpha}{2}}\varphi_R(x_0)\right| &\leqslant c_{d,\alpha}\|D\varphi_R\|_\infty\int_{\mathbb{R}^d\setminus B_1(0)}\frac{1}{|z|^{d+\alpha-1}}dz + \frac{c_{d,\alpha}}{2}\|D^2\varphi_R\|_\infty\int_{B_1(0)}\frac{1}{|z|^{d+\alpha-2}}dz \\ &\leqslant c_{d,\alpha}\frac{K_1}{R}\int_{\mathbb{R}^d\setminus B_1(0)}\frac{1}{|z|^{d+\alpha-1}}dz + \frac{c_{d,\alpha}}{2}\frac{K_2}{R^2}\int_{B_1(0)}\frac{1}{|z|^{d+\alpha-2}}dz.\end{aligned}$$

Since both integrals in the last expression are finite, it follows that $\left|(-\Delta)^{\alpha/2}\phi_R(x_0)\right|$ goes to zero when $R \to \infty$. Furthermore, since this holds for all $x \in \mathbb{R}^d$, we get that $\lim_{R\to\infty}\|\left(-\Delta^{\frac{\alpha}{2}}\right)\varphi_R\|_\infty = 0$.

By now letting $R \to \infty$ in (102), we deduce that $(-\Delta)^{\frac{\alpha}{2}} v_d(t_0,x_0) \leqslant -q$. However, from (103), we get that $(-\Delta)^{\frac{\alpha}{2}} v_d(t_0,x_0) \geqslant 0$ which is a contradiction since $q > 0$.

It follows that there cannot exist any global maximum (t_0,x_0) of $\tilde{v}_d$ for $t_0 > 0$. Since the existence of a global maximum is assumed, it follows that $\tilde{v}_d$ must have a global maximum $(0,x_0)$ for some $x_0 \in \mathbb{R}^d$. Remembering that the contradiction is only fullfilled as $R \to \infty$, we get by similar calculations as in (100) that for any $x \in \mathbb{R}^d$ and any $t \in [0,T_0]$,

$$\begin{aligned}v_d(t,x) &= \lim_{R\to\infty}\tilde{v}_d(t,x) + qt + \varphi_R(x) \leqslant \lim_{R\to\infty}\tilde{v}_d(0,x_0) + qt + \varphi_R(x) \\ &= \lim_{R\to\infty} v_d(0,x_0) - \varphi_R(x_0) + qt + \varphi_R(x) \leqslant \lim_{R\to\infty}\varphi_R(x) - \varphi_R(x_0) + qt.\end{aligned}$$

Notice that $\varphi_R(x) - \varphi_R(x_0) = |x - x_0|\,D\varphi_R(\xi)$ for some $\xi \in (x_0, x)$. Since $\|D\phi_R\|_\infty \to 0^+$ as $R \to \infty$, we get that $\lim_{R\to\infty}(\varphi_R(x) - \varphi_R(x_0)) = 0$. Finally, since $q > 0$ can be chosen arbitrarily small, we get that $v_d(t,x) \leqslant 0$.

Since $v_d(t,x) \leqslant 0$ in $[0,T_0] \times \mathbb{R}^d$ both in the case where a global maximum exists, and when we only have a supremum, it holds in general. We get that $v^-(t,x) \leqslant v^+(t,x)$ in $[0,T_0] \times \mathbb{R}^d$, and the proof is complete. □

By combining this lemma with an argument similar to Lemma 2.42, uniqueness follows directly.

Theorem 3.7. *Let $\alpha \in (1,2)$, $\beta \in (0,1)$ and $\varepsilon > 0$. Furthermore, let v_0, f, H and T_0 be as in Theorem 3.1. Then, there exists a unique classical solution $v \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$ to the HJB equation (37) which satisfies the regularity results in Theorem 3.1 and 3.3.*

Proof. By Theorem 3.1, 3.3 and 3.4, there exists a classical solution v to the HJB equation that satisfies the regularity assumptions. Since any solution also is a sub- and supersolution, the existence of sub/supersolutions $v^-, v^+ \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ is evident. By Lemma 3.6, it follows that $v^- \leqslant v^+$ in the entire $[0,T_0] \times \mathbb{R}^d$. Uniqueness of our solution v then follows directly by an argument similar to Lemma 2.42. □

3.5 Long time existence

We proceed by proving existence of a unique classical solution in $(0,T]$ for any terminal time $T > 0$. This is done through a patching argument where we derive short time solutions in overlapping time intervals. By utilizing the uniqueness result from Theorem 3.7, we can then conclude that these short time solutions are part of the same solution.

Theorem 3.8 (Long time existence)**.** *Let $\alpha \in (1,2)$, $\beta \in (0,1)$, $\varepsilon > 0$ and $T > 0$. Given functions v_0, f and H as in Theorem 3.1, there exists a unique classical solution $v \in C_b\left((0,T] \times \mathbb{R}^d\right)$. Furthermore, v satisfies the regularity results in Theorem 3.1 and 3.3 in the entire $(0,T] \times \mathbb{R}^d$.*

Proof. By Theorem 3.1, 3.3, 3.4 and 3.7, there exists a unique classical solution v in $(0,T_0] \times \mathbb{R}^d$ which satisfies the regularity assumptions. Recall from Theorem 3.1 that T_0 only depends on α, β, λ and L_H. Since these are time independent constants, T_0 will not depend on the initial time. Thus, by translating our short time existence proof in time, we achieve solutions on intervals of equal length.

By $v_0 \in C^{0,\beta}\left(\mathbb{R}^d\right)$ and Theorem 3.1, there exists a unique classical solution v on the time interval $(0,T_0]$. Let $v_1(x) := v(T_0/2, x)$. By Theorem 3.3, we know that $v_1 \in C^{0,\beta}\left(\mathbb{R}^d\right)$. Using v_1 as initial data in Theorem 3.1 gives us existence of a solution on the time interval $(T_0/2, 3T_0/2]$. Notice that the length of the time interval is T_0 here as well, since T_0 is independent on the initial time. Since our solutions are unique and overlapping on $(T_0/2, T_0]$, they have to be part of the same solution. It follows that there exists a unique classical solution on $(0, 3T_0/2]$.

Now, let $N \in \mathbb{N}$ be the largest integer such that $NT_0/2 < T$. By iteratively showing existence of solutions on time intervals $(0,T_0], (T_0/2, 3T_0/2], (T_0, 2T_0], \ldots, ((N-2)T_0/2, NT_0/2]$ and arguing as above, there exists a solution $v \in C_b\left((0, NT_0/2] \times \mathbb{R}^d\right)$. For the last iteration, we need to make sure that we do not exceed the terminal time $T > 0$. This is because our source term $f(t,x)$ is only defined on the time interval $[0,T]$. Let therefore $T_1 := T - (N-1)T_0/2$ and $v_N(x) := v(NT_0/2, x)$. By using v_N as initial data in Theorem 3.1, there exists a solution on the time interval $((N-1)T_0/2, T]$. The fact that $T_1 < T_0$ is not problematic, as it still provides us with a contraction map in Theorem 3.1. We conclude that there exists a unique classical solution $v \in C_b\left((0,T] \times \mathbb{R}^d\right)$. The regularity results in Theorem 3.1 and 3.3 are satisfied in $(0,T]$ by the same patching argument. □

3.6 Uniform continuity

In the last section of the chapter, we study uniform continuity of the solution v and its derivatives. This is an important step towards proving existence of classical solutions to the MFG system, as we will discuss in Chapter 5. In order to avoid time blowup, we will only work locally in time and consider domains of the form $[t_1, t_2] \times \mathbb{R}^d$ for any $0 < t_1 < t_2 \leqslant T$.

Theorem 3.9 (Uniform continuity)**.** *Let $\alpha \in (1,2)$, $\beta \in (0,1)$, $\varepsilon > 0$ and $T > 0$. Furthermore, let assumptions on v_0, f and H be as in Theorem 3.1, and assume in addition that $f(\cdot, x)$ is uniformly continuous in $[0,T]$ for any $x \in \mathbb{R}^d$. Let $v \in C_b\left((0,T] \times \mathbb{R}^d\right)$ be the corresponding classical solution from Theorem 3.8 and choose t_1, t_2 such that $0 < t_1 < t_2 \leqslant T$. Then, there exists a modulus of continuity ω such that for any $(t,x), (s,y) \in [t_1, t_2] \times \mathbb{R}^d$, the following statements hold:*

If $\alpha + \beta \leqslant 2$:

$$\begin{aligned} &|v(t,x) - v(s,y)| + |Dv(t,x) - Dv(s,y)| \\ &+ |\partial_t v(t,x) - \partial_t v(s,y)| + |(-\Delta)^{\frac{\alpha}{2}} v(t,x) - (-\Delta)^{\frac{\alpha}{2}} v(s,y)| \leqslant \omega(|t-s|, |x-y|). \end{aligned} \tag{105}$$

If $\alpha + \beta > 2$:

$$\begin{aligned} &|v(t,x) - v(s,y)| + |Dv(t,x) - Dv(s,y)| + \left|D^2 v(t,x) - D^2 v(s,y)\right| \\ &+ |\partial_t v(t,x) - \partial_t v(s,y)| + |(-\Delta)^{\frac{\alpha}{2}} v(t,x) - (-\Delta)^{\frac{\alpha}{2}} v(s,y)| \leqslant \omega(|t-s|, |x-y|). \end{aligned} \tag{106}$$

Furthermore, ω only depends on v through uniform bounds on v and Dv as well as $[v]_{C^{\alpha+\beta-\varepsilon}(\mathbb{R}^d)}$. If $\alpha + \beta > 2$, ω also depends on uniform bounds on $D^2 v$.

Proof. We will only prove the case where $\alpha + \beta > 2$. For $\alpha + \beta \leqslant 2$, the proof is exactly the same but without the estimates for $D^2 v$.

Since the time interval $[t_1, t_2]$ is strictly away from zero, we can ignore any time blowup at $t = 0$. By Theorem 3.1, 3.3 and 3.8, we then have that $v, Dv, D^2 v \in C_b\left([t_0, t_1] \times \mathbb{R}^d\right)$. Furthermore, $[v(t,\cdot)]_{C^{\alpha+\beta-\varepsilon}}$ is bounded uniformly in $[t_1, t_2]$. Then, there exists a constant $C \geqslant 0$ such that for any $t \in [t_1, t_2]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &|v(t,x) - v(t,y)| + |Dv(t,x) - Dv(t,y)| \\ &\quad + \left|D^2 v(t,x) - D^2 v(t,y)\right| \leqslant C\left(|x-y| + |x-y|^{\alpha+\beta-\varepsilon-1} + |x-y|^{\alpha+\beta-\varepsilon-2}\right). \end{aligned} \tag{107}$$

For the time regularity, we use an analogous approach to the time continuity proofs in Theorem 3.1 and 3.3. By calculations similar to (48), (49) and (50), there exist constants $c_1, c_2, c_3, c_4 \geqslant 0$ independent of v such that for any $x \in \mathbb{R}^d$, $t_0 \in [t_1, t_2)$ and $\tau \in (0, t_1 - t_0]$,

$$|v(t_0 + \tau, x) - v(t_0, x)| \leqslant c_1 \|v\|_\infty \tau^{\frac{1}{2(d+1)}} + c_2 \|Dv\|_\infty \tau^{\frac{1}{2\alpha}} + (c_3 \|Dv\|_\infty + c_4)\, \tau, \tag{108}$$

where we let $\|\cdot\|_\infty$ denote the L^∞-norm over $[t_1, t_2] \times \mathbb{R}^d$. The exponents on τ are different than the ones in Theorem 3.1 due to the absence of time blowup on Dv.

Similarly, by (51) and (53) we get that

$$|Dv(t_0 + \tau, x) - Dv(t_0, x)| \leqslant c_1 \|Dv\|_\infty \tau^{\frac{\beta}{2(d+\beta)}} + c_2 [Dv]_{C^{0,\beta}} \tau^{\frac{\beta}{2\alpha}} + (c_3 \|Dv\|_\infty + c_4)\, \tau, \tag{109}$$

for different constants also independent of v. The second derivative is estimated similarly to (77) and (78) such that

$$\left|D^2 v(t_0 + \tau, x) - D^2 v(t_0, x)\right| \leqslant c_1 \|Dv\|_\infty \tau^{\frac{\eta}{2(d+\eta)}} + c_2 [Dv]_{C^{0,\eta}} \tau^{\frac{\eta}{2\alpha}} + c_3 \|Dv\|_\infty \tau^{\frac{\eta}{2\alpha}} + c_4 \tau^{\frac{\alpha-1}{\alpha}}, \tag{110}$$

where $\eta = \alpha + \beta - \varepsilon - 2$. By (107)-(110), there exists a modulus of continuity $\tilde{\omega}$ such that for any $(t,x), (s,y) \in [t_1, t_2] \times \mathbb{R}^d$,

$$|v(t,x) - v(s,y)| + |Dv(t,x) - Dv(s,y)| + \left|D^2 v(t,x) - D^2 v(s,y)\right| \leqslant \tilde{\omega}(|t-s|, |x-y|). \tag{111}$$

We proceed with estimating space and time regularity of $(-\Delta)^{\alpha/2} v$. Let $t, s \in [t_1, t_2]$ and $h \in \mathbb{R}^d$ and notice that $v(t, \cdot + h) - v(t, \cdot) \in C^{\alpha+\beta-\varepsilon}\left(\mathbb{R}^d\right)$. We can then use Proposition 2.15 to deduce that

$$\begin{aligned} \left|(-\Delta)^{\frac{\alpha}{2}} v(t, x+h) - (-\Delta)^{\frac{\alpha}{2}} v(s,x)\right| &\leqslant \left\|-(-\Delta)^{\frac{\alpha}{2}} (v(t, \cdot + h) - v(t, \cdot))\right\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant C_1 \|v(t, \cdot + h) - v(s, \cdot)\|_{L^\infty(\mathbb{R}^d)} + C_2 [Dv(t, \cdot + h) - Dv(s, \cdot)]_{C^{0,\alpha+\delta-1}(\mathbb{R}^d)}, \end{aligned} \tag{112}$$

for constants $C_1, C_2 \geqslant 0$, where δ is chosen such that $0 < \delta < \beta - \varepsilon$ and $\alpha + \delta < 2$. By Theorem 2.33, we can estimate the Hölder seminorm by

$$\widehat{C}_2 \|Dv(t, \cdot + h) - Dv(s, \cdot)\|_{L^\infty(\mathbb{R}^d)}^{1-\zeta} [v(t, \cdot + h) - v(s, \cdot)]_{C^{\alpha+\beta-\varepsilon}(\mathbb{R}^d)}^{\zeta}, \tag{113}$$

where $\zeta := \frac{\alpha+\delta-1}{\alpha+\beta-\varepsilon-1}$ and $\widehat{C}_2 \geqslant 0$. Since (111) holds for all $x, y \in \mathbb{R}^d$, we can bound the L^∞-norm in (113) by the modulus $\tilde{\omega}$. We let $y = x + h$ and deduce that

$$\begin{aligned} &\left|(-\Delta)^{\frac{\alpha}{2}} v(t,x) - (-\Delta)^{\frac{\alpha}{2}} v(s,y)\right| \leqslant C_1 \tilde{\omega}(|t-s|, |x-y|) \\ &\qquad + \widehat{C}_2 \left([v(t, \cdot + h)]_{C^{\alpha+\beta-\varepsilon}} + [v(s, \cdot)]_{C^{\alpha+\beta-\varepsilon}}\right)^{\zeta} \tilde{\omega}^{1-\zeta}(|t-s|, |x-y|). \end{aligned} \tag{114}$$

Finally, we derive estimates for $\partial_t v$. Recall that v is a classical solution to the HJB equation (37) in $(0, T] \times \mathbb{R}^d$. For any $(t, x), (s, y) \in [t_1, t_2] \times \mathbb{R}^d$, it follows that

$$\begin{aligned} &|\partial_t v(t,x) - \partial_t v(s,y)| \\ &\leqslant \left|(-\Delta)^{\frac{\alpha}{2}} v(t,x) - (-\Delta)^{\frac{\alpha}{2}} v(s,y)\right| + |H(Dv(t,x)) - H(Dv(s,y))| + |f(t,x) - f(s,y)| \\ &\leqslant |f(t,x) - f(s,y)| + (C_1 + L_H)\tilde{\omega}(|t-s|, |x-y|) \\ &+ \widehat{C}_2 \left([v(t, \cdot + h)]_{C^{\alpha+\beta-\varepsilon}} + [v(s, \cdot)]_{C^{\alpha+\beta-\varepsilon}}\right)^{\zeta} \tilde{\omega}^{1-\zeta}(|t-s|, |x-y|), \end{aligned} \tag{115}$$

where we used (111), (114) and that H is globally Lipschitz. Furthermore, since f is β-Hölder continuous in space and uniformly continuous in time, there exists a modulus of continuity ω_f such that

$$|f(t,x) - f(s,y)| \leqslant \omega_f(|t-s|, |x-y|), \quad \forall (t,x) \in [t_1, t_2] \times \mathbb{R}^d. \tag{116}$$

By combining (111), (114), (115) and (116), there exists a modulus of continuity ω such that (106) holds. Furthermore, ω will only depend on v through uniform bounds on v, Dv, $D^2 v$ and $[v]_{C^{\alpha+\beta-\varepsilon}}$. □

4 The Fokker-Planck Equation

This chapter studies the Fokker-Planck equation (FP), which was briefly introduced in Section 2.3. We consider the system

$$\begin{cases} \partial_t m(t,x) + (-\Delta)^{\frac{\alpha}{2}} m(t,x) + \nabla \cdot (b(t,x) m(t,x)) = 0, & (t,x) \in (0,T] \times \mathbb{R}^d, \\ m(0,x) = m_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{117}$$

where $\alpha \in (1,2)$. Since the FP equation generally studies probabilistic distributions, we assume that $0 \leqslant m_0 \in L^1(\mathbb{R}^d)$ and $\|m_0\|_{L^1(\mathbb{R}^d)} = 1$. In addition, we let $m_0 \in C^{0,\nu}(\mathbb{R}^d)$ for some $\nu \in (0,1)$. In order to motivate our regularity assumptions for the drift term $b(t,x)$, we need to take a brief look at the Mean Field Game system (2).

In (2), the HJB equation moves backwards in time. This means that any time blowup occuring at $t=0$ in Chapter 3 now occurs at the terminal time. In other words, we have that $(T-t)^{\frac{1-\beta}{\alpha}} Dv \in C_b([0,T) \times \mathbb{R}^d)$ instead of $t^{\frac{1-\beta}{\alpha}} Dv \in C_b((0,T] \times \mathbb{R}^d)$ and so forth. Looking at the MFG system, we observe that $b(t,x) := D_p H(Dv(t,x))$. The time blowup on Dv at $t=T$ will therefore transfer to the FP equation through b, complicating our existence proof drastically.

We circumvent this issue by working with a terminal time $T_\varepsilon < T$ instead. Notice that for any $t \in [0,T_\varepsilon]$, we have $(T-t)^{\frac{1-\beta}{\alpha}} \leqslant (T-T_\varepsilon)^{\frac{1-\beta}{\alpha}} < \infty$, and it follows that $Dv \in C_b([0,T_\varepsilon] \times \mathbb{R}^d)$. Similar observations hold also for the second derivative and the Hölder seminorms considered in Theorem 3.3. For simplicity, we denote the terminal time by T for the remainder of this chapter, and revisit the issue when discussing the coupled MFG system in Chapter 5.

In order to impose sufficient regularity on b, we will here assume that $H \in C^3(\mathbb{R}^d)$, where H, $\partial_p H$ and $\partial_p^2 H$ are globally Lipschitz with corresponding Lipschitz constants $L_H, L'_H, L''_H \geqslant 0$. It follows that

$$|b(t,x+h) - b(t,x)| \leqslant L'_H |Dv(t,x+h) - Dv(t,x)|, \quad \forall x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}. \tag{118}$$

Furthermore, if $D^2 v$ exists, we can use the chain rule to deduce that

$$\begin{aligned} |Db(t,x+h) - Db(t,x)| &\leqslant \left| D^2 v(t,x+h) D_p^2 H(Dv(t,x+h)) - D^2 v(t,x) D_p^2 H(Dv(t,x)) \right| \\ &\leqslant \|D^2 v(t,\cdot)\|_\infty \left| D_p^2 H(Dv(t,x+h)) - D_p^2 H(Dv(t,x)) \right| \\ &+ \|D_p^2 H(Dv(t,\cdot))\|_\infty \left| D^2 v(t,x+h) - D^2 v(t,x) \right| \\ &\leqslant L'_H \|D^2 v(t,\cdot)\|_\infty |Dv(t,x+h) - Dv(t,x)| + L''_H \left| D^2 v(t,x+h) - D^2 v(t,x) \right|. \end{aligned} \tag{119}$$

Assume now that $v(t,\cdot)$ is $(\mu+1)$-Hölder continuous in space for every $t \in [0,T]$ where $\mu \in (0,2)$. Dividing (118) by $|h|^\mu$ if $\mu \leqslant 1$, or (119) by $|h|^{\mu-1}$ if $\mu > 1$, we deduce by taking the supremum over h that $b(t,\cdot)$ is μ-Hölder continuous in space for every $t \in [0,T]$. Furthermore, by instead considering a time difference $b(t+\tau,x) - b(t,x)$ in (118) and letting $\tau \to 0^+$, we deduce that b is continuous in time. This follows from $Dv \in C_b([0,T] \times \mathbb{R}^d)$. A similar argument holds for Db in (119) when $\mu > 1$ and uses $D^2 v \in C_b([0,T] \times \mathbb{R}^d)$.

This leads us to the assumption we will impose, namely that $b \in C_b([0,T]; C^\mu(\mathbb{R}^d))$. In addition, if $\mu > 1$, we assume $Db \in C_b([0,T]; C^{\mu-1}(\mathbb{R}^d))$. By recalling that $v(t,\cdot)$ is $(\alpha+\beta-\varepsilon)$-Hölder continuous in space for every $t \in (0,T]$ (see Theorem 3.3), we get that $\mu = \alpha+\beta-\varepsilon-1$. It follows that $\mu \in (0,\alpha)$ since $\alpha \in (1,2)$ and $\beta \in (0,1)$.

The outline for this chapter is quite similar to Chapter 3. We begin with showing short time existence of a unique mild solution in Section 4.1. Regularity estimates are for the most part proven simultaneously, in contrast to the HJB case where we considered existence of $D^2 v$ and Hölder regularity in seperate arguments (see Section 3.2). Since the FP equation describes probabilistic distributions, we need to show that $m \geqslant 0$ almost everywhere and that $m(t,\cdot) \in L^1(\mathbb{R}^d)$ with $\|m(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$ for every $t \in [0,T]$. This is dealt with in Section 4.2 where we show positivity and mass preservation of so-called very weak solutions. By utilizing these properties, uniqueness follows immediately. Section 4.3 provides a long time existence result for very weak solutions by an identical patching argument as in Theorem 3.8. In Section 4.4, we prove that our unique very weak solution is classical whenever $\mu > 1$. Finally, we finish the chapter with a uniform continuity result for m and its derivatives in Section 4.5.

4.1 Short time existence and regularity estimates

We begin with proving short time existence of a unique mild solution $m \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ where $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for every $t \in (0,T_0]$. The approach is quite similar to Theorem 3.1, but differs in some key aspects. In contrast to the HJB equation, we will not address existence of the second derivative in a seperate result. As discussed in Chapter 3, D^2v was treated seperately due to difficulties when differentiating H in the contraction argument. Since the Hamiltonian is not present in the FP equation, however, we will not encounter this problem when including D^2m in the fixed point argument. Unlike the HJB equation, we may not have existence of the first derivative. This requires us to consider three different cases in Theorem 4.1, depending on the existence of Dm and D^2m. Other differences from Theorem 3.1 include that we require $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for every $t \in [0,T]$, and that we show $(\alpha+\mu-\varepsilon-1)$-Hölder continuity of $m(t,\cdot)$ in space. This is analogous to the $(\alpha+\beta-\varepsilon)$-Hölder continuity of $v(t,\cdot)$ in Theorem 3.3.

Theorem 4.1 (Short time existence of mild solutions)**.** *Let $\alpha \in (1,2)$, $\mu \in (0,\alpha)$, $\nu \in (0,1)$, $\varepsilon > 0$ and suppose that $m_0 \in C^{0,\nu}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$ where $\|m_0\|_1 = 1$. Define $\eta := \alpha+\mu-\varepsilon-1$ and let λ be the constant from Theorem 2.26. Given a terminal time $T > 0$, suppose that $b \in C_b\left([0,T];C^{\mu}\left(\mathbb{R}^d\right)\right)$. If $\mu > 1$, assume also that $Db \in C_b\left([0,T];C^{\mu-1}\left(\mathbb{R}^d\right)\right)$. Then, there exists $T_0 \in (0,T)$ only depending on α, μ, ν, λ, interpolation constants from Section 2.8 and uniform bounds on b such that the following statements hold:*

(a) If $\eta < 1$, there exists a unique mild solution $m \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ to the FP equation (117) where $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for every $t \in [0,T_0]$. Furthermore, if $\nu < \eta$, we have uniform boundedness of $t^{\frac{\eta-\nu}{\alpha}}\left[m(t,\cdot)\right]_{C^{0,\eta}}$ in $(0,T_0]$. If $\nu \geqslant \eta$, this holds for $\left[m(t,\cdot)\right]_{C^{0,\eta}}$ instead.

(b) If $1 < \eta < 2$, there exists a unique mild solution $m \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ to the FP equation (117) where $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for every $t \in [0,T_0]$, and $t^{\frac{1-\nu}{\alpha}}Dm \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$. Furthermore, $t^{\frac{\eta-\nu}{\alpha}}\left[Dm(t,\cdot)\right]_{C^{0,\eta-1}}$ is bounded uniformly in $(0,T_0]$.

(c) If $\eta > 2$, there exists a unique mild solution $m \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ to the FP equation (117) where $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for every $t \in [0,T_0]$, and $t^{\frac{1-\nu}{\alpha}}Dm, t^{\frac{2-\nu}{\alpha}}D^2m \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$. Furthermore, $t^{\frac{\eta-\nu}{\alpha}}\left[D^2m(t,\cdot)\right]_{C^{0,\eta-2}}$ is bounded uniformly in $(0,T_0]$.

The proofs are similar to each other and the short time existence proof in Theorem 3.1. We will therefore only provide a complete proof for (c), and comment on the differences in (a) and (b).

Proof of Theorem 4.1 (c). Define the Banach space X by

$$X = \left\{m : m, t^{\frac{1-\nu}{\alpha}}Dm, t^{\frac{2-\nu}{\alpha}}D^2m \in C_b\left((0,T_0] \times \mathbb{R}^d\right), m \in B\left((0,T_0];L^1\left(\mathbb{R}^d\right)\right) \text{ and } |||m|||_X < \infty\right\}, \tag{120}$$

where $|||m|||_X = \sup_{t\in(0,T_0]} \|m(t,\cdot)\|_X$ and

$$\begin{aligned}\|m(t,\cdot)\|_X &= \|m(t,\cdot)\|_1 + \|m(t,\cdot)\|_\infty \\ &\quad + t^{\frac{1-\nu}{\alpha}}\|Dm(t,\cdot)\|_\infty + t^{\frac{2-\nu}{\alpha}}\|D^2m(t,\cdot)\|_\infty + t^{\frac{\eta-\nu}{\alpha}}\left[D^2m(t,\cdot)\right]_{C^{0,\eta-2}}.\end{aligned} \tag{121}$$

We note that the norm $\|\cdot\|_X$ depends on t, but we skip it in the notation for the sake of readability. Recall the definition of the Duhamel map $\psi(m)$ in Definition 2.28.

$$\psi(m)(t,x) = K(t,\cdot) * m_0(\cdot)(x) - \sum_{i=1}^{d}\int_0^t \partial_{x_i}K(t-s,\cdot) * \left(b_i(s,\cdot)\,m(s,\cdot)\right)(x)\,ds. \tag{122}$$

Similarly to Theorem 3.1, we need to show that $\psi : X \to X$ in order to use Banach's fixed point theorem. We begin by showing that $\|\psi(m)\|_X$ is bounded whenever $m \in X$, implying that $\psi(m)(t,\cdot), t^{\frac{1-\nu}{\alpha}} D\psi(m)(t,\cdot), t^{\frac{2-\nu}{\alpha}} D^2\psi(m)(t,\cdot) \in C_b(\mathbb{R}^d)$ and $\psi(m)(t,\cdot) \in L^1(\mathbb{R}^d)$ for any $t \in (0, T_0]$. Our approach consists of using interpolation inequalities and heat kernel estimates on the Duhamel map, similar to Theorem 3.1. We will, however, need some new results when working with the product $b_i m$. Specifically, we require estimates for the L^1-norm and Hölder seminorms of a product of functions. The latter is presented in Lemma 2.37 and the L^1-estimate is a special case of Hölder's inequality (Lemma 2.39).

We estimate each part of $\|\psi(m)(t,\cdot)\|_X$ seperately, starting with the L^1-norm. By Young's inequality, heat kernel estimates and Hölder's inequality with $p = 1$ and $q = \infty$, we then get that

$$\begin{aligned}
\|\psi(m)(t,\cdot)\|_1 &\leqslant \|K(t,\cdot)\|_1 \|m_0\|_1 + \sum_{i=1}^d \int_0^t \|\partial_{x_i} K(t-s,\cdot)\|_1 \|b_i(s,\cdot) m(s,\cdot)\|_1 ds \\
&\leqslant 1 \cdot 1 + d\lambda \left(\int_0^t (t-s)^{-\frac{1}{\alpha}} \|b(s,\cdot)\|_\infty \|m(s,\cdot)\|_1 ds \right) \leqslant 1 + d\lambda \frac{\alpha}{\alpha-1} t^{\frac{\alpha-1}{\alpha}} \sup_{s\in(0,t)} \|b(s,\cdot)\|_\infty \|m(s,\cdot)\|_1 \\
&\leqslant 1 + d\lambda \frac{\alpha}{\alpha-1} T_0^{\frac{\alpha-1}{\alpha}} \sup_{s\in(0,T_0]} \|b(s,\cdot)\|_\infty \|m(s,\cdot)\|_1, \qquad (123)
\end{aligned}$$

where the last inequality holds since $(\alpha-1)/\alpha$ is positive. It follows that $\psi(m)(t,\cdot) \in L^1(\mathbb{R}^d)$ for any $t \in (0, T_0]$ since $m \in X$ and b is bounded. By a similar calculation, we get that

$$\begin{aligned}
\|\psi(m)(t,\cdot)\|_\infty &\leqslant \|K(t,\cdot)\|_1 \|m_0\|_\infty + \sum_{i=1}^d \int_0^t \|\partial_{x_i} K(t-s,\cdot)\|_1 \|b_i(s,\cdot)\|_\infty \|m(s,\cdot)\|_\infty ds \\
&\leqslant \|m_0\|_\infty + d\lambda \frac{\alpha}{\alpha-1} T_0^{\frac{\alpha-1}{\alpha}} \sup_{s\in(0,T_0]} \|b(s,\cdot)\|_\infty \|m(s,\cdot)\|_\infty, \qquad (124)
\end{aligned}$$

and it follows that $\psi(m)(t,\cdot)$ is bounded in $(0, T_0] \times \mathbb{R}^d$.

When estimating $\|t^{\frac{1-\nu}{\alpha}} D\psi(m)(t,\cdot)\|_\infty$, we need to differentiate the convolutions. This requires additional regularity on $b_i m$, since we lose integrability if we differentiate K twice in the integral. By $\alpha \in (1,2)$ and $\eta > 2$, we have $\mu > 1$. This means that Db is bounded since $b(t,\cdot) \in C^\mu(\mathbb{R}^d)$. Furthermore, $t^{\frac{1-\nu}{\alpha}} Dm$ is bounded since $m \in X$. We can then put the derivative on $b_i m$ in the integral. For the initial data term, we interpolate as in (45) and get that

$$\begin{aligned}
\|t^{\frac{1-\nu}{\alpha}} D\psi(m)(t,\cdot)\|_\infty &\leqslant C_I \lambda^{1-\nu} [m_0]_{C^{0,\nu}} + t^{\frac{1-\nu}{\alpha}} \sum_{i=1}^d \int_0^t \|\partial_{x_i} K(t-s,\cdot)\|_1 \|D(b_i(s,\cdot) m(s,\cdot))\|_\infty ds \\
&\leqslant C_I \lambda^{1-\nu} [m_0]_{C^{0,\nu}} + d\lambda t^{\frac{1-\nu}{\alpha}} \int_0^t (t-s)^{-\frac{1}{\alpha}} \left(s^{-\frac{1-\nu}{\alpha}} \|b\|_\infty \|s^{\frac{1-\nu}{\alpha}} Dm\|_\infty + \|Db\|_\infty \|m\|_\infty \right) ds,
\end{aligned}$$

where we have taken the supremum over $i \in \{1, \dots, d\}$ and $s \in (0, T_0]$ in the $\|\cdot\|_\infty$-terms. The L^∞-norms are bounded and independent of t and s. Since the exponents on $t-s$ and s are greater than -1, we have integrability and it follows that

$$\|t^{\frac{1-\nu}{\alpha}} D\psi(m)(t,\cdot)\|_\infty \leqslant C_I \lambda^{1-\nu} [m_0]_{C^{0,\nu}} + c_1 t^{\frac{\alpha-1}{\alpha}} \|b\|_\infty \|s^{\frac{1-\nu}{\alpha}} Dm\|_\infty + c_2 t^{\frac{\alpha-\nu}{\alpha}} \|Db\|_\infty \|m\|_\infty, \quad (125)$$

for some $c_1, c_2 \geqslant 0$. The supremum over t is attained at $t = T_0$, and we get that $t^{\frac{1-\nu}{\alpha}} D\psi(m)$ is uniformly bounded in $(0, T_0] \times \mathbb{R}^d$.

For $\|t^{\frac{2-\nu}{\alpha}} D^2\psi(m)(t,\cdot)\|_\infty$, we need to use interpolation in the integral as well. By using Theorem 2.35 (c) as well as Young's inequality on the integrand, we get that

$$\begin{aligned}
&\|t^{\frac{2-\nu}{\alpha}} D^2\psi(m)(t,\cdot)\|_\infty \leqslant C_I \lambda [m_0]_{C^{0,\nu}} \\
&\quad + t^{\frac{2-\nu}{\alpha}} C_I \sum_{i=1}^d \int_0^t \|\partial_{x_i} K(t-s,\cdot)\|_1^{\mu-1} \|D\partial_{x_i} K(t-s,\cdot)\|_1^{2-\mu} [D(b_i(s,\cdot) m(s,\cdot))]_{C^{0,\mu-1}} ds. \qquad (126)
\end{aligned}$$

The initial data term was estimated similarly to what we did for $t^{\frac{1-\nu}{\alpha}}D\psi(m)$, and is bounded independently of t. It remains to estimate the second term in (126). By taking the supremum over $i\in\{1,\dots,d\}$ and using the product rule on $D(b_i m)$, we get that

$$\begin{aligned}&\|t^{\frac{2-\nu}{\alpha}}D^2\psi(m)(t,\cdot)\|_\infty \leqslant C_I\lambda[m_0]_{C^{0,\nu}}\\&+t^{\frac{2-\nu}{\alpha}}C_I d\lambda\int_0^t (t-s)^{-\frac{3-\mu}{\alpha}}\left([m(s,\cdot)Db(s,\cdot)]_{C^{0,\mu-1}}+[b(s,\cdot)Dm(s,\cdot)]_{C^{0,\mu-1}}\right)ds.\end{aligned}\tag{127}$$

Notice that $-(3-\mu)/\alpha>-1$. We can now use Lemma 2.37 on the Hölder seminorms in (127). Omitting $(s,\cdot)$ for simplicity, we have that

$$\begin{aligned}[mDb]_{C^{0,\mu-1}}+[bDm]_{C^{0,\mu-1}}\leqslant{}&\|m\|_\infty[Db]_{C^{0,\mu-1}}+[m]_{C^{0,\mu-1}}\|Db\|_\infty\\&+\|b\|_\infty[Dm]_{C^{0,\mu-1}}+[b]_{C^{0,\mu-1}}\|Dm\|_\infty.\end{aligned}$$

Boundedness of $\|b\|_\infty$, $[b]_{C^{0,\mu-1}}$, $\|Db\|_\infty$ and $[Db]_{C^{0,\mu-1}}$ follows from $b(s,\cdot)\in C^\mu\left(\mathbb{R}^d\right)$. Furthermore, since $m\in X$, we know that $\|m\|_\infty$ and $\|s^{\frac{1-\nu}{\alpha}}Dm\|_\infty$ are bounded. For the Hölder seminorms on m and Dm, we need to use interpolation. By Theorem 2.34, we have that

$$[m]_{C^{0,\mu-1}}\leqslant C_{\mu-1,1}\|m\|_\infty^{2-\mu}\|Dm\|_\infty^{\mu-1}\leqslant C_{\mu-1,1}s^{-\frac{(1-\nu)(\mu-1)}{\alpha}}\|m\|_\infty^{2-\mu}\|s^{\frac{1-\nu}{\alpha}}Dm\|_\infty^{\mu-1}.\tag{128}$$

Similarly,

$$[Dm]_{C^{0,\mu-1}}\leqslant C_{\mu-1,1}\|Dm\|_\infty^{2-\mu}\|D^2m\|_\infty^{\mu-1}\leqslant C_{\mu-1,1}s^{-\frac{\mu-\nu}{\alpha}}\|s^{\frac{1-\nu}{\alpha}}Dm\|_\infty^{2-\mu}\|s^{\frac{2-\nu}{\alpha}}Dm\|_\infty^{\mu-1}.\tag{129}$$

By combining these estimates and taking the supremum over $s\in(0,T_0]$ in the norms and seminorms, there exist constants $c_1,c_2,c_3,c_4\geqslant 0$ independent of t and s such that

$$[mDb]_{C^{0,\mu-1}}+[bDm]_{C^{0,\mu-1}}\leqslant c_1+c_2s^{-\frac{(1-\nu)(\mu-1)}{\alpha}}+c_3s^{-\frac{\mu-\nu}{\alpha}}+c_4s^{-\frac{1-\nu}{\alpha}}.\tag{130}$$

All exponents on s are greater than -1, and by inserting (130) into (127), we have integrability. By calculating the resulting integral, we get that

$$\|t^{\frac{2-\nu}{\alpha}}D^2\psi(m)(t,\cdot)\|_\infty\leqslant C_I\lambda[m_0]_{C^{0,\nu}}+\tilde{c}_1t^{\frac{\alpha+\mu-\nu-1}{\alpha}}+\tilde{c}_2t^{\frac{\alpha+\nu(\mu-2)}{\alpha}}+\tilde{c}_3t^{\frac{\alpha-1}{\alpha}}+\tilde{c}_4t^{\frac{\alpha+\mu-2}{\alpha}},\tag{131}$$

for constants $\tilde{c}_1,\tilde{c}_2,\tilde{c}_3,\tilde{c}_4\geqslant 0$. All exponents on t are positive, and we can let $t\to T_0$. It follows that $t^{\frac{2-\nu}{\alpha}}D^2\psi(m)$ is uniformly bounded in $(0,T_0]\times\mathbb{R}^d$.

We estimate $[t^{\frac{\eta-\nu}{\alpha}}D^2\psi(m)(t,\cdot)]_{C^{0,\eta-2}}$ by a similar calculation as in (126) and (127), using Theorem 2.35 (b) instead of (c) for the interpolations.

$$\begin{aligned}&\left[t^{\frac{\eta-\nu}{\alpha}}D^2\psi(m)(t,\cdot)\right]_{C^{0,\eta-2}}\leqslant C_I\lambda[m_0]_{C^{0,\nu}}\\&+C_Id\lambda t^{\frac{\eta-\nu}{\alpha}}\int_0^t(t-s)^{-\frac{\eta+1-\mu}{\alpha}}\left([m(s,\cdot)Db(s,\cdot)]_{C^{0,\mu-1}}+[b(s,\cdot)Dm(s,\cdot)]_{C^{0,\mu-1}}\right)ds.\end{aligned}\tag{132}$$

Combining this estimate with (130), we reach integrability and it follows that

$$\left[t^{\frac{\eta-\nu}{\alpha}}D^2\psi(m)(t,\cdot)\right]_{C^{0,\eta-2}}\leqslant C_I\lambda[m_0]_{C^{0,\nu}}+c_1't^{\frac{\alpha+\mu-\nu-1}{\alpha}}+c_2't^{\frac{\alpha+\nu(\mu-2)}{\alpha}}+c_3't^{\frac{\alpha-1}{\alpha}}+c_4't^{\frac{\alpha+\mu-2}{\alpha}},$$

which is similar to (131). All exponents are positive, and by letting $t\to T_0$, we get that $[t^{\frac{\eta-\nu}{\alpha}}D^2\psi(m)(t,\cdot)]_{C^{0,\eta-2}}$ is uniformly bounded in $(0,T_0]$. It follows that $\|\psi(m)\|_X<\infty$.

We proceed with showing $\psi(m)$, $t^{\frac{1-\nu}{\alpha}}D\psi(m)$, $t^{\frac{2-\nu}{\alpha}}D^2\psi(m)\in C_b\left((0,T_0]\times\mathbb{R}^d\right)$. This is very similar to Theorem 3.1. By fixing $t\in(0,T_0]$ and noting that $t^{\frac{1-\nu}{\alpha}}D\psi(m)(t,\cdot)$ and $t^{\frac{2-\nu}{\alpha}}D^2\psi(m)(t,\cdot)$ are bounded in space, it follows that $\psi(m)(t,\cdot)$ and $t^{\frac{1-\nu}{\alpha}}D\psi(m)(t,\cdot)$ are Lipschitz continuous. Furthermore, $t^{\frac{2-\nu}{\alpha}}D^2\psi(m)(t,\cdot)$ is $(\eta-2)$-Hölder continuous in space, as shown above.

Time continuity is proven as in Theorem 3.1. By Lemma 2.30, we have that

$$\begin{aligned}\psi(m)(t_0+\tau,x)-\psi(m)(t_0,x)={}&(K(\tau,\cdot)*\psi(m)(t_0,\cdot))(x)-\psi(m)(t_0,x)\\&-\sum_{i=1}^d\int_{t_0}^{t_0+\tau}\partial_{x_i}K(t_0+\tau-s,\cdot)*(b_i(s,\cdot)m(s,\cdot))(x)\,ds.\end{aligned}$$

This is similar to (48). By Lemma 2.23 and arguments similar to the proof in Theorem 3.1, it follows that $\psi(m) \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$. For $t^{\frac{1-\nu}{\alpha}} D\psi(m)$, we argue as in (51) and use that $t^{\frac{1-\nu}{\alpha}} D\psi(m)$ is Lipschitz continuous in space to conclude with Lemma 2.23. For $t^{\frac{2-\nu}{\alpha}} D^2\psi(m)$, we argue as in (77), and use instead the spatial $(\eta-2)$-Hölder continuity of $t^{\frac{2-\nu}{\alpha}} D^2\psi(m)(t,\cdot)$. It follows that $\psi(m), t^{\frac{1-\nu}{\alpha}} D\psi(m), t^{\frac{2-\nu}{\alpha}} D^2\psi(m) \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$, and we can conclude that $\psi : X \to X$.

It only remains to prove that ψ is a contraction map. Notice that

$$\|\psi(m_1)(t,\cdot) - \psi(m_2)(t,\cdot)\|_X \leqslant \sum_{i=1}^d \int_0^t \|\partial_{x_i} K(t-s,\cdot) * (b_i(s,\cdot)(m_1(s,\cdot) - m_2(s,\cdot)))\|_X,$$

for any $m_1, m_2 \in X$. By letting $m := m_1 - m_2$ in calculations similar to (123)-(129) and (132), and omitting $(t,\cdot)$ for simplicity, we get that

$$\begin{aligned}
|||\psi(m_1) - \psi(m_2)|||_X &\leqslant C(T_0) \sup_{t\in(0,T_0]} \Big(\|m_1-m_2\|_1 + \|m_1-m_2\|_\infty + \|t^{\frac{1-\nu}{\alpha}} D(m_1-m_2)\|_\infty \\
&+ \|m_1-m_2\|_\infty^{2-\mu} \|t^{\frac{1-\nu}{\alpha}} D(m_1-m_2)\|_\infty^{\mu-1} + \|t^{\frac{1-\nu}{\alpha}} D(m_1-m_2)\|_\infty^{2-\mu} \|t^{\frac{2-\nu}{\alpha}} D^2(m_1-m_2)\|_\infty^{\mu-1} \Big) \\
&\leqslant 5C(T_0)\, |||m_1-m_2|||_X. \qquad (133)
\end{aligned}$$

Notice that the last inequality holds since the exponents $(2-\mu)$ and $(\mu-1)$ sum up to 1. Here, $C(T_0) \geqslant 0$ is the maximum of the coefficients that appear when estimating the different parts of $|||\cdot|||_X$. It depends solely on α, μ, ν, λ, T_0, interpolation constants and uniform bounds on b. Furthermore, $C(T_0)$ is strictly decreasing as $T_0 \to 0$. Thus, by choosing $T_0 \in (0,T)$ sufficiently small, it follows from (133) that there exists $L < 1$ such that $|||\psi(m_1) - \psi(m_2)|||_X \leqslant L|||m_1 - m_2|||_X$ for any $m_1, m_2 \in X$. Then, ψ is a contraction map, and by Banach's fixed point theorem there exists a unique fixed point $m \in X$ such that $\psi(m) = m$. This is a mild solution to the FP equation (117). By also noting that $m(0,\cdot) = m_0 \in C^{0,\nu}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$, we get $m \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ and $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for every $t \in [0,T_0]$. Finally, since T_0 has the same dependencies as $C(T_0)$, the requirements for T_0 are satisfied. ☐

Proof of Theorem 4.1 (a),(b). The proofs are essentially the same as in Theorem 4.1 (c), only with fewer terms.

For Theorem 4.1 (a), we define the Banach space X as in (120), but without the regularity requirements on $t^{\frac{2-\nu}{\alpha}} D^2 m$ and $t^{\frac{1-\nu}{\alpha}} Dm$. If $\nu < \eta$, the corresponding norm is given by $|||m|||_X = \sup_{t\in(0,T_0]} \|m(t,\cdot)\|_X$ where

$$\|m(t,\cdot)\|_X = \|m(t,\cdot)\|_1 + \|m(t,\cdot)\|_\infty + [t^{\frac{\eta-\nu}{\alpha}} m(t,\cdot)]_{C^{0,\eta}}.$$

If $\nu \geqslant \eta$, we omit the time blowup in the Hölder seminorm. We show $\psi(m)(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ as in Theorem 4.1 (c). For $\psi(m) \in C_b\left((0,T_0] \times \mathbb{R}^d\right)$, we derive η-Hölder continuity of $\psi(m)$, similar to (132). Time continuity follows by Lemma 2.23 and that $[\psi(m)(t,\cdot)]_{C^{0,\eta}}$ is bounded. This is similar to the time continuity proof for $t^{\frac{2-\nu}{\alpha}} D^2\psi(m)$ in Theorem 4.1 (c). Then, $\psi : X \to X$. The contraction argument follows as in Theorem 4.1 (c), only with fewer terms. We conclude the proof by Banach's fixed point theorem.

The approach is similar for Theorem 4.1 (b). Define the Banach space X as in (120), but without the regularity requirements on $t^{\frac{2-\nu}{\alpha}} D^2 m$. The corresponding norm is given by $|||m|||_X = \sup_{t\in(0,T_0]} \|m(t,\cdot)\|_X$ where

$$\|m(t,\cdot)\|_X = \|m(t,\cdot)\|_1 + \|m(t,\cdot)\|_\infty + \|t^{\frac{1-\nu}{\alpha}} Dm(t,\cdot)\|_\infty + [t^{\frac{\eta-\nu}{\alpha}} Dm(t,\cdot)]_{C^{0,\eta-1}}.$$

We show $\psi(m)(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ as in Theorem 4.1 (c). Similar to (132), we show $(\eta-1)$-Hölder continuity of $t^{\frac{\eta-\nu}{\alpha}} D\psi(m)$ and conclude the continuity arguments. It follows that $\psi : X \to X$. By a contraction argument as in Theorem 4.1 (c) and Banach's fixed point theorem, the proof is complete. ☐

We finish the section with a spatial Hölder regularity estimate for $m(t,\cdot)$ without time blowup. This is analogous to the β-Hölder continuity of $v(t,\cdot)$ in Theorem 3.3.

Lemma 4.2. *Let $\alpha \in (1,2)$, $\mu \in (0,\alpha)$ and $\nu \in (0,1)$. Furthermore, let assumptions on m_0 and b be as in Theorem 4.1 and let $m \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$ be the solution obtained in the theorem. If $\nu < \alpha + \mu - 1$, there exists a constant $C_1 > 0$ such that $[m(t,\cdot)]_{C^{0,\nu}} \leqslant C_1$ for any $t \in (0,T_0]$.*

Proof. By Theorem 4.1, m is a mild solution to the FP equation and a fixed point of the Duhamel map in (28). Using heat kernel estimates from Theorem 2.26 and an interpolation as in Theorem 2.35, we get that

$$[m(t,\cdot)]_{C^{0,\nu}} \leqslant [m_0]_{C^{0,\nu}} + \lambda C_I \int_0^t (t-s)^{-\frac{\nu+1-\mu}{\alpha}} [b(s,\cdot)\, m(s,\cdot)]_{C^{\mu}}\, ds.$$

Notice that by $\nu < \alpha + \mu - 1$, and since $\varepsilon > 0$ is chosen arbitrarily small in Theorem 3.1, the exponent on $(t-s)$ is less negative than the one in (132). We can then argue similarly to (132), and deduce that $[m(t,\cdot)]_{C^{0,\nu}} \leqslant C_1$ in $(0,T_0] \times \mathbb{R}^d$ for some constant $C_1 > 0$. For $t = 0$, we have that $[m(0,\cdot)]_{C^{0,\nu}} = [m_0]_{C^{0,\nu}} \leqslant C_1$, and the proof is complete. □

4.2 Uniqueness results for very weak solutions

In this section, we study properties of so-called very weak solutions to the FP equation (see Lemma 4.3). This includes positivity and L^1-regularity of solutions, as well as mass preservation. These are essential properties of probabilistic distributions, and are thus expected for solutions to the FP equation. The underlying purpose of this section, however, is to show uniqueness, which follows quite directly from the properties mentioned above (see Theorem 4.7). Our approach is heavily inspired by the work of Espen R. Jakobsen and Artur Rutkowski in [13].

We begin our analysis with a result stating that mild solutions are very weak solutions for a class of sufficiently regular test functions. The following Lemma is a simplified version of Lemma C.1 in [13], and is rewritten in terms of notation. Notice, however, that we allow for a larger class of test functions. Specifically, we assume $\phi \in C\left([0,T_0]; C^{\alpha+\delta}\left(\mathbb{R}^d\right)\right)$ for some $\delta > 0$ instead of only $\phi \in C\left([0,T_0]; C_b^2\left(\mathbb{R}^d\right)\right)$. This is necessary in order for Proposition 4.5 to hold. Although C_b^2 is required in [13], the proof only uses that $\phi(t,\cdot) \in C^{\alpha+\delta}\left(\mathbb{R}^d\right)$. Hence, our Lemma follows directly from this proof.

Lemma 4.3 (Inspired by Lemma C.1 in [13])**.** *Let $T_0 > 0$, $m_0 \in L^1\left(\mathbb{R}^d\right)$ and $b \in L^\infty\left([0,T_0] \times \mathbb{R}^d\right)$ and assume that $m \in B\left([0,T_0]; L^1\left(\mathbb{R}^d\right)\right)$ is a mild solution to the FP equation (117). Furthermore, let $\delta > 0$. Given any $t \in [0,T_0]$ and any function $\phi \in C\left([0,T_0]; C^{\alpha+\delta}\left(\mathbb{R}^d\right)\right)$ where $\partial_t \phi \in C_b\left([0,T_0] \times \mathbb{R}^d\right)$, it follows that*

$$\int_{\mathbb{R}^d} m(t,x)\,\phi(t,x)\,dx = \int_{\mathbb{R}^d} m_0(x)\phi(0,x)\,dx + \int_0^t \int_{\mathbb{R}^d} m(s,x)\left(\partial_t\phi - (-\Delta)^{\frac{\alpha}{2}}\phi - bD\phi\right)(s,x)\,dxds. \tag{134}$$

We say that $m(t,x)$ is a *very weak* solution to the FP equation if it solves (134) for all test functions described in the Lemma.

Proof. The proof is given for $\phi \in C\left([0,T_0]; C_b^2\left(\mathbb{R}^d\right)\right)$ in [13]. The spatial regularity of ϕ is however only used to ensure that $\mathcal{L}^*\phi(s,x)$ is well defined, where $\mathcal{L}^*$ is the operator, in our case $-(-\Delta)^{\alpha/2}$. By Proposition 2.15, we have that $-(-\Delta)^{\alpha/2}\phi(s,x)$ is well defined if $\phi(s,\cdot) \in C^{\alpha+\delta}\left(\mathbb{R}^d\right)$ for any $\delta > 0$. Thus, the proof in [13] is valid even for $(\alpha+\delta)$-Hölder continuous test functions. □

By applying different test functions $\phi(t,x)$ to Lemma 4.3, we are able to derive mass preservation and positivity. We begin with the following Proposition.

Proposition 4.4. *Let assumptions on m_0, b and m be as in Lemma 4.3. It follows that for any $t \in [0, T_0]$,*

$$\int_{\mathbb{R}^d} m(t,x)\,dx = \int_{\mathbb{R}^d} m_0(x)dx. \tag{135}$$

Proof. We define a penalty function somewhat similarly to what we did for the comparison principle proof in Lemma 3.6. In contrast to this proof, however, we now let ϕ_R be nonzero inside some compact instead of outside. Define $\phi_R \in C_b^\infty(\mathbb{R}^d)$ by $\phi_R(x) := \phi_1(x/R)$ where

$$\phi_1(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases}$$

By letting $\phi = \phi_R$ in Lemma 4.3, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^d} m(t,x)\,\phi_R(t,x)\,dx = &\int_{\mathbb{R}^d} m_0(x)\,\phi_R(0,x)\,dx \\ &+ \int_0^t \int_{\mathbb{R}^d} m(s,x)\left(\partial_t\phi_R - (-\Delta)^{\frac{\alpha}{2}}\phi_R - bD\phi_R\right)(s,x)\,dxds. \end{aligned} \tag{136}$$

Since there is no time dependence in ϕ_R, we have $\partial_t\phi_R(x) = 0$ for all $x \in \mathbb{R}^d$ and $R > 0$. Furthermore, $\lim_{R\to\infty} bD\phi_R(x) = \lim_{R\to\infty}(-\Delta)^{\alpha/2}\phi_R(x) = 0$ by a similar proof as in Lemma 3.6, where we use that b is bounded, and that $\phi_R = C(1-\varphi_R)$ for some $C > 0$ where φ_R is the penalty function from Lemma 3.6. In addition, $\lim_{R\to\infty}\phi_R(x) = 1$ for any $x \in \mathbb{R}^d$.

Notice that all integrands in (136) are absolutely integrable in their respective domains since $m_0, m(s,\cdot) \in L^1(\mathbb{R}^d)$ for any $s \in (0, T_0]$, and all terms with ϕ_R are bounded uniformly in $\mathbb{R}^d$. Taking the limit as $R \to \infty$, we can move the limits inside the integrals by dominated convergence. It follows that for any $t \in [0, T_0]$,

$$\int_{\mathbb{R}^d} m(t,x)\,dx = \int_{\mathbb{R}^d} m_0(x)\,dx, \tag{137}$$

and the proof is complete. □

We will now show that $m \geqslant 0$ almost everywhere. This seems harder, since we only want to use the very weak formulation which considers integrals over the entire $\mathbb{R}^d$. In order to conclude with positivity, we therefore have to show positivity of $\int_{\mathbb{R}^d} m(t,x)\,\phi(t,x)\,dx$ for all sufficiently regular positive functions ϕ. We also want the last integral in (134) to be zero, thus requiring ϕ to be a classical solution of a specific PDE. We present our requirements for ϕ in the following Proposition, inspired heavily by Lemma 3.3 in [13]. Our result is however slightly different since we need to consider $b(t,\cdot) \in C^\mu(\mathbb{R}^d)$ for any $\mu \in (0,\alpha)$.

Proposition 4.5 (Inspired by Lemma 3.3 in [13])**.** *Assume that $b \in C_b([0,T_0]; C^\mu(\mathbb{R}^d))$ for some $\mu \in (0,\alpha)$ and let $\delta \in (0,1)$ with $\delta < \mu$. Furthermore, let $t_0 \in [0,T_0]$ and $0 \leqslant \phi_{t_0} \in C^{\alpha+\delta}(\mathbb{R}^d)$. Then, there exists a unique classical solution $\phi \in C([0,t_0]; C^{\alpha+\delta}(\mathbb{R}^d))$ to the following problem:*

$$\begin{aligned} \partial_t\phi(t,x) - (-\Delta)^{\frac{\alpha}{2}}\phi(t,x) - b(t,x)\,D\phi(t,x) = 0, &\qquad (t,x) \in [0,t_0) \times \mathbb{R}^d, \\ \phi(t_0,x) = \phi_{t_0}(x), &\quad x \in \mathbb{R}^d. \end{aligned} \tag{138}$$

Furthermore, $\phi(0,x) \geqslant 0$ for any $x \in \mathbb{R}^d$.

Proof. Notice that the PDE moves backwards in time. By defining $z(t,x) := \phi(t_0 - t, x)$, we are able to formulate our problem in a more well-known setting.

$$\begin{aligned} \partial_t z(t,x) + (-\Delta)^{\frac{\alpha}{2}} z(t,x) + b(t,x)\,Dz(t,x) = 0, &\qquad (t,x) \in (0,t_0] \times \mathbb{R}^d, \\ z(0,x) = \phi_{t_0}(x), &\quad x \in \mathbb{R}^d. \end{aligned} \tag{139}$$

The change of signs in (139) follows when shifting from backwards to forwards in time. Since this is an equivalent problem, it suffices to show that there exists a unique classical solution

$z \in C\left([0, t_0]; C^{\alpha+\delta}\left(\mathbb{R}^d\right)\right)$ where $z(t_0, x) \geqslant 0$. We begin by proving short time existence with a similar approach as in Theorem 4.1. Let $t_1 \in (0, t_0)$ and define the Banach space

$$X = \left\{z : z, Dz \in C_b\left((0, t_1] \times \mathbb{R}^d\right), \|\!|z|\!\|_X < \infty\right\}, \tag{140}$$

where we let $\|\!|z|\!\|_X := \sup_{t\in(0,t_1]} \|z(t,\cdot)\|_X$ with

$$\|z(t,\cdot)\|_X = \|z(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + \|Dz(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + [Dz(t,\cdot)]_{C^{0,\alpha+\delta-1}(\mathbb{R}^d)}. \tag{141}$$

The Duhamel map for (139) is of the form

$$S(z) = K(t) * \phi_{t_0}(x) - \int_0^t K(t-s,\cdot) * (b(s,\cdot)\, Dz(s,\cdot))(x)\, ds. \tag{142}$$

In order to use Banach's fixed point theorem, we need to show that $S : X \to X$. We begin with showing boundedness of $\|\!|S(z)|\!\|_X$ whenever $z \in X$. Since our Duhamel map is very similar to the Duhamel map for the FP equation in (28), we will only show boundedness of $[DS(z)(t,\cdot)]_{C^{0,\alpha+\delta-1}}$. For the other terms in (141), we refer to Theorem 4.1.

By heat kernel estimates from Theorem 2.26, Lemma 2.37 and an interpolation as in Theorem 2.35 (a) in the integral, we deduce that for any $t \in (0, t_1]$,

$$\begin{aligned}
[DS(z)(t,\cdot)]_{C^{0,\alpha+\delta-1}} &\leqslant \|K(t,\cdot)\|_1 [\phi_{t_0}]_{C^{0,\alpha+\delta-1}} + C_I\lambda \int_0^t (t-s)^{-\frac{\alpha-\varepsilon}{\alpha}} [b(s,\cdot)\, Dz(s,\cdot)]_{C^{0,\delta+\varepsilon}}\, ds \\
&\leqslant [\phi_{t_0}]_{C^{0,\alpha+\delta-1}} + C_I\lambda \frac{\alpha}{\varepsilon} t_1^{\frac{\varepsilon}{\alpha}} \left(\|b\|_\infty [Dz]_{C^{0,\delta+\varepsilon}} + [b]_{C^{0,\delta+\varepsilon}} \|Dz\|_\infty\right),
\end{aligned} \tag{143}$$

where we let $0 < \varepsilon < \delta$ and $\varepsilon < \mu - \delta$. Furthermore, we have let $t \to t_1$ since its exponent is positive. All terms on the right hand side are bounded independently of t, and we conclude that $\|\!|S(z)|\!\|_X < \infty$. This immediately implies that $z(t,\cdot)$ is Lipschitz continuous, and that $Dz(t,\cdot)$ is $(\alpha+\delta-1)$-Hölder continuous in space.

Time continuity of z and Dz follows by a simpler version of the proof in Theorem 4.1. The main difference is that we do not have time blowup on Dz. By using Lemma 2.23 with $\gamma = 1$ and $\gamma = \alpha+\delta-1$ for z and Dz respectively, we get that $z, Dz \in C_b\left((0, t_1] \times \mathbb{R}^d\right)$. Hence, $S : X \to X$.

By a similar proof as in Theorem 4.1, we know that S is a contraction map when choosing t_1 sufficiently small. In addition, we see that t_1 depends solely on α, δ, ε, λ, interpolation constants and uniform bounds on b. This is similar to the dependencies for T_0 in Theorem 4.1. By Banach's fixed point theorem, there exists $z \in X$ such that $S(z) = z$.

We proceed with showing that our solution is classical. Notice that our Duhamel map in (142) becomes similar to the one in Theorem 3.4 by letting $g(s,\cdot) := b(s,\cdot)\, Dz(s,\cdot)$. The β-Hölder continuity of v_0 and $g(s,\cdot)$ in Theorem 3.4 is analogous to the $(\delta+\varepsilon)$-Hölder continuity of ϕ_{t_0} and $g(s,\cdot)$, and allows us to arrive at an $(\alpha+\delta)$-Hölder continuous classical solution. By noticing that $\delta + \varepsilon < \mu$, it follows that

$$g(s,\cdot) \in C^\mu\left(\mathbb{R}^d\right) \subset C^{\delta+\varepsilon}\left(\mathbb{R}^d\right) \quad \text{and} \quad \phi_0 \in C^{\alpha+\delta}\left(\mathbb{R}^d\right) \subset C^{\delta+\varepsilon}\left(\mathbb{R}^d\right).$$

Thus, by a similar argument as in Theorem 3.4, we can conclude that the fixed point $z \in C_b\left((0, t_1]; C^{\alpha+\delta}\left(\mathbb{R}^d\right)\right)$ indeed is a classical solution to (139). For a more detailed proof of a similar fashion, we refer to the classical solution proof for the FP equation in Section 4.3.

Uniqueness of $z(t,x)$ follows from the comparison principle. This is very similar to what we did in Lemma 3.6 for the HJB equation and we omit the proof. Finally, we deduce that $z \geqslant 0$ by noticing that the zero function is a subsolution to (139).

Lastly, we use a patching argument similar to Theorem 3.8. We get that t_1 is independent of the initial time when translating the short time existence proof above. Thus, by letting $N \in \mathbb{N}$ be the largest integer such that $Nt_1/2 \leqslant t_0$, we can iteratively find unique classical solutions on time intervals $(0, t_1]$, $(t_1/2, 3t_1/2]$, …, $((N-2)\,t_1/2, Nt_1/2]$ and $((N-1)\,t_1/2, t_0]$. Since these solutions are unique and overlapping in $(t_1/2, t_1], (t_1, 3t_1/2], \ldots ((N-1)\,t_1/2, Nt_1/2]$ respectively, we conclude that there exists a unique classical solution $z(t,x)$ in the entire $(0, t_0] \times \mathbb{R}^d$. We refer to Theorem 3.8 for a more comprehensive proof. □

Lemma 4.6 (Positivity of very weak solutions)**.** *Let assumptions on m_0 and m be as in Lemma 4.3, and assume that $m_0 \geqslant 0$. For $\mu \in (0, \alpha)$, let $b \in C_b\left([0, T_0]; C^{\mu}\left(\mathbb{R}^d\right)\right)$. It follows that $m(t, \cdot) \geqslant 0$ almost everywhere in $\mathbb{R}^d$ for all $t \in [0, T_0]$.*

Proof. Let $t_0 \in (0, T_0]$ be fixed, and consider the very weak formulation in (134) with $t = t_0$. For some $\delta \in (0, 1)$ with $\delta < \mu$, let $\phi_{t_0} \in C^{\alpha+\delta}\left(\mathbb{R}^d\right)$ be nonnegative and let $\phi(t, \cdot)$ be the corresponding solution from Lemma 4.5. By (134), we then get that

$$\int_{\mathbb{R}^d} m(t_0, x)\, \phi_{t_0}(x)\, dx = \int_{\mathbb{R}^d} m_0(x)\, \phi(0, x)\, dx \geqslant 0, \tag{144}$$

where the inequality holds since both m_0 and $\phi(0, \cdot)$ are nonnegative functions. By letting ϕ_{t_0} approximate unity at each $x \in \mathbb{R}^d$, it follows that $m(t_0, x) \geqslant 0$ almost everywhere in $\mathbb{R}^d$ for all $t_0 \in (0, T_0]$. Finally, since $m(0, \cdot) = m_0 \geqslant 0$ almost everywhere, the result holds in the entire $[0, T_0] \in \mathbb{R}^d$. ☐

By combining Proposition 4.4 and Lemma 4.6, we can quite easily prove uniqueness of very weak solutions. We summarize our results in the following theorem.

Theorem 4.7. *For $\mu \in (0, \alpha)$, let $b \in C_b\left([0, T_0]; C^{\mu}\left(\mathbb{R}^d\right)\right)$ and $0 \leqslant m_0 \in L^1\left(\mathbb{R}^d\right)$ with $\|m_0\|_1 = 1$. Suppose that $m \in B\left([0, T_0]; L^1\left(\mathbb{R}^d\right)\right)$ is a very weak solution to the FP equation (117), meaning that it satisfies (134) for test functions described in Lemma 4.3. Then, m is a unique very weak solution where $m \geqslant 0$ almost everywhere and $\|m(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1$ for every $t \in [0, T_0]$.*

Proof. By Lemma 4.6, we know that $m(t, \cdot) \geqslant 0$ almost everywhere in $\mathbb{R}^d$ for every $t \in [0, T_0]$. Combining this property with Proposition 4.4, it follows that

$$\|m(t, \cdot)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} m(t, x)\, dx = \int_{\mathbb{R}^d} m_0(x)\, dx = \|m_0\|_{L^1(\mathbb{R}^d)} = 1, \tag{145}$$

for any $t \in [0, T_0]$. We proceed with proving that m is unique. Assume by contradiction that m_1 and m_2 are distinct very weak solutions to the FP equation. It follows that $m_d := m_1 - m_2$ is a very weak solution with $m_0 \equiv 0$. Since the mass preservation property in (145) holds for all very weak solutions, it follows that

$$\int_{\mathbb{R}^d} |m_d(t, x)|\, dx = 0,$$

for any $t \in [0, T_0]$. The integrand is nonnegative and the equation is therefore only satisfied if $m_d(t, x) = 0$ for almost every $x \in \mathbb{R}^d$ and every $t \in [0, T_0]$. This implies $m_1(t, \cdot) = m_2(t, \cdot)$ almost everywhere in $\mathbb{R}^d$, which is a contradiction. It follows that any very weak solution m to the FP equation (117) is unique. ☐

We finish the section by combining our results in a Corollary.

Corollary 4.8. *Let $\alpha \in (1, 2)$, $\mu \in (0, \alpha)$, $\nu \in (0, 1)$ and suppose that $0 \leqslant m_0 \in C^{0,\nu}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$ with $\|m_0\|_1 = 1$. Furthermore, let λ be the constant from Theorem 2.26. For a terminal time $T > 0$, suppose that $b \in C_b\left([0, T]; C^{\mu}\left(\mathbb{R}^d\right)\right)$. If $\mu > 1$, assume also that $Db \in C_b\left([0, T]; C^{\mu-1}\left(\mathbb{R}^d\right)\right)$. Then, there exists a unique very weak solution $m \in C_b\left([0, T_0] \times \mathbb{R}^d\right)$ where $T_0 \in (0, T)$ is given by α, μ, ν, λ, interpolation constants from Section 2.8 and uniform bounds on b. Furthermore, $m \geqslant 0$ almost everywhere, and for any $t \in [0, T_0]$, $m(t, \cdot) \in L^1\left(\mathbb{R}^d\right)$ with $\|m(t, \cdot)\|_1 = 1$. The regularity results from Theorem 4.1 and Lemma 4.2 are also satisfied in $(0, T_0] \times \mathbb{R}^d$.*

Proof. The existence and regularity results for m follows from Theorem 4.1 and Lemma 4.2. Furthermore, m is unique and satisfies the positivity and L^1-regularity requirements by Theorem 4.7. ☐

4.3 Long time existence

We proceed with showing existence of a unique very weak solution in the entire $(0,T]$ for any terminal time $T>0$. This is done similarly to Theorem 3.8 and uses a patching argument.

Theorem 4.9. *Let $\alpha \in (1,2), \mu \in (0,\alpha), \nu \in (0,1)$ and suppose that $0 \leqslant m_0 \in C^{0,\nu}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$ with $\|m_0\|_1 = 1$. Given a terminal time $T>0$, suppose that $b \in C_b\left([0,T]; C^\mu\left(\mathbb{R}^d\right)\right)$. If $\mu > 1$, assume also that $Db \in C_b\left([0,T]; C^{\mu-1}\left(\mathbb{R}^d\right)\right)$. Then, there exists a unique very weak solution $m \in C_b\left([0,T] \times \mathbb{R}^d\right)$ where $m \geqslant 0$ almost everywhere, and for any $t \in [0,T]$, $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ with $\|m(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$. Furthermore, m satisfies the regularity results from Theorem 4.1 and Lemma 4.2 in the entire $(0,T] \times \mathbb{R}^d$.*

Proof. By Theorem 4.1 and Corollary 4.8, there exists a unique very weak solution $m(t,x)$ in $[0,T_0] \times \mathbb{R}^d$ for some $T_0 \in (0,T)$ which satisfies the positivity and regularity requirements. It then follows that $m(T_0/2,\cdot) \in C^{0,\nu}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$ with $\|m(T_0/2,\cdot)\|_1 = 1$, hence satisfying the regularity requirements for the initial data m_0. By the dependencies of T_0 in Theorem 4.1, we know that T_0 is independent of the initial time in the short time existence proof. Thus, by an identical patching argument as in Theorem 3.8, we get that there exists a unique very weak solution m in the entire $[0,T] \times \mathbb{R}^d$. The positivity and regularity results from Theorem 4.1, Lemma 4.2 and Corollary 4.8 hold in $(0,T] \times \mathbb{R}^d$ by the same patching argument. □

4.4 Existence of a unique classical solution when $\mu > 1$

In this section, we will investigate whether the unique very weak solution in Theorem 4.9 is classical. For the HJB equation, we considered only classical solutions (see Theorem 3.4). This is however not the case for the FP equation. In order to have a classical solution, we need $\nabla \cdot (bm)$ to be well-defined which requires bm to be differentiable in space. By the product rule for differentiation, we then need Db to be well-defined. This is only the case when $\mu > 1$, since we are considering $b \in C_b\left([0,T]; C^\mu\left(\mathbb{R}^d\right)\right)$.

Notice also that any classical solution to the FP equation is a very weak solution. This follows from integrating by parts in (134) and is similar to Lemma 6.3 in [8]. By the uniqueness result for very weak solutions in Theorem 4.7, we then know that classical solutions are unique as well, given that they exist.

Theorem 4.10 (Existence of a unique classical solution)**.** *Let $\alpha \in (1,2)$, $\mu \in (1,\alpha)$, $\nu \in (0,1)$ and $m_0 \in C^{0,\nu}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$ with $\|m_0\|_1 = 1$. Given a terminal time $T > 0$, suppose that $b \in C_b\left([0,T]; C^\mu\left(\mathbb{R}^d\right)\right)$ and $Db \in C_b\left([0,T]; C^{\mu-1}\left(\mathbb{R}^d\right)\right)$. Then, there exists a unique classical solution $m \in C_b\left([0,T] \times \mathbb{R}^d\right)$ to the FP equation. Furthermore, $m \geqslant 0$ almost everywhere, and for every $t \in [0,T]$, $m(t,\cdot) \in L^1\left(\mathbb{R}^d\right)$ with $\|m(t,\cdot)\|_1 = 1$. The regularity results from Theorem 4.1 and Lemma 4.2 are also satisfied in $(0,T] \times \mathbb{R}^d$.*

Proof. By Theorem 4.9, there exists a unique very weak solution $m \in C_b\left([0,T] \times \mathbb{R}^d\right)$ which satisfies the positivity and regularity requirements. It remains to show that the solution is classical.

The proof is very similar to Theorem 3.4, and we will therefore only comment on the differences. Instead of working with β-Hölder continuous initial data, we now have $m_0 \in C^{0,\nu}\left(\mathbb{R}^d\right)$. This does not alter the proof in any major way, since we assume $\nu \in (0,1)$ which is analogous to $\beta \in (0,1)$. Instead of letting $g(s,x) := H(Dv(s,x)) - f(s,x)$, we define functions $g_i(s,x) := \partial_{x_i}(b_i(s,x)\, m(s,x))$ for $i \in \{1,\dots d\}$. By the spatial regularity assumptions on b and m, we then get that $g_i(s,\cdot) \in C^{0,\mu-1}\left(\mathbb{R}^d\right)$ for any $s \in (0,T]$. Furthermore, we deduce that

$$\begin{aligned}
\|g_i(s,\cdot)\|_\infty &\leqslant C_{g_0} + \tilde{C}_{g_0} s^{-\frac{1-\nu}{\alpha}}, \\
[g_i(s,\cdot)]_{C^{0,\mu-1}} &\leqslant C_{g_\mu} + \tilde{C}_{g_\mu} s^{-\frac{(1-\nu)(\mu-1)}{\alpha}} + C'_{g_\mu} s^{-\frac{1-\nu}{\alpha}} + C''_{g_\mu} s^{-\frac{\mu-\nu}{\alpha}},
\end{aligned} \tag{146}$$

for constants $C_{g_0}, \tilde{C}_{g_0}, C_{g_\mu}, \tilde{C}_{g_\mu}, C'_{g_\mu}, C''_{g_\mu} \geqslant 0$. The Hölder estimate is given by (130), and the L^∞-estimate follows from the product rule, $s^{\frac{1-\nu}{\alpha}} Dm \in C_b\left((0,T] \times \mathbb{R}^d\right)$ and that b and Db are uniformly bounded. The estimates in (146) are similar to the corresponding bounds in (81). Since all exponents on s in (146) are greater than -1, we ensure that for any $t \in (0,T]$ and $\varepsilon > 0$, $\|K(t-s,\cdot) * g_i(s,\cdot)\|_\infty$ and $[K(t-s,\cdot) * g_i(s,\cdot)]_{C^{\alpha+\mu-1-\varepsilon}}$ are integrable in $(0,t)$. This is similar to (84), and will be sufficient to complete the proof since $\alpha + \mu - 1 - \varepsilon > \alpha$. Finally, we require $g_i(t,x)$ to be continuous in time for any $(t,x) \in (0,T] \times \mathbb{R}^d$ (see (97)). This holds since m, Dm, b and Db are continuous in time in this domain.

Based on the observations above, we are able to repeat the proof of Theorem 3.4 with $g_i(s,x) := \partial_{x_i} b_i(s,x)\, m(s,x)$. By Theorem 4.1, m is a mild solution and a fixed point of the Duhamel map in (28). We get that

$$
\begin{aligned}
m(t,x) &= (K(t,\cdot) * m_0)(x) - \sum_{i=1}^{d} \int_0^t \partial_{x_i} K(t-s,\cdot) * (b_i(s,\cdot)\, m(s,\cdot))(x)\, ds \\
&= (K(t,\cdot) * m_0)(x) - \sum_{i=1}^{d} \int_0^t K(t-s,\cdot) * (\partial_{x_i}(b_i(s,\cdot)\, m(s,\cdot)))(x)\, ds \\
&= (K(t,\cdot) * m_0)(x) - \sum_{i=1}^{d} \int_0^t K(t-s,\cdot) * g_i(s,\cdot)(x)\, ds.
\end{aligned}
$$

This is analogous to the Duhamel map considered in Theorem 3.4, except for the summation. Since the arguments for the initial and integral terms in Theorem 3.4 are independent, however, we can argue seperately for each integral in the sum. It follows that

$$
\begin{aligned}
\partial_t m(t,x) + (-\Delta)^{\frac{\alpha}{2}} m(t,x) + \sum_{i=1}^{d} g_i(t,x) &= 0 \qquad \text{in } (0,T] \times \mathbb{R}^d, \\
m(0,x) &= m_0(x) \qquad \text{in } \mathbb{R}^d.
\end{aligned} \tag{147}
$$

By the definition of g_i, we have

$$
\sum_{i=1}^{d} g_i(t,x) = \sum_{i=1}^{d} \partial_{x_i}(b_i(t,x)\, m(t,x)) = \nabla \cdot (b(t,x)\, m(t,x)). \tag{148}
$$

Finally, by combining (147) and (148), m is a classical solution to the FP equation in $(0,T] \times \mathbb{R}^d$. By Theorem 4.7, we know that very weak solutions to the FP equation are unique. Since any classical solution is a very weak solution, we conclude that our classical solution is unique. □

4.5 Uniform continuity

We finish the chapter by showing uniform continuity of m and its derivatives in $[t_1,t_2] \times \mathbb{R}^d$ for $0 < t_1 < t_2 \leqslant T$. This is very similar to Theorem 3.9.

Theorem 4.11 (Uniform continuity). *Let $\alpha \in (1,2)$, $\mu \in (1,\alpha)$, $\nu \in (0,1)$, $\varepsilon > 0$ and $T > 0$. Furthermore, let assumptions on m_0 and b be as in Theorem 3.1, and assume in addition that $b(\cdot,x)$ and $Db(\cdot,x)$ are uniformly continuous in $[0,T]$ for any $x \in \mathbb{R}^d$. Let m be the corresponding classical solution from Theorem 4.10 and choose t_1, t_2 such that $0 < t_1 \leqslant t_2 \leqslant T$. Then, there exists a modulus of continuity ω such that for any $(t,x),(s,y) \in [t_1,t_2] \times \mathbb{R}^d$, the following statements hold:*

If $\alpha + \mu \leqslant 3$:

$$
\begin{aligned}
&|m(t,x) - m(s,y)| + |Dm(t,x) - Dm(s,y)| \\
&+ |\partial_t m(t,x) - \partial_t m(s,y)| + |(-\Delta)^{\frac{\alpha}{2}} m(t,x) - (-\Delta)^{\frac{\alpha}{2}} m(s,y)| \leqslant \omega(|t-s|, |x-y|).
\end{aligned} \tag{149}
$$

If $\alpha + \mu > 3$:

$$
\begin{aligned}
&|m(t,x) - m(s,y)| + |Dm(t,x) - Dm(s,y)| + \left|D^2 m(t,x) - D^2 m(s,y)\right| \\
&+ |\partial_t m(t,x) - \partial_t m(s,y)| + |(-\Delta)^{\frac{\alpha}{2}} m(t,x) - (-\Delta)^{\frac{\alpha}{2}} m(s,y)| \leqslant \omega(|t-s|, |x-y|).
\end{aligned} \tag{150}
$$

Furthermore, ω only depends on m through uniform bounds on m and Dm as well as $[m]_{C^{\alpha+\mu-\varepsilon-1}}$. If $\alpha+\mu>3$, ω also depends on uniform bounds on D^2m.

Proof. The proof is very similar to Theorem 3.9, and we will be brief. Only the case where $\alpha+\mu>3$ is considered. For $\alpha+\beta\leqslant 3$, the proof is exactly the same, but without the estimates for D^2m.

The time interval $[t_1,t_2]$ is strictly away from zero, and we can ignore blowup at $t=0$. By Theorem 4.1, 4.2 and 4.9, we then have that $m, Dm, D^2m\in C_b\left([t_0,t_1]\times\mathbb{R}^d\right)$. Furthermore, $[m(t,\cdot)]_{C^{\alpha+\mu-\varepsilon-1}}$ is bounded uniformly in $[t_1,t_2]$. By a similar approach as in Theorem 3.9, there exists $\tilde\omega$ satisfying the dependence assumptions such that for any $(t,x),(s,y)\in[t_1,t_2]\times\mathbb{R}^d$,

$$|m(t,x)-m(s,y)|+|Dm(t,x)-Dm(s,y)|+\left|D^2m(t,x)-D^2m(s,y)\right|\leqslant\tilde\omega(|t-s|,|x-y|). \tag{151}$$

The regularity estimate for $(-\Delta)^{\alpha/2}m$ is also similar to Theorem 3.9. For $t,s\in[t_1,t_2]$ and $h\in\mathbb{R}^d$, we have $m(t,\cdot+h)-m(t,\cdot)\in C^{\alpha+\mu-\varepsilon-1}\left(\mathbb{R}^d\right)$. We use Proposition 2.15 to deduce that

$$\begin{aligned}&\left|(-\Delta)^{\frac{\alpha}{2}}m(t,x+h)-(-\Delta)^{\frac{\alpha}{2}}m(s,x)\right|\\&\leqslant C_1\|m(t,\cdot+h)-m(s,\cdot)\|_{L^\infty(\mathbb{R}^d)}+C_2\left[m(t,\cdot+h)-m(s,\cdot)\right]_{C^{0,\alpha+\delta-1}(\mathbb{R}^d)},\end{aligned} \tag{152}$$

for constants $C_1,C_2\geqslant 0$ where δ is chosen such that $0<\delta+\mu-\varepsilon-1$ and $\alpha+\delta<2$. We interpolate the Hölder seminorm between $\|Dm(t,\cdot+h)-Dm(s,\cdot)\|_\infty$ and $[m(t,\cdot+h)-m(s,\cdot)]_{C^{\alpha+\mu-\varepsilon-1}}$ and use Theorem 2.33. By (151) and a calculation similar to (114), we get that

$$\begin{aligned}\left|(-\Delta)^{\frac{\alpha}{2}}m(t,x)-(-\Delta)^{\frac{\alpha}{2}}m(s,y)\right|&\leqslant C_1\tilde\omega(|t-s|,|x-y|)\\&+\widehat{C}_2\left([m(t,\cdot+h)]_{C^{\alpha+\mu-\varepsilon-1}}+[m(s,\cdot)]^{\zeta}_{C^{\alpha+\mu-\varepsilon-1}}\right)\tilde\omega^{1-\zeta}(|t-s|,|x-y|),\end{aligned} \tag{153}$$

where $\zeta:=\frac{\alpha+\delta-1}{\alpha+\mu-\varepsilon-2}$ and $\widehat{C}_2\geqslant 0$.

For $\partial_t m$, we recall that m is a classical solution to the FP equation (117) in $(0,T]\times\mathbb{R}^d$. For any $(t,x),(s,y)\in[t_1,t_2]\times\mathbb{R}^d$, we then have

$$\begin{aligned}&|\partial_t m(t,x)-\partial_t m(s,y)|\leqslant\left|(-\Delta)^{\frac{\alpha}{2}}m(t,x)-(-\Delta)^{\frac{\alpha}{2}}m(s,y)\right|\\&+\sum_{i=1}^d\left(|m(t,x)\,\partial_{x_i}b(t,x)-m(s,y)\,\partial_{x_i}b(s,y)|+|b(t,x)\,\partial_{x_i}m(t,x)-b(s,y)\,\partial_{x_i}m(s,y)|\right).\end{aligned} \tag{154}$$

Since b and Db are uniformly continuous in time, and Lipschitz and $(\mu-1)$-Hölder continuous in space respectively, there exists a modulus of continuity ω_b such that

$$|b(t,x)-b(s,y)|+\sum_{i=1}^d|\partial_{x_i}b(t,x)-\partial_{x_i}b(s,y)|\leqslant\omega_b(|t-s|,|x-y|),\quad\forall(t,x)\in[t_1,t_2]\times\mathbb{R}^d. \tag{155}$$

By (151) and (155), it then follows that for any $i\in\{1,\dots,d\}$,

$$\begin{aligned}&|m(t,x)\,\partial_{x_i}b(t,x)-m(s,y)\,\partial_{x_i}b(s,y)|\\&\leqslant\|m\|_\infty|\partial_{x_i}b(t,x)-\partial_{x_i}b(s,y)|+\|Db\|_\infty|m(t,x)-m(s,y)|\\&\leqslant\|m\|_\infty\omega_b(|t-s|,|x-y|)+\|Db\|_\infty\tilde\omega(|t-s|,|x-y|).\end{aligned} \tag{156}$$

Similarly, we get that

$$\begin{aligned}&|b(t,x)\,\partial_{x_i}m(t,x)-b(s,y)\,\partial_{x_i}m(s,y)|\\&\qquad\leqslant\|b\|_\infty\tilde\omega(|t-s|,|x-y|)+\|Dm\|_\infty\omega_b(|t-s|,|x-y|).\end{aligned} \tag{157}$$

By inserting (156) and (157) into (154), and by (151) and (153), there exists a modulus of continuity ω such that (150) holds. Furthermore, ω will only depend on m through uniform bounds on m, Dm, D^2m and $[m]_{C^{\alpha+\mu-\varepsilon-1}}$. ☐

5 Mean Field Games and Further Work

We finish the thesis with a brief discussion on how our results from Chapter 3 and 4 contribute to the study of the Mean Field Game system of the form

$$\begin{cases} -\partial_t v + (-\Delta)^{\frac{\alpha}{2}} v + H(Dv) = F(m(t), x) & \text{in } [0,T) \times \mathbb{R}^d, \\ v(T, \cdot) = G(m(T), \cdot) & \text{in } \mathbb{R}^d, \\ \partial_t m + (-\Delta)^{\frac{\alpha}{2}} m + \nabla \cdot (D_p H(Dv) m) = 0 & \text{in } (0,T] \times \mathbb{R}^d, \\ m(0, \cdot) = m_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{158}$$

As mentioned in the introduction, our main deviation from earlier work is the imposition of spatial Hölder regularity on the initial and source terms. In order to be consistent with the analysis earlier in the thesis, we assume $F(m(t), \cdot)$ and $G(m(T), \cdot)$ to be β-Hölder continuous in space for some $\beta \in (0,1)$. In addition, we let $0 \leqslant m_0 \in C^{0,\nu}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with $\|m\|_{L^1(\mathbb{R}^d)} = 1$.

A promising approach for showing existence of solutions to the MFG system in (158) is presented in Theorem 3.2 in [8]. In order to relate this to our results in Chapter 3 and 4, we give a brief review.

The proof uses a fixed point argument in $C([0,T]; \mathcal{P}(\mathbb{R}^d))$, where $\mathcal{P}(\mathbb{R}^d)$ denotes the space of Borel probability measures on $\mathbb{R}^d$. The corresponding metric is $d(\mu_1, \mu_2) = \sup_{[0,T]} d_0(\mu_1(t), \mu_2(t))$, where d_0 is the Kantorovich-Rubinstein distance defined by

$$d_0(\mu_1, \mu_2) := \sup_{\phi \in \mathrm{Lip}_{1,1}(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} \phi(x) \, d(\mu_1 - \mu_2)(x) \right\},$$

where $\mathrm{Lip}_{1,1}(\mathbb{R}^d) = \{\phi : \phi \text{ is Lipschitz continuous and } \|\phi\|_\infty, \|D\phi\|_\infty \leqslant 1\}$.

Unlike the fixed point arguments in our analysis, the existence proof for the MFG system is based on Schauder's fixed point theorem (see Theorem 11.1 in [9]). This is more complicated than Banach, since it requires us to to show compactness of a subset in $C([0,T]; \mathcal{P}(\mathbb{R}^d))$. These arguments are often quite technical, and utilize a combination of Arzelà-Ascoli and Prokhorov's theorem.

The general outline of the proof involves defining a map $S(\mu) := m$, where m is given by μ through the following system of equations:

$$\begin{cases} -\partial_t v + (-\Delta)^{\frac{\alpha}{2}} v + H(Dv) = F(\mu(t), x) & \text{in } [0,T) \times \mathbb{R}^d, \\ v(T, \cdot) = G(\mu(T), \cdot) & \text{in } \mathbb{R}^d. \end{cases} \tag{159}$$

$$\begin{cases} \partial_t m + (-\Delta)^{\frac{\alpha}{2}} m + \nabla \cdot (D_p H(Dv) m) = 0 & \text{in } (0,T] \times \mathbb{R}^d, \\ m(0, \cdot) = m_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{160}$$

The map is defined on some compact convex subset $\mathcal{C} \subset C([0,T]; \mathcal{P}(\mathbb{R}^d))$, and fixed points of the map will be solutions to the MFG system in (158). Existence can therefore be proven by Schauder's fixed point theorem, provided that S is continuous and that $S : \mathcal{C} \to \mathcal{C}$.

Showing that S is a self map requires $S(\mu) := m \in C([0,T]; \mathcal{P}(\mathbb{R}^d))$ whenever $\mu \in C([0,T]; \mathcal{P}(\mathbb{R}^d))$. This means that our solution m must be defined at the terminal time. Recall from the introduction in Chapter 4 that we only showed existence of solutions to the FP equation in $[0, T_\varepsilon] \times \mathbb{R}^d$ for some $T_\varepsilon < T$. This was done in order to avoid the time blowup on $b := D_p H(Dv)$ at $t = T$, which stems from the HJB equation.

In order to circumvent this problem, notice that $\|b\|_{L^\infty([0,T] \times \mathbb{R}^d)} = \|D_p H(Dv)\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leqslant L_H$ since H is globally Lipschitz, even at the terminal time. The blowup at $t = T$ is therefore only problematic when working with derivatives or Hölder seminorms of b. By repeating the short time existence proof in Theorem 4.1 with only $b \in L^\infty([0,T] \times \mathbb{R}^d)$, we are then able to show that there exists a solution at the terminal time. Since we assume less regularity on b, however, m will not be regular enough to be a classical solution at $t = T$. Therefore, it seems that we only have classical solutions in $(0,T) \times \mathbb{R}^d$ for the FP equation in the MFG system.

Our results from Chapter 3 and 4 are essential steps towards deriving existence of solutions to the MFG system. Firstly, note that S is well-defined only if there exist solutions v to (159) and m to (160) seperately. Furthermore, we will only have existence of a classical solution (v, m) if v and m are classical solutions to the HJB and FP equations respectively. Finally, our uniform continuity results in sections 3.6 and 4.5 prove useful when showing that (v, m) is classical. This is evident by the proof of Theorem 3.2 in [8], and stems from a compactness argument that requires equicontinuity of a family of functions.

Bibliography

[1] Yves Achdou et al. *Mean Field Games: Cetraro, Italy 2019*. Vol. 2281. Springer Nature, 2021.

[2] Amund Skretting Bergset. 'Existence and uniqueness results for the fractional Hamilton-Jacobi-Bellman equation in Hölder spaces'. Project thesis. NTNU, 2023.

[3] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. *Measure theory*. Vol. 1. Springer, 2007.

[4] Peter S Bullen. *Handbook of means and their inequalities*. Vol. 560. Springer Science & Business Media, 2003.

[5] Pierre Cardaliaguet. *Notes on mean field games*. Tech. rep. Technical report, 2010.

[6] Zhen-Qing Chen, Panki Kim and Renming Song. 'Heat kernel estimates for the Dirichlet fractional Laplacian'. In: *Journal of the European Mathematical Society* 12.5 (2010), pp. 1307–1329.

[7] Xuban Diez-Izagirre and Carlota M Cuesta. 'Vanishing viscosity limit of a conservation law regularised by a Riesz–Feller operator'. In: *Monatshefte für Mathematik* 192.3 (2020), pp. 513–550.

[8] Olav Ersland and Espen R. Jakobsen. 'On fractional and nonlocal parabolic mean field games in the whole space'. In: *Journal of Differential Equations* 301 (2021), pp. 428–470. ISSN: 0022-0396. DOI: https://doi.org/10.1016/j.jde.2021.08.026. URL: https://www.sciencedirect.com/science/article/pii/S0022039621005374.

[9] David Gilbarg et al. *Elliptic partial differential equations of second order*. Vol. 224. 2. Springer, 1977.

[10] Rudolf Gorenflo et al. *Mittag-Leffler functions, related topics and applications*. Springer, 2020.

[11] Minyi Huang, Roland P Malhamé and Peter E Caines. 'Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle'. In: (2006).

[12] Cyril Imbert. 'A non-local regularization of first order Hamilton–Jacobi equations'. In: *Journal of Differential Equations* 211.1 (2005), pp. 218–246.

[13] Espen Robstad Jakobsen and Artur Rutkowski. 'The master equation for mean field game systems with fractional and nonlocal diffusions'. In: *arXiv preprint arXiv:2305.18867* (2023).

[14] Wolfram Koepf. 'The gamma function'. In: *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities* (2014), pp. 1–10.

[15] Nikolaı Vladimirovich Krylov. *Lectures on elliptic and parabolic equations in Holder spaces*. 12. American Mathematical Soc., 1996.

[16] Mateusz Kwaśnicki. 'Ten equivalent definitions of the fractional Laplace operator'. In: *Fractional Calculus and Applied Analysis* 20.1 (2017), pp. 7–51.

[17] Jean-Michel Lasry and Pierre-Louis Lions. 'Mean field games'. In: *Japanese journal of mathematics* 2.1 (2007), pp. 229–260.

[18] Gary M Lieberman. *Second order parabolic differential equations*. World scientific, 1996.

[19] John Nash. 'Non-Cooperative Games'. In: *Annals of Mathematics* 54.2 (1951), pp. 286–295. ISSN: 0003486X. URL: http://www.jstor.org/stable/1969529 (visited on 21st June 2023).

[20] Jet Nestruev, AV Bocharov and S Duzhin. *Smooth manifolds and observables*. Springer, 2003.

[21] Louis Nirenberg. 'On elliptic partial differential equations'. In: *Il principio di minimo e sue applicazioni alle equazioni funzionali*. Springer, 2011, pp. 1–48.

[22] Terence Tao. *An introduction to measure theory*. Vol. 126. American Mathematical Soc., 2011.

[23] John Von Neumann. 'On the theory of parlor games'. In: *Mathematische Annalen* 100.1 (1928), pp. 295–320.

[24] Haiping Ye, Jianming Gao and Yongsheng Ding. 'A generalized Gronwall inequality and its application to a fractional differential equation'. In: *Journal of Mathematical Analysis and Applications* 328.2 (2007), pp. 1075–1081.

Appendix

A Proof of Lemma 2.25 and Theorem 2.26

Proof of Lemma 2.25 (Corrected from Lemma 3.3 in [2]). Let $\widehat{D^k K}(1,\omega)$ denote the Fourier transform of $D^k K(1,u)$, and recall that differentiating $\widehat{D^k K}(1,\omega)$ with respect to ω_j on the Fourier side corresponds to multiplying $D^k K(1,u)$ by iu_j. We can then express the m^{th}-order derivative of $\widehat{D^k K}(1,\omega)$ with respect to ω_j as

$$(iu_j)^m D^k K(1,u) = \int_{\mathbb{R}^d} e^{iu\cdot\omega}\left(\frac{\partial^m}{\partial\omega_j^m}\widehat{D^k K}(1,\omega)\right)d\omega.$$

By combining the absolute values of this expression for all $j \in \{1,\dots,d\}$, and using that $|ab| = |a|\cdot|b|$ for all $a, b \in \mathbb{C}^d$, we get that

$$\left(|u_1|^m + \ldots + |u_d|^m\right)\left|D^k K(1,u)\right| = \sum_{j=1}^{d}\left|\int_{\mathbb{R}^d} e^{iu\cdot\omega}\left(\frac{\partial^m}{\partial\omega_j^m}\widehat{D^k K}(1,\omega)\right)d\omega\right|. \tag{161}$$

Furthermore, the power mean inequality (see Theorem 1, Section 3.1 in [4]) states that

$$\left(\frac{|u_1|^2 + \ldots + |u_d|^2}{d}\right)^{\frac{1}{2}} \leqslant \left(\frac{|u_1|^m + \ldots + |u_d|^m}{d}\right)^{\frac{1}{m}}.$$

Combining this with (161) yields

$$|u|^m\left|D^k K(1,u)\right| \leqslant d^{\frac{m-2}{2}}\sum_{j=1}^{d}\left|\int_{\mathbb{R}^d} e^{iu\cdot\omega}\left(\frac{\partial^m}{\partial\omega_j^m}\widehat{D^k K}(1,\omega)\right)d\omega\right|. \tag{162}$$

In order to prove the Lemma, we need to bound the right hand side of (162) by a constant $C \geqslant 0$ for $m = |k|$. This is possible whenever $\widehat{D^k K}(1,\cdot) \in W^{|k|,1}\left(\mathbb{R}^d\right)$. Since repeated differentiation of $\widehat{D^k K}(1,\omega)$ results in an impractical number of terms as the differentiation progresses, we need to look at the derivatives in a more general sense. We propose that any derivative $D^s(\widehat{D^k K}(1,\omega))$ can be written as

$$D^s(\widehat{D^k K}(1,\omega)) = e^{-|\omega|^\alpha}\sum_{j=1}^{n}\left(c_j\omega_1^{a_{j,1}}\ldots\omega_d^{a_{j,d}}|\omega|^{b_j}\right), \tag{163}$$

for constants $n \in \mathbb{N}$, $a_{j,1},\dots,a_{j,d} \in \mathbb{N}_0$, $b_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$. Since $D^s(\widehat{D^k K}(1,\omega))$ in (163) has exponential decay as $|\omega| \to \infty$, it will be absolutely integrable over $\mathbb{R}^d$ as long as it does not attain a singularity at $\omega = 0$. This singularity can only occur if $a_{j,1} + a_{j,2} + \ldots + a_{j,d} + b_j < 0$ for some $j \in \{1,\dots,n\}$. We can easily see this by noting that $\omega_j = |\omega|\,\sigma_j$ for some $|\sigma_j| = 1$, and letting $|\omega| \to 0$. To investigate when this occurs, we define the function

$$\kappa(s) = \min_{j\in\{1,\dots n\}}\left(a_{j,1} + a_{j,2} + \ldots + a_{j,d} + b_j\right),$$

where the constants are given by the relation in (163) and s corresponds to the derivative order. If $\kappa(s) \geqslant 0$ for all multi-indices $|s| \leqslant |k|$, it follows that $\widehat{D^k K}(1,\cdot) \in W^{|k|,1}\left(\mathbb{R}^d\right)$. We proceed with showing that (163) indeed is a general form for the derivatives $D^s(\widehat{D^k K}(1,\omega))$, and that $\kappa(s+1) \geqslant \kappa(s) - 1$ by induction.

First, observe that $\widehat{D^k K}(1,\omega)$ is on the form given in (163). Assume now that (163) holds for some multi-index s. We include the case where $|s| = 0$ by letting D^0 denote the identity operator.

By differentiating with respect to ω_1, we get that

$$\frac{\partial}{\partial \omega_1} D^s \left(\widehat{D^k K}(1,\omega)\right) = \frac{\partial}{\partial \omega_1}\left(e^{-|\omega|^\alpha}\sum_{j=1}^{n}\left(c_j \omega_1^{a_{j,1}}\omega_2^{a_{j,2}}\dots\omega_d^{a_{j,d}}|\omega|^{b_j}\right)\right)$$

$$= e^{-|\omega|^\alpha}\sum_{j=1}^{n}\left(-\alpha c_j \omega_1^{a_{j,1}+1}\omega_2^{a_{j,2}}\dots\omega_d^{a_{j,d}}|\omega|^{b_j+(\alpha-2)}\right) + e^{-|\omega|^\alpha}\sum_{j=1}^{n}\left(c_j b_j \omega_1^{a_{j,1}+1}\omega_2^{a_{j,2}}\dots\omega_d^{a_{j,d}}|\omega|^{b_j-2}\right)$$

$$+ e^{-|\omega|^\alpha}\sum_{j=1}^{n}\left(c_j\left(a_{j,1}-1\right)\omega_1^{a_{j,1}-1}\omega_2^{a_{j,2}}\dots\omega_d^{a_{j,d}}|\omega|^{b_j}\right), \tag{164}$$

with the second and third terms only present when $b_j \neq 0$ and $a_{j,1} \geqslant 1$ respectively. The resulting expression is on the form presented in (163). Since we get similar expressions when differentiating with respect to any ω_j, we see that (163) must hold for any multi-index $s+1$, and by induction that (163) holds for all derivatives. The positivity requirement on $a_{j,1},\dots a_{j,d}$ holds since the term with $a_{j,1}-1$ in the exponent above vanishes in the case where $a_{j,1}=0$. Furthermore, recalling that $\alpha > 1$, we can calculate the sum of the exponents in the different terms in (164).

$$\min_{j\in\{1,\dots,n\}} a_{j,1}+1+(a_{j,2}+\dots a_{j,d})+b_j+(\alpha-2) = \kappa(s)+\alpha-1 \geqslant \kappa(s).$$

$$\min_{j\in\{1,\dots,n\}} a_{j,1}+1+(a_{j,2}+\dots a_{j,d})+b_j-2 = \kappa(s)-1.$$

$$\min_{j\in\{1,\dots,n\}} a_{j,1}-1+(a_{j,2}+\dots a_{j,d})+b_j = \kappa(s)-1.$$

Taking the minimum over the terms above, we get that $\kappa(s+1) \geqslant \kappa(s)-1$. By observing that $\kappa(0)=|k|$, it follows that

$$\kappa(|k|) \geqslant \kappa(0)-|k| \geqslant 0. \tag{165}$$

Thus, by the argumentation above, $\widehat{D^k K}(1,\omega) \in W^{|k|,1}\left(\mathbb{R}^d\right)$. Finally, by (162), there exists a constant $C \geqslant 0$ such that $\left|D^k K(1,u)\right| \leqslant C/|u|^{|k|}$ for any $u \in \mathbb{R}^d\setminus\{0\}$. □

Proof of Theorem 2.26 (Corrected from Theorem 3.1 in [2]). By the self-similarity result in Lemma 2.24,

$$\left\|D^k K(t,\cdot)\right\|_{L^1(\mathbb{R}^d)} = t^{-\frac{d+|k|}{\alpha}}\int_{\mathbb{R}^d}\left|D^k K\left(1, xt^{-\frac{1}{\alpha}}\right)\right|dx = t^{-\frac{|k|}{\alpha}}\int_{\mathbb{R}^d}\left|D^k K(1,u)\right|du. \tag{166}$$

Hence, it suffices to show that $D^k K(1,\cdot) \in L^1\left(\mathbb{R}^d\right)$ to complete our proof. By the exponential decay of $\widehat{D^k K}(1,\omega)$, there exists a constant $C_0 > 0$ such that for any $u \in \mathbb{R}^d$,

$$\left|D^k K(1,u)\right| = \left|\int_{\mathbb{R}^d} i^{|k|}\omega_1^{k_1}\dots\omega_d^{k_d}e^{-|\omega|^\alpha}e^{iu\cdot\omega}d\omega\right| \leqslant \int_{\mathbb{R}^d}\left|\omega_1^{k_1}\dots\omega_d^{k_d}e^{-|\omega|^\alpha}\right|d\omega \leqslant C_0.$$

Combining this with our pointwise estimate in Lemma 2.25, we get that

$$\|D^k K(1,\cdot)\|_{L^1(\mathbb{R}^d)} = \int_{B_1(0)}\left|D^k K(1,u)\right|du + \int_{\mathbb{R}^d\setminus B_1(0)}\left|D^k K(1,u)\right|du \leqslant V_d C_0 + \int_{\mathbb{R}^d\setminus B_1(0)}\frac{C}{|u|^{|k|}}du,$$

where V_d is the volume of the d-dimensional unit ball. The last integral is finite whenever $|k| \geqslant d+1$, and it follows that $D^k K(1,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for these multi-indices k.

When $|k| < d+1$, we use the Gagliardo-Nirenberg inequality (see p.125 in [21]). For any $1 \leqslant |k| \leqslant d$, there then exists a constant $\tilde{C} > 0$ such that

$$\|D^k K(1,\cdot)\|_{L^1(\mathbb{R}^d)} \leqslant \tilde{C}\left(\|D^{d+1}K(1,\cdot)\|_{L^1(\mathbb{R}^d)}\right)^{\frac{k}{d+1}}\left(\|K(1,\cdot)\|_{L^1(\mathbb{R}^d)}\right)^{\left(1-\frac{k}{d+1}\right)}. \tag{167}$$

Since $\|K(1,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$ by Proposition 2.22 and $\|D^{d+1}K(1,\cdot)\|_{L^1(\mathbb{R}^d)}$ is bounded by the argumentation above, it follows by (167) that $\|D^k K(1,\cdot)\|_{L^1(\mathbb{R}^d)}$ is bounded as well. Having shown that $D^k K(1,\cdot) \in L^1\left(\mathbb{R}^d\right)$ for any multi-index k, we deduce by (166) that there exists a constant $\lambda > 0$ such that

$$\|D^k K(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leqslant \lambda t^{-\frac{|k|}{\alpha}},$$

and the proof is complete. □

NTNU

Norwegian University of
Science and Technology