Elias Klakken Angelsen

On Differential Cohomology and Geometric Hodge-filtered *K*-theory

Master's thesis in Mathematical Sciences (MSMNFMA)
Supervisor: Professor Gereon Quick
June 2023

Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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\begin{tikzcd}
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& H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr, "j"] & & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{H}^k(M;\mathbb{Z}) \arrow[ur, "\mathrm{cc}"] \arrow[dr, "\mathrm{curv}"] & & H^k_{dR}(M) \\
& \Omega^{k-1}(M)/\Omega^{k-1}_0(M) \arrow[ur, "\iota"] \arrow[rr, "d"] & & \Omega^k_0(M) \arrow[ur] \arrow[dr] & \\
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Abstract

The thesis has two main goals.

First, we present an overview of *differential cohomology*, focusing on existing approaches to the topic and important results. This starts with the differential characters of Cheeger-Simons ([CS85]), which are generalized by the differential function complexes of Hopkins-Singer ([HS05]). The approach by Bunke-Nikolaus-Völkl ([BNV16]) using spectral sheaves is presented, and we decompose the ∞-category of differential cohomology theories by using known results on stable ∞-categories with certain subcategory structures. We study the axioms for smooth extensions and a corresponding uniqueness theorem proven by Bunke-Schick ([BS10]) before we investigate how differential cohomology theories respect homotopy invariance and Landweber exact formal group laws ([Bun+09]). Furthermore, we investigate explicit descriptions of differential cohomology theories, such as smooth Deligne cohomology for ordinary differential cohomology. After developing a geometric suspension construction, we briefly explain how the structured bundles of Simons-Sullivan ([SS08b]) define differential K-theory.

Secondly, we study an analog of differential cohomology for complex manifolds, namely *Hodge-filtered cohomology theories*, where we work towards understanding (geometric) Hodge-filtered K-theory. We study Deligne cohomology and the work of Hopkins-Quick ([HQ15]), and see how they fit into the axioms of Hodge-filtered extensions by Haus-Quick ([Hau22], [HQ22]). We investigate differences between differential- and Hodge-filtered cohomology, most notably by discussing a conjecture that Hodge-filtered extensions respect Landweber-theories similarly to differential cohomology, despite differences between the two. After a discussion on multiplicative K-theory in the sense of Karoubi ([Kar90]), we prove that *multiplicative K-theory is a geometric model of Hodge-filtered K-theory* in the sense of Haus-Quick.

Sammendrag

Oppgaven har to hovedmål.

Først presenterer vi en oppsumering av fagfeltet *differensiell kohomologi*, med hovedfokus på eksisterende tilnærminger og viktige resultater. Vi starter med differensialkarakterene til Cheeger og Simons ([CS85]), som kan generaliseres til differensialfunksjonskomplekser etter Hopkins-Singer ([HS05]). Vi presenterer tilnærmingen til Bunke, Nikolaus og Völkl ([BNV16]) gjennom spektralknipper, og vi dekomponerer uendeligkategorien av differensielle kohomologiteorier gjennom kjente metoder for stabile uendeligkategorier som utnytter spesifikke underkategoristrukturer. Vi studerer aksiomene til glatte utvidelser og unikhetsteoremet for disse, som bevist av Bunke og Schick ([BS10]), før vi undersøker hvordan disse respekterer homotopiinvarians og landwebereksakte formelle gruppelover. Videre betrakter vi eksplisitte beskrivelser av spesifikke differensielle kohomologiteorier, slik som glatt delignekohomologi for ordinær differensiell kohomologi. Etter vi studerer en geometrisk suspensjonskonstruksjon, forklarer vi kort hvordan arbeidet til Simons og Sullivan ([SS08b]) på strukturerte bunter brukes til å definere differensiell K-theori.

Deretter studerer vi en analog til differensielle kohomologiteorier for komplekse mangfoldigheter, nemlig *hodgefiltrerte kohomologiteorier*, hvor hovedfokuset etterhvert flyttes til å forstå (geometrisk) hodgefiltrert K-teori. Motivert av delignekohomologi og arbeidet til Hopkins og Quick ([HQ15]), studerer vi aksiomene for hodgefiltrerte utvidelser som presentert av Haus og Quick ([Hau22], [HQ22]). Vi studerer forskjeller mellom differensiell- og hodgefiltrert kohomologiteori, og vi diskuterer en formodning om at hodgefiltrerte utvidelser respekterer landweberteorier på samme måte som differensiell kohomologi, tross subtile forskjeller mellom de to. Fokuset rettes mot hodgefiltrert K-teori, og etter en diskusjon av Karoubis "multipliativ K-teori", beviser vi at *multiplikativ K-teori er en geometrisk modell for hodgefiltrert K-teori*, basert på aksiomene til Haus og Quick.

Acknowledgements

In many ways, this thesis marks the end of five years of mathematical studies at NTNU, Trondheim.

First and foremost, I want to thank my supervisor Professor Gereon Quick for his help and support. He has been patient with my many questions, notational misunderstandings, and nonsensical ideas, and he has always encouraged me to investigate my own questions when working on this thesis. Furthermore, I wish to thank him for suggesting the underlying topic of this thesis and for pushing me in the right direction(s) toward the end. Thank you for believing in me.

Next, I want to thank Thomas Wilskow Thorbjørnsen and Tallak Manum for their proof-reading, their opinions, and their patience that resulted in filling my categorically challenged mind with much knowledge I would not be able to learn on my own. I want to thank Gløshaugen Revy- og Teaterlag for endless hours of theater, laughter, and frustration the last year. It has been a pleasure being distracted from thesis work with you. Many individuals deserve to be mentioned, and to avoid forgetting anyone, I extend my deepest thanks to the amazing mathematical community at Matteland for helping me grow both as a person and as a mathematician the last years. In particular, I need to thank Lesesal 395B for much quizzing, many discussions (both academic and non-academic), and a bunch of ice creams (and beers) throughout the last year. You have made the challenging affair of writing a master's thesis in mathematics a pleasure.

I also wish to thank my family, and my dog, for their support and love throughout the years. Last, but definitely not least, I wish to deeply thank Nora for providing a steady flow of love, support, and coffee the last year. It has been a pleasure sharing this challenging adventure with you, and I look forward to many more!

Initially, I thought of thanking myself for being the one who actually wrote this thesis, but I have realized this thesis would never have been finished if not for the amazing people around me. Thank you!

Elias Klakken Angelsen
Trondheim, Norway
May 2023

Contents

Introduction

In the field of algebraic topology, one broadly aims to study and classify topological spaces and their properties. Many important tools are homotopy invariants, such as homology, cohomology, and homotopy groups. Established theorems from differential geometry, such as the Gauss-Bonnet theorem and the de Rham theorem, assert that constructions that are not homotopy invariant, such as closed differential forms, curvatures, and integrals, can be used to construct homotopy invariants, such as de Rham cohomology, or the Euler characteristic. Despite their power, much valuable information is lost when considering homotopy invariants of smooth manifolds, as they rarely respect the geometric notions of lengths, angles, and curvatures.

However, results such as the de Rham theorem may be used as motivation to study more refined constructions. By specializing the discussion of homotopy invariants to cohomology, the more "refined" constructions we study are called differential cohomology. Some informal questions that subsume the entire thesis, will therefore be the following:

The underlying questions:

1. How, if at all, can we refine cohomology on smooth manifolds to include "geometric information"?
2. Which axioms define these "differential" cohomology theories, and how unique are they?
3. How much homotopy invariance is "traded away" for "geometric information"?
4. Which explicit constructions make up geometric models for these theories?

The first goal of the thesis is to answer these questions based on already existing papers. We briefly start with the approach of Cheeger-Simons [CS85], based on the de Rham theorem, and see how they "blend" the "geometric" and "homotopic" information into a hexagon diagram. That is, given a smooth manifold M, we have a construction $\hat{H}^*(M;\mathbb{Z})$ that fits into a diagram,

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr, "j"] & & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{H}^k(M;\mathbb{Z}) \arrow[ur, "\mathrm{cc}"] \arrow[dr, "\mathrm{curv}"] & & H^k_{dR}(M) \\
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0 \arrow[ur] & & & & 0.
\end{tikzcd}
$$

The groups $\hat{H}^*(M;\mathbb{Z})$ are the ordinary differential cohomology groups of a smooth manifold M, and the above illustrates the interplay between the homotopy invariant picture (found in the top half of the diagram) and the "geometric" (e.g. non-homotopy invariant) picture (found in the bottom half).

The differential characters of Cheeger-Simons can be vastly generalized through differential function complexes, and we may use this generalization to construct differential E-cohomology for any spectrum E, which is based on the work of Hopkins-Singer [HS05].

Based on the maps in and out of $\hat{H}^*(M;\mathbb{Z})$, there is an axiomatic way to study differential cohomology theories, and this axiomatic framework allows for a uniqueness theorem, under certain technical assumptions. This is the work of [BS10], and in the axiomatic framework, it is possible to classify "how much homotopy invariance is traded away for geometric information", resulting in a homotopy formula. We can even find explicit constructions that fit the axioms. For example, smooth Deligne cohomology defines ordinary differential cohomology and therefore defines differential cohomology groups isomorphic to the groups of differential characters.

The naive idea of adding geometric information to topological information is especially clear when considering differential K-theory as defined by Simons-Sullivan [SS08b], where the vector bundles are replaced by "structured bundles", which are vector bundles with an appended (class of) connection(s).

We can also consider an extremely general approach by [BNV16] based on defining an ∞-category $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, which becomes the ∞-category of differential cohomology theories. This ∞-category can be decomposed through the theory of stable recollements. This allows us to retrieve a hexagon diagram as above, although it is too general to satisfy the axioms.

Furthermore, differential cohomology inherits classic algebraic constructions. For example, given a formal group law that is Landweber exact, (R, g), we can construct a cohomology theory by $R^*(X) = MU^*(X) \otimes_{MU^*} R$. The axiomatic picture of differential cohomology can be used to show that we can consider differential complex cobordism $\hat{MU}^*$ to obtain a differential R-theory as $\hat{R}^*(M) = \hat{MU}^*(M) \otimes_{MU^*} R$.

Differential cohomology theories are only defined for smooth manifolds. Although complex manifolds are smooth manifolds, differential cohomology is not made to respect the complex structure of complex manifolds. Motivated by Deligne cohomology, which was originally formulated for complex manifolds, we move on to study Hodge-filtered cohomology theories. The general construction showing the existence of Hodge-filtered cohomology theories for an arbitrary spectrum E is the Hopkins-Quick construction ([HQ15]). An axiomatic picture slightly different to differential cohomology was established by Haus-Quick in [Hau22] and [HQ22]. However, there does not exist a uniqueness theorem for Hodge-filtered extensions (yet).

After discussing the basic properties and results for Hodge-filtered cohomology theories in light of the preceding discussion of differential cohomology, we conjecture that despite subtle, problematic differences in their definitions, we expect Landweber exact formal group laws to produce Hodge-filtered cohomology theories similarly to differential cohomology. We did not manage to prove this, which resulted in Conjecture 4.6.2.

The novelty in the thesis is mainly found in Chapter 5, where we specialize the discussion to Hodge-filtered K-theory. We discuss potential approaches to Hodge-filtered K-theory and find a geometric model for Hodge-filtered K-theory by showing that multiplicative K-theory in the sense of Karoubi [Kar90] can be modified to satisfy the axioms of a Hodge-filtered K-theory.

Theorem 1 (Geometric Hodge-filtered K-theory, Theorem 5.2.3).

With certain maps specified in Section 5.2, Multiplicative K-theory is a Hodge-filtered K-theory with twist $p = 0$.

Our construction is motivated by a geometric suspension construction, bypassing the problem that ΣM does not need to be a smooth manifold, even though M is. This lets us specify the maps and check the axioms only in cohomological degree 0, as this approach induces maps to higher degrees by being describing higher groups as certain kernels.

Hodge-filtered cohomology theories depend on a prespecified twist p. The generality of multiplicative K-theory lets us define a p-shift, which induces the corresponding result for general p.

Corollary 2 (Geometric Hodge-filtered K-theory, Corollary 5.2.6.1).

The p-shifted multiplicative K-theory in the sense of Definition 5.2.5 is a Hodge-filtered K-theory.

More intuitively speaking, the two last results combine to say that multiplicative K-theory defines a (geometric model for) Hodge-filtered K-theory.

These last results are also original to the thesis, as far as we are aware, although the idea is not new to this thesis, as the possible connection between multiplicative K-theory and Hodge-filtered K-theory is briefly mentioned in [HQ22].

Overview of the thesis

Chapter 1 discusses the necessary preliminaries to reading this thesis, and we briefly discuss connection theory, complex geometry, stable homotopy theory, simplicial sets, and ∞-categories (quasi-categories).

Chapter 2 is dedicated to understanding the main approaches to differential cohomology. After investigating how the de Rham theorem can be used to "blend" differential forms with singular cocycles, we consider ordinary differential cohomology through differential characters, and develop their hexagon diagram. These can be generalized vastly, ending in the definition of differential E-cohomology for any spectrum E. Furthermore, we study differential cohomology through spectral sheaves on manifolds and use theory on recollements for stable ∞-categories to reprove the hexagon diagram in a general framework.

Chapter 3 builds on the axiomatic framework summarizing (most of) the approaches in Chapter 2, and aims at investigating central results in the field. We discuss the uniqueness of smooth extensions under technical assumptions, and quantify the failure of homotopy invariance for differential cohomology theories, resulting in a homotopy formula. Explicit descriptions of ordinary differential cohomology and differential K-theory are discussed after investigating a geometric suspension construction, and we briefly discuss a Landweber-type result.

Chapter 4 marks the transition from smooth manifolds to complex manifolds, and we consider an axiomatic framework for Hodge-filtered cohomology theories based on Deligne cohomology, Hopkins-Quick theories and differential cohomology theories. All of these can be used to produce constructions satisfying the axioms of Hodge-filtered extensions. We briefly discuss results from differential cohomology, ending in a homotopy formula and a conjecture on a Landweber-type result for Hodge-filtered cohomology theories.

Chapter 5 specializes the discussion to Hodge-filtered K-theory, establishing that multiplicative K-theory works as a geometric model for Hodge-filtered K-theory.

How to read this thesis

As readers might be aware, this thesis is quite long. This is simply because we intended the thesis to *not be too short*. We believe the field of differential cohomology is quite dispersed in the sense that there are many articles, approaches, results, and beautiful concepts, but few papers, books, or theses that aim to gather the work done in the field. This makes it especially hard for unguided students to get an overview.

A treatment of differential cohomology through the eyes of homotopical tools can be found in [ADH23]. They also cover topics we have not mentioned in this thesis, and in this thesis, we mention many topics not covered in [ADH23]. A thorough overview of any mathematical field requires many ways of viewing the field as a whole, and what we present here can be thought of as complementary to [ADH23] or similar treatments of differential cohomology. Alternatively, this thesis can be thought of as "a companion to exploring differential cohomology", where we aim to introduce sufficient motivation and alternative approaches, develop sound intuition, and elaborate on the main results in the field. Presenting a coherent story of differential cohomology has been the main goal of the thesis since the start, as the extension to Hodge-filtered cohomology and Hodge-filtered K-theory came later.

Experienced readers may skip much material found in the thesis. For example, a reader aware of the main approaches to differential cohomology may skip Chapter 2 to study the axiomatic picture and its consequences in Chapter 3, or the extensions to Hodge-filtered cohomology theories (Chapter 4) and Hodge-filtered K-theory (Chapter 5). To guide such readers, we present a summary of what we consider the most important parts of the thesis. This is simply one of many "short tables" one can make.

For differential cohomology:

1. Section 2.2 (Motivation and initial approach),
2. Definition 2.3.1 and Theorem 2.3.8 (Differential characters and their hexagon),
3. Section 2.5 (Interpretations of ordinary differential cohomology),
4. Definition 2.7.5 (Filtered differential function complexes),
5. Definition 2.8.4 (Differential E-cohomology groups),
6. Theorem 2.7.6 (Differential function complexes generalize differential characters),
7. Definition 2.11.5 (The ∞-category of differential cohomology theories),
8. Theorem 2.13.7 (Rediscovering the hexagon through stable recollements),
9. Definition 3.1.1 (Axioms of a smooth extension),
10. Theorem 3.2.9 (On uniqueness of smooth extensions),
11. Theorem 3.3.1 (The homotopy formula),
12. Remark 3.4.23 (Deligne cohomology as a homotopy pullback),
13. Theorem 3.4.24 (Deligne cohomology is ordinary differential cohomology),
14. Proposition 3.5.1 (The geometric suspension),
15. Definitions 3.6.4 and 3.6.6 (Structured bundles and differential K-theory).

For Hodge-filtered cohomology:

1. Definition 4.1.1 (The Deligne complex and Deligne cohomology),
2. Definition 4.2.2 (Hopkins-Quick construction of Hodge-filtered cohomology theories),
3. Definition 4.3.1 (Axioms of Hodge-filtered extensions),
4. Section 4.4 (Deligne cohomology, Hopkins-Quick theories, and differential cohomology produce Hodge-filtered cohomology theories),
5. Definitions 5.1.1 and 5.1.4 (Multiplicative bundles and multiplicative K-theory),
6. Theorem 5.2.3 (Multiplicative K-theory is a Hodge-filtered K-theory for $p = 0$),
7. Definition 5.2.5 (p-shifted multiplicative K-theory),
8. Corollary 5.2.6.1 (p-shifted multiplicative K-theory defines a Hodge-filtered K-theory).

To further explore the mathematical world and connect this thesis to ideas and results outside the scope of the thesis, we have used an exploratory type of remark called a Queequeg.

Queequeg 0.0.1. The term "Queequeg" was first (as far as we are aware) coined for mathematical terms by the author in [Ang21], from which we quote:

> "If the remarks are exploratory digressions that take us way outside the scope of the thesis, we will call them Queequegs. This word comes from the brilliant book "Moby Dick" by Herman Melville, where Queequeg shows up as an easy-going son of a tribal chief leaving his island society to explore and experience the world, just out of pure curiosity. Therefore, when we encounter a Queequeg, readers should be aware that these are remarks meant to open doors to further exploration, deeper connections, and perhaps even to point readers to topics way beyond the author's knowledge."

Additionally, we include a section on "other topics" in each chapter. These serve many purposes. They are meant to refer readers to important papers and concepts we have not discussed, to gather answers to questions the author asked throughout the process of writing this thesis, and to elaborate briefly on unexplored ideas. For example, natural questions such as "do we have a Künneth theorem for ordinary differential cohomology?", or "do we have a way to refine cohomology operations?" certainly deserve an answer, or at least a reference to where the answer can be found.

Furthermore, some questions, such as "can the filtration on filtered differential complexes from Section 2.6 be used to construct a spectral sequence for ordinary differential cohomology?", were not addressed in the preexisting literature (as far as we know). These questions do not belong in the main matter of the thesis, but addressing them and providing a brief discussion may be the starting point of future research.

All mistakes in the thesis can be blamed on the author, and all diagrams presented are made by the author.

Chapter 1

Preliminaries and Important Results

Although the ultimate goal would be to write this thesis in a way accessible to ambitious third-year students of topology, this has not been a priority. This thesis tries to collect ideas and survey the scattered world of differential and Hodge-filtered cohomology. Addressing all the necessary concepts, results, and detailed proofs would be a herculean task, and hence we merely focus on conveying the ideas found in each section, outsourcing much to the reader and the papers/books on display.

As one might expect, this thesis builds on a large set of preliminaries. We will throughout assume, both fairly and unfairly, that a reader has a general overview of main concepts in algebraic topology and homotopy theory. The same can be said for differential forms, smooth manifolds, and homological algebra. Knowledge about generalized cohomology theories is a must, and readers should know complex K-theory.

Much of the work in this thesis is based on either stable homotopy theory, some basic theory on quasi-categories (∞-categories), and/or sheaf theory and -cohomology. The necessary theory on sheaf cohomology will be introduced where needed. Some pages require knowledge of chromatic homotopy theory, but we place warnings throughout where necessary. For these sections, we do not require much preliminary knowledge about complex cobordism other than knowing it is an important generalized cohomology theory. Even though it will be of relevance in Chapter 2, we will not discuss sheaves of ∞-categories. We refer to [Lur09] and [Lur17], for the general theory, and we mostly refer to [BNV16] and [ADH23] for the material we present on the topic.

The above list of preliminaries is not at all exhaustive. We recommend reading this thesis with a textbook, a search engine, or a topologist nearby.

Assuming many preliminaries from potential readers might be ambitious, but this thesis is written in the spirit that *"the thesis should be accessible (and interesting) to the author, a year* *ago".* However, connection theory and complex geometry are topics the author had not formally learned before embarking on the thesis work, and hence we will allow some basic definitions and results. To fill the gap slightly, we include a section on stable homotopy theory, and a section on simplicial sets.

Note that these sections are meant only to showcase the relevant ideas, definitions, and main results. We thus omit almost all proofs and refer extensively to the relevant sources.

1.1 Connections and Chern-Weil theory

There are many ways to give a short introduction to differential geometry and connection theory. We will try to keep it simple, but we will skip many foundational concepts, to focus mostly on what we need for this thesis. This subsection is inspired by [Tu17]. Unless otherwise mentioned, M is a smooth manifold, and **Man** denotes the category of smooth manifolds. We will later require manifolds to be complex, but this assumption will be stated, or be clear from context.

Classically, differential geometry can be viewed as a generalization of calculus in several variables to smooth manifolds. If we try to understand something as simple as a directional derivative on a general smooth manifold, problems arise already when we move from smooth functions to vector fields. For example, given a direction $X_p \in T_pM$ and a smooth function f on M, we can form the directional derivative of f in direction X_p as

$$\nabla_{X_p} f = X_p f.$$

However, if we try to replace f with a vector field $Y \in \mathfrak{X}(M)$, we have a problem. In $\mathbb{R}^n$, we may pick a canonical basis $x_1, \dots x_n$ and write the vector field as $Y = \sum a^i \partial_i$, where $\partial_i = \partial/\partial x_i$. The directional derivative of Y in direction X_p is then given by

$$\nabla^{\mathbb{R}^n}_{X_p} Y = \sum (X_p a^i) \partial_i \mid_p .$$

However, the canonical choice of basis of $\mathbb{R}^n$ is vital to this formula. For an arbitrary smooth manifold M, the tangent space T_pM does not necessarily have such a canonical choice of basis. Furthermore, if we have a basis on T_pM, we should be able to globalize it to a spanning frame on M, that is, a set of vector fields $X^1, \dots, X^n \in \mathfrak{X}(M)$ such that $X^1_p, \dots, X^n_p$ forms a basis for T_pM at each $p \in M$. This leads to the following definition.

Definition 1.1.1. (Affine connections).

An affine connection on a manifold M is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$

A pair (X, Y) of vector fields is mapped to $\nabla_X Y$, which we can read as "the directional derivative of Y with respect to (the directions given by) X". We require $\nabla_X Y$ to be $C^\infty(M)$-linear in X, and to satisfy the Leibniz rule with respect to $C^\infty(M)$ in Y.

Note that an affine connection on M can equivalently be specified as a linear map

$$\mathfrak{X}(M) \to \mathrm{End}_{\mathbb{R}}(\mathfrak{X}(M)).$$

We can also construct a Lie bracket of connections, $[\nabla_X, \nabla_Y]$, by $\nabla_X\nabla_Y - \nabla_Y\nabla_X$. This again defines a new connection. As we have a Lie bracket at the level of vector fields, given by $[X,Y]_p(-) = X_p(Y(-)) - Y_p(X(-))$, we may measure how much $\nabla_{(-)}$ strays from respecting the Lie bracket. This is measured by its curvature.

Definition 1.1.2. (Curvature of an affine connection).

For an affine connection ∇, its curvature R^∇ is given by

$$\begin{aligned} \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathrm{End}_{\mathbb{R}}(\mathfrak{X}(M)) \\ (X,Y) &\mapsto R^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \end{aligned}$$

By recalling that vector fields are sections of the tangent bundle on M, that is, $\mathfrak{X}(M) = \Gamma(TM)$, we can generalize the above definitions to vector bundles. However, the second vector field does not indicate any direction, as it is merely the "function" to be differentiated. We therefore replace the second $\mathfrak{X}(M)$ with sections of the bundle in question.

Definition 1.1.3. (Connection on a smooth vector bundle).

A connection on a C^∞ vector bundle $E \to M$ is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that $\nabla_X s$ is $\mathbb{R}$-linear in the section s and C^∞-linear in the vector field X, and we require ∇_X to satisfy the Leibniz rule with respect to $C^\infty(M)$ in s. We say a section s is flat if $\nabla_X s = 0$ for all $X \in \mathfrak{X}(M)$.

By local trivializations, we can always define a connection locally. After showing convex linear combinations of connections produce connections, these local connections can be stitched together by using a partition of unity to obtain the following result.

Theorem 1.1.4. *(Smooth vector bundles always admit a connection).*

All smooth vector bundles E over M admit a connection.

As convex linear combinations of connections produce connections, we can, given two connections ∇ and ∇', consider $\gamma(t) = t\nabla + (1-t)\nabla'$ for $t \in [0,1]$. This is a path in the space $\mathrm{Conn}(E)$ of connections on E connecting ∇ and ∇', implying that the space $\mathrm{Conn}(E)$ is contractible. We call $\mathrm{Conn}(E)$ a space as it is possible to show that it forms an affine space over $\Omega^1(M; \mathrm{End}(E))$ ([Tu17]). The contractability of $\mathrm{Conn}(E)$ hints to the possibility of defining topological invariants based on the geometry found in connections. This is the purpose of the Chern-Weil homomorphism, which we will soon define.

For a given connection ∇, we can define its curvature as before.

Definition 1.1.5. (Curvature of a connection).

If ∇ is a connection on a smooth vector bundle $E \to M$, its curvature R^∇ is given by

$$\begin{aligned}\mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathrm{End}(\Gamma(E)) \\ (X, Y) &\mapsto R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.\end{aligned}$$

We often omit the superscript ∇ if the choice of connection is clear from context.

As connections and curvatures "eat" vector fields on M, they should intuitively be related to differential forms. Indeed, by working locally with respect to a local frame $e_1, \ldots e_n$, we describe ∇_X by

$$\nabla_X e_j = \sum \omega_j^i(X) e_i.$$

The 1-forms ω_j^i (defined locally) are called the connection forms of ∇. Similarly, for the curvature,

$$R^\nabla(X, Y) e_j = \sum \Omega_j^i(X, Y) e_i.$$

The 2-forms Ω_j^i are called the curvature forms of ∇.

Theorem 1.1.6. *(The second structural equation).*

These forms are related by

$$\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k.$$

This is called the *second* structural equation, because there exists a similar equation for the torsion of an affine connection, referred to as the *first* structural equation. This can be found in [Tu17].

A connection ∇ on a bundle $E_0 \oplus E_1 \to M$ induces connections ∇_i on E_i. Similarly, connections ∇_i on bundles $E_i \to M$ for $i = 0, 1$ define a connection on $E_0 \oplus E_1 \to M$ as connection forms can be defined in terms of matrices $[\omega_j^i]$. Furthermore, we can define a product connection $\nabla_0 \otimes \nabla_1$ on $E_0 \otimes E_1 \to M$ by

$$(\nabla_0 \otimes \nabla_1)_X (s \otimes t) = \nabla_{0,X}(s) \otimes t + s \otimes \nabla_{1,X}(t).$$

We will need to consider forms that take values in some vector space V, and not just in $\mathbb{R}$ (or $\mathbb{C}$). Since the vector space of smooth k-forms is given by $\Omega^k(M) = \Gamma(\wedge^k T^*M)$, and we have

$$\mathrm{Hom}_\mathbb{R}(\wedge^k TM, V) \cong (\wedge^k TM)^* \otimes V \cong (\wedge^k TM^*) \otimes V,$$

we define

$$\Omega^k(M; V) = \Gamma((\wedge^k T^*M) \otimes V).$$

A product of a V-valued form with a W-valued form can become a Z-valued form for appropriate vector spaces V, W and Z if we have a bilinear map $\mu : V \times W \to Z$. In the special case where $V = \mathfrak{g}$ is a Lie algebra, we may use the Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as the bilinear map μ.

Similarly, by considering a vector bundle E over M, we can define

$$\Omega^k(M; E) = \Gamma((\wedge^k T^*M) \otimes E)$$

to be the E-valued k-forms on M. An immediate example of bundle-valued forms are the curvature forms, which can be thought of as sections of the bundle $(\wedge^2 T^*M) \otimes End(E)$.

The curvature forms locally depend on a choice of frame $e = (e_1, \dots, e_n)$, consisting of vector fields e_i on an open U of M. Assume a is a change of basis matrix for such a frame, which can be thought of as a smooth function $a : U \to \mathrm{GL}(n, \mathbb{R})$, and assume we have fixed a connection ∇. If we consider the new frame given by $\bar{e} = ea$, the curvature forms $\Omega = [\Omega^i_j]$ change as $\bar{\Omega} = a^{-1}\Omega a$.

Considering so-called $\mathrm{Ad}\,\mathrm{GL}(n, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(n, \mathbb{R})$, which are polynomials $P(X)$ on $\mathfrak{gl}(n, \mathbb{R})$ such that

$$P(A^{-1}XA) = P(X)$$

for all $A \in \mathrm{GL}(n, \mathbb{R})$, we see that $P(\bar{\Omega}_p) = P(a_p^{-1}\Omega_p a_p) = P(\Omega_p)$. Hence, when we let p vary, we can omit any choice of local frame.

Theorem 1.1.7. *(Chern-Weil).*

Assume $E \to M$ is a smooth vector bundle of rank n, P an $\mathrm{Ad}\,\mathrm{GL}(n, \mathbb{R})$-invariant and homogeneous polynomial of degree k on $\mathfrak{gl}(n, \mathbb{R})$, and ∇ a connection on E. Then

1. *the $2k$-form $P(\Omega)$ on M is closed, and*
2. *the cohomology class $[P(\Omega)] \in H^{2k}(M)$ is independent of the connection.*

The map $\varphi : \mathrm{Inv}(\mathfrak{gl}(n, \mathbb{R})) \to H^*(M)$ given by $P \mapsto [P(\Omega)]$ is called the Chern-Weil homomorphism.

By defining P by certain formulas of traces, determinants, and Pfaffians, we can obtain the Pontryagin classes, Chern classes and Euler classes, respectively. We refer to [Tu17] for the explicit construction of these P's.

Working with $\mathrm{GL}(n, \mathbb{R})$ is a special case of a more general phenomenon. Recall that any local frame $e = (e_1, \dots, e_n)$ as above can be thought of as a local section of the frame bundle of E, $\mathrm{Fr}(E)$. The frame bundle $\mathrm{Fr}(E) \to M$ consists of all ordered bases, or frames, for the fibers of $E \to M$, and it is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle over M.

Connections of principal G-bundles in general, however, are more complicated than for vector bundles. Before we delve into this, we need to remark that the connection theory on vector bundles could have been (equivalently) formulated through a less explicit approach, using horizontal distributions. The idea is that for a bundle $\pi : E \to M$, we may consider the differential of π at a point p of E to obtain $(\pi_*)_p : T_pE \to T_{\pi(p)}M$. By the local trivializations, this map is surjective. We may therefore consider a short exact sequence

$$0 \to \mathcal{V}_p \to T_pE \to T_{\pi(p)}M \to 0.$$

$\mathcal{V}$, as defined point-wise above, is often called the *vertical subbundle of TE.*

A splitting $TM \to TE$ is equivalent to a subbundle H such that $\mathcal{V} \oplus H = TE$. Such an H is called a horizontal distribution, and we say it is linear if, for each $p \in E$, we have $((S_\lambda)_*)_p(H_p) = H_{\lambda p}$, where $S_\lambda : E \to E$ denotes scalar multiplication by λ. A choice of horizontal distribution on E can be shown to be equivalent to a choice of connection on E. We refer to [Tu17] for the details.

If we now move to principal G-bundles and consider a principal G-bundle $\pi : P \to M$, we can approach connections on P as above. Let $\mathcal{V}_p$ denote $\ker(\pi_*)_p$, and assume that we have a horizontal decomposition $T_pP \cong \mathcal{V}_p \oplus H_p$ of TP. To try to define a connection through its connection form, we can use the canonical projection $v_p : T_pP \to \mathcal{V}_p$, but to obtain a form, we need to understand $\mathcal{V}_p$ a bit better.

A central lemma in principal bundle theory asserts that $\mathcal{V}_p \cong \mathfrak{g}$, which can be shown by studying the differential of a map $j_p : G \to P$ at the identity $e \in G$. The map j_p is defined by $j_p(g) = p \cdot g$, and $((j_p)_*)_e$ can be shown to induce an isomorphism onto $\mathcal{V}_p$.

We can therefore define a $\mathfrak{g}$-valued 1-form $\omega_p : T_pP \to \mathcal{V}_p \to \mathfrak{g}$. This form is smooth, satisfies G-equivariance in the sense that $r_g^*\omega = (\operatorname{Ad} g^{-1})\omega$ for any $g \in G$, and preserves fundamental vector fields in the sense that for any vector field $X \in \mathfrak{g}$, we have $\omega_p(X^\#) = X$ for all p. The adjoints representation $\operatorname{Ad} g : \mathfrak{g} \to \mathfrak{g}$ used above is defined for a $g \in G$ by the differential $((c_g)_*)_e$, where $c_g : G \to G$ is the conjugation $c_g(x) = gxg^{-1}$. The fundamental vector field $X^\#$ of X is defined through j by setting $X^\# = ((j_p)_*)_e(X)$. Therefore, the last statement means that the composition

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "((j_p)_*)_e"] & \mathcal{V}_p \arrow[r, hook] & T_pP \arrow[r, "v"] \arrow[rr, bend right, "\omega_p"'] & \mathcal{V}_p \arrow[r, "((j_p)_*)_e^{-1}"] & \mathfrak{g}
\end{tikzcd}$$

maps $X \in \mathfrak{g}$ to X.

This gives the following definition.

Definition 1.1.8. (Connections on principal G-bundles).

A connection on a principal G-bundle $\pi : P \to M$ (also called an Ehresmann connection) is defined as a smooth $\mathfrak{g}$-valued 1-form on P that satisfies

- $r_g^*\omega = (\operatorname{Ad} g^{-1})\omega$ for all $g \in G$ (it is G-equivariant), and
- $\omega_p(X_p^\#) = X$ for all $X \in \mathfrak{g}$ (it preserves fundamental vector fields).

Defining the curvature of such a connection can be done by following Formula 1.1.6, giving the following definition of the curvature.

Definition 1.1.9. (Curvature of an Ehresmann connection).

Given a connection ω on a principal G-bundle $P \to M$,its curvature is the $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

Similarly to the case of vector bundles, we obtain a Chern-Weil homomorphism.

Theorem 1.1.10. *(Chern-Weil for principal G-bundles).*

If $\pi : P \to M$ is a principal G-bundle with connection ω, Ω its curvature form, and if f is an $\mathrm{Ad}(G)$-invariant polynomial of degree k on $\mathfrak{g}$, then the following hold:

1. *The $2k$-form $f(\Omega)$ on P is basic in the sense that there exists a $2k$-form Λ on M with $f(\Omega) = \pi^*\Lambda$,*
2. *Λ is a closed form, and*
3. *the cohomology class $[\Lambda]$ is independent of the connection.*

1.2 Complex geometry

After considering differential cohomology in Chapters 2 and 3, which focuses on smooth manifolds, we move on to Hodge-filtered cohomology theories in Chapters 4 and 5, which involves complex manifolds.

We will not need a lot of theory on complex manifolds, and we focus on the main definitions and examples. Even though we will be talking about Hodge-filtered cohomology theories, we will not need much Hodge theory. We refer interested readers to [Voi02] and [Huy05] for a discussion of Hodge theory. This subsection is mainly based on [Huy05].

Simply speaking, a complex manifold differs from a smooth manifold in the sense that we promote the smooth atlas in a smooth manifold to be a holomorphic atlas. More concretely, we assume the atlas $\{(U_i, \varphi_i)\}$ is of the form $\varphi_i : U_i \xrightarrow{\cong} \varphi_i(U_i) \subset \mathbb{C}^n$, and we require the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ to be holomorphic. Two holomorphic atlases $\{(U_i, \varphi_i)\}$ and $\{(U'_j, \varphi'_j)\}$ are equivalent if all combinations $\varphi_i \circ \varphi_j'^{-1} : \varphi'_j(U_i \cap U'_j) \to \varphi_i(U_i \cap U'_j)$ are holomorphic.

A complex n-manifold M is a smooth $2n$-manifold with an equivalence class of holomorphic atlases. It will be important to note that all complex manifolds are smooth. The category of complex manifolds is denoted $\mathbf{Man}_{\mathbb{C}}$.

Holomorphic functions on M are defined as for the smooth case, but replacing the word "smooth" with "holomorphic". Usually, such a simple change works to construct new definitions, but one should be aware that this sometimes changes the theory substantially! Maps $f : M \to \mathbb{C}$ are said to be holomorphic if for all charts (U_i, φ_i) in the equivalence class defining M, the map $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic. Replacing $\mathbb{C}$ with a complex manifold N defined by charts (U', φ'), we instead require the maps $\varphi' \circ f \circ \varphi^{-1} : \varphi(f^{-1}(U') \cap U) \to \varphi'(U')$ to be holomorphic.

Naturally, the isomorphisms in $\mathbf{Man}_{\mathbb{C}}$ are the holomorphic homeomorphisms with holomorphic inverses. We denote by $\mathcal{O}$ the sheaf of holomorphic functions. That is, for any open subset U of M, we write $\mathcal{O}_M(U) = \Gamma(U, \mathcal{O}_M) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$.

Remark 1.2.1. It turns out that we only need to require maps to be holomorphic homeomorphisms in order to be isomorphisms in $\mathbf{Man}_{\mathbb{C}}$, as the inverse will automatically be holomorphic by results from complex analysis ([Huy05]).

Indeed, one could ask how complex geometry differs from smooth geometry. Some of the rigidity of complex analysis is inherited to complex geometry. For example, it is possible to show that $\mathcal{O}(M) = \mathbb{C}$ for any compact complex manifold M. That is, any global holomorphic function on M is constant.

Furthermore, local arguments (e.g through partitions of unity) do not always work that well. This is due to the identity theorem ([Huy05]) stating that if we have holomorphic functions $f, g : U \to \mathbb{C}$, where U is a connected open subset of $\mathbb{C}$, such that $f(z) = g(z)$ for all z in some non-empty open subset of U, we have $f = g$. Any smooth vector bundle admits a connection, but the same does not hold for holomorphic connections on holomorphic vector bundles. An obstruction to the existence of a holomorphic connection is found in the Atiyah classes ([Huy05]).

Submanifolds of complex manifolds are also defined as in the smooth case. If M is a smooth $2n$-manifold and $N \subseteq M$ a smooth $2k$-submanifold, N is a complex submanifold if there is a holomorphic atlas $\{(U_i, \varphi_i)\}$ of M such that $\varphi_i : U_i \cap N \xrightarrow{\cong} \varphi_i(U_i) \cap \mathbb{C}^k$.

There are many examples of complex manifolds. Any open subset of $\mathbb{C}^n$ is a complex manifold. However, where open disks of $\mathbb{R}^n$ are diffeomorphic to $\mathbb{R}^n$, bounded open disks of $\mathbb{C}^n$ are not biholomorphic to $\mathbb{C}^n$. Despite this, since the definition of a manifold is a local definition, $\mathbb{C}^n$ is a complex manifold as well. Complex projective space $\mathbb{C}P^n$, defined as $(\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*$, is also a complex manifold. Any complex manifold isomorphic to a closed submanifold of complex projective space is called projective. These make up an important class of manifolds.

Other examples include complex tori, which can be constructed as $\mathbb{C}^n/\mathbb{Z}^{2n}$ (more generally $\mathbb{C}^n/\Gamma$ for a lattice $\Gamma \subset \mathbb{C}^n$), complex Lie groups such as $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ (where we require $(g, h) \mapsto g \cdot h^{-1}$ to be holomorphic), and hypersurfaces $f^{-1}(0)$ defined by holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$ such that $0 \in \mathbb{C}$ is a regular value. Furthermore, we can construct complex Grassmannians as, $\mathrm{Gr}_k(\mathbb{C}^n) = \{V \subseteq \mathbb{C}^n \mid \dim V = k \text{ as a subspace of the vector space } \mathbb{C}^n\}$. This is a complex manifold of dimension $k(n-k)$, as established in [Huy05].

To study the geometry of complex manifolds, we first need to understand bundles.

Definition 1.2.2. (Holomorphic vector bundles).

A holomorphic vector bundle of rank r on a complex manifold M is a complex manifold E and a holomorphic map $\pi : E \to M$ such that each fiber $\pi^{-1}(x)$ is a complex vector space of dimension r for each $x \in M$. We require the existence of an open covering $M = \bigcup U_i$, and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$. commuting with the projection onto U_i. The induced map on the fiber $\pi^{-1}(x) \to \{x\} \times \mathbb{C}^r$ needs to be $\mathbb{C}$-linear.

Maps of holomorphic bundles are bundle maps that are required to be holomorphic. Note that complex vector bundles and holomorphic vector bundles are not the same. Complex vector bundles are simply fiber bundles where the fibers are required to be vector spaces over $\mathbb{C}$. These do not involve any notion of holomorphicity.

We can still consider pullbacks of holomorphic bundles along holomorphic maps, as before.

To consider differential forms on complex manifolds, we must understand how the tangent bundles of complex manifolds work. We consider them locally. If $U \subseteq \mathbb{C}^n$ is an open subset, $x \in U$, and $z_1 = x_1 + y_1, \dots, z_n = x_n + iy_n$ are the standard coordinates of $\mathbb{C}^n$, the tangent vectors

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n},$$

form a canonical basis of T_xU. The map $I : T_xU \to T_xU$ defined by

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i} \quad \text{and} \quad \frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i},$$

has the property $I^2 = -\operatorname{Id}$. This way of mimicking the complex unit i turns out to be essential.

Definition 1.2.3. (Almost complex structures).

An almost complex structure I on a smooth manifold M is a vector bundle endomorphism $I : TM \to TM$ such that $I^2 = -\operatorname{Id}$. A smooth manifold with an almost complex structure is called an almost complex manifold.

By a local argument with the explicit maps defined above, any complex manifold is indeed an almost complex manifold.

It turns out that most of the definitions depend only on the almost complex structure. Assume therefore that M is an almost complex manifold with almost complex structure given by I. To study complex differential forms, we naively try to complexify its tangent bundle by considering $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. This leads to the following result.

Proposition 1.2.4. *(Decomposition of the complexified tangent bundle).*

The complexified tangent bundle of M decomposes as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ (resp. $T^{0,1}M$) is the eigenspace of I with eigenvalue i (resp. $-i$). We call $T^{1,0}M$ the holomorphic tangent bundle and $T^{0,1}M$ the antiholomorphic tangent bundle on M.

This motivates the following definition.

Definition 1.2.5. (Complex differential forms).

For an almost complex manifold M, we have complex vector bundles

$$\bigwedge_{\mathbb{C}}^{k} M = \bigwedge^{k}(T_{\mathbb{C}}M)^* \quad \text{and} \quad \bigwedge_{\mathbb{C}}^{p,q} M = \bigwedge^{p}(T^{0,1}M)^* \otimes_{\mathbb{C}} \bigwedge^{q}(T^{1,0}M)^*.$$

Their (sheaves of) sections are denoted $\mathcal{A}^k_{\mathbb{C}}$ and $\mathcal{A}^{p,q}$, respectively, and are called complex differential forms of degree k, or of type (p, q), respectively.

By the above decomposition, we have

$$\mathcal{A}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}.$$

It looks like $\mathcal{A}^k_{\mathbb{C}}$ is the total complex of an underlying double complex $\mathcal{A}^{*,*}$. By defining suitable differentials, this happens.

Definition 1.2.6. (The differentials ∂ and $\bar{\partial}$).

If $d : \mathcal{A}^k_{\mathbb{C}} \to \mathcal{A}^{k+1}_{\mathbb{C}}$ denotes the $\mathbb{C}$-linear extension of the exterior differential, then we may define differentials ∂ and $\bar{\partial}$ by

$$\partial := \pi^{p+1,q} \circ d = \mathcal{A}^{p,q} \to \bigoplus_{p+q=k+1} \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q},$$
$$\bar{\partial} := \pi^{p,q+1} \circ d = \mathcal{A}^{p,q} \to \bigoplus_{p+q=k+1} \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}.$$

The pullback of complex forms along holomorphic maps is well-defined and commutes with the differentials.

For $U \subseteq \mathbb{C}^n$, the bases of $T^{1,0}U$ and $T^{0,1}U$ can be defined to be

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}),$$

respectively.

If dz^i and $d\bar{z}^i$ denote the dual bases of the above, we can write differential forms in $\mathcal{A}^{p,q}(M)$ locally as a sum of expressions

$$f dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}.$$

A holomorphic form is exactly such a form where f is a holomorphic function.

If we consider holomorphic forms, we can restrict the index q notably. By the Cauchy-Riemann equations, a function $f : U \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial}{\partial \bar{z}_i} f = 0$ for all i. Hence, by a purely local argument, we can see that the holomorphic p-forms on M are the forms $\omega \in \mathcal{A}^{p,0}$ such that $\bar{\partial}\omega = 0$. The holomorphic p-forms on M are denoted by $\Omega^p(M)_{hol}$.

We use Ω^* for the smooth de Rham complex when considered on a smooth manifold, and $\mathcal{A}^*$ for the complex case (without the subscript $\mathbb{C}$). If we use $\mathcal{A}^*$, Ω^* denotes the holomorphic de Rham complex, without the subscript *hol*. This should be clear from context, and we will warn readers where necessary in order to avoid confusion.

On a complex manifold, the differentials ∂ and $\bar{\partial}$ are related to d through $d\omega = \partial\omega + \bar{\partial}\omega$. On an almost complex manifold in general, this is equivalent to the statement that $\pi^{0,2} \circ d = 0$ on $\mathcal{A}^{2,0}(M)$. However, as there exist almost complex manifolds where these do not hold (such as the 6-sphere, see [Huy05]), this is a non-trivial criterion. Any almost complex structure that satisfies one of the above is called an *integrable almost complex structure*.

Theorem 1.2.7. *(The Newlander-Nirenberg theorem).*

Any integrable almost complex structure is induced by a complex structure.

If we have an almost complex structure with $\bar{\partial}^2 = 0$, the almost complex structure is integrable.

By allowing for more structure (e.g. a Riemannian metric) on a complex manifold M (with canonically induced almost complex structure I), we run into interesting classes of manifolds.

Definition 1.2.8. (Hermitian structures and fundamental forms).

Assume g is a Riemannian metric on M. g is called an hermitian structure on M if for each $x \in M$, the inner product g_x on T_xM is compatible with the almost complex structure I_x in the sense that for all $v, w \in T_xM$, $g_x(v,w) = g_x(I_x(v), I_x(w))$.

We may obtain a canonical $(1,1)$-form by defining $\omega = g(I(-), -)$. This is called the fundamental form of the hermitian structure. A complex manifold with an hermitian structure is called an hermitian manifold.

Locally, the fundamental form can be written as $\omega = \frac{i}{2}\sum_{i,j=1}^n h_{ij} dz^i \wedge \bar{z}^j$, for some h_{ij} that defines a positive definite hermitian matrix when evaluated at any point x.

This leads to the notion of a Kähler manifold.

Definition 1.2.9. (Kähler structures).

A Kähler structure on a complex manifold M is an hermitian structure g such that the induced fundamental form ω is closed. We also call g a Kähler metric, and ω is often referred to as the Kähler form. A complex manifold with a Kähler structure is called a Kähler manifold.

By restricting a Kähler metric g on a Kähler manifold M to a complex submanifold N, we obtain a Kähler structure on N, as well.

Remark 1.2.10. It is possible to show that Kähler manifolds also possess canonical symplectic structures. We won't delve into this, although we remark that Kähler manifolds have rick geometries. Details can be found in [Huy05].

As mentioned, we will not focus on the theoretical aspects of complex geometry here. We should have covered a bunch of interesting operators and decompositions in order to present a thorough treatment of complex geometry, but we omit this and refer to [Huy05], as we will soon have covered everything we need for this thesis. We give some examples.

Simple examples of Kähler manifolds include the unit disc of $\mathbb{C}^n$, $\mathbb{C}^n$ itself, any complex torus $\mathbb{C}^n/\Gamma$ for a lattice $\Gamma \subset \mathbb{C}^n$, and any complex curve. We refer to [Huy05] for constructions of their Kähler metrics. The complex projective spaces are also Kähler manifolds, as they can be endowed with Kähler metrics called the Fubini-Study metrics. Hence, any projective manifold is Kähler, as these are isomorphic to submanifolds of suitable complex projective spaces. A topological obstruction to the converse, whether or not a Kähler manifold is projective, is found in [Huy05] under the section on the Kodaira embedding theorem.

Kähler manifolds are surprisingly topological as well. For example, compact Kähler manifolds are formal, in the sense that for a compact Kähler manifold M, $(\mathcal{A}^*(M), d)$ is a formal differential graded algebra. It is equivalent (as a differential graded algebra) to its cohomology $(H^*(A^*(M), d), 0)$.

1.3 Stable homotopy theory

To account for the homotopical concepts of this thesis, we need stable homotopy theory. We briefly mention some important results and refer to [BR20] for proofs and details.

The idea of stability in topology can best be seen through the Freudenthal suspension theorem.

Theorem 1.3.1. *(Freudenthal suspension theorem).*

For a natural number k and a k-connected topological space X with a non-degenerate basepoint, the map

$$\pi_n(X) = [S^n, X] \xrightarrow{\Sigma} [\Sigma S^n, \Sigma X] \cong \pi_{n+1}(X),$$
$$[f] \mapsto [\Sigma f],$$

is an isomorphism for $n < 2k + 1$ and a surjection if $2k + 1$ (i.e. it is a $(2k + 1)$-equivalence).

As the suspension raises the connectivity of the space by 1, the homotopy groups of $\Sigma^n X$ will stabilize when we take the limit $n \to \infty$. This allows us to define the stable homotopy groups.

Definition 1.3.2. (Stable homotopy groups of X).

Given a pointed CW-complex X, the n'th stable homotopy group of X is

$$\pi_n^S(X) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X) = \pi_{2n+2}(\Sigma^{n+2} X).$$

This stable behavior does not only arise at the level of homotopy groups. The same result happens for the suspension functor itself. Recall that the suspension $\Sigma = S^1 \wedge -$ has a right adjoint, given by forming the loop space Ω. If X is a k-connected space, then the unit $\eta_X : X \to \Omega\Sigma X$ of this adjunction is a $(2k+1)$-equivalence, and the unit $\varepsilon_X : \Sigma\Omega X \to X$ is a $2k$-equivalence. Hence, the stability above essentially amounts to inverting the suspension Σ. If we set $QX = \operatorname{hocolim}_n \Omega^n \Sigma^n X$, the stable homotopy groups of X are the same as the homotopy groups of QX.

There have been many attempts at defining a "stable homotopy category" over the years. Many have failed. We won't delve into the attempts on this as our main motivation for studying stable homotopy theory is to study cohomology theories. However, as we will see, this actually results in constructing the stable homotopy category.

A fundamental result on cohomology theories is the Brown representability theorem, from which the following is a special case.

Theorem 1.3.3. *(Brown representability).*

If $\tilde{E}^*$ is a reduced, generalized cohomology theory, we can find a sequence of connected, pointed CW-complexes E_n, unique up to homotopy equivalence, such that for each connected, pointed CW-complex X,

$$\tilde{E}^n(X) \cong [X, E_n].$$

For any such X, we have

$$[X, E_n] \cong \tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}].$$

The above chain of isomorphisms is natural in X. By setting $X = E_n$, we see that the identity map in $[E_n, E_n]$ corresponds to a map $E_n \to \Omega E_{n+1}$. This can be done for each n. The maps $s_n^E : E_n \to \Omega E_{n+1}$ (or by adjunction $\Sigma E_n \to E_{n+1}$) are called structure maps, and they are weak homotopy equivalences (as can be seen by setting $X = S^k$ for each k). A map f in $[X, E_n]$ is simply mapped to $s_n^E \circ f$ in $[X, \Omega E_{n+1}]$ under the isomorphisms above.

Definition 1.3.4. (Sequential spectra).

A sequential spectrum E is a sequence of (pointed) spaces E_n with structure maps $s_n^E : \Sigma E_n \to E_{n+1}$. We say E is an Ω-spectrum if the adjoint structure maps $E_n \to \Omega E_{n+1}$ are weak homotopy equivalences.

By CW-approximation, we can always assume the spaces E_n are CW-complexes.

Maps of sequential spectra E, F are defined to be sequences of maps $E_n \to F_n$ that commute with the structure maps.

Brown representability ensures that sequential spectra represent (reduced) cohomology theories. The converse also holds. By defining $\tilde{E}^n(-)$ as $[-, E_n]$, we obtain a reduced cohomology theory.

Intuitively, we already know some spectra. The sphere spectrum $\mathbb{S}$ is defined by $\mathbb{S}_n = S^n$. If we are given a pointed topological space K, we can form its suspension spectrum, $\Sigma^\infty K$ given by $(\Sigma^\infty K)_n = \Sigma^n K$.

For an abelian group G, we can combine Eilenberg-MacLane spaces to form the Eilenberg-MacLane spectrum HG, where $HG_n = K(G, n)$ is the n'th Eilenberg-MacLane space of G. The structure maps are induced as earlier, by the fact that Eilenberg-MacLane spaces represent ordinary cohomology with coefficients in G.

Complex K-theory also has a spectrum, which often is called KU. We will, for simplicity, often just write K. Due to Bott periodicity, the form of K_n only depends on whether n is even or odd. We can consider the unitary group $U(n)$ as a subgroup of $U(n+1)$. By forming the colimit, we obtain the infinite unitary group $U = \operatorname{colim}_n U(n)$. The K-theory spectrum is given by U if n is odd, and $BU \times \mathbb{Z}$ if n is even.

We want to study sequential spectra up to homotopy. That is, we need to take the homotopy category of the category of sequential spectra. In a model-categorical fashion, we define the cylinder spectrum of a given spectrum E to be $\mathrm{Cyl}(E)_n = E_n \times [0,1]/\sim$, where $\sim$ relates (x_0, t) for all $t \in [0,1]$, and x_0 denotes the basepoint of E_n.

Definition 1.3.5. (Homotopies of maps of sequential spectra).

Let E and F be sequential spectra. If we let $i_0, i_1 : E \to \mathrm{Cyl}(E)$ denote the maps including each E_n to the cylinder at $t = 0$ and $t = 1$, respectively, we say that maps $f, g : E \to F$ are homotopic if there exists a map $h : \mathrm{Cyl}(E) \to F$ such that

$$\begin{tikzcd}
E \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(E) \arrow[d, "h"] & E \arrow[l, "i_1"'] \arrow[dl, "g"] \\
& F &
\end{tikzcd}$$

commutes.

Definition 1.3.6. (The stable homotopy category).

The stable homotopy category is the homotopy category of sequential spectra with respect to the notion of homotopies defined above. That is, the objects are sequential spectra, and the morphisms are homotopy classes of maps of sequential spectra. We will refer to objects of the stable homotopy category as *spectra*, and the category is denoted **SHC**.

As expected from a homotopy category, it is indeed triangulated. Strictly speaking, a thorough understanding of the stable homotopy category requires more machinery. We refer to [BR20], as our goals are merely to present the fundamentals.

It turns out that the suspension-loop adjunction is promoted to a Quillen equivalence when passing to sequential spectra. We have not covered the stable model structure that allows for this, nor do we plan to, but the important thing to note is that in the stable homotopy category, the loop and suspension functors are inverse equivalences. Hence, we have achieved stability.

However, we are not done. Some cohomology theories are multiplicative, but by now, we have not turned the stable homotopy category into a symmetric monoidal category. That is, we have no useful product of spectra that can define ring objects in the stable homotopy category. The solution is to consider other models of spectra than sequential spectra, i.e. other stable model categories $\mathcal{C}$ such that $ho\mathcal{C}$ is equivalent to the stable homotopy category.

Luckily, the following theorem by Schwede ensures us that working with other models gives us a coherent theory.

Theorem 1.3.7. *(Rigidity of spectra).*

If $\mathcal{C}$ is a stable model category such that its homotopy category is equivalent to the stable homotopy category, then $\mathcal{C}$ is Quillen equivalent to sequential spectra.

For our purposes, we will only use the symmetric spectra Sp^{Σ}. A symmetric spectrum is a sequence E_n of pointed topological spaces with structure maps $s_n^E : S^1 \wedge E_n \to E_{n+1}$, where E_n has a continuous action from the symmetric group Σ_n that fixes the basepoint, and the composition of structure maps

$$S^k \wedge E_n \to S^{k-1} \wedge E_{n+1} \to \ldots \to E_{n+k}$$

is compatible with the action of $\Sigma_k \times \Sigma_n$ on the domain and target. The action on S^k is given by viewing S^k as the one-point compactification of $\mathbb{R}^k$, and letting Σ_k act by permuting its coordinates. Given symmetric spectra E and F, a map of symmetric spectra is a collection of Σ_n-invariant maps $E_n \to F_n$ commuting with the structure maps.

The reason this gives us a symmetric product of spectra is quite complicated. It is based on re-writing the above definition into appropriate functors, which can be Day-convoluted into forming an abstract definition of the smash product of symmetric spectra, with the sphere spectrum $\mathbb{S}$ as the monoidal unit. Explaining this properly would require much more machinery than what this thesis aims to cover, and we thus refer to [BR20] for the details.

We end by defining the homotopy groups of a spectrum and how to consider (co-)homology theories on spectra.

Definition 1.3.8. (Homotopy groups of spectra).

Given a spectrum E, we define its homotopy groups to be

$$\pi_n(E) = \operatorname{colim}_k \pi_{n+k}(E_k).$$

A map $f : E \to F$ is a π_*-isomorphism if $\pi_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

Note that the homotopy groups of the sphere spectrum coincides with the stable homotopy groups of the spheres. Similarly to how a spectrum E defines a cohomology theory $\tilde{E}^n(-)$ by $[-, E_n]$, we can define a reduced homology theory (on spectra) by

$$E_n(F) = \pi_n(E \wedge F).$$

For a space X, we this can be done by considering the homotopy groups of the spectrum $E \wedge \Sigma^\infty X$ given by $(E \wedge \Sigma^\infty X)_k = E_k \wedge X$.

This short survey is not at all enough to display the depth, beauty, and power of stable homotopy theory. Luckily, this was not our goal. We advise readers to consult [BR20] if more homotopy theory is needed.

1.4 Simplicial sets and quasi-categories

Although we will assume basic knowledge about simplicial sets and quasi-categories, we allow for a short summary to establish important results, such as the Dold-Kan correspondence. For this short treatise, we follow [GJ99] and [Lan21].

Recall that a simplicial set is a contravariant functor $X : \Delta^{op} \to \mathbf{Set}$, where Δ denotes the full subcategory of the category of posets with objects $[n]$ for all $n \geq 0$. Here $[n] = \{0, 1, \ldots, n\}$ denotes the linearly ordered set $0 \leq 1 \leq \ldots \leq n$. Maps of simplicial sets are natural transformations.

We certainly know some examples already. Let Δ^n be the simplicial set represented by $[n]$. We may also, for a space X, consider the singular simplicial set $\mathrm{sing}(X)$ defined by mapping $[n]$ to $\mathrm{sing}_n(X) = \mathrm{Hom}(\Delta^n_{Top}, X)$. For simplicial sets X, Y, we can define the function complex $\mathrm{Hom}(X, Y)$ by $\mathrm{Hom}(X, Y)_n = \mathrm{Hom}(X \times \Delta^n, Y)$.

The geometric realization $| - | : \mathbf{sSet} \to \mathbf{Top}$ can be defined as the unique colimit-preserving functor that sends Δ^n to the topological n-simplex Δ^n_{Top}. It can explicitly be constructed for a simplicial set X as

$$|X| = (\coprod_n X_n \times \Delta^n_{Top})/\sim,$$

where the relation $\sim$ relates $(f^*(x), t)$ to $(x, f_*(t))$ for $f : [m] \to [n], x \in X_n$ and $t \in \Delta^m_{Top}$. The geometric realization preserves colimits as it is a left-adjoint. Its right adjoint is the singular complex functor described above.

A simplicial abelian group is a simplicial object in $\mathbf{Ab}$, which means it is contravariant functor $\Delta^{op} \to \mathbf{Ab}$. By replacing $\mathbf{Ab}$ with any other category C, we can define simplicial objects of C, and a corresponding theory of simplicial objects. The geometric realizations of abstract simplicial objects can be defined, but they are complicated and requires us to develop ends and coends. We refer to [GJ99].

If A is a simplicial abelian group, we can define its normalized chain complex NA by letting $NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n$, where d_i are the face maps induced by the maps $\delta^i : [n-1] \to [n]$ that does not hit $i \in [n]$. We define the differential $NA_n \to NA_{n-1}$ to be $(-1)^n d_n$, which makes NA into a complex by the simplicial identities. This lands in $\mathrm{Ch}_{\geq 0}(\mathbf{Ab})$. In fact, if we start with a $C \in \mathrm{Ch}_{\geq 0}(\mathbf{Ab})$, it is possible to construct a simplicial abelian group $\Gamma(C)$ (see [GJ99]).

Theorem 1.4.1. *(The Dold-Kan correspondence).*

The functors $N : \mathbf{sAb} \longleftrightarrow \mathrm{Ch}_{\geq 0}(\mathbf{Ab}) : \Gamma$ *define inverse equivalences of categories.*

For our purposes, the explicit construction of Γ is not needed, as we only need the correspondence.

There are many interesting types of simplicial sets. Some of the most important are defined by certain extension properties.

Definition 1.4.2. (S-horns).

For a subset $S \subset [n]$, we define the S-horn $\Lambda^n_S \subseteq \Delta^n$ to consist of the k-simplices $f : [k] \to [n]$ that are not surjective on the complement of S. That is, we can find an $i \in [n]\backslash S$ such that the image of f does not contain i. We call $\Lambda^n_{\{j\}}$ (often just written Λ^n_j) a left (resp. right) outer horn if $j = 0$ (resp. $j = n$). If $1 \leq j \leq n-1$, we call Λ^n_j an inner horn.

Definition 1.4.3. (Kan complexes).

A simplicial set X is a Kan complex if it has the extension property with respect to all horn inclusions $\Lambda^n_j \to \Delta^n$, for $0 \leq j \leq n$. That is, the lifting problem

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

given by the solid part of the diagram has a solution (given by the dashed arrow).

The singular complex functor defines a Kan complex as we can use the adjunction with geometric realization to retract Δ^n_{top} to a horn. Any simplical abelian group can be shown to form a Kan complex (this is non-trivial, and is shown in [GJ99])! The latter will be used later, as some statements can be shown to hold only for Kan complexes.

Definition 1.4.4. (Homotopies in simplicial sets).

If $f, g : X \to Y$ are maps of simplicial sets, we say f is homotopic to g with homotopy h if we can form a commutative diagram

$$\begin{tikzcd} X \times \Delta^0 \arrow[r, "\mathrm{Id}\times d^1"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "h"] & X \times \Delta^0 \arrow[l, "\mathrm{Id}\times d^0"'] \arrow[dl, "g"] \\ & Y. & \end{tikzcd}$$

If X is a Kan complex with a designated basepoint (0-simplex) x_0, we may form its simplicial homotopy groups $\pi_n^{\Delta}(X, x_0)$ to be homotopy classes of maps $[(\Delta^n, \partial\Delta^n), (X, x_0)]$. The boundary $\partial\Delta^n$ is defined as the subsimplicial set of Δ^n where the k-simplices are non-surjective maps $[k] \to [n]$.

For a simplicial abelian group A, we have $\pi_n(A) = H_i(N(A))$.

The theory of simplicial sets may look a lot like the theory of categories, where we can view the objects of a category as 0-simplices and morphisms as 1-simplices. This can be made formal by the nerve functor. If C is a category, define its nerve to be the simplicial set $N(C) : \Delta^{op} \to \mathbf{Set}$ given by $[n] \mapsto \mathrm{Fun}([n], C)$.

We immediately note that Δ^n is the nerve of $[n]$. A more exciting example comes from considering a group G as a groupoid with one object. The nerve $N(G)$ is a simplicial set whose geometric realization is BG, the classifying space of G. One can show that the nerve of a category C is a Kan complex if and only if C is a groupoid ([Lan21]), and hence the classifying simplicial set $N(G)$ is a Kan complex.

It is possible to "unnerve" a simplicial set by forming its homotopy category. Let us make this precise.

If X is a simplicial set, we construct its homotopy category hX as follows: Let the objects of hX be given by X_0 and morphisms generated by X_1 in the sense that a 1-simplex $\Delta^1 \to X$ can be thought of as a morphism from $d^1(f)$ to $d^0(f)$. We say that the 1-simplex $s_0(x)$ for an object x is the identity on x, and for any 2-simplex $\Delta^2 \to X$ with boundary given by a triple (f, g, h), we require that $h = g * f$, where the right-hand side denotes the formal composition of g and f. To make this coherent, we say that 1-simplices f and g (that go from x to y) are equivalent, denoted $f \sim g$, if we can find a 2-simplex $\sigma : \Delta^2 \to X$ such that

$$\sigma \mid_{\Delta^{0,1}} = f, \quad \sigma \mid_{\Delta^{1,2}} = \mathrm{Id}_y, \quad \sigma \mid_{\Delta^{0,2}} = g.$$

This relation is not an equivalence relation. We require that if $f \sim f'$, then $f * g \sim f' * g$ and $g' * f \sim g' * f'$.

The above construction defines the homotopy category of a simplicial set, which yields a functor $h : \mathbf{sSet} \to \mathbf{Cat}$. Note that $hN(C) \cong C$. However, we do not have $Nh(X) \cong X$. We should not have this either, as forgetting the higher simplices does not vouch for a reconstruction of the simplicial set in its entire complexity. However, the nerve functor N is right adjoint to h.

By the left and right (outer) horn liftings in Kan complexes, any morphism in the homotopy category of a Kan complex is invertible, making the Kan complexes "the groupoids" of simplicial sets. By not requiring the lifting of outer horns, we form special simplicial sets that we call quasi-categories, as they behave more like categories, although with "higher" information contained in their higher n-simplices.

Definition 1.4.5. (Quasi-categories).

A simplicial set X is a quasi-category if it has the extension property with respect to all inner horn inclusions $\Lambda^n_j \to \Delta^n$. We call quasi-categories ∞-categories, and we say that a morphism in an ∞-category is an equivalence if its image in the homotopy category is an isomorphism. An ∞-category is an ∞-groupoid if every morphism is an equivalence.

The above leads us to the general "philosophy" of ∞-categories. An (n, k)-category can be thought of as a category where we need to specify n "levels" of morphisms between morphisms. That is, the 0-morphisms are the objects, 1-morphisms are the morphisms between the objects (0-morphisms), and inductively, the m-morphisms are the morphisms between the $(m-1)$-morphisms, up to $m = n$. The k states that all of morphisms of "level" greater than k are invertible (equivalences). Groupoids are $(1, 0)$-categories as we only specify the objects and the morphisms, but all morphisms between objects are invertible. Ordinary categories are $(1, 1)$-categories, often just called 1-categories. Letting n be ∞, we obtain (∞, k)-categories. In this fashion, ∞-groupoids are $(\infty, 0)$-categories, and quasi-categories are $(\infty, 1)$-categories.

As (briefly) argued above, Kan complexes are ∞-groupoids. The converse also holds, but it is not as simple to prove. One can also consider the maximal subgroupoid $\mathcal{C}^{\simeq}$ of an ∞-category $\mathcal{C}$ by picking out the isomorphisms in its homotopy category and forming the pullback (of simplicial sets)

$$\begin{tikzcd}
\mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
N(h\mathcal{C}^{\simeq}) \arrow[r] & N(h\mathcal{C}).
\end{tikzcd}$$

The nerve can be enhanced to a "homotopy coherent nerve". If we let J be a finite, non-empty, linearly ordered sets containing elements i, j, we can form $P_{i,j}$ to be

$$P_{i,j} = \{I \subseteq J : i, j \in I \text{ and } i \leq k \leq j \text{ for all } k \in I\}.$$

Using these, we may form a simplicially enriched category $\mathfrak{C}[\Delta^J]$, whose objects are elements of J and morphisms $i \to j$ are given by $\emptyset$ if $j < i$, and $N(P_{i,j})$ for $i \leq j$. It is possible to show that $N(P_{0,n})$ is isomorphic as a simplicial set to $(\Delta^1)^{n-1}$, and that $P_{i,j} \cong P_{0,j-i}$. This implies that $N(P_{i,j})$ is isomorphic to $(\Delta^1)^{j-i-1}$. These can be shown to be contractible, in the sense that their identity maps are homotopic to a constant map.

We form the homotopy-coherent nerve of a simplicially enriched category $\mathcal{C}$ by mapping $[n]$ to $N(\mathcal{C})_n = \mathrm{Hom}(\mathfrak{C}[\Delta^n], \mathcal{C})$. The 0-simplices of $N(\mathcal{C})$ is given by the objects of $\mathcal{C}$ and the 1-simplices are given by the 1-simplices of $\mathcal{C}$. This is not new. Essentially, what's new compared to the "old" nerve functor, is that 2-simplices informally consists of two composable morphisms $X \to Y$ and $Y \to Z$ for objects X, Y, Z associated to $0, 1$ and 2, respectively. There is also a morphism $X \to Z$ and a homotopy between this morphism and the composition $X \to Y \to Z$. This nerve construction is "coherent" with the relation of composing maps (up to homotopy). One can show that if $\mathcal{C}$ is enriched over Kan complexes, its homotopy-coherent nerve is an ∞-category.

We can define the ∞-category of spaces, denoted **Spc**, by considering the simplicially enriched category that has CW-complexes as objects and hom-simplicial sets given by considering the singular simplicial set on mapping spaces. The homotopy-coherent nerve of this simplicial category is an ∞-category, as it is enriched in Kan complexes, and it is called the ∞-category of spaces.

It is possible to show that for simplicial sets X, Y, if Y is an ∞-category (resp. a Kan complex), the hom-simplicial set $\mathrm{Hom}(X, Y)$ becomes an ∞-category (resp. a Kan complex). This allows us to form a ∞-category of functors between two given ∞-categories. Furthermore, this implies that Kan complexes themselves are enriched over Kan complexes, and hence defines an ∞-category through the homotopy-coherent nerve. This gives an equivalent definition of the ∞-category of spaces.

Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we can form $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$, which we think of as the functors between the objects $\mathcal{C}$ and $\mathcal{D}$ in a suitable ∞-category of ∞-categories. However, if we want to consider $\mathcal{C}$ to be some higher notion of a category, we need to have a mapping space $\mathrm{map}_{\mathcal{C}}(x, y)$ for objects (0-simplices) $x, y \in \mathcal{C}$. Often, when thinking of quasi-categories as higher categories, we write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the Hom-simplicial set. We define $\mathrm{map}_{\mathcal{C}}(x, y)$ by the pullback

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "(\mathrm{dom},\mathrm{codom})"] \\
* \arrow[r, "(x,y)"] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

This is not just an ∞-category, it is an ∞-groupoid, which aligns quite well with the philosophy of ∞-categories, as (∞, k)-categories should be enriched over $(\infty, k-1)$-categories.

Although we could have formulated the theory of adjunctions through bicartesian fibrations, we only need to know that the 1-categorical ways of thinking about adjoints still hold. We may consider $f : \mathcal{C} \to \mathcal{D}$ to be left adjoint to $g : \mathcal{D} \to \mathcal{C}$ if and only if we get an equivalence of mapping spaces similar to the 1-categorical theory. This is again equivalent to considering suitable unit and counit functors satisfying familiar triangle identities.

By constructing a suitable model structure on the category of simplicial sets, called the Joyal model structure, one can show that all simplicial sets are weakly equivalent to quasi-categories by the generalized Whitehead theorem. The notion of weak equivalence used above is the notion a Joyal equivalence, where we say $f : X \to Y$ is a Joyal equivalence if it is an inner fibration (that is, it satisfies the right lifting property with respect to inner horn inclusions), and it has the right lifting property with respect to the map $\Delta^0 \to J$, where J is the nerve of the category with two objects x and x' and a unique isomorphism $x \to x'$. J is called the walking equivalence.

An important construction we will need is the stable ∞-category of spectra, for which an in-depth treatment can be found in [Lur17]. Recall that a pointed ∞-category $\mathcal{C}$ is an ∞-category $\mathcal{C}$ with a 0-object, and that a 0-object is an object 0 (that will be unique up to equivalence) such that $\mathrm{map}_{\mathcal{C}}(0, X)$ and $\mathrm{map}_{\mathcal{C}}(X, 0)$ are contractible for all X. We may define a (co-)fiber sequence to be a pullback (pushout)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z, \end{tikzcd}$$

often just written

$$X \to Y \to Z.$$

To account for stability, we need fiber and cofiber sequences to be the same. This leads to the definition of a stable ∞-category.

Definition 1.4.6. (Stable ∞-categories).

A stable ∞-category is a pointed ∞-category $\mathcal{C}$ such that all morphisms have a fiber and a cofiber, and a sequence $X \to Y \to Z$ is a fiber sequence if and only if it is a cofiber sequence.

Homotopy categories of stable ∞-categories are triangulated, with triangles induced by the fiber/cofiber sequences.

To move toward spectrum objects and the stable homotopy category of spectra, we define the loop functor as a pullback and the suspension functor as a pushout, by

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X. \end{tikzcd}$$

The most powerful way of defining spectrum objects are through reduced and excisive functors, but this does not easily translate to sequential spectra. However, we can approach these more intuitively through prespectrum objects. If $\mathcal{C}$ is an ∞-category, a prespectrum object is a functor $E : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ such that $E(i, j) = 0$ when $i \neq j$. The full subcategory of $\mathrm{Fun}(N(\mathbb{Z} \times \mathbb{Z}), \mathcal{C})$ spanned by the prespectrum objects of $\mathcal{C}$ is denoted $\mathrm{PSp}(\mathcal{C})$. Evaluation at (n, n) gives a functor $\mathrm{PSp}(\mathcal{C}) \to \mathcal{C}$ called the n'th space functor. We denote $E_n = E(n, n)$. This essentially gives a diagram

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & E_n \arrow[ur] \arrow[dr] & & E_{n+1} \arrow[ur] \arrow[dr] & & \cdots \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] &
\end{tikzcd}$$

The above does not involve any requirement about the commuting squares being pullbacks or pushouts. We can include the pushout defining the suspension functor in this diagram to obtain

$$\begin{tikzcd}
 & 0 \arrow[dr] \arrow[drr, bend left] & & & 0 \arrow[dr] \arrow[drr, bend left] & & & 0 \arrow[dr] \arrow[drr, bend left] & & \\
\cdots \arrow[ur] \arrow[dr] & & \Sigma E_{n-1} \arrow[r, dashed] & E_n \arrow[ur] \arrow[dr] & & \Sigma E_n \arrow[r, dashed] & E_{n+1} \arrow[ur] \arrow[dr] & & \Sigma E_{n+1} \arrow[r, dashed] & \cdots \\
 & 0 \arrow[ur] \arrow[urr, bend right] & & & 0 \arrow[ur] \arrow[urr, bend right] & & & 0 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

The same can be done for the loop functor. We obtain structure maps $\Sigma E_n \to E_{n+1}$ and $E_n \to \Omega E_{n+1}$.

If $\mathcal{C}$ is a pointed ∞-category, we say that a prespectrum object in $\mathcal{C}$ is a spectrum object if the structure map $E_n \to \Omega E_{n+1}$ is an equivalence for all n. The full subcategory of spectrum objects in $\mathcal{C}$ is denoted $\mathrm{Sp}(\mathcal{C})$. This is also called the stabilization, $\mathrm{Stab}(\mathcal{C})$, of $\mathcal{C}$, as it is possible to show it defines a stable ∞-category.

Definition 1.4.7. (Stable ∞-category of spectra).

The stable ∞-category of spectra, denoted **Spt**, is defined as the spectrum objects of the ∞-category of spaces. That is,

$$\mathbf{Spt} = \mathrm{Sp}(\mathbf{Spc}).$$

An essential feature of **Spt** is that

$$h\mathbf{Spt} = \mathbf{SHC}.$$

1.5 Homotopy pullbacks

We should also address homotopy pullbacks at some point in the preliminaries. Although a general theory of homotopy limits and colimits may be formulated, we will be brief and cover the case of homotopy pullbacks. Based on [BR20, Appendix A.7], we give a short definition of homotopy pullbacks for an arbitrary model category $\mathcal{C}$. There is a classical example concerning simple topological spaces showing that the theory of limits and colimits do not respect homotopy equivalences. motivating the need for homotopy limits and colimits.

Consider the two diagrams

$$\begin{tikzcd}
D^n \arrow[d, dotted] & S^{n-1} \arrow[l] \arrow[r] \arrow[d, dotted] & D^n \arrow[d, dotted] \\
* & S^{n-1} \arrow[l] \arrow[r] & *.
\end{tikzcd}$$

The pushout of the top diagram yields the n-sphere S^n as a pushout, and pushout of the bottom diagram yields the point. Although the vertical maps are homotopy equivalences, the pushouts S^n and $*$ are not homotopy equivalent. As explained in [DS95], any "seasoned" topologist will argue that the top diagram yields the correct homotopy type, as the maps in the diagram are cofibrations. This idea can be formalized, and it gives the correct notion of a homotopy pushout. We work dually, and give the definition of a homotopy pullback.

Let I be the diagram category $I = (2 \to 1 \leftarrow 3)$, and let $\mathcal{C}$ be any model category.

Lemma 1.5.1. *(Injective model structure).*

There is a model structure on the category $\mathcal{C}^I$ of I-diagrams in $\mathcal{C}$, called the injective model structure, such that a morphism $f : X \to Y$ in $\mathcal{C}^I$ is a weak equivalence (resp. cofibration) if and only if $f_i : X_i \to Y_i$ is a weak equivalence (resp. cofibration) in $\mathcal{C}$ for all $i \in I$.

A similar model category can be found for more general index categories I, given a reasonable simplicity requirement on I. We refer to [BR20, Appendix A.7] and the references therein.

Definition 1.5.2. (Homotopy pullback).

For an object $X \in \mathcal{C}^I$, we define the *homotopy pullback* $\operatorname{holim} X$ of X to be the limit of a fibrant replacement of X in the model structure mentioned above.

Remark 1.5.3. The above definition defines a homotopy limit if we simply allow for more general index categories I. With our choice of I, the homotopy limit is called a homotopy pullback. There is a comparison morphism $\lim X \to \operatorname{holim} X$, although this is not a weak equivalence in general.

Remark 1.5.4. The homotopy pullbacks respect weak equivalences in the sense that if we are given $X, Y \in \mathcal{C}^I$ and a vertex-wise weak equivalence $F : X \to Y$ of diagrams, there exists a weak equivalence $\operatorname{holim} X \to \operatorname{holim} Y$.

Remark 1.5.5. Theorem A.7.17. in [BR20] gives a way of checking whether or not a diagram is fibrant in $\mathcal{C}^I$.

For our purposes, we will consider homotopy pullbacks mainly as an underlying technicality ensuring the constructions we mention are coherent and respect weak equivalences. We will therefore rarely delve into the model categorical technicalities regarding the homotopy pullbacks we consider. For more theory on homotopy pullbacks, we refer to [BR20, Appendix A.7.], [DS95], and [Mat76]. For homotopy pullbacks in ∞-categories, the canonical refrence is [Lur17].

Chapter 2

Approaching Differential Cohomology

2.1 Historical motivation: Geometry interacts with Topology

The conceptual motivation for differential cohomology is quite simple. Topology, with its stretchable notions of homotopies and homotopy invariant constructions, often neglects geometrical notions, such as curvature, angles, length, and so on. For not-so-structured compactly generated spaces[1], it is fine to trade away geometric information for homotopic information, but if we restrict ourselves to more structured spaces, such as smooth manifolds, valuable information may be lost in the homotopical processes.

A running example will be the 2-sphere of radius r. Considered as a geometric object, focusing on lengths and curvature, it is well-known that its principal curvatures are both given by $\kappa_1 = \kappa_2 = 1/r$. This yields Gaussian curvature $\kappa = \kappa_1\kappa_2 = 1/r^2$. If we deform the sphere to an ellipsoid, we obtain the same object up to homotopy, but the curvature has changed. The shape now has a flatter region and a more curved region depending on the "direction of stretching".

This example tells us that if we allow the topological mantra of working up to homotopy equivalence, we may change the geometry of the surface in question. If we want to study homotopy invariants, such as cohomology theories, we can try to amend the above problem by including differential information, such as curvature forms and connections. A natural slogan to keep in mind may therefore be:

"Don't be a fool and neglect geometry,
respect smooth manifolds by using differential cohomology!"

[1]The category of compactly generated (weak Hausdorff) spaces is the subcategory of topological spaces we use in homotopy theory, as the additional structure defines an adjunction between $-\times Y$ and $(-)^{(Y)}$ and a homeomorphism $Z^{(X\times Y)} \cong (Z^Y)^X$ of mapping spaces (with the compact-open topology).

Before embarking on this journey, we need to clarify why we expect there to be clear connections between geometry and topology. We do this in two ways, first through established theorems from differential geometry, and then using de Rham cohomology.

Several classic theorems from differential geometry yield clear connections to homotopy invariant constructions. In the following, we could have illustrated the point using other theorems, such as Hopfs Umlaufsatz ([Tu17]), but we will use the Gauss-Bonnet theorem.

We again consider the 2-sphere of radius r. If we integrate the Gaussian curvature κ of the 2-sphere S^2, called M for simplicity, we obtain

$$\int_M \kappa dS = \int_M \frac{1}{r^2} dS = \frac{1}{r^2}\int_M 1 dS = \frac{1}{r^2}(\text{surface area of} M) = \frac{1}{r^2}4\pi r^2 = 4\pi.$$

The integral of the Gaussian curvature, or more intuitively the "sum of the geometrical data of curvature", is independent of the radius of the sphere! In other words, stretching the sphere by perturbing the radius does not change the integral. It turns out that this is true for other types of stretching as well.

Theorem 2.1.1. *(Gauss-Bonnet theorem for a surface, [Tu17]).*

If M is a compact, oriented, Riemannian 2-manifold[2] *with Gaussian curvature K,*

$$\int_M K \,\mathrm{vol} = 2\pi\chi(M).$$

As the Euler characteristic of the 2-sphere is 2, this is essentially what we found. What's remarkable is that this old theorem incorporates geometric, metric-dependent information on the left-hand side and homotopy invariant information on the right-hand side.

To be able to refine the idea of mixing geometry and topology, we turn to differential forms. We refer to [Tu17] or any other textbook on differential geometry for the results below.

Differential forms, which are quite geometric and combinatorial in nature, are key players in the de Rham theory of cohomology. Given a smooth manifold M, one can form the de Rham complex $\Omega^*(M)$ consisting of vector spaces of differential forms in each degree, with coboundary given by exterior differentiation. The functoriality of Ω^* is given by pullbacks of forms.

Taking the homology of this cochain complex yields de Rham cohomology, or ordinary cohomology with real coefficients, which can be explicitly expressed as closed forms modulo exact forms in each degree. As the name suggests, this is a cohomology theory, and hence homotopy invariant, even though it is built up from differential forms in each degree. This is summarized in the following theorem, and we refer to any textbook containing de Rham cohomology (e.g. [BT82]) for a proof.

[2] It is possible to generalize the Gauss-Bonnet theorem quite a bit, but we won't dive into this. This can be found in [Tu17].

Theorem 2.1.2. *(The explicit homotopy invariance of de Rham cohomology).*

Let M and N be smooth manifolds with smooth maps $f, g : M \to N$ and let $\Psi : M \times [0,1] \to N$ be a smooth homotopy.

There is a chain homotopy between the induced morphisms at the level of de Rham complexes, which ensures the homotopy invariance of de Rham cohomology. The chain homotopy can be explicitly given by the homotopy operator $\psi : \Omega^*(N) \to \Omega^{*-1}(M)$ sending

$$\psi : \omega \mapsto \left(m \mapsto \int_{[0,1]} \iota_{\partial_t} \left(\Psi^* \omega\right)_m dt \right).$$

The integration of forms does not only aid in constructing an explicit formula for this homotopy operator. It also yields an intuitive connection to singular cohomology.

To see this, recall that the universal coefficient theorem implies that singular cohomology with real coefficients, $H^p(M;\mathbb{R})$, can be expressed as a dual of singular homology. More precisely,

$$\begin{aligned} H^p(M;\mathbb{R}) &\cong \mathrm{Hom}(H_p(M;\mathbb{Z}),\mathbb{R}) \bigoplus \mathrm{Ext}^1(H_{p-1}(M;\mathbb{Z}),\mathbb{R}) \\ &\cong \mathrm{Hom}(H_p(M;\mathbb{Z}),\mathbb{R}), \end{aligned}$$

where the Ext vanishes as $\mathbb{R}$ is a divisible group, and hence injective, by the Baer Criterion.

If we now want to move from $H^p_{dR}(M)$ to $H^p(M;\mathbb{R})$, we can take the approach of creating a functional on the integral homology groups of M. If we take a differential form ω, we may consider $\int_{\_} \omega$, and intuitively think of cycles in homology as relevant subspaces of M to integrate ω over.

More formally, a p-form may be integrated over a p-cycle by considering a p-cycle $\sigma : \Delta^p \to M$ as a p-dimensional subspace $c = \sigma(\Delta^p)$ of M. We can include c into M. Integrating the p-form over σ is done by pulling back ω from M to c through the inclusion.

Remark 2.1.3. Even though our manifolds may not be orientable, we will still use Stokes' theorem. Cycles, if considered as above, are orientable. The applications of Stokes' theorem on non-orientable manifolds will be only in the context of integrating forms on cycles.

We end this section by stating the de Rham theorem, which is vital to this thesis.

Theorem 2.1.4. *(The de Rham theorem, [BT82]).*

Integrating forms on cycles yields a natural isomorphism between de Rham cohomology $H^*_{dR}(M)$ and singular cohomology $H^*(M;\mathbb{R})$.

2.2 Refined motivation: de Rham as unification

Even though de Rham cohomology is homotopy invariant, it contains geometrical information, although not explicitly at the level of cohomology. The de Rham theorem (2.1.4) lets us view two identical stories of cohomology through the eyes of differential forms and singular cochains. If we want to understand the blending of the geometrical and topological information, it can be useful to view both of these in parallel, and then modify the part coming from de Rham cohomology to contain more geometrical information. We follow the description given in [ADH23] for this section.

Fix a smooth manifold M and consider the following sequences, connected by the de Rham isomorphism,

$$\begin{tikzcd}
\dots \arrow[r] & H^{k-1}(M;\mathbb{R}) \arrow[r] \arrow[d, "\cong"] & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r, "\mathrm{Bock}"] & H^k(M;\mathbb{Z}) \arrow[r] & H^k(M;\mathbb{R}) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & H^{k-1}_{dR}(M) \arrow[r] & \Omega^{k-1}(M)/\operatorname{im}(d) \arrow[r, "d"] & \Omega^k_{cl}(M) \arrow[r] & H^k_{dR}(M) \arrow[r] & \dots .
\end{tikzcd}$$

The top sequence is induced by a short exact sequence of coefficients, while the bottom sequence is made by doing the appropriate inclusions and maps we have available for differential forms.

A key point is that the top sequence is homotopy invariant, and hence resembles the homotopy theoretic part, while the lower sequence resembles the more geometric component, as the functor Ω^k_{cl} is not even homotopy invariant.

Why is it not homotopy invariant? By restricting to closed forms, the differentials in question will be zero, and hence, at the level of complexes, we may see that homotopic maps must be equal. That is, a chain homotopy $d\psi_n + \psi_{n+1}d = f - g$ between homotopic maps f and g yield $f = g$ when restricting to closed forms. Therefore, for Ω^*_{cl} to be homotopy invariant, we need $\Omega^i_{cl}(M) \cong \Omega^i_{cl}(N)$ for homotopy equivalent M and N. This is rarely the case, as illustrated by $0 \cong \Omega^2(pt) \ncong \Omega^2(\mathbb{R}^2) \ni dx \wedge dy \neq 0$.

One may therefore use this diagram as motivation for a naive blending of geometry and topology. To "blend" geometry and topology using de Rham cohomology, we are after invariants $\hat{H}^k(M;\mathbb{Z})$ that fit into the following type of commuting hexagon diagram,

$$\begin{tikzcd}
 & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "Bock"] \arrow[dr] & & H^k(M;\mathbb{Z}) \arrow[dr] & \\
H^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{H}^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & & H^k_{dR}(M). \\
 & \Omega^{k-1}(M)/\operatorname{im}(d) \arrow[rr, "d"] \arrow[ur] & & \Omega^k_{cl}(M) \arrow[ur] &
\end{tikzcd}$$

When referring to *a hexagon*, we mean a diagram of the above type, often with similar entries.

Even though this should be sufficient motivation for seeking $\hat{H}$, the diagram we put up is slightly too naive to describe $\hat{H}$, at least if we want the diagonal maps to form short exact sequences. In addition, this diagram does not yet tell us how we can differentially refine other cohomology theories, such as complex K-theory.

To briefly motivate what we will do in the next section, we give a short argument for what happens when $k = 1$.

Recall that $H^1(M;\mathbb{Z}) = \pi_0 \operatorname{Hom}(M, K(\mathbb{Z}, 1)) = \pi_0 \operatorname{Hom}(M, \mathbb{R}/\mathbb{Z})$, where $\mathbb{R}/\mathbb{Z}$ now denotes the circle. A compatible $\hat{H}^1$ should be related to H^1 through some map in the hexagon diagram, and it should hopefully contain some geometric information as well. Luckily, we have a result connecting $\operatorname{Hom}(M, \mathbb{R}/\mathbb{Z})$ to the more geometric analog of smooth maps, $C^\infty(M, \mathbb{R}/\mathbb{Z})$.

Proposition 2.2.1. *(Representing continuous maps by smooth maps, [BT82, Theorem 17.8]).*

Any continuous map between smooth manifolds is homotopic to a smooth map.

To rephrase, we have $H^1(M;\mathbb{Z}) = \pi_0 \operatorname{Hom}(M, \mathbb{R}/\mathbb{Z}) \cong \pi_0 C^\infty(M, \mathbb{R}/\mathbb{Z})$. Returning to the hexagon, this aligns quite well with the fact that we have $\Omega^0(M) = C^\infty(M, \mathbb{R})$. Post-composing with the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ yields a (non-injective) map

$$\iota : \Omega^0(M) = C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}/\mathbb{Z})$$

and we have a surjection

$$\pi_0 : C^\infty(M, \mathbb{R}/\mathbb{Z}) \twoheadrightarrow H^1(M;\mathbb{Z}) = \pi_0 C^\infty(M, \mathbb{R}/\mathbb{Z}).$$

These maps may work as the diagonal

$$\Omega^0(M)/\operatorname{im}(d) \to C^\infty(M, \mathbb{R}/\mathbb{Z}) \twoheadrightarrow H^1(M;\mathbb{Z})$$

in the hexagon diagram. For our first (naive) attempt at differential cohomology, we just use $\hat{H}^1(M;\mathbb{Z}) = C^\infty(M, \mathbb{R}/\mathbb{Z})$. To obtain the rest of the maps, as well as the exactness of the diagonals, we would need quite a lot of further work. Details can be found in [ADH23].

The key point is that $C^\infty(M, \mathbb{R}/\mathbb{Z})$ may work as a valid model for the first ordinary differential cohomology group. Unfortunately, the entries in the bottom sequence fail to yield short exact sequences on the diagonals of the hexagon. A solution is the differential characters of [CS85], which generalizes the role $C^\infty(M, \mathbb{R}/\mathbb{Z})$ plays for $k = 1$ to arbitrary k.

2.3 The Differential Characters of Cheeger-Simons

We follow [BB14] for our introduction of differential characters. This section is mainly included for completion, as differential characters are quite explicit and useful when wanting to show that a relevant construction is ordinary differential cohomology. They also motivate the generalization done by [HS05], which we will see in Section 2.6.

We restrict the discussion to smooth simplices, chains, and cycles. We will emphasize this temporarily by letting $C_k^{sm}(M;\mathbb{Z})$ denote the abelian group of smooth k-chains in M and consequently, let $Z_k^{sm}(M;\mathbb{Z})$ denote the smooth k-cycles in M. If we do not work with smooth manifolds, or the sub-/superscript is forgotten or omitted, it should be clear from context whether we consider smooth cycles or not.

The defining idea is that a class in $\hat{H}^k(M;\mathbb{Z})$ should contain information about the (smooth) cycles of M, as well as information about differential forms on M. Note that by the degeneracy of the hexagon diagram for $k=0$, we might as well set $\hat{H}^0(M;\mathbb{Z}) = H^0(M;\mathbb{Z})$.

Definition 2.3.1. (Differential characters, [CS85]).

A differential character on M of degree k is a homomorphism $h : Z_{k-1}^{sm}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ with the property that there exist a differential form $\omega \in \Omega^k(M)$ such that for all $c \in C_k^{sm}(M;\mathbb{Z})$,

$$h(\partial c) = \exp(2\pi i \int_c \omega).$$

The abelian group of differential characters of degree k is written $\hat{H}^k(M;\mathbb{Z})$, indicating that it is a model for ordinary differential cohomology. We will see that this is indeed true.

Remark 2.3.2. In the above definition, the differential characters contain information about k-forms and smooth k-chains, with a homomorphism on $(k-1)$-cycles blending them together. The differential characters of degree k is a subgroup of $\mathrm{Hom}_{Ab}(Z_{k-1}^{sm}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$.

Remark 2.3.3. In the original paper of [CS85], these are called differential characters of degree $(k-1)$. The index convention adopted later was to call these the differential characters of degree k, as this ensures information about k-forms and k-chains to be found in the differential characters of degree k, as explained above.

Remark 2.3.4. One should indeed be aware that we write the group operation additively, even though the operation on $\mathbb{R}/\mathbb{Z}$ is multiplication, i.e. $(h+h')(c) = h(c)h'(c)$. This is because there is an additional multiplicative structure on the group of differential characters that make up its ring structure. Note that this means that the identity is $0(c) = 1$, which has caused confusion.

Definition 2.3.5. (Curvature of a differential character).

The k-form connected to h in Definition 2.3.1 is called the curvature of h, which we write $\omega = \mathrm{curv}(h)$. In this language, a differential character h is called flat if $\mathrm{curv}(h) = 0$.

The curvature form ω can be shown to be unique. It is closed (which one can see by evaluating on boundaries), and it has integral periods.

Remark 2.3.6. Strictly speaking, we should think of a differential character as a pair (h, ω) as above, but due to the uniqueness of ω given h, we only consider a differential character to be an h that satisfies Definition 2.3.1.

In addition to the map picking out the curvature form, we can construct a map picking out a cohomology class, called the *characteristic class* of h. Let $h \in \mathrm{Hom}_{Ab}(Z^{sm}_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ be a differential character. Since $Z^{sm}_{k-1}(M;\mathbb{Z})$ is free, it is projective, and we can find a (not necessarily unique) real lift of h, called $\tilde{h} : Z^{sm}_{k-1}(M;\mathbb{Z}) \to \mathbb{R}$ such that post-composing $\tilde{h}$ with the projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ yields h, i.e. $h(c) = \exp(2\pi i \tilde{h}(c))$.

Definition 2.3.7. (Characteristic class of a differential character).

We follow [BB14] and define

$$\mu^{\tilde{h}} : C^{sm}_k(M;\mathbb{Z}) \to \mathbb{Z}, \quad c \mapsto -\tilde{h}(\partial c) + \int_c \mathrm{curv}(h).$$

We put $\mathrm{cc}(h) = [\mu^{\tilde{h}}] \in H^k(M;\mathbb{Z})$. This is the characteristic class of h. A differential character h is called topologically trivial if $\mathrm{cc}(h) = 0$.

We check that $\mu^{\tilde{h}}$ is a cocycle given a fixed real lift $\tilde{h}$ and that it takes integral values. This follows as the crux is to remember that $\mathrm{curv}(h)$ is closed and then that

$$\exp(2\pi i \int_c \mathrm{curv}(h)) = h(\partial c) = \exp(2\pi i \tilde{h}(\partial c)).$$

Since the fraction $h(\partial c)/h(\partial c)$ is 1, we may use both equalities above to obtain $\mu^{\tilde{h}}(c) \in \mathbb{Z}$.

It can also be shown that the cocycle is independent of which choice of lift $\tilde{h}$ we pick, implying cc is well-defined. The map cc can also be shown to be surjective onto $H^k(M;\mathbb{Z})$. We refer to [BB14].

Furthermore, any representative of the cohomology class $\mathrm{cc}(h)$ is of the form $\mu^{h'}$ for some real lift h' of h. Let α be a representative cocycle of the class. Then since we constructed $\mathrm{cc}(h)$ as $\mu^{\tilde{h}}$ for some fixed real lift $\tilde{h}$, we know that there exists a cochain s such that $\alpha = \mu^{\tilde{h}} - \delta s$. Setting $h' = \tilde{h} + s$ yields

$$\mu^{h'} := \int_- \mathrm{curv}(h) - \delta(\tilde{h} + s) = \int_- \mathrm{curv}(h) - \delta\tilde{h} - \delta s = \mu^{\tilde{h}} - \delta s = \alpha.$$

Next we define the map $\iota : \Omega^{k-1}(M)/\operatorname{im}(d) \to \hat{H}^k(M;\mathbb{Z})$ by

$$\eta \mapsto \exp(2\pi i \int_- \eta) =: h(-),$$

which is a differential character by Stokes' theorem, ensuring

$$h(\partial c) = \exp(2\pi i \int_{\partial c} \eta) = \exp(2\pi i \int_c d\eta).$$

We immediately obtain $\mathrm{curv}(\iota(\eta)) = d\eta$, which proves the commutativity of the lower triangle in the hexagon diagram. By picking a canonical real lift $\widetilde{\iota(\eta)} = \int_{\_} \eta$, this map can be shown to induce exactness in the upwardsgoing diagonal of $\hat{H}^k(M;\mathbb{Z})$. Be aware that $\mathrm{cc} \circ \iota = 0$ easily follows from Stokes' theorem.

To further work towards short exact diagonals, we must change the domain of ι. We can see that the kernel of ι must be the $(k-1)$-forms of integral periods, as if c is an arbitrary chain, $\exp(2\pi i \int_c \eta) = 1 = 0(c)$ forces $\int_c \eta \in \mathbb{Z}$.

The last map to define is the map $j : H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \to \hat{H}^k(M;\mathbb{Z})$. We use the pairing between homology and cohomology to define j. If $u \in H^{k-1}(M;\mathbb{R}/\mathbb{Z})$ is any cohomology class, define

$$j_u : [z] \mapsto \langle u, [z] \rangle \in \mathbb{R}/\mathbb{Z}.$$

This is indeed a differential character, as

$$j_u(\partial c) = \langle u, [\partial c] \rangle = \langle \delta u, [c] \rangle = \langle 0, [c] \rangle = 0(c) = 1 = \exp(0).$$

We immediately see that $\mathrm{curv}(j(u)) = 0$. By further work, we do in fact have a downward-going diagonal short exact sequence as well. The map j is often referred to as the inclusion of flat characters, as these are the ones with curvature 0. The fact that the $\mathbb{R}/\mathbb{Z}$-theory accounts for the "flat part" of differential cohomology, is a more general result than indicated here, as we'll briefly see in Section 3.3.

With the maps defined above, it is possible to obtain a new, refined version of the hexagon diagram. We refer to [ADH23], [BB14], and [SS08a] for the missing details.

Theorem 2.3.8. *(The hexagon diagram of differential characters).*

We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & & & 0 \\
& H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rrrr] \arrow[drr, "j"] & & & & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & & \hat{H}^k(M;\mathbb{Z}) \arrow[urr, "\mathrm{cc}"] \arrow[drr, "\mathrm{curv}"] & & & H^k_{dR}(M) \\
& \Omega^{k-1}(M)/\Omega^{k-1}_0(M) \arrow[urr, "\iota"] \arrow[rrrr, "d"] & & & & \Omega^k_0(M) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & & & 0
\end{tikzcd}$$

with exact diagonals.

2.4 A first definition of (ordinary) differential cohomology

The above maps and their relations define ordinary differential cohomology.

Definition 2.4.1. (Ordinary differential cohomology theory, [SS08a]).

An (ordinary) differential cohomology theory is a contravariant functor

$$\tilde{H}^*(-;\mathbb{Z}) : \mathbf{Man} \to \mathbf{grAb}$$

with four natural transformations;

- inclusion of flat classes, $\tilde{j} : H^{*-1}(-;\mathbb{R}/\mathbb{Z}) \to \tilde{H}^*(-;\mathbb{Z})$,
- topological trivializations, $\tilde{\iota} : \Omega^{*-1}(-)/\Omega_0^{*-1}(-) \to \tilde{H}^*(-;\mathbb{Z})$,
- curvatures, $\widetilde{\mathrm{curv}} : \tilde{H}^*(-;\mathbb{Z}) \to \Omega_0^*(-)$,
- and characteristic classes, $\tilde{\mathrm{cc}} : \tilde{H}^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z})$.

These are required to fit together in a hexagon diagram with exact diagonals, as in Theorem 2.3.8.

Theorem 2.4.2. *(Ordinary differential cohomology is unique, [SS08a, Theorem 1.1]).*

Ordinary differential cohomology exists (as shown by differential characters) and is unique up to unique isomorphism.

Remark 2.4.3. The unique isomorphism mentioned above can be constructed explicitly. See everything that leads up to Definition 24 in [BB14].

Remark 2.4.4. [SS08a] argues for the above theorem by using an axiomatic approach through so-called character functors. They also show that in such an axiomatic framework, the map cc is redundant, as it can be derived uniquely by the other maps and the existence of the hexagon.

As we will see in Section 3.2, generalized differential cohomology theories are not as strongly unique. Uniqueness up to an explicitly constructed unique isomorphism is true in most cases.

2.5 Ring structure and examples

Before examining a more general approach by Hopkins-Singer, we take a slight detour to interpret and compute examples of some differential cohomology groups, and see a curious consequence of the ring structure. We refer to [BB14] for missing details.

Example 2.5.1. (Differential characters when $k = 0$).

Due to the degeneracy of the hexagon diagram, we take $\hat{H}^0(M;\mathbb{Z}) = H^0(M;\mathbb{Z})$.

Example 2.5.2. (Differential characters when $k = 1$).

We have $\hat{H}^1(M;\mathbb{Z}) = C^\infty(M;\mathbb{R}/\mathbb{Z})$ as $\mathbb{Z}_0^{sm}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ yields a map $M \to \mathbb{R}/\mathbb{Z}$, and the relation to its curvature shows that it must be smooth.

Example 2.5.3. (Differential characters when $k = 2$).

Recall that $H^2(M;\mathbb{Z}) \cong [M, K(\mathbb{Z}, 2)]$. As we have the fibration $S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$, the limit $\mathbb{C}P^\infty$ is homotopy equivalent to $K(\mathbb{Z}, 2)$ by checking its homotopy groups through the long exact sequence in homotopy. As $\mathbb{C}P^\infty$ is $BU(1)$, the classifying space of principal $U(1)$-bundles, we do indeed obtain that

$$H^2(M;\mathbb{Z}) \cong [M, K(\mathbb{Z}, 2)] \cong [M, \mathbb{C}P^\infty] \cong [M, BU(1)] \cong \mathrm{Bun}(U(1)).$$

Therefore, we hope that $\hat{H}^2(M;\mathbb{Z})$ describes principal $U(1)$-bundles with some geometric information, such as principal $U(1)$-connections. This is indeed the case. The group consists of principal $U(1)$-bundles with connection, (E, ∇). For such a bundle, we can assign to a smooth 1-cycle z an element $h(z) \in U(1)$ by utilizing some sort of holonomy map (see [BB14]). In fact, all differential characters of degree 2 arise this way. The interesting thing is that the curvature of such a differential character, $\mathrm{curv}(h)$ is $\frac{-1}{2\pi i} R^\nabla$, where R^∇ is the curvature of the connection. The characteristic class of h, $\mathrm{cc}(h)$, is the first Chern class of the bundle.

Queequeg 2.5.4. (Differential characters for higher k).

The reader interested in higher analogs of differential geometry may be interested in connections in principal G-k-bundles. Just as $\hat{H}^2(M;\mathbb{Z})$ classifies $U(1)$-1-bundles with connections, $\hat{H}^k$ can be thought of as classifying principal $U(1)$-$(k-1)$-bundles with connection, where the index $(k-1)$ indicates that we are thinking of $U(1)$ as a $(k-1)$-group. This can be made specific by understanding the Deligne complex as a model for ordinary differential cohomology (which we do in Section 3.4) and by studying higher parallel transport. We will not meet the latter idea, nor spend a substantial amount of time on differential characters. See [BB14] and the references therein for more details.

There is a ring structure on differential characters. One can explicitly define an exterior product

$$\times : \hat{H}^k(X) \times \hat{H}^l(X') \to \hat{H}^{k+l}(X \times X')$$

and refine this to a (graded commutative) interior product

$$* : \hat{H}^k(X) \times \hat{H}^l(X) \to \hat{H}^{k+l}(X)$$

by $\Delta_X^*(h \times h')$ where Δ_X is the diagonal on X.

We will not construct this product for differential characters, as it becomes quite involved, but the rather hairy formulas can be found in [BB14]. The product can be shown to satisfy some wanted properties. For example, we have

$$\mathrm{curv}(h * h') = \mathrm{curv}(h) \wedge \mathrm{curv}(h') \qquad \text{and} \qquad \mathrm{cc}(h * h') = \mathrm{cc}(h) \cup \mathrm{cc}(h').$$

Remark 2.5.5. Using their axiomatic character diagram framework, [CS85] shows that this product is unique.

Remark 2.5.6. There is a framework in which the internal product becomes more intuitive, as it uses the wedge product of forms, the cup product of cocycles, and an explicit formula for blending them. We return to it in the next section.

The reason we mention the ring structure, apart from completeness, is the curious construction it yields in low degrees. This is Example 34 in [BB14].

Construction 2.5.7. (Constructing a principal $U(1)$-bundle from smooth functions).

Consider two differential characters h_1 and h_2 of degree $k = 1$ on M. These correspond to smooth functions $\bar{h_i} : M \to U(1)$. The internal product $h_1 * h_2$ of these should by our interpretation of $\hat{H}^2(M;\mathbb{Z})$ be a principal $U(1)$-bundle with connection on M.

To find this bundle, consider the differential character $i \in \hat{H}^1(U(1);\mathbb{Z})$ that corresponds to the smooth function given by the identity on $U(1)$. Since we have $\bar{h_j} = \mathrm{id}_{U(1)} \circ \bar{h_j}$, we have $\bar{h}_j = \bar{h_j}^* i$. Set $\bar{h} = (\bar{h_1}, \bar{h_2}) : M \to U(1) \times U(1)$ and let $\Delta : U(1) \to U(1) \times U(1) =: T^2$ be the diagonal map, which goes from $U(1)$ to the 2-torus.

We compute

$$\begin{aligned} h_1 * h_2 = \Delta^*(h_1 \times h_2) &= \Delta^*(\bar{h_1}^* i \times \bar{h_2}^* i) \\ &= \Delta^*(\bar{h_1} \times h_2)^*(\bar{i} \times i) = \bar{h}^*(i \times i). \end{aligned}$$

This construction may seem a bit ad hoc, but the takeaway is beautiful. Note that $i \times i$ lives in $\hat{H}^2(T^2;\mathbb{Z})$, which means that this is a principal $U(1)$-bundle with connection on T^2. Our product $h_1 * h_2$, which is a principal $U(1)$-bundle with connection on M "generated" by the smooth functions $\bar{h_i}$, is in fact a pullback along $\bar{h}$ of a universal principal $U(1)$-bundle with connection on T^2.

The universal bundle (and the connection) can be explicitly constructed, as done in [BB14]. There are deeper reasons for being interested in this bundle, see for example [BB14, P.37.].

As we have seen, differential characters are technical in nature, but they yield an explicit description of differential cohomology. However, it is not yet clear how to generalize differential characters to arbitrary cohomology theories.

2.6 Generalization through homotopy pullbacks

In their important paper ([HS05]), Hopkins-Singer find a way to generalize the differential characters of [CS85], both to obtain another view of ordinary differential cohomology, and as a framework of differentially refining other generalized cohomology theories. Instead of the requirement involving maps and the hexagon diagram, they ask for a theory to sit in the corner of the rightmost square of the hexagon.

To be more precise, Hopkins and Singer define a new complex $\hat{C}(q)^*(M)$ by a homotopy pullback

$$\begin{tikzcd} \hat{C}(q)^*(M) \arrow[r] \arrow[d] & \Omega^{*\geq q}(M) \arrow[d, "j"] \\ C^*(M;\mathbb{Z}) \arrow[r, "i"] & C^*(M;\mathbb{R}), \end{tikzcd}$$

where $\Omega^{*\geq q}(M)$ is the complex Ω^*, but with $\Omega^n(M) = 0$ if $n < q$.

Remark 2.6.1. As a generalization inspired by Deligne cohomology, the filtration of forms is included. One could do a similar construction without the filtration on forms, but that would just amount to constructing $\hat{C}^* := \hat{C}(0)^*(M)$. If $q \leq q'$, we do indeed have $\hat{C}(q)(M) \supseteq \hat{C}(q')(M)$ by the inclusion of the geometric data.

In the category of cochain complexes, a homotopy pullback need not be unique. It is however unique up to quasi-isomorphism. Using known results on the topic, such as [BR20, Lemma A.7.21], we can find an explicit model for the complex in question. The above diagram is a homotopy pullback if and only if

$$\begin{tikzcd} \hat{C}(q)^*(M;\mathbb{Z}) \arrow[r] \arrow[d] & C^*(M;\mathbb{Z}) \times \Omega^{*\geq q}(M) \arrow[d] \\ 0 \arrow[r] & C^*(M;\mathbb{R}) \end{tikzcd}$$

is a fiber sequence, where $C^*(M;\mathbb{Z}) \times \Omega^{*\geq q}(M) \to C^*(M;\mathbb{R})$ is the map $-i + j$.

There is a canonical way to construct such homotopy fibers. By shifting $\hat{C}(q)^*(M)$, we can define

$$\hat{C}(q)^*(M)[1] = \mathrm{cone}((-i, j)) = \mathrm{cone}(C^*(M;\mathbb{Z}) \times \Omega^{*\geq q}(M) \to C^*(M;\mathbb{R})).$$

Definition 2.6.2. (The filtered differential complex).

More explicitly, we call the complex

$$\hat{C}(q)^n(M) = C^n(M;\mathbb{Z}) \times \Omega^{n\geq q}(M) \times C^{n-1}(M;\mathbb{R})$$

the filtered differential complex, where $\Omega^{n\geq q}(M)$ is either 0 or $\Omega^n(M)$ if $n < q$ or $n \geq q$, respectively.

The differential d is given by

$$d(c, \omega, h) = (\delta c, d\omega, \omega - c - \delta h).$$

Remark 2.6.3. To ensure $\omega - c - \delta h$ makes sense, we use the de Rham map. This may be an abuse of notation, but the true home of these cocycles should be clear from the context. We should not forget, however, that ω considered as a singular cocycle means $\int_{\_} \omega$.

By the long exact sequence in cohomology, we obtain a Mayer-Vietoris sequence

$$\begin{tikzcd} \cdots \arrow[r] & \hat{H}(q)^n(M) \arrow[r] & H^n(M;\mathbb{Z}) \times H^n(\Omega^{*\geq q}) \arrow[r] & H^n(M;\mathbb{R}) \arrow[dll, "I"] \\ & \hat{H}(q)^{n+1}(M) \arrow[r] & H^{n+1}(M;\mathbb{Z}) \times H^{n+1}(\Omega^{*\geq q}) \arrow[r] & \cdots, \end{tikzcd}$$

where $\Omega^{*\geq q}$ is short for the complex $\Omega^{*\geq q}(M)$.

If $n > q$, the de Rham isomorphism implies that we can identify $\hat{H}(q)^n(M)$ with $H^n(M;\mathbb{Z})$. In the case where $n < q$, we have $H^n(\Omega^{*\geq q}) = 0$, which allows us to use the 5-lemma to argue that $\hat{H}(q)^n(M)$ is $H^{n-1}(M;\mathbb{R}/\mathbb{Z})$.

What about $n = q$? The group $\hat{H}(q)^q(M)$ consists of equivalence classes of cocycles (c, ω, h) where c is a singular cocycle, ω is a closed form and h satisfies $\delta h = \omega - c$.

Remark 2.6.4. One should note that the above requirement for δh heuristically works as the embodiment of our goal to "blend" geometry (the ω's) with topology (the c's).

Theorem 2.6.5. *(Filtered differential complexes generalize differential characters).*

We have an isomorphism

$$\hat{H}(q)^q(M) \cong \hat{H}^q(M),$$

where the $\hat{H}$ on the right hand side denotes the group of differential characters of [CS85].

Proof. We sketch the proof. The cocycle (c, ω, h) can be made into a differential character by mapping $(c, \omega, h) \mapsto \chi = \exp(2\pi i h)$, which lives in $\mathrm{Hom}_{Ab}(Z^{sm}_{q-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$. This is a differential character since if we are given an arbitrary smooth q-cycle z, we have

$$\chi(\partial z) = \exp(2\pi i h(\partial z)) = \exp(2\pi i(\delta h)(z)) = \exp(2\pi i(\omega(z) - c(z))) = \exp(2\pi i(\int_z \omega)).$$

Above, we used the fact that c is integer-valued. It is immediate that χ has curvature ω, which is unique, indicating that we can construct an inverse map. The cocycle c is determined by the requirement in cohomology that $\delta h = \omega - c$. The inverse map is given by $(\chi, \omega) \mapsto (\omega - \delta h, \omega, h)$, where h can be chosen to be a real lift of χ, similarly to what we did in the previous section. These can be shown to be group homomorphisms that are natural in M. See [BB14]. □

Remark 2.6.6. There are short exact sequences that ensure $\hat{H}(q)^q$ fits into the hexagon diagram. We could also have proven that $\hat{H}(q)^q$ simply satisfies the axioms of an ordinary cohomology theory. This would have sufficed to prove uniqueness, by Theorem 2.4.2.

Remark 2.6.7. One should note that [HS05] uses the index convention of [CS85] for differential characters.

The product of differential characters becomes prettier in this framework, as we'll briefly mention. For two cocycles (c, ω, h) and (c', ω', h'), one can make the product component-wise for the first two components using the cup product $c \cup c'$ and the wedge product $\omega \wedge \omega'$. The problem arises when considering the product of h and h', as the de Rham map does not necessarily map $\wedge$ to $\cup$.

Luckily, as [CS85] points out, Kevaire presented a way make the difference $\omega \wedge \omega' - \omega \cup \omega'$ exact in a canonical way. The term $\omega \wedge \omega'$ should by abuse of notation be interpreted as the cocycle associated to the differential form $\omega \wedge \omega'$. We can choose a natural chain homotopy B, unique up to homotopy, between $\cup$ and $\wedge$ to obtain

$$\omega \cup \omega' - \omega \wedge \omega' = \delta B(\omega, \omega') + B(d\omega, \omega') + (-1)^{\deg \omega} B(\omega, d\omega').$$

Proposition 2.6.8. *(The generalized product).*

The product of (c, ω, h) *and* (c', ω', h') *is given by*

$$(c \cup c', \omega \wedge \omega', h \cup \omega' + (-1)^{\deg c} c \cup h' + B(\omega, \omega')).$$

We refer to [CS85] and the references therein on how to construct B, and we continue to follow [HS05] to further generalize their construction to arbitrary cohomology theories.

2.7 The Differential Function Complexes of Hopkins-Singer

We worked with triples (c, ω, h) in order to generalize the differential characters. By Brown's representability theorem, we can represent cocycles in a cohomology theory by maps into a suitable set of spaces, which lead to the notion of sequential spectra. We have not yet come to link differential cohomology to spectra, but we may still include the idea of having a function representing the cocycle c. This observation leads to a refined notion of differential cocycles.

Throughout this section, fix an integer n. We also need to fix a topological space X and a cocycle $\iota \in Z^n(X; \mathbb{R})$. Neither X nor ι needs to be smooth. We will in Section 2.8 define a canonical class of choices for such a cocycle, called *a fundamental cocycles.*

Definition 2.7.1. (Differential functions).

We say a differential function $t : M \to (X, \iota)$ is a triple (c, ω, h) as before, where $\omega \in \Omega^n(M)$ and $h \in C^{n-1}(M)$, but now we require

$$c : M \to X \qquad \text{such that} \qquad \delta h = \omega - c^* \iota.$$

Example 2.7.2. (Putting a connection on a $U(1)$-bundle).

If we choose $X = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty = BU(1)$ and let L be the universal line bundle on $\mathbb{C}P^\infty$. Pick $\iota_\mathbb{Z}$ to be the integral two-cocycle representing the first Chern-class of L. A cocycle $c \in Z^2(M)$, which can be viewed as a map $c : M \to K(\mathbb{Z}, 2)$, induces a line bundle c^*L on M. Refining c to a differential function is equivalent to putting a connection on c^*L, where the $U(1)$-bundle with connection is the one corresponding to the class of

$$(c^*\iota_\mathbb{Z}, \omega, h) \in Z(2)^2(M).$$

To make a more versatile object, we construct a simplicial complex based on these differential functions.

Definition 2.7.3. (Differential function complex).

The differential function complex $(X; \iota)^M$ is the simplicial set where the k-simplices are the differential functions

$$M \times \Delta^k \to (X; \iota)$$

To align this approach with the filtered differential complex of Definition 2.6.2, we can force a filtration of $\Omega^*(\Delta^k)$ by

$$\operatorname{filt}_s \Omega^*(\Delta^k) = (\Omega^0(\Delta^k) \to \cdots \to \Omega^s(\Delta^k)).$$

If we want to define this filtration on $\Omega^*(M \times \Delta^k)$, we simply let

$$\operatorname{filt}_s \Omega^*(M \times \Delta^k) = \Omega^*(M) \otimes \operatorname{filt}_s \Omega^*(\Delta^k).$$

This filtration can be used to construct a filtration of the simplicial complex $(X; \iota)^M$. A key observation is that the exterior derivative induces a shift in the filtration, $d : \operatorname{filt}_s \to \operatorname{filt}_{s+1}$.

Definition 2.7.4. (Weight filtration on the differential function complex).

We say that a k-simplex $(c, \omega, h) \in (X; \iota)^M_k$ has weight filtration $\leq s$ if

$$\omega \in \operatorname{filt}_s \Omega^*(M \times \Delta^k)_{cl}.$$

Definition 2.7.5. (Filtered differential function complex).

The simplicial set whose k-simplices are $(c, \omega, h) \in (X; \iota)^M_k$ that has weight filtration $\leq s$ is denoted by $\operatorname{filt}_s(X; \iota)^M$, and is called the filtered differential function complex, or often just *the differential function complex.*

We introduce a convention to generalize to the case where we have cocycles of various degrees. If $\mathcal{V}$ is a graded vector space, we define the following:

$$
\begin{aligned}
C^*(X;\mathcal{V})^n &= \oplus_{i+j=n} C^i(X;\mathcal{V}^j)\\
\Omega^*(M;\mathcal{V})^n &= \oplus_{i+j=n} \Omega^i(M;\mathcal{V}^j)\\
Z^*(X;\mathcal{V})^n &= \oplus_{i+j=n} Z^i(X;\mathcal{V}^j)\\
H^n(X;\mathcal{V}) &= \oplus_{i+j=n} H^i(X;\mathcal{V}^j).
\end{aligned}
$$

If the subscript j is lowered, as in $\Omega^i(X;\mathcal{V}_j)$, we use the convention that this has degree $(i-j)$ obtained by setting $\mathcal{V}_j = \mathcal{V}^{-j}$.

Using the well-known yoga of simplicial objects, the Dold-Kan correspondence, and a whole lot of lemmas, one can show the following results. We refer to Appendices A and D in [HS05] for the details.

Theorem 2.7.6. *(The differential function complexes generalize Differential Characters).*

Let $X = K(\mathbb{Z}, n)$ and let ι correspond to the inclusion $\mathbb{Z} \subset \mathbb{R}$ under the isomorphism $H^n(K(\mathbb{Z}, n);\mathbb{R}) \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, \mathbb{R})$.

The simplicial set $\mathrm{filt}_s(X;\iota)^M$ is a simplicial abelian group homotopy equivalent to the simplicial abelian group associated to the chain complex

$$\hat{C}(n-s)^0(M) \to \cdots \to \hat{C}(n-s)^{n-1}(M) \to \hat{Z}(n-s)^n(M)$$

through the Dold-Kan correspondence.

The homotopy groups of the simplicial set is given by

$$\pi_i \,\mathrm{filt}_s(X;\iota)^M = \hat{H}(n-s)^{n-i}(M).$$

For general X and ι, the differential function complex fits into a homotopy pullback as

$$
\begin{tikzcd}
\mathrm{filt}_s(X;\iota)^M \arrow[r] \arrow[d] & \mathrm{filt}_s\,\Omega^*(M\times\Delta^*;\mathcal{V}_*)^n_{cl} \arrow[d] \\
\mathrm{sing}(X^M) \arrow[r] & Z^*(M\times\Delta^*;\mathcal{V}_*).
\end{tikzcd}
$$

Remark 2.7.7. As the Dold-Kan correspondence maps into simplicial abelian groups, and simplicial abelian groups are Kan complexes, the differential function complexes are in fact Kan complexes.

By Appendices A, D, and page 36 of [HS05], it is possible to show the following. We omit the proof, although we include the result for completion, hoping to inspire readers to do (and publish) computations of these groups for interesting manifolds M.

Proposition 2.7.8. *(Explicit homotopy groups for differential function complexes).*

We have the following homotopy groups.

$$
\begin{aligned}
\pi_m Z(M \times \Delta^*; \mathcal{V}_*)^n &\cong H^{n-m}(M; \mathcal{V}_*). && \\
\pi_m \operatorname{filt}_s \Omega^*(M \times \Delta^*; \mathcal{V}_*)^n_{cl} &\cong H^{n-m}_{dR}(M; \mathcal{V}_*) && m < s. \\
\pi_m \operatorname{filt}_s \Omega^*(M \times \Delta^*; \mathcal{V}_*)^n_{cl} &\cong \Omega^*(M; \mathcal{V}_*)^{n-m}_{cl} && m = s. \\
\pi_m \operatorname{filt}_s \Omega^*(M \times \Delta^*; \mathcal{V}_*)^n_{cl} &\cong 0 && m > s. \\
\pi_k \operatorname{filt}_s(X; \iota)^M &\cong \pi_k(X; \iota)^M && k < s. \\
\pi_k \operatorname{filt}_s(X; \iota)^M &\cong H^{n-k-1}(M; \mathcal{V}_*)/\pi_{k+1}(X; \iota)^M && k > s.
\end{aligned}
$$

For $k = s$, we define $A^{n-s}(M; X, \iota)$ by

$$
\begin{tikzcd}
A^{n-s}(M; X, \iota) \arrow[r] \arrow[d] & \Omega^*(M; \mathcal{V}_*)^{n-s}_{cl} \arrow[d] \\
\pi_s(X; \iota)^M \arrow[r] & H^{n-s}(M; \mathcal{V}_*).
\end{tikzcd}
$$

$\pi_s \operatorname{filt}_s(X; \iota)^M$ sits in a short exact sequence

$$H^{n-s-1}(M; \mathcal{V}_*)/\pi_{s+1}(X; \iota)^M \to \pi_s \operatorname{filt}_s(X; \iota)^M \to A^{n-s}(M; X, \iota).$$

A natural question to ask is how the simplicial set$(X; \iota)^M$ depends on X, ι, and M, and which functorial or homotopy preserving properties filt_s has.

Assume we are given a smooth map $g : M \to N$ of smooth manifolds. By considering the mapping $(c, \omega, h) \mapsto (c \circ g, g^*\omega, g^*h)$, we obtain a map

$$g^* : \operatorname{filt}_s(X; \iota)^N \to \operatorname{filt}_s(X; \iota)^M.$$

Assume we have a continuous map $f : X \to Y$. Since the cocycle depends on the underlying space, this requires a change in ι. By letting $\iota_Y \in Z^n(Y; \mathcal{V}_*)$, we obtain

$$\hat{f} = \operatorname{filt}_s(f) : \operatorname{filt}_s(X; f^*\iota_Y)^M \to \operatorname{filt}_s(Y; \iota_Y)^M$$

by mapping $(c, \omega, h) \mapsto (c \circ f, \omega, h)$.

Remark 2.7.9. Note that filt_s is functorial in the manifold M, but not in the space X due to the cocycle ι. This could be expected, as X in some sense represents the differential cohomology theory.

This construction has important properties.

Proposition 2.7.10. *((weak) homotopy equivalences are preserved by* filt_s*).*

If $f : X \to Y$ *is a (weak) homotopy equivalence, then*

$$\hat{f} = \mathrm{filt}_s(f) : \mathrm{filt}_s(X; f^*\iota_Y)^M \to \mathrm{filt}_s(Y; \iota_Y)^M$$

is a (weak) homotopy equivalence.

Proof. If f is a homotopy equivalence, we can consider the following diagram,

$$\begin{tikzcd} \operatorname{sing} X^M \arrow[r] \arrow[d] & Z^*(M \times \Delta^*; \mathcal{V}_*)^n \arrow[d] & \mathrm{filt}_s\, \Omega^*(M \times \Delta^*; \mathcal{V}_*)^n_{cl} \arrow[l] \arrow[d] \\ \operatorname{sing} Y^M \arrow[r] & Z^*(M \times \Delta^*; \mathcal{V}_*)^n & \mathrm{filt}_s\, \Omega^*(M \times \Delta^*; \mathcal{V}_*)^n_{cl}. \arrow[l] \end{tikzcd}$$

Note that the two rightmost vertical maps are identity maps and that the leftmost vertical map is a homotopy equivalence since sing preserves homotopy equivalences. For weak homotopy equivalences, one must use bifibrant approximation and the generalized Whitehead theorem to show that these can be promoted to homotopy equivalences. Taking the homotopy pullback preserves vertex-wise (weak) equivalences, yielding the wanted result. □

Different choices of the cocycle ι may yield different differential function complexes. Nevertheless, the resulting differential function complex only depends on the cohomology class of ι.

Proposition 2.7.11. *(Invariance under cohomologous changes of* ι*).*

If $\iota_1, \iota_2 \in Z^*(X; \mathcal{V}_*)^n$ *and* $b \in C^*(X; \mathcal{V}_*)^{n-1}$ *satisfies* $\delta b = \iota_1 - \iota_2$*, the map*

$$\begin{aligned} (X; \iota_1)^M &\to (X; \iota_2)^M, \\ (c, \omega, h) &\mapsto (c, \omega, h + c^* b), \end{aligned}$$

is an isomorphism.

Proof. The inverse is made by replacing b by $-b$. □

Assume we have a map of graded vector spaces $t : \mathcal{V}_* \to \mathcal{W}_*$ and a cocycle $\iota \in Z^n(X; \mathcal{V}_*)$ as before. Then we get an induced cocycle $t \circ \iota \in Z^n(X; \mathcal{W}_*)$ and a map $(X; \iota)^M \to (X; t \circ \iota)^M$. Assume we are given a map $f : X \to Y$, cocycles $\iota_X \in Z^n(X; \mathcal{V}_*)$ and $\iota_Y \in Z^n(Y; \mathcal{W}_*)$, and a cochain $b \in C^{n-1}(X; \mathcal{W}_*)$ with $\delta b = t \circ \iota_X - f^*\iota_Y$. Then we get a map $(X; \iota_X)^M \to (Y; \iota_Y)^M$ by mapping

$$(c, \omega, h) \mapsto (f \circ c, t \circ \omega, t \circ h + c^* b).$$

The conclusion from these investigations is the following, which combines the above results.

Theorem 2.7.12. *(On* $\mathrm{filt}_s(X;\iota)^M$ *and its homotopy type).*

Assume we fix a smooth manifold M*. The homotopy type of* $\mathrm{filt}_s(X;\iota)^M$ *only depends on the cohomology class of* ι *and the homotopy type of* X*.*

2.8 Differential E-cohomology and fundamental cocycles

We have not yet had any definition of differential E-cohomology groups for a spectrum E, although we recovered the definition of ordinary differential cohomology in Theorem 2.7.6 in terms of the homotopy groups of a differential function complex.

Assume now that we are given a spectrum E with structure maps

$$s_n^E : \Sigma E_n \to E_{n+1}.$$

We seek a construction $\mathrm{filt}_s(E;\iota)_n^M$, where $(E;\iota)_n$ denotes (E_n, ι_n). For that we need a suitable choice of cocycle ι_n. We define the singular (co)chain complex of E in the spirit of stability.

Definition 2.8.1. (The singular (co)chain complex of a spectrum).

The singular chain complex of E is defined as the complex

$$C_*(E) = \mathrm{colim}_n\, C_{*+n}E_n,$$

where the defining maps $C_{*+n}E_n \to C_{*+n+1}E_{n+1}$ are given by the horizontal composition

$$\begin{tikzcd} C_{*+n}(E_n) \arrow[rr] \arrow[dr, "-\times Z_{S^1}"'] & & C_{*+n+1}(\Sigma E_n) \arrow[r, "(s_n)_*"] & C_{*+n+1}(E_{n+1}) \\ & C_{*+n+1}(E_n \times S^1). \arrow[ur, "(q_n)_*"'] & & \end{tikzcd}$$

Here q_n is the map collapsing the wedge sum $E_n \vee S^1$, inducing $E_n \times S^1 \to \Sigma E_n$, and Z_{S^1} is a fundamental cycle of S^1. The differential is induced by the limit.

We define the singular cochain complex of E with coefficients in an abelian group A to be

$$C^*(E;A) = \mathrm{Hom}(C_*(E), A) = \lim C^{*+n}(E_n, A).$$

The induced map $\mathrm{Hom}(- \times Z_{S^1}, A)$ defines the slant product $/Z_{S^1}$.

Remark 2.8.2. The slant product can be linked to integration by [HS05, Section 3.5].

Remark 2.8.3. The (co)homology of the spectrum E can be defined as the homology of the (co)chain complex of E. Due to the stable nature of these complexes, we may have nonzero (co)homology groups even at negative indices.

A cocycle $\iota \in Z^p(E; \mathcal{V}_*)$ is, by the above definition, a list of cocycles $\iota_n \in Z^{p+n}(E_n; \mathcal{V}_*)$ compatible by $\iota_n = ((s_n^E)^* \iota_{n+1})/Z_{S^1}$. Assume that we have fixed such a cocycle $\iota \in Z^p(E; \mathcal{V}_*)$ for any fixed p, which we continue to omit from notation. The natural thing to do is to have $p = 0$, but for generality, it is possible to have the cocycle shifted.

Definition 2.8.4. (Differential E-cohomology groups).

The differential E-cohomology groups of a smooth manifold M are given by

$$E(n-s)^n(M; \iota) = \pi_0 \operatorname{filt}_s(E; \iota)_n^M = \pi_0 \operatorname{filt}_s(E_n; \iota_n)^M.$$

If er only write $\hat{E}^n(M)$, we mean $\hat{E}^n(M) = \pi_0 \operatorname{filt}_0(E, \iota)_n^M = E(n)^n(M)$.

Remark 2.8.5. Be aware that even though we write $E(n-s)^n(M; \iota)$, ι is a cocycle on E, not on M, as previous notation may hint to.

Example 2.8.6. As shown in [HS05, Appendix D], if E is the Eilenberg-MacLane spectrum $H\mathbb{Z}$, we obtain

$$H\mathbb{Z}(k)^n(M) \cong \hat{H}(k)^n(M),$$

if we use *the fundamental cocycle*, which we will now define.

There is a canonical choice of cohomology class to choose cocycle representatives from, which we call the fundamental cocycles. The fundamental observation is that of Dold ([Dol72]).

Proposition 2.8.7. *(Chern-Dold isomorphism).*

If we are given a compact space X, then we have an isomorphism

$$E^*(X) \otimes \mathbb{R} \cong H^*(X; \pi_* E \otimes \mathbb{R}).$$

This can be viewed as coming from a generalized Chern character, called the Chern-Dold character. The Chern-Dold character is induced by the "realification" of the spectrum E, where realification is the real analogue of rationalization coming from a smashing localization of spectra obtained by smashing with $H\mathbb{R}$ instead of $H\mathbb{Q}$. The Chern-Dold character specializes to the Chern character if E is complex K-theory.

By this observation, we are motivated to choose a cocycle of $H^*(E; \pi_* E \otimes \mathbb{R})$, and we seek to choose a cocycle of degree 0 for simplicity.

Note that

$$\begin{aligned} H^0(E; \pi_* E \otimes \mathbb{R}) &= H^0(C^*(E; \pi_* E \otimes \mathbb{R})) \\ &= H^0(\lim C^{*+n}(E_n; \pi_{*+n} E \otimes \mathbb{R})) \\ &= \lim H^n(E_n; \pi_{*+n} E \otimes \mathbb{R}). \end{aligned}$$

The "realification" map $\pi_* E \to \pi_* E \otimes \mathbb{R}$ given by $a \mapsto a \otimes 1$ lives in $\operatorname{Hom}(\pi_* E, \pi_* E \otimes \mathbb{R})$, which corresponds to $H^0(E, \pi_* E \otimes \mathbb{R})$ under a graded Hurewicz isomorphism. The map therefore represents a cohomology class $[\iota^E]$.

Definition 2.8.8. (A fundamental cocycle).

Let E be a spectrum. A fundamental cocycle on E is a cocycle

$$\iota^E \in Z^0(E; \pi_* E \otimes \mathbb{R})$$

that represents the cohomology class defined above.

If the cocycle is not specified, as in Example 2.8.6, we mean the differential cohomology theory associated to fundamental cocycles. As shown earlier, this is well-defined, as cohomologous cocycles yield isomorphic differential cohomology theories.

Remark 2.8.9. (A fundamental cocycle specializes to other cocycles, [HS05, Remark 4.58]).

Given a cocycle $\iota \in Z^*(E; \mathcal{V}_*)^0$, we can find a map $t : \pi_* E \otimes \mathbb{R} \to \mathcal{V}_*$ and a cochain $b \in C^{-1}(E; \mathcal{V}_*)$ such that $\delta b = \iota - t \circ \iota^E$.

Remark 2.8.10. (Explicit Chern-Dold character).

The Chern-Dold character

$$\mathrm{chd} : E^*(X) \to H^*(X; \pi_* E \otimes \mathbb{R})$$

can be given by sending $x \in E^n(X)$ to the cohomology class $f^* \iota_n$, where $f : X \to E_n$ represents x, and ι_n is the projection of

$$\iota^E \in H^0(E; \pi_* E \otimes \mathbb{R}) = \lim_n H^n(E_n; \pi_{*+n} E \otimes \mathbb{R})$$

down to $H^n(E_n; \pi_{*+n} E \otimes \mathbb{R})$.

2.9 The Differential Function Spectra of Hopkins-Singer

It is natural to wonder if the differential function complexes form a spectrum. In this section, we will briefly outline the main idea of differential function spectra, which is included for completeness. Many lengthy proofs can be found in [HS05].

One can define a spectrum object $\mathrm{filt}_s(E; \iota)^M$, where we set

$$(\mathrm{filt}_s(E; \iota)^M)_n = \mathrm{filt}_{s+n}(E; \iota)^M_n = \mathrm{filt}_{s+n}(E_n; \iota_n)^M.$$

Remark 2.9.1. By the definition of differential E-cohomology, the higher homotopy groups of such a spectrum is

$$\pi_t \, \mathrm{filt}_s(E_n, \iota_n)^M = \pi_0 \, \mathrm{filt}_{s-t}(E_{n-t}; \iota_{n-t})^M = E(n-t-s+t)^{n-t}(M; \iota) = E(n-s)^{n-t}(M; \iota).$$

Remark 2.9.2. By the Dold-Kan correspondence, the simplicial abelian group $\mathrm{filt}_s(E_n, \iota_n)^M$ yields a cochain complex, and hence cohomology groups.

To "spectrify" this constriction, we must acknowledge that the statement "we have a homotopy equivalence $\mathrm{filt}_{s+n}(E_n, \iota_n)^M \to \Omega\,\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M$" is slightly ambiguous, as the object $\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M$ is a simplicial abelian group, not a topological space. One can indeed argue that $\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M$ is a space, as it is a simplicial group, hence a Kan complex, and therefore corresponds homotopically to a CW-complex (or an ∞-groupoid). This is not the point. The problem is that we have not specified a simplicial version of Ω.

Definition 2.9.3. (Simplicial loop space, Ω^{simp}).

For a simplicial set X_*, dthe simplicial loop space $\Omega^{simp}X_*$ is the simplicial set where the k-simplicies are maps $h : \Delta^k_* \times \Delta^1_* \to X_*$ of simplicial sets where we collapse Δ^1_* into a loop by requiring $h(x, 1) = h(x, 0) = pt$.

Remark 2.9.4. Note that this is the function complex from the simplicial circle to X.

To begin, one can show there is a canonical map

$$|\Omega^{simp}X_*| \to \Omega|X_*|.$$

If X_* is a Kan complex, this can be shown to be a homotopy equivalence. In particular, this will be a homotopy equivalence for differential function complexes.

The existence of a simplicial homotopy equivalence

$$\mathrm{filt}_{s+n}(E_n, \iota_n)^M \to \Omega^{simp}\,\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M,$$

is equivalent to the existence of a homotopy equivalence

$$|\,\mathrm{filt}_{s+n}(E_n, \iota_n)^M| \to |\Omega^{simp}\,\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M| \to \Omega|\,\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M|.$$

The construction of such a homotopy equivalence, as found in [HS05], is quite technical. The idea is to define and equivalent model for the loop space ΩE^M_{n+1}, which is a space $E^{M\times\mathbb{R}}_{n+1,c}$ of "compactly supported functions", and its corresponding differential function complex. Then it is possible to produce a zigzag of homotopy equivalences

$$\mathrm{filt}_{s+n}(E_n, \iota_n)^M \leftarrow \mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^{M\times\mathbb{R}}_c \to \Omega^{simp}\,\mathrm{filt}_{s+n+1}(E_{n+1}, \iota_{n+1})^M.$$

We omit any further details. Interested readers are refered to [HS05].

Similarly to the case of differential function complexes, we have the following proposition.

Proposition 2.9.5. *(The induced differential function spectrum preserves (weak) homotopy equivalences).*

Assume $f : E \to F$ is a (weak) homotopy equivalence, that $\iota \in Z^p(F; \mathbb{R})$ is a cocycle, and that M is a smooth manifold. Then the map

$$\hat{f} : \mathrm{filt}_s(E; f^*\iota)^M \to \mathrm{filt}_s(F; \iota)^M$$

is a (weak) homotopy equivalence.

2.10 Differential K-theory through Hopkins-Singer theory

With the general approach to differential cohomology groups, it would be a shame not to include a brief section on differential K-theory. We will meet differential K-theory and explicit models again in Section 3.6, but for now, we use the Hopkins-Singer definition to define its differential cohomology groups.

Differential K-theory is not only motivated by the above work, where it serves as a simple example of a differential cohomology theory. It is in fact quite important in Type II string theory, where differential K-theory was discovered to be closely linked to RR-fields and anomaly cancellation on D-branes in M-theory ([FW99]). Due to the bundle-theoretic nature of K-theory, and the close link between differential forms and field strengths in physics, it is not surprising that differential K-theory should be of relevance in physics, but the beauty resides in how much power and relevance the abstractions of mathematics have.

To construct differential K-theory, choose any model for the classifying space of K-theory. For notational simplification, we choose the space of Fredholm operators on some Hilbert space, $\mathcal{F}$. These operators are bounded linear operators whose kernel and cokernel are finite-dimensional. $\mathcal{F}$ classifies K-theory by constructing every virtual vector bundle as an index bundle. With $\mathcal{F}$, we only need a cocycle to construct $\hat{K}$. Let $\mathcal{V}_*$ be given by $\mathbb{R}$ in even degrees and 0 in odd degrees, and let us consider

$$\iota = (\iota_n)_n \in Z^0(\mathcal{F}; \mathcal{V}_*)$$

to be a set of representative cocycles for the universal Chern character classes. That is, any vector bundle V is classified by a map $f : M \to \mathcal{F}$, and we pick ι such that $f^*\iota_n$ represents $\mathrm{ch}_n(V) \in H^{2n}(M; \mathbb{R})$. Even though these usually live in a world with integral coefficients, they can be viewed as cycles and cohomology classes with $\mathbb{R}$-coefficients through the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

To find a cocycle in the iterated loop space $\Omega^i\mathcal{F}$, consider the evaluation map

$$S^i \times \Omega^i\mathcal{F} \to \mathcal{F}$$

and consider the cocycle obtained by pulling back ι. This is a cocycle in $Z^0(S^i \times \mathcal{F}; \mathcal{V}_*) = \bigoplus Z^{2n}(S^i \times \mathcal{F}; \mathbb{R})$. By taking the slant product with the fundamental class of S^i, which can be thought of as integrating along S^i, we obtain a cocycle $\iota^{-i} = (\iota^{-i}_{2n-i})_n \in Z^{2n-i}(\Omega^i\mathcal{F}; \mathbb{R})$. This explicit description is the same as picking a fundamental cocycle.

Definition 2.10.1. (Differential K-theory).

Given a smooth manifold M, the differential K-group $\hat{K}^0(M)$ is

$$\hat{K}^0(M) = \pi_0 \, \mathrm{filt}_0(\mathcal{F}; \iota)^M.$$

For the higher K-groups, $\hat{K}^{-i}(M)$ is given by

$$\hat{K}^{-i}(M) = \pi_0 \, \mathrm{filt}_0(\Omega^i\mathcal{F}; \iota^{-i})^M.$$

Before we move on, we make a few important remarks.

Remark 2.10.2. The K-theory spectrum is given by choosing $E_{2n} = \mathcal{F}$ and $E_{2n-1} = \Omega\mathcal{F}$, and Bott periodicity ensures that the maps $E_k \to \Omega E_{k+1}$ are all homotopy equivalences. By the construction of the cocycles of $\Omega^i\mathcal{F}$ as above, the differential cohomology groups induced by the induced differential function spectrum equivalently define differential K-theory.

Remark 2.10.3. We have not yet mentioned any results on uniqueness for differential E-cohomology for general E. This is postponed to Section 3.2. However, with some technical additional assumption about the existence of an integration map or multiplicativity of $\hat{K}$, differential K-theory is unique up to a unique isomorphism.

Remark 2.10.4. An element of $\hat{K}^{-i}(M)$ is a triple (c, ω, h) where $c : M \to \Omega^i\mathcal{F}$, ω is a sequence $(\omega_n)_n$ of $(2n-i)$-forms and h is a sequence $(h_n)_n$ $(2n-i-1)$-cochains such that

$$\delta h = \omega - c^*\iota^{-i}.$$

Due to Bott periodicity, providing a homotopy equivalence $\Omega^2\mathcal{F} \simeq \mathcal{F}$ respecting the cocycles ι_n, we obtain the following proposition by passing the equivalence on through differential function complexes.

Theorem 2.10.5. *(Differential Bott periodicity).*

There is an isomorphism of differential K-groups,

$$\hat{K}^{-n}(M) \cong \hat{K}^{-n-2}(M).$$

Proof. All levels in the construction from the space $\Omega^n\mathcal{F}$ to $\pi_0 \operatorname{filt}_0(\Omega^n\mathcal{F}; \iota^{-n})^M$ preserves the equivalence $\Omega^2\mathcal{F} \simeq \mathcal{F}$, as seen by Theorem 2.7.12. □

Remark 2.10.6. The awkward choice of indexing as $\hat{K}^{-i}$ is due to the fact that we slanted away i dimensions when constructing the cocycles on $\Omega^i\mathcal{F}$. By Differential Bott periodicity, we can reindex, preserving the parity, to define

$$\hat{K}^n(M) = \hat{K}^{n-2N}(M)$$

where N is chosen large enough to ensure $n - 2N \leq 0$.

The only important part when specifying K-groups is whether or not the index is even or odd, meaning we may be sloppy in our indexing, as long as the parity is clear.

As we'll briefly see in Section 3.6, there is an explicit model for differential K-theory, where the only subtle change is replacing vector bundles with a "differential" notion of structured bundles. We come back to this model, results on uniqueness, and other things in later sections.

The Hopkins-Singer constructions arose from generalizing the differential characters of [CS85] in several steps, ending at differential function spectra. With K-theory as an example to define differential cohomology groups, we move on from their work to a more abstract approach to differential cohomology.

2.11 Sheaf Theoretic Differential Cohomology

The idea of exploring differential cohomology through sheaves is not new, as we will see when we meet Deligne cohomology in Section 3.4. From the work of [HS05] in Section 2.9, we have seen that we can differential cohomology groups through differential function spectra. Due to the locality of differential forms, it is natural to be thinking about a sheaf theoretic version of differential cohomology.

The results in the next sections can be found in the work of [BNV16], but we will follow the exposition given by [ADH23]. In the latter reference, much of the theory is written out for some arbitrary presentable ∞-category $\mathcal{C}$. We will work with the stable ∞-category of spectra, **Spt**. Recall that **Spt** is given by the stabilization of the ∞-category of spaces, **Spc**. As long as we have certain technical requirements concerning compact generation, stability, and presentability, we can replace **Spt** with our favorite ∞-category $\mathcal{C}$. Recall that **Spt** is both compactly generated and presentable. For details on the underlying topos theoretic problems and results, we refer to Appendix A in [ADH23].

Definition 2.11.1. (Presheaves and sheaves on manifolds).

We let

$$\mathrm{PSh}(\mathbf{Man};\mathbf{Spt}) := \mathrm{Fun}(\mathbf{Man}^{op},\mathbf{Spt})$$

denote presheaves of spectra on manifolds, and let

$$\mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \subset \mathrm{PSh}(\mathbf{Man},\mathbf{Spt})$$

denote the full subcategory of spectra-valued sheaves on manifolds.

A **Spt**-valued presheaf E is a sheaf if and only if for each manifold M, restricting E to the site $\mathrm{Open}(M)$ of open submanifolds of M yields a sheaf on M considered as a topological space.

Remark 2.11.2. The requirement of being a sheaf when restricted to $\mathrm{Open}(M)$ for each manifold M is equivalent to being a sheaf on the site **Man**, where **Man** is endowed with the Grothendieck topology of open covers.

As in the classical theory of sheaves, the inclusion $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \hookrightarrow \mathrm{PSh}(\mathbf{Man};\mathbf{Spt})$ admits a left adjoint, given the familiar name sheafification.

Definition 2.11.3. (Sheafification).

The left adjoint to the inclusion is denoted $\mathrm{S}_{\mathbf{Man}} : \mathrm{PSh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, and is called sheafification.

Remark 2.11.4. Since **Spt** is found in the second argument of Fun, $\mathrm{PSh}(\mathbf{Man};\mathbf{Spt})$ becomes an ∞-category as well, if we view **Man** as a slightly degenerate ∞-category by the nerve construction. Sheafification can be shown to be a Bousfield localization, as its right adjoint is fully faithful. See for example Proposition 5.1.23. in [Lan21] for further details.

Definition 2.11.5. (The ∞-category of differential cohomology theories).

The ∞-category $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ of sheaves of spectra on **Man** is called the ∞-category of differential cohomology theories.

Definition 2.11.6. (The stalk of E at $x \in M$).

Let $M \in \mathbf{Man}$, fix a point $x \in M$, and let $E \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$. The stalk of E at $x \in M$ is defined as the filtered colimit

$$x^*(E) := \operatorname*{colim}_{U \in \mathrm{Open}_x(M)^{op}} E(U).$$

For any positive real number r, consider the ball $B_{\mathbb{R}^n}(r)$ of radius r around the origin $0_n \in \mathbb{R}^n$, and define

$$0_n^*(E) = \operatorname*{colim}_{k \in \mathbb{Z}_{\geq 1}} E(B_{\mathbb{R}^n}(1/k)).$$

Proposition 2.11.7. *(Equivalent conditions for equivalences in* $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$*).*

The following are equivalent:

1. *A morphism* $f : E \to E'$ *in* $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ *is an equivalence,*
2. *for each integer* $n \geq 0$*, the map* $f(\mathbb{R}^n) : E(\mathbb{R}^n) \to E'(\mathbb{R}^n)$ *is an equivalence in* **Spt**,
3. *for each integer* $n \geq 0$*, the map* $0_n^*(f)$ *is an equivalence in* **Spt**,
4. *for each* $M \in \mathbf{Man}$ *and* $x \in M$*,* $x^*(f)$ *is an equivalence in* **Spt**.

Definition 2.11.8. (Global sections and constant sheaves).

Define the global sections functor $\Gamma_* : \mathrm{PSh}(\mathbf{Man};\mathbf{Spt}) \to \mathbf{Spt}$ by $\Gamma_*(E) = E(*)$, and set $\Gamma^* : \mathbf{Spt} \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ to denote the constant sheaf functor.

It is possible to show that Γ^* is left adjoint to Γ_*, when the latter is restricted to sheaves. The global sections functor, Γ_*, also admits a right adjoint.

Proposition 2.11.9. *(Right adjoint to* Γ_**, [ADH23, Section 4.]).*

Define the functor $\Gamma^! : \mathbf{Spt} \to \mathrm{PSh}(\mathbf{Man};\mathbf{Spt})$ *by*

$$\Gamma^!(X)(M) := \prod_{m \in M} X.$$

This functor is fully faithful, right adjoint to Γ_**, and it factors through* $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$*. The adjunction* $\Gamma_* \dashv \Gamma^!$ *also holds when restricted to sheaves.*

We are now in a situation where we have an adjoint triple

$$\begin{tikzcd} \mathbf{Spt} \arrow[r, bend left, "\Gamma^*"] \arrow[r, bend right, "\Gamma^!"'] & \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt}). \arrow[l, "\Gamma_*"'] \end{tikzcd}$$

We state the following well-known property for such adjoint triples, which also holds for 1-categories.

Proposition 2.11.10. *(Fully faithful maps in adjoint triples).*

In such an adjoint triple, the upper map Γ^* is fully faithful if and only if the lower map $\Gamma^!$ is fully faithful.

This is the essence of why we need $\Gamma^!$. It does not play an important role in differential cohomology other than ensuring useful properties, such as showing Γ^* is fully faithful or that Γ_* is a left adjoint and hence preserves colimits.

We also obtain this useful corollary.

Corollary 2.11.10.1. *(Sheafification does not change global sections).*

Given $E \in \mathrm{PSh}(\mathbf{Man}; \mathbf{Spt})$, the map $E \mapsto \mathrm{S}_{\mathbf{Man}}\, E$ induces an equivalence on global sections.

Proof. For all $X \in \mathbf{Spt}$, we can show that the morphism

$$\mathrm{map}_{\mathbf{Spt}}(\Gamma_* \, \mathrm{S}_{\mathbf{Man}}\, E, X) \to \mathrm{map}_{\mathbf{Spt}}(\Gamma_* E, X)$$

is an equivalence. This follows from the $\Gamma_* - \Gamma^!$-adjunction and the fact that $\Gamma^!$ factors through $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$, since the above is equivalent to

$$\mathrm{map}_{\mathrm{PSh}(\mathbf{Man};\mathbf{Spt})}(\mathrm{S}_{\mathbf{Man}}\, E, \Gamma^! X) \to \mathrm{map}_{\mathrm{PSh}(\mathbf{Man};\mathbf{Spt})}(E, \Gamma^! X).$$

Now, since $\Gamma^! X$ already is a sheaf, the universal property of sheafification tells us that this is an equivalence. □

To study $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$ as differential cohomology, it will be important to understand the homotopy invariant part of $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$.

Definition 2.11.11. (Homotopy invariant presheaves).

We say an $F \in \mathrm{PSh}(\mathbf{Man}; \mathbf{Spt})$ is homotopy invariant if for all manifolds M, the projection $\mathrm{pr}_M : M \times \mathbb{R} \to M$ induces an equivalence

$$\mathrm{pr}_M^* : F(M) \simeq F(M \times \mathbb{R}).$$

We denote by $\mathrm{PSh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt}) \subseteq \mathrm{PSh}(\mathbf{Man}; \mathbf{Spt})$ and $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt}) \subseteq \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$ the homotopy invariant presheaves and sheaves of spectra on manifolds.

Proposition 2.11.12. *(Homotopy invariance captures homotopy invariance).*

A (pre)sheaf F is homotopy invariant in the sense of the above definition if and only if for each homotopy equivalence $M \to N$ of manifolds, the map $F(N) \to F(M)$ is an equivalence in **Spt***.*

Proof. If F preserves homotopy equivalences, then we certainly have homotopy invariance, as $M \simeq M \times \mathbb{R}$. If F is homotopy invariant in the above sense, then for maps $f : M \longleftrightarrow N : g$ that are homotopic to the identity when composed, we obtain the following diagrams witnessing the homotopy.

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "id_M \times \{0\}"'] \arrow[dr, "f \circ g"] & N \arrow[d, "f"] & & F(M) & \\
M \times \mathbb{R} \arrow[r, "H"'] & M & \Longrightarrow & F(M \times \mathbb{R}) \arrow[u] \arrow[d] & F(M) \arrow[l, "H^*"] \arrow[ul, "(f \circ g)^*"'] \arrow[dl, "(id_M)^*"] \\
M \arrow[u, "id_M \times \{1\}"] \arrow[ur, "id_M"'] & & & F(M) &
\end{tikzcd}$$

Since the vertical inclusions are sections of pr_M, i.e. $\mathrm{pr}_M \circ (id_M \times \{x\}) = id_M$, we obtain that $(id_M \times \{x\})^*$ must also be an equivalence by the two-out-of-three-property after applying F and our assumption. This implies H^* is an equivalence, and consequently that $(f \circ g)^*$ is an equivalence. By a similar argument for N instead of M, we conclude that both $(f \circ g)^*$ and $(g \circ f)^*$ are equivalences. $\square$

These homotopy invariant sheaves have the nice property that global sections describes equivalences.

Proposition 2.11.13. *(Global sections and homotopy invariant sheaves).*

A morphism f in $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt})$ is an equivalence if and only if $\Gamma_*(f)$ is an equivalence in **Spt***.*

Proof. This follows from Proposition 2.11.7 and the contractibility of $\mathbb{R}^n$. $\square$

The subcategories $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt}) \subseteq \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$ and $\mathrm{PSh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt}) \subseteq \mathrm{PSh}(\mathbf{Man}; \mathbf{Spt})$ are presentable (see [Lur17]), implying the existence of both left and right adjoints to these inclusions by the Adjoint Functor Theorem ([Lan21]).

Definition 2.11.14. ($\mathbb{R}$-localization and left/right homotopification).

Denote by $L_{\mathbb{R}} : \mathrm{PSh}(\mathbf{Man}; \mathbf{Spt}) \to \mathrm{PSh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt})$ and $L_{hi} : \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt}) \to \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt})$ the left adjoints to the inclusions of homotopy invariant sheaves, $\mathrm{PSh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt}) \subseteq \mathrm{PSh}(\mathbf{Man}; \mathbf{Spt})$

and $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \subseteq \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, respectively. The functor

$$R_{hi} : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$$

is the right adjoint to the latter inclusion.

These functors are called $\mathbb{R}$-localization and left/right homotopification in the case of presheaves and sheaves, respectively.

As we seek to avoid too many technical results regarding these functors and their properties, we will rush through some important results and their consequences to understand the sheaf theoretic picture better. All results can be found in Sections 4 and 5 of [ADH23] and the references therein.

Proposition 2.11.15. *(The constant sheaf, global sections and* $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$*).*

The constant sheaf functor $\Gamma^* : \mathbf{Spt} \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ *factors through* $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$*. Moreover, when restricting the global sections functor to* $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$*,* $\Gamma_* : \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \to \mathbf{Spt}$ *is an equivalence with* Γ^* *as its inverse.*

Remark 2.11.16. From the equivalence $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \simeq \mathbf{Spt}$, we can think of $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ as the correct homotopical part of sheaf theoretic differential cohomology.

We are now in the following scenario,

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[r, bend left, "L_{hi}"] \arrow[r, bend right, "R_{hi}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \arrow[l, hook'] \arrow[r, bend left, "\Gamma_*"] \arrow[r, phantom, "\simeq"] & \mathbf{Spt}. \arrow[l, bend left, "\Gamma^*"]
\end{tikzcd}$$

Remark 2.11.17. We can write

$$\mathrm{map}_{\mathbf{Spt}}(\Gamma_* L_{hi} E, X) \cong \mathrm{map}_{\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})}(L_{hi}E, \Gamma^* X) \cong \mathrm{map}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})}(E, \Gamma^* X)$$

for any $E \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ and $X \in \mathbf{Spt}$ to see that $\Gamma_* L_{hi}$ is left adjoint to Γ^*, when Γ^* is viewed as a $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ functor through the inclusion $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \hookrightarrow \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$.

This motivates the following definition.

Definition 2.11.18. (The extreme left adjoint $\Gamma_!$).

Define the functor

$$\Gamma_! := \Gamma_* L_{hi} : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathbf{Spt},$$

which is left adjoint to the constant sheaf functor, Γ^*.

Remark 2.11.19. We have $L_{hi} \simeq \Gamma^*\Gamma_!$ and by a similar computation, $R_{hi} \simeq \Gamma^*\Gamma_*$.

If we view the Γ's as functors between $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ and $\mathbf{Spt}$, we are now in the following scenario, where all maps are written over their right adjoints,

$$\begin{tikzcd} \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[rr, bend left=50, "\Gamma_!"] \arrow[rr, bend right=20, "\Gamma_*"'] & & \mathbf{Spt}. \arrow[ll, bend right=20, "\Gamma^*"'] \arrow[ll, bend left=50, "\Gamma^!"] \end{tikzcd}$$

Queequeg 2.11.20. (Towards Cohesive ∞-topoi). Interested readers may take a look at [Sch13] and the one-paragraph summary found in subsection 2.14 for the importance of this adjoint quadruple when defining differential cohomology theories in cohesive ∞-topoi.

The functors $L_{\mathbb{R}}$ and $\Gamma_!$ can in fact be given explicit formulae. Do note that an explicit formula for $\Gamma_!$ will induce an explicit formula for L_{hi} as well.

Proposition 2.11.21. *(Morel-Suslin-Voevodsky construction, explicit formula for $L_{\mathbb{R}}$, [ADH23, Proposition 5.1.2.]).*

The functor

$$L_{\mathbb{R}} : \mathrm{PSh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{PSh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$$

is explicitly given by

$$L_{\mathbb{R}}(F)(M) = |F(M \times \Delta^*)|,$$

where $|-|$ denotes the geometric realization of the abstract simplicial object $F(M \times \Delta^*)$ in $\mathbf{Spt}$.

Using the above result by Morel-Suslin-Voevodsky, originally formulated in a very general manner, it is possible to show how sheafification interacts with $L_{\mathbb{R}}$. We gather all the results in [ADH23] regarding this interaction into a single proposition and omit the details.

Proposition 2.11.22. *(Sheafification and $\mathbb{R}$-localization).*

1. *For $F \in \mathrm{PSh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$, the counit $\Gamma^*\Gamma_* \mathrm{S}_{\mathbf{Man}} F \to \mathrm{S}_{\mathbf{Man}} F$ is an equivalence.*
2. *In particular, $\mathrm{S}_{\mathbf{Man}} F$ is homotopy invariant, i.e. $\mathrm{S}_{\mathbf{Man}} F \in \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$.*
3. *The composition $\mathrm{S}_{\mathbf{Man}} L_{\mathbb{R}} : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ factors through $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$.*
4. *The above composition is left-adjoint to the inclusion $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \subseteq \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$.*
5. *In particular, $L_{hi} \simeq \mathrm{S}_{\mathbf{Man}} L_{\mathbb{R}}$.*

The last part of the above proposition yields the following explicit formula.

Proposition 2.11.23. *(Explicit formula for $\Gamma_!$).*

The functor $\Gamma_! : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathbf{Spt}$, which is left adjoint to the constant sheaf functor Γ^*, can be explicitly given by

$$\Gamma_!(E) \simeq |E(\Delta^*)|.$$

Proof. Since $L_{hi} \simeq \mathrm{S}_{\mathbf{Man}}\, L_{\mathbb{R}}$, an explicit formula for $\Gamma_!$ is given by $\Gamma_*\, \mathrm{S}_{\mathbf{Man}}\, L_{\mathbb{R}}$. It is therefore enough to show that

$$\Gamma_*\, \mathrm{S}_{\mathbf{Man}}\, L_{\mathbb{R}}(E) \simeq |E(\Delta^*)|.$$

Due to Corollary 2.11.10.1, we know $\Gamma_*\, \mathrm{S}_{\mathbf{Man}}\, L_{\mathbb{R}}(E) \simeq \Gamma_* L_{\mathbb{R}}(E)$. From the explicit formula of $L_{\mathbb{R}}$ (Proposition 2.11.21) we obtain

$$\Gamma_!(E) \simeq \Gamma_*\, \mathrm{S}_{\mathbf{Man}}\, L_{\mathbb{R}}(E) \simeq \Gamma_* L_{\mathbb{R}}(E) \simeq L_{\mathbb{R}}(E)(*) \simeq |E(* \times \Delta^*)|.$$

$\square$

Corollary 2.11.23.1. *(Explicit formula for L_{hi}).*

Since $L_{hi} \simeq \Gamma^*\Gamma_!$, left homotopification $L_{hi} : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ is given by the constant sheaf $E \mapsto \Gamma^*|E(\Delta^*)|$.

If we take a step back and breathe out after this adjoint bonanza, we can summarize the above by saying we have an ∞-category of differential cohomology theories, $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, with a concrete subcategory $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \simeq \mathbf{Spt}$ explaining the homotopical part of differential cohomology. In a more diagrammatic manner, we have

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[r, bend left, "L_{hi}"] \arrow[r, bend right, "R_{hi}"'] & \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \arrow[l, hook', "\perp"', "\perp"] \arrow[r, bend left, "\Gamma_*"] \arrow[r, phantom, "\simeq"] & \mathbf{Spt}. \arrow[l, bend left, "\Gamma^*"]
\end{tikzcd}$$

2.12 Stable Recollements

Can we use the subcategory $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ and the maps above to say something about the "non-homotopical" part of $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$? The theory of stable recollements will luckily help us to answer this question.

The theory of stable recollements can be formulated quite abstractly, and there are many technicalities related to this theory. As we want to apply it to the scenario described above, we will more or less run through the most relevant results and definitions. For more general or in-depth treatments, consult [Lur17, Appendix A.8.] or [BG16] for more (∞-)categorical technicalities, and [ADH23] or [BNV16] for its consequences for differential cohomology.

As motivation for recollements, we consider the following simple scenario. Let X be a topological space and let U be an open subset of X. Denote by Y the complement of U in X, i.e. $Y = X\backslash U$. We have inclusions $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$. If we are given a sheaf (of sets) on X, say $\mathcal{F}$, we can pull $\mathcal{F}$ back to sheaves on U and Y by forming $\mathcal{F}_U = j^*\mathcal{F}$ and $\mathcal{F}_Y = i^*\mathcal{F}$, respectively. A natural question one can ask is if we can reattach $\mathcal{F}_U$ to $\mathcal{F}_Y$ to build $\mathcal{F}$. This is precisely the idea of recollements[3].

The pullbacks i^* and j^* have well-known right adjoints given by forming the direct images, yielding functors i_* and j_*, respectively. Consider for example the unit of the last adjunction, i.e. the natural transformation. $\eta^j : \mathrm{Id}_{\mathrm{Sh}(X)} \to j_*j^*$. This can be used to "relate $\mathcal{F}_Y$ to $\mathcal{F}_U$", which will be needed as a tool to glue them together to obtain $\mathcal{F}$. The way we relate $\mathcal{F}_Y$ to $\mathcal{F}_U$ is by applying i^* to the unit to obtain a morphism

$$u = i^*\eta^j_{\mathcal{F}} : \mathcal{F}_Y \to (i^*j_*)\mathcal{F}_U.$$

Amazingly enough, this is enough data to recover $\mathcal{F}$ from $\mathcal{F}_U$ and $\mathcal{F}_Y$, as it is possible to show that the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\eta^i_{\mathcal{F}}"] \arrow[d, "\eta^j_{\mathcal{F}}"'] & i_*\mathcal{F}_Y \arrow[d, "i_*(u)"] \\
j_*\mathcal{F}_U \arrow[r, "\eta^i_{j_*\mathcal{F}_U}"'] & i_*i^*j_*\mathcal{F}_U
\end{tikzcd}$$

is a pullback diagram.

Remark 2.12.1. As we will see, this is not just a sheafwise phenomenon. It is in fact possible to reconstruct the category of sheaves on X from the categories of sheaves on the subspaces, as long as we have the gluing functor i^*j_*.

The main takeaway is that we have inclusions of U and Y into X, which yield covariant and fully faithful maps on sheaf categories through direct images. The direct images i_*, j_* admit left adjoints, i^*, j^*, which are both left exact, i.e. they preserve limits, and these maps relate the sheaf categories of U and Y through a gluing map.

The above can be summarized (a bit more generally) in the following definition.

Definition 2.12.2. (Recollements).

If $\mathcal{C}$ is an ∞-category with finite limits, and $\mathcal{C}_0, \mathcal{C}_1$ are full subcategories of $\mathcal{C}$. We say $\mathcal{C}$ is a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$ if:

[3]The french word *recollements* encapsulates this quite well, as it translates to *reattachments*.

1. $\mathcal{C}_0$ and $\mathcal{C}_1$ are stable under equivalence ($\mathcal{C} \ni X \simeq Y \in \mathcal{C}_i \implies X \in \mathcal{C}_i$).
2. The inclusion functors $\mathcal{C}_i \hookrightarrow \mathcal{C}$ admits left adjoints L_i.
3. The functors L_i are left exact.
4. The functor L_1 carries $\mathcal{C}_0$ to the terminal object of $\mathcal{C}$.
5. A morphism α in $\mathcal{C}$ is an equivalence if and only if the morphisms $L_i(\alpha)$ in $\mathcal{C}_i$ are equivalences.

Remark 2.12.3. [Lur17] and others often say the localization L_0 is *complementary* to the localization L_1.

Remark 2.12.4. Due to the notation in our motivational example and our future use case, we will not use L_i to denote these left adjoint functors anywhere else than in this definition. When the adjoints are present, it will be clear from context. We will continue to use the notation $\mathcal{C}_i$ from [Lur17] and the notation i_*, i^*, j_*, j^* from [ADH23].

Remark 2.12.5. Note that the definition of a recollement is not symmetric. If $\mathcal{C}$ is the recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$ as above, $\mathcal{C}$ is not necessarily the recollement of $\mathcal{C}_1$ and $\mathcal{C}_0$.

Remark 2.12.6. As the section title may indicate, we will be interested in the theory of recollements for stable ∞-categories, as this is much nicer than the general theory. Many definitions and results still hold either way, but readers should be cautious whether a statement is given in the stable case or not.

Example 2.12.7. (Sheaves revisited, [Lur17, A.8.4]).

If we are given a topological space X with a closed embedding $i : Y \to X$ and an open embedding $j : U \to X$, we can construct $\mathrm{Sh}(X), \mathrm{Sh}(U)$ and $\mathrm{Sh}(Y)$ to be sheaves of sets on the respective spaces. Consider $\mathcal{C} = N(\mathrm{Sh}(X))$ to be the ∞-category given by taking the nerve of $\mathrm{Sh}(X)$.

If we set $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{C}_1 \subseteq \mathcal{C}$ to be the induced image of the direct image functors i_* and j_*, respectively, it can be shown that $\mathcal{C}$ is a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$ as long as we assume $Y \cap U = \emptyset$. In particular, if $Y = X \backslash U$ as in our initial example, $N(\mathrm{Sh}(X))$ is a recollement of $N(\mathrm{Sh}(Y))$ and $N(\mathrm{Sh}(U))$.

Remark 2.12.8. A similar example can be made for schemes and quasicoherent sheaves. See [ADH23, Example 6.1.6.].

After seeing the example of sheaves above, we long back to sheaf theoretic differential cohomology. However, it is not obvious how to relate this decomposition of sheaves to the setting of differential cohomology. Luckily, the following result in [Lur17] solves our problems.

Theorem 2.12.9. *(Relating adjoints to recollements, [Lur17, A.8.20.]).*

Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory closed under equivalence. Then the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits left and right adjoints if and only if there exists a full subcategory $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $\mathcal{C}$ is a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$.

Moreover, we can identify $\mathcal{C}_1$ with the full subcategory spanned by the right orthogonal objects to $\mathcal{C}_0$. That is, $\mathcal{C}_1 = \mathcal{C}_0^{\perp} \subseteq \mathcal{C}$, where $\mathcal{C}_0^{\perp}$ is spanned by those objects Y such that the mapping space $\mathrm{map}_{\mathcal{C}}(X, Y)$ is contractible for each $X \in \mathcal{C}_0$.

We often call $\mathcal{C}_1$ the right Hom*-complement of $\mathcal{C}_0$.*

Since $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ is a stable ∞-category and $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ is a full subcategory closed under equivalences and the functors L_{hi} and R_{hi} are left and right adjoints to the inclusion, we can find a full subcategory $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})^{\perp}$ such that $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ is a recollement of $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ and $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})^{\perp}$. We return to differential cohomology in the next section.

In a decomposition, such as the one we intuitively obtain from a recollement, questions about equivalences and important properties can often be checked "locally". We informally summarize some results from [Lur17] in the following remark, explaining the relevance of recollements to "local" questions.

Remark 2.12.10. (Proto-theorems on local conditions, [Lur17, A.8.14. and A.8.17.]).

Let $\mathcal{C}$ and $\mathcal{C}'$ be ∞-categories that can be exhibited as the recollements of $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}'_0, \mathcal{C}'_1$, respectively. With the appropriate assumptions, a functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence if it restricts to equivalences $\mathcal{C}_0 \simeq \mathcal{C}'_0$ and $\mathcal{C}_1 \simeq \mathcal{C}'_1$. Similarily, with some additional assumptions, $\mathcal{C}$ can be shown to be a stable ∞-category if the ∞-category $\mathcal{C}_0$ and $\mathcal{C}_1$ are stable.

As mentioned, we want to study the complement to $\mathcal{C}_0 = \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$, given by the right orthogonal objects, $\mathcal{C}_0^{\perp}$ to obtain an explicit decomposition of differential cohomology. Before we do this, we must state the main results concerning stable recollements. We omit most proofs and refer to [ADH23, Section 6.], [Lur17, Appendix A.8.] and the references therein for details.

Theorem 2.12.11. *(Fracture squares from recollements).*

If $i_* : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $j_* : \mathcal{C}_1 \hookrightarrow \mathcal{C}$ exhibit $\mathcal{C}$ as the recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$, we have a pullback square of ∞-categories,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "j^*"'] & \mathrm{Fun}([1], \mathcal{C}_0) \arrow[d, "target"] \\ \mathcal{C}_1 \arrow[r, "i^* j_*"] & \mathcal{C}_0. \end{tikzcd}$$

The top horizontal arrow is given by mapping $X \in \mathcal{C}$ to the morphism in $\mathcal{C}$ defined by applying i^* to the unit $X \to j_* j^*(X)$. Furthermore, by using the units of the adjunctions, we have a pullback square of endofunctors on $\mathcal{C}$, called the fracture square of the recollement. The fracture square is the pullback square

$$\begin{tikzcd} \mathrm{Id}_{\mathcal{C}} \arrow[r] \arrow[d] & j_* j^* \arrow[d] \\ i_* i^* \arrow[r] & i_* i^* j_* j^*. \end{tikzcd}$$

By further investigation of the maps i_*, i^*, j_* and j^*, it is possible to show that they have further adjoints, and that they fit in suitable fibre sequences, given suitable assumptions on $\mathcal{C}_0$ and $\mathcal{C}$. First we recall the following well-known facts, which also holds for 1-categories.

Proposition 2.12.12. *(Units and counits for fully faithful adjoints).*

If $L : \mathcal{C} \to \mathcal{D}$ is left adjoint to $R : \mathcal{D} \to \mathcal{C}$, then:

1. *R is fully faithful if and only if the counit $\epsilon : L \circ R \to \mathrm{Id}$ is a natural isomorphism.*
2. *L is fully faithful if and only if the unit $\eta : \mathrm{Id} \to R \circ L$ is a natural isomorphism.*

Theorem 2.12.13. *(Further adjoints and fiber sequences in recollements).*

1. *If $\mathcal{C}_0$ has an initial object, then $j^* : \mathcal{C} \to \mathcal{C}_1$ admits a fully faithful left adjoint $j_! : \mathcal{C}_1 \to \mathcal{C}$.*
2. *$j_* : \mathcal{C}_1 \to \mathcal{C}$ is fully faithful, and the counit $j^* j_* \to \mathrm{Id}_{\mathcal{C}_1}$ is an equivalence.*
3. *If in addition $\mathcal{C}$ has a zero object, $i_* : \mathcal{C}_0 \to \mathcal{C}$ admits a right adjoint $i^! : \mathcal{C} \to \mathcal{C}_0$.*
4. *The right adjoint $i^!$ satisfies*
$$i_* i^! \simeq \mathrm{fib}(\eta : \mathrm{Id}_{\mathcal{C}} \to j_* j^*)$$
as an endofunctor on $\mathcal{C}$ and sits in a fiber sequence
$$i^! \to i^* \to i^* j_* j^*$$
when pulled back to an endofunctor on $\mathcal{C}_0$.
5. *If $\mathcal{C}$ is stable, then $\mathcal{C}_0$ and $\mathcal{C}_1$ are stable, and we have a fiber sequence with the counit and unit, respectively,*
$$j_! j^* \to \mathrm{Id}_{\mathcal{C}} \to i_* i^*.$$

Remark 2.12.14. In the stable case, we therefore obtain

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C}_1. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

As in Theorem 2.12.9, the structure in the left-hand side of this diagram is equivalent to the existence of a $\mathcal{C}_1$ such that $\mathcal{C}$ is the recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$. The right hand side of the diagram, i.e. the adjoints related to $\mathcal{C}_1$, follows in the stable case from $\mathcal{C}_1$ being a component in the recollement of $\mathcal{C}$.

Queequeg 2.12.15. (Stable recollements, local duality, and chromatic homotopy theory).

Strictly speaking, we obtain "another $\mathcal{C}_1$" in the sense that we could have formed $\mathcal{C}_1$ through taking left Hom-complements. However, this will be equivalent, unifying the two to obtain the scenario presented above. This can be found outlined in [ADH23]. The interesting thing is that each Hom-complement contribute with one adjoint each. With our notation, the right Hom-complement contributes with the left adjoint, and the left Hom-complement contributes to the right adjoint, as can be seen from [ADH23].

In [BHV18a], [BHV18b] and the references therein, this is closely related to the notion of a local duality context (which we have not defined). In a stable recollement-inducing manner, these local duality contexts yield diagrams as below. In the leftmost diagram $\mathcal{C}$ is a stable ∞-category and $\mathcal{T}$ is a localizing subcategory of $\mathcal{C}$ generated by compact objects of $\mathcal{C}$. In the rightmost diagram, we have an example where $\mathcal{C}$ is the $E(n)$-local spectra and $\mathcal{T}$ is the the monochromatic layer at height n.

$$\begin{tikzcd}
 & \mathcal{T}^{\perp} \arrow[dd, dashed, bend left, "(-)^{\perp}"] \arrow[d, shift left] & \\
 & \mathcal{C} \arrow[u, shift left] \arrow[u, shift left=3] \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathcal{T} \arrow[uur, dashed, bend left, "(-)^{\perp}"] \arrow[ur, shift left] \arrow[rr, dash, "\simeq"'] & & \mathcal{T}^{\perp\perp} \arrow[ul, shift left]
\end{tikzcd}
\qquad
\begin{matrix} \mathcal{C} = \mathbf{Spt}_{E(n)} \\ \Longrightarrow \\ \mathcal{T} = \mathcal{M}_n \end{matrix}
\qquad
\begin{tikzcd}
 & \mathbf{Spt}_{E(n-1)} \arrow[dd, dashed, bend left, "(-)^{\perp}"] \arrow[d, shift left] & \\
 & \mathbf{Spt}_{E(n)} \arrow[u, shift left] \arrow[u, shift left=3] \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathcal{M}_n \arrow[uur, dashed, bend left, "(-)^{\perp}"] \arrow[ur, shift left] \arrow[rr, dash, "\simeq"'] & & \mathbf{Spt}_{K(n)} \arrow[ul, shift left]
\end{tikzcd}$$

These local duality contexts induce two equivalent stable recollements of $\mathcal{C}$ in terms of $(\mathcal{T}, \mathcal{T}^{\perp})$ and $(\mathcal{T}^{\perp}, \mathcal{T}^{\perp\perp})$. The fracture square obtained from the latter stable recollement in the case of $E(n)$-local spectra realizes $E(n)$-localization as a suitable pullback of $E(n-1)$-localization and $K(n)$-localization. More informally, this means we "reattach" information about chromatic height $(\leq n-1)$ to information about chromatic height n in order to obtain information about chromatic height $(\leq n)$, as found in $E(n-1), K(n)$, and $E(n)$, respectively. This fracture square is a classic result in chromatic homotopy theory, as illustrated in [Lur10].

With this digressive illustration of the power of stable recollements, we return to understand the fiber sequences in the above theorem.

From now on, assume that we work in the stable case, i.e. the case where $i_* : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $j_* : \mathcal{C}_1 \hookrightarrow \mathcal{C}$ exhibit a stable ∞-category $\mathcal{C}$ as a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$.

The preceding theorem contained several fiber sequences, without any clear instruction on how they are related. It turns out the fracture square can be extended, but we first need to define the norm map related to the right hand side of the above diagram.

Definition 2.12.16. (Norm map in a fully faithful adjoint triple).

Consider an adjoint triple

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "j^*"] & \mathcal{C}_1, \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

where the extreme adjoints are fully faithful.

As j_* is fully faithful, the counit $j^*j_* \to \mathrm{Id}_{\mathcal{C}_1}$ is an equivalence. We can compute

$$\mathrm{map}(j_!, j_*) \simeq \mathrm{map}(\mathrm{Id}_{\mathcal{C}_1}, j^*j_*) \simeq \mathrm{map}(\mathrm{Id}_{\mathcal{C}_1}, \mathrm{Id}_{\mathcal{C}_1}).$$

It is now possible to pick a map between the extreme adjoints $j_! \to j_*$, that corresponds to the identity $\mathrm{Id}_{\mathcal{C}_1} \to \mathrm{Id}_{\mathcal{C}_1}$. This is the natural transformation called the *norm* of the adjoint triple, Nm.

The above construction yields enough data to staple the fiber sequences together to an extended fracture square.

Theorem 2.12.17. *(An extended fracture square).*

The norm fits in the fibre sequence

$$j_!j^* \xrightarrow{\mathrm{Nm}\, j^*} j_*j^* \longrightarrow i_*i^*j_*j^*.$$

Moreover, all the previously defined fiber sequences fit in an extended fracture square,

$$\begin{tikzcd}
 & j_!j^* \arrow[r, "\mathrm{Id}"] \arrow[d] & j_!j^* \arrow[d] \\
i_*i^! \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathrm{Id}_{\mathcal{C}} \arrow[r] \arrow[d] & j_*j^* \arrow[d] \\
i_*i^! \arrow[r] & i_*i^* \arrow[r] & i_*i^*j_*j^*,
\end{tikzcd}$$

where all sequences in the rows and columns are fiber sequences.

With the extended fracture square, we wrap up this brief section on (stable) recollements, and return to study the scenario

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[r, bend left, "L_{hi}"] \arrow[r, bend right, "R_{hi}"'] & \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \arrow[l, hook', "\perp" description] \arrow[r, bend left, "\Gamma_*"] \arrow[r, phantom, "\simeq"] & \mathbf{Spt}. \arrow[l, bend left, "\Gamma^*"]
\end{tikzcd}$$

2.13 Decomposing Differential Cohomology and rediscovering the Hexagon

In the above setup we have an inclusion $\mathcal{C}_0 = \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \hookrightarrow \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) = \mathcal{C}$, admitting a left and a right adjoint. These adjoints are the homotopification functors L_{hi} and R_{hi}, which we have explicit formulas for. From Theorem 2.12.9, we know that the subcategory $\mathcal{C}_1 = \mathcal{C}_0^{\perp} \hookrightarrow \mathcal{C}$ of right ortogonal objects to $\mathcal{C}_0$ will contribute to exhibiting $\mathcal{C}$ as a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$. Intuitively, $\mathcal{C}_1$ makes up the "geometric part" of differential cohomology, or at least contain information about differential forms.

Remark 2.13.1. (Warning: Homotopy invariant geometry).

Although we want differential cohomology to intuitively split into a geometric and a homotopical part, we must acknowledge that there is "geometric info" in $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \simeq \mathbf{Spt}$ as well.

Consider for example the sheaf $\Omega^* \in \mathrm{Sh}(\mathbf{Man}; D(\mathbb{R}))$ of differential forms, and recall that for each manifold M we may use $\Omega^*(M)$ to make an Eilenberg-MacLane spectrum $H(\Omega^*(M))$. By blanking out the manifold, we obtain a sheaf of spectra. The sheaf of spectra given by $H\Omega^* \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ is homotopy invariant, which means that it lies in $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$, and hence on the "wrong side" of the decomposition, if it meant to resemble "geometric information".

However, as we will see, the sheaf of spectra given by filtered forms, $H\Omega^{*\geq q}$ for $q \geq 1$, will lie in the "geometric" part of the recollement, i.e. in $\mathcal{C}_1$. This is rather counter-intuitive, as $H\Omega^{*\geq q}(M)$ lies in $\mathbf{Spt} \simeq \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ when evaluated at a manifold.

To understand the wanted recollement, we define $\mathcal{C}_1$.

Definition 2.13.2. (Pure sheaves as $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})^{\perp}$).

The sheaves $E \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ that are right ortogonal to $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ are called *pure* sheaves. Denote the full subcategory of pure sheaves by

$$\mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt}) := \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})^{\perp} \subset \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}).$$

We have the following simple condition for being a pure sheaf.

Proposition 2.13.3. *(Pure sheaves as the kernel of global sections).*

A sheaf E is pure if and only if $\Gamma_* E = E(*) \simeq 0$

Proof. A sheaf E is pure if and only if $\mathrm{map}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})}(F, E) = 0$ for all $F \in \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$. Since $\mathbf{Spt}$ is equivalent to $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ through Γ^*, we may consider this as a question in $\mathbf{Spt}$. Since $F \simeq \Gamma^*(X)$ for some $X \in \mathbf{Spt}$, we compute

$$\mathrm{map}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})}(\Gamma^*(X), E) \simeq \mathrm{map}_{\mathbf{Spt}}(X, \Gamma_*(E)).$$

This is contractible for all X if and only if $\Gamma_*(E)$ is the terminal object in **Spt**, i.e. $\Gamma_*(E) \simeq 0$, which proves the statement. $\square$

Example 2.13.4. (Purity of $H\Omega^*$ and $H\Omega^{*\geq q}$).

From the above condition for being a pure sheaf, we immediately obtain that $H\Omega^*$ is not pure due to the copy of $\mathbb{R}$ coming from the 0-forms, but $H\Omega^{*\geq q}$ is pure as long as $q \geq 1$.

As $\mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt})$ plays the role of $\mathcal{C}_1$ in the recollement, we know there should be adjoints and a fracture square involved. To compare notation, we are applying the theory of stable recollements where we set $i^* = L_{hi}$ and $i^! = R_{hi}$ and i_*, j_* are the inclusions of homotopy invariant and pure sheaves, respectively. We then know there are fiber sequences consisting of units and counits

$$i_*i^! \to \mathrm{Id}_{\mathcal{C}} \to j_*j^*$$

and

$$j_!j^* \to \mathrm{Id}_{\mathcal{C}} \to i_*i^*.$$

By comparing notation and considering all functors to be endofunctors on $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, we only lack the following constructions.

Definition 2.13.5. (Differential cycles and curvature).

Let the functor $\mathrm{Cyc} : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ and the natural transformation

$$\mathrm{curv} : \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to \mathrm{Cyc}$$

be given by the cofiber sequence

$$R_{hi} \to \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to \mathrm{Cyc}\,.$$

The first map is the counit $R_{hi} \simeq \Gamma^*\Gamma_* \to \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})}$.

Definition 2.13.6. (Differential deformations).

Let the functor

$$\mathrm{Def} : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$$

be given by the fiber sequence

$$\mathrm{Def} \to \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to L_{hi}.$$

The last map is the unit $\mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to \Gamma^*\Gamma_! \simeq L_{hi}$.

For a given sheaf $\hat{E} \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, the above functors are called the sheaves of differential cycles and differential deformations associated to $\hat{E}$, respectively. Cyc factors through $\mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt})$ and has $\mathrm{Def} : \mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt}) \to \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ as its left adjoint.

These functors sit in an adjoint triple

$$\begin{tikzcd} \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[r, dashed, "\mathrm{Cyc}"] & \mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt}), \arrow[l, bend right, "\mathrm{Def}"'] \arrow[l, bend left, "incl"] \end{tikzcd}$$

which implies that Def is fully faithful. In the notation used earlier, $j_! = \mathrm{Def}$ and $j^* = \mathrm{Cyc}$.

We now have everything we need to compute the fracture square, as we have

$$\begin{tikzcd} \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}) \arrow[r, dashed, "incl"] & \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[l, bend right, "L_{hi}"'] \arrow[l, bend left, "R_{hi}"] \arrow[r, dashed, "\mathrm{Cyc}"] & \mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt}). \arrow[l, bend right, "\mathrm{Def}"'] \arrow[l, bend left, "incl"] \end{tikzcd}$$

For reasons that will be clear soon, we write $d : \mathrm{Def} \to \mathrm{Cyc}$ for the composition

$$d : \mathrm{Def} \to \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to \mathrm{Cyc},$$

where the last map is the curvature, curv. This takes the place of the norm map in the fracture square.

By comparing notation to Theorem 2.12.17, we get the fracture square for differential cohomology.

Corollary 2.13.6.1. *(An extended fracture square in differential cohomology).*

We have a commutative diagram of endofunctors on $\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$*,*

$$\begin{tikzcd} & \mathrm{Def} \arrow[r, "\mathrm{Id}"] \arrow[d] & \mathrm{Def} \arrow[d, "d"] \\ R_{hi} \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \arrow[r] \arrow[d] & \mathrm{Cyc} \arrow[d] \\ R_{hi} \arrow[r] & L_{hi} \arrow[r] & L_{hi}\,\mathrm{Cyc}, \end{tikzcd}$$

where the lower right square is a pullback square. All sequences found in the rows and columns are fiber sequences.

By rearranging this diagram, we should be able to recover a general hexagon diagram, as we have divided $\mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})}$ into two parts. These are intuitively the "geometric part" given by $\mathcal{C}_1 = \mathrm{Sh}_{pu}(\mathbf{Man};\mathbf{Spt})$ and the corresponding functors, and the "homotopical part" given by $\mathcal{C}_0 = \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt})$ and the corresponding functors.

Theorem 2.13.7. *(A general hexagon diagram in differential cohomology).*

For any $\hat{E} \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, we have a differential cohomology hexagon,

$$\begin{tikzcd}
 & R_{hi}(\hat{E}) \arrow[rr] \arrow[dr] & & L_{hi}(\hat{E}) \arrow[dr] & \\
\Omega L_{hi}\,\mathrm{Cyc}(\hat{E}) \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur] \arrow[dr, "\mathrm{curv}"] & & L_{hi}\,\mathrm{Cyc}(\hat{E}), \\
 & \mathrm{Def}(\hat{E}) \arrow[ur] \arrow[rr, "d"] & & \mathrm{Cyc}(\hat{E}) \arrow[ur] &
\end{tikzcd}$$

where the diagonals are fiber sequences, the outer sequences are looped fiber sequences and the left- and right squares are pullback squares.

Proof. We use the diagram in Corollary 2.13.6.1. Note that the upper rightmost and lower leftmost squares of the extended fracture square are redundant, as they have two copies of the same endofunctor and maps defined to commute. By removing the outer maps and swapping the place of Def and R_{hi}, we obtain

$$\begin{tikzcd}
 & R_{hi} \arrow[d] \arrow[dd, dotted, bend right] & \\
\mathrm{Def} \arrow[r] \arrow[rr, dotted, bend left, "d"] & \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Cyc} \arrow[d] \\
 & L_{hi} \arrow[r] & L_{hi}\,\mathrm{Cyc},
\end{tikzcd}$$

where the dotted lines are the maps $d : \mathrm{Def} \to \mathrm{Cyc}$ and $R_{hi} \to L_{hi}$ we just omitted. The square is a pullback square, and hence also a pushout square due to stability, as indicated by the box.

Since Cyc and Def were defined from the cofiber sequence

$$R_{hi} \to \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to \mathrm{Cyc}$$

and the fiber sequence

$$\mathrm{Def} \to \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \to L_{hi},$$

we obtain a similar diagram, but extended.

That is, we have a diagram

$$\begin{tikzcd}
 & R_{hi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\mathrm{Def} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Cyc} \arrow[d] \\
0 \arrow[r] & L_{hi} \arrow[r] & L_{hi}\,\mathrm{Cyc}\,.
\end{tikzcd}$$

Since pullback squares compose to an outer pullback square, we obtain the following.

$$\begin{tikzcd}
\Omega L_{hi}\,\mathrm{Cyc} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R_{hi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\mathrm{Def} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Cyc} \arrow[d] \\
0 \arrow[r] & L_{hi} \arrow[r] & L_{hi}\,\mathrm{Cyc}\,.
\end{tikzcd}$$

The only thing left is to rearrange this to the diagram in the theorem statement. This is simply done by forgetting the outer 0's, swapping R_{hi} and Def in the upper leftmost square, and add the last maps to be the compositions through $\mathrm{Id}_{\mathrm{Sh}(\mathbf{Man};\mathbf{Spt})}$. □

Remark 2.13.8. Note that the only essential part of the hexagon is the original fracture square, as the rest of the diagram can be computed by forming certain fiber sequences and pullbacks.

Remark 2.13.9. The hexagon becomes even more explicit by remembering that $L_{hi} \simeq \Gamma^*\Gamma_!$ and $R_{hi} \simeq \Gamma^*\Gamma_*$ or that we have explicit formulae for the homotopification functors.

Remark 2.13.10. Recall that we stated that most of the theorems would hold if we replaced **Spt** with a sufficiently nice ∞-category $\mathcal{C}$. There even exists a differential cohomology-type hexagon diagram when we work with $\mathrm{Sh}(\mathbf{Man};\mathcal{C})$. This can be found in [ADH23].

Remark 2.13.11. For ordinary differential cohomology, it is shown in [BNV16] that we recover the classical hexagon diagram of Section 2.2 from sheaf theoretic differential cohomology.

We are finally ready to define what a differential refinement of a spectrum should be, following [BNV16]. Recall that we have $L_{hi} \simeq \Gamma^*\Gamma_!$ since $\Gamma_! : \mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \to \mathbf{Spt}$ was defined as $\Gamma_* L_{hi}$.

Definition 2.13.12. (A differential refinement of a spectrum).

A differential refinement of a spectrum $E \in \mathbf{Spt}$ is a pair $(\hat{E}, \varphi)$, where $\hat{E} \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ and $\varphi : \Gamma_!(\hat{E}) \to E$ is an equivalence.

Remark 2.13.13. By replacing **Spt** with a sufficiently nice ∞-category $\mathcal{C}$, a similar definition defines differential refinements of the objects of $\mathcal{C}$.

Remark 2.13.14. Intuitively, the refinement of E is an $\hat{E} \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ such that $\hat{E}$ aligns with E-cohomology through φ.

Remark 2.13.15. (Recipe for computing a differential refinement, [ADH23]).

Finding a differential refinement of E is equivalent to the procedure of choosing a pure sheaf $P \in \mathrm{Sh}_{pu}(\mathbf{Man}; \mathbf{Spt})$ and a map $E \to \Gamma_!(P)$, and then defining $\hat{E}$ as the pullback

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] & P \arrow[d] \\
\Gamma^*(E) \arrow[r] & \Gamma^*\Gamma_!(P).
\end{tikzcd}$$

The recipe for constructing a differential refinement can thus be summarized as:

1. Choose a pure sheaf P.
2. Compute $\Gamma_! P$ using Formula 2.11.23.
3. Find a map in **Spt**, $f : E \to \Gamma_! P$.
4. Define $\hat{E}$ as the pullback above.

It is possible to show that in the case above, $\mathrm{Def}(\hat{E}) \simeq \mathrm{Def}(P)$, $\mathrm{Cyc}(\hat{E}) \simeq P$ and the global sections of $\hat{E}$ sit in a fiber sequence

$$\Gamma_*(\hat{E}) \to E \to \Gamma_!(P).$$

If we have a map of spectra, $f : E \to E'$ and we have a differential refinement of E', it is possible to pull back the differential refinement to E as well, just by considering the pullback

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] & \hat{E}' \arrow[d] \\
\Gamma^*(E) \arrow[r] & \Gamma^*(E').
\end{tikzcd}$$

Applying $\Gamma_!$ to this pullback will induce the needed equivalence to argue $\hat{E}$ is a differential refinement of E.

This construction preserves the differential deformations and the differential cycles. The global sections of the pulled-back refinement are found as a pullback

$$\begin{tikzcd}
\Gamma_*(\hat{E}) \arrow[r] \arrow[d] & \Gamma_*(\hat{E}') \arrow[d] \\
\hat{E} \arrow[r, "f"] & \hat{E}'.
\end{tikzcd}$$

For a differential cohomology theory $\hat{E} \in \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$, its differential cohomology groups can be defined as

$$\hat{E}^k(M) = \pi_{-k}\hat{E}(M).$$

Important examples can be found in both [BNV16] and [ADH23], but there are are some examples worth mentioning.

Example 2.13.16. (E-cohomology as differential cohomology).

By choosing pure sheaf $\hat{P} = 0$ and spectrum E, the defining pullback collapses to define $\hat{E} = \Gamma^* E$. The hexagon becomes degenerate, we obtain E-cohomology, and we are merely reformulating the fact that $\Gamma^* : \mathbf{Spt} \to \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt})$ is an equivalence.

Example 2.13.17. (Differential characters as sheaf theoretic differential cohomology).

By choosing pure sheaf $\hat{P} = H\Omega^{*\geq k}$ for $k \geq 1$, and spectrum $E = H\mathbb{Z}$, the defining pullback yields ordinary differential cohomology $\hat{H\mathbb{Z}}(k)$. The differential cohomology groups are defined as

$$\hat{H}^k(M) = \pi_{-k}\hat{H\mathbb{Z}}(k)(M).$$

The above example generalizes to Hopkins-Singer theories.

2.14 Other topics:

A spectral sequence for ordinary differential cohomology

Recall that in Section 2.6, we defined a complex $\hat{C}(q)(M)$ as a homotopy pullback to generalize differential characters. More explicitly, we constructed the complex

$$\hat{C}(q)^n(M) = C^n(M; \mathbb{Z}) \times \Omega^{n \geq q}(M) \times C^{n-1}(M; \mathbb{R}).$$

The differential d is given by

$$d(c, \omega, h) = (\delta c, d\omega, \omega - c - \delta h).$$

If $q \leq q'$, we have $\hat{C}(q)(M) \supseteq \hat{C}(q')(M)$ by the inclusion of the geometric data. We'll use this to obtain a spectral sequence.

Recall from [McC00] that for a filtered graded module (A, d, F^*) with F^* a decreasing filtration $A = F^0 A \supset F^1 A \supset \cdots$, we may define a spectral sequence $\{E_r^{*,*}, d_r\}$ with $|d_r| = (r, 1-r)$ such that $E_1^{p,q} = H^{p+q}(F^p A / F^{p+1} A)$. If the filtration is bounded in the sense that for each n, we can find values $s = s(n)$ and $t = t(n)$ such that $A^n = F^t A^n \supset F^{t+1} A^n \supset \cdots \supset F^{s-1} A^n \supset F^s A^n \supset 0$, the spectral sequence converges to the cohomology $H^*(A, d)$ in the sense that $E_\infty^{p,q} \cong F^p H^{p+q}(A, d) / F^{p+1} H^{p+q}(A, d)$.

Proposition 2.14.1. *(A spectral sequence for $\hat{H}(s)^*(M)$).*

Fix a smooth manifold M. For any s, we have a spectral sequence

$$E_1^{p,q}(s)(M) = H^{p+q}(\hat{C}(s+p)^*(M))/\hat{C}(s+p+1)^*(M)).$$

that converges to $\hat{H}(s)^*(M)$ in the sense that

$$E_\infty^{p,q}(s)(M) \cong F^p\hat{H}(s)^{p+q}(M)/F^{p+1}\hat{H}(s)^{p+q}(M).$$

Proof. To construct such a spectral sequence, consider $A = \hat{C}^*(M)$ and $F^p\hat{C}^*(M) = \hat{C}(p)^*(M)$. This is a decreasing filtration of $\hat{C}^*(M)$. The differential d on $\hat{C}^*(M)$ respects the filtration in the sense that $d : \hat{C}(p)^*(M) \to \hat{C}(p)^{*+1}(M)$ does not change the level p of the filtration. This induces a filtration on the cohomology. We set

$$E_1^{p,q} = H^{p+q}(F^p\hat{C}^*(M)/F^{p+1}\hat{C}^*(M)) = H^{p+q}(\hat{C}(p)^*(M)/\hat{C}(p+1)^*(M)).$$

Since we assume M is finite dimensional, the filtration is bounded. This spectral sequence will therefore converge to $\hat{H}(0)^*(M)$ in the sense that

$$E_\infty^{p,q} \cong F^pH^{p+q}(\hat{C}^*(M))/F^{p+1}H^{p+q}(\hat{C}^*(M)) \cong F^p\hat{H}(0)^{p+q}(M)/F^{p+1}\hat{H}(0)^{p+q}(M).$$

By shifting the procedure and setting $A = \hat{C}(s)^*(M)$ and $F^p\hat{C}(s)^*(M) = \hat{C}(s+p)^*(M)$, the spectral sequence

$$E_1^{p,q} = H^{p+q}(F^p\hat{C}(s)^*(M))/F^{p+1}\hat{C}(s)^*(M)) = H^{p+q}(\hat{C}(s+p)^*(M))/\hat{C}(s+p+1)^*(M)).$$

converges to $\hat{H}(s)^*(M)$ in the sense that $E_\infty^{p,q} \cong F^p\hat{H}(s)^{p+q}(M)/F^{p+1}\hat{H}(s)^{p+q}(M)$. □

Although we have the existence and convergence of this spectral sequence, we have not yet had the time to compute anything with it. A future focus would be to find a simpler form of the E_1-page, or perhaps passing on to find a simpler form of the E_2-page. We'll briefly mention an Atiyah-Hirzebruch spectral sequence in Section 3.8, but apart from this, we are not aware of any discussion of "refined" spectral sequences in the literature.

Application: Differential Cohomology and Dirac Quantization

In physics, forces change the paths of physical objects by giving them new "paths of least resistance" to move along. In mathematics, the notion of straight line on a manifold, or path of least resistance, is the one of a geodesic. We refer to [Tu17] for the needed differential geometry. The underlying mechanic that can be modified to change geodesics, is curvature, which intuitively means that forces and curvature do the same thing. The idea of having curvature explain fields of forces and their strengths is at the heart of mathematical physics, where differential geometry is used as a powerful language to describe reality.

A simple example on how differential geometry explicitly interacts with physics, is the formulation of Maxwell's equations through forms. Maxwell's equations are equivalent to some 2-form $F := B - dt \wedge E \in \Omega^2(M)$ being closed under d, and yielding a predefined form under $d*$, where $*$ denotes the Hodge star operator. Note that if we want to do electromagnetism on a contractible manifold (like $\mathbb{R}^4$), we may always assume $F = dA$ for some suitable "potential" A.

In [Fre02], two interesting consequences of this philosophy is discussed which are related to differential cohomology.

First of all, any gauge theory can be associated a group of charges. This group of charges is postulated in [Fre02] to be a differential cohomology group for some suitable differential cohomology theory. We have not formally defined any of the words above, but we think of them informally as connections A on suitable principal G-bundles and their transformations, e.g. transformations $A \mapsto A + dh$, which preserves physics by implying $F = dA = d(A + dh)$.

Secondly, ordinary differential cohomology seems to be a suitable arena for (Dirac) quantization. The idea of Dirac quantization is captured by ordinary differential cohomology, as ordinary differential cohomology marries differential forms with integral cohomology.

Equivariant Differential Cohomology

As with any construction in algebraic topology, it would be interesting to study how an equivariant version of differential cohomology would work. In the preprint [Ort09], a model for equivariant differential K-theory is suggested, and a procedure for obtaining equivariant differential cohomology theories in general are given, when restricting to actions of finite groups. The main idea is to use the pullback procedure of Hopkins-Singer as in Section 2.6 to define $\hat{K}_G^*$, but with equivariant models for cohomology and differential forms. This returns $\hat{K}^*$ as $\hat{K}^*_{\{e\}}$. The construction also respects free actions in the sense that $\hat{K}_G^*(M) = \hat{K}^*(M/G)$, it yields Bott periodicity, and it returns known connections to the representation ring of G in the sense that $\hat{K}_G^*(M) = \hat{K}^*(M) \otimes R(G)$ when G acts trivially. Interested readers are refered to [Ort09] and the references therein for more on equivariant differential cohomology. Some topics on equivariant differential cohomology and equivariant de Rham cohomology can also be found in [ADH23].

Differential Cohomology in a Cohesive $(\infty, 1)$-topos

In [Sch13], differential cohomology in the general context of a Cohesive ∞-topos is formulated. The essence is that most of the structure of differential cohomology (or in the words of [Sch13], a cohesive ∞-topos), depends mainly on the existence of four adjoints, as in Section 2.11. This allows for a quite formal treatment of higher structures in geometry and physics, such as higher bundles, higher parallel transport, higher Chern-Weil theory and higher Chern-Simons theory.

Chapter 3

On Main Results and Models in Differential Cohomology

Throughout the previous chapter, we studied several approaches to both ordinary and generalized differential cohomology.

With plenty of historical motivation, the story started with the observation that the de Rham theorem could be used to naively blend geometric and homotopical information. This idea was refined through the story of differential characters of [CS85]. As these are not directly easy to generalize further to differential E-cohomology, another approach was needed. The work of [HS05] gave not only an alternative approach to differential characters, but also several generalizations. Through the main idea of differential function complexes, differential E-cohomology groups could be made for any spectrum E. The construction depended on a fixed integer $p \geq 0$ and some prespecified cocycle, but by using fundamental cocycles, we have a canonical choice of cocycle with $p = 0$. As expected from a construction trying to explain anything cohomology, the differential function complexes could be pieced together to form a differential function spectra. Not only do spectra show up as above, as a version of differential cohomology through spectral sheaves on manifolds by [BNV16] could be made! Using stable recollements, we managed to decompose differential cohomology through a fracture square to rediscover the Hexagon diagram we started with.

This completes the circle, in some sense, but there are still many questions that require answers. The most pressing questions are the following:

- What is a differential cohomology theory, axiomatically?
- How do Hopkins-Singer theories and sheaf theoretic differential cohomology relate to this axiomatic framework?
- How unique are these axiomatic theories, and what can we say about homotopy invariance?
- If differential characters define ordinary differential cohomology, do we have other explicit models, e.g. for complex K-theory?

The goal of this chapter is to answer these questions, although not in their entirety, and give an overview of the papers these answers originated from.

3.1 Axioms of Differential Cohomology and Integration

In the case of ordinary differential cohomology, the axioms and uniqueness up to a unique natural transformation, were established by [SS08a]. In a similar manner, axioms for a generalized differential cohomology theory, or for a smooth extension of the generalized cohomology theory in question, can be defined. The main reference for this section is [BS10], where axioms and uniqueness results for smooth extensions are developed. We advise readers to think of Hopkins-Singer theory (Section 2.7) and differential characters (Section 2.3) rather than sheaf theoretic differential cohomology. This is due to Remarks 3.1.3 and 3.1.6.

Before we state the axioms of a smooth extension, we must fix a graded (abelian) group of coefficients. We fix a cohomology theory E throughout the section.

Inspired by the Chern-Dold character and the isomorphism of Proposition 2.8.7, we denote by $E^* = E^*(*) = \pi_* E$ the coefficient group of E, and define $\mathcal{V}_* = E^* \otimes_{\mathbb{Z}} \mathbb{R}$ to be the graded real vector space of coefficients.

Definition 3.1.1. (A smooth extension).

A smooth extension of E is a contravariant functor $\hat{E} : \mathbf{Man} \to \mathbf{grAb}$ with three transformations, natural in the manifold M,

- $R : \hat{E}^*(M) \to \Omega^*_{cl}(M; \mathcal{V}_*)$,
- $I : \hat{E}^*(M) \to E^*(M)$,
- $a : \Omega^{*-1}(M; \mathcal{V}_*)/\operatorname{im}(d) \to \hat{E}^*(M)$.

These are related by three requirements. Firstly, we require $R \circ a = d$. Secondly, the diagram

$$\begin{array}{ccc} \hat{E}^*(M) & \xrightarrow{R} & \Omega^*_{cl}(M; \mathcal{V}_*) \\ \big\downarrow{\scriptstyle I} & & \big\downarrow \\ E^*(M) & \xrightarrow{\text{chd}} & H^*(M; \mathcal{V}_*), \end{array}$$

where chd is the Chern-Dold character, should commute. Thirdly, we have an exact sequence

$$E^{*-1}(M) \longrightarrow \Omega^{*-1}(M; \mathcal{V}_*)/\operatorname{im}(d) \xrightarrow{a} \hat{E}^*(M) \xrightarrow{I} E^*(M) \longrightarrow 0.$$

For $x \in \hat{E}^*(M)$, $R(x)$ is called the *curvature* of x, $I(x)$ is called the *underlying cohomology class* or the *characteristic class* of x, while a is often referred to as the *action of forms* on $\hat{E}^*(M)$.

Remark 3.1.2. There is a variant of the above definition where smooth extensions are only defined on the category of compact manifolds. This is not important for the sake of the axiomatic definition, but for the uniqueness theorem of Section 3.2, it enables us to employ alternative assumptions.

Remark 3.1.3. (Hopkins-Singer theory as a smooth extension).

The Hopkins-Singer construction produces a differential cohomology theory as we obtain a diagram

$$\begin{tikzcd}
\hat{E}^*(M) \arrow[r, "R"] \arrow[d, "I"] & \Omega^*_{cl}(M;\mathcal{V}_*) \arrow[d] \\
E^*(M) \arrow[r, "\mathrm{chd}"] & H^*(M;\mathcal{V}_*),
\end{tikzcd}$$

which defines the maps R and I (see e.g. [HS05, Page 48.]). The map a, as well as the rest of the axioms, can be shown by defining a to be the map induced by the pullback,

$$\begin{tikzcd}
\Omega^{*-1}(M;\mathcal{V}_*)/\operatorname{im}(d) \arrow[dr, dashed, "a"] \arrow[drr, bend left, "d"] \arrow[ddr, bend right, "0"'] & & \\
& \hat{E}^*(M) \arrow[r, "R"] \arrow[d, "I"] & \Omega^*_{cl}(M;\mathcal{V}_*) \arrow[d] \\
& E^*(M) \arrow[r, "\mathrm{chd}"] & H^*(M;\mathcal{V}_*).
\end{tikzcd}$$

Remark 3.1.4. Note that in the above, we only used $E(q)^n$ with $n = q$. This has been a trend since Theorem 2.7.6, and indeed, this was how we defined differential K-theory in Section 2.10. For differential cohomology, most of the interesting information lives in the groups with matching indices, i.e. $E(n)^n$.

Remark 3.1.5. We should note that we only have three maps, although the hexagon certainly has four. This could naively be thought of as a similar degeneracy as in the definition of ordinary differential cohomology by [SS08a], but this can only be taken as inspiration. The difference is that in [SS08a], it is shown that the map cc, which corresponds to I, is induced by specifying the three other maps in the hexagon. As we will briefly touch upon, in the axiomatic picture, the "missing map in the hexagon" is trivially there, as the fourth entry is the flat theory. i.e. the kernel of R. It is highly nontrivial to show that the flat theory often corresponds to the $\mathbb{R}/\mathbb{Z}$-theory, as classically found in the hexagon.

Remark 3.1.6. (Problems with sheaf theoretic differential cohomology).

Although it is natural to think the sheaf theoretic differential cohomology of Section 2.11 will fit into the axioms, there are some cases in which it does not. The maps we want to define from the generalized hexagon diagram of Theorem 2.13.7 work as decoys, making sheaf theoretic differential cohomology a promising approach in its generality, but too general to satisfy the axioms. In [BNV16], the Hopkins-Singer definition of differential E-cohomology can be retrieved for the correct choices of spectrum and pure sheaf, and it spits out the associated hexagon diagram from Theorem 2.13.7. However, this depends on computations showing $\mathrm{Def}(\hat{E})$ and $\mathrm{Cyc}(\hat{E})$ being suitable spaces of differential forms. In general, there is no need for $\mathrm{Def}(\hat{E})$ or $\mathrm{Cyc}(\hat{E})$ to be related to differential forms, nor even real vector spaces.

This can be shown by using an extreme right adjoint of R_{hi}. Recall that $R_{hi} \simeq \Gamma^*\Gamma_*$. We compute

$$\mathrm{map}(R_{hi}(-), -) \cong \mathrm{map}(\Gamma^*\Gamma_*(-), -) \cong \mathrm{map}(\Gamma_*(-), \Gamma_*(-)) \cong \mathrm{map}(-, \Gamma^!\Gamma_*(-)).$$

That is, we define $\mathcal{G} = \Gamma^!\Gamma_* : \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt}) \to \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$. We use the letter $\mathcal{G}$ as this is called *the Godement functor*, as it resembles the Godement construction from sheaf theory. Recall that the functor $\Gamma^! : \mathbf{Spt} \to \mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$ was defined (manifold-wise) by

$$\Gamma^!(X)(M) := \prod_{m \in M} X,$$

where the product is taken over the underlying set of the smooth manifold M. The diagram of adjoins between $\mathrm{Sh}(\mathbf{Man}; \mathbf{Spt})$ and $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt})$ can therefore be extended to

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{Man};\mathbf{Spt}) \arrow[r, bend left=60, "L_{hi}"] \arrow[r, bend left=30, phantom, "\perp"] \arrow[r, shift right, "R_{hi}"'] \arrow[r, phantom, "\perp"] & \mathrm{Sh}_{\mathbb{R}}(\mathbf{Man};\mathbf{Spt}). \arrow[l, hook', shift right] \arrow[l, bend left=60, "\mathcal{G}"] \arrow[l, bend left=30, phantom, "\perp"]
\end{tikzcd}$$

Viewing $\mathrm{Sh}_{\mathbb{R}}(\mathbf{Man}; \mathbf{Spt})$ as $\mathbf{Spt}$ through Γ^*, we obtain $\mathcal{G}(E)(-) \simeq \prod_{(-)} E$ for an $E \in \mathbf{Spt}$.

By considering the differential cohomology theory given by

$$\hat{E}(-) = \mathcal{G}(H(\mathbb{Z}/2\mathbb{Z}))(-) \simeq \prod_{(-)} H(\mathbb{Z}/2\mathbb{Z}),$$

we obtain a non-example. As shown in [BNV16], this yields a differential cohomology theory in which its differential deformations has non-trivial torsion in the sense that

$$\mathrm{Def}(\hat{E}) \not\simeq 0, \quad \text{but rationally}, \quad \mathrm{Def}(\hat{E}) \wedge H\mathbb{Q} \simeq 0.$$

Although sheaf theoretic differential cohomology is a bit too general to fit the axioms, it yields an interesting approach showcasing which properties arise from the explicit details or underlying structures.

We return to the axiomatic approach, were maps of smooth extensions are defined as expected.

Definition 3.1.7. (Natural transformations of smooth extensions).

If $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ are smooth extensions, a natural transformation of smooth extensions is a natural transformation $\Phi : \hat{E}^* \to \hat{E}'^*$ such that the diagram

$$\begin{tikzcd}
 & \hat{E}^*(M) \arrow[d, "\Phi"] \arrow[dr, "I"] \arrow[drr, bend left, "R"] & & \\
\Omega^{*-1}(M;\mathcal{V}_*)/\operatorname{im}(d) \arrow[ur, "a"] \arrow[dr, "a'"'] & {} & E^*(M) & \Omega^*_{cl}(M;\mathcal{V}_*), \\
 & \hat{E}'^*(M) \arrow[ur, "I'"'] \arrow[urr, bend right, "R'"'] & &
\end{tikzcd}$$

commutes for all M.

We also want a multiplicative structure, if we assume the underlying cohomology theory E to be multiplicative.

Definition 3.1.8. (Multiplicative smooth extensions).

A smooth extension $(\hat{E}, R, I, a)$ is multiplicative if $\hat{E}$ takes values in graded commutative rings, R and I are both multiplicative, and the identity

$$x \cup a(\omega) = a(R(x) \wedge \omega), \quad x \in \hat{E}^*(M) \quad \omega \in \Omega^*(M;\mathcal{V}_*)/\operatorname{im}(d)$$

holds.

Remark 3.1.9. The above is well-defined. If we choose a representative of the same equivalence class as ω, say $\omega + d\alpha$ for some α, we have

$$\begin{aligned}
x \cup a(\omega + d\alpha) &= a(R(x) \wedge \omega + R(x) \wedge d\alpha) \\
&= a(R(x) \wedge \omega + d(R(x) \wedge \alpha)) \\
&= a(R(x) \wedge \omega) = x \cup a(\omega).
\end{aligned}$$

The last structure we need to define is a smooth extension with integration. Although it may not be too intuitive yet, the integration is quite vital to ensure uniqueness under certain assumptions, as we will see in the next section. Nevertheless, we should expect to have an integration map for smooth extensions, as we have notions of integration both for differential forms and E-cohomology. We start with the fiber integration of differential forms.

Consider a trivial bundle $\pi : E = M \times \mathbb{R}^l \to M$. If we are given a k-form ω on E, we may construct a $(k - l)$-form on M, called the pushforward of ω along π, or the integral of ω along the fiber (or along π), denoted $\pi_*\omega$ or $\int_{E/M}\omega$. The form ω can be written as a sum of forms on the form $(\pi^*\tilde{\alpha})\wedge f(x,t_1,\ldots,t_l)dt_{i_1}\wedge\ldots\wedge dt_{i_r}$ for some $r<l$, or $(\pi^*\alpha)\wedge f(x,t_1,\ldots,t_l)dt_1\wedge\ldots\wedge dt_l$. Here, the α's are suitable forms on M, $t_1,\ldots t_l$ are the coordinates of $\mathbb{R}^l$ and f is a function that has compact support on each fiber, that is, compact support for each fixed $x\in M$. We define the map $\pi_*\omega$ by

$$\begin{aligned} f(x,t_1,\ldots,t_l)(\pi^*\alpha)\wedge dt_{i_1}\wedge\ldots\wedge dt_{i_r} &\longmapsto 0, \\ f(x,t_1,\ldots,t_l)(\pi^*\alpha)\wedge dt_1\wedge\ldots\wedge dt_l &\longmapsto \alpha\int_{\mathbb{R}^l} f(x,t_1,\ldots,t_l)dt_1\ldots dt_l. \end{aligned}$$

For arbitrary vector bundles, we define π_* locally as above. It is also possible to consider a general integration of forms of fiber bundles with compact oriented fibers. This is done similarly, although we will not cover it here. We will however briefly use the existence of fiber integration on the trivial circle bundle $M\times S^1$. By using the atlas of S^1 as a smooth 1-manifold, we may intuitively consider the case to follow from the above discussion.

To construct some integration map for E-cohomology, consider the projection $p: M\times S^1\to M$ and the inclusion $i: M\to S^1\times M$. We have a short exact sequence given by

$$\begin{tikzcd} 0 \arrow[r] & \ker(i^*) \arrow[r, hook] & E^*(S^1\times M) \arrow[r, "i^*"'] & E^*(M) \arrow[r] \arrow[l, bend right, dotted, "p^*"'] & 0, \end{tikzcd}$$

where the splitting comes from the fact that $p\circ i=\mathrm{Id}$, inducing $i^*\circ p^*=(p\circ i)^*=\mathrm{Id}$. This induces a splitting

$$E^*(S^1\times M)\cong\ker(i^*)\oplus\operatorname{im}(p^*).$$

Recall now that the pointed suspension ΣM_+ of M, considered as a space, can be realized as $S^1\wedge M$. Let q be the quotient map $q: S^1\times M\to S^1\times M/(S^1\vee M)\simeq S^1\wedge M\simeq\Sigma M_+$. The map q can be shown to induce an isomorphism $\tilde{E}^{*+1}(\Sigma M_+)\cong\ker(i^*)$.

Then we can define the integration map as the composition

$$\int : E^{*+1}(S^1\times M)\to\ker(i^*)\cong\tilde{E}^{*+1}(\Sigma M_+)\to E^*(M),$$

where we have utilized the canonical projection, the inverse of q^*, and the suspension isomorphism, respectively.

Remark 3.1.10. Note that we constructed an integration for E-cohomology only in the case of the trivial S^1-bundle. We could have done this for more general bundles (see [BT82]), with extra assumptions and enough work on Thom spaces, Thom maps, and the Pontryagin-Thom construction. As we will only need the S^1-case, we limit ourselves to the ad hoc derivation above.

With the integration of forms and integration in E-cohomology, we can combine these two to a notion of integration for smooth extensions. We define $I_{S^1}\hat{E}^*(M)=\hat{E}^*(S^1\times M)$.

Definition 3.1.11. (Smooth extensions with integration).

A smooth extension with integration is a smooth extension $(\hat{E}, R, I, a)$ of E with a natural transformation

$$\int : I_{S^1}\hat{E}^{*+1} \to \hat{E}^*.$$

Let $p : S^1 \times M \to M$ be the canonical projection onto M and $t : S^1 \to S^1$ be given by complex conjugation, $z \mapsto \bar{z}$, where S^1 is considered to live in $\mathbb{C}$. The integration should satisfy $\int \circ p^* = 0$ and $\int \circ (t \times \mathrm{Id})^* = -\int$, and we should have a commutative diagram

$$\begin{tikzcd}
I_{S^1}\Omega^*(M;\mathcal{V}_*)/\operatorname{im}(d) \arrow[r, "a"] \arrow[d, "\int"] & I_{S^1}\hat{E}^{*+1}(M) \arrow[r, "I"] \arrow[d, "\int"] \arrow[rr, bend left, "R"] & I_{S^1}E^{*+1}(M) \arrow[d, "\int"] & I_{S^1}\Omega^{*+1}_{cl}(M;\mathcal{V}_*) \arrow[d, "\int"] \\
\Omega^{*-1}(M;\mathcal{V}_*)/\operatorname{im}(d) \arrow[r, "a"] & \hat{E}^*(M) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & E^*(M) & \Omega^*_{cl}(M;\mathcal{V}_*).
\end{tikzcd}$$

Definition 3.1.12. (Natural transformations of extensions with integration).

A natural transformation between smooth extensions $(\hat{E}, R, I, a, \int)$ and $(\hat{E}', R', I', a', \int')$ of E with integration is a natural transformation $\Phi : \hat{E}^* \to \hat{E}'^*$ of smooth extensions with the additional requirement that for all manifolds M, the diagram

$$\begin{tikzcd}
I_{S^1}\hat{E}^{*+1}(M) \arrow[r, "\Phi"] \arrow[d, "\int"] & I_{S^1}\hat{E}'^{*+1}(M) \arrow[d, "\int'"] \\
\hat{E}^*(M) \arrow[r, "\Phi"] & \hat{E}'^*(M)
\end{tikzcd}$$

commutes.

Remark 3.1.13. Under an additional (weak) assumption, a multiplicative extension will have a canonical choice of integration. See Corollary 4.3. in [BS10]. This will be important when proving uniqueness in Theorem 3.2.9.

Remark 3.1.14. The above remark ensures that multiplicative extensions (in most cases) have an integration. In geometric descriptions, the tendency is that there is a trade-off between how simply the integration and the product can be described. For example, in ordinary differential cohomology, the fiber integration of differential characters can be "easily" constructed as we have explicit components making up a differential character, but the product is a bit involved. In the sheaf theoretic approach of Deligne cohomology (Section 3.4), the product is quite simple (as can be seen in [Bry93]), but the integration is terribly complicated.

3.2 Results on uniqueness

In order to have a result regarding uniqueness, we first need a natural transformation of smooth extensions. Assume that $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ are smooth extensions (possibly with integration). Then we need a natural transformation Φ between these extensions. The main problem in [BS10] is exactly how to ensure that such a natural transformation always exists between two arbitrary smooth extensions of the same cohomology theory E.

In the case that E can be represented by a sequence of smooth manifolds, e.g. if E_k is a smooth manifold that represents E^k, this problem becomes tangible. In this case, $\hat{E}^k(E_k)$ and $\hat{E'}^k(E_k)$ are both defined, which turns out to be quite handy.

How do we use this to construct $\Phi_M^k : \hat{E}^k(M) \to \hat{E'}^k(M)$? Assume that $\hat{v} \in \hat{E}^k(M)$. We want to define $\Phi_M^k(\hat{v}) \in \hat{E'}^k(M)$. To do this, let's investigate the universal case.

In the case of the representing manifold E_k, we may consider $E^k(E_k)$. This has a canonical choice of cohomology class. Let $u \in E^k(E_k)$ be the class that corresponds to the identity $E_k \to E_k$. From the axioms of a smooth extension, we know $I : \hat{E}^k(E_k) \to E^k(E_k)$ and $I : \hat{E'}^k(E_k) \to E^k(E_k)$ are surjections. Therefore, we may find lifts $\hat{u} \in \hat{E}^k(E_k)$ and $\hat{u'} \in \hat{E'}^k(E_k)$ through I and I', respectively.

If we now consider $\hat{v} \in \hat{E}^k(M)$, we can consider $v = I(\hat{v}) \in E^k(M)$. Naively, we could employ a similar argument as above and say there exists a lift $\hat{v'}$ of v through I' and map $\hat{v} \mapsto \hat{v'}$, but this does not work. There are too many choices to be made and additional requirements for compatibility with both R, R', a and a'. On the other hand, there exists a map $f : M \to E_k$ such that $f^*u = v$. At the level of $\hat{E}^k(M)$, we may not have $f^*\hat{u} = \hat{v}$, but we know that $\hat{v} - f^*\hat{u} \in \ker(I) = \operatorname{im}(a)$. Hence, there exists an $\alpha \in \Omega^{k-1}(M; \mathcal{V}_*)/\operatorname{im}(d)$ such that

$$a(\alpha) = \hat{v} - f^*\hat{u} \iff \hat{v} = a(\alpha) + f^*\hat{u}.$$

We can therefore make the following definition.

Definition 3.2.1. (Φ_M^k when the E^k is represented by a smooth manifold).

In the case where each E^k is represented by a smooth manifold E_k, we can for each element $\hat{v} \in \hat{E}^k(M)$ write $\hat{v} = a(\alpha) + f^*\hat{u}$ for some $\alpha \in \Omega^{k-1}(M; \mathcal{V}_*)/\operatorname{im}(d)$ and $f : M \to E_k$. Then $\Phi_M^k : \hat{E}^k(M) \to \hat{E'}^k(M)$ is given by

$$\Phi_M^k(\hat{v}) = a'(\alpha) + f^*(\hat{u'}).$$

Remark 3.2.2. We have not established that anything in this definition is well-defined. We have simply constructed a map (not yet natural nor additive), given some choices. To check that the map is independent of the choices made, we would need extra work that turns out to be a special case of the work in [BS10]. A proof simply involves checking every single choice and is thus omitted. The most interesting part is the explicit dependency on differential cohomology classes of E_k, which ultimately depends on E_k being a smooth manifold, a problem we will soon return to.

The above definition (if well-defined) is enough to ensure a natural transformation $\Phi : \hat{E}^* \to \hat{E'}^*$.

Proposition 3.2.3. *(Naturality of Φ).*

If $\varphi : M \to N$ is a map of smooth manifolds,

$$\begin{tikzcd} \hat{E}^k(N) \arrow[r, "\hat{E}^k(\varphi)"] \arrow[d, "\Phi_N^k"'] & \hat{E}^k(M) \arrow[d, "\Phi_M^k"] \\ \hat{E'}^k(N) \arrow[r, "\hat{E'}^k(\varphi)"'] & \hat{E'}^k(M) \end{tikzcd}$$

commutes for all k.

Proof. For suitable α and f, we have $\Phi_M^k(\hat{v}) = a'(\alpha) + f^*(\hat{u'})$ and $\hat{v} = a(\alpha) + f^*\hat{u}$. The latter yields $\varphi^*\hat{v} = \varphi^*a(\alpha)+\varphi^*f^*\hat{u} = a(\varphi^*\alpha)+\varphi^*f^*\hat{u}$, which simply implies that $\Phi_N^k(\varphi^*\hat{v}) = a'(\varphi^*\alpha)+\varphi^*f^*\hat{u'} = \varphi^*a'(\alpha) + \varphi^*f^*\hat{u'} = \varphi^*\Phi_M^k(\hat{v})$. $\square$

The map Φ is not additive in general, which is needed to preserve the group structure of the differential cohomology groups.

The deviance from additivity can be measured by a natural transformation $B : \hat{E}^k \times \hat{E}^k \to \hat{E'}^k$, through the formula

$$\Phi^k(\hat{v} + \hat{w}) = \Phi^k(\hat{v}) + \Phi^k(\hat{w}) + B(\hat{v}, \hat{w}).$$

It is possible (see [BS10]) to show that the following requirement on the cohomology theory E ensures the vanishing of B for even k. Although it includes close to all important examples, it would be interesting to see how the assumption could be weakened.

Definition 3.2.4. (Rationally even cohomology theories).

The cohomology theory E is rationally even if $E^n \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for odd n, where $E^n = E^n(*)$.

Example 3.2.5. Ordinary cohomology theories are all rationally even. The same also holds for important generalized cohomology theories such as K-theory (real, complex, ...), complex cobordism, the Morava K-theories, the Johnson-Wilson theories, and many others.

Before we continue, we must acknowledge the main problem. In everything we did above, we assumed that the representing space E_k of E^k was a smooth manifold. This is rarely true, especially if we want the representing manifold to be finite-dimensional.

The main technicality in [BS10] is to amend this problem, and the idea is simple. If E_k is not a smooth manifold, can we approximate E_k in a suitable way by a sequence of smooth manifolds $E_{k,i}$?

Indeed, we may find a sufficiently nice sequence of smooth manifolds $E_{k,i}$ (even compact, depending on the assumption) such that

$$\operatorname{colim}_i[M, E_{k,i}] = [M, E_k].$$

This intuitively ensures that the work above remains valid, although we would need to make everything respect the approximating sequence $\{E_{k,i}\}_i$ of E_k.

Although we refer interested readers to [BS10] for the details, the above idea does not work unless we have one of two assumptions. The assumption merely depends on whether or not we are able to define the smooth extensions on the entire category of smooth manifolds, or just on the subcategory of compact manifolds, as mentioned in Remark 3.1.2.

Assumption 3.2.6. We assume one of the following:

1. If the smooth extensions are only defined on the category of compact manifolds, we assume that the groups $E^n = E^n(*)$ are finitely generated for all $n \in \mathbb{Z}$.
2. If the smooth extensions are defined on the category of all smooth manifolds, we assume that the groups $E^n = E^n(*)$ are countably generated for all $n \in \mathbb{Z}$.

Remark 3.2.7. If the smooth extensions are only defined on compact manifolds, the approximating objects $E_{k,i}$ must be compact manifolds in order to employ the ideas of our previous work, while if the smooth extensions are defined on all smooth manifolds, we merely need the existence of these approximating objects. Note that the first assumption is stronger than the second.

Remark 3.2.8. The above assumptions can be changed if our only interest is to construct an approximating sequence $\{E_{k,i}\}_i$ of E_k and construct Φ^k. The assumptions can then be weakened to saying "finitely/countably generated for all $n \leq k$", as long as we add the requirement $E^{k-1} = 0$ for the first assumption. Our goal is the uniqueness theorem, and hence we assume one of the above.

As we have only explained the main idea of the construction, we are in no position to prove the uniqueness theorem. We will simply just state the theorem and explain why the assumptions are needed. Interested readers should take a look at the proof in [BS10], as well as the technical details leading up to the proof.

Theorem 3.2.9. *(Uniqueness of smooth extensions).*

Let E be a rationally even generalized cohomology theory. Assume one of the assumptions in 3.2.6 hold, and let $(\hat{E}, R, I, a, \int)$ and $(\hat{E}', R', I', a', \int')$ be smooth extensions of E with integration, defined on the appropriate category of manifolds (from 3.2.6). Then there exists a unique natural isomorphism

$$\Phi : \hat{E} \to \hat{E}'$$

of smooth extensions with integration.

Remark 3.2.10. (Unraveling the assumptions of the uniqueness theorem).

The assumptions in this theorem are quite important, and due to their explicit impact on Φ, it is not easy to see how one could weaken the assumptions. It would be interesting to see which construction could extend the uniqueness theorem beyond the above assumptions.

As explained throughout the first half of this section, there is an explicit way to define the map Φ (see Definition 3.2.1) if the representing spaces E_k for E^k are smooth manifolds. If the representing spaces are not smooth manifolds, we need more assumptions.

If the smooth extensions are defined on all smooth manifolds, Assumption 2. of 3.2.6 is sufficient to ensure there exists an approximating sequence $\{E_{k,i}\}_i$ of E_k with the right properties to fix the above definition. If the smooth extensions are defined only on the compact manifolds, we need a slightly stronger assumption, namely Assumption 1. from 3.2.6 to ensure the same. In this case, the approximating objects are compact manifolds.

The naturality of Φ is obtained similarly to Proposition 3.2.3, by doing the proof in an arbitrary degree i of the approximating sequence $\{E_{k,i}\}_i$ of E_k. To ensure the additivity of Φ in even degrees (i.e. the additivity of Φ^{2k}), the assumption that E is rationally even is needed. This ensures that the deviance of additivity vanishes in even degrees.

The requirement that the smooth extensions should have an integration, is perhaps the most mysterious one. It is simply necessary to shift the index of the above arguments by 1 and extend the results from even degrees to odd degrees. Similarly to how the deviance B to additivity in even degrees vanished due to the cohomology theory being rationally even, the terms that may cause problems will also vanish.

Remark 3.2.11. The importance of integration in the uniqueness theorem is not only to extend the additive transformation to odd degrees. If we omit the integration and consider E to be complex K-theory, we may construct infinitely many non-equivalent smooth extensions. This is Theorem 6.2. of [BS10].

Multiplicative extensions often have a canonical choice of integration. Corollary 4.3. of [BS10] ensures this, as long as we assume $E^{-1} = E^{-1}(*) = E^0(\Sigma *)$ to be a torsion group. This is much weaker than the assumptions of the uniqueness theorem, implying the following corollary.

Corollary 3.2.11.1. *(Uniqueness of multiplicative smooth extensions).*

Let E be a rationally even generalized cohomology theory. Assume one of the assumptions in 3.2.6, and let $(\hat{E}, R, I, a)$ and $(\hat{E}', R', I', a')$ be multiplicative smooth extensions of E, defined on the appropriate category of manifolds (from 3.2.6). Then there exists a unique natural isomorphism

$$\Phi : \hat{E} \to \hat{E}'$$

of smooth extensions preserving the induced integration. Moreover, the transformation is multiplicative.

Proof. Given Corollary 4.3. in [BS10], this easily follows from the uniqueness theorem. Theorem 4.6. in [BS10] proves the transformation is multiplicative. □

Remark 3.2.12. We can obtain uniqueness up to a unique natural isomorphism for multiplicative extensions, without bothering to define the integration.

The power of the uniqueness theorem is clear. If we manage to define geometric models that satisfy the axioms of a smooth extension of a cohomology theory E, there are only a few assumptions to check to obtain a unique natural isomorphism to all other smooth extensions.

3.3 On the failure of homotopy invariance

A defining property of generalized cohomology theories is their homotopy invariance. One of the fundamental ideas we investigated when approaching differential cohomology was whether or not geometric information could be blended into the homotopic nature of cohomology. As established throughout several approaches in the previous chapter, we managed to do just that, but we have not yet understood how homotopy invariant these constructions are. This is the goal of this brief section.

We will assume that we have fixed a generalized cohomology theory E, and a smooth extension $(\hat{E}, R, I, a)$ of E.

Assume that $i_0, i_1 : M \to M \times [0,1]$ are induced from the inclusions at the endpoints, and let $p : M \times [0,1] \to M$ be the projection. If $\hat{x} \in \hat{E}^{*+1}(M \times [0,1])$, we want to study the difference $i_1^*\hat{x} - i_0^*\hat{x}$.

It is useful to note that the difference $\hat{x} - p^*i_0^*\hat{x}$ lies in the kernel of I, and hence in $\mathrm{im}(a)$. We may therefore write $\hat{x} = p^*i_0^*\hat{x} + a(\omega)$ for some $\omega \in \Omega^*(M \times [0,1]; \mathcal{V}_*)$. Intuitively, we want to integrate out $[0,1]$ by fiber integration, and to do that, we can push the form ω through i_0^* and i_1^*. We obtain

$$\begin{aligned} i_1^*(\omega) - i_0^*(\omega) &= \int_{M\times[0,1]/M} d\omega = \int_{M\times[0,1]/M} R(a(\omega)) = \int_{M\times[0,1]/M} R(\hat{x} - p^*i_0^*\hat{x}) \\ &= \int_{M\times[0,1]/M} R(\hat{x}) - \int_{M\times[0,1]/M} R(p^*i_0^*\hat{x})) = \int_{M\times[0,1]/M} R(\hat{x}) - p^*R(i_0^*\hat{x})) \\ &= \int_{M\times[0,1]/M} R(\hat{x}). \end{aligned}$$

By noting that $i_1^*(p^*z) - i_0^*(p^*z)$ vanishes as $p \circ i = \mathrm{Id}$, we simply obtain

$$i_1^*\hat{x} - i_0^*\hat{x} = i_1^*(a(\omega)) - i_0^*(a(\omega)) = a(i_1^*(\omega) - i_0^*(\omega)) = a(\int_{M\times[0,1]/M} R(\hat{x})).$$

The above computations just proved the following, which is Lemma 5.1. in [BS10].

Theorem 3.3.1. *(The homotopy formula).*

If $\hat{x}$, i_0 and i_1 are as above, we have

$$i_1^*\hat{x} - i_0^*\hat{x} = a(\int_{M\times[0,1]/M} R(\hat{x})),$$

where the integral denotes fiber integration of forms.

Remark 3.3.2. The homotopy formula can be proved in the context of spectral sheaves (Section 2.11) by a heavy diagram chase. This can be found in [BNV16].

Remark 3.3.3. An interesting takeaway is that, as initially motivated, the geometric information found in the curvature R decides whether or not we have homotopy invariance!

A theory that will certainly be homotopy invariant, is the flat theory.

Definition 3.3.4. (The flat theory).

Define the flat theory of E to be the subfubctor of $\hat{E}^*$ defined by

$$\hat{E}^*_{flat}(M) = \ker(R : \hat{E}^*(M) \to \Omega^*_{cl}(M; \mathcal{V}_*)).$$

Intuitively, from the original hexagon (see e.g. Section 2.2), the flat theory takes the place of cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$. This can be made precise.

Recall that for a spectrum E, we can define the spectrum of E with coefficients in an abelian group G by considering $EG = E \wedge MG$, where MG denotes the Moore spectrum of G. We say the flat theory is *topological* if there exists a map $\Phi_{flat} : \hat{E}^*_{flat} \to E\mathbb{R}/\mathbb{Z}^{*-1}$. By an appropriate application of the five lemma to the following diagram,

$$\begin{tikzcd}
E^{*-1}(M) \arrow[r] & E\mathbb{R}^{*-1}(M) \arrow[r] & E\mathbb{R}/\mathbb{Z}^{*-1}(M) \arrow[r] & E^*(M) \arrow[r] & E\mathbb{R}^*(M) \\
E^{*-1}(M) \arrow[r] \arrow[u, equal] & H^{*-1}(M; \mathcal{V}_*) \arrow[r] \arrow[u, "\cong"] & \hat{E}^*_{flat}(M) \arrow[r] \arrow[u, "\Phi_{flat}"] & E^*(M) \arrow[u, equal] & H^*(M; \mathcal{V}_*), \arrow[u, "\cong"]
\end{tikzcd}$$

the map Φ_{flat} is an isomorphism, if it exists. Criteria for the existence of such a map can be formulated through similar assumptions as for the uniqueness theorem (3.2.9), as one may construct such a map via the unique isomorphism of smooth extensions from $\hat{E}$ to the Hopkins-Singer theory $\hat{E}_{HS}$, which has the $\mathbb{R}/\mathbb{Z}$-theory as the flat part. Similar arguments can be made without mentioning the Hopkins-Singer construction, but at the cost of restricting to compact manifolds. The details and results of the above can be found in [BS10], but it is worth mentioning that the intuition saying that the flat theory replaces the $\mathbb{R}/\mathbb{Z}$-theory is valid.

With this, we end this brief section and move on to the last, perhaps most daunting question; do we have any geometric descriptions of smooth extensions?

3.4 Smooth Deligne Cohomology as Ordinary Differential Cohomology

An important way of viewing ordinary differential cohomology is through smooth Deligne cohomology. As it will be important for our later discussions on Hodge-filtered cohomology in Chapter 4, we will spend some time understanding it thoroughly. Smooth Deligne cohomology is best formulated through sheaf cohomology. We therefore begin with an interlude on sheaf cohomology. If not otherwise specified, all sheaves are simply considered as sheaves on a topological space X, $U \subseteq X$ will always be an arbitrary open set of X.

We will first recall standard facts and introduce interesting examples, before we state the classical homological algebra of sheaves and the properties of sheaf cohomology. Our treatise will follow [Bry93]. For details on sheaf cohomology, see [Ive86], and for details on homological algebra, consult [Rot09].

First, recall that the category of sheaves on a space X is abelian. The kernel and cokernel of a map $\varphi : A \to B$ of sheaves is defined as the (pre)sheaves $\ker(\varphi)$ and $\operatorname{cok}(\varphi)$ sending an open U to $\ker(\varphi(U))$ and $\operatorname{cok}(\varphi(U))$. If these fail to be sheaves, which may be the case for both cok and im, sheafification will yield the appropriate objects in the category of sheaves.

The exactness of a sequence
$$A \to B \to C$$
of sheaves is defined through exactness at the stalks of the sheaves,
$$A_x \to B_x \to C_x,$$
for all $x \in X$.

This leads us to an interesting example. For reasons that will be clear later, let $\mathbb{Z}(1)$ denote the (cyclic) subgroup $(2\pi i) \cdot \mathbb{Z}$ of $\mathbb{C}$. If G is a Lie group, we let $\underline{G_X}$ be the sheaf of locally constant smooth G-valued functions on X (assuming X is smooth).

Example 3.4.1. If X is smooth, there is a short exact sequence of sheaves of abelian groups
$$0 \to \mathbb{Z}(1) \to \underline{\mathbb{C}_X} \to \underline{\mathbb{C}^*_X} \to 0.$$
The first sheaf is the subsheaf of $\underline{\mathbb{C}_X}$ consisting of those functions that take values in $\mathbb{Z}(1)$. The latter map is induced by the exponential map. The only nontrivial part is to show that the last map is surjective. This can be done at the level of stalks. For an $x \in X$, pick $f_x \in \underline{\mathbb{C}^*_X}$. Since we work stalkwise, there exists a contractible open neighborhood U around x and a smooth function $f : U \to \mathbb{C}^*$ such that f_x is the germ at x of f. Due to the contractability of U, it is possible to (locally) define a complex logarithm. Hence, we may find a smooth $g : U \to \mathbb{C}$ such that $\exp(g) = f$. By considering a germ g_x of g at x, we obtain $\exp(g_x) = f_x$, proving the claim.

When working with sheaves, we have an important (covariant) functor, namely the global sections functor Γ. For a sheaf $\mathcal{F}$ of abelian groups, we define $\Gamma(X, -) : \mathrm{Sh}(X) \to \mathbf{Ab}$ sending $\mathcal{F}$ to $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

Remark 3.4.2. Recall from Section 2.11 that we worked with a functor $\Gamma_* : \mathcal{F} \mapsto \mathcal{F}(*)$ on spectral presheaves on manifolds. This was also called global sections, but in the notation above, it is the functor $\Gamma(*, -)$. Although it should be possible to compare these two definitions through the map $X \to *$, we warn about the ambiguity when we talk about global sections. The intention should always be clear from context and notation. In this section, we mean $\Gamma(X, -)$.

A key feature of $\Gamma(X, -)$ is that it is left exact, i.e. it preserves everything except the surjectivity of a short exact sequence. A counterexample to right exactness can be found by perturbing Example 3.4.1. It is well known to complex analysts that it is not possible to define a global complex logarithm, so by omitting the "locally constant" part of the example, surjectivity fails.

Any mathematician fluent in the yoga of homological algebra sees opportunity in the failure of right exactness. It is well known that the category of sheaves has enough injectives (see for example Lemma 1.1.13. in [Bry93]). Hence, constructing a right derived functor of $\Gamma(X, -)$ could be a possible candidate for sheaf cohomology.

This requires a definition of injective resolutions, which again involves a notion of acyclicity. A simple approach to constructing acyclicity could be first to consider a complex K^* of sheaves and construct so-called *cohomology sheaves* $H^n(K^*)$ defined by $U \mapsto H^n(K^*(U))$, where the latter is defined by sheafification of the quotient presheaf $\ker(d^n)/\operatorname{im}(d^{n-1})$.

We recall the definition of an injective resolution of sheaves.

Definition 3.4.3. (Injective resolution of sheaves).

Let $\mathcal{F}$ be a sheaf. A resolution of $\mathcal{F}$ is a complex K^* and a map $i : \mathcal{F} \to K^0$ of sheaves such that i is a monomorphism with image $\ker(d^0)$, and for $n \geq 1$, we have $\ker(d^n) = \operatorname{im}(d^{n-1})$.

If each sheaf K^q in K^* is injective, we call this an injective resolution.

Remark 3.4.4. The last requirement in the definition of a resolution is equivalent to the vanishing of the cohomology sheaves for degree $n \geq 0$, while the zero'th cohomology sheaf will be $\mathcal{F}$ itself.

Remark 3.4.5. Since the category of sheaves has enough injectives, we can always construct an injective resolution of a sheaf $\mathcal{F}$. Injectivity of the resolution is important to ensure our construction of sheaf cohomology is homotopy invariant.

This gives a definition of sheaf cohomology.

Definition 3.4.6. (Sheaf cohomology).

If $\mathcal{F}$ is a sheaf of abelian groups on a space X, then the sheaf cohomology groups $H^n(X; \mathcal{F})$ are defined as follows. Pick an injective resolution I^* of $\mathcal{F}$. This yields a complex $\Gamma(X, I^*)$, and $H^n(X; \mathcal{F})$ is the n'th cohomology group of this complex.

Remark 3.4.7. This is well-defined and independent of the choice of resolution. For $n = 0$, $H^0(X; \mathcal{F}) = \Gamma(X, \mathcal{F})$.

This formulation of sheaf cohomology is powerful in the sense that we obtain all the tools from homological algebra for free, such as a long exact sequence in sheaf cohomology. However, it is often hard to compute sheaf cohomology groups using this approach, unless we have extra results. For this purpose, one would develop Čech cohomology, which approximates the sheaf cohomology of a space X by constructing a resolution based on open covers of X. Due to the injectivity of the resolutions used in sheaf cohomology, natural maps from Čech cohomology to sheaf cohomology are obtained, which can be shown to be an isomorphism in many cases.

For our purposes, we do not need Čech cohomology, but we need to develop hypercohomology. The idea of hypercohomology is to take the sheaf cohomology of not just one sheaf, but of a complex of sheaves. For example, it would be interesting to take the sheaf cohomology of the de Rham complex. The de Rham complex is a suitable complex to consider, as it is *bounded below*. We say a complex K^* of sheaves is bounded below if there exists a $k \in \mathbb{Z}$ such that $K^p = 0$ for all $p \leq k$. Intuitively, by doing the approach through the injective resolution approach above, we should be able to obtain a double complex constructed through level-wise injective resolutions.

Lemma 3.4.8. *(Injective resolutions of complexes).*

Assume (K^*, d_K) is a (bounded below) complex of sheaves on a space X. Then there exists a double complex $(I^{*,*}, \delta, d)$ with $I^{p,*} = 0$ for $p < 0$ and a map of complexes $u : K^* \to I^{0,*}$ such that the complex $I^{*,q}$ is an injective resolution of K^q. Furthermore, morphisms of the K^q's can be extended uniquely (up to homotopy) to construct double complexes (i.e. the morphisms can be extended to the injective resolutions).

Proof. For a proof, consult either [Bry93] or [Rot09]. □

Remark 3.4.9. It is possible to show that the resolutions respect both the kernel and the image of the differential d_K in the sense that $d(I^{*,q-1})$ is an injective resolution of $d_K(K^{q-1})$. A similar statement can be shown for the kernel of d_K.

If K^* is a (bounded below) complex of sheaves on X, we can consider the double complex $\Gamma(X, I^{*,*})$ as before. Intuitively, the n'th cohomology of this double complex should be what we call the n'th hypercohomology group of K^*.

To formalize what we mean by the n'th cohomology of a double complex, recall that we may form the total complex. Depending on the conventions specified for the differentials when defining double complexes, it may be a bit technical to define the total complex. However, following the conventions of [Rot09], it is quite simple. For a double complex $I^{*,*}$ (not necessarily injective), define $(\operatorname{Tot} I^{*,*})_n = \bigoplus_{p+q=n} I^{p,q}$, and consider $D_n : (\operatorname{Tot} I^{*,*})_n \to (\operatorname{Tot} I^{*,*})_{n+1}$ to be the differential given by the sum of the vertical and horizontal differentials,

$$\sum_{p+q=n} (d_{p,q} + \delta_{p,q}).$$

This defines a complex (given the correct conventions on how the differentials d and δ should compose), and lets us complete the definition of hypercohomology.

Definition 3.4.10. (Hypercohomology).

Given a (bounded below) complex K^* of sheaves on a space X, the n'th hypercohomology group $H^n(X;K^*)$ is the n'th cohomology group of the complex

$$H^n(X;K^*) = H^n(\operatorname{Tot}\Gamma(X, I^{*,*}))_*,$$

where $I^{*,*}$ is an injective resolution as above. This is well-defined, and a morphism $K^* \to L^*$ of complexes of sheaves induce group homomorphisms $H^n(X;K^*) \to H^n(X;L^*)$.

Remark 3.4.11. By simply constructing a complex of 0's with a sheaf $\mathcal{F}$ in degree 0, we may recover the sheaf cohomology of $\mathcal{F}$ through hypercohomology.

Remark 3.4.12. For computational purposes, it is possible to develop Čech hypercohomology.

Through certain double complex spectral sequence arguments, it is possible to conclude with the following result, yielding a simple way of understanding hypercohomology, if we work with a complex of acyclic sheaves.

Proposition 3.4.13. *(Hypercohomology using acyclic sheaves).*

Let K^* be a (bounded below) complex of sheaves on a space X, and assume each sheaf K^p is acyclic in the sense that its higher sheaf cohomology vanishes, i.e. $H^q(X;K^p) = 0$ for $q > 0$.

Then the hypercohomology groups $H^n(X;K^*)$ is isomorphic to the cohomology groups of the complex

$$\ldots \to \Gamma(X;K^{p-1}) \to \Gamma(X;K^p) \to \Gamma(X;K^{p+1}) \to \ldots.$$

We refer to [Bry93] for the proof, which is quite short, if the reader is familiar with hypercohomology spectral sequences (also found in [Bry93]).

Recalling our original motivation of differential cohomology, we hope that the de Rham sheaves Ω^* on a smooth manifold M satisfy the assumptions of Proposition 3.4.13. Note that this would immediately show the hypercohomology of the de Rham complex on M and the de Rham cohomology of M are isomorphic.

There is a well-known chain of implications regarding injective and acyclic sheaves. The following results can be shown for more general cases, such as for paracompact spaces. We will formulate the theory for smooth manifolds.

We define a sheaf $\mathcal{F}$ on M to be *soft* if for all closed subsets Z of M, the induced restriction $\Gamma(M;\mathcal{F}) \to \Gamma(Z;\mathcal{F})$ is surjective. We say $\mathcal{F}$ is *flabby* a similar condition holds for all open subsets U of M. These classes of sheaves are related through the following result, which summarizes Theorem 1.4.6. in [Bry93] and important facts from its proof.

Theorem 3.4.14. *(Injective, flabby, soft, and acyclic sheaves).*

On a smooth manifold M, injective sheaves are flabby, flabby sheaves are soft, and soft sheaves are acyclic.

In light of the above theorem, the power of the following is clear.

Theorem 3.4.15. *(de Rham sheaves are soft).*

On a smooth manifold M, the de Rham sheaves Ω^n are soft.

This is Theorem 1.4.15. in [Bry93], and it follows partially from the fact that all smooth manifolds are Hausdorff and admit a partition of unity subordinate to any open cover. This lets us use Proposition 3.4.13.

Theorem 3.4.16. *(de Rham cohomology as hypercohomology).*

The hypercohomology groups $H^n(M;\Omega^*)$ are isomorphic to the de Rham cohomology groups $H^n_{dR}(M)$.

This connection can be rephrased, without using hypercohomology.

Proposition 3.4.17. *(de Rham sheaves as a resolution of $\mathbb{R}$).*

For a smooth manifold M, the complex of de Rham sheaves, Ω^*, is an acyclic resolution of the constant sheaf $\mathbb{R}$.

Proof. Since each de Rham sheaf is soft, and hence acyclic, we only need to check if we obtain a resolution. This is verified at the level of stalks. Pick an $x \in M$. Working on a smooth manifold, which is locally Euclidean, we may find a sequence of contractible neighborhoods U_i around x such that

$$\Omega^n_x = \operatorname*{colim}_i \Omega^n(U_i).$$

By the Poincaré-lemma, the complex

$$\mathbb{R} \to \Omega^0(U_i) \to \Omega^1(U_i) \to \dots$$

is acyclic for each i. Since the colimit respects exactness, we obtain the wanted resolution at the level of stalks at an arbitrary x, proving the claim. □

Theorem 3.4.18. *(de Rham cohomology as sheaf cohomology).*

On a smooth manifold M, the de Rham cohomology groups $H^n_{dR}(M)$ are isomorphic to the sheaf cohomology groups $H^n(M;\mathbb{R})$, where $\mathbb{R}$ is the constant sheaf.

It is now clear that the complex of sheaves

$$\mathbb{R} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \dots$$

is important in our cause, as it has geometric entries (in terms of differential forms), and it yields de Rham cohomology in a variety of different ways.

An idea of Deligne was to perturb this complex to amend an annoying problem; The interesting part of the Chern classes is not integer-valued, as we need to normalize it with an appropriate constant $(2\pi i)^p$, where $2p$ is the degree of the Chern class in question. The Chern classes (without normalization) lie in cohomology groups with coefficients in $\mathbb{Z}(p) := (2\pi i)^p \cdot \mathbb{Z} \subseteq \mathbb{C}$. To change the above complex of sheaves so it is compliant with the change $\mathbb{Z} \mapsto \mathbb{Z}(p)$, we complexify the de Rham sheaves. Write $\Omega^n_{\mathbb{C}}$ for the sheaf $\Omega^n(M)_{\mathbb{C}} := \Omega^n(M) \otimes \mathbb{C}$. Note also that the sheaf $\mathbb{Z}(p)$ can be viewed as a subsheaf of $\Omega^0_{\mathbb{C}}$ taking values in $\mathbb{Z}(p) \subset \mathbb{C}$.

Definition 3.4.19. (The smooth Deligne complex and smooth Deligne cohomology).

Given a smooth manifold M and a $p \geq 0$, the smooth Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$ is the complex

$$\mathbb{Z}(p) \to \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}} \to \ldots \to \Omega^{p-1}_{\mathbb{C}}$$

of sheaves.

The smooth Deligne cohomology groups $H^n_{\mathcal{D}}(M; \mathbb{Z}(p))$ is defined as the hypercohomology groups of the smooth Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$.

Note that the complex is truncated as it ends at $\Omega^{p-1}_{\mathbb{C}}$. It is clear that the sheaves $\Omega^n_{\mathbb{C}}$ are soft. This simplifies the explicit construction of smooth Deligne cohomology.

We can define a similar complex by for any subring $B \subseteq \mathbb{C}$, such as $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. The connections to de Rham cohomology can be made more explicit if we choose $B = \mathbb{R}$ and consider $B(p) = \mathbb{R}(p)$. However, we restrict ourselves to $B = \mathbb{Z}$.

Example 3.4.20. $\mathbb{Z}_{\mathcal{D}}(0)$ is just $\mathbb{Z}(0) = \mathbb{Z}$. $H^n_{\mathcal{D}}(M; \mathbb{Z}(0))$ is therefore just the sheaf cohomology $H^n(M; \mathbb{Z})$.

Example 3.4.21. $\mathbb{Z}_{\mathcal{D}}(1)$ is the complex $\mathbb{Z}(1) \to \underline{\mathbb{C}_M}$, as in Example 3.4.1. This complex is quasi-isomorphic to the complex induced from the sheaf $\underline{\mathbb{C}^*_M}[-1]$. This is simply because the short exact sequence of Example 3.4.1 induces a square

$$\begin{tikzcd}
\mathbb{Z}(1) \arrow[r] \arrow[d] & \underline{\mathbb{C}_M} \arrow[d, "exp"] \\
0 \arrow[r] & \underline{\mathbb{C}^*_M}.
\end{tikzcd}$$

The vertical maps make up a quasi-isomorphism of complexes. This means that

$$H^n_{\mathcal{D}}(M; \mathbb{Z}(p)) \cong H^{n-1}(M; \underline{\mathbb{C}^*_M}).$$

For general p, it is possible to understand the smooth Deligne cohomology groups $H^*_{\mathcal{D}}(M; \mathbb{Z}(p))$. Let $\Omega^n_{\mathbb{C},0}(M)$ be the subgroup of $\Omega^n_{\mathbb{C}}(M)$ containing the closed n-forms with cohomology classes in the image of $H^n(M; \mathbb{Z}(p)) \to H^n(M; \mathbb{C})$. Note the similarity to Section 2.3.

Theorem 3.4.22. *(Smooth Deligne cohomology groups).*

For a smooth manifold M, the smooth Deligne cohomology groups $H^*_{\mathcal{D}}(M;\mathbb{Z}(p))$ can be found as follows:

1. *If $n > p$, we have $H^n_{\mathcal{D}}(M;\mathbb{Z}(p)) \cong H^n(M;\mathbb{Z}(p))$, the sheaf cohomology of M with respect to the constant sheaf $\mathbb{Z}(p)$.*
2. *If $n = p$, $H^p_{\mathcal{D}}(M;\mathbb{Z}(p))$ is found in a short exact sequence*
$$0 \longrightarrow \Omega^{p-1}_{\mathbb{C}}(M)/\Omega^{p-1}_{\mathbb{C},0}(M) \longrightarrow H^p_{\mathcal{D}}(M;\mathbb{Z}(p)) \longrightarrow H^p(M;\mathbb{Z}(p)) \longrightarrow 0,$$
where the latter map is induced by projecting the smooth Deligne complex down on its first entry.
3. *If $n < p$, $H^n_{\mathcal{D}}(M;\mathbb{Z}(p))$ is found in a short exact sequence*
$$0 \longrightarrow H^{n-1}(M;\mathbb{C})/H^{n-1}(M;\mathbb{Z}(p)) \longrightarrow H^n_{\mathcal{D}}(M;\mathbb{Z}(p)) \longrightarrow \mathrm{Tors}(H^n(M;\mathbb{Z}(p))) \longrightarrow 0,$$
where Tors *denotes the torsion subgroup of $H^n(M;\mathbb{Z}(p))$.*

Proof. The smooth Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$ sits in a short exact sequence

$$0 \longrightarrow \Omega^{*\leq p-1}_{\mathbb{C}}[-1] \longrightarrow \mathbb{Z}(p)_{\mathcal{D}} \longrightarrow \mathbb{Z}(p) \longrightarrow 0,$$

where the notation $\Omega^{*\leq p-1}_{\mathbb{C}}$ is inspired by Hopkins-Singer (Section 2.7) and refers to the complex

$$\Omega^0_{\mathbb{C}} \longrightarrow \Omega^1_{\mathbb{C}} \longrightarrow \ldots \longrightarrow \Omega^{p-1}_{\mathbb{C}}.$$

Simply considering the cohomology of the complex $\Omega^{*\leq p-1}_{\mathbb{C}}$ lets us find

$$\begin{aligned} H^n(M;\Omega^{*\leq p-1}_{\mathbb{C}}) &= 0 && \text{if} \quad n \geq p, \\ H^n(M;\Omega^{*\leq p-1}_{\mathbb{C}}) &= \Omega^{p-1}_{\mathbb{C}}(M)/d\Omega^{p-2}_{\mathbb{C}}(M) && \text{if} \quad n = p-1, \\ H^n(M;\Omega^{*\leq p-1}_{\mathbb{C}}) &= H^n_{dR}(M) \otimes \mathbb{C} && \text{if} \quad n \leq p-2. \end{aligned}$$

The short exact sequence for the smooth Deligne complex yields a long exact sequence

$$\begin{tikzcd} \ldots \arrow[r] & H^{n-1}(M;\mathbb{Z}(p)) \arrow[r] & H^{n-1}(M;\Omega^{*\leq p-1}_{\mathbb{C}}) \arrow[r] & H^n_{\mathcal{D}}(M;\mathbb{Z}(p)) \arrow[dll, out=-10, in=170] \\ & H^n(M;\mathbb{Z}(p)) \arrow[r] & H^n(M;\Omega^{*\leq p-1}_{\mathbb{C}}) \arrow[r] & \ldots. \end{tikzcd}$$

Note the shift in the cohomology of $\Omega_{\mathbb{C}}^{*\leq p-1}$, stemming from the shift in the short exact sequence above.

If $n > p$, the isomorphism $H^n_{\mathcal{D}}(M;\mathbb{Z}(p)) \cong H^n(M;\mathbb{Z}(p))$ is evident from the long exact sequence.

If $n = p$, the above sequence becomes

$$\begin{tikzcd} \ldots \arrow[r] & H^{p-1}(M;\mathbb{Z}(p)) \arrow[r] & \Omega^{p-1}_{\mathbb{C}}(M)/d\Omega^{p-2}_{\mathbb{C}}(M) \arrow[r] & H^p_{\mathcal{D}}(M;\mathbb{Z}(p)) \arrow[dll, out=-10, in=170] \\ & H^p(M;\mathbb{Z}(p)) \arrow[r] & 0 \arrow[r] & \ldots. \end{tikzcd}$$

The map $H^{p-1}(M;\mathbb{Z}(p)) \to \Omega^{p-1}_{\mathbb{C}}(M)/d\Omega^{p-2}_{\mathbb{C}}(M)$ is constructed as the composition

$$H^{p-1}(M;\mathbb{Z}(p)) \to H^{p-1}(M,\mathbb{C}) \cong H^{p-1}(\Omega^*_{\mathbb{C}}(M)) \to H^{p-1}(M;\Omega_{\mathbb{C}}^{*\leq p-1}) \cong \Omega^{p-1}_{\mathbb{C}}(M)/d\Omega^{p-2}_{\mathbb{C}}(M).$$

Its image is exactly $\Omega^{p-1}_{\mathbb{C},0}(M)/d\Omega^{p-2}_{\mathbb{C}}(M)$ by the definition of of $\Omega_{\mathbb{C},0}$, which indicates that the cokernel is

$$\frac{\Omega^{p-1}_{\mathbb{C}}(M)/d\Omega^{p-2}_{\mathbb{C}}(M)}{\Omega^{p-1}_{\mathbb{C},0}(M)/d\Omega^{p-2}_{\mathbb{C}}(M)} \cong \Omega^{p-1}_{\mathbb{C}}(M)/\Omega^{p-1}_{\mathbb{C},0}(M).$$

Hence we obtain a short exact sequence as wanted.

Finally, if $n < p$, we simply have $H^{n-1}(M;\Omega_{\mathbb{C}}^{*\leq p-1}) \cong H^{n-1}(M;\mathbb{C})$, yielding

$$\begin{tikzcd} \ldots \arrow[r] & H^{n-1}(M;\mathbb{Z}(p)) \arrow[r] & H^{n-1}(M;\mathbb{C}) \arrow[r] & H^n_{\mathcal{D}}(M;\mathbb{Z}(p)) \arrow[dll, out=-10, in=170] \\ & H^n(M;\mathbb{Z}(p)) \arrow[r, "f_n"] & H^n(M;\Omega_{\mathbb{C}}^{*\leq p-1}) \arrow[r] & \ldots. \end{tikzcd}$$

Let i_n denote the map $i_n : H^n(M;\mathbb{Z}(p)) \to H^n(M;\mathbb{C}) \to H^n(M;\Omega_{\mathbb{C}}^{*\leq p-1})$. Since the situation $n = p-1$ does not necessarily yield $H^n(M;\Omega_{\mathbb{C}}^{*\leq p-1}) \cong H^n(M;\mathbb{C})$, the best we can say is that the kernel of i_n must be the kernel of $j_n : H^n(M;\mathbb{Z}(p)) \to H^n(M;\mathbb{C})$ as $H^n(M;\mathbb{C}) \cong H^n_{dR}(M) \to H^n(M;\Omega_{\mathbb{C}}^{*\leq p-1})$ is injective (simply by inclusion at the level of complexes for degrees $n < p$). Hence, the above sequence may be rewritten as

$$0 \longrightarrow H^{n-1}(M;\mathbb{C})/H^{n-1}(M;\mathbb{Z}(p)) \longrightarrow H^n_{\mathcal{D}}(M;\mathbb{Z}(p)) \longrightarrow \mathrm{im}(f_n) \cong \ker(j_n) \longrightarrow 0,$$

since $\mathrm{im}(f_n) \cong \ker(i_n)$ by exactness, where f_n is the map found in the long exact sequence above. This yields the wanted result, as $\ker(i_n)$ is exactly the torsion subgroup $\ker(i_n) \cong \mathrm{Tors}(H^n(M;\mathbb{Z}(p)))$. $\square$

An interesting observation for the case $n = p$ is that the cohomology classes in $H^p(M;\mathbb{Z}(p))$ can be lifted to $H^p_{\mathcal{D}}(M;\mathbb{Z}(p))$. If we lifted a class that vanishes in $H^p(M;\mathbb{Z}(p))$, its lift will lie in the image of $\Omega^{p-1}_{\mathbb{C}}(M)/\Omega^{p-1}_{\mathbb{C},0}(M) \to H^p_{\mathcal{D}}(M;\mathbb{Z}(p))$. Hence, vanishing cohomology classes will have underlying geometric information describing it, as found in $\Omega^{p-1}_{\mathbb{C}}(M)/\Omega^{p-1}_{\mathbb{C},0}(M)$. This aligns with the mantra of differential cohomology as a tool to refine topological invariants, as it detects information that the cohomology groups neglect. The above idea will be central in our discussion of secondary characteristic classes and multiplicative K-theory in Chapter 5.

Remark 3.4.23. (Deligne cohomology, differential cohomology and homotopy pullbacks).

The connection to differential cohomology can be more explicit. Since the important matter is to consider smooth Deligne cohomology, and not the smooth Deligne complex itself, we may work towards finding an equivalent formulation in the derived category of complexes of sheaves. That is, we seek to find a quasi-isomorphic complex to the smooth Deligne complex.

Essentially, the smooth Deligne complex consists of a combination of $\mathbb{Z}(p)$ and some $\Omega^*_{\mathbb{C}}[-1]$. Consider the map $\mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \to \Omega^*_{\mathbb{C}}$ and its (shifted) cone as follows, where we are keeping track of the maps/complexes vertically and the degrees horizontally.

$$\begin{tikzcd}
 & 0 & 1 & 2 & \dots & p & p+1 & \\
\mathrm{cone}(h)[-1] \arrow[d] & \mathbb{Z}(p) \arrow[r] \arrow[d] & \Omega^0_{\mathbb{C}} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{C}} \arrow[r] \arrow[d] & \dots \arrow[r] & \Omega^{p-1}_{\mathbb{C}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \arrow[d, "h"] & \mathbb{Z}(p) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & \Omega^p_{\mathbb{C}} \arrow[r] \arrow[d] & \Omega^{p+1}_{\mathbb{C}} \arrow[r] \arrow[d] & \dots \\
\Omega^*_{\mathbb{C}} & \Omega^0_{\mathbb{C}} \arrow[r] & \Omega^1_{\mathbb{C}} \arrow[r] & \Omega^2_{\mathbb{C}} \arrow[r] & \dots \arrow[r] & \Omega^p_{\mathbb{C}} \arrow[r] & \Omega^{p+1}_{\mathbb{C}} \arrow[r] & \dots
\end{tikzcd}$$

The shifted cone above is the smooth Deligne complex.

In the derived category, $\mathrm{cone}(\mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \to \Omega^*_{\mathbb{C}})$ sits in a triangle

$$\mathrm{cone}(\mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \to \Omega^*_{\mathbb{C}})[-1] \longrightarrow \mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \longrightarrow \Omega^*_{\mathbb{C}} \longrightarrow \mathrm{cone}(\mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \to \Omega^*_{\mathbb{C}}).$$

That is, the smooth Deligne complex is a weak kernel of the map $\mathbb{Z}(p) \oplus \Omega^{*\geq p}_{\mathbb{C}} \to \Omega^*_{\mathbb{C}}$ (as long as we specify the signs correctly). This is quasi-isomorphic to a homotopy pullback of the diagram

$$\begin{tikzcd}
 & \Omega^{*\geq p}_{\mathbb{C}} \arrow[d] \\
\mathbb{Z}(p) \arrow[r] & \Omega^*_{\mathbb{C}}.
\end{tikzcd}$$

This aligns perfectly with the Hopkins-Singer approach and the Theorems 2.4.2 and 3.2.9 on the uniqueness of ordinary differential cohomology.

The above is a vital observation when further developing Hodge-filtered cohomology theories (see [HQ15]) in Chapter 4.

The above motivates the following theorem, which was known long before the Hopkins-Singer approach or the uniqueness theorems were developed in [HS05], [SS08a], and [BS10], respectively. See for example [Esn88] for a proof.

Theorem 3.4.24. *(Smooth Deligne cohomology is ordinary differential cohomology).*

The group of differential characters $\hat{H}^p(M)$ by Cheeger-Simons (2.3) is isomorphic to the smooth Deligne cohomology group $H^p_{\mathcal{D}}(M;\mathbb{Z}(p))$.

Remark 3.4.25. Surprisingly enough, we would have obtained a quasi-isomorphic complex if we used $\mathrm{cok}(h)[-1]$ instead of $\mathrm{cone}(h)[-1]$ above.

Remark 3.4.26. There exists a product on smooth Deligne cohomology much simpler than the product on differential characters. We refer to [ADH23], [Bry93], or [Esn88] for its construction.

3.5 Interlude: The Geometric Suspension Construction

Other explicit models for differential cohomology such as the differential K-theory of [SS08b] (Section 3.6), are usually just defined in degree 0. For a cohomology theory E, we can define E^n from E^0 by considering the suspension $E^n = E^0 \circ \Sigma^n$. In differential cohomology, this does not work. The suspension ΣM of a smooth manifold M is rarely a smooth manifold, which means that $\hat{E}^n = \hat{E}^0 \circ \Sigma^n$ rarely is defined. However, as briefly touched upon in Section 2.4, we can obtain such a shift in the cohomological degree by considering the map $i : X \to X \times S^1$ and the kernel of $i^* = E^0(i)$. Again, this is proved (see for example [BNV16]) merely for topological spaces without addressing how it works with more structured spaces, such as smooth manifolds.

What we present here is based on the abstract machinery we met in Section 2.11, explaining that there is a solution that agrees with the above for topological spaces. Following Section 3.1, define I_{S^1} by $I_{S^1}\hat{E}(M) = \hat{E}(M \times S^1)$ for a spectral sheaf $\hat{E} \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$ and a smooth manifold M. Define now $I_\Sigma \hat{E}$ by the fiber sequence

$$I_\Sigma \hat{E} \to I_{S^1}\hat{E} \to \hat{E},$$

where the last map is induced by $i : \mathrm{Id}_{\mathbf{Man}} \to \mathrm{Id}_{\mathbf{Man}} \times S^1$. By a result of [BNV16], we have the following.

Proposition 3.5.1. *(Geometric Suspension).*

Given an $\hat{E} \in \mathrm{Sh}(\mathbf{Man};\mathbf{Spt})$, the canonical map $\hat{E} \to \Sigma I_\Sigma \hat{E}$ is an equivalence if and only if $\hat{E}$ is homotopy invariant.

Essentially, this lets us define $\hat{E}^n(M) = \ker(i^* : E^0(M \times S^n) \to E^0(M))$, as this induces the correct shift in degree. This is not new from [BNV16], as it has been used at least since [Kar90].

If we now consider $\hat{E}$ to be a "trivial" differential refinement in the sense of Example 2.13.16 that returns E-cohomology, we see that E^n can be defined from E^0 as above for smooth manifolds. This agrees with the definition $E^n = E^0 \circ \Sigma^n$ as topological spaces.

This way of considering the consequences of taking the suspension without actually forming the suspension of the manifold or topological space, will be referred to as the *geometric suspension* or *the geometric suspension construction*. For our purposes, it will be used in defining a geometric model of Hodge-filtered K-theory in Chapter 5.

3.6 The Differential K-theory of Simons-Sullivan

It is not only just ordinary differential cohomology that has been described explicitly. The structured bundles of Simons-Sullivan [SS08b] turn out to define differential K-theory. We will briefly cover the main definitions.

Recall that complex K-theory on a smooth manifold X is defined by considering the monoid of (isomorphism classes of) complex vector bundles on M and then applying the Grothendieck construction to obtain an abelian group. Naively, the same idea should hold for differential K-theory, but perhaps by considering bundles with connection to "force" geometric information into the picture. However, considering just tuples (E, ∇) where E is a complex vector bundle om M and ∇ is a connection on E, is slightly too naive.

A vital idea to the construction is to establish an equivalence relation on the set of connections on E. This would have been simple if we had already covered the Chern-Simons forms and secondary characteristic classes, but these are postponed until our discussion on multiplicative K-theory. The idea is to describe the underlying geometric information of the Chern characters of connections. This will be made precise in Section 5.1. The idea of trivializing characteristic classes was developed in [CS74].

If not otherwise mentioned, E is a complex vector bundle on M, ∇ is a connection on E, and $R \in \Omega^2(M; \mathrm{End}(E))$ is its curvature. Recall that the Chern character may be defined through the Chern-Weil homomorphism as

$$\mathrm{ch}(\nabla) = \sum_{j=0} \frac{1}{(2\pi i)^j} \frac{1}{j!} \mathrm{tr}(R \wedge \ldots \wedge R),$$

where we wedge the R's together j times. The above defines a closed form of even degree taking complex values.

Consider $\gamma(t) = \nabla^t$ to be a smooth curve of connections on E for $t \in [0, 1]$. Let $\pi : M \times [0, 1] \to M$ be the projection, and set $\bar{E} = \pi^* E$ to be the pullback bundle. The coordinate of $[0, 1]$ will always be referred to as t. Consider the slice map, $\psi_t : M \to M \times [0,1]$, defined by $m \mapsto \psi_t(m) = (m,t)$. From the path γ of connections, we may form a connection $\bar{\nabla}$ on $\bar{E}$. For a section s of E, set $\bar{\nabla}_{\partial/\partial t}(\pi^*(s)) = 0$. For a tangent vector X to the slice at coordinate t, set $\bar{\nabla}_X = \nabla^t_{\pi_*(X)}$.

Definition 3.6.1. (Defining the form $\mathrm{cs}(\gamma)$).

Given the setup above, we may define a form $\mathrm{cs}(\gamma)$ as

$$\mathrm{cs}(\gamma) = \int_0^1 \psi_t^*(i_{\partial/\partial t}\,\mathrm{ch}(\bar{\nabla})).$$

An important property, as originally motivated from [CS74], is that $d\,\mathrm{cs}(\gamma) = \mathrm{ch}(\nabla^1) - \mathrm{ch}(\nabla^0)$. In [SS08b], the proof becomes quite simple after formulating an equivalent expression for $\mathrm{cs}(\gamma)$ to obtain a form more similar to $\mathrm{ch}(\nabla)$. We refer to [SS08b] for the details.

It is also possible to show that if γ is a closed path of connections, $\mathrm{cs}(\gamma)$ integrates to 0 on every cycle of M. From this, one can deduce that if γ and γ' are two paths connecting two connections ∇^0 and ∇^1,

$$\mathrm{cs}(\gamma) - \mathrm{cs}(\gamma') \in \mathrm{im}(d).$$

As two connections always may be joined by a smooth path, we can define the following form, CS.

Definition 3.6.2. (The form CS).

For connections ∇^0 and ∇^1, and a path γ connecting them, set

$$\mathrm{CS}(\nabla^0, \nabla^1) = \mathrm{cs}(\gamma) \mod \mathrm{im}(d).$$

This construction induces an equivalence relation.

Definition 3.6.3. (Equivalence relation on connections).

We say $\nabla^0 \sim \nabla^1$ if $\mathrm{CS}(\nabla^0, \nabla^1) = 0$. This defines an equivalence relation by [SS08b].

Definition 3.6.4. (Structured bundles).

We call a pair $\mathcal{E} = (E, [\nabla])$ a structured bundle on M if E is a complex vector bundle on M and $[\nabla]$ is an equivalence class of connections on E.

A structured bundle is called *flat* if some $\nabla \in [\nabla]$ is flat in the sense that $R^\nabla = 0$.

A map $\psi : N \to M$ of base spaces induces connections $\psi^*\nabla^0$ and $\psi^*\nabla^1$ on the pullback bundle ψ^*E over N, and the CS-form respects this in the sense that $\mathrm{CS}(\psi^*\nabla^0, \psi^*\nabla^1) = \psi^*(\mathrm{CS}(\nabla^0, \nabla^1))$.

We say that structured bundles $\mathcal{E}$ and $\mathcal{E}'$ are *isomorphic* if there exists a bundle isomorphism $\tau : E \to E'$ covering the identity on M such that $\tau^*([\nabla^{E'}]) = [\nabla^E]$.

The construction of structured bundles is functorial, but we have not yet understood its target category.

Proposition 3.6.5. *($\oplus$ and $\otimes$ of structured bundles).*

Let $\mathcal{E} = (E, [\nabla^E])$ and $\mathcal{E}' = (E', [\nabla^{E'}])$ be structured bundles on a smooth manifold M. The following operations yield well-defined operations on structured bundles,

$$\mathcal{E} \oplus \mathcal{E}' = (E \oplus E', [\nabla^E \oplus \nabla^{E'}]), \quad \textit{and}$$
$$\mathcal{E} \otimes \mathcal{E}' = (E \otimes E', [\nabla^E \otimes \nabla^{E'}]),$$

where $\oplus$ and $\otimes$ of connections are defined as before. That is, for a tangent vector X to M and sections s_E and $s_{E'}$ of E and E', respectively, we set

$$(\nabla^E \oplus \nabla^{E'})_X(s_E, s_{E'}) = (\nabla^E_X s_E, \nabla^{E'}_X s_{E'}), \quad \textit{and}$$
$$(\nabla^E \otimes \nabla^{E'})_X(s_E \otimes s_{E'}) = \nabla^E_X(s_E) \otimes s_{E'} + s_E \otimes \nabla^{E'}_X s_{E'}.$$

Proof. Note that we have

$$\mathrm{ch}\left(\nabla^E \oplus \nabla^{E'}\right) = \mathrm{ch}\left(\nabla^E\right) + \mathrm{ch}\left(\nabla^{E'}\right), \quad \text{and}$$
$$\mathrm{ch}\left(\nabla^E \otimes \nabla^{E'}\right) = \mathrm{ch}\left(\nabla^E\right) \wedge \mathrm{ch}\left(\nabla^{E'}\right).$$

Using the above and Formula 1.2 in [SS08b], direct computations yield that for connections ∇^E_0, ∇^E_1 on E and $\nabla^{E'}_0, \nabla^{E'}_1$ on E', we have

$$\mathrm{CS}\left(\nabla^E_0 \oplus \nabla^{E'}_0, \nabla^E_1 \oplus \nabla^{E'}_1\right) = \mathrm{CS}\left(\nabla^E_0, \nabla^E_1\right) + \mathrm{CS}\left(\nabla^{E'}_0, \nabla^{E'}_1\right), \quad \text{and}$$
$$\mathrm{CS}\left(\nabla^E_0 \otimes \nabla^{E'}_0, \nabla^E_1 \otimes \nabla^{E'}_1\right) = \mathrm{ch}\left(\nabla^E_0\right) \wedge \mathrm{CS}\left(\nabla^{E'}_0, \nabla^{E'}_1\right) + \mathrm{ch}\left(\nabla^{E'}_1\right) \wedge \mathrm{CS}\left(\nabla^E_0, \nabla^E_1\right).$$

This means that if $\nabla^E_0 \sim \nabla^E_1$ and $\nabla^{E'}_0 \sim \nabla^{E'}_1$, the right-hand sides vanish, implying the operations are well-defined. $\square$

From the above, we can define the functor Struct from the category of smooth manifolds to commutative semi-rings. By using K, the Grothendieck construction, which sends abelian semi-groups to abelian groups, we can form differential K-theory. Recall that the Grothendieck group of a monoid S is constructed by considering $S \times S$ with the relation that $(s_1, s_2) \sim (s'_1, s'_2)$ if there exists a $k \in S$ such that $s_1 + s'_2 + k = s_2 + s'_1 + k$.

Definition 3.6.6. (Differential K^0-theory).

For a smooth manifold M, we define $\hat{K}^0(M) := K(\mathrm{Struct}(M))$. Equivalently, $\hat{K}^0$ may be defined as the free abelian group on (isomorphism classes of) structured bundles where we identify $(\mathcal{E} \oplus \mathcal{E}')$ with $\mathcal{E} + \mathcal{E}'$.

Remark 3.6.7. To see the description of $\hat{K}^0$ as "formal differences" of structured bundles, note that the above is equivalent to considering pairs $(\mathcal{E}, \mathcal{E}')$ of structured bundles modulo all pairs $(\mathcal{E}, \mathcal{E})$. Hence, $(0, \mathcal{E})$ is the additive inverse of $(\mathcal{E}, 0)$, letting us write $(\mathcal{E}, \mathcal{E}')$ as $\mathcal{E} - \mathcal{E}'$.

Remark 3.6.8. From differential Bott periodicity, we know that we only need to define $\hat{K}^0$ and $\hat{K}^1$. The first is already defined, and the second can be defined by the geometric suspension.

Readers interested in the details regarding structured bundles, and technicalities related to constructing the map a, are referred to [SS08b]. The map I is given by sending a structured bundle to the K-theory class of the bundle, and the map R is given by the Chern character of the connection.

As K-theory gives a rationally even spectrum, the structured bundles of Simons-Sullivan define differential K-theory up to a unique isomorphism. This displays how explicitly the geometry is forced into the picture, as we can think of the structured bundles as bundles with a (class of) connection(s).

3.7 Differential Complex Cobordism and Landweber Theories

In this section, we will briefly investigate differential complex cobordism and the differential version of its role in chromatic homotopy theory.

Warning 3.7.1. Much of the motivation and main ideas of this section require some knowledge of chromatic homotopy theory. We refer to [Lur10].

Recall that complex cobordism MU plays a universal role in chromatic homotopy theory. If we consider the product of complex line bundles, $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, at the level of classifying spaces, we can produce a formal group law over $E^{even}(*)$, where E is required to be a complex-orientable cohomology theory. The map $i : S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$ induces a map $i^* : \tilde{E}^2(\mathbb{C}P^\infty) \to \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^0(S^0) \cong E^0(*)$. If the identity element of $E^0(*)$ is in the image of i^*, we say that E is a complex-oriented cohomology theory (after we have chosen a preimage of 1, called the complex orientation of E). A motivating example is ordinary cohomology with coefficients in $\mathbb{Z}_2$, which yields the classical result that all spaces are $\mathbb{Z}_2$-oriented. E being a complex-orientabed cohomology theory essentially ensures that the Atiyah-Hirzebruch spectra sequence for E,

$$E_2^{p,q} = H^p(X, E^q(*)) \implies E^{p+q}(X)$$

degenerates at the E_2-page for any topological space X. Computations yield isomorphisms $E^*(\mathbb{C}P^\infty) \cong E^*(*)[[t]]$ and $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(*)[[x, y]]$, where the generators are of degree 2. Furthermore, if E is rationally even (which holds for any cohomology theory we have a unique differential refinement of), the Atiyah-Hirzebruch spectral sequence can be used to show E is complex-orientable.

The formal group laws are produced by considering $\mu^* t \in E^*(*)[[x, y]]$. This defines E-Chern classes (also called Conner-Floyd Chern classes) of line bundles by letting $f : X \to \mathbb{C}P^\infty$

represent a line bundle L and considering $c_1^E(L) := f^*t \in E^2(X)$. These formal group laws aim to express how Conner-Floyd classes of a product of line bundles behave. For example, given line bundles L and L' over X, ordinary Chern classes satisfy

$$c_1(L \otimes L') = c_1(L) + c_1(L') = f(c_1(L), c_1(L')).$$

In the language of formal group laws, this is to say that ordinary cohomology can be assigned the additive group law $f(x, y) = x + y$. Complex K-theory does not satisfy this, as the Bott element introduces a shift, yielding the multiplicative group law $f(x, y) = x + y + xy$.

There exists a universal ring L and a universal formal group law f over L. The ring L is called the Lazard ring, and it is universal in the sense that any other ring R and a formal group law g over R can be classified by a ring homomorphism $L \to R$. Quillens theorem on MU shows that the coefficient ring of MU is precisely the Lazard ring L, which can be shown to be $\mathbb{Z}[c_1, c_2, \ldots]$, where c_i is of degree $2i$. The map $L \to R$ induces an $L = MU^* = MU^*(*)$-module structure on R, and we can form a functor $R^*(-) = MU^*(-) \otimes_{MU^*} R$. A natural question is to ask whether or not the above functor is a cohomology theory. An important theorem in chromatic homotopy theory, Landwebers exact functor theorem, quantifies this into an algebraic criterion, as the main problem is the exactness of the new sequence due to R-torsion. A formal group law (R, g) that yields a cohomology theory ${}^*R(-)$ as above is called a *Landweber exact formal group law.* We also say R is *Landweber flat.*

If we take R to be the integers with the multiplicative formal group law, Landwebers theorem returns complex K-theory as the cohomology theory. This was originally called the Conner-Floyd theorem. In light of our overarching topic, it is reasonable to ask how these Landweber theories adapt to differential cohomology.

First, note that in [Bun+09], an explicit geometric model for differential complex cobordism is developed. We will not cover its construction here, as we merely need the existence of differential complex cobordism to understand differential Landweber exact cohomology theories. The geometric model of complex cobordism is essential to the work of [HQ22] and [Hau22].

To develop one of the main results in [Bun+09], the following lemma of Landweber [Lan76] is useful. We state it for the discussion of Conjecture 4.6.2 in Section 4.6.

Lemma 3.7.2. *(Landweber flatness).*

Assume M is a finitely presented MU^*-module which is also a comodule over MU_*MU. If (R, g) is a Landweber exact formal group law, then $\mathrm{Tor}_i^{MU^*}(M, R) = 0$ for $i \geq 1$.

From the above statement yielding vanishing R-torsion, the following result can be shown.

Theorem 3.7.3. *(Differential Landweber Theories).*

Let (R, g) be a Landweber exact formal group law, and let $R^*(-) = MU^*(-) \otimes_{MU^*} R$ denote the Landweber theory associated to R as defined on the category of finite CW-complexes. Then $\hat{R}^*(-) = \hat{MU}^*(-) \otimes_{MU^*} R$ is a multiplicative smooth extension of R^*.

Remark 3.7.4. The above Landweber theory was defined on finite CW-complexes, and although it can be extended to arbitrary CW-complexes by an inductive limit, we merely need to consider finite CW-complexes as these are the homotopy types of all smooth (and complex) manifolds.

Remark 3.7.5. Due to the uniqueness of differential K-theory, we may see that the Conner-Floyd-type differential K-theory obtained by the above theorem is canonically isomorphic to both Hopkins-Singer K-theory and the differential K-theory of structured bundles by Simons-Sullivan. We have not investigated how the geometric description of differential complex cobordism (from [Bun+09]) is transferred to K-theory.

Remark 3.7.6. The vanishing condition in Lemma 3.7.2 is easily applicable to differential cohomology, as the last axiom in Section 3.1 can be reformulated to a short exact sequence, yielding an explicit long exact sequence of Tor-terms when tensored with a ring R. The case for Hodge-filtered theories is not as simple, due to a slight change in the sequence axiom, as we will see in Section 4.6.

3.8 Other topics:

Differential index theory

The well-celebrated Atiyah-Singer index theorem of [AS63] connects the analysis and the topology of certain differential operators. The first proof of the index theorem was based on K-theoretic arguments, and hence it should be possible to construct a differential analog of the index theorem using differential K-theory. This is done in [FL10], and is based on further study of the flat theory, which is found in [Lot94].

Classification Sheaves for Principal G-Bundles with Connection

Principal G-bundles over a space X can be classified in terms of homotopy classes of maps $X \to BG$, where BG denotes the classifying space of G. BG can be constructed as the geometric realization of the simplicial space given by $[n] \mapsto (BG)_n = G^{\times n}$ with the group operation and the unit map inducing the face and degeneracy maps, respectively. Finding a classifying space $B_\nabla G$ for principal G-bundles with connection was first done in [FH13].

The paper builds on the observation that $\mathrm{Hom}(\mathbf{Man}(-, M), \Omega^n(-)) \cong \Omega^n(M)$ by the Yoneda lemma, or more generally that we may pass from smooth manifolds to simplicial presheaves on manifolds. For a fixed Lie group G, the simplicial presheaf $B_\nabla G$ maps a smooth manifold M to the simplicial set associated to the groupoid $\mathcal{G}(M)$ of principal G-connections. That is, the groupoid $\mathcal{G}(M)$ consists of objects (π, ∇), where $\pi : P \to M$ is a principal G-bundle and $\nabla \in \Omega^1(P; \mathfrak{g})$ denotes a G-connection. The morphisms are principal G-bundle maps covering the base M respecting the connections. This is a groupoid as all principal G-bundle maps covering the base are isomorphisms. Given a groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$, its associated simplicial set has 0-simplices $\mathcal{G}_0$, 1-simplices $\mathcal{G}_1$ and n-simplicies the collection of compositions of n arrows in $\mathcal{G}_1$.

The two face maps we can apply to a 1-simplex are the source and target morphisms. We can always associate the identity morphism (a 1-simplex) to any 0-simplex, yielding the degeneracy map on 0-simplices. One can show that $B_\nabla G$ is a sheaf, not only a presheaf.

With more work, one can define a simplicial presheaf $E_\nabla G$ and a map of simplicial presheaves $E_\nabla G \to B_\nabla G$ such that $E_\nabla G \to B_\nabla G$ takes the role as a universal principal G-bundle with connection. The universal G-connection on $E_\nabla G$ is also defined in [FH13].

Furthermore, the above is used to show that the characteristic classes coming from Chern-Weil theory are the only "natural" differential forms it is possible to construct from a G-connection.

The question of extending the above results to the holomorphic case, with holomorphic principal G-bundles and holomorphic connections was discussed in [And21]. Note however that admitting a holomorphic connection is more strict than simply admitting a connection (which always exists in the case of smooth manifolds). Obstructions to admitting a holomorphic connection can be found in Atiyah classes (see [Huy05]). However, as argued in [And21], there are reasons to expect a universal holomorphic connection on the holomorphic analogue of $E_\nabla G$.

In [ADH23] one can find a survey on differential characteristic classes. It is based on using the classifying sheaf $B_\nabla G$ to reformulate the results of Cheeger-Simons on trivializations of the Chern-Weil classes. As the original construction of [CS85] used n-classifying spaces, this can be thought of as a unified approach to the content of [CS85].

Differentially Refined Cohomology Operations and Massey Products

Cohomology operations are important tools in algebraic topology, and they have plenty of applications. We refer to [MT08] for a thorough treatment of the topic. Classically, they are defined as natural transformations $H^n(-;G) \to H^m(-;H)$, which can be represented as natural transformations $[-, K(G,n)] \to [-, K(H,m)]$. By the Yondea lemma, the set of such cohomology operations is in bijection to $H^m(K(G,n);H)$. We only consider the transformations in **Set**, and not in **Ab**, as many interesting examples fail to yield group homomorphisms. Examples of such operations include any square operation (in the case $G = H$ and $m = 2n$), the Steenrod squares (for $G = H = \mathbb{Z}/2\mathbb{Z}, m = n + i, i \geq 0$) and the Steenrod reduced p'th powers (for $G = H = \mathbb{Z}/p\mathbb{Z}, m = n + 2i(p-1), i \geq 0$).

We know from Section 2.5 that the higher differential cohomology groups are related to the classification of higher principal $U(1)$-bundles with connection. The relevant higher analog for bundles with connection can be expressed through stacks, and works as one of the main tool used to define and characterize differential cohomology operations in [GS18b].

For differential cohomology, we need the refined Steenrod squares to live in the middle of a hexagon diagram, and their image under the mod 2-reduced characteristic class map $\hat{H}^n(M;\mathbb{Z}) \to H^n(M;\mathbb{Z}) \to H^n(M;\mathbb{Z}/2\mathbb{Z})$ should be related to the ordinary Steenrod operations. More formally, we should at least have integral lifts to be able to define differential Steenrod squares.

Integral lifts are possible to construct for odd Steenrod squares Sq^{2i+1}, but not for the even case. If ρ_2 denotes the mod 2-reduction map and β denotes the Bockstein map associated to the long exact sequence coming from the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\rho_2} \mathbb{Z}/2\mathbb{Z} \to 0$, the existence of an integral lift of Sq^{2i} implies that $\beta(Sq^{2i}) = 0$. However, it has been shown that $\rho_2\beta(Sq^{2i}) = Sq^{2i+1}$. This tells us that no integral lift of even Steenrod squares exist, but they do for odd Steenrod squares, as $\beta(Sq^{2i})$ is an integral lift of Sq^{2i+1}.

The results on differential Steenrod squares follow the same pattern. Even Steenrod squares cannot be differentially refined, but the odd ones can. We may even express the refined odd squares through the ordinary even squares, as above. The unique way to differentially refine odd Steenrod squares Sq^{2i+1} to $\hat{Sq}^{2i+1}$ is through the composition

$$\begin{tikzcd}
\hat{H}^n(-;\mathbb{Z}) \arrow[r, "I"] & H^n(-;\mathbb{Z}) \arrow[r, "\rho_2"] & H^n(-;\mathbb{Z}/2\mathbb{Z}) \arrow[dll, "Sq^{2i}"] \\
H^{n+2i}(-;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\Gamma_2"] & H^{n+2i}(-;U(1)) \arrow[r, "j"] & \hat{H}^{n+2i+1}(-;\mathbb{Z}),
\end{tikzcd}$$

where Γ_2 is induced by the representation of $\mathbb{Z}/2\mathbb{Z}$ in $U(1)$ as roots of unity and j is the inclusion of flat characters from Section 2.3.

The differential Steenrod squares satisfy some of the classical relations. They do satisfy Adem relations as before, but due to the non-existence of even differential Steenrod squares, we do not have an identity square (which is normally Sq^0). This is just one of the problems obstructing the existence of a differential Steenrod algebra. We refer to [GS18b] for details, applications, and a thorough discussion.

Although the Steenrod squaring $H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{2n}(-;\mathbb{Z}/2\mathbb{Z})$ cannot always be differentially refined, other types of interesting product constructions in cohomology have differential analogs, such as Massey products. For a discussion of differential Massey products, see [GS18a].

A Künneth theorem for Ordinary Differential Cohomology

In [GS18b], a differential Künneth theorem was developed.

Proposition 3.8.1. *(Differential Künneth theorem).*

Let M and N be compact, smooth manifolds. Then there exists a short exact sequence,

$$\begin{tikzcd}
0 \arrow[r] & \hat{H}^n(M;\mathbb{Z}) \oplus \bigoplus_{i=1}^n H^i(N;\mathbb{Z}) \otimes H^{n-i-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r] & \hat{H}^n(M \times N;\mathbb{Z}) \arrow[dl] \\
& \mathrm{Tor}(\hat{H}^n(M;\mathbb{Z}), H^1(N;\mathbb{Z})) \oplus \bigoplus_{i=1}^n \mathrm{Tor}(H^{n-i-1}(M;\mathbb{R}/\mathbb{Z}), H^{i+1}(N;\mathbb{Z})) \arrow[r] & 0,
\end{tikzcd}$$

that splits (although not naturally).

The proof and its theoretical applications can be found in [GS18b]. It would be of interest to compute differential cohomology groups of certain simple product spaces.

One could also ask whether there exists a universal coefficient-type theorem for differential cohomology. It is not clear to us how we would even define differential cohomology with (general) coefficients, as the axioms fix $\mathcal{V}_* = E^*(*) \otimes_{\mathbb{Z}} \mathbb{R}$. One could perhaps try to change the axioms for ordinary differential cohomology (Section 3.1 or Section 2.4) to include coefficients, but as integer coefficients are integral to the main philosophy and motivation of ordinary differential cohomology, it is not clear what could be obtained from such a change. We are not aware of any papers that work with other coefficients.

Differentially Refined Atiyah-Hirzebruch Spectral Sequence

The Atiyah-Hirzebruch spectral sequence we briefly mentioned in Section 3.7 is a powerful tool for computing generalized cohomology groups ([McC00]). A differential analog was developed in [GS17], where it is stated as the following proposition.

Theorem 3.8.2. *(Atiyah-Hirzebruch Spectral Sequence for Differential E-cohomology).*

Let $\hat{E}^*$ be a differential cohomology theory and let M be a compact, smooth manifold. Then there is a convergent spectral sequence with

$$E_2^{p,q} = H^p(M; \hat{E}^q) \implies \hat{E}^{p+q}(M),$$

where H^p denotes the p'th sheaf cohomology of the sheaf $\hat{E}^q$.

Certain examples in [GS17] for differential K-theory and differential Morava K-theory use the differential Steenrod squared discussed earlier to obtain expressions of certain differentials in the spectral sequence.

Application: Differential K-theory and RR-fields in String Theory

It is not just ordinary differential cohomology that has applications in physics. Important quantities in string theory, such as so-called D-brane charges and RR-fields, can be classified by certain associated K-groups. This is discussed and developed in [MW00]. As shown in [FH00], the geometric nature of mathematical physics implies the proper homes of RR-fields are certain differential K-groups.

Chapter 4

Hodge-filtered Cohomology Theories

In this section, we make the move from differential cohomology to Hodge-filtered cohomology theories. As we will see, the conceptual change is not large, as most of the work was done in the earlier chapters. However, there are some subtle changes that impact the theory. We start by discussing the original Deligne cohomology, before we move on to the construction of Hopkins-Quick (see [HQ15]). The axiomatics of Hodge-filtered cohomology theories was formulated in [Hau22] and [HQ22], and we briefly connect differential cohomology theories, Hopkins-Quick theories and Deligne cohomology to this framework, closely following the explanations found in [HQ22]. Furthermore, we delve into questions that arise naturally from the similarities to differential cohomology.

Compared to the previous chapters, each section will be shorter, as much of the intuition we built in Chapters 2 and 3 can still be used.

4.1 Motivation: Returning to Deligne Cohomology

Originally, Deligne cohomology was used to study questions related to algebraic, arithmetic, and complex geometry, as mentioned in [HQ15], and it is related to number theoretic concepts, such as the Beilinson conjectures [RSS88]. For our purposes, we briefly follow [Bry93] on Deligne cohomology for complex manifolds. An in-depth treatment of Hodge theory and complex algebraic geometry can be found in [Voi02].

From now on, when we write Ω^*, we refer to the holomorphic de Rham complex. The complex de Rham complex is denoted by $\mathcal{A}^*$.

Instead of using the smooth de Rham complex as a foundation for the Deligne complex on a complex manifold M, we use the holomorphic de Rham complex, denoted $\Omega^*(M)$, where $\Omega^n(M) \subseteq \mathcal{A}^{n,0}(M)$ is given by the $(n,0)$-forms that are in the kernel of $\bar{\partial} : \mathcal{A}^{n,0}(M) \to \mathcal{A}^{n,1}(M)$. Note that by Proposition 2.6.11. of [Huy05], this is equivalent to the $(n,0)$-forms that locally have holomorphic coordinate functions.

Definition 4.1.1. (Deligne complex and Deligne cohomology).

Let M be a complex manifold and let $p \geq 0$. Define the Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$ as the complex

$$\mathbb{Z}(p) \to \Omega^0 \to \Omega^1 \to \ldots \to \Omega^{p-1}$$

of sheaves.

The Deligne cohomology groups $H^n_{\mathcal{D}}(M;\mathbb{Z}(p))$ are defined as the hypercohomology groups of the Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$.

Remark 4.1.2. We again ask readers to be aware that we use the same notation for both Deligne cohomology and smooth Deligne cohomology, although the complexes are different, working with truncated holomorphic and smooth de Rham complexes, respectively.

Example 4.1.3. Similarly to Example 3.4.21, the Deligne complex $\mathbb{Z}(1)_{\mathcal{D}}$ is quasi-isomorphic to $\mathcal{O}^*[-1]$, where $\mathcal{O}^*$ denotes the sheaf of non-vanishing holomorphic functions. Hence

$$H^n_{\mathcal{D}}(M;\mathbb{Z}(1)) = H^{n-1}(M;\mathcal{O}^*).$$

A priori, the holomorphic de Rham sheaves Ω^* can be slightly complicated due to the restrictivity of being holomorphic. However, from the Dolbeault theorem (Corollary 2.6.20. in [Huy05]), the complexes of sheaves Ω^p and $\mathcal{A}^{p,*}$ are quasi-isomorphic, and $\mathcal{A}^{p,*}$ is an acyclic resolution of Ω^p (see [Voi02]).

By the classical decompositions of complex structures, we may decompose the spaces of forms on M as

$$\Omega^n(M) = \bigoplus_{p+q=n} \Omega^{p,q}(M) \quad \text{and} \quad \mathcal{A}^n(M) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}(M).$$

If we want to extend the filtration $\Omega^{*\geq p}$ on holomorphic forms to $\mathcal{A}^*$, we may consider a total complex-type filtration, called the Hodge filtration.

Definition 4.1.4. (The Hodge filtration).

For a complex manifold M, we define the Hodge-filtration F^p on $\mathcal{A}^n(M)$ to be the filtration

$$F^p\mathcal{A}^n(M) = \bigoplus_{i\geq p} \mathcal{A}^{i,n-i}(M).$$

Remark 4.1.5. This is constructed so that $\Omega^{*\geq p} \subseteq F^p\mathcal{A}^*$.

Remark 4.1.6. It turns out that both the sheaves $F^p\mathcal{A}^n$ and $\frac{\mathcal{A}^n}{F^p}$ are acyclic, yielding simple computations of hypercohomology.

There are some interesting cohomology classes related to Deligne cohomology, when restricting to compact Kähler manifolds. These are the Hodge classes. We refer to both [Bry93] and [Voi02] for details on these topics.

Definition 4.1.7. (Hodge cohomology classes).

For a compact Kähler manifold M, the Hodge group, $\mathrm{Hdg}^p(M)$, is the subgroup of $H^{2p}_{\mathcal{D}}(M;\mathbb{Z}(p))$ of classes with image in $F^pH^{2p}(M;\mathbb{C}) := \bigoplus_{i\geq p} H^{i,2p-i}(M;\mathbb{C}) \subseteq H^{2p}(M;\mathbb{C})$ under map induced by $\mathbb{Z}(p) \hookrightarrow \mathbb{C}$.

Deligne showed the following interesting connection between these classes and Deligne cohomology.

Theorem 4.1.8. *(The fundamental short exact sequence).*

There is a short exact sequence

$$0 \to J^p \to H^{2p}_{\mathcal{D}}(M;\mathbb{Z}(p)) \to \mathrm{Hdg}^p(M) \to 0,$$

where J^p is Griffiths intermediate Jacobian. That is, J^p is defined as

$$J^p = H^{2p-1}(M;\mathbb{C})/(F^pH^{2p-1}(M;\mathbb{C}) + H^{2p-1}(M;\mathbb{Z}(p)).$$

To see the relevance of Hodge classes, we do not need to venture further than $p = 1$. The Hodge classes classify the first Chern classes over compact Kähler manifolds in the sense that for a compact Kähler manifold M, the image of the first Chern class $c_1 : \operatorname{Pic} M \to H^2(M;\mathbb{Z}(1))$ corresponds to $\mathrm{Hdg}^2(M)$. Here, $\operatorname{Pic} M$ is the (Picard) group of holomorphic line bundles over M up to isomorphism. This is Theorem 11.30 in [Voi02].

Much motivation for studying Deligne cohomology, Hodge-filtered cohomology theories, and Hodge classes has been omitted here. For our purposes, we simply consider Hodge-filtered cohomology to be an interesting analog to differential cohomology for complex manifolds, but we refer to [HQ15] for more of the original motivation.

Remark 4.1.9. Note the subtle difference compared to differential cohomology. This theorem shows that there is interesting info to be obtained for non-matching indices, such as $H^{2p}_{\mathcal{D}}(M;\mathbb{Z}(p))$, whereas for differential cohomology, Proposition 2.7.8 and Theorem 2.7.6 hint that the most interesting cases are for matching indices, i.e. $E(p)^p$.

Since we know from Section 3.4 that smooth Deligne cohomology is ordinary differential cohomology, it is natural to ask what Deligne cohomology models. More concretely, given a cohomology theory E, can we make a complex geometric version of differential cohomology such that Deligne cohomology will be an ordinary theory? Using the Hodge-filtration instead of the $\Omega^{*\geq p}$-filtration of Hopkins-Singer in Section 2.7, Hopkins-Quick [HQ15] defined Hodge-filtered cohomology theories, a complex geometric analog of differential cohomology.

Remark 4.1.10. Just as in the smooth case (Remark 3.4.23), the Deligne complex is quasi-isomorphic to a homotopy pullback of the diagram

$$\begin{tikzcd} & \Omega^{*\geq p} \arrow[d] \\ \mathbb{Z}(p) \arrow[r] & \Omega^{*}. \end{tikzcd}$$

4.2 Hodge-filtered Cohomology Theories of Hopkins–Quick

Hopkins–Quick defined Hodge-filtered cohomology theories motivated by the Hopkins-Singer approach (2.7) to differential cohomology and from the fact that the Deligne cohomology arises from an appropriate homotopy pullback.

Let E be a rationally even spectrum and let $\mathcal{V}_* = \pi_* E \otimes \mathbb{C}$ (as compared to $\pi_* E \otimes \mathbb{R}$ in the smooth case). By considering the complexification instead of the realification when defining the fundamental cocycle of Section 2.8, we obtain a map $E \to H\mathcal{V}_*$, where H is the Eilenberg-MacLane functor.

Remark 4.2.1. In the following section, we use spectra and presheaves. Compared to the ∞-categorical context of Section 2.11, we now fix a 1-categorical[1] model of spectra (e.g. symmetric spectra) and consider sheaves on complex manifolds with values in spectra.

In the spirit of Deligne cohomology, we fix an integer p and make sense of a diagram of the form

$$\begin{tikzcd} & E \arrow[d] \\ F^p\mathcal{A}^* \arrow[r] & \mathcal{A}^*, \end{tikzcd}$$

using the Hodge-filtration. One problem is that this nonsensical diagram mixes a spectrum E with sheaves $\mathcal{A}^*$. The solution is to consider everything to live in the category $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$ of symmetric spectrum objects in pointed simplicial presheaves on complex manifolds. We will not delve into the yoga of symmetric sequences in the category of pointed simplicial presheaves, as it is quite technical and thoroughly explained in [HQ15].

A symmetric spectrum E can be viewed to live in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$ by considering the trivial simplicial presheaf coming from the representable presheaf of E.

By defining $\mathcal{A}^n(\mathcal{V}_*) = \mathcal{A}^n(-;\mathcal{V}_*)$ by the convention

$$\mathcal{A}^n(-;\mathcal{V}_*) = \bigoplus_j \mathcal{A}^{n+j}(-) \otimes_{\mathbb{C}} \mathcal{V}_j,$$

[1] The extension to stable ∞-categories could be of interest, as it could be interesting to investigate how it yields similar results to Section 2.11.

we can construct a map $\mathcal{V}_* \to \mathcal{A}^0(\mathcal{V}_*) \to \mathcal{A}^*(\mathcal{V}_*)$ of presheaves by considering $\mathcal{V}_*$ as a constant presheaf. Applying the Eilenberg-MacLane functor H will turn these into objects of $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ (see [HQ15]). Picking out the fundamental cocycle will therefore yield a map

$$\psi_p^E : E \to H\mathcal{V}_* \to H\mathcal{A}^0(\mathcal{V}_*) \to H\mathcal{A}^*(\mathcal{V}_*)$$

in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$. The inclusion $F^p\mathcal{A}^*(\mathcal{V}_*) \to \mathcal{A}^*(\mathcal{V}_*)$ induces a morphism $HF^p\mathcal{A}^*(\mathcal{V}_*) \to H\mathcal{A}^*(\mathcal{V}_*)$ in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ as well. These maps yield enough to define the Hodge-filtered symmetric spectrum object $E_\mathcal{D}(p)$ from E, and its corresponding Hodge-filtered E-cohomology.

Definition 4.2.2. (Hodge-filtered E-spectrum objects and cohomology).

For a fixed integer p, let $E_\mathcal{D}(p)$ be given by the homotopy pullback of

$$\begin{tikzcd}
E_\mathcal{D}(p) \arrow[r] \arrow[d] & E \arrow[d] \\
HF^p\mathcal{A}^*(\mathcal{V}_*) \arrow[r] & H\mathcal{A}^*(\mathcal{V}_*),
\end{tikzcd}$$

with the maps outlined above.

For a complex manifold M, set

$$E^n_\mathcal{D}(p)(M) = \mathrm{Hom}_{\mathrm{ho}\,\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(\Sigma^\infty_+\mathcal{F}_M, \Sigma^n E_\mathcal{D}(p)),$$

where $\mathcal{F}_M = \mathrm{Hom}_{\mathbf{Man}}(-, M)$ is the representable presheaf of M.

Remark 4.2.3. Behind the scenes, the definition involves choices of maps $E \to H(\mathcal{V}_*)$, but as remarked in [HQ15, Remark 4.10.], the assumptions on E being rationally even implies that the space of such choices is simply connected.

Remark 4.2.4. It is possible to show that there exists a functorial construction of the Eilenberg-MacLane functor H and that $H\mathcal{A}^*(\mathcal{V}_*)$ represents the hypercohomology $H^*(-;\mathcal{A}^*(\mathcal{V}_*))$. Similar statements hold for $HF^p\mathcal{A}^*(\mathcal{V}_*)$ and $H(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*))$. We refer to [HQ15] for the technicalities on regarding the Eilenberg-MacLane functor H.

Remark 4.2.5. Although quite abstract, the Eilenberg-MacLane spaces of (filtered) forms can be made concrete, as done in [HQ22]. For a fixed n, the simplicial presheaf $K(\mathcal{A}^*(\mathcal{V}_*), n)$ is weakly equivalent to the simplicial presheaf $\mathcal{A}^n(-\times\Delta^*; \mathcal{V}_*)$. For filtered forms, $K(F^p\mathcal{A}^*(\mathcal{V}_*), n)$ is weakly equivalent to $F^p\mathcal{A}^n(-\times\Delta^*; \mathcal{V}_*)_{cl}$. To compute the homotopy pullbacks, explicit (fibrant) Ω-spectrum constructions are found in [HQ22, Section 2.13.].

Note however that this requires a way to multiply complex manifolds with possibly non-complex manifolds. This is addressed in [HQ22] as a motivation for defining Hodge-filtered cohomology theories on a suitable category $\mathbf{Man}_F$ of manifolds with (manifold-wise) filtrations on $\mathcal{A}^*$. We will discuss this briefly.

4.3 The Axioms of Hodge-filtered Cohomology of Haus-Quick

Based on the axioms for differential cohomology of Section 3.1 and the Hopkins-Quick theories of [HQ15], Haus-Quick developed axioms for Hodge-filtered cohomology theories for the PhD-thesis of Haus, see [Hau22] and [HQ22]. As discussed thoroughly in [Hau22], Hodge-filtered cohomology theories are not yet shown to be unique, due to the rigidity of complex geometry. For example, analogs of Proposition 2.2.1 for holomorphic functions are still open problems. This section is dedicated to the axioms of Hodge-filtered cohomology, as our work on Hodge-filtered K-theory in Chapter 5 amounts to finding an explicit construction satisfying these axioms.

Definition 4.3.1. (Axioms of Hodge-filtered Cohomology).

Let $\mathcal{V}_*$ be an evenly graded vector space (e.g. $\mathcal{V}_* = \pi_* E \otimes \mathbb{C}$) and let E^* be a rationally even cohomology theory. For $p \in \mathbb{Z}$, let

$$c(p) : E^* \to H^*(-; \mathcal{A}^*(\mathcal{V}_*))$$

be a map of cohomology theories (e.g. induced by the Chern-Dold character).

A Hodge-filtered cohomology theory over $(E^*, c(p))$ is a contravariant functor,

$$E^*_{\mathcal{D}}(p) : \mathbf{Man}_{\mathbb{C}} \to \mathbf{grAb},$$

with three natural transformations (natural in M),

- $R : E^*_{\mathcal{D}}(p)(M) \to H^*(M; F^p\mathcal{A}^*(\mathcal{V}_*))$,
- $I : E^*_{\mathcal{D}}(p)(M) \to E^*(M)$,
- $a : H^{*-1}(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \to E^*_{\mathcal{D}}(p)(M)$.

These should be related by three requirements. Firstly, we should have $R \circ a = d$, where d is the connecting homomorphism in the long exact sequence induced by the short exact sequence

$$F^p\mathcal{A}^*(\mathcal{V}_*) \to \mathcal{A}^*(\mathcal{V}_*) \to \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)$$

of sheaves. Secondly, the diagram

$$\begin{tikzcd}
E^*_{\mathcal{D}}(p)(M) \arrow[r, "I"] \arrow[d, "R"'] & E^*(M) \arrow[d, "c(p)"] \\
H^*(M; F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] & H^*(M; \mathcal{A}^*(\mathcal{V}_*))
\end{tikzcd}$$

should commute. Lastly, we need a (horizontal) long exact sequence

$$
\begin{tikzcd}
 & & H^{n-1}(M; \mathcal{A}^*(\mathcal{V}_*)) \arrow[d, dashed] & \\
\cdots \arrow[r] & E^{n-1}(M) \arrow[ur, dashed, "c(p)"] \arrow[r, "c(\bar{p})"] & H^{n-1}(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \arrow[r, "a"] & E^n_{\mathcal{D}}(p)(M) \arrow[dll, "I"] \\
 & E^n(M) \arrow[r] & H^n(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \arrow[r] & \cdots .
\end{tikzcd}
$$

The map $c(\bar{p})$ is the composition shown in the (dashed) commuting triangle.

Remark 4.3.2. (Differences between differential cohomology and Hodge-filtered cohomology).

There are some differences we must address, compared to the axioms of differential cohomology. Most notably, the integer p in Definition 4.3.1 is not present in the axiomatic definition of differential cohomology. This is a fundamental difference, which may be a symptom of the goal that Hodge-filtered theories should encapsulate interesting information in non-matching degrees, such as in Theorem 4.1.8 and Remark 4.1.9. In addition, the codomain of R and the domain of a in Definition 4.3.1 are cohomology groups, compared to certain groups of differential forms in Definition 3.1.1. The long exact sequence is more explicit in the above definition than for differential cohomology, which can be useful. However, sometimes a "shorter" exact sequence behaves better, e.g. when working with tensor products and Landweber theories (Sections 3.7 and 4.6). The assumption of E being rationally even is a prerequisite for defining Hodge-filtered cohomology theories, as compared to an additional assumption contributing towards uniqueness of differential cohomology.

Definition 4.3.3. (Multiplicative Hodge-filtered cohomology).

Let E be a multiplicative cohomology theory. We say that

$$E^*_{\mathcal{D}}(*) = \bigoplus_{p,n} E^n_{\mathcal{D}}(p)$$

is a multiplicative Hodge-filtered theory if for each p, $E^*_{\mathcal{D}}(p)$ is a Hodge-filtered cohomology theory, and we have with an exterior multiplication

$$\mu : E^n_{\mathcal{D}}(p)(M) \otimes E^{n'}_{\mathcal{D}}(p')(M') \to E^{n+n'}_{\mathcal{D}}(p+p')(M \times M').$$

This exterior multiplication induces an interior product $\cdot$ through the diagonal map. As before, we require that the maps I and R are multiplicative with respect to $\cdot$, and that a satisfies

$$a(\omega) \cdot x = a(\omega \wedge R(x))$$

for each $\omega \in H^n(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*))$ and $x \in E^*_{\mathcal{D}}(*)(M)$.

Remark 4.3.4. Definition 4.3.1 does not depend much on working in $\mathbf{Man}_{\mathbb{C}}$. Following [HQ22], we can rephrase Hodge-filtered theories for the category $\mathbf{Man}_F$ instead of $\mathbf{Man}_{\mathbb{C}}$, where $\mathbf{Man}_F$

denotes the category of pairs (M, F^*), where M is a smooth manifold and F^* is a descending filtration of $\mathcal{A}^*(M)$. Note that we have canonical inclusions $\mathbf{Man} \to \mathbf{Man}_F$ and $\mathbf{Man}_{\mathbb{C}} \to \mathbf{Man}_F$ given by including smooth manifolds with the trivial filtration and complex manifolds with the Hodge-filtration, respectively.

If we let $M, M' \in \mathbf{Man}_F$, we may consider the smooth product manifold $M \times M'$. To obtain an object $M \times M'$ in $\mathbf{Man}_F$, consider the filtration

$$F^p\mathcal{A}^*(M \times M') = \bigoplus_{p_1+p_2=p} \overline{F^{p_1}\mathcal{A}^*(M) \otimes F^{p_2}\mathcal{A}^*(M')}.$$

The closure is taken in $\mathcal{A}^*(M \times M')$, which is viewed as a topological space with the smooth compact-open topology. Explicitly, a form in $F^p\mathcal{A}^*(M \times M')$ is a finite sum of forms ω, locally written $\omega = f \cdot \omega_M \otimes \omega_{M'}$ for a smooth function f on $M \times M'$ and forms $\omega_M, \omega_{M'}$ on M and M', respectively. For $p_1 + p_2 = p$, ω is in $F^{p_1+p_2}\mathcal{A}^*(M \times M')$ when $\omega_M \in F^{p_1}\mathcal{A}^*(M)$ and $\omega_{M'} \in F^{p_2}\mathcal{A}^*(M')$.

As remarked in [HQ22], this is not the categorical product in $\mathbf{Man}_F$, but in the full subcategory where $\Delta : M \to M \times M$ is a morphism.

Remark 4.3.5. As the sheaves $F^p\mathcal{A}^*$ and $\frac{\mathcal{A}^*}{F^p}$ are acyclic, their hypercohomology groups become quite simple. In [Hau22], these are computed, to obtain

$$H^n(M; F^p\mathcal{A}^*(\mathcal{V}_*)) \cong \frac{F^p\mathcal{A}^n(M;\mathcal{V}_*)_{cl}}{dF^p\mathcal{A}^{n-1}(M;\mathcal{V}_*)} \quad \text{and}$$
$$H^n(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \cong \frac{d^{-1}(F^p\mathcal{A}^{n+1}(M;\mathcal{V}_*)^n}{(F^p\mathcal{A}^n(M;\mathcal{V}_*) + d\mathcal{A}^{n-1}(M;\mathcal{V}_*)}.$$

In the axioms of a Hodge-filtered cohomology theory, the statement $R \circ a = d$ means that the composition $R \circ a$ should be equal to the connecting homomorphism

$$d : H^{n-1}(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \to H^n(M; F^p\mathcal{A}^*(\mathcal{V}_*)).$$

By the explicit forms of these groups, it can be shown that the connecting homomorphism is induced by the exterior derivative $d : d^{-1}(F^p\mathcal{A}^n(M;\mathcal{V}_*))^{n-1} \to F^p\mathcal{A}^n(M;\mathcal{V}_*)_{cl}$.

4.4 Returning to Differential Cohomology and Hopkins-Quick

Since the axioms often extract the essence from the examples, we should ask ourselves how they hold up to the previously studied approaches to Hodge-filtered cohomology. We ask;

1. are the Hopkins-Quick theories Hodge-filtered?
2. is Deligne cohomology an ordinary Hodge-filtered cohomology?
3. can differential cohomology theories construct Hodge-filtered cohomology theories?

This section aims to illustrate a positive answer to these questions. We refer to [Hau22, Chapter 3.] for the proofs. We will mainly restrict ourselves to understanding what the maps R, I, and a look like, restricting the number of pages in this section dramatically.

For Deligne Cohomology

The first important thing to note is that all the maps R, I, and a can be induced from similar maps at the level of sheaves.

To obtain a map $\mathbb{Z}_{\mathcal{D}}(p) \to \Omega^{*\geq p}$, construct R to be the map of complexes of sheaves given by

$$\begin{tikzcd} \mathbb{Z}_{\mathcal{D}}(p) \arrow[d] & \mathbb{Z}(p) \arrow[r] \arrow[d] & \Omega^0 \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \Omega^{p-1} \arrow[r] \arrow[d, "d"] & 0 \arrow[r] \arrow[d] & \cdots \\ \Omega^{*\geq p} & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \Omega^p \arrow[r, "d"] & \Omega^{p+1} \arrow[r] & \cdots . \end{tikzcd}$$

The map a can be defined as the inclusion $\frac{\Omega^*}{\Omega^{*\geq p}}[-1] \to \mathbb{Z}_{\mathcal{D}}(p)$, and by setting $I : \mathbb{Z}_{\mathcal{D}}(p) \to \mathbb{Z}$ be $(-1)^p\pi$, where we let π be the natural projection (of complexes of sheaves), we have specified all maps needed to show Deligne cohomology is an ordinary Hodge-filtered cohomology. The remaining details are found in [Hau22].

For Hopkins-Quick theories

Similarly to Deligne cohomology, the maps for Hopkins-Quick theories can be induced from similar maps in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$. From the definition of $E_{\mathcal{D}}(p)$ as a homotopy pullback, we have canonical maps $I : E_{\mathcal{D}}(p) \to E$ and $R : E_{\mathcal{D}}(p) \to HF^p\mathcal{A}^*(\mathcal{V}_*)$, which induce the corresponding maps I and R in cohomology.

The process of finding a is a bit more involved. Consider the composed diagram

$$\begin{tikzcd} E_{\mathcal{D}}(p) \arrow[r] \arrow[d] & E \arrow[d] \\ HF^p\mathcal{A}^*(\mathcal{V}_*) \arrow[r] \arrow[d] & H\mathcal{A}^*(\mathcal{V}_*) \arrow[d] \\ 0 \arrow[r] & H(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)). \end{tikzcd}$$

The bottom square is a homotopy pushout square. We have stacked a homotopy pullback square in a stable model category (and hence a homotopy pushout square) on top of a homotopy pushout square that we pushed through H. Since the Eilenberg-MacLane functor H is a Quillen equivalence of stable model categories (see [HQ15] and [Hau22]), the pushout is preserved. By composing the top horizontal map with the map $E \to 0$, we may form the homotopy pushout defining $\Sigma E_D(p)$, yielding

$$\begin{tikzcd}
E_{\mathcal{D}}(p) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] \arrow[ddr, dashed, bend left] & 0 \arrow[dd] \\
0 \arrow[r] \arrow[drr, bend right] & H(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \arrow[dr, dashed, "a"] & \\
 & & \Sigma E_D(p).
\end{tikzcd}$$

The map $a : H(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \to \Sigma E_D(p)$ is exactly the one inducing the wanted map.

Remark 4.4.1. Note how this motivates the role of a as the connecting homomorphism in the long exact sequence of Definition 4.3.1.

Complexified Differential Cohomology from Differential Cohomology

When constructing Hodge-filtered cohomology theories from differential cohomology, a natural first step is to try to complexify the differential cohomology theory.

Recall the axioms of a differential cohomology theory in the sense of Section 3.1 and [BS10]. In this definition, Ω^* refers to the smooth de Rham complex, and not the holomorphic de Rham complex.

Definition 4.4.2. (A smooth extension, again).

A smooth extension of E is a contravariant functor $\hat{E} : \mathbf{Man} \to \mathbf{grAb}$ with three transformations natural in the manifold M,

- $R : \hat{E}^*(M) \to \Omega^*_{cl}(M; \mathcal{V}_*)$,
- $I : \hat{E}^*(M) \to E^*(M)$,
- $a : \Omega^{*-1}(M; \mathcal{V}_*)/\operatorname{im}(d) \to \hat{E}^*(M)$.

These are related by three requirements. Firstly, we have $R \circ a = d$. Secondly, the diagram

$$\begin{tikzcd}
\hat{E}^*(M) \arrow[r, "R"] \arrow[d, "I"] & \Omega^*_{cl}(M; \mathcal{V}) \arrow[d] \\
E^*(M) \arrow[r, "\mathrm{chd}"] & H^*(M; \mathcal{V}_*),
\end{tikzcd}$$

where chd is the Chern-Dold character, should commute. Thirdly, we have an exact sequence

$$E^{*-1}(M) \longrightarrow \Omega^{*-1}(M;\mathcal{V}_*)/\operatorname{im}(d) \xrightarrow{a} \hat{E}^*(M) \xrightarrow{I} E^*(M) \longrightarrow 0.$$

Following [Hau22], let $\mathcal{V}_{*,\mathbb{C}} := \mathcal{V}_* \otimes_{\mathbb{R}} \mathbb{C}$ and define the complexified differential cohomology $\hat{E}^n_{\mathbb{C}}$ as

$$\hat{E}^n_{\mathbb{C}}(M) = \{(\hat{x},\phi) \in \hat{E}^n(M) \times \Omega^{n-1}(M;\mathcal{V}_{*,\mathbb{C}}) \quad | \quad (\hat{x}+a(\omega),\phi) = (\hat{x},\phi+\omega), \quad \forall \omega \in \Omega^{n-1}(M;\mathcal{V}_*)\}.$$

Note that we identify $\Omega^{n-1}(M;\mathcal{V}_*)$ with a subspace of $\Omega^{n-1}(M;\mathcal{V}_{*,\mathbb{C}})$ through the natural map of coefficients.

With this definition, we obtain canonical maps and a simple phrasing of complexified axioms. We set

$$R_{\mathbb{C}}(\hat{x},\phi) = R(\hat{x}) + d\phi, \quad I_{\mathbb{C}}(\hat{x},\phi) = I(\hat{x}), \quad a_{\mathbb{C}}(\omega) = (0,\omega).$$

These maps satisfy the same axioms as before, but with appropriate subscripts. An important remark made by [Hau22] is that the uniqueness theorem applies without change to complex differential cohomology theories as we have merely changed the coefficients in the axioms, and not promoted any smooth structure to a complex or holomorphic structure.

Hodge-filtered Cohomology from Complex Differential Cohomology

Assume now that we are given an integer p and a (rationally even) complex differential cohomology theory. Let M be a complex manifold. Since M has the underlying structure of a smooth manifold, we consider $\hat{E}^*(M)$ and try to force in a (p) by constructing a Hodge-type filtration on $\hat{E}^*$. We use the filtration on $\mathcal{A}(M;\mathcal{V}_*)$ and the curvature map R to define $F^p\hat{E}^*(M) \subseteq \hat{E}^*(M)$ as

$$F^p\hat{E}^*(M) = R^{-1}(F^p\mathcal{A}^*(M;\mathcal{V}_*)).$$

To ensure that the maps R, I and a stay well-defined, we set

$$\hat{E}^*_{\mathcal{D}}(p)(M) := F^p\hat{E}^*(M)/a(F^p\mathcal{A}^{*-1}(M;\mathcal{V}_*) + d\mathcal{A}^{*-2}(M;\mathcal{V}_*)).$$

With the above definition, the maps R, I and a induce the maps needed for $\hat{E}^*_{\mathcal{D}}(p)$ to be a Hodge-filtered cohomology theory. It is non-trivial, as the axioms of Definition 4.3.1 do not follow automatically, and the construction relies on explicit computations of the hyper-cohomology groups $H^n(M;F^p\mathcal{A}^*(\mathcal{V}_*))$ and $H^n(M;\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*))$. The proof and remaining details can be found in [Hau22].

4.5 On Hodge-filtered homotopy invariance (in $\mathbf{Man}_F$)

Since we had the homotopy formula from Theorem 3.3.1, we expect something similar for Hodge-filtered cohomology theories. However, to be able to consider products of complex and smooth manifolds more easily, we will state the result in $\mathbf{Man}_F$ (see Remark 4.3.4). As we are working with smooth manifolds, we can consider fiber integration in $\mathbf{Man}_F$ as before. With this setup, the homotopy formula still holds.

The proof is similar, but with some slight changes, to the one of Theorem 3.3.1, and is stated as Lemma 5.9 of [HQ22].

Theorem 4.5.1. *(The homotopy formula for Hodge-filtered cohomology theories).*

Let $M \in \mathbf{Man}_F$ and $x \in E_{\mathcal{D}}^{*+1}(p)(M \times \mathbb{R})$. Denote by $i_0, i_1 : M \to M \times \mathbb{R}$ the inclusions at real coordinates 0 and 1, respectively, and let $p : M \times \mathbb{R} \to M$ denote the projection. Then we have

$$i_1^* x - i_0^* x = a\left(\int_{M \times \mathbb{R}/M} R(x)\right),$$

where the integral denotes fiber integration of forms.

4.6 A dream of Hodge-filtered Landweber theories

Warning 4.6.1. Much of the motivation and main ideas of this section requires some knowledge of chromatic homotopy theory. We refer to [Lur10].

This section is dedicated to discussing a Hodge-filtered version of Theorem 3.7.3. There is no evident reason why the transition from differential cohomology to Hodge-filtered cohomology theories should obstruct the existence of Hodge-filtered Landweber theories. We can construct a differential extension of Landweber theories, and these can be used to construct a Hodge-filtered cohomology theory by Section 4.4.

To be more formal, we state the following conjecture, that we have not yet been able to prove.

Conjecture 4.6.2. *(Hodge-filtered Landweber theories).*

Let (S, g) be a Landweber exact formal group law. For a fixed p, let $S_{\mathcal{D}}^*(p)$ be the functor $S_{\mathcal{D}}^*(p)(-) = MU_{\mathcal{D}}^*(p)(-) \otimes_{MU^*} S$. This is a multiplicative Hodge-filtered cohomology theory over $(S, c_S(p))$.

We'll try to explain some thoughts on our most serious attempt at a proof. To show such a theorem, we start with the fact that $MU_{\mathcal{D}}^*(p)(-)$ exists by the Hopkins-Quick construction, and there exists a geometric model from [Hau22] and [HQ22] that is isomorphic to the Hopkins-Quick construction of $MU_{\mathcal{D}}^*(p)(-)$. To obtain a long exact sequence for $S_{\mathcal{D}}^*(p)$, we start with the long exact sequence of $MU^*_{\mathcal{D}}(p)$. However, compared to the proof of Theorem 3.7.3 as found in [Bun+09], it is not clear how long exact sequences act when tensored with a ring. An idea could be to split it up into short exact sequences and tensor these with S, but is is not clear how to staple them together to a long exact sequence again, nor if (or how) the vanishing condition of Lemma 3.7.2 can be used. Questions related to derived functors and long exact sequences may be highly non-trivial, and even invoke problems from higher homological algebra. However, we believe there is a way to bypass working with the long exact sequence. The exactness of

$$\begin{tikzcd}
 & & H^{n-1}(M;\mathcal{A}^*(\mathcal{V}_*)) \arrow[d, dashed] & \\
\cdots \arrow[r] & E^{n-1}(M) \arrow[r, "c(\bar{p})"] \arrow[ur, dashed, "c(p)"] & H^{n-1}(M;\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \arrow[r, "a"] & E^n_{\mathcal{D}}(p)(M) \arrow[dll, "I"] \\
 & E^n(M) \arrow[r] & H^n(M;\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \arrow[r] & \cdots
\end{tikzcd}$$

should be equivalent to the combined exactness of

$$E^{n-1}(M) \xrightarrow{c(\bar{p})} H^{n-1}(M;\tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \xrightarrow{a} E^n_{\mathcal{D}}(p)(M) \xrightarrow{I'} F^pE^n(M) \longrightarrow 0$$

for all n, where

$$F^pE^n(M) = c(p)^{-1}(H^n(M;F^p\mathcal{A}^*(\mathcal{V}_*)))$$

denotes the pullback

$$\begin{tikzcd}
F^pE^n(M) \arrow[r] \arrow[d] & E^n(M) \arrow[d, "c(p)"] \\
H^n(M;F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r, "inc"] & H^n(M;\mathcal{A}^*(\mathcal{V}_*)).
\end{tikzcd}$$

This statement is heavily inspired by ideas and remarks from [Hau22].

To prove Conjecture 4.6.2, we would start with a Landweber exact formal group law (S, g), which means that $S^*(-) = MU^*(-) \otimes_{MU^*} S$ is a cohomology theory, and set $\mathcal{V}_* = \pi_*MU \otimes_{MU^*} \mathbb{C} \cong L \otimes_{MU^*} \mathbb{C}$, where L denotes the Lazard ring (Section 3.7). As S is by assumption an L-module, we only need to check if $MU^*_{\mathcal{D}}(p)(-)$ is an L-module to form $S^*_{\mathcal{D}}(p)(-) = MU^*_{\mathcal{D}}(p)(-) \otimes_L S$. This can probably be done similarly to the proof of Theorem 3.7.3, which is found in [Bun+09]. Since $MU^n(*) = 0$ for odd n, a way of showing this could be by showing $MU^{even}_{\mathcal{D}}(p)(*) \cong MU^*(*) \cong L$, where $MU^{even}_{\mathcal{D}}(p)(*)$ denotes graded abelian group $MU^*_{\mathcal{D}}(p)(*)$, but with $MU^n_{\mathcal{D}}(p)(*)$ set to 0 for n odd. This is sufficient, as for any complex manifold M, $MU^*_{\mathcal{D}}(p)(M)$ is a module over $MU^*_{\mathcal{D}}(p)(*)$. By restricting the action to $MU^{even}_{\mathcal{D}}(p)(*)$, we see that $MU^*_{\mathcal{D}}(p)(M)$ becomes a module over $MU_{\mathcal{D}}^{even}(p)(*) \cong MU^{even}(*) \cong MU^*(*) \cong L$. This can be proved for even $n \geq 2$ by using the long exact sequence for $MU_{\mathcal{D}}^*(p)(M)$ and letting M be a point. However, for $n = 0$, one would need to study the maps in the sequence a bit further to see how $MU_{\mathcal{D}}^0(p)(*)$ relates to $MU^0(*)$ and the cohomology groups involved.

As we assume $MU_{\mathcal{D}}^*(p)(-)$ is an Hodge-filtered cohomology theory over MU^*, there are maps $c(p), a, R$, and I. After checking that the domains and codomains these maps are L-modules, we can define $c(p)_S, a_S, I_S$ and R_S by appying $- \otimes_L S$. One would also need to show that $H^*(-;\mathcal{A}^*(\mathcal{V}_*)) \otimes_L S$ can be rewritten to the form $H^*(-;\mathcal{A}^*(\mathcal{V}_*^S))$ for some suitable $\mathcal{V}_*^S$, which is necessary to have the correct domains and codomains.

The most involved problem to check is if we obtain a long exact sequence. In light of the above discussion, we hope to rephrase the long exact sequence for $MU_{\mathcal{D}}^*(p)(-)$ to

$$MU^{n-1}(M) \xrightarrow{\bar{c(p)}} H^{n-1}(M; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \xrightarrow{a} MU_{\mathcal{D}}^n(M) \xrightarrow{I'} F^pMU^n(M) \longrightarrow 0$$

for each n. This can equivalently be written as the short exact sequence

$$0 \longrightarrow H^{n-1}(M; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*))/\bar{c(p)}(MU^{n-1}(M)) \xrightarrow{a} MU_{\mathcal{D}}^n(M) \xrightarrow{I'} F^pMU^n(M) \longrightarrow 0.$$

Next, we apply $- \otimes_L S$ and identify $MU_{\mathcal{D}}^*(p)(-) \otimes_L S$ with $S_{\mathcal{D}}^*(p)(-)$ and use that $- \otimes_L S$ respects quotients to obtain

$$\begin{tikzcd} \ldots \arrow[r] & \mathrm{Tor}_1^L(F^pMU^n(M), S) \arrow[r] & H^{n-1}(M; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \otimes_L S/\bar{c(p)}(MU^{n-1}(M)) \otimes_L S \arrow[dll] \\ S_{\mathcal{D}}^n(M) \arrow[r] & F^pMU^n(M) \otimes_L S \arrow[r] & 0. \end{tikzcd}$$

There are now several things to check. First, it is not clear if $F^pMU^n(M)$ satisfies Lemma 3.7.2 or if $\mathrm{Tor}_1^L(F^pMU^n(M), S) = 0$ by another argument. Secondly, we would need to identify $F^pMU^n(M) \otimes_L S$ with $F^pS^n(M)$, and $H^{n-1}(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \otimes_L S/\bar{c(p)}(MU^{n-1}(M)) \otimes_L S$ with $H^{n-1}(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*^S))/c_S\bar{(p)}(S^{n-1}(M))$, if this even makes sense to.

However, we have not had the time to check all of these details. Therefore, we do not (yet) know if Conjecture 4.6.2 holds, although we expect it to.

We have not checked if a similar result can be induced from differential cohomology. This may be simpler to do, although it does not prove Conjecture 4.6.2 due to the lack of a uniqueness theorem for Hodge-filtered cohomology theories.

4.7 Other topics:

Generalized fundamental short exact sequences

For Deligne cohomology, which is an ordinary Hodge-filtered cohomology theory, we have a fundamental short exact sequence with Hodge classes, as in Theorem 4.1.8. Given a cohomology theory E^*, and a Hodge-filtered Hopkins-Quick theory $E^*_{\mathcal{D}}(p)$ over E, it is possible to find a similar short exact sequence when working over compact Kähler manifolds, involving $E^*_{\mathcal{D}}(p)$, E-Hodge classes and an E-Jacobian. Readers with background from complex algebraic geometry may consider this as motivation to study Hodge-filtered cohomology theories. Although our motivation has been to study an extension of differential cohomology for complex manifolds (or even for objects of $\mathbf{Man}_F$), we find it mandatory to mention this, as it is much of the motivation for the work found in [HQ15]. It would be interesting to delve into which properties Hodge-filtered K-theory has, when viewed through the lens of complex algebraic geometry.

Currential Hodge-filtered Cohomology

An interesting observation from [Hau22] and [HQ22] is that the theory of currents can be used to define Hodge-filtered cohomology theories. Recall that the support of a differential form ω on M, $\mathrm{supp}(\omega)$, is defined the smallest closed subset where ω vanishes on the complement. That is, one can define $\mathrm{supp}(\omega) = \bigcap_{\text{open } U \text{ s.t. } \omega|_U = 0}(M \backslash U)$. Considering $\mathcal{A}^n(M)$ with the smooth compact-open topology, we can consider the space of compactly supported forms. That is, we write $\mathcal{A}^n_c(M) = \{\omega \in \mathcal{A}^n(M) \mid \mathrm{supp}(\omega) \text{ compact }\}$. The dual (in the sense of functional analysis) is defined to be the space of currents on M, $\mathcal{D}^n(M)$. It can be topologized by the weak*-topology. As explained in [Hau22], the complex $\mathcal{A}^*(M)$ is quasi-isomorphic to $\mathcal{D}^*(M)$, giving the same theory, although this is not the case for currential versions of differential cohomology.

Potential uses in physics

Although it has not yet been investigated, we believe that Hodge-filtered cohomology theories can be used in physics as well. Since certain types of complex manifolds, such as Calabi-Yau manifolds, are essential to theoretical physics, it would be interesting to see how Hodge-filtered cohomology theories can play the role as an intricate complex variant of differential cohomology. We are not aware of any papers that discuss this, but we hope there will be some in the future.

Chapter 5

Towards (Geometric) Hodge-filtered K-Theory

In this chapter, we specialize to Hodge-filtered K-theory. Initially, we hope that Hodge-filtered cohomology theories are unique. If this is not true in general, we hope at least that Hodge-filtered K-theory is uniquely characterized by the axioms (Definition 4.3). Thoughts on how to work towards partial results for this is discussed in [Hau22, Section 3.5.].

Although we do not have a uniqueness theorem for Hodge-filtered K-theory, we investigate several approaches to Hodge-filtered K-theory. Most notably, we find a geometric model for Hodge-filtered K-theory by establishing an interesting new relation between multiplicative K-theory in the sense of [Kar90] and Hodge-filtered K-theory in Theorem 5.2.3 and Corollary 5.2.6.1. Multiplicative K-theory in the sense of [Kar90] and secondary characteristic classes in the sense of [CS74] are therefore briefly discussed in Section 5.1.

Note that this chapter does not contain a section on ”other topics”, as it is simply a chapter of many open problems and few answers. We will however provide a section on open problems and questions we have not yet been able to answer.

5.1 Multiplicative K-theory

The most promising approach to a geometric model for Hodge-filtered K-theory seems to be through multiplicative K-theory in the sense of [Kar90]. The idea is not new to this thesis, as it is briefly mentioned in [HQ22], and folklore says it has been an unexplored path since the work of [HQ15]. However, we have not found any source that proves, or even discusses, the relation between multiplicative K-theory and Hodge-filtered K-theory. We dedicate the next two subsections to establishing the relation formally, starting with understanding multiplicative K-theory and secondary characteristic classes.

5.1.1 The Secondary Characteristic Classes of Chern-Simons

Recall that characteristic classes can be constructed through the Chern-Weil homomorphism. Although the Chern-Weil construction is independent of the choice of connection, the geometric information should be hidden somewhere. The main idea of [CS74] is to pull the Chern-Weil forms back to a bundle where it becomes exact, and study its trivializations. This does not only allow for secondary (underlying) characteristic classes, it also refines certain classical applications to include more structure.

The Chern-Simons forms

Consider a principal G-bundle $\pi : P \to M$ and a connection ∇ with curvature Ω. The Chern-Weil morphism maps $P \in \mathrm{Inv}(\mathfrak{g})$ to a cohomology class $[\Lambda]$ on M that satisfies the relation $P(\Omega) = \pi^*\Lambda$. The $2k$-form $P(\Omega)$ need not be exact. However, we can pull $P(\Omega)$ back through $\pi : P \to M$ to obtain a diagram

$$\begin{tikzcd} \pi^*P \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"] & M. \end{tikzcd}$$

The bundle $\pi^*P \to P$ has a canonical global section by using the diagonal on P, and is therefore trivial ([Tu17]). The trivial bundle π^*P therefore has a trivial classifying map, which in turn implies that its characteristic classes are all trivial, yielding exact characteristic forms.

Essentially, the form $\pi^*P(\Omega)$ on $\pi^*P \cong P \times G$ is $P(\Omega)$, which can be trivialized by some form $TP(\nabla)$ such that $dTP(\nabla) = P(\Omega)$. Details are found in [CS74]. The expression $TP(\nabla)$ could be read as "trivialization of P depending on the connection ∇".

If we set $\varphi_t = t\Omega + \frac{1}{2}(t^2 - t)[\nabla, \nabla]$, where $[.,.]$ denotes the Lie bracket of connections, the $(2k-1)$-form $TP(\nabla)$ can be defined as

$$TP(\nabla) = k \int_0^1 P(\nabla \wedge \varphi_t^{k-1})dt.$$

This trivializes $P(\Omega)$ uniquely (up to an exact term) by [CS74]. We will not delve into the above formula, the construction, or any of the details related to this, as it is quite technical in nature and not too relevant for our purposes.

It should however be noted that what we briefly mentioned in Section 3.6 was similar to a trivialization of the Chern character in the sense of [CS74]. We refer to [SS08b] and [CS74] for the missing details. The idea of seeking "the underlying geometric information" aligns quite well with our goal of finding a geometric model for Hodge-filtered K-theory.

Ordinary characteristic classes (e.g. Stiefel-Whitney classes) can often be thought of as obstructions to geometric-topological problems, such as if a n-manifold M can be embedded into $\mathbb{R}^{n+k}$ for some $k < n$.

In a similar manner, the trivializing forms above can be used to obstruct conformal immersions of 3-manifolds. We'll make it slightly more concrete. Smooth manifolds always admit Riemannian metrics, as these can be stapled together from local metrics due to partitions of unity. Two Riemannian metrics h and g on a smooth manifold M are conformally equivalent if we can find a function $\lambda : M \to \mathbb{R}$ such that $h = \lambda^2 g$. If g is a Riemannian metric on M and h is a Riemannian metric on $\mathbb{R}^k$, a conformal immersion of M is a smooth immersion $i : M \to \mathbb{R}^k$ such that i^*h is conformally equivalent to g.

Using secondary characteristic classes, [CS74] constructs a conformal invariant $\Phi(M)$ which obstructs conformal immersions of M into $\mathbb{R}^4$. They consider the case where $M = \mathbb{R}P^3$ and show that $\Phi(\mathbb{R}P^3) = \frac{1}{2}$. Hence, $\mathbb{R}P^3$ cannot be conformally immersed into $\mathbb{R}^4$, which is quite surprising as it can (at least) be smoothly immersed into $\mathbb{R}^4$.

Not only does this align quite well with the philosophy of differential and Hodge-filtered cohomology, as we are able to use underlying geometric information, but a central result of [CS74] is to show that these Chern-Simons forms are differential characters in the sense of [CS85]. This was one of the main motivations why differential characters were made, as there was a clear need of a framework to capture the "underlying geometric information".

5.1.2 Multiplicative K-theory

In the world of secondary characteristic classes, the Chern-Simons forms were alone for a long time. As the main (geometric) way of constructing charactersitic classes is through the Chern-Weil homomorphism, it is not too easy to construct new secondary characteristic classes. As a solution, Karoubi constructed a general framework for secondary characteristic classes, namely Multiplicative K-theory.

Multiplicative K-theory in the sense of Karoubi ([Kar90]) is naturally defined on $\mathbf{Man}_F$. Even though it can be phrased quite generally, allowing for flexibility in choice of a Lie group G, a series of "invariant polynomials" and a (descending) filtration, we will merely restrict ourselves to stable Multiplicative K-theory. That is, we follow [Kar90, Section 5.], where G is set to be $GL(k)$ for $k = \mathbb{R}$ or $\mathbb{C}$. The series of invariant polynomials is given by the Chern characters.

This leaves us with the following definition. Unless otherwise stated, assume that M is in $\mathbf{Man}_F$, and let $F^i = F^i\mathcal{A}^*(M)$.

Definition 5.1.1. (Multiplicative vector bundles).

A multiplicative vector bundle on M is a triple (E, ∇, ω), where E is a smooth vector bundle on M, ∇ is a connection on E and ω is a sum $\omega = \omega_1 + \omega_2 + \ldots$ of differential forms of odd degrees ($\omega_i \in \mathcal{A}^{2i-1}(M)$) such that $\mathrm{ch}_i(\nabla) - d\omega_i \in F^i$.

A morphism $(E, \nabla, \omega) \to (E', \nabla', \omega')$ of multiplicative bundles over M consists of an isomorphism $\varphi : E \to E'$ such that for all i,

$$\omega'_i - \omega_i = \int_{t=0}^{1} \mathrm{ch}_i((1-t)\nabla + t\varphi^*\nabla') \mod (F^i + dF^{i-1}).$$

Remark 5.1.2. Although phrased differently in [Kar90], the essence is as above, as we only require smoothness and a filtration on forms. Hence, working in $\mathbf{Man}_F$ is suitable.

To form a category of multiplicative vector bundles over M, $\mathbf{MBun}(M)$, we need to make sure composition provides new morphisms of multiplicative bundles. Assume we are given maps of multiplicative bundles

$$(E^0, \nabla^0, \omega^0) \xrightarrow{\varphi} (E^1, \nabla^1, \omega^1) \xrightarrow{\psi} (E^2, \nabla^2, \omega^2).$$

First, note that we have $E^0 \cong E^1 \cong E^2$. Hence, we will be sloppy denoting the pullbacks of connections. Let

$$\theta^1_r(\nabla, \nabla') = \int_{t=0}^{1} \mathrm{ch}_r((1-t)\nabla + t\varphi^*\nabla')$$

as above. To ensure that $\omega^2_r - \omega^1_r = \theta^1_r(\nabla^0, (\psi\varphi)^*\nabla^2) \mod (F^r + dF^{r-1})$ for all r, consider

$$\theta^2_r(\nabla^0, \nabla^1, \nabla^2) = \iint_{\Delta^2} \mathrm{ch}_r(t^0\nabla^0 + t^1\varphi^*\nabla^1 + t^2(\psi\varphi)^*\nabla^3).$$

By Stokes' theorem,

$$d\theta^2_r(\nabla^0, \nabla^1, \nabla^2) = \theta^1_r(\nabla^0, \nabla^1) + \theta^1_r(\nabla^1, \nabla^2) - \theta^1_r(\nabla^0, \nabla^2),$$

which means

$$\theta^1_r(\nabla^0, \nabla^2) = \omega^2_r - \omega^0_r - d\theta^2_r(\nabla^0, \nabla^1, \nabla^2) = \omega^2_r - \omega^0_r \mod (F^r + dF^{r-1}).$$

Hence, $\mathbf{MBun}(M)$ forms a category.

Remark 5.1.3. It is almost mandatory to remark, as done in [Kar90], how beautiful it is that we just used Stokes' theorem to show that something is a category. That is quite rare.

We may also form a Whitney-type sum of multiplicative bundles, namely

$$(E, \nabla, \omega) + (E', \nabla', \omega') = (E \oplus E', \nabla \oplus \nabla', \omega + \omega').$$

The definition of multiplicative vector bundles naturally gives us a definition of multiplicative K-theory.

Definition 5.1.4. (Multiplicative K-theory).

For a smooth manifold M, let $MK(M)$ denote the abelian group obtained by forming the Grothendieck group of the commutative monoid of (isomorphism classes of) multiplicative vector bundles on M. That is, $MK(M) = K(\mathbf{MBun}(M)/iso)$.

Remark 5.1.5. Note that by the classical properties of the Grothendieck construction (and Lemma 5.2 of [Kar90]), a class $[(E, \nabla, \omega)]$ is a formal difference $[E, \nabla, \omega] - [T, d, 0]$, for a trivial bundle T of dimension $\dim E$. This is often suppressed to writing $[E, \nabla, \omega]$.

Remark 5.1.6. By [Kar90], MK is homotopy invariant in the sense that $MK(M) \cong MK(M \times I)$. Karouibi also defines an algebraic notion of homotopies in which multiplicative bundles are homotopic if and only if they are isomorphic.

Remark 5.1.7. MK does not only define an abelian group, but also a commutative ring. By using the tensor product of bundles, we define

$$[(E, \nabla, \omega)] \otimes [(E', \nabla', \omega')] := [(E \otimes E', \nabla \otimes \nabla', \omega \wedge d\omega' + (\mathrm{ch}(\nabla) - d\omega) \wedge \omega' + \omega \wedge (\mathrm{ch}(\nabla') - d\omega')].$$

Karoubi also defines a set of Adams-type operations on MK, enhancing it to a λ-ring as well.

To define the higher multiplicative K-groups, we are motivated by the geometric suspension and consider the kernel of the map induced by $M \to M \times S^1$. This was discussed in Section 3.5.

Definition 5.1.8. (Higher Multiplicative K-groups).

For a smooth manifold M, the higher multiplicative K-groups are given by

$$MK^n(M) = \ker\left(MK(S^n \times M) \to MK(M)\right).$$

These sit in a long exact sequence involving hypercohomology and complex K-theory.

Theorem 5.1.9. *(Long exact sequence for multiplicative K-groups, [Kar90, Theorem 5.3 and Discussion 5.10.])*

The higher multiplicative K-groups sit in a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & K^{n-1}(M) \arrow[r] & \bigoplus_i H^{n-1+2i}(M; \frac{\mathcal{A}^*}{F^i}) \arrow[r] & MK^n(M) \arrow[dll, "I"'] \\
& K^n(M) \arrow[r] & \bigoplus_i H^{n+2i}(M; \frac{\mathcal{A}^*}{F^i}) \arrow[r] & \dots .
\end{tikzcd}$$

Essentially, the above sequence lets us see how multiplicative K-theory can work as a framework for secondary characteristic classes. For $n = 0$, this essentially yields [Kar90, Theorem C.2.], which relates the above to the long exact sequence in cohomology.

Theorem 5.1.10. *(Long exact sequence of MK and cohomology, [Kar90, Theorem C.2.]).*

Mapping $(E, \nabla, \omega) \mapsto \mathrm{ch}_i(\nabla) - d\omega_i$ for each i induces a map $MK(M) \to \bigoplus_i H^{2i}(M; F^i\mathcal{A}^*)$ such that the diagram

$$\begin{tikzcd}
K^1(M) \arrow[r] \arrow[d] & \bigoplus_i H^{2i-1}(M; \frac{\mathcal{A}^*}{F^i}) \arrow[r] \arrow[d, equal] & MK(M) \arrow[r] \arrow[d] & K^0(M) \arrow[r] \arrow[d] & \bigoplus_i H^{2i}(M; \frac{\mathcal{A}^*}{F^i}) \arrow[d, equal] \\
\bigoplus_i H^{2i-1}(M; \mathcal{A}^*) \arrow[r] & \bigoplus_i H^{2i-1}(M; \frac{\mathcal{A}^*}{F^i}) \arrow[r] & \bigoplus_i H^{2i}(M; F^i\mathcal{A}^*) \arrow[r] & \bigoplus_i H^{2i}(M; \mathcal{A}^*) \arrow[r] & \bigoplus_i H^{2i}(M; \frac{\mathcal{A}^*}{F^i}),
\end{tikzcd}$$

commutes. The maps from K-theory to cohomology are given by the Chern character.

As secondary characteristic classes are often constructed by moving the characteristic class to a trivial bundle (as done in [CS74]), they can be thought of as the classes of $MK(M)$ that map to 0 in $K^0(M)$, picking out the "underlying" geometric information that K^0 doesn't see. The kernel of $MK(M) \to K^0(M)$ lives in $MK(M)$, and by the above exactness, this corresponds to the image of $\bigoplus_i H^{2i-1}(M; \frac{\mathcal{A}^*}{F^i})$. More explicitly, any class of $MK(M)$ causing a 0 in $K^0(M)$ should come from $\bigoplus_i H^{2i-1}(M; \frac{\mathcal{A}^*}{F^i}) / \operatorname{im}(K^1(M))$, which is referred to as the secondary characteristic classes in [Kar90]. Karoubi refers to the classes in

$$\operatorname{im}\left(MK(M) \to \bigoplus_i H^{2i}(M; F^i\mathcal{A}^*)\right) \subseteq \bigoplus_i H^{2i}(M; F^i\mathcal{A}^*)$$

as the primary classes.

The above philosophy was used to construct new secondary characteristic classes in [Kar94].

5.2 Geometric Hodge-filtered K-theory

Note that by setting $\mathcal{V}_* = K^*(*) \otimes_{\mathbb{Z}} \mathbb{C}$, we can rewrite the long exact sequence in multiplicative K-theory.

Theorem 5.2.1. *(Long exact sequence for multiplicative K-theory, revisited).*

The higher multiplicative K-groups sit in a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1}(M) \arrow[r] & H^{n-1}(M; \frac{\mathcal{A}^*}{F^0}(\mathcal{V}_*)) \arrow[r] & MK^n(M) \arrow[dll, "I"'] \\
& K^n(M) \arrow[r] & H^n(M; \frac{\mathcal{A}^*}{F^0}(\mathcal{V}_*)) \arrow[r] & \cdots .
\end{tikzcd}$$

This hints clearly that $MK^n(M)$ should be a Hodge-filtered K-theory with $p = 0$.

As the idea is to connect multiplicative K-theory to Hodge-filtered K-theory, we will cherry-pick the relevant constructions and theorems of [Kar90]. To align multiplicative K-theory with the axiomatics of Hodge-filtered cohomology, we slightly modify the constructions later, e.g. to account for the integer p floating around.

5.2.1 Axioms of $K^*_{\mathcal{D}}(0)$ and Multiplicative K-theory

First of all, we start by assuming $p = 0$. As mentioned by [HQ15], we should not expect $K_{\mathcal{D}}(0)$ to return K-theory when working with Hopkins-Quick theories. In light of a potential uniqueness theorem, we conjecture that $K_{\mathcal{D}}(0)$ actually becomes multiplicative K-theory. However, this statement requires us to show that MK^* gives a Hodge-filtered K-theory.

Definition 5.2.2. (The maps $c(0), R, I$ and a for MK^n).

We define $c(0)$ for all degrees as below. R, I and a are first defined in degree 0, before we extend them to all degrees. The maps are defined as

$$\begin{aligned} c(0) : K^0(M) &\longrightarrow H^*(M; \mathcal{A}(\mathcal{V}_*)), \\ [E] &\longmapsto [\mathrm{ch}(E)], \end{aligned}$$

$$\begin{aligned} R : MK(M) &\longrightarrow H^0(M; F^0\mathcal{A}(\mathcal{V}_*)), \\ [E, \nabla, \omega] &\longmapsto [(\mathrm{ch}(\nabla) - d\omega)], \end{aligned}$$

$$\begin{aligned} I : MK(M) &\longrightarrow K^0(M), \\ [E, \nabla, \omega] &\longmapsto [E], \end{aligned}$$

$$\begin{aligned} a : H^{-1}(M; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) &\longrightarrow MK(M), \\ \omega &\longmapsto [T, d, -\omega]. \end{aligned}$$

To extend R, I, and a to degree n, let $i : M \to M \times S^n$ be the inclusion. The maps R^n, I^n, a^n in degree n are induced by considering the kernels, as in the diagrams

$$\begin{tikzcd} K^0(M) & MK(M) \arrow[l, "I"'] \arrow[r, "R"] & H^0(M; F^0\mathcal{A}^*(\mathcal{V}_*)) \\ K^0(M \times S^n) \arrow[u, "i_K^*"] & MK(M \times S^n) \arrow[l, "I"'] \arrow[r, "R"] \arrow[u] & H^0(M \times S^n; F^0\mathcal{A}^*(\mathcal{V}_*)) \arrow[u, "i_H^*"'] \\ K^n(M) \arrow[u] & MK^n(M) \arrow[l, dashed, "I^n"'] \arrow[r, dashed, "R^n"] \arrow[u] & H^n(M; F^0\mathcal{A}^*(\mathcal{V}_*)) \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd}
H^{n-1}(M;\frac{\mathcal{A}^*}{F^0}(\mathcal{V}_*)) \arrow[r] \arrow[d, dashed, "a^n"] & H^{-1}(M\times S^n;\frac{\mathcal{A}^*}{F^0}(\mathcal{V}_*)) \arrow[r, "i^*_{H'}"] \arrow[d, "a"] & H^{-1}(M;\frac{\mathcal{A}^*}{F^0}(\mathcal{V}_*)) \arrow[d, "a"] \\
MK^n(M) \arrow[r] & MK(M\times S^n) \arrow[r] & MK(M).
\end{tikzcd}$$

The maps will be denoted by a, I and R, without referring to the degree.

The kernels of of i^*_H and $i^*_{H'}$ are the cohomology groups above by the geometric suspension (Proposition 3.5.1) as the cohomology groups are homotopy invariant and are represented by certain spectrum objects (see Remark 4.2.4 and [HQ15, Proposition 2.12.]). Due to the setup above, the axioms for higher degrees will be induced from the case $n = 0$.

We start by checking if $R \circ a = d$. We have

$$H^{-1}(M;\tfrac{\mathcal{A}^*}{F^0}(\mathcal{V}_*)) \xrightarrow{a} MK(M) \xrightarrow{R} H^0(M;F^0\mathcal{A}(\mathcal{V}_*)),$$

$$[\omega] \xmapsto{\quad a \quad} [T,d,-\omega] \xmapsto{R} [(\mathrm{ch}(d)+d\omega)] = [d\omega],$$

which by Remark 4.3.5 is what we want. Above, we have used that $[\mathrm{ch}(d)] = [\mathrm{ch}(T)] = [0]$, as T is a trivial bundle. Secondly, consider the following diagram,

$$\begin{tikzcd}
{[E,\nabla,\omega]} \arrow[rrr, mapsto, "I"] \arrow[ddd, mapsto, "R"] & & & {[E]} \arrow[dd, mapsto, "c(0)"] \\
& MK(M) \arrow[r, "I"] \arrow[d, "R"'] & K^0(M) \arrow[d, "c(0)"] & \\
& H^0(M;F^0\mathcal{A}(\mathcal{V}_*)) \arrow[r, "inc"'] & H^0(M;\mathcal{A}(\mathcal{V}_*)) & {[\mathrm{ch}(E)]} \arrow[dl, equal, "\text{ch indep. of connection}"] \\
{[(\mathrm{ch}(\nabla)-d\omega)]} \arrow[rr, mapsto, "\text{since } [d\omega]=0"] & & {[\mathrm{ch}(\nabla)]}. &
\end{tikzcd}$$

This diagram commutes as characteristic classes are independent of choice of connection. Lastly, the long exact sequence is exact as this is precisely Theorem 5.1.9 (with commuting triangle implied from Theorem 5.1.10).

We summarize this as a theorem.

Theorem 5.2.3. *(Multiplicative K-theory is a Hodge-filtered $K(0)$-theory).*

With the maps $c(0), a, R,$ and I defined above, Multiplicative K-theory is a Hodge-filtered $K(0)$-theory in the sense of Definition 4.3.1 with $p = 0$.

5.2.2 Extending the result to $MK^*(p)$

To extend Theorem 5.2.3 from $p = 0$ to arbitrary p, we can simply introduce a shift in the filtration.

Definition 5.2.4. (p-shifted filtration).

If F is a filtration, we can define the p-shifted filtration of F, denoted $F(p)$, by

$$F(p)^i = F^{p+i}.$$

As remarked earlier, multiplicative K-theory does not depend on any specific filtration. We may therefore include a p into the above theorem by considering multiplicative K-theory with respect to the filtration $F(p)$ instead of F. If we fix a descending filtration F of $\mathcal{A}^*$ (e.g. the Hodge-filtration), we may define p-shifted multiplicative K-theory.

Definition 5.2.5. (p-shifted multiplicative K-theory).

We define $MK(p)(M)$ to be the multiplicative K-groups with respect to the filtration $F(p)$.

That is, we consider (p-shifted) multiplicative bundles on M, consisting of triples (E, ∇, ω), where E is a smooth vector bundle on M, ∇ is a connection on E, and ω is a sum $\omega = \omega_1 + \omega_2 + \ldots$ of differential forms of odd degrees ($\omega_i \in \mathcal{A}^{2i-1}(M)$) such that $\mathrm{ch}_i(\nabla) - d\omega_i \in F(p)^i$. If (E, ∇, ω) and (E', ∇', ω') are two multiplicative vector bundles, a morphism $(E, \nabla, \omega) \to (E', \nabla', \omega')$ consists if an isomorphism $\varphi : E \to E'$ such that for all i,

$$\omega_i' - \omega_i = \int_{t=0}^{1} \mathrm{ch}_i((1-t)\nabla + t\varphi^*\nabla') \mod (F(p)^i + dF(p)^{i-1}).$$

$MK(p)(M)$ is defined to be the Grothendieck group of isomorphism classes of (p-shifted) multiplicative bundles on M.

Remark 5.2.6. Going from MK to $MK(p)$ does not change anything about the construction of MK^n or the results we know about multiplicative K-theory, as the work done in [Kar90] simply works with "a filtration". The p comes from defining MK with respect to a fixed filtration, and then considering $MK(p)$ to arise from a p-shift of the fixed filtration.

The maps R, I, and a are defined similarly, and we set $c(p) = c(0)$, as there is no shift needed in the Chern character.

Corollary 5.2.6.1. *(Geometric Hodge-filtered K-theory).*

p-shifted multiplicative K-theory in the sense of Definition 5.2.5 is a Hodge-filtered $K(p)$-theory.

More simply stated:

Multiplicative K-theory is a Hodge-filtered K-theory.

5.3 Open questions and further thoughts on Hodge-filtered K-theory

Uniqueness of Hodge-filtered K-theory

In this chapter, we have found a geometric model for Hodge-filtered K-theory. However, we can only say multiplicative K-theory is *a* Hodge-filtered K-theory, and not that it *is* Hodge-filtered K-theory, due to the lack of a uniqueness theorem. Having a geometric model is still a triumph, but it would be stronger if it was a geometric way of viewing *the* Hodge-filtered K-theory. In general, it does not seem possible to transfer the proof of Theorem 3.2.9 to Hodge-filtered cohomology theories, although there may still be some hope to prove the uniqueness of Hodge-filtered K-theory through this method. All the thoughts presented on the uniqueness of Hodge-filtered theories in this thesis are also found in [Hau22, Section 3.5]. A natural next project would be to study uniqueness further. More concretely, it would be interesting to see how the explicitly constructed isomorphism between geometric Hodge-filtered complex cobordism and Hopkins-Quick complex cobordism in [Hau22] translates to K-theory.

Hodge-filtered K-theory from differential K-theory

As in Section 4.4, we know that we may obtain a Hodge-filtered K-theory from differential K-theory.

Following [Hau22], we let $\mathcal{V}_{*,\mathbb{C}} := \mathcal{V}_* \otimes_{\mathbb{R}} \mathbb{C}$ and define the complexified differential K-theory $\hat{K}^n_{\mathbb{C}}$ as

$$\hat{K}^n_{\mathbb{C}}(M) = \{(\hat{x},\phi) \in \hat{K}^n(M)\times\Omega^{n-1}(M;\mathcal{V}_{*,\mathbb{C}}) \quad | \quad (\hat{x}+a(\omega),\phi) = (\hat{x},\phi+\omega), \quad \forall\omega \in \Omega^{n-1}(M;\mathcal{V}_*)\},$$

where Ω^* denotes the smooth de Rham complex, and not the holomorphic de Rham complex. We identify $\Omega^{n-1}(M;\mathcal{V}_*)$ with a subspace of $\Omega^{n-1}(M;\mathcal{V}_{*,\mathbb{C}})$ through the natural map of coefficients.

As in Section 4.4, we set

$$R_{\mathbb{C}}(\hat{x},\phi) = R(\hat{x}) + d\phi, \quad I_{\mathbb{C}}(\hat{x},\phi) = I(\hat{x}), \quad a_{\mathbb{C}}(\omega) = (0,\omega).$$

Assume now that we are given an integer p, and let M be a complex manifold. Since M has the underlying structure of a smooth manifold, define $F^p\hat{K}^*(M) \subseteq \hat{K}^*(M)$ as

$$F^p\hat{K}^*(M) = R^{-1}(F^p\mathcal{A}^*(M;\mathcal{V}_*)).$$

To ensure that the maps R, I and a stay well-defined, we set

$$\hat{K}^*_{\mathcal{D}}(p)(M) := F^p\hat{K}^*(M)/a(F^p\mathcal{A}^{*-1}(M;\mathcal{V}_*) + d\mathcal{A}^{*-2}(M;\mathcal{V}_*)).$$

With the above definition, the maps R, I and a of complexified differential K-theory induce the maps needed for $\hat{K}^*_{\mathcal{D}}(p)$ to be a Hodge-filtered K-theory.

It would be interesting to understand exactly how this structure works. For example, can we obtain a Bott periodicity theorem from the differential Bott periodicity of Theorem 2.10.5? It is not clear to us how we do this, due to the complexified differential K-theory

$$\hat{K}^n_{\mathbb{C}}(M) = \{(\hat{x}, \phi) \in \hat{K}^n(M) \times \Omega^{n-1}(M; \mathcal{V}_{*,\mathbb{C}}) \quad | \quad (\hat{x}+a(\omega), \phi) = (\hat{x}, \phi+\omega), \quad \forall \omega \in \Omega^{n-1}(M; \mathcal{V}_*)\}$$

having an n in $\Omega^{n-1}(M; \mathcal{V}_{*,\mathbb{C}})$ as well. The simple replacement $n \mapsto n+2$ does not immediately yield Bott periodicity.

Additionally, we have not yet written out what this construction does to the structured bundles of [SS08b] (Section 3.6). We believe that we retrieve multiplicative K-theory from the above procedure, although this is not something we have investigated. This would support the uniqueness of Hodge-filtered cohomology theories, at least for the case of complex K-theory.

It could be interesting to see how the constructions of [SS08b] relate to holomorphic bundles and complex structures, in a wide sense. We suspect that the paper [BC70] could be of interest when investigating this questions, as it is related to the work found in [CS74] (and hence [SS08b]) for the holomorphic case.

On Bott periodicity

We indeed suspect to have a Bott periodicity theorem for Hodge-filtered K-theory as well. However, we have not been able to prove this. Our most serious attempt at this was to prove Bott periodicity for Hopkins-Quick theories, where we attempted to prove that $\Omega^2 K_{\mathcal{D}}(p) \simeq K_{\mathcal{D}}(p)$ in $ho\,\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$. During the proofreading of this thesis, we found vital flaws to the proof, as we had been too naive when switching back and forth between $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$ and $ho\,\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$.

We have not found any sources that mention Bott periodicity for multiplicative K-theory, but this does not bother us too much, as it is indeed difficult to prove such a theorem directly for a geometric model. If we manage to prove a (model dependent) Bott periodicity theorem, we suspect it to be through differential K-theory, as discussed above, or by using the abstract machinery of Hopkins-Quick theories.

Hodge-filtered Conner-Floyd K-theory

In the light of Section 4.6, we hope that we can realize a Hodge-filtered Conner-Floyd-type K-theory as $K^*_{\mathcal{D}}(p)(-) \cong MU^*_{\mathcal{D}}(p)(-) \otimes_L K^*(*)$. This would possibly let us translate many results from [Hau22] and [HQ22] to the case of Hodge-filtered K-theory. Hopefully, we are able to prove Conjecture 4.6.2 in the future.

Universal computations of Hodge-filtered K-groups

The long exact sequence of the axioms (Definition 4.3.1) opens for doing computations that will hold true for all Hodge-filtered K-theories, independently of a result on uniqueness. Unfortunately, we have not had the time to attempt these computations, but we suspect it to be possible to compute some groups for simple complex manifolds, such as S^2.

Mayer-Vietoris sequence for Hopkins-Quick K-theory

In addition to the long exact sequence aligning with the axioms of Definition 4.3.1, Hopkins-Quick establishes a Mayer-Vietoris sequence for their Hodge-filtered cohomology theories in [HQ15]. We state the result for complex K-theory.

Theorem 5.3.1. *(Mayer-Vietoris sequence of Hopkins-Quick K-theory).*

For a complex manifold M, its Hodge-filtered K-groups (in the sense of Hopkins-Quick) sits in a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}(M;\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] & K^n_{\mathcal{D}}(p)(M) \arrow[r] & K^n(M)\oplus H^n(M;\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) \arrow[dll, out=-10, in=170] \\ & H^n(M;\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] & K^{n+1}_{\mathcal{D}}(p)(M) \arrow[r] & \cdots \end{tikzcd}$$

Proof. The idea is to use a Puppe-type sequence coming from $K_{\mathcal{D}}(p)$ as a homotopy pullback and determine the hypercohomology groups. Details are in [HQ15]. $\square$

If universal computations for Hodge-filtered K-theory as explained above is too difficult, it would be interesting to use this Mayer-Vietoris sequence to compute simple Hopkins-Quick K-groups.

Other

There are many interesting topics we could study related to Hodge-filtered K-theory, many of which we have not pursued in this thesis. However, we choose to mention some of them, as they certainly deserve a mention, if only to show potential topics of study in the future.

K-theoretic Abel-Jacobi maps

As mentioned in Section 4.7, much of the original motivation in [HQ15] was to construct generalized Abel-Jacobi maps and generalized fundamental short exact sequences. As these were constructed for all differential cohomology theories E, it could be interesting to see what happens when we specialize to Hodge-filtered K-theory.

Spectral sequences for Hodge-filtered cohomology theories

In [GS17], the differentially refined Atiyah-Hirzebruch spectral sequence was constructed. The construction is mostly based on constructing a suitable exact couple. We have not studied the paper enough to say what would go wrong by passing to Hodge-filtered cohomology theories, but it could be of great interest to try to translate the proof and find a Hodge-filtered Atiyah-Hirzebruch spectral sequence.

Equivariant Hodge-filtered K-theory

It would be interesting to construct equivariant Hodge-filtered K-theory. A possible approach would be to ask how the axioms need to change to comply with the group action of a group G. This must be done in a way that retrieves the ordinary axioms when faced with the trivial group. Due to the simple nature of Multiplicative K-theory, we suspect there to be an equivariant version of multiplicative K-theory that satisfies the (not yet developed) axioms of equivariant Hodge-filtered K-theory.

5.4 Summary and Last Thoughts

In this thesis, we have covered much material, but have we been able to answer our initial question? We asked how it is to refine cohomology on smooth manifolds to include "geometric information", if possible at all.

Simply speaking, we started with the de Rham theorem and discovered that it should be possible to find a construction $\hat{H}^k(-;\mathbb{Z})$ fitting into the middle of the hexagon diagram. Even though our idea was slightly too naive, the differential characters gave us as a solution, and there were even interesting interpretations of the differential character groups that showed how geometric info was "added".

We generalized differential characters further, eventually enabling the construction of differential E-cohomology for some spectrum E. This answered our initial question, not only for ordinary cohomology, but also for E-cohomology. We *can* refine cohomology to include geometric information.

We also studied differential cohomology through spectral sheaves. Even though we "just" considered the ∞-category of spectral sheaves on manifolds, we ended up seeing that homotopy invariant sheaves corresponded to spectra, and that it was possible to decompose the ∞-category into homotopy invariant sheaves and pure sheaves through stable recollements. This gave another approach to differential cohomology, although a bit too general to fit into the axiomatic framework.

The axiomatic framework was used to answer our final questions regarding uniqueness and how to quantify the homotopy invariance of a differential cohomology theory through a homotopy formula. Furthermore, we saw that smooth Deligne cohomology was ordinary differential cohomology, that structured bundles could be used to define differential K-theory, and that given a Landweber exact formal group law (S, g), the cohomology theory $S^*(-) = MU^*(-) \otimes_L S$ could be refined to a model for differential S-theory, by using differential complex cobordism to form $\hat{S}^*(-) = M\hat{U}^*(-) \otimes_L S$.

At this point, we had to make a choice. Do we continue study differential cohomology, or should we be inspired by Deligne cohomology to study an analog of differential cohomology for complex manifolds. As all of our initial questions were answered, we moved to Hodge-filtered cohomology theories. The existence of Hodge-filtered E-theory for a spectrum E was constructed similarly to differential cohomology, although using much heavier machinery.

Even though an axiomatic framework exists, we met a wall quite quickly. Many of the ideas we had met through our discussion of differential cohomology, are not yet transferred to Hodge-filtered cohomology, apart from the homotopy formula and the fact that Deligne cohomology models ordinary Hodge-filtered cohomology. For example, Hodge-filtered cohomology theories do not yet have a uniqueness theorem. We discussed the differences between differential cohomology and Hodge-filtered cohomology. The subtle differences prevented us from proving that given a Landweber exact formal group law (S, g), the cohomology theory $S^*(-) = MU^*(-) \otimes_L S$ produces a Hodge-filtered S-theory, by using Hodge-filtered complex cobordism as above. This discussion resulted in a conjecture.

As ordinary Hodge-filtered cohomology is already modeled by Deligne cohomology, we specialized the discussion to Hodge-filtered K-theory and managed to find a geometric model for Hodge-filtered K-theory through multiplicative K-theory. Furthermore, we discussed several other approaches and open problems related to Hodge-filtered cohomology and K-theory.

If our primary goal was to study Hodge-filtered K-theory, this thesis is definitely way too long. However, as the intention was to showcase and collect important and interesting ideas from differential cohomology, we should rather criticize the content of the thesis. Summarizing a research field is an herculean task, and we have omitted much material. In retrospect, it would be natural to include more information of differential- and Hodge-filtered complex cobordism, or more of the motivation for Hodge-filtered cohomology theories stemming from complex algebraic geometry.

We hope that this thesis contributes to the fields of differential- and Hodge-filtered cohomology not just by finding a geometric model of Hodge-filtered K-theory, but also by providing an alternative text to the (few) existing sources providing an overview of the fields, and perhaps by addressing interesting questions.

This thesis was written with the philosophy that it should be accessible and interesting to the author a year ago. Luckily, it is not possible to evaluate if we reached this goal, although we believe the author (a year ago) would greatly appreciate the work done here. Hopefully, this thesis will help students of differential- and Hodge-filtered cohomology reach a bit further in the future.

We end with a (surprisingly relevant) poem taken from [Ang21].

A small poem

Now my thesis has started to merge
And the timing is right on the verge
But I have delivered
You can say I have shivered
Because the number of pages diverge

- *The author.*

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