Master's thesis

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Norwegian University of Science and Technology
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Department of Mathematical Sciences

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Implementing a Heath, Jarrow & Morton model for estimating counterparty credit exposure in interest rate derivatives

Master's thesis in Industrial Mathematics
Supervisor: Jacob Kooter Laading
June 2023

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Preface

This thesis concludes my Master of Science in Industrial Mathematics as a part of the five-year Applied Physics and Mathematics programme at the Norwegian University of Science and Technology. This thesis is the culmination of the work conducted in my final year, where I have directed my interest in statistics to the exciting world of quantitative finance.

I would like to thank my supervisor Jacob Kooter Laading for his guidance, patience and extensive knowledge, and DNB for providing the data used in the thesis. Finally, I would like to thank my parents for unfailing support throughout my studies, and my classmates for five unforgettable years.

William Rom
Trondheim, June 2023

Abstract

This thesis explores the implementation of a Heath, Jarrow and Morton interest rate model for assessing counterparty credit risk associated with interest rate derivatives across different periods and market conditions. The model is built using principal component analysis to capture the structure of interest rates in historical data and uses these components to generate forward curves. The forward curves are used to value derivatives from their initialization to maturity. Starting from the last data point in the calibration data, concurrent with contract initialization, forward curves are simulated until the contract's maturity under the real-world measure to generate a distribution of future market scenarios. Risk-neutral probabilities are used to price contracts, which together with the distribution of future market scenarios are used to build exposure profiles for interest rate derivatives. The model is evaluated on interest rate swaps and an interest rate cap by comparing percentiles of simulated exposure with true exposure in different periods. The volatile period around the covid-19 pandemic is of special interest and is compared to periods in the 2010s. Satisfactory estimates are made in times of economic stability, while market-disrupting events and consistent monotonous changes in interest rates following the calibration period of the model result in actual exposure deviating from the simulated confidence intervals to different extents.

Sammendrag

Denne oppgaven utforsker implementasjonen av en Heath, Jarrow og Morton rentemodell for å beregne motpartsrisiko assosiert med rentederivater i forskjellige perioder og markedsforhold. Modellen bruker prinsipalkomponentanalyse for å fange opp den underliggende strukturen i historisk rentedata, og bruker disse komponentene til å generere forwardkurver. Fra et kjent markedsscenario simuleres forwardkurver frem til en kontrakts løpetid under et faktisk sannsylighetsmål for å generere distribusjoner av fremtidige markedsscenerioer. Et risikonøytralt mål blir brukt for å prise, som sammen med distribusjonene av markedsscenarioer brukes til å bygge risikoprofiler for rentederivater. Modellen evalueres på renteswapper og rentecaps ved å sammenligne persentiler i simulerte eksponering med faktisk eksponering i forskjellige perioder. Den volatile perioden rundt covid-19 pandemien er spesielt interessant og sammenlignes med perioder i 2010-årene. Gode estimater blir gjort i tider karakterisert av økonomisk stabilitet, mens intreffelsen av markedsforstyrrende hendelser og konsistente monotone renteendringer etter modellens kalibreringsperiode resulterer i faktisk eksponering som avviker fra de simulerte konfidensintervellene i varierende grad.

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1 Introduction

The early 2020s have been marked by significant events sending shock waves though the world economy, starting with the covid-19 pandemic bringing economic turmoil and a dim outlook for growth worldwide. This was followed by heightened geopolitical tension caused by an escalation of the Russo-Ukrainian war as well as strong global inflation, which reached a peak of around 8.7% in 2022 [5]. These events have caused an atmosphere of uncertainty and have been reflected in the financial markets.

The regulation of interest rates is a crucial tool employed by central banks to steer economic growth, and have been a central part of the response to these challenges. The pandemic brought with it global recession and market crashes across various sectors, and was countered with lowered interest rates to boost economic activity. The economic stimuli provided to counter the pandemic, combined with global supply-chain problems and geopolitical tension have led to a surge in inflationary pressure in economies worldwide. To counter this inflation, central banks globally have embarked on series of aggressive interest rate hikes. The turbulent beginning of the 2020s marks an intriguing economic phase and sets the stage for this thesis.

Interest rates not only affects the economic climate as a whole, but also plays a vital role in the valuation of financial instruments, including fixed-income securities and interest rate derivatives. The market for interest rate derivatives is huge, as exemplified by the Bank of International Settlements estimating the daily over-the-counter turnover to 5.2 trillion in April 2022 [11]. When assessing the risk associated with interest rate derivatives, interest rate changes is the most significant risk factor. Interest rate modelling plays a fundamental role in making these measurements, and has been an area of study since the 1970s. Early models focused on the evolution of individual rates and employed a single source of randomness, resulting in relatively simple frameworks. In 1987, Heath, Jarrow, and Morton (HJM) published "Bond pricing and the term structure of interest rates: a new methodology" and proposed a general framework for modelling the evolution of forward rate curves. With advancements in technology and the increased power of computers, mathematical methods such as Monte Carlo simulation and numerical integration have become increasingly viable, strengthening the case for the HJM approach.

Counterparty credit risk (CCR) is defined by the bank of international settlements as the risk arising from the probability of a counterparty in a transaction defaulting before the final settlement date, leaving the other party with a potential loss [10]. Modelling of CCR has become a considerable focus in financial markets, especially after the global financial crisis of 2008. It is mandatory to estimate and is used to determine the capital requirements of banks. Managing CCR is particularly important when trading in interest rate derivatives, where long maturity profiles and large national amounts are common. The CCR stemming from changes in interest rates is the focus of this thesis.

This thesis starts in chapter 2 by introducing some basic financial concepts and mathematics used in quantitative finance. Chapter 3 describes central concepts related to interest rates and some financial contracts related to these. Chapter 4 defines counterparty credit risk, how to measure it and outlines how it is used in banking. Chapter 5 describes the dataset of interest rates and some properties of the data, setting the stage for chapter 6 which delves into the theoretical foundation for interest rate models before diving deeper into the focus of the thesis, the HJM model. Chapter 7 deals with the implementation of the HJM model and how principal components are used to capture the volatility of interest rates. Chapter 8 cover the results, the simulated exposure profiles compared with the actual exposure of interest rate derivatives and discussion on model feasibility under different market conditions. Chapter 9 concludes the thesis with a summary and some notes on further work of interest.

This thesis is a continuation of the work done in my project thesis "An implementation of the Heath, Jarrow and Morton model applied to valuation and risk management for interest rate derivatives". The sections regarding financial concepts, interest rate and interest rate derivatives, interest rate models and model implementation, is based on the project thesis, with some extensions and elaborations.

2 Theory and financial concepts

2.1 Financial markets

Financial markets are places and systems where financial instruments are traded. These systems are the key to smooth deal-making and the allocation of capital where it is needed. There is a wide variety of financial markets, each distinguished by the type of securities that are traded within them. The stock market is an example of such a financial market and is the place where shares of publicly traded companies are bought and sold. There exist markets for bonds which will be discussed in section 3.3, commodities, currencies and many others catering to specific asset classes. Each market operates with its own set of rules and regulations providing sustainable and safe frames for conducting trades. Some financial instruments are traded without being listed on a centralized exchange, but rather over-the-counter, directly between parties or through broker-dealer networks. Over-the-counter trading allows for tailor-made contracts that can be specialized to fit the need of any business.

2.2 Financial derivatives

Most commonly associated with assets on financial markets are stocks and commodities. Stocks represent an ownership share of a company, while commodities are physical assets. Both relate to direct ownership of something physical and their basic concepts are easy to understand.

Financial derivatives are more abstract contracts and do not deal with direct ownership of a commodity or company. Derivatives are contracts based on an underlying asset such as a stock, a bond or an interest rate. For example, a European call option is a popular derivative that grants the holder the right, but not the obligation to buy an underlying asset at a specified price at a predetermined date in the future. Derivatives offer investors the opportunity to speculate in price movements, manage risk or gain exposure to an asset without direct ownership. These contracts vary in sophistication and can be suited to the complex needs of investors. Derivatives can serve as powerful tools for market participants but require careful consideration and risk management to ensure effective and responsible use.

2.3 The efficient market hypothesis

The efficient market hypothesis suggests that markets are efficient, and that prices reflect all available information. When new information is released, asset values will immediately shift to compensate for it. The hypothesis does not take into account cognitive and emotional biases or informational asymmetry [8].

2.4 Arbitrage

An important assumption for derivative pricing, and a fundamental financial concept is the one of no-arbitrage. No arbitrage implies that it is impossible to earn a riskless profit by exploiting market inefficiencies. That is, there does not exist, for a prolonged time, any opportunities for investors to make a profit without taking on any risk. The principle is based on the assumption that financial markets are efficient, and that any such opportunity to make a risk-free profit will be eliminated quickly. The way these opportunities are eliminated is by the actions of arbitrageurs, like quantitative hedge funds, whose job is to find these market mispricings.

The risk-free interest rate is a theoretical rate of return on an investment that carries no risk. The risk-free rate is used as a benchmark for other rates of varying risk and to find the time-value of money. Under the assumption of no-arbitrage, the return on an investment approaches the risk-free rate as the risk goes to zero. The rate is often in practice derived from short-maturity government bonds, as sovereign defaults are rare.

2.5 Hedging

Hedging is the reduction of risk associated with the changes in an asset's price by taking a position in another financial asset whose value is negatively correlated with the original asset. An example of this would be buying put options for an underlying asset which one have invested in. If the price of the asset sinks, the value of the put option would rise, reducing the overall loss. This strategy would also limit the potential profits in the event of an increasing asset price.

A common hedge is called the delta-hedge which consists of one long option with value V and a short position in the underlying asset with value ΔS, where S is the value of the underlying and Δ is the amount of asset held. Thus, the value of the portfolio is $\Pi = V - \Delta S$ and by letting $\Delta = \frac{\partial V}{\partial S}$, the negative correlation would be perfectly exploited resulting in a portfolio insensitive to change in the underlying asset price. This insensitivity is only instantaneous and the hedge would have to be continuously re-balanced to keep it risk neutral.

This way of staying risk neutral depends on assuming volatility and interest rates remaining unchanged and that fractions of the underlying can be bought. The cost of making transactions must also be taken into account which removes the feasibility of continuous re-hedging.

The delta-hedge eliminates the largest stochastic component of the portfolio. A second sensitivity is the gamma $\Gamma = \frac{\partial^2 V}{\partial S^2}$. It is defined as the second derivative of the portfolio value with respect to the underlying, and takes into account higher order correlation between the underlying and the option.

2.6 Random walk asset price

Asset prices are usually modelled as stochastic processes due to the random nature of their movements. For an asset S, the change in value over a time period of dt is usually expressed in terms of a predictable, deterministic part and a random part as

$$dS = \mu dt + \sigma dW \tag{2.1}$$

where μ is the drift of the price, σ the volatility or standard deviation of returns and dW is a Wiener process. As the evolution of asset prices is considered random, it is important to note that predicting it from past data is impossible. However, the past data is not useless as it can be used to estimate general behaviour and distributions. This approach is used to model stock prices and other financial instruments because it is simple, yet flexible and captures many of the key features of financial markets.

2.7 Martingales

A martingale is a bounded stochastic process in which the expected value of the next value in the sequence is equal to the current value. In other words, given the current information available, there is no way to predict the future value of the process, expressed mathematically in discrete time as

$$\mathbf{E}(X_{n+1}|X_1, \ldots, X_n) = X_n, \qquad \mathbf{E}(|X_n|) < \infty \tag{2.2}$$

2.8 Risk-neutral pricing

Risk-neutral pricing is a concept used to value financial assets and derivatives. By assuming that investors are completely indifferent to the risk of investments and that the price of an asset should be based only on the expected return and the risk-free rate. Risk-neutral pricing can be explained using the simple binomial model to price an option. This example is inspired by R. Sundarams expository note in the Journal of Derivatives [15] and gives an informal introduction to the concept of risk-neutral pricing.

To introduce the concept of risk-neutral pricing, the binomial model is used as an example. In the binomial model, two scenarios are possible. The stock price increasing or decreasing with exactly two different rates

$$S_1 = \begin{cases} uS & \text{with probability } q \\ dS & \text{with probability } (1-q). \end{cases} \tag{2.3}$$

The option will thus have either a value of V_u or V_d depending on the direction of the stock price. A replicating portfolio, a portfolio requiring the same investment and that has the same payoff at maturity, can be constructed to price this option. This is a consequence of the assumption of no arbitrage, because such a portfolio must have the same price as the option. The replicating portfolio consists of two primitive securities. The stock and a risk-free bond with the risk-free rate of return. The bond has an initial value of 1 and a value of r after one period of time. A portfolio consisting of Δ stock and Θ units of the bond would then have the payoff of

$$\begin{aligned} \Delta uS + \Theta r &= V_u \\ \Delta dS + \Theta r &= V_d \end{aligned} \tag{2.4}$$

in the respective scenarios. To replicate an option with a binary payoff V_u and V_d , the number of stock and bond units can be set to

$$\Delta = \frac{V_u - V_d}{uS - dS} \qquad\qquad \Theta = \frac{uV_d - dV_u}{r(u-d)} \tag{2.5}$$

which will give the same payoff as the original claim. Thus, the arbitrage-free price of the option can be found by setting it equal to the initial price of the replicating portfolio.

$$V_0 = \Delta S + \Theta \cdot 1 = \frac{1}{r}\left(\frac{r-d}{u-d}V_u + \frac{u-r}{u-d}V_d\right) \tag{2.6}$$

Pricing any derivative by this method can lead to substantial work, as different replicating portfolios must be created for all the different derivatives. Risk-neutral pricing is another way to derive the price, and is based on the adaptation of a new probability measure, the risk-neutral probability measure. Using this measure, the expected discounted payoff of the derivative is taken, where the discounting is done using the risk-free rate. This approach has the benefit of not needing to construct a new portfolio for every different derivative, but rather using the same risk-neutral probability only depending on the primitive assets used. The integral step of this approach is to identify the risk-neutral probability. This can often be daunting, but is for the binomial model a simple task.

As all returns are expected to be equal in the risk-neutral world, the expected returns of both the stock and the bond must be equal, and the probabilities must satisfy

$$qu + (1-q)d = r \tag{2.7}$$

which gives a unique q of

$$q = \frac{r-d}{u-d}. \tag{2.8}$$

With this risk-neutral probability, the discounted payoffs of the option would be V_u/q and V_d/q respectively. Thus the expected discounted value of the option would be

$$q\frac{V_u}{r} + (1-q)\frac{V_d}{r}, \tag{2.9}$$

which after inserting the risk-free probability given in equation (2.8) gives the initial value specified in equation (2.6). This is the case for all replicable claims [15].

2.9 The Monte Carlo method

Monte Carlo methods encompass a broad class of numerical algorithms using repeated random sampling to solve mathematical problems. They can be used for pricing derivatives and risk management by simulating relevant market factors, computing payoffs and taking expectations. A simple case is the appliance to a European option based on an asset that follows the random walk asset price described in section 2.6, where a random term accompanies a deterministic model to incorporate volatility into the asset price evolution. Given the historical volatility of the asset, a number of asset value paths can be simulated to generate possible future option outcomes and an option price can be obtained by looking at the average payoff.

As more samples are generated, the standard deviation of the solution decreases. The convergence rate of a Monte Carlo method is of order $O(n^{-1/2})$ [1], where n denotes the number of simulations and should be chosen according to the precision needed for the problem at hand.

3 Interest rate and financial instruments

Interest rates are key components of the financial system and have a profound impact on economic activity, investment decisions, and asset valuation. In essence, an interest rate represents the cost of borrowing or the return on lending money over a specified period and serves as a mechanism for balancing the demand for and supply of funds in an economy.

Within the field of quantitative finance, interest rates are of particular importance due to their role in pricing many financial instruments. By understanding and modelling interest rate dynamics, informed investment and risk management decisions can be made.

3.1 Central concepts

Interest rates appear in many forms, but can be generalized by the ratio that the lender charges the borrower for using their assets. This rate depends first and foremost on the length of the loan and the risk of default, the chance of the loan principal not being repaid. An initial deposit of U_0 made with the interest rate r_t would then grow exponentially as

$$U(t) = U_0 e^{\int_0^t r_u du}. \tag{3.1}$$

The time value of money is a fundamental principle in finance that asserts the greater value of a cashflow received today compared to the same cashflow received at a future date. This principle recognizes that the passage of time carries financial implications, such as the potential for earning returns and inflation. As a result, the concept of present values of future cashflows arises. Analogously to a deposit growing at an exponential rate, a discount factor between time t and T is defined as

$$D(t,T) = e^{-\int_t^T r_u du} \tag{3.2}$$

which can be multiplied with a future cashflow that will be received at time T to find its value at time $t \leq T$. Solving equation (3.2) for r_u gives the continuously compounded spot rate at time t until maturity T and is denoted

$$R(t,T) = -\frac{ln(D(t,T)}{T-t}. \tag{3.3}$$

For a given t and a set of maturities T, a curve of $R(t,T)$ can be constructed. This curve is called the yield curve, or the term structure of interest rates, and is often used as a broad indicator for the market's expectation of future interest rates and thus a general benchmark for the economy as a whole. A steep yield curve typically indicates expectations of future economic growth, whereas a flatter or inverted yield curve may suggest an economic slowdown or recession.

The instantaneous spot rate, or short rate is found by letting $T \to t$ in equation (3.3) and is denoted

$$r(t) = \lim_{T \to t} R(t,T). \tag{3.4}$$

This is the rate of return on a deposit of infinitesimal duration. Totally infinitesimal rates are not available in the market, and the short rate is thus commonly represented by a short maturity rate like a 3-month rate, as done in this thesis with the 3-month NIBOR.

3.2 The forward rate

A forward rate is an interest rate applicable to a financial transaction that will take place in the future, and is derived from current market information. At time t, given two riskless strategies with maturity T_1 and $T_2 \geq T_1$, the forward rate between the two maturities, $F(t, T_1, T_2)$, can be extracted. Assuming no arbitrage, two riskless investment strategies with the same initial deposit should yield the same return. Thus, investing first with interest rate $R(t, T_1)$ and then reinvesting the returns with rate $R(T_1, T_2)$ should be equivalent to simply investing in the strategy yielding the rate $R(t, T_2)$. Under continuous compounding, this equates to

$$e^{R(t,T_2)\cdot(T_2-t)} = e^{R(t,T_1)\cdot(T_A-t)}e^{R(T_1,T_2)\cdot(T_2-t_1)}. \tag{3.5}$$

Solving this equation for $R(T_1, T_2)$ gives the forward rate under continuous compounding contracted at time t , which is denoted

$$F(t, T_1, T_2) = \frac{r_2 T_2 - r_1 T_1}{T_2 - T_1}. \tag{3.6}$$

This forward rate can be expressed in terms of discount factors as

$$F(t, T_1, T_2) = \frac{ln(D(t, T_1) - ln(D(t, T_2)}{T_1 - T_2}. \tag{3.7}$$

3.2.1 Instantaneous forward rate

Letting the distance between the two future maturities tend to zero, the instantaneous forward rate $f(t, T)$ arises. This is the rate that applies for an infinitesimal period in the future and can be written as

$$f(t, T) = \lim_{\delta \to 0} F(t, T, T + \delta) = -\frac{\partial}{\partial T} ln(D(t, T) \tag{3.8}$$

Here it can be observed that the instantaneous forward rate at time t for maturity $T = t$, $f(t, t)$, is the instantaneous spot rate at time t, $r(t)$. The interest rate model used in this thesis produces instantaneous forward rates. To find discount factors from these, equation (3.8) can be solved for $D(t, T)$, which gives

$$D(t, T) = e^{-\int_t^T f(t,s)ds}. \tag{3.9}$$

Combining this with equation (3.2), the relationship between the spot rate and the instantaneous forward rates can be written as

$$R(t, T) \cdot (T - t) = \int_t^T f(t, u)du \tag{3.10}$$

It should also be noted that given the instantaneous forward rates, forward rates can be found by integrating between the desired periods

$$F(t, T, T + \delta) = \frac{1}{\delta} \int_T^{T+\delta} f(t, s)ds. \tag{3.11}$$

3.3 Bonds and interest rate derivatives

Unlike equities, interest rates can not be bought directly but can be traded indirectly through interest rate derivatives such as swaps and caps or through fixed-income securities such as bonds.

3.3.1 Bonds

A bond is not a derivative, but a contract that yields a specified amount in the future, the principal. The simplest bond, the zero coupon bond (ZCB), simply pays a known amount at its expiry date. Some bonds may also pay dividends during their lifetime at fixed discrete times, known as bonds with coupons.

The issuers of bonds are typically governments and corporations that want to raise money. They can thus be thought of as a loan from the buyer to the issuer. The price of a bond must then depends on what the believed future worth of money is and the probability that the payments will be made as promised. The zero-coupon bonds used in the rest of this paper have a principal value of 1. As a ZCB yields a single cashflow in the future, its value is comparable to the corresponding discount factor and is denoted

$$Z(t,T) = e^{-R(t,T)\cdot(T-t)} \tag{3.12}$$

3.3.2 Swaps

An interest rate swap is a popular derivative based on interest rates. Most swaps are based around a notional principal amount that the parties pay their interest rate on, to each other. This principal is however usually not traded. Swaps are not traded on exchanges like bonds or equities, but are over-the-counter and are usually contracts between larger companies and financial institutions. They are used to hedge against interest rate risk, to lock in debt cost or asset return at a fixed interest rate and can convert floating rate debt/assets into fixed rate debt/assets. As with other financial instruments they can also be used to speculate.

The most commonly traded swap is the vanilla interest rate swap. It is a contract where one party agrees to pay at a fixed rate in exchange for a variable rate, usually an interbank offered rate or another reference rate like a secured overnight rate. The cashflows occur on agreed-upon times, usually semi-annually. When such a swap contract is initialized, the value for both parties is typically set to zero. This is achieved by choosing a fixed rate such that the expected present value of all future payments of the floating leg is equal to the fixed ones. The rate used in the floating leg of the swap is decided a given period before the payment and is for all examples in this thesis chosen to be 6 months prior.

For pricing a swap Wilmott [16] uses the route through zero-coupon bonds. The fixed leg of the swap can be seen as a sum of these, and with a fixed rate r_f, N total payments and time between payments τ, the value of this leg is

$$r_f \tau \sum_{i=1}^{N} Z(t, T_i) \tag{3.13}$$

where t is the current date and payments happen on the dates T_i.

For the floating leg at time T_i, a payment of $r_{i-1}\tau$ is made, where r_{i-1} is the rate decided in the previous period. Since the floating rate is the same as the interest received on a fixed-term deposit, this part of the floating leg is equivalent to a deposit of the notional at T_{i-1} and a withdrawal of the same notional at time T_i. Summing the terms gives a floating leg value of

$$1 - Z(t, T_N).$$

Knowing that the value of the swap at initialization should be zero, the legs can be set equal and then solved for r_f which gives the swap rate

$$r_f = \frac{1 - Z(t, t_N)}{\tau \sum_{i=1}^{N} Z(t, T_i)} \tag{3.14}$$

As the swap value is decided by interest rates settled in the future, forward rates are naturally convenient when pricing these instruments. Given the instantaneous forward rate curve over the swap lifetime, the applicable forward rates can be extracted and used to calculate the market's expectation of future cashflows and thus the value of the swap at any date.

3.3.3 Caps and floors

A cap is an interest rate derivative that guarantees its holder that the floating interest rate on a loan will not exceed a specified limit. This is achieved by a payment being received every period that the interest rate exceeds the agreed-upon upper limit, effectively capping the rate. As opposed to plain vanilla interest rate swaps, caps are not usually initialized with zero value. To achieve a fair contract, a premium is instead paid at initialization based on the capping rate and current market conditions. A cap with a notional value of N, a day count fraction τ and a fixed rate r_f can be divided into caplets whose payoff given the floating rate L is

$$V_c(L, r_f) = N \cdot \tau \cdot \max(L - r_f, 0). \tag{3.15}$$

A floor works the same way, but here the interest rate is bounded below, and the floorlet payoff function is similarly

$$V_f(L, r_f) = N \cdot \tau \cdot \max(r_f - L, 0). \tag{3.16}$$

Interest rate caps and floors are very efficient tools for hedging against changes in interest rates. It offers an advantage over swaps, as the initial purchase price is the only cost for the buyer and can ensure future expenses. Companies with floating-rate loans or issuers of floating-rate bonds may use caps to protect themselves against interest rate hikes that would cause their costs to rise. Similarly, lenders whose income relies on a floating rate may use floors to protect their investments.

4 Counterparty credit risk

Counterparty credit risk (CCR) is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flow [10]. Whenever a defaulting partner is left unable to meet their obligations, it can leave the other involved parties with potential losses. If the contract at the time of counterparty default had a positive economic value, an economic loss would have occurred for the other part. If a bank enters into a swap arrangement with a counterparty, the value of this contract would vary with the time to maturity and the interest rate. As the market value of the contract can be both positive and negative to both the bank and the counterparty, the risk is bilateral. This is different from credit risk arising from a loan, where only the lending bank is exposed to loss in the case of the borrower defaulting.

CCR is particularly interesting in the context of OTC interest rate derivatives. They are frequently traded in large notional amounts and have long maturity profiles, this results in large potential loss magnitudes and the need to account for factors such as the long-term financial health of the counterparty and possible changes in market conditions when managing risk. After the financial crisis of 2008, legal obligations to facilitate these derivative trades through central clearing houses have increased. Some non-standardized OTC derivatives are still traded without the involvement of a central clearinghouse leaving the parties directly exposed to risk. In these cases, additional precautions should be taken.

The modelling of CCR for interest rate derivatives such as the ones described in section 3.3, as a result of interest rate changes, can be done through risk-neutral valuation in the HJM framework. The exposure from a portfolio with a value V_t at time t with components maturing no later than T, can be simulated over the entire duration of the portfolio through a combination of real-world and risk-neutral interest rate simulations. At initialization, to calculate the exposure at $0 < t \leq T$, real-world evolutions of the forward curve are simulated until t followed by a number of risk-neutral simulations covering the interval $[t, T]$. The real-world simulation until time t is conducted to simulate a distribution of potential future market scenarios, while the risk-neutral simulations are used to price the instrument given the realized market scenario. The concept is shown in figure 1 where a scaled-down simulation example is illustrated by extracting the 3-month forward rate from the simulated forward curves. The valuation of the portfolio is done for a select set of dates to reduce runtime, and results in a distribution of values for these dates. This distribution shows the possible values of the portfolio according to the interest rates simulated by the HJM model, and can be used in combination with various risk measures to quantify risk.

For a bank, understanding the potential future exposure of the contracts it issues is crucial for decision-making and risk management. When a bank deals in contracts that have the potential for both negative and positive cash flows, it is important to assess both the scenario where the buyer may owe significant amounts to the bank and vice versa. In both cases, the size of the credit line must be evaluated. Consideration should also be taken when entering into contracts where the counterparty has a high probability of being unable to meet its obligations, leading to counterparty default and economic loss. Other than determining the credit line size needed, exposure estimates are also used for calculating the regulatory capital requirements related to each contract.

4.1 Risk measures

Once a distribution of future values of a contract has been simulated, a number of statistics can be used to quantify the risk embedded in it. Different risk measures have different domains of application, where some are used for calculating capital requirements, others for internal decision-making and to estimate lines of credit.

4.1.1 Potential future exposure

The potential future exposure (PFE) is a statistic applied to the distribution of portfolio values at time t and is defined as

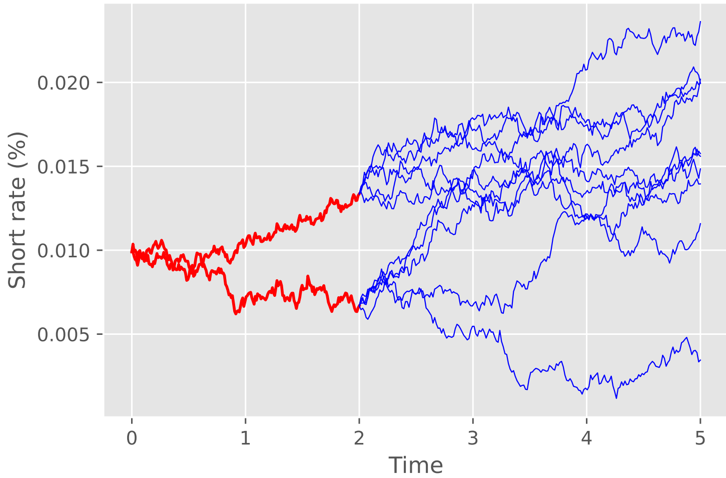


Figure 1: Schematic plot of CCR exposure simulation concept. Simulation of real-world(red) short rate paths until $t = 2$, the time where exposure is to be measured, followed by risk-neutral(blue) short rate paths from each starting point to maturity for the sake of valuing each cashflow with transaction date $t > 2$.

$$PFE_{\alpha,t} = q_{\alpha,t} = inf\{x : \mathbb{P}(V_t \leq x) \geq \alpha\} \tag{4.1}$$

where V_t denotes the distribution of values, and α the level of confidence. This statistic gives rise to a function of time that characterises the quantiles of the exposure. For the purpose of deciding the size of credit lines to their counterparties, banks are often also interested in a single measure for the exposure related to each contract. In this case, the maximum PFE can be used with a certain threshold selected according to the risk the bank is willing to take. The maximum 95% PFE is a common choice and is denoted

$$\max PFE_{95} = \max_t(PFE_{95\%,t}). \tag{4.2}$$

4.1.2 Expected positive exposure

The expected positive exposure (EPE) is another measure applied to the portfolio value distribution, and is the expected value over the positive values V_t^+ defined by Cesari [2] as

$$EPE_t = \mathbb{E}[V_t^+]. \tag{4.3}$$

When used to calculate the minimum capital requirement, the first year of exposure is taken into account [10].

4.2 Regulatory requirements and the internal approach

For calculating the capital requirements associated with CCR, all banks can use a standardised approach outlined by the Bank of International Settlements. The approach is defined in the Basel framework [12] and defines the exposure at default as

$$EAD = \alpha(RC + PFE), \tag{4.4}$$

where α is a constant, RC is the replacement cost of the derivative and PFE is the standardised potential future exposure.

A bank meeting the requirements stated in the Basel framework can seek approval for using an internal approach, rather than the standardised one for calculating their capital requirements. Internal models can be adapted to individual banks and ensure that risk is measured more accurately. Banks with experience in certain sectors often have large amounts of data and can use their information to account for correlations between assets to a higher degree than what is possible with the standardised approach. Other internal models, separate from the one related to the size of exposure and the focus of this thesis, can be used to estimate probabilities of default which is essential in calculating capital requirements. The rationale behind this is that banks with good risk management would need to hold less capital. Such an internal approach could be based on more complex modelling of portfolio values and is in this thesis explored using a HJM model.

For the derivation of the regulatory measures used in the internal model, a few metrics are needed and described in the Basel framework for credit risk exposure. The following definitions are taken from chapter 53 [9]. Expected exposure (EE) is the average simulated exposure at any future date and is denoted EE_t. Effective EE is the maximum exposure reached up until a specified time and is computed recursively as

$$\text{Effective } EE_{t_k} = max(EE_{t_k}, EE_{t_{k-1}}). \tag{4.5}$$

Effective EPE is defined to be

$$\sum_{k=1}^{1year} \text{Effective } EE_{t_k} \cdot \Delta t_k, \tag{4.6}$$

where Δt_k is used to weight the exposure and thus allows for dates not equally spaced, and where the summation is done for all measurements up to one year. Using the metrics defined above, EAD can be calculated using effective EPE scaled with a constant α,

$$EAD = \alpha \cdot \text{Effective } EPE. \tag{4.7}$$

The scaling coefficient α is generally set to 1.4. Based on the bank's CCR exposure, supervisors can adjust the α. This is typically done whenever the CCR is increased by effects not captured by the Effective EPE such as credit exposure that is positively correlated with the default probability of the counterparty or market value correlation between different counterparties [9].

4.3 Managing CCR

CCR is inevitable, but measures can be taken to keep it tolerable. Reducing this risk can be done by performing thorough analysis of potential counterparties to assess their financial health and creditworthiness.

Entities concerned with credit exposure to a specific counterparty are often engaged in multiple contracts. For any particular development of underlying risk factors, the values of the contracts with the particular counterparty do not need to be correlated. This gives rise to the concept of netting, where in the case of a defaulting counterparty, the loss corresponding to the positively valued contracts is covered by the negatively valued contracts.

Reducing CCR can be done by demanding collateral, the use of credit derivatives such as credit default swaps and diversification of counterparties. Another way to mitigate risk is by involving a central counterparty, a clearing house, in the contract. The clearinghouse can act as the counterparty for both the original parties in the contract, removing all exposure between them. The clearinghouse collects collateral and should have a low probability of default.

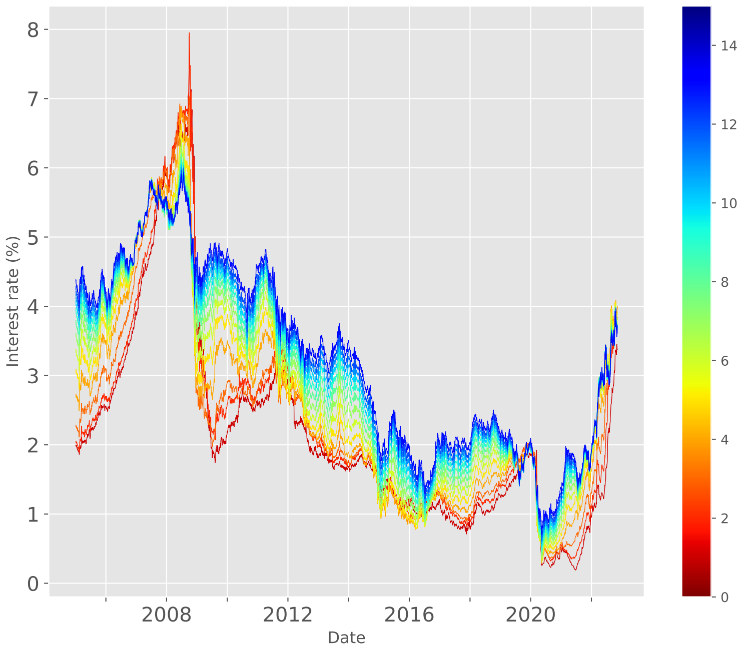


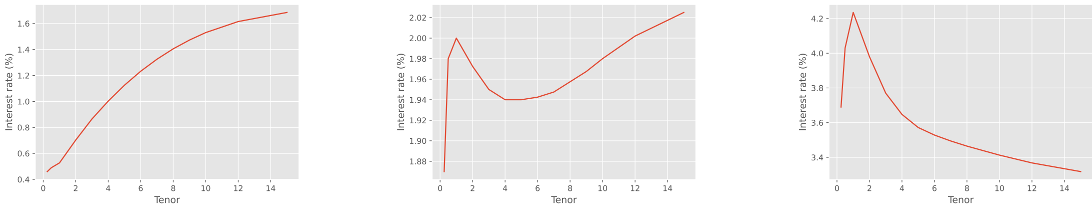
Figure 2: Historical spot rates from the Norwegian swap market. Tenors ranging from 3 months to 15 years, indicated by colours ranging from dark blue(15 years) to dark red(3 months) as depicted by the colourbar. Major economic events are easily observable from significant shifts where the 2007-2008 financial crisis is the most evident.

5 Data

The dataset used in this thesis is delivered by DNB and consists of Norwegian Interbank offered rates(NIBOR) and Norwegian swap rates totalling 14 different maturities for a period ranging from 01.04.2005 until 24.04.2023. There are 4612 NIBOR and 4776 swap data points in the period. For use with the HJM model, the historical spot rates are converted into forward rates. This is done by using the connection between forward rates and spot rates described in section 3.2. The spot rates are shown in figure 2.

The interest rates reflect the economic climate and the expectations of future interest rates and are thus influenced by events that affect the outlook on growth and economic activity. Of special interest in this thesis is the period of economic insecurity following the COVID-19 pandemic. The period starts in early 2020 and is characterized by rates being brought down to extreme lows with the intention of increasing economic activity, followed by substantial hikes to counterweight the surge in inflation that followed.

From every date in the dataset, a yield curve can be constructed as described in section 3.1. This curve holds information about the market's expectation of future rates, and thus the economy as a whole. Figure 3 shows three yield curves extracted from different dates in the last four years. Early 2021 experienced record-low rates with Norges Bank keeping the policy rate at 0 to stimulate economic activity. A yield curve from this period can be seen in figure 3a. It is a normal yield curve, characterized by a monotonic increase corresponding to short-term instruments having lower yields than longer-term equivalents and is associated with a positive view on future growth. Figure 3b shows a pre-pandemic yield curve from late 2019 and is characterized by an initial peak on the short end followed by a decrease in the medium-term rates and a steady increase in the medium to the long end of the curve. This curve indicates a strong belief in short-term growth, uncertainty in the intermediate future followed by a more positive outlook for the longer term. Figure 3c



(a) 08.02.21.

(b) 16.12.19.

(c) 24.04.23.

Figure 3: Three yield curves extracted from the dataset at different dates. Each holding information about their respective periods.

is extracted from 24.04.2023, the last data point of the dataset. This date follows a period of aggressive rate hikes made by Norges Bank to tame inflation and marks a time of great economic uncertainty. The curve is near-inverted, with the exception of a notable peak around the 1-year rate. The curve relates a positive outlook on the intermediate future but signifies an economic downturn in the long term. It should be noted that the longer end of the yield curve carries a lot of uncertainty, and should not be considered an indicator of long-term growth with complete certainty.

5.1 Preprocessing

Before deriving the volatility structure, a differencing of the time series is conducted. This is done to eliminate seasonal effects and correlation that originates from constant terms. It should also be noted that periods of up to three consecutive days are missing from the original dataset for the 3,6 and 12-month rates. This is due to the shorter rates being NIBOR rates, only being available on Norwegian trading days. These missing values are added through linear interpolation, and is assumed not to affect any further analysis owing to the somewhat stable nature of interest rates.

5.2 Descriptive statistics

The behaviour of interest rates is driven by a complex interplay of factors, including economic conditions, monetary policy and market expectations. A prominent characteristic of interest rates is their tendency to fluctuate over time. During periods of economic expansion and strong growth, central banks often raise interest rates to slow down the economy to control inflation. On the other hand, during economic downturns or recessions, central banks may lower interest rates to stimulate borrowing and investment, thereby encouraging economic activity. This behaviour can be observed in the dataset, where sharp drops in interest rates were made to counteract the pandemic in 2020 followed by a series of interest rate hikes starting in 2022 to fight the inflation that followed. Furthermore, it is common to observe clustering of volatility in interest rate data, where periods of high volatility are followed by subsequent periods of high volatility. This phenomenon, documented in studies like den Haan and Levin (1998) [4], is an important observation and a key assumption in interest rate modelling.

The rates of different maturities are highly correlated. This is shown in figure 4, where Pearson correlation coefficients for the 14 interest rate time series are displayed. The minimum correlation is found between the rates furthest apart, the 3-month and 15-year tenors, measuring 0.835. As shown in table 1, the spot rate volatility is inversely correlated with increasing tenor length while the mean for each rate taken over the whole period is increasing with tenor length.

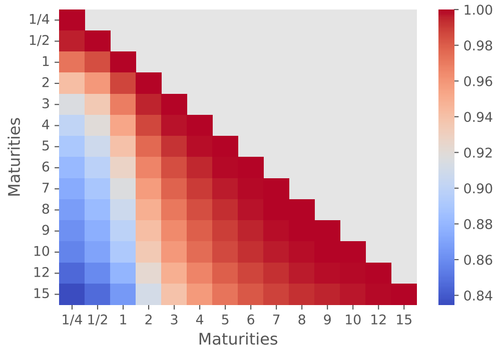


Figure 4: Matrix depicting the correlations between interest rate time series using Pearson correlation coefficients.

Tenor (Year)	volatility (%)	mean
1/4	0.01498	0.02193
1/2	0.01491	0.02343
1	0.01479	0.02388
2	0.01479	0.02529
3	0.01446	0.02634
4	0.01421	0.02735
5	0.01386	0.02833
6	0.01370	0.02921
7	0.01357	0.03000
8	0.01346	0.03068
9	0.01338	0.03177
10	0.01332	0.03178
12	0.01323	0.03251
15	0.01317	0.03310

Table 1: The volatility and mean of the spot rate time series from 04.01.2005 to 24.04.2023.

6 Interest rate modelling

Modelling interest rates is essential for accurately valuing and managing the risk associated with interest rate derivatives and fixed-income securities. Different models exist, ranging from those focusing on modelling the short rate using a single stochastic factor to more complex ones incorporating multiple factors and sources of randomness.

Since interest rates are inherently random, interest rate models aim to capture their general behaviour rather than providing precise predictions. When developing an interest rate model, certain properties are typically sought after. These can include the positivity of interest rates or mean-reversion with a specific speed. Being able to calibrate the model to observed market data is another important property. By fitting the model to current yield curve data, simulated and theoretical rates can be aligned, as demonstrated in practice in section 7.2. Given the stochastic nature of interest rates, it is natural to model them as random variables. By treating interest rates as random variables, models can incorporate uncertainty and account for the randomness in interest rate movements.

6.1 One-factor interest rate models

A simple one-factor interest rate model is typically on the form

$$dr = u(r,t)dt + \omega(r,t)dW \tag{6.1}$$

where dr denotes the change in what is typically the short rate over the time period dt. The Wiener process dW introduces stochasticity to the differential equation and the functions $\omega(r,t)$ and $u(r,t)$ are the volatility and drift respectively.

These functions have been thoroughly studied and are found to produce tractable solutions when on the form

$$\omega(r,t) = \sqrt{\alpha(t)r - \beta(t)} \tag{6.2}$$

$$u(r,t) = -\gamma(t)r + \eta(t) + \lambda(r,t)\sqrt{\alpha(t)r - \beta(t)} \tag{6.3}$$

By choosing suitable parameters α, β, γ and η the data can be fitted and give the model economically plausible properties. An example of a model on this form is the Vasicek model which simplifies the calculations by setting $\alpha = 0$, requiring $\beta > 0$, and using time-independent parameters. In this case, the stochastic differential equation is

$$dr = (\eta - \gamma r)dt + \sqrt{\beta}dW. \tag{6.4}$$

This model has properties like mean reversion towards $\frac{\eta}{\gamma}$ with speed γ. It can give negative interest rates, which have been viewed as a negative property of the Vasicek model. However, negative interest rates have been observed in various markets, particularly over the past 15 years, challenging the significance of positivity.

One-factor models are easy to understand and to estimate parameters for, where a simple maximum likelihood estimation is sufficient. A drawback of the one-factor models is the amount of information they can embed and the types of structures they are able to model given their simplicity. Being based on one source of randomness, they can be good at modelling the overall level of the yield curve but have trouble modelling nonlinear effects such as twisting and bending of the yield curve which comes from different behaviour of different maturity rates.

6.2 The Heath, Jarrow & Morton framework

Multi-factor models utilize multiple sources of randomness to capture information and enable the modelling of the more complex structures found within the term structure of interest rates. For derivatives that only depend on the level of the yield curve, a single source of randomness might be sufficient. However, for more complex contracts depending on the curvature of the yield curve or the difference of yields at multiple maturities, multi-factor models are better suited.

In a general multi-factor model using N different factors, $x_1, \dots x_N$, their development is modelled as

$$dx_i = \mu_i(\mathbf{x}, \mathbf{t}) + \sigma_{\mathbf{i}}(\mathbf{x}, \mathbf{t})\mathbf{dW_i} \tag{6.5}$$

where W_i denotes the different Wiener processes, $\mathbf{x}$ the vector of factors and μ_i, σ_i the drift and volatility terms respectively. A 2-factor model can for example utilizes the short rate and a longer term rate as its second factor. Other factors such as the curvature at a specific point, a volatility or a spread between two rates can also be used.

The focus of this project is on a multi-factor model based on the Heath, Jarrow & Morton(HJM) framework. This framework originates from their work on bond pricing [3] published in 1987. They chose to model the evolution of the whole forward rate curve, which at time t with maturity T is denoted $f(t,T)$. This differs from other models which instead derive the forward rates from the spot rates. The evolution of the forward rate curve in a d-factor HJM model is derived from the discount factors as stated in chapter 3.2. The framework is based on no-arbitrage and the stochastic nature of interest rates. An informal derivation of the framework is given and is based on the one provided by Glasserman [6]. See the original paper by Heath, Jarrow and Morton for a thorough account.

The starting point for the derivation of the HJM centrepiece equation is the relation between the instantaneous forward rate and discount factors as described in section 2.6,

$$f(t,T) = -\frac{\partial}{\partial T} ln(D(t,T). \tag{6.6}$$

As with the asset in section 2.6, also the risk-neutral forward curve in the HJM framework is modelled using the stochastic differential on the form

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)^T dW(t) \tag{6.7}$$

where σ is a d-dimensional volatility vector, dW is a d-dimensional Wiener process and both the drift μ and the volatility are assumed deterministic. In reality, capturing the full complexity of the forward curve requires treating it like an infinite-dimensional object. However, modelling it using a low-dimensional Brownian motion gives sufficient properties empirically as shown later.

As shown in the original paper, to impose the martingale property and prohibit arbitrage, the discount factor dynamics are modelled as

$$\frac{dD(t,T)}{D(t,T)} = r(t)dt + v(t,T)^T dW(t) \tag{6.8}$$

where $v(t,T)$ is the discount factor volatility and $r(t)$ is the risk-free rate. Using Itôs lemma, the evolution of the discount factor prices is dictated by

$$d(lnD(t,T)) = (r(t) - \frac{1}{2}v(t,T)^T v(t,T))dt + v(t,T)^T dW(t) \tag{6.9}$$

Now differentiating equation (6.6) with respect to t gives

$$df(t,T) = -\frac{\partial}{\partial T} d(log(D(t,T)) \tag{6.10}$$

which when combined with the discount factor dynamics from equation (6.9) gives

$$df(t,T) = -\frac{\partial}{\partial T}(r(t) - \frac{1}{2}v(t,T)^T v(t,T))dt - \frac{\partial}{\partial T}v(t,T)^T dW(t) \tag{6.11}$$

Comparing this with equation 6.7 imposes a volatility on the form

$$\sigma(t,T) = -\frac{\partial}{\partial T}v(t,T) \tag{6.12}$$

and a drift

$$\mu(t,T) = -\frac{\partial}{\partial T}(r(t) - \frac{1}{2}v(t,T)^T v(t,T)) = \left(\frac{\partial}{\partial T}v(t,T)\right)^T v(t,T). \tag{6.13}$$

Knowing that the discount factor converges to 1 as $t \to T$, the discount factor volatility at maturity $v(T,T) = 0$ implies that

$$v(t,T) = -\int_t^T \sigma(t,u)du \tag{6.14}$$

which when inserted in equation (6.13) leads to the expression for the drift entirely expressed through the forward rate volatility

$$\mu(t,T) = \sigma(t,T)^T \int_t^T \sigma(t,u)du. \tag{6.15}$$

Now substituting the drift in equation (6.7) with (6.15) leads to the centrepiece of the HJM framework, the forward curve evolution expressed as

$$df(t,T) = \left(\sigma(t,T)^T \int_t^T (\sigma(t,u)du\right) dt + \sigma(t,T)^T dW(t). \tag{6.16}$$

7 Implementation

Equation (6.16) is the centrepiece of the HJM framework, but several steps must be completed before it can be applied to the data described in section 5. First of all, it operates in the continuous world, while financial data is discretized, often reported on a daily basis. The volatility term, used both to add stochasticity and to calculate the drift, must also be calculated. Lastly, a way to parameterize the equation such that it can be specified according to each maturity instead of each maturity date will prove useful.

7.1 The volatility and principal component analysis

Solving stochastic differential equation (6.16) analytically is often unfeasible, except for specific choices of $\sigma(t, T)$. Choosing a variance structure is thus an integral part of constructing a model in the HJM framework. This can be done either by assuming a sufficiently simple tractable model or by matching the data. For example, a constant σ leads to a tractable Ho-Lee model with calibrated drift, while the Vasicek model with time-varying drift can be derived through an exponential volatility parameterization [6]. In this thesis principle component analysis (PCA) is used to find a volatility structure matching historical data, leading to a more flexible model.

Forward rates move with market expectation, and the different maturity rates are as seen in figure 4 highly collinear. This suggests that applying PCA will result in a low-dimensional set of vectors explaining large parts of the variance to be modelled. This method starts by determining the covariance matrix of the differences of the historical interest rate data, Σ. Then, PCA is utilized on Σ to capture the common factors that are driving the movements in the rates. By the properties of the principal components, the eigenvectors ξ_k with corresponding eigenvalues λ_k satisfies

$$\boldsymbol{\Sigma}\boldsymbol{\xi_k} = \lambda_k \boldsymbol{\xi_k}. \tag{7.1}$$

The eigenvalues represent the amount of variance explained by its corresponding eigenvector. Assuming that the volatility structure only depends on the time to maturity, $\tau = T - t$, the volatility factors scaled to yearly rates can then be expressed as

$$\boldsymbol{\sigma}_k(\tau_j) = \sqrt{\lambda_k}(\boldsymbol{\xi}_k)_j. \tag{7.2}$$

The subscript j represents tenors of different lengths and ξ_k , λ_k is the k^{th} eigenvector and eigenvalue respectively. For a dataset with N different rates, $k = 1, \ldots, N$. However, as the reason for applying PCA is dimensionality reduction, a much smaller number of factors is used. As repeatedly found by researchers, such as Heidari [7] and Soto [14], three or fewer principal components (PC) are empirically sufficient to explain more than 95% of yield curve variability under most market conditions. This applies for both low and high volatility periods, with a few exceptions, and is demonstrated in section 8.1 of the results. When simulating the evolution of forward rates, the PCs shape the forward curve evolution by sending stochastic impulses through a number of random walks. This is exemplified in figure 5 which depicts the first three PCs extracted from the period stretching from the start of the covid-19 pandemic in 2020 to 2023.

7.2 Discretization of the HJM model

To build a model capable of handling arbitrary volatility structures, a discrete approximation of both the time and maturity is introduced. The discretization is based on the description of the HJM framework given by Glasserman [6]. For a time grid $0 = t_0 < t_1 < \ldots < t_M$, the forward curve $f(t_i, T)$ is estimated on a grid of maturities. Letting the grid of maturities equal the time grid simplifies the notation with little loss of generality. In the following derivation, hats are used to indicate a variable being discretized, and thus discrete discount factors are expressed as

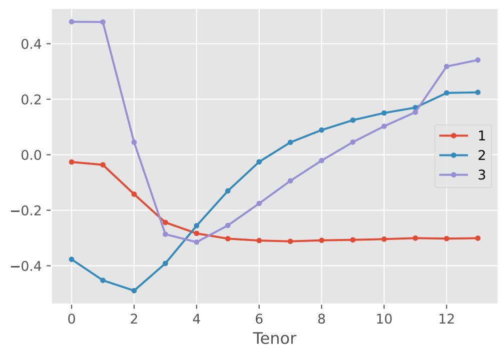


Figure 5: The three first principal components capturing the period of fluctuations caused by the pandemic stretching from April 2020 to April 2023.

$$\hat{D}(t_i, t_j) = exp\left(-\sum_{l=i}^{j-1} \hat{f}(t_i, t_l)[t_{l+1} - t_l]\right). \tag{7.3}$$

To minimize discretization error, the initial values of the discretized discount factors are chosen to correspond to the values as extracted from the interest rate dataset through equation (3.9) at time 0. This is the case whenever

$$\sum_{l=0}^{j-1} \hat{f}(0, t_l)[t_{l+1} - t_l] = \int_0^{t_j} f(0, u)du. \tag{7.4}$$

For each forward rate given by $l = 0, 1, \ldots, M-1$, this expression becomes

$$\hat{f}(0, t_l) = \frac{1}{t_{l+1} - t_l} \int_{t_l}^{t_l+1} f(0, u)du. \tag{7.5}$$

Discretizing using equation (7.5) initializes the forward rate as the average over the interval $[t_l, t_{l+1}]$. From this initial forward curve, the evolution of the curve on the time and maturity grid given by $i, j = 1, \ldots, M$ respectively, can be written as

$$\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_i, t_j)[t_i - t_{i-1}] + \sum_{k=1}^{M} \hat{\sigma}_k(t_{i-1}, t_j)\sqrt{t_i - t_{i-1}} W_{ik}, \tag{7.6}$$

where $W_1, \ldots, W_M$ are vectors of samples from a standard normal distribution and $\hat{\sigma}$ and $\hat{\mu}$ are the discrete volatility and drift. The discrete approximation of the risk-neutral drift is as its continuous counterpart found in equation (6.15) specified by the volatility. The discrete drift term found in (7.6) can be expressed as

$$\hat{\mu}(t_i, t_j)[t_i - t_{i-1}] = \sum_{k=1}^{M} \hat{\mu}_k(t_i, t_j). \tag{7.7}$$

The discretized drift components $\hat{\mu}_k$ are found through the assumption of no arbitrage. One way of obtaining a drift for an arbitrage-free model is to make the drift preserve the martingale property for the discount factors. As the instantaneous spot rate at time $r(t) = f(t, t)$ for a given time t, having an $\hat{\mu}$ that makes

$$\hat{D}(t_i, t_j) exp(\left(\sum_{l=i}^{j-1} \hat{f}(t_k, t_k)[t_{k+1} - t_k]\right) \tag{7.8}$$

a martingale in i for each j would imply an arbitrage-free model. By studying the expected value of the above expression conditioned on the previous values and solving for $\hat{\mu}$, the expression for the drift factors is found to be preserving the martingale property when

$$\hat{\mu}_k(t_i, t_j)[t_i - t_{i-1}] = \frac{1}{2}\left(\sum_{l=1}^{j} \hat{\sigma}_k(t_i, t_l)[t_{l+1} - t_l]\right)^2 - \frac{1}{2}\left(\sum_{l1=1}^{j-1} \hat{\sigma}_k(t_i, t_l)[t_{l+1} - t_l]\right)^2. \tag{7.9}$$

7.3 Musiela parameterization

In practice, it is easier to model the volatility function for each maturity, instead of at each maturity date, as described by Wilmott in [16]. Given maturity date T and current date t we define $\tau = T - t$ to find

$$\sigma(t, T) = \bar{\sigma}(t, \tau). \tag{7.10}$$

The forward rate using this parameterization is then

$$d\bar{F}(t, \tau) = \bar{m}(t, \tau)dt + \bar{\sigma}(t, \tau)dX \tag{7.11}$$

where

$$\bar{m}(t, \tau) = \bar{v}(t, \tau)\int_t^{\tau} \bar{v}(t, s)ds + \frac{\partial}{\partial \tau}\bar{F}(t, \tau) \tag{7.12}$$

7.4 Real-world measure

When simulating exposure at a future time $t > 0$ as described in section 4, a real-world simulation is conducted from the current time, until t. To do this, the risk-neutral drift of the forward rates must be replaced with a real-world drift. To find this real-world drift the market price of risk must first be found. This is a daunting task, and the drift in this thesis relies on the expectations hypothesis when simulating real-world rates. The hypothesis states that the current forward rates hold information about the market's expectations, and in the purest form, all the information required to determine future spot rates [13]. This can be expressed mathematically as

$$E[R(t_0, T)] = F(0, t_0, T) = \sum_{i=1}^{l} f(t_{i-1}, t_i) \cdot (t_{i-1} - t_i). \tag{7.13}$$

where $t_l = T$. Thus, to simulate real-world spot rates, the HJM drift is simply set equal to zero. The pure expectations hypothesis assumes forward rates are unbiased estimators of future spot rates, which may not always hold true in practice.

7.5 Monte Carlo simulation

Now that the drift and volatility can be found for all maturities, simulation of forward rates can be carried out using a Monte Carlo approach in both risk-neutral and real-world settings to price interest rate derivatives and generate future market scenarios.

At the current date t_0 the forward curve $F(t_0, T)$ is known. For each tenor the forward rate is simulated for a given number of steps, until a specified time, T_{end} as seen in figure 6. For each step, the forward interest rate is the sum of three terms. The deterministic drift, the random term

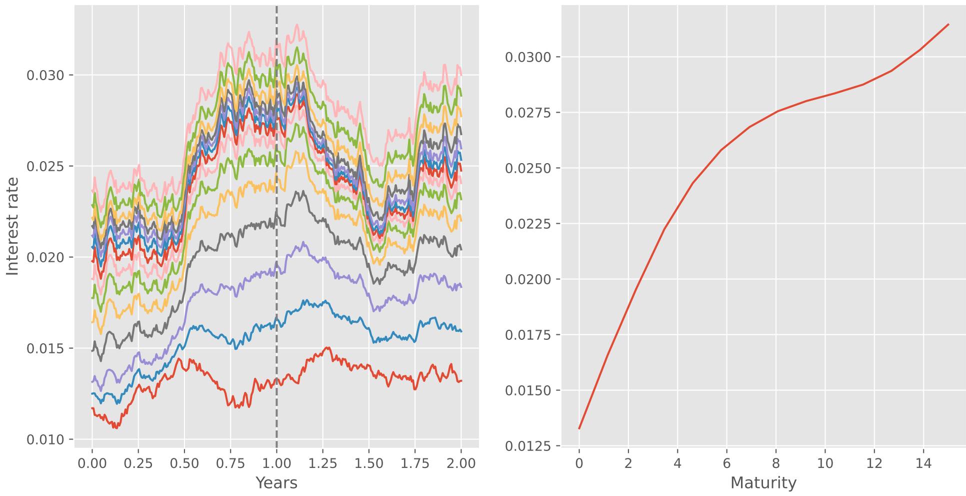


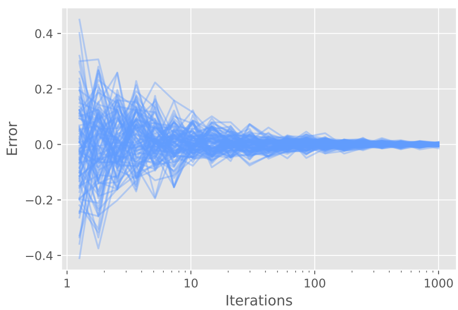
Figure 6: **Left:** A single realization of the simulated forward rates starting from a known forward curve at $t = 0$, evolving two years into the future. The 14 time series correspond to rates with maturities ranging from 3 months to 15 years. **Right:** A forward curve extracted from this single realization at $t = 1$ year, highlighted by the vertical dotted line in the left plot.

consisting of a sample from a normal distribution multiplied by the volatility stemming from each of the fitted volatility functions and lastly the term added by the Musiela parameterization. By iteratively applying these steps, a realization of the forward rate $F(t, T)$ is obtained for $t_0 \leq t \leq T_{end}$ where $T \geq t$. Consequently, the simulation provides prices of all zero-coupon bonds for all t with maturity no later than T.

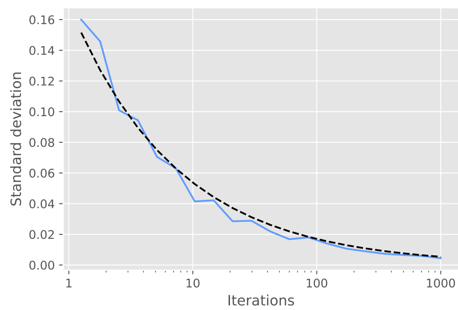
7.6 Convergence

Using Monte Carlo methods, the accuracy of the estimates depend on the number of simulations performed, which should be chosen sufficiently large based on the specific problem at hand. To analyze the convergence of the method, simply looking at a simulated forward rate is insufficient as the pricing algorithm depends on the forward rate at multiple dates. Some metric measuring the total difference in the produced forward curves could be used, but a more natural approach is the one of pricing a swap at initialization in a risk-neutral setting.

As seen in figure 7, the convergence is as described in section 2.9. When generating derivative exposure profiles, a compromise between accuracy and performance is made, with the number of risk-neutral simulations used in the thesis chosen to 200, giving a standard deviation of around 0.009. This choice is made based on the need for accuracy when producing risk profiles and efficiency when producing these. The number of real-world simulations used is 2500, which is chosen empirically in a way that produces smooth distributions.



(a) The error in the initial swap price.



(b) The standard deviation corresponding to the 300 simulations. The theoretical convergence of $O(n^{-1/2})$ is shown by a black dotted line.

Figure 7: To analyze the convergence of the algorithm, the initial price of a 5-year swap with semiannual payments and a notional amount of 100 is simulated 300 times. The estimates are done on a logarithmic scale ranging from 1 to 1000 with 20 samples.

8 Results

Once the discretized version of the model is implemented, drift and volatility are estimated for the periods of interest. In the simulations provided below, the parameters used are estimated based on historical data from the two years preceding the initialization period, ensuring that the simulations align with the observed behaviour leading up to the initialization period while at the same time capturing a large period of possible changes. This period was chosen based on empirical evidence gained throughout model testing and proved to give generally good results.

8.1 Swaps

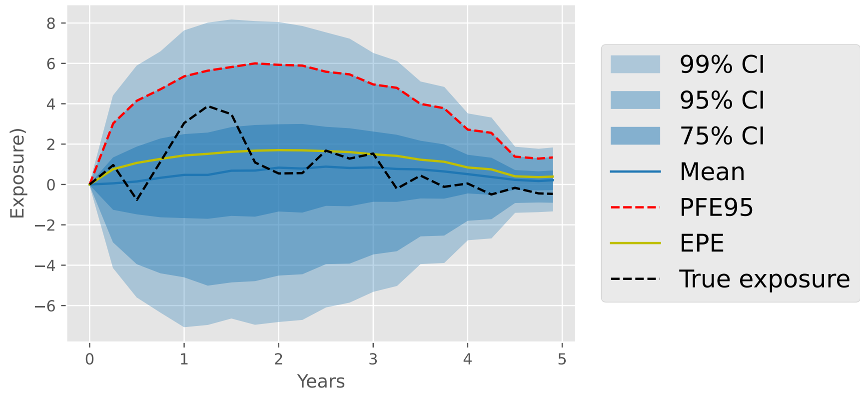
As described in section 3.3.2, an interest swap can be initialized with zero value by choosing a fixed rate in accordance with the forward rates at initialization. After choosing the fixed rate, exposure for a set of future dates can be simulated as described in section 4.

The swap exposure profiles for three equivalent 5-year swaps are found in figure 8. These distributions are generated by measuring the exposure every 3 months until maturity, starting at $t = 0$. The swaps are equivalent, except for the fixed rate which is set to correspond to an initial value of zero. Common for all the swaps is the rapid increase in distribution width with a peak around the two-year mark followed by a steady decline as the contract nears maturity. This is characteristic of swaps and can be attributed to two opposite effects. Firstly, the value of the contract is derived from the forward curve for any given time, and the deviation of the value of the swap from 0 is due to possible deviations from the initial market expectation. The reason for the rapid initial growth is the number of cashflows remaining in the swap agreement. At initialization, 10 payments are remaining and are subject to these deviations. The further into the future the exposure is simulated, the larger deviations from the initial forward curve are expected. However, at the end of the contract lifetime, the exposure variance decreases as a result of fewer payments remaining. This results in peak exposure being reached around $t = 2$, where the number of remaining payments and possible forward curve changes reach an inflexion point. Thus, to interpret the exposure profile of a swap, two effects are fundamental. The first being the positive correlation between future spot rate uncertainty and distance to measuring date, while the second is the declining number of remaining cashflows.

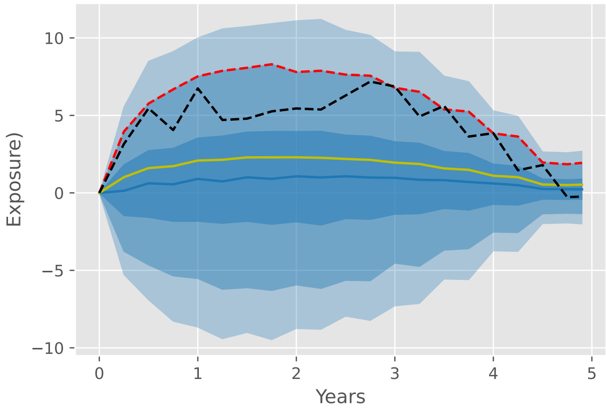
Figure 8b shows an exposure profile for a swap whose lifetime is characterized by a slow decline in interest rates. From initialization with a 6 month NIBOR of 2.68% and an upward sloping yield curve, the key rate declines steadily to 1.13% over the swap duration. The two years prior to initialization are characterized by higher rates, and thus, the model is in this scenario not able to capture the actual exposure within its 75% confidence interval, and at several points bordering the 95% PFE. The exposure at default can be calculated using the method given by the Bank of International Settlements, described in section 4.2. For this swap with a notional of 100, with $\alpha = 1.4$, the EAD is 1.95.

Figure 8c concerns the period of interest rate changes related to the covid-19 pandemic. The period is characterized by a sharp drop in the Norges Bank's policy rate around $t = 2$ to counter an economic slowdown, followed by rapid increases beginning around $t = 3$ to counter the inflation that followed. The actual exposure is roughly contained within the 75% simulated interval until $t = 2$, where shortly after, it exceeds both the 95% PFE and even the 99% simulation interval. The effects of the rapid increases in interest rates are also not reflected by the exposure profile, and more severe divergences would be noted for a swap with a longer duration, having additional transactions subject to these changes. This suggests that the model is unable to account for sudden drops and hikes in interest rates like the ones seen in the beginning of the 2020s. These changes are the results of serious market-disrupting events and were not expected to be predicted by the model. Special notice should be taken of the period used to calibrate the model. This is a period of economic stability and low volatility, causing narrower simulated distributions. The EAD using the BIS approach is 1.18

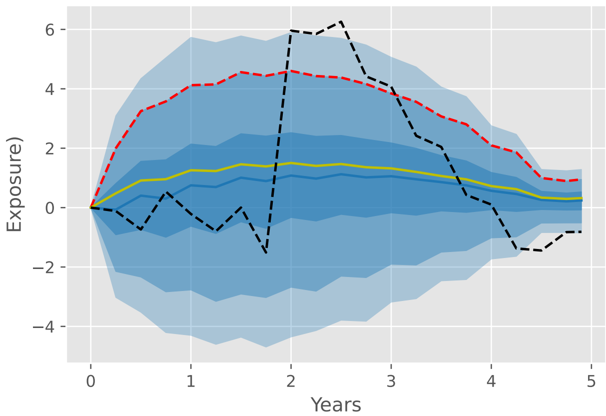
The swap in figure 8a concerns a period of economic stability where interest rates are close to mean



(a) Swap initialized 22.12.2014 with maturity 23.12.2019 spanning a period of economic stability.



(b) Swap initialized 16.04.2012 with maturity 17.04.2017, a period characterized by a slow but steady decline in interest rates.



(c) Swap initialized 09.04.2018 with maturity 10.04.2023, capturing both the rapid drop and increase in interest rates associated with the covid-19 pandemic.

Figure 8: Exposure profiles for three equivalent 5-year swaps for three different periods, with a notional amount of 100. The swaps are based on the 6 months NIBOR, have semi-annual payments and reset dates 6 months prior to the transaction dates. For each swap, the model is calibrated using historical data from the last 2 years preceding the initialization date. The 99, 95 and 75 simulated intervals are shown through increasingly opaque blue distributions. The 95% PFE and EPE are highlighted by red and yellow lines respectively, while the black dotted line depicts the true exposure and is used to indicate the viability of the model subject to the respective market conditions.

Corresponding swap	PC1	PC2	PC3	Total
a	88.0	7.2	2.3	97.5
b	79.5	10.2	3.7	93.4
c	85.2	7.6	3.0	95.8

Table 2: Percent of volatility explained by the three first principal components in four different periods. Each period spans 2 years from period start. The first three rows each corresponds to a calibration period for the swaps.

reverting both in the calibration and simulation period. This causes the actual exposure to stay within the 75% simulated intervals for most of the swaps duration, close to the expected positive exposure. The EAD using the BIS approach is 1.56

These examples highlight the need for special care concerning periods of shifting market conditions and market-disrupting events, while the model seems adequate in periods characterized by slow shifts or fluctuations around some mean.

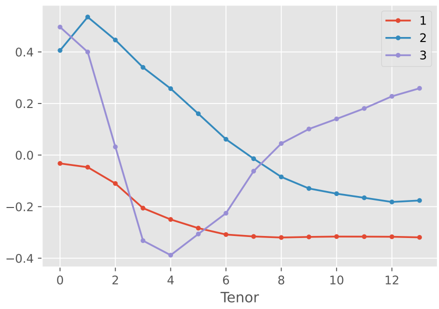
The PCs extracted from the calibration period hold information about the historical volatility and are used to modulate the source of stochasticity in the generation of forward curves. The three plots in figure 9 corresponds to the calibration period used in generating the three profiles shown in figure 8. The same general structure with slight variations is observed in all three periods. The first PC is generally characterized by an initial decrease before flatting out, resulting in a steepening effect of the short end of the curve while the longer rates move together. As seen in figure 9c, the first PC never flattens out completely resulting in a slight steepening effect also for the longer-term rates. The second PC is generally characterized by a close to monotone decrease with slight variations in the shortest rates and one change of sign. This results in the rates on each side of the zero moving in opposite directions, giving a twisting effect. The maturity where the change of sign occurs determines the centre of twisting. The third PC has a parabolic shape and changes sign twice in all the periods. This gives a bending effect where short and long-term rates move in the opposite direction of the medium-term ones. Also here the position of the two zeros determines the specifics of the bending effect.

Interestingly, the overall structure of the three different calibration periods is highly similar but results in different simulated distributions. Even not visually obvious, the net volatility magnitude of the principal components varies from period to period. If each PC is multiplied with their respective eigenvalues, a clear relationship between the magnitude of the total volatility from the aggregated principal components and the distribution width is observed. This suggests that when using the model to estimate future exposure, the shape of the principal components need not be detrimental and that the total volatility embedded in them is more important.

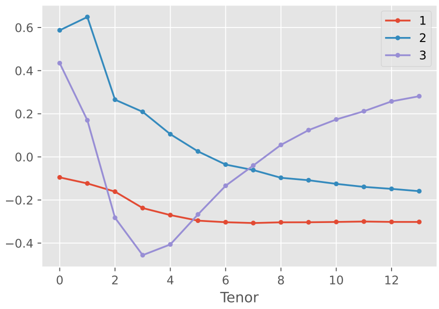
8.2 Cap

Interest rate caps are not concerned with symmetric risk like an interest swap is, but simulating distributions of their values are still relevant from a risk management perspective. These distributions can be created using the same general simulation approach as for the swaps, described in section 4, now used in combination with the caplet payoff function (3.15).

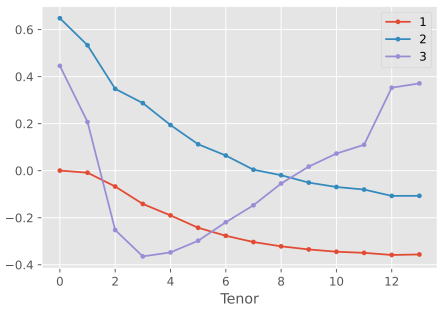
The distribution of simulated values of a 5-year cap with semi-annual payments is shown in figure 10, together with the actual contract value. The future cashflows are, as with the swap cashflows, subject to the increasing spot rate uncertainty with increasing distance to measuring time. The cap is not initialized with value 0, but as done in practice, set to some specified value by the choice of the strike rate. In this example, the strike rate is set equal to 1.8%. Devoid of any premium, the initial price of this cap would be its value of 1.75. Unlike the swap, the floating rate for this cap is not set months prior to the transaction date, but is chosen to coincide with it. The exposure profile is also characterized by the absence of negative values due to the nature of caplet payments being made only in the case of the reference rate exceeding the strike rate. The shape of the distribution is as with swaps characterized by two opposing effects. The increased uncertainty with increased



(a) From 20.12.2012 to 22.12.2014



(b) From 15.04.2010 to 16-04-2012.



(c) From 07.04.2016 to 09.04.2018.

Figure 9: The principal components corresponding to the calibration period of the three swaps.

distance from the known interest rate and the number of caplets remaining. The peak exposure is also in this case reached around 2 years after initialization.

8.3 Re-estimation and re-calibration

As described earlier, figure 8c shows the actual value of a 5-year swap compared with the simulated distributions. The actual value stays largely within the 75% confidence interval until around $t = 2$. The estimation of parameters is done using the historical data from the two years prior to initialization, 11.04.2018, as illustrated in figure 11.

Throughout the lifetime of any interest rate derivative, a re-estimation of the exposure or a specific metric like the 95% PFE may be of interest. Such a re-estimation is done using the same model parameters and is shown in figure 12a. Here the actual exposure quickly departs from the expected exposure

Figure 12b shows the exposure profile of the same period, but now simulated with a re-calibration of model parameters. At $t = 2$, after the major drop in interest rates, new parameters are estimated using again historical rates from the last two years, but this time including the data leading up to $t = 2$ instead of $t = 0$. Looking at the volatility of the 6-month NIBOR, a sharp increase from 0.00149 to 0.00299 is noted. The general increase in volatility in the model affects the exposure profile, widening the simulated distribution, ensuring that the actual exposure stays within simulated confidence intervals to a higher degree. This highlights the need for re-calibration of models after market-disrupting events, where earlier estimation of exposure becomes outdated. Even with re-calibration, the re-estimation is unable to account for the rapid increase in interest rates beginning in early 2022 that continues until maturity.

The difference in the distributions can be attributed to the difference in calibration period market conditions, highlighted by the change in term structure volatility observed in the principle components displayed in figure 13. The initial PCs illustrated in figure 13a are characterized by two

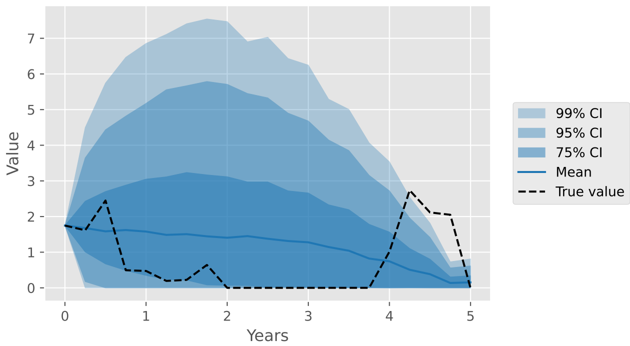


Figure 10: Distribution of simulated values for an interest rate cap with semi-annual payments, based on the 6-month NIBOR. The contract is initialized the 09.04.2018 and matures after 5 years. The 99, 95 and 75 simulated intervals are shown through increasingly opaque blue distributions, while the actual value of the contract to the counterparty is illustrated by the black dotted line.

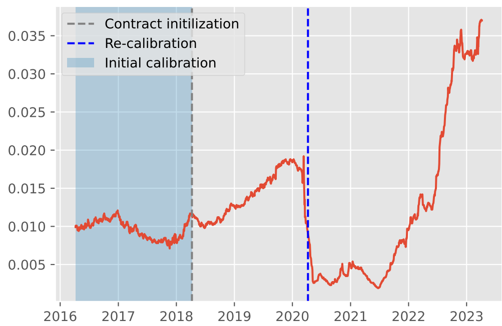
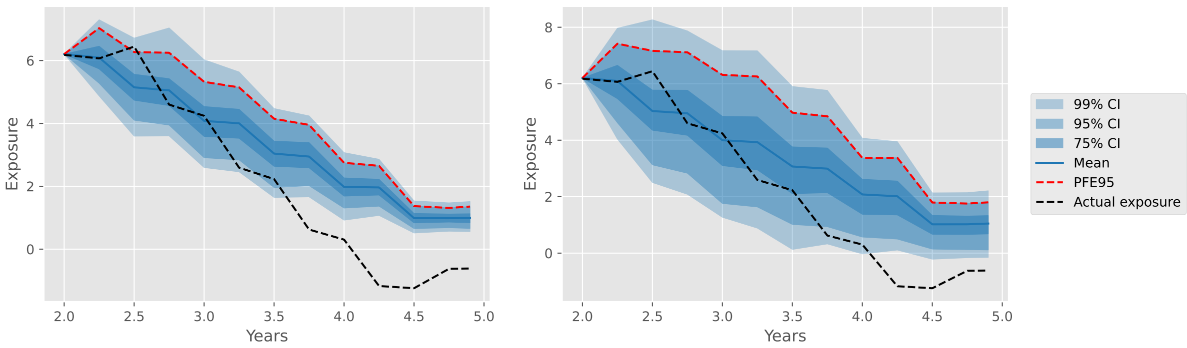


Figure 11: The 6-month NIBOR plotted from 11.04.2016 to maturity, 11.04.2023. The swap initialization date is shown as a grey dotted line, which is preceded by the period used for model calibration marked by a blue shaded area. The blue dotted line coincides with the fourth transaction date of the swap and is the point where a re-estimation of remaining exposure is carried out.

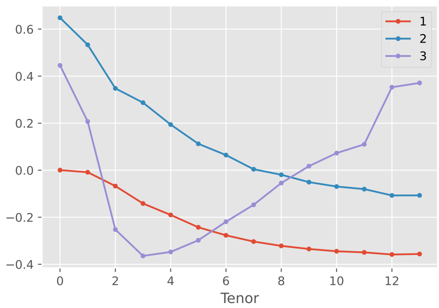


(a) Re-estimation of remaining exposure after $t = 2$ using original model parameters estimated in the

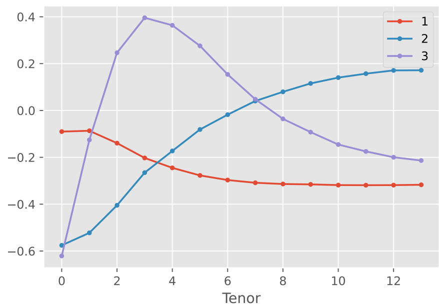
(b) Re-estimation of remaining exposure after $t = 2$, now with the re-calibrated model using historical data up until the re-estimation point.

Figure 12: Exposure profiles of the swap from shown in figure 8c after re-estimation at t=2, 09.04.2020. The $99, 95$ and 75 simulated intervals are shown through increasingly opaque blue distributions. The 95% PFE and EPE are highlighted by red and yellow lines respectively, and the actual exposure is shown with the black dotted line.

main effects. Both the first and second PC are monotonously decreasing and represent a steepening of the forward curve, while the third PC has a parabolic shape and represents a bending effect, making short and long rates move in the opposite direction of the medium rates. As seen in table 13c, the two first PCs explain 92.8% of the variance, and serves as the dominating driver of model output. The re-calibrated simulation shown in figure 12b uses the PCs from figure 13b. The first PC has an initial drop followed by a flat part. This relates to a steepening of the shorter rates and a parallel shift of the medium to long rates. The second PC is monotonously increasing and has a steepening effect opposite to the one associated with the first PC. The third PC has a parabolic shape and is associated with a bending effect. The original PCs are obtained from a period of economic stability, an absence of inverted yield curves, and little volatility. This results in problems encapsulating the actual value within the simulated distributions once the market conditions change. Another clear difference from the original PCs is the larger amount of variance explained by the second PC, confirming the more complex volatility structure in the re-calibration data.



(a) Original principal components found at initialization used in figure 12a



(b) Re-estimation of remaining exposure after $t = 2$, now with the re-calibrated model using historical data up until the re-estimation point, used in figure 12b

	PC1	PC2	PC3
Original	85.2	7.6	3.0
Re-calibration	79.4	16.6	2.2

(c) Explained variance by PC for both training periods.

Figure 13: The first three principal components corresponding to the re-estimations described in section 8.3.

9 Conclusion and further work

The purpose of this thesis was to explore the feasibility and implementation of a model based on the Heath, Jarrow and Morton framework in the context of counterparty credit risk. This has been done by generating and analyzing simulated exposure profiles corresponding to common interest rate derivatives, such as interest rate caps and swaps.

Throughout the thesis, supporting evidence is gained for utilizing the Heath, Jarrow, and Morton model in risk management and for the calculation of capital requirements. The model is able to capture and quantify interest rate dynamics, can be calibrated to market observations and produces reasonable results and exposure profile shapes consistent with existing literature. For the purposes of counterparty credit risk and calculating capital requirements, the model appears to be computationally feasible.

To evaluate the model, simulated exposure profiles are compared with actual exposure. The results are satisfactory under normal market conditions, but the model fails to deliver adequately when the duration of the derivative is affected by market-disrupting events, such as the covid-19 pandemic. The simulated instrument values are determined by the initial market conditions and the historical term structure variability. The distribution width is thus directly related to the calibration period volatility, and the choice of calibration period should be taken into account when evaluating the model output. The simulated exposure appears especially unreliable for contracts initialized in low-volatile periods, with either large increases in volatility or consistent increases or decreases in rates over time.

Future work of interest includes evaluating the model on the upcoming period where a sustained level of higher rates might be the norm, and on longer-term duration contracts initialized before the pandemic or before the rapid rate hikes of 2022.

Bibliography

[1] Russel E Caflisch. 'Monte carlo and quasi-monte carlo methods'. In: *Acta numerica* 7 (1998), pp. 1–49.

[2] Giovanni Cesari et al. *Modelling, pricing, and hedging counterparty credit exposure: A technical guide*. Springer Science & Business Media, 2009.

[3] R. Jarrow D. Heath and A. Morton. 'Bond pricing and the term structure of interest rates: A new methodology'. In: *Econometrica: Journal of the Econometric Society* (1992), pp. 77–105.

[4] Wouter J Den Haan and Scott A Spear. 'Volatility clustering in real interest rates theory and evidence'. In: *Journal of Monetary Economics* 41.3 (1998), pp. 431–453.

[5] International monetary fund. *World Economic Outlook - April 2023*. URL: https://www.imf.org/en/Publications/WEO (visited on 19th May 2023).

[6] Paul Glasserman. *Monte Carlo methods in financial engineering*. Vol. 53. Springer, 2004.

[7] M. Heidari and L. Wu. 'Are Interest Rate Derivatives Spanned by the Term Structure of Interest Rates?' In: *The Journal of Fixed Income* Summer (2003), pp. 75–78.

[8] CFA institute. *EMH*. URL: https://www.cfainstitute.org/en/research/cfa-digest/2003/11/the-efficient-market-hypothesis-and-its-critics-digest-summary (visited on 3rd Apr. 2023).

[9] Bank for International Settlement. *Approval to adopt internal models*. URL: https://www.bis.org/basel_framework/chapter/CRE/53.htm (visited on 13th Apr. 2023).

[10] Bank for International Settlement. *CCR definition*. URL: https://www.bis.org/basel_framework/chapter/CRE/50.htm (visited on 4th Apr. 2023).

[11] Bank for International Settlement. *Interest rate derivative turnover*. URL: https://www.bis.org/statistics/rpfx22_ir.htm (visited on 29th Nov. 2022).

[12] Bank for International Settlement. *Standardised approach to CCR*. URL: https://www.bis.org/basel_framework/chapter/CRE/52.htm (visited on 13th Apr. 2023).

[13] Nasdaq. *PET*. URL: https://www.nasdaq.com/glossary/p/pure-expectations-theory (visited on 4th Apr. 2023).

[14] G Soto. 'Using Principal Component Analysis to Explain Term Structure Movements Performance and Stability'. In: *Progress in economics research* 8 (2004).

[15] Rangarajan K Sundaram. 'Equivalent martingale measures and risk-neutral pricing: An expository note'. In: *Journal of Derivatives* 5.1 (1997), p. 85.

[16] Paul Wilmott. *Paul Wilmott on quantitative finance*. John Wiley & Sons, 2013.

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